Option pricing

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1 Binomial model

We assume the stock price at time t is S_t. The probability that the stock prices rises is p and when it rises it will rise to uS. The probability that the stock price drops is $1-p$ and when it drops it will drop to $\frac{1}{u}S$. We will also assume that there is a risk-free asset with growth rate R. The question is how to set the price for such a stock.

It might be tempting to price the stock by its (discounted) expected value at time T. But the following paradox shows this is not the correct way. Consider a game with a fair coin and the gambler will be paid \$$2^n$ if head shows up at the n^{th} coin toss. It is easy to calculate that the expected return is

$$E = \sum_{i=1}^{\infty} 2^n \times \frac{1}{2^n} \sim \infty.$$

If we use E to price the stock (in this case, the game), then we will be asking an infinity amount of money for anyone who wants to join which is apparently absurd.

To resolve this paradox Daniel Bernouilli purposed a so-called utility function $u(x)$ which is monotonically increasing and concave. Therefore we have

$$E = \sum_{i=1}^{\infty} u(2^n) \frac{1}{2^n}.$$

In this case a suitable choice of $u(x)$ could be the logarithmic function, hence

$$E = \sum_{i=1}^{\infty} \frac{\log(2^n)}{2^n} < \infty.$$

Therefore to price the stock, it seems that we could define a suitable utility function and apply it upon each outcome in the calculation of the expectation E. We will soon see there is a much simpler way to solve this problem without specifying the exact form of $u(x)$.

1.1 One-period binomial model

Let us consider a one-period binomial model. We assume that the initial price of stock is S_0 and the probability of S_0 rising to uS_0 is p and the probability of S_0 dropping to dS_0 is $1-p$ where $d = 1/u$. We also assume that there is a risk-free rate $R-1$ and short-sales are allowed. Finally we assume that there is no transaction fee.

First let us notice that $d < R < u$ otherwise it violates the no-arbitrage hypothesis. Now we consider the price of an option that pays $\max(S_1 - C, 0)$ where $dS_0 < C < uS_0$. Here C is the strike. We will simultaneously buy x shares of stock and invest y amount of money in the risk-free account. Therefore at $t = 1$ the portfolio is worth

$$\begin{aligned} &uS_0x + Ry, \text{ when } S_1 = uS_0, \\ &dS_0x + Ry, \text{ when } S_1 = d_0. \end{aligned}$$

We will choose x and y such that the portfolio equals the payoff at $t = 1$. Therefore we have

$$\begin{cases} uS_0x + Ry = uS_0 - C \\ dS_0x + Ry = 0 \end{cases}$$

Therefore we have

$$\begin{cases} x = \frac{uS_0 - C}{(u-d)S_0} \\ y = -\frac{d(uS_0 - C)}{R(u-d)} \end{cases}$$

Here we have constructed a so-called *replicating portfolio*. If $x < 0$ it means we are short-selling the stock and if $y < 0$ it means that we are borrowing money. Therefore using the replicating portfolio the fair price of the option is

$$P = S_0x + y = \left(1 - \frac{d}{R}\right) \frac{uS_0 - C}{u - d}.$$

More generally we can assume that a derivative of the stock pays off C_u when the price rises and C_d when the price drops. In this situation we have the following set of equations

$$\begin{cases} uS_0x + Ry = C_u \\ dS_0x + Ry = C_d \end{cases}$$

whose solution will give the fair price of the derivative to be

$$P = S_0x + y.$$

In fact we have

$$\begin{aligned} P = & \frac{1}{R}\left(\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right) \\ = & \frac{1}{R}\left(qC_u + (1-q)C_d\right) \\ = & \frac{1}{R}\mathbb{E}^q[C_1]. \end{aligned}$$

We will call q the risk-neutral probability and P is called risk-neutral price.

Obviously we see that P is independent of p and this conclusion might seem quite counterintuitive. We will discuss this issue in a moment.

1.2 Multi-period binomial model

We will view the multi-period binomial model as a composition of several one-period binomial models. Consider the diagram in Figure 1 which describes a 3-period binomial model. Note that $ud = 1$.

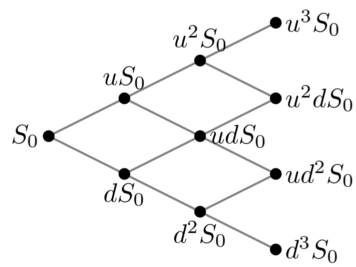

Figure 1: A 3-period binomial model.

Consider the following subprocess shown in Figure 2.

The subprocess described in Figure 2 is a 1-period binomial model. The fair price of the derivative at $t = 1$ when the stock price is uS_0 is $\frac{1}{R}\mathbb{E}^Q[C_1]$ as we have learned from Section 1.1. Therefore by simple induction, the fair price of the derivative for this 3-period binomial model will be

$$P = \frac{1}{R^3}\mathbb{E}^q[C_3].$$

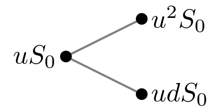

Figure 2: A subprocess of the 3-period binomial model.

Suppose the strike is S, we have

$$P = \frac{1}{R^3}\mathbb{E}^q[\max(S_3 - S, 0)].$$

For a general T-period binomial model with strike S we have

$$P = \frac{1}{R^T}\mathbb{E}^q[\max(S_T - S, 0)].$$

In the above formula the risk-neutral probability of each event is $C_T^k q^{T-k}(1-q)^k$ where events are ordered from top to bottom as in Figure 1.

1.3 The replicating portfolio of a multi-period model

We consider a time-ordered set of portfolios $\theta_t = (x_t, y_t)$ which is held between time t and $t+1$. We will again consider the process in Figure 1 as a concrete example. We define the value process $V(\theta)$ to be

$$V_t = \begin{cases} x_1S_0 + y_1B_0, \ t = 1 \\ x_tS_t + y_tB_t, \ t \geq 1 \end{cases}$$

1.4 Some comments

Consider the following two models shown in Figure 3.

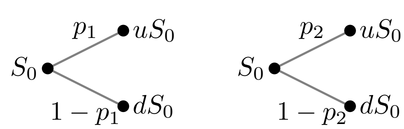

Figure 3: Two 1-period binomial models with same u, d, R and S_0 but different probabilities of price rising or dropping.

From the calculation in Section 1.1 we know that the fair prices of the above two derivatives are the same as the risk-neutral probability is determined solely by u, d and R. Though intuitively one would think that the price of the derivative with a larger p will have a higher price. This is not mysterious at all if we notice that we are not balancing the events with different probabilities at all. What we have done is to construct a portfolio with both stock and cash that effectively reproduces the outcome of holding an option and in constructing this replicating portfolio nowhere have we ever used the physical probability p.

Another seemingly controversial point is that since usually $C_d = 0$, we have

$$P = \frac{1}{R}\frac{R-d}{u-d}C_u$$

for the 1-period binomial model. It is easily seen that when R increases, the call option price P increases as well. This is a bit counterintuitive as well since in general one would expect a cash flow in the future will be discounted more with a higher risk-free rate R. The point is that for a fixed strike S at time T, a higher risk-free rate R will lead to a lower discounted value for S at time 0 henceforth a lower buy price which is certainly beneficial therefore leads to a higher call option price.

Finally we want to discuss an important point, that is the equivalent between no-arbitrage and the existence of risk-neutral probability. Recall that the derivative can be priced by

$$P = \frac{1}{R^T}\mathbb{E}^q[C_i]. \tag{1}$$

Suppose that $C_i = C(w_i)$ where w_i is a state in which the system can be at time T. We will further assume that $C_i \geq 0$, $\forall i$ and there exists q_i such that Eq. 1 holds. It is straightforward to show that $\mathbb{E}^q[C_i] \geq 0$. Therefore there is not room for any type of arbitrage. The reverse is also true, i.e., if there is no arbitrage then there exists a risk-neutral probability.

First fundamental theorem of assent pricing There is no arbitrage iff there exists a risk-neutral probability.

2 The price of an American option

Recall that it never optimal to early exercise an American option therefore we consider an American put option. For example we will consider the following 3-period binomial model as shown in Figure 4.

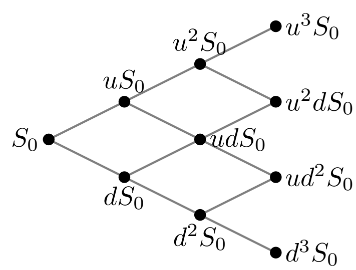

Figure 4: A 3-period binomial put model.

Recall that the payoff of a put option is $\max(S - S_T, 0)$ where S is the strike. Now we will work backwards and check at each node if an early exercise is optimal. Now consider the subprocess shown in Figure 5.

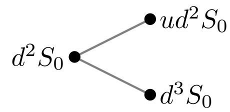

Figure 5: A subprocess of the 3-period binomial model.

We have two choices, that is, we could either continue or exercise at $t = 2$. Therefore for this subprocess we have

$$P_s = \max(S - d^2S_0, \frac{1}{R}(q(S - ud^2S_0) + (1-q)(S - d^3S_0)))$$

where we suppose that $S > ud^2S_0$. Therefore we could work backwards in this manner for each node and that leads to a fair value P at the very first node at $t = 0$.

2.1 Optimal stopping strategy: an example

In this subsection we discuss a classical game. Suppose there is fair die and we could roll it for at most three times. We could choose to stop the game after any roll and earn an \$$x$ reward where x is the number of the roll.

We can consider this problem backwards. Suppose there is one roll left, then apparently the expectation value is $E_1 = \frac{1}{6}(1+2+3+4+5+6) = 3.5$. Now let us consider the case where there are two rolls left. Since when there is one roll left the expectation is 3.5, then if the second last roll turns out to be smaller than 4 we will definitely choose to roll one more time. Therefore the total expectation value is

$$E_2 = \frac{1}{6}(4+5+6) + \frac{1}{2}E_1 = 4.25.$$

Now consider when there are three rolls left. Since the expectation value when there are two rolls left is 4.25, we will definitely choose to continue to roll if the third to last roll is smaller than 5. Therefor the expectation value is

$$E_3 = \frac{1}{6}(5+6) + \frac{2}{3}E_2 \sim 4.667.$$

Similarly if we are allowed to roll at most four times we have

$$E_4 = \frac{1}{6}(5+6) + \frac{2}{3}E_3 \sim 4.94$$

and if we are allowed to roll at most five times we have

$$E_5 = \frac{1}{6}(5+6) + \frac{2}{3}E_4 \sim 5.13.$$

From here we can proceed to have

$$E_n = \frac{1}{6} \times 6 + \frac{5}{6} E_{n-1} = 1 + \frac{5}{6} \left(1 + \frac{5}{6} E_{n-2}\right) = \dots$$

We have

$$E_n = 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \dots + \left(\frac{5}{6}\right)^{n-6} + \left(\frac{5}{6}\right)^{n-5} E_5.$$

As $n \to \infty$ we have

$$E_\infty = 6$$

which it certainly a sensible result.

Here we use this example to illustrate the importance and the flavor a set of problems called optimal stopping problems and finding the best strategy of when to exercise an American option falls into this category.

3 Option with dividends

We will consider the following 1-period binomial model shown in Figure 6.

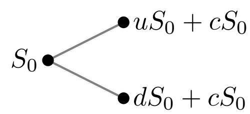

Figure 6: A 1-period binomial model with dividends. cS_0 is the dividend to be paid to the shareholders.

For the above process the no-arbitrage condition is $d + c < R < u + c$ where R is the risk-free rate. We could construct the following replicating portfolio

$$\begin{cases} (uS_0 + cS_0)x + Ry = C_u \\ (dS_0 + cS_0)x + Ry = C_d \end{cases}$$

Here we will find again

$$P = \frac{1}{R} \mathbb{E}^q [C_1]$$

with $q = \frac{R-d-c}{u-d}$.

4 Black-Scholes formula

We assume that the stock price follows a geometric Brownian motion

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

where W_t is a Brownian motion. The stock pays a dividend c and there is no transaction fee and short-selling is allowed.

The Black-Scholes formula for the price of European call option with strike K and maturity T is given by

$$C_0 = S_0 e^{-cT} N(d_1) - K e^{-rT} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\log(S_0/K) + (r - c + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

and $N(d) = P(N(0,1) \leq d)$. The European put option price can then be computed via the put-call parity

$$P_0 + S_0 e^{-cT} = C_0 + K e^{-rT}.$$