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Chern-Simons Theory

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Abstract: Note on Chern-Simons theory.

Contents

This is the note written by the author of Witten's lecture on Chern-Simons theory at IAS in September, 2024 [1]. All possible errors and typos are of course by the author of this note.

1 3D TFT without impurities

Chern-Simons theory is a topological field theory, which is a theory defined on a manifold M with on Riemannian metric. We assume that M is oriented. There are other interesting topological properties like spin, framing, etc. The most interesting case for us is $D = 3$, hence the space dimension is 2.

Let us quantize the theory on a spatial manifold Σ, as illustrated in Figure 1. We

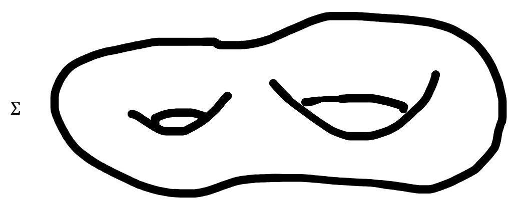

Figure 1. A spatial manifold Σ.

associated to Σ a Hilbert space $\mathcal{H}_\Sigma$ obtained by quantizing the field theory on Σ.

Let us ask how to prepare a state in a general QFT. We will treat Σ as the *future boundary* of a spacetime manifold M as illustrated in Figure 2. We then do a path integral on M as a function of boundary values on Σ. More precisely, we integrate over all the fields on M while keeping the boundary values of these fields fixed:

$$\Psi(\Phi_\Sigma) := \int_{\Phi_M|_\Sigma = \Phi_\Sigma} \mathcal{D}\Phi_M \; e^{-I(\Phi_M)} \in \mathcal{H}_\Sigma. \tag{1.1}$$

In general one can think of the path integral in one-dimensional higher as a way to prepare a state in the Hilbert space.

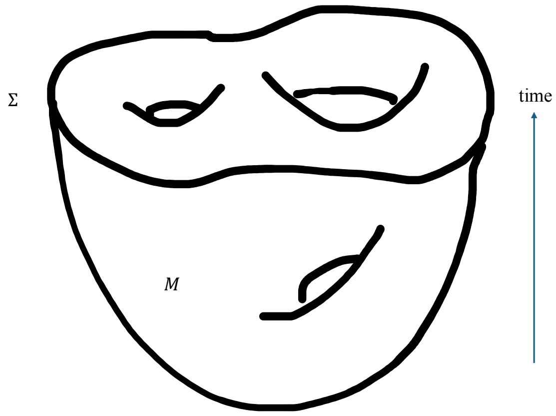

Figure 2. The spatial manifold Σ as the future boundary of M.

We then ask what is a transition amplitude, i.e. what operator can one define to map a state from $\mathcal{H}_{\Sigma_1}$ to $\mathcal{H}_{\Sigma_2}$:

$$\Theta_M : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}. \tag{1.2}$$

We need to find an M with past boundary Σ_1 and future boundary Σ_2 as in Figure 3 The

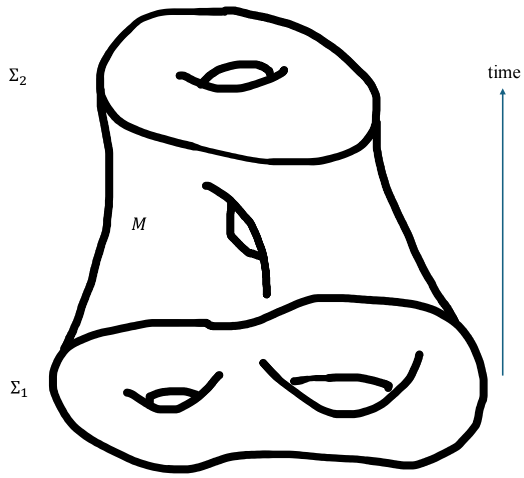

Figure 3. Transitioning from Σ_1 to Σ_2 via M.

path integral on M with boundary values defined on Σ_1 and Σ_2 can be written as:

$$\chi(\Phi_{\Sigma_2}) = \int_{\Phi_M|_{\Sigma_2}=\Phi_{\Sigma_2}} \mathcal{D}\Phi_M \; e^{-I(\Phi_M)}\Psi(\Sigma_1). \tag{1.3}$$

Formally the composition of transitions can be represented pictorially as shown in Figure 4 with:

$$M = M_1 \cup M_2. \tag{1.4}$$

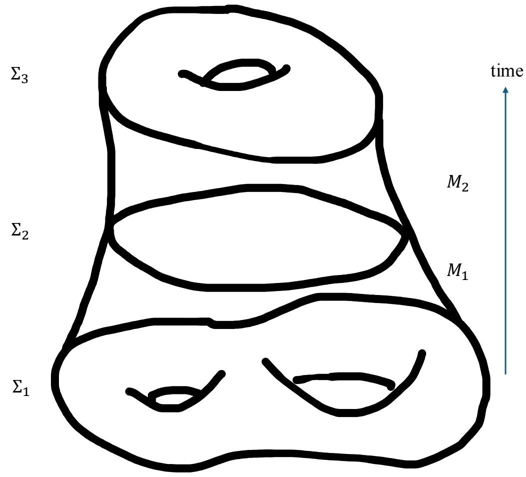

Figure 4. Composition of the transitions Θ_{M_1} and Θ_{M_2}.

There are several general things we could say. We consider a closed D-manifold M and there is a path integral on M:

$$Z_M = \int \mathcal{D}\Phi_M \; e^{-I(\Phi_M)}. \tag{1.5}$$

Since we do not assume any extra structures beyond the topological data, Z_M must be a topological invariant of M with certain other topological quantities specified (for us it is orientation).

2 3D TFT with impurities

We ask a physical question: how do TFTs arise in condensed matter physics? They usually arise in *gapped systems*, where typically there are no local degrees of freedom, in the sense that nothing happens when one perturbs the system.

A gapped system can have impurities in it, hence a manifold Σ can have labeled points in it, see Figure 5. In this case we will denote by $\mathcal{H}_{\Sigma;R_1,\cdots,R_n}$ the Hilbert space associated

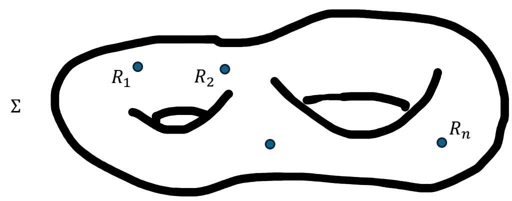

Figure 5. Σ with impurities.

to the spatial manifold M with the impurities $R_1, \cdots, R_n$.

An important special case is $\mathbb{R}^2$ with impurities, see Figure 6. These impurities in Σ

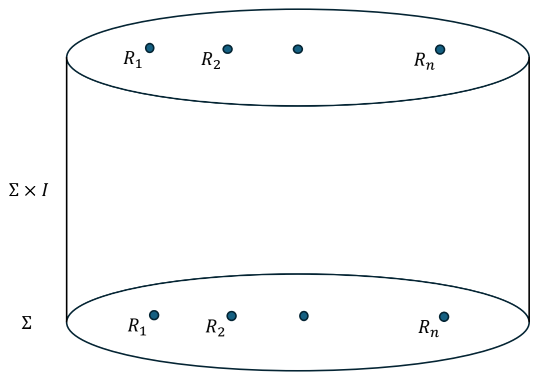

Figure 6. $\mathbb{R}^2$ transition with impurities.

will become lines in spacetime $\Sigma \times I$. Therefore, allowing the impurities in space will force us to introduce line operators in the field theory, which can braid in spacetime as illustrated in Figure 7. A line operator in field theory represents the world line of a quasi-particle in

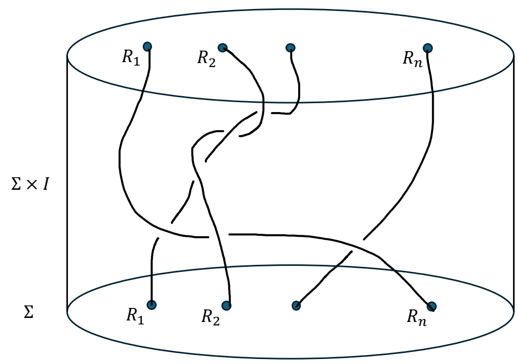

Figure 7. Lifting impurities in spatial Σ to lines in spacetime $\Sigma \times I$ with braidings.

Σ. In addition to the initial and the final states living on Σ in the past and in the future, we now also have to include the world lines of these quasi-particles.

One important feature of these line operators is that they form a group called the *braid group* (though it is a slightly non-trivial fact that these braids actually form a group). A braid B defines a *unitary* operator acting on the Hilbert space:

$$U_B : \mathcal{H}_{\Sigma;R_1,\cdots,R_n} \to \mathcal{H}_{\Sigma;R_1,\cdots,R_n}. \tag{2.1}$$

The braid group is a highly non-abelian infinite discrete group acting unitarily on the space of states for any finite set of quasi-particles.

The situation becomes more interesting when all the impurities are identical particles. Usually we simply exchange the particles and this gives us bosons and fermions given their statistics. Now we can enrich the exchanging of the particles by braids. Hence the idea of particle statistics gets drastically generalized in this situation to the infinite discrete group

$\widehat{B}$, which can be encoded in the following exact sequence:

$$1 \to B \to \widehat{B} \to \text{Permutation group} \to 1. \tag{2.2}$$

In words, if one forgets about the braiding, the particle statistics is governed solely by the permutation group which is the ordinary bosons vs. fermions story. Given a fixed permutation of the impurities, different configurations will differ by different braids in B. Therefore, the statistics of quasi-particles in a $(2+1)$-D QFT are given by the unitary representations of $\widehat{B}$ which is an extension of the ordinary permutation group by the braid group B. $\widehat{B}$ has irreducible non-abelian representations, and the quasi-particles are called *non-abelian anyons*.

Now we see another reason that $D = 3$ is the most interesting case: such an enrichment of particle statistics is only possible in $(2+1)$-D. This is because the braids can disentangle by sliding through each other without crossing in higher dimension. Therefore, the braids are non-trivial only in $(2+1)$-D.

3 Path integral with quasi-particles

Recall that for a general 3-manifold M, we have:

$$Z_M = \int \mathcal{D}\Phi_M \; e^{-I(\Phi_M)} \tag{3.1}$$

is a topological invariant of M. A more general statement holds for M with quasi-particle orbits in it. These quasi-particle orbits are known as *knots* in a 3-manifold M. A simple example of a trefoil in S^3 (or in $\mathbb{R}^3$) is illustrated in Figure 8. We will have the following

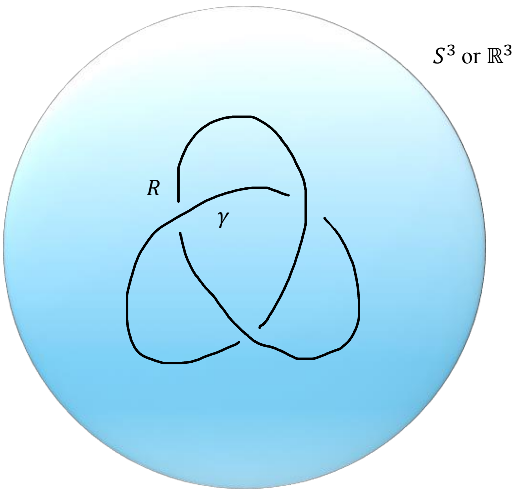

Figure 8. A quasi-particle R along a trefoil γ in S^3 (or in $\mathbb{R}^3$).

topological invariant:

$$Z = \int \mathcal{D}\Phi_M \; e^{-I(\Phi_M)} \; W_R(\gamma) \tag{3.2}$$

where $W_R(\gamma)$ is the operator insertion associate with the quasi-particle trajectory.

4 Chern-Simons gauge theory

We take a *compact* gauge group G (could be $U(1)$, $SU(n)$, etc.). There is a gauge field A_μ. When $G = U(1)$ one can construct the field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.1}$$

which is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \epsilon$ for a scalar function ϵ. For non-abelian G, the gauge field carries an extra index labelling the Lie algebra of G, i.e. we have the gauge field A^a_μ. The gauge transformation has an extra term:

$$A^a_\mu \to A^a_\mu - D_\mu \epsilon^a = A^a_\mu - (\partial_\mu \epsilon^a + [A_\mu, \epsilon^a]). \tag{4.2}$$

There is a gauge *covariant* field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{4.3}$$

It transforms under gauge transformation as:

$$\delta_\epsilon F_{\mu\nu} = [\epsilon, F_{\mu\nu}]. \tag{4.4}$$

Usually, in order to discuss a theory one needs to construct a gauge-invariant action. For our purpose we not only want it to be gauge-invariant, also we want it to be defined without metric. Precisely in 3D there is an action for gauge fields on any oriented 3-manifold which can moreover be quantized [1]. The action is called the *Chern-Simons action*:

$$\mathrm{CS}(A) = \frac{1}{4\pi} \int_M d^3x \; \epsilon^{ijk} \mathrm{Tr} \left(A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k \right). \tag{4.5}$$

One immediately sees that $\mathrm{CS}(A)$ is generally covariant even without a metric. Slightly more non-trivial, one can check that $\mathrm{CS}(A)$ is invariant up to a surface term, hence is invariant under infinitesimal transformations.

It is very interesting that when G is non-abelian one can find a topologically non-trivial transformation such that under which we have:

$$\mathrm{CS}(A) \to \mathrm{CS}(A) + 2\pi r, \; r \in \mathbb{Z}. \tag{4.6}$$

This is closely related to the fact that $\pi_3(SU(2)) = \mathbb{Z}$ for example when $G = SU(2)$. Therefore, $\mathrm{CS}(A)$ is actually not gauge invariant. Since the partition function Z is defined as $\int \mathcal{D}\Phi_M \; e^{iI(\Phi_M)}$ in Minkowskian signature, the partition function:

$$Z = \int \mathcal{D}\Phi_M \; e^{ik\mathrm{CS}(A)} \tag{4.7}$$

will make sense for any $k \in \mathbb{Z}$ to be potentially the partition function of a TFT. Here k is called the *level*. Since an orientation reversal changes the sign of k, we shall simply

[1] E.g., there is no way to construct a quantizable gauge-invariant action without using a Riemannian metric in 4D. In higher odd dimensions one can write down an action but they cannot be quantized since there is no good linearization.

assume that $k > 0$. Therefore, we have a family of TFTs labeled by G and k. Here we have implicitly assumed that G is simple.

Now we wish to do gauge theory with impurities. We will label these quasi-particles by representations R of G. In gauge theory, we have:

$$W_R(\gamma) = \mathrm{Tr}_R \, \mathcal{P} \, \exp\left(i \int_\gamma A\right) \tag{4.8}$$

which is also called the *holonomy* of A around γ. We will claim without proving that this theory is anomaly-free therefore makes sense as a TFT.

An important example is when $\Sigma = S^2$ with impurities, see Figure 9. We want to

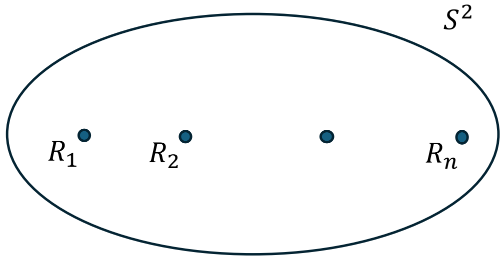

Figure 9. Spatial manifold S^2 with impurities.

describe the Hilbert space $\mathcal{H}_{S^2;R_1,\cdots,R_n}$. We will quantize the theory in the semi-classical limit, i.e. for large k. One can justify that by looking at the exponential term $e^{ik\mathrm{CS}(A)}$ in the path integral where $1/k$ plays the role of coupling constant.

Let us first study the case when there are no impurities. In the classical limit one naturally asks what is the Euler-Lagrange equation of this theory. One can find that the *classical equation of motion* is:

$$F_{ij} = 0. \tag{4.9}$$

Therefore, by gauge transformation, the gauge field can be set to zero (though probably not always). Consequently, the classical solutions contain only global information (since the curvature is always zero hence there can be no local fluctuations). For example, when $\Sigma = T^2$, gauge fields with vanishing curvature can have non-trivial holonomy around the A and B-cycles of Σ. In any case, the quantum states can only depend on global information of Σ. But when $\Sigma = S^2$, the states can depend only on trivial topological information since S^2 is simply-connected. Therefore on S^2 the gauge field can do nothing in the classical limit.

By Gauss law, we know that in a closed universe the total charge must vanish, since "the flux has nowhere to go". Before we impose this constraint, the Hilbert space would be:

$$R_1 \otimes R_2 \otimes \cdots \otimes R_n. \tag{4.10}$$

In an abelian theory the charge is an integer so one simply adds them up. In a non-abelian field theory the charges are representations so one has to determine what total charge means. In this case we require that the charges must be projected to the trivial piece, therefore the Hilbert space is:

$$(R_1 \otimes R_2 \otimes \cdots \otimes R_n)^G \tag{4.11}$$

where the superscript G means that we project to the G-invariant piece.

Having obtained the Hilbert space in the classical limit, we ask what happens when k is finite. In the $k \to \infty$ limit the braid group representation will be trivial as moving quasi-particles around results in nothing. There is a more dramatic modification of the theory for finite k: *most representations are lost.* For example, for $G = SU(2)$ only representations of spin $\leq \frac{k}{2}$ are "allowed".

Let us fix $G = SU(2)$ and consider quasi-particles all of spin-$\frac{1}{2}$. What does the Hilbert space look like when $k \to \infty$? We need to look for all different ways of combining spin-$\frac{1}{2}$ into trivial representation of $SU(2)$. This is not very hard by recursion and we will not show the details here.

We now consider $k = 2$. In this case spin $k > 1$ states are not allowed. For a long chain of l quasi-particles, *classically* the number of states is approximately:

$$N \sim C \times 2^l. \tag{4.12}$$

Quantum mechanically, for $k = 2$ we have:

$$N \sim C \times \sqrt{2}^l. \tag{4.13}$$

The $\sqrt{2}$ appearing in the above equation is called the *quantum dimension* of the representation at $k = 2$.

5 A theorem in knot theory

We fix $G = SU(n)$ and consider two representations R and $\overline{R}$ and $R \neq \overline{R}$ when $n > 2$. We study the configuration shown in Figure 10.

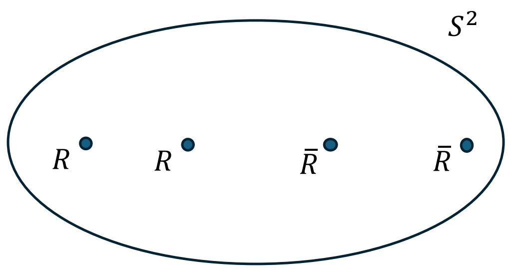

Figure 10. Four quasi-particles R, R, $\overline{R}$ and $\overline{R}$ on S^2.

For sufficiently large k, we have:

$$\mathcal{H}_{S^2;R,R,\overline{R},\overline{R}} = \left(R \otimes R \otimes \overline{R} \otimes \overline{R}\right)^{SU(n)}. \tag{5.1}$$

To find the $SU(n)$-invariant piece, recall that:

$$R \otimes R = \text{ASym} \oplus \text{Sym} \tag{5.2}$$

and one can find that $\dim\left(\mathcal{H}_{S^2;R,R,\overline{R},\overline{R}}\right) = 2$. An almost-trivial consequence is that any three vectors in $\mathcal{H}_{S^2;R,R,\overline{R},\overline{R}}$ must have a linear relation.

Let us recall how do we prepare the states. We find a 3-manifold M whose future boundary is S^2 with the impurities and prepare a state which we call $|3\rangle$ as illustrated in Figure 11. Similarly one can prepare two other states which we call $|1\rangle$ and $|2\rangle$ as illustrated

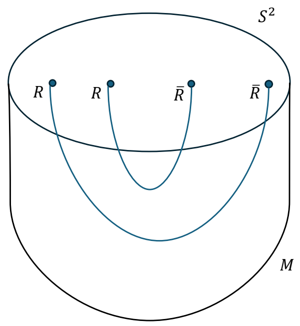

Figure 11. Prepare a state in $\mathcal{H}_{S^2;R,R,\overline{R},\overline{R}}$.

in Figure 12. The three states (vectors) $|1\rangle$, $|2\rangle$ and $|3\rangle$ must obey a linear relation since

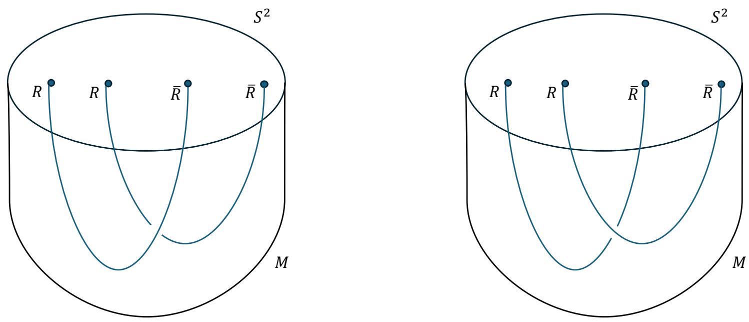

Figure 12. Prepare two other states in $\mathcal{H}_{S^2;R,R,\overline{R},\overline{R}}$.

they live in a 2-dimensional Hilbert space. More precisely, there is a relation:

$$\alpha\,|1\rangle + \beta\,|2\rangle + \gamma\,|3\rangle = 0 \tag{5.3}$$

where the coefficients α, β and γ depend on the CS level k and the number of impurities N.

We now introduce *knot invariants*. Basically, a knot invariant assigns a number to a knot illustrated in Figure 13 that does not depend on how it is drawn. It is far from trivial that such an invariant does exist. It does exist and is related to the braid group.

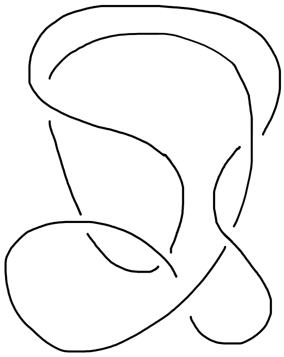

Figure 13. A knot.

Actually, at each "crossing", we have three possibilities when projected onto a disk and we have a linear relation as shown in Figure 14. One should compare Figure 14 with

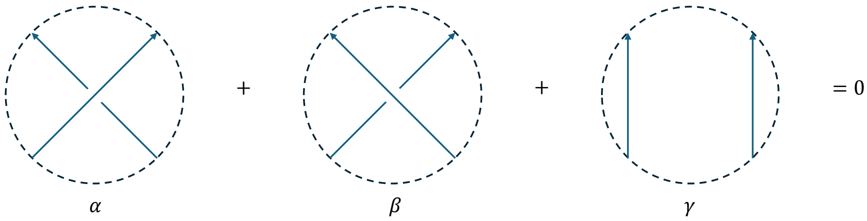

Figure 14. A relation between crossings.

Figure 11 and Figure 12. Therefore we expect to find a knot invariant:

$$F\left(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}\right). \tag{5.4}$$

Since we do care about the normalization of an unknot, we do the following trick as shown in Figure 15. Denote by $l = \langle O \rangle$ the VEV of an unknot, from Figure 15 we have:

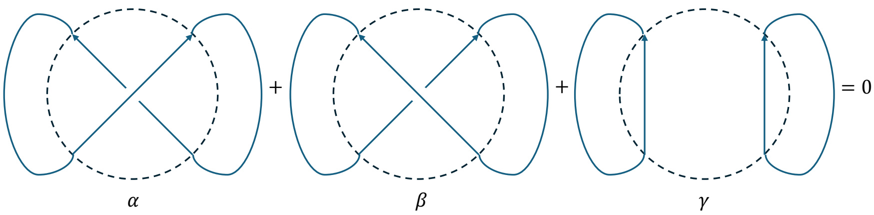

Figure 15. A relation between knots.

$$\alpha l + \beta l + \gamma l^2 = 0 \tag{5.5}$$

Therefore, we have:

$$l = \frac{\alpha + \beta}{\gamma}. \tag{5.6}$$

References

[1] Edward Witten, *Introduction to Chern Simons Theory and Topology*, https://www.youtube.com/watch?v=uIWUya3QfOQ&ab_channel=InstituteforAdvancedStudy