
My Second Report

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1 Introduction

Nothing to introduce here. Just a collection of maths.

2 Products

2.1 Inner product / Dot product

From [8, 6, 5].

For real numbers, inner product is the standard multiplication:

$$\langle x, y \rangle := xy \quad (1)$$

For vectors or column matrices:

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle := x^T y = \sum_{i=1}^n x_i y_i = x_1 y_1 + \cdots + x_n y_n \quad (2)$$

where $x, y \in \mathbb{R}^{n \times 1}$

The scalar product or dot product, written $x \cdot y$, is a common special case of inner product. Geometrically, it is the product of the Euclidean magnitudes of the two vectors and the cosine of the angle between them. These definitions are equivalent when using Cartesian coordinates.

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta \quad (3)$$

2.2 Outer product / Tensor product

From [9].

14 For vectors or column matrices, it is equivalent to a matrix multiplication xy^T :

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1v_1 & u_1v_2 & \dots & u_1v_n \\ u_2v_1 & u_2v_2 & \dots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_mv_1 & u_mv_2 & \dots & u_mv_n \end{bmatrix} \quad (4)$$

15 where $u, v \in \mathbb{R}^{n \times 1}$

16 For higher order tensors:

$$\mathbf{T} = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{d}, \quad T_{ijk} = a_ib_jd_k \quad (5)$$

17 For example, if \mathbf{A} is of order 3 with dimensions $(3, 5, 7)$ and \mathbf{B} is of order 2 with dimensions
18 $(10, 100)$, their outer product \mathbf{C} is of order 5 with dimensions $(3, 5, 7, 10, 100)$.

19 Tensor product is a generalisation of outer product [10]. The outer product of two vectors \mathbf{u} and
20 \mathbf{v} is their tensor product $\mathbf{u} \otimes \mathbf{v}$ [9].

21 2.3 Hadamard product

22 From [7].

23 The Hadamard product (also known as the Schur product or the entrywise product) is a binary
24 operation that takes two matrices of the same dimensions and produces another matrix where
25 each element i, j is the product of elements i, j of the original two matrices.

26 For two matrices \mathbf{A} and \mathbf{B} of the same dimension $m \times n$, the Hadamard product $\mathbf{A} \circ \mathbf{B}$ is a
27 matrix of the same dimension as the operands:

$$(\mathbf{A} \circ \mathbf{B})_{ij} = (\mathbf{A})_{ij}(\mathbf{B})_{ij} \quad (6)$$

28 For example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix} \quad (7)$$

29 2.4 Circular correlation

30 From [2, 3], related: [4].

31 Circular correlation between two vectors $\mathbf{a} \star \mathbf{b}$ is defined as:

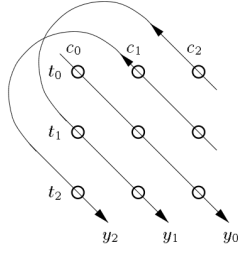
$$y_k = [\mathbf{c} \star \mathbf{t}]_k = \sum_{i=0}^{d-1} c_i t_{(k+i) \bmod d} \quad \text{for } i = 0, \dots, d-1 \quad (8)$$

32 where $\star : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$

33 Properties include:

34 i) Non Commutative

35 Correlation, unlike convolution, is not commutative, i.e. $\mathbf{a} \star \mathbf{b} \neq \mathbf{b} \star \mathbf{a}$. Non-
36 commutativity is necessary to model asymmetric relations (directed graphs).



$$\begin{aligned}
 \mathbf{y} &= \mathbf{c} \star \mathbf{t} \\
 y_0 &= c_0 t_0 + c_1 t_1 + c_2 t_2 \\
 y_1 &= c_2 t_0 + c_0 t_1 + c_1 t_2 \\
 y_2 &= c_1 t_0 + c_2 t_1 + c_0 t_2
 \end{aligned} \tag{9}$$

Figure 1: Circular correlation. Figure from [3], equations from [3, 2].

ii) Similarity Component

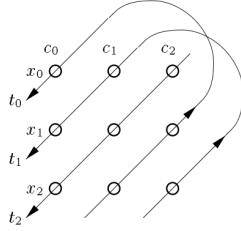
In the correlation $\mathbf{a} \star \mathbf{b}$, a single component $[\mathbf{a} \star \mathbf{b}]_0 = \sum_i a_i b_i$ corresponds to the dot product $\langle \mathbf{a}, \mathbf{b} \rangle$. The existence of such a component can be helpful to model relations in which the similarity of entities is important. No such component exists in circular convolution $\mathbf{a} \circledast \mathbf{b}$.

2.5 Circular convolution

From [3, 2, 1].

$$t_k = [\mathbf{c} \circledast \mathbf{x}]_k = \sum_{i=0}^{d-1} c_i x_{(k-i) \bmod d} \quad \text{for } i = 0, \dots, d-1 \tag{10}$$

where $\circledast : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$



$$\begin{aligned}
 \mathbf{t} &= \mathbf{c} \circledast \mathbf{x} \\
 t_0 &= c_0 x_0 + c_2 x_1 + c_1 x_2 \\
 t_1 &= c_1 x_0 + c_0 x_1 + c_2 x_2 \\
 t_2 &= c_2 x_0 + c_1 x_1 + c_0 x_2
 \end{aligned} \tag{11}$$

Figure 2: Circular convolution. Figure and equations from [3].

Properties include:

i) Commutative

Convolution (cyclic or acyclic) is commutative, i.e. $\mathbf{a} \circledast \mathbf{b} = \mathbf{b} \circledast \mathbf{a}$.

References

- [1] DuBois, G. M., Phillips, J. L., 2017. Working memory concept encoding using holographic reduced representations. In: MAICS. pp. 137–144.
- [2] Nickel, M., Rosasco, L., Poggio, T., 2016. Holographic embeddings of knowledge graphs. In: Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence. AAAI Press, pp. 1955–1961.
- [3] Plate, T. A., 1995. Holographic reduced representations. IEEE Transactions on Neural networks 6 (3), 623–641.
- [4] Tay, Y., Phan, M. C., Tuan, L. A., Hui, S. C., 2017. Learning to rank question answer pairs with holographic dual lstm architecture. In: Proceedings of the 40th International ACM SIGIR Conference on Research and Development in Information Retrieval. ACM, pp. 695–704.
- [5] Wikipedia contributors, 2019. Cosine similarity — Wikipedia, the free encyclopedia. [Online; accessed 28-March-2019].
URL https://en.wikipedia.org/w/index.php?title=Cosine_similarity&oldid=886018786
- [6] Wikipedia contributors, 2019. Dot product — Wikipedia, the free encyclopedia. [Online; accessed 28-March-2019].
URL https://en.wikipedia.org/w/index.php?title=Dot_product&oldid=887553882
- [7] Wikipedia contributors, 2019. Hadamard product (matrices) — Wikipedia, the free encyclopedia. [Online; accessed 28-March-2019].
URL [https://en.wikipedia.org/w/index.php?title=Hadamard_product_\(matrices\)&oldid=884602156](https://en.wikipedia.org/w/index.php?title=Hadamard_product_(matrices)&oldid=884602156)
- [8] Wikipedia contributors, 2019. Inner product space — Wikipedia, the free encyclopedia. [Online; accessed 28-March-2019].
URL https://en.wikipedia.org/w/index.php?title=Inner_product_space&oldid=889220737
- [9] Wikipedia contributors, 2019. Outer product — Wikipedia, the free encyclopedia. [Online; accessed 28-March-2019].
URL https://en.wikipedia.org/w/index.php?title=Outer_product&oldid=886026441
- [10] Wikipedia contributors, 2019. Tensor product — Wikipedia, the free encyclopedia. [Online; accessed 29-March-2019].
URL https://en.wikipedia.org/w/index.php?title=Tensor_product&oldid=889702125