Maths Maths Maths Maths Maths

Your Name
Project started on 31 March 2019

1 Introduction

Nothing to introduce here. Just a collection of maths.

2 Products

2.1 Inner product / Dot product

From [8, 6, 5].

For real numbers, inner product is the standard multiplication:

$$\langle x, y \rangle := xy \tag{1}$$

For vectors or column matrices:

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle := x^T y = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + \cdots + x_n y_n \tag{2}$$

where $x, y \in \mathbb{R}^{n \times 1}$

The scalar product or dot product, written $x \cdot y$, is a common special case of inner product. Geometrically, it is the product of the Euclidean magnitudes of the two vectors and the cosine of the angle between them. These definitions are equivalent when using Cartesian coordinates.

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta \tag{3}$$

2.2 Outer product / Tensor product

From [9].

For vectors or column matrices, it is equivalent to a matrix multiplication xy^T:

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^T = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \dots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \dots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_m v_1 & u_m v_2 & \dots & u_m v_n \end{bmatrix} \tag{4}$$

where $u, v \in \mathbb{R}^{n \times 1}$

For higher order tensors:

$$\mathbf{T} = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{d}, \quad T_{ijk} = a_i b_j d_k \tag{5}$$

For example, if $\mathbf{A}$ is of order 3 with dimensions $(3, 5, 7)$ and $\mathbf{B}$ is of order 2 with dimensions $(10, 100)$, their outer product $\mathbf{C}$ is of order 5 with dimensions $(3, 5, 7, 10, 100)$.

Tensor product is a generalisation of outer product [10]. The outer product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is their tensor product $\mathbf{u} \otimes \mathbf{v}$ [9].

2.3 Hadamard product

From [7].

The Hadamard product (also known as the Schur product or the entrywise product) is a binary operation that takes two matrices of the same dimensions and produces another matrix where each element i, j is the product of elements i, j of the original two matrices.

For two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimension $m \times n$, the Hadamard product $\mathbf{A} \circ \mathbf{B}$ is a matrix of the same dimension as the operands:

$$(A \circ B)_{ij} = (A)_{ij}(B)_{ij} \tag{6}$$

For example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix} \tag{7}$$

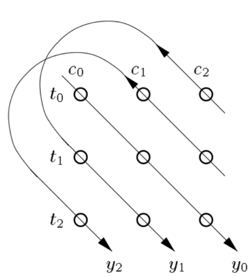
2.4 Circular correlation

From [2, 3], related: [4].

Circular correlation between two vectors $\mathbf{a} \star \mathbf{b}$ is defined as:

$$y_k = [\mathbf{c} \star \mathbf{t}]_k = \sum_{i=0}^{d-1} c_i \, t_{(k+i) \bmod d} \qquad \text{for } i = 0, \ldots, d-1 \tag{8}$$

where $\star : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$



$$\begin{aligned} \mathbf{y} &= \mathbf{c} \star \mathbf{t} \\ y_0 &= c_0 t_0 + c_1 t_1 + c_2 t_2 \\ y_1 &= c_2 t_0 + c_0 t_1 + c_1 t_2 \\ y_2 &= c_1 t_0 + c_2 t_1 + c_0 t_2 \end{aligned} \tag{9}$$

Figure 1: Circular correlation. Figure from [3], equations from [3, 2].

Properties include:

i) **Non Commutative**
Correlation, unlike convolution, is not commutative, i.e. $\mathbf{a} \star \mathbf{b} \neq \mathbf{b} \star \mathbf{a}$. Non-commutativity is necessary to model asymmetric relations (directed graphs).

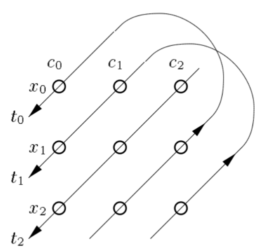
ii) **Similarity Component**
In the correlation $\mathbf{a} \star \mathbf{b}$, a single component $[\mathbf{a} \star \mathbf{b}]_0 = \sum_i a_i b_i$ corresponds to the dot product $\langle \mathbf{a}, \mathbf{b} \rangle$. The existence of such a component can be helpful to model relations in which the similarity of entities is important. No such component exists in circular convolution $\mathbf{a} \circledast \mathbf{b}$.

2.5 Circular convolution

From [3, 2, 1].

$$t_k = [\mathbf{c} \circledast \mathbf{x}]_k = \sum_{i=0}^{d-1} c_i \, x_{(k-i) \bmod d} \qquad \text{for } i = 0, \ldots, d-1 \tag{10}$$

where $\circledast : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$



$$\begin{aligned} \mathbf{t} &= \mathbf{c} \circledast \mathbf{x} \\ t_0 &= c_0 x_0 + c_2 x_1 + c_1 x_2 \\ t_1 &= c_1 x_0 + c_0 x_1 + c_2 x_2 \\ t_2 &= c_2 x_0 + c_1 x_1 + c_0 x_2 \end{aligned} \tag{11}$$

Figure 2: Circular convolution. Figure and equations from [3].

Properties include:

i) **Commutative**
Convolution (cyclic or acyclic) is commutative, i.e. $\mathbf{a} \circledast \mathbf{b} = \mathbf{b} \circledast \mathbf{a}$.

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