Maths Maths Maths Maths

Your Name

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1 Introduction

2 Nothing to introduce here. Just a collection of maths.

3 2 Products

4 2.1 Inner product / Dot product

- 5 From [8, 6, 5].
- 6 For real numbers, inner product is the standard multiplication:

$$\langle x, y \rangle := xy \tag{1}$$

7 For vectors or column matrices:

$$\left\langle \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\rangle := x^T y = \sum_{i=1}^n x_i y_i = x_1 y_1 + \dots + x_n y_n \tag{2}$$

- 8 where $x, y \in \mathbb{R}^{n \times 1}$
- 9 The scalar product or dot product, written $x \cdot y$, is a common special case of inner product.
- Geometrically, it is the product of the Euclidean magnitudes of the two vectors and the cosine
- of the angle between them. These definitions are equivalent when using Cartesian coordinates.

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta \tag{3}$$

12 2.2 Outer product / Tensor product

- 13 From [9].
- For vectors or column matrices, it is equivalent to a matrix multiplication xy^T :

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^{T} = \begin{bmatrix} u_{1}v_{1} & u_{1}v_{2} & \dots & u_{1}v_{n} \\ u_{2}v_{1} & u_{2}v_{2} & \dots & u_{2}v_{n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m}v_{1} & u_{m}v_{2} & \dots & u_{m}v_{n} \end{bmatrix}$$
(4)

- where $u, v \in \mathbb{R}^{n \times 1}$
- 16 For higher order tensors:

$$\mathbf{T} = \mathbf{a} \otimes \mathbf{b} \otimes \mathbf{d}, \quad T_{ijk} = a_i b_j d_k \tag{5}$$

For example, if **A** is of order 3 with dimensions (3,5,7) and **B** is of order 2 with dimensions (10,100), their outer product **C** is of order 5 with dimensions (3,5,7,10,100).

Tensor product is a generalisation of outer product [10]. The outer product of two vectors \mathbf{u} and \mathbf{v} is their tensor product $\mathbf{u} \otimes \mathbf{v}$ [9].

21 2.3 Hadamard product

22 From [7].

23 The Hadamard product (also known as the Schur product or the entrywise product) is a binary

24 operation that takes two matrices of the same dimensions and produces another matrix where

each element i, j is the product of elements i, j of the original two matrices.

For two matrices A and B of the same dimension $m \times n$, the Hadamard product $A \circ B$ is a

27 matrix of the same dimension as the operands:

$$(A \circ B)_{ij} = (A)_{ij}(B)_{ij} \tag{6}$$

28 For example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{12}b_{12} & a_{13}b_{13} \\ a_{21}b_{21} & a_{22}b_{22} & a_{23}b_{23} \\ a_{31}b_{31} & a_{32}b_{32} & a_{33}b_{33} \end{bmatrix}$$
(7)

29 2.4 Circular correlation

30 From [2, 3], related: [4].

Circular correlation between two vectors $\mathbf{a} \star \mathbf{b}$ is defined as:

$$y_k = [\mathbf{c} \star \mathbf{t}]_k = \sum_{i=0}^{d-1} c_i t_{(k+i) \bmod d}$$
 for $i = 0, \dots, d-1$ (8)

32 where $\star: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$

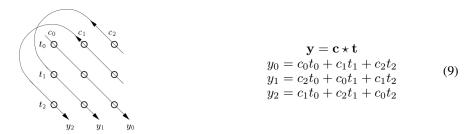


Figure 1: Circular correlation. Figure from [3], equations from [3, 2].

33 Properties include:

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i) Non Commutative

Correlation, unlike convolution, is not commutative, i.e. $\mathbf{a} \star \mathbf{b} \neq \mathbf{b} \star \mathbf{a}$. Non-commutativity is necessary to model asymmetric relations (directed graphs).

ii) Similarity Component

In the correlation $\mathbf{a} \star \mathbf{b}$, a single component $[\mathbf{a} \star \mathbf{b}]_0 = \sum_i a_i b_i$ corresponds to the dot

product $\langle \mathbf{a}, \mathbf{b} \rangle$. The existence of such a component can be helpful to model relations in which the similarity of entities is important. No such component exists in circular convolution $\mathbf{a} \circledast \mathbf{b}$.

42 2.5 Circular convolution

43 From [3, 2, 1].

$$t_k = [\mathbf{c} \circledast \mathbf{x}]_k = \sum_{i=0}^{d-1} c_i \, x_{(k-i) \bmod d} \quad \text{for } i = 0, \dots, d-1$$
 (10)

44 where $\circledast: \mathbb{R}^d \times \mathbb{R}^d o \mathbb{R}^d$

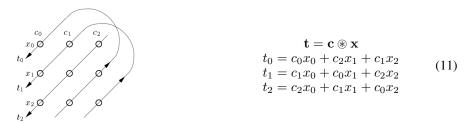


Figure 2: Circular convolution. Figure and equations from [3].

45 Properties include:

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i) Commutative

Convolution (cyclic or acyclic) is commutative, i.e. $\mathbf{a} \circledast \mathbf{b} = \mathbf{b} \circledast \mathbf{a}$.

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