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解決數學物理方程

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Contents

前言

本资料主要是数理方程李书敏老师课程的听课笔记，兼有公式题型梳理和概念理解

第一章　基本数学工具

1　一阶非齐次线性常微分方程 (1st-ODE)

$$\frac{dy(t)}{dt}+p(t)y(t)=q(t),\quad p(t),q(t)\in C(\mathbb{R})$$

$$y(t)=\mathrm{e}^{-\int p(t)\mathrm{d}t}\left\{\int q(t)\mathrm{e}^{\int p(t)\mathrm{d}t}\,\mathrm{d}t+C\right\}=\frac{\int qe^{\int pdt}dt+C}{e^{\int pdt}},C\text{ 是任意常数.}$$

2　Euler 方程

$$r^2R''(r)+prR'(r)+qR(r)=0$$

$$\text{当}r>0\text{ 时, 令}r=\mathrm{e}^S,S=\ln r,$$

$$R'(r)=\frac{dR}{dS}\frac{dS}{dr}=\frac{1}{r}\frac{dR}{dS}$$

$$R''(r)=\frac{d}{dr}\left(\frac{1}{r}\frac{dR}{dS}\right)=-\frac{1}{r^2}\frac{dR}{ds}+\frac{1}{r}\frac{d^2R}{ds^2}\frac{ds}{dr}=\frac{1}{r^2}\left(\frac{d^2R}{ds^2}-\frac{dR}{dS}\right)$$

$$\text{Euler 方程化为：}\frac{\mathrm{d}^2R(S)}{\mathrm{d}s^2}+(p-1)\frac{\mathrm{d}R(S)}{\mathrm{d}s}+qR(S)=0\text{ 后求解}$$

3　常系数二阶线性齐次常微分方程 (2nd-ODE)

$y''(t)+by'(t)+cy(t)=0,\quad(b,c$ 常数, 与 t 无关.) 特征方程 $\mu^2+b\mu+c=0.$

1. 当 $b^2-4c>0$ 时, 特征方程有两互异实根: $\mu_{1,2}=\dfrac{-b\pm\sqrt{b^2-4c}}{2},\mu_1\neq\mu_2$, 且 $\mu_1,\mu_2\in\mathbb{R}$.
通解 $y(t)=A\mathrm{e}^{\mu_1t}+B\mathrm{e}^{\mu_2t},A,B$ 是任意常数, 下同.
2. 当 $b^2-4c=0$ 时, 特征方程有两相同实根: $\mu_1=\mu_2\triangleq\mu_0=-\frac{b}{2}$.
通解 $y(t)=(At+B)\mathrm{e}^{\mu_0t}=(At+B)\mathrm{e}^{-\frac{b}{2}t}$.
3. 当 $b^2-4c<0$ 时, 特征方程有一对共轭复根: $\mu_1=\dfrac{-b+i\sqrt{4c-b^2}}{2},\quad\mu_2=\dfrac{-b-i\sqrt{4c-b^2}}{2}$.
通解 $y(t)=\mathrm{e}^{(\mathrm{Re}\,\mu_1)t}\left\{A\cos\left(\mathrm{Im}\,\mu_1\right)t+B\sin\left(\mathrm{Im}\,\mu_1\right)t\right\}$. (即 $y(t)=A\,\mathrm{Re}\,\mathrm{e}^{\mu_1t}+B\,\mathrm{Im}\,\mathrm{e}^{\mu_1t}$.)

注: 常用二阶线性常微分方程的解

$$\frac{\mathrm{d}^2R}{dt^2}=0\text{ 的通解：}R(t)=A+Bt,A,B\text{ 是任意常数.}$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}t^2}+\omega^2R=0(\omega>0)\text{ 的通解: }R(t)=A\cos\omega t+B\sin\omega t,A,B\text{ 是任意常数.}$$

$$\frac{d^2R}{dt^2}-\omega^2R=0(\omega>0)\text{的通解}R(t)=A\mathrm{e}^{\omega t}+B\mathrm{e}^{-\omega t}=R(t)=A'\cosh\omega t+B'\sinh\omega t$$

4 Fourier 变换

$$\mathscr{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-2\pi\omega it}\,\mathrm{d}t$$

注: 常用积分等式

$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} = \frac{1}{a^2+b^2}\begin{vmatrix} \sin bx & e^{ax} \\ b\cos bx & ae^{ax} \end{vmatrix}$$

$$\int e^{ax}\cos bx dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} = \frac{1}{a^2+b^2}\begin{vmatrix} \cos bx & e^{ax} \\ -b\sin bx & ae^{ax} \end{vmatrix}$$

$$\int \sin(ax)\cos(bx)dx = -\frac{b\sin(ax)\sin(bx) + a\cos(ax)\cos(bx)}{a^2-b^2} = \frac{1}{a^2-b^2}\begin{vmatrix} \sin ax & \cos bx \\ a\cos ax & -b\sin bx \end{vmatrix}$$

$$\int \sin(ax)\sin(bx)dx = \frac{b\sin(ax)\cos(bx) - a\cos(ax)\sin(bx)}{a^2-b^2} = \frac{1}{a^2-b^2}\begin{vmatrix} \sin ax & \sin bx \\ a\cos ax & b\cos bx \end{vmatrix}$$

$$\int \cos(ax)\cos(bx)dx = \frac{a\sin(ax)\cos(bx) - b\cos(ax)\sin(bx)}{a^2-b^2} = \frac{1}{a^2-b^2}\begin{vmatrix} \cos ax & \cos bx \\ -a\sin ax & -b\sin bx \end{vmatrix}$$

5 Laplace 变换

$$\mathscr{L}[f(t)] = \int_0^{\infty} f(t)e^{-(\sigma+i\omega)t}dt$$

$$f(t) = \mathscr{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)\mathrm{d}s$$

性质 1.0.1: Laplace 变换

1. 线性叠加：$\mathscr{L}[af(t)+bg(t)] = a\mathscr{L}[f(t)] + b\mathscr{L}[g(t)]$
2. 微分运算：$\mathscr{L}\left[f^{(n)}(t)\right] = s^n\mathscr{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$
3. 积分运算：$\mathscr{L}\left[\underbrace{\int_0^t dt\int_0^t dt..\int_0^t}_{n\text{重}} f(t)dt\right] = \frac{1}{s^n}\mathscr{L}[f(t)]$
4. 延迟定理：$\mathscr{L}[f(t-\tau)u(t-\tau)] = e^{-\tau s}F(s)$
5. 卷积定理：$\mathscr{L}[f(t)*g(t)] = \mathscr{L}[\int_0^t f(\tau)g(t-\tau)d\tau] = \mathscr{L}[f(t)]\cdot\mathscr{L}[g(t)]$

第二章 偏微分方程定解问题

第 1 节 导入方程

1. 三个常系数线性：

$$(1)\text{ 弦振动方程: } u_{tt}=\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}+f(t,\vec{x})$$
$$(2)\text{ 热传导方程: } u_t=\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+f(t,\vec{x})$$
$$(3)\text{ 拉普拉斯方程 (场位方程): } \Delta_3 u=f(x,y,z).$$

$$\text{其中}\vec{x}=(x_1,x_2,\ldots,x_n)\,,\quad \Delta u=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\ldots+\frac{\partial^2 u}{\partial x_n^2}$$

2. 非常系数线性方程：Schrödinger 方程：

$$i\hbar\psi_t=(u-\frac{\hbar}{2m})\psi$$

3. 非线性方程：如 KDV 方程

$$\frac{\partial\eta}{\partial\tau}+\frac{\partial^3\eta}{\partial\xi^3}+\eta\frac{\partial\eta}{\partial\xi}=0$$

第 2 节 定解问题

注: 通解和特解

- 通解：m 阶 PDE 的含有 m 个任意函数的解. 找通解极其困难.
- 特解：不含任意函数和任意常数的解. 故绝大多数是找特解，需要对方程附加定解条件

$$\begin{cases}\text{泛定方程: 反映系统内部作用机制的方程.}\\ \text{定解条件}\begin{cases}\text{初始条件: 历史的影响;}\\ \text{边界条件: 环境的影响.}\end{cases}\end{cases}$$

1. 初始条件：对于一般发展型方程，

$$F\left(x,u(x),\cdots,\frac{\partial^n u}{\partial t^n},\cdots,\frac{\partial^m u}{\partial t^{m_0}\partial x_1^{m_1}\cdots\partial x_1^{m_l}},\cdots\right)=0$$

给出 $u|_{t=t_0},\ \frac{\partial u}{\partial t}\big|_{t=t_0},\ \frac{\partial^2 u}{\partial t^2}\Big|_{t=t_0},\cdots,\ \frac{\partial^{n-1}u}{\partial t^{n-1}}\Big|_{t=t_0}$. $t=t_0$: 初始时刻 (唯一).

2. 边界条件：（右侧为 0 则是齐次的）线性边界：$(\alpha u+\beta\frac{\partial u}{\partial n})|_{\partial V}=F(t,x,y,z)|_{\partial V}$

   (a) 第 I 类边界条件：

$$u|_{x=x_1}=f_1(t),\quad u|_{x=x_2}=f_2(t)$$

   (b) 第 II 类边界条件：

$$\frac{\partial u}{\partial x}(t,x_2)=\frac{F_2(t)}{T_1}$$

(c) 第 III 类边界条件:

$$ku(t,x_2)+T_1\frac{\partial u}{\partial x}(t,x_2)=F_2(t)$$

(d) 总结边界条件:

$$\left.\left(\alpha u+\beta\frac{\partial u}{\partial n}\right)\right|_{\partial V}=F(t,x,y,z)|_{\partial V}$$

第 3 节 一阶线性偏微分方程通解

1 两自变量的一阶线性 PDE

$$a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u=f(x,y)$$

步骤 1 寻找两个函数使得右侧 $f=0$,

$$\begin{cases}\xi=\varphi(x,y), & \text{满足}J(\varphi,\psi)=\varphi_x\psi_y-\psi_x\varphi_y\neq 0\\ \eta=\psi(x,y).\end{cases}$$

步骤 2 发现要求

$$a(x,y)\frac{\partial\varphi}{\partial x}+b(x,y)\frac{\partial\varphi}{\partial y}=0$$

得到特征线方程:

步骤 3 解特征线方程:

$$\frac{dx}{a(x,y)}=\frac{dy}{b(x,y)}$$

在D 内求它隐式通解$\varphi(x,y)=h$.

步骤 4 满足雅克比行列式不为 0 前提下任取 ψ

$$\because\left(a\frac{\partial\psi}{\partial x}+b\frac{\partial\psi}{\partial y}\right)\frac{\partial u}{\partial\eta}+cu=f$$

$$p=c/\left(a\frac{\partial\psi}{\partial x}+b\frac{\partial\psi}{\partial y}\right),q=f/\left(a\frac{\partial\psi}{\partial x}+b\frac{\partial\psi}{\partial y}\right)$$

$$u=\mathrm{e}^{-\int p(\xi,\eta)\mathrm{d}\eta}\left\{\int q(\xi,\eta)\mathrm{e}^{\int p(\xi,\eta)\mathrm{d}\eta}\mathrm{d}\eta+G(\xi)\right\}$$

步骤 5

用$\xi=\varphi(x,y),\eta=\psi(x,y)$ 将解换回原变量x,y

若 $c\equiv f\equiv 0$ 时, 则方程化为, $\frac{\partial u}{\partial\eta}=0$, 则 $u=G(\xi)=G(\varphi(x,y))$

检验: 将解代入方程和初始条件检验.

例 2.3.1: 二自变量

$$\text{求解}\begin{cases} t\frac{\partial u}{\partial t}=-\frac{\partial u}{\partial x}-3x^2u, t>1,-\infty<x<\infty, \\ u|_{t=1}=x^3. \end{cases}$$

$$\frac{\mathrm{d}x}{1}=\frac{\mathrm{d}t}{t}=\mathrm{d}(\ln t),\quad \ln t-x=h$$

$$\text{令}\begin{cases} \xi=te^{-x}, \\ \eta=x(\text{ 任取}) \end{cases}\quad J=\begin{vmatrix} -te^{-x} & e^{-x} \\ 1 & 0 \end{vmatrix}=e^{-x}\neq 0,\ \text{反变换}\begin{cases} x=\eta, \\ t=\xi\mathrm{e}^{\eta} \end{cases}$$

由链式法则,

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}=\mathrm{e}^{-x}\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=-t\mathrm{e}^{-x}\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}$$

代入方程得 $-\dfrac{\partial u}{\partial \eta}-3x^2u=0.\dfrac{\partial u}{\partial \eta}+3\eta^2u=0.\dfrac{\partial}{\partial \eta}\left(\mathrm{e}^{\int 3\eta^2 d\eta}u\right)=0$, 故

$$u|_{t=1}=e^{-x^3}G\left(e^{-x}\right)=x^3, G\left(e^{-x}\right)=x^3e^{x^3}$$

$$\text{令}p=\mathrm{e}^{-x},\text{ 则}x=-\ln p.\text{ 故}G(p)=-(\ln p)^3\mathrm{e}^{-(\ln p)^3}$$

$$u=\mathrm{e}^{-x^3}G\left(t\mathrm{e}^{-x}\right)=-\mathrm{e}^{-x^3}(\ln t-x)^3\mathrm{e}^{-(\ln t-x)^3}$$

2 n 个自变量的一阶线性 PDE

$$\sum_{j=1}^{n}b_j\frac{\partial u}{\partial x_j}+cu=f$$

特征线方程:

$$\frac{\mathrm{d}x_1}{b_1}=\frac{\mathrm{d}x_2}{b_2}=\cdots=\frac{\mathrm{d}x_n}{b_n}=\mathrm{d}t$$

得到有且仅有 $n-1$ 个相互独立的首次积分, 记为:

$$\varphi_1\left(x_1,\cdots,x_n\right)=h_1,\varphi_2\left(x_1,\cdots,x_n\right)=h_2,\cdots,\varphi_{n-1}\left(x_1,\cdots,x_n\right)=h_{n-1}$$

$$\frac{\partial u}{\partial \xi_n}+p\left(\xi_1,\cdots,\xi_n\right)u=q\left(\xi_1,\cdots,\xi_n\right),\quad \text{其中}\quad p=\frac{c}{\sum_{j=1}^{n}b_j\frac{\partial \varphi_n}{\partial x_j}},q=\frac{f}{\sum_{j=1}^{n}b_j\frac{\partial \varphi_n}{\partial x_j}}$$

$$u=\mathrm{e}^{-\int p\left(\xi_1,\cdots,\xi_n\right)\mathrm{d}\xi_n}\left\{\int q\left(\xi_1,\cdots,\xi_n\right)\mathrm{e}^{\int p\left(\xi_1,\cdots,\xi_n\right)\mathrm{d}\xi_n}\mathrm{d}\xi_n+G\left(\xi_1,\cdots,\xi_{n-1}\right)\right\}$$

例 2.3.2: 三自变量

$$\text{求解}\begin{cases} x^2\frac{\partial u}{\partial x}+2\sqrt{y}\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}, y>0, x\neq 0, \\ u|_{z=1}=x+y^2 \end{cases}$$

特征方程为 $\frac{\mathrm{d}x}{x^2}=\frac{\mathrm{d}y}{2\sqrt{y}}=\frac{\mathrm{d}z}{z}$. 由 $\frac{\mathrm{d}x}{x^2}-\frac{\mathrm{d}y}{2\sqrt{y}}=0$ 积分可得一个首次积分

$$-\frac{1}{x}-\sqrt{y}=h_1,\quad \frac{1}{x}+\sqrt{y}=-h_1.$$

由 $\frac{\mathrm{d}x}{x^2}-\frac{\mathrm{d}z}{z}=0$ 得 $-\frac{1}{x}-\ln z=h_2,\mathrm{e}^{\frac{1}{x}}+\ln z=\mathrm{e}^{-h_2},\mathrm{e}^{\frac{1}{x}}z=\mathrm{e}^{-h_2}$.

$$令\begin{cases}\xi_1=\frac{1}{x}+\sqrt{y},\\ \xi_2=\mathrm{e}^{\frac{1}{x}}z, \qquad J=e^{\frac{1}{x}}\frac{1}{2\sqrt{y}}\neq 0\\ \xi_3=x\end{cases}$$

因方程中无 cu 和 f, 故方程化为 $\frac{\partial u}{\partial \xi_3}=0$.

$$u=G(\xi_1,\xi_2)=G\left(\frac{1}{x}+\sqrt{y},\mathrm{e}^{\frac{1}{x}}z\right)$$

其中 $G\in C^1(\mathbb{R}^2)$.

$$u|_{z=1}=G\left(\frac{1}{x}+\sqrt{y},\mathrm{e}^{\frac{1}{x}}\right)=x+y^2.令p=\frac{1}{x}+\sqrt{y},q=\mathrm{e}^{\frac{1}{x}},$$

$$则x=\frac{1}{\ln q},y=(p-\ln q)^2.\ 故得G(p,q)=\frac{1}{\ln q}+(p-\ln q)^4.$$

$$故解u=G\left(\frac{1}{x}+\sqrt{y},\mathrm{e}^{\frac{1}{x}}z\right)=\frac{1}{\frac{1}{x}+\ln z}+(\sqrt{y}-\ln z)^4,\quad x\neq 0$$

解答: 巧妙得到首次积分的情况

$$(y+z)\frac{\partial u}{\partial x}+(z+x)\frac{\partial u}{\partial y}+(x+y)\frac{\partial u}{\partial z}=0$$

$$\frac{dz}{x+y}=\frac{dy-dz}{(z+x)-(x+y)}=-\frac{d(y-z)}{y-z}=-\frac{d(z-x)}{z-x},$$

$$\frac{dy}{z+x}=\frac{dz}{x+y}=\frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)}=\frac{d(x+y+z)}{2(x+y+z)}$$

$$\xi_1=\frac{z-x}{y-z},\quad \xi_2=(y-z)^2(x+y+z)$$

解答: 积分因子 $e^{\int pdt}$ 的使用。相当于 u,u' 组合

$$\frac{\partial u}{\partial \xi_3}-u=\xi_1\xi_2\mathrm{e}^{-2\xi_3}\cdot\ 乘以\mathrm{e}^{-\int 1\,\mathrm{d}\xi_3}=\mathrm{e}^{-\xi_3}\ 得\frac{\partial}{\partial \xi_3}\left(\mathrm{e}^{-\xi_3}u\right)=\xi_1\xi_2\mathrm{e}^{-3\xi_3}$$

3 波动方程行波解 & d'Alembert 公式

波动方程可以分为两个行波：左、右方程

$$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}\Longrightarrow\left(\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-a\frac{\partial}{\partial x}\right)u=0$$

$$\text{令}\frac{\partial u}{\partial t}-a\frac{\partial u}{\partial x}=v,\ \text{则}\frac{\partial v}{\partial t}+a\frac{\partial v}{\partial x}=0\Longrightarrow -4a^2\frac{\partial^2 u}{\partial\xi\partial\eta}=0$$

$$\Rightarrow\quad u=F(x-at)+G(x+at),\quad F,G\text{分别为左，右行波，是任意}C^2(\mathbb{R})\text{ 函数}.$$

例 2.3.3: d'Alembert 公式

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2},\quad t>0,\quad -\infty<x<\infty,\quad a>0\text{ 是常数},\\ u|_{t=0}=\varphi(x),\quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x).\end{cases}$$

$$u|_{t=0}=G(x)+F(x)=\varphi(x)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=aG'(x)-aF'(x)=\psi(x)$$

▶ $$\boxed{u=\frac{1}{2}\{\varphi(x+at)+\varphi(x-at)\}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi}$$

4 d'Alembert 公式拓展

确定区域内每点 u 值仅由 $\varphi(x),\psi(x)$ 在区间 $[h_1,h_2]$ 上值完全确定

影响区域内每点 u 值都跟 $\varphi(x),\psi(x)$ 在区间 $[h_1,h_2]$ 值有关

例 2.3.4: 半无界问题

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}, t>0, x>0, a>0\\ u|_{x=0}=0,\quad \text{I 类齐次边界条件}\\ u|_{t=0}=\varphi(x),\quad \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x),\quad x>0.\end{cases}$$

为保证解满足 $u|_{x=0}=0$, 对φ,ψ 奇延拓:

$$u=\frac{1}{2}\{\varphi(x+at)\pm\varphi(\pm x\mp at)\}+\frac{1}{2a}\int_{|x-at|}^{x+at}\psi(\xi)\mathrm{d}\xi,\quad x-at\geqslant 0\text{取正常形式}$$

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2},\quad t>0,x>0,a>0\text{ 是常数},\\ \frac{\partial u}{\partial x}\big|_{x=0}=0,\quad \text{II 类齐次边界条件}\\ u|_{t=0}=\varphi(x),\quad \frac{\partial u}{\partial t}\big|_{t=0}=\psi(x),x>0.\end{cases}$$

$$u=\begin{cases}\dfrac{1}{2}\{\varphi(x+at)+\varphi(x-at)\}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)d\xi,\quad x\geqslant at\\ \dfrac{1}{2}\{\varphi(x+at)+\varphi(at-x)\}+\dfrac{1}{2a}\left\{\displaystyle\int_0^{x+at}\psi(\xi)\mathrm{d}\xi+\int_0^{at-x}\psi(\eta)\mathrm{d}\eta\right\},\quad x<at\end{cases}$$

例 2.3.5: 中心对称球面波

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=a^2\Delta_3 u, \quad t>0, \quad r=\sqrt{x^2+y^2+z^2}>0, \\ u|_{t=0}=\varphi(r), \quad \frac{\partial u}{\partial t}\big|_{t=0}=\psi(r). \end{cases}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2}=a^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}\right)$$

为了消去一阶导数项，一种可能的想法：寻找过渡函数，不妨

$$u(t,r)=e^{\alpha(r)}v(t,r)$$

一阶导数为 0 得到 $\alpha(r)=\ln\dfrac{1}{r}, \quad v=ru$

$$\begin{cases} \frac{\partial^2 v}{\partial t^2}=a^2\frac{\partial^2 v}{\partial r^2}, \quad t>0, r>0, \quad (a>0 \text{ 是常数}), \\ v|_{t=0}=(ru)|_{t=0}=r\varphi(r), \ \frac{\partial v}{\partial t}\big|_{t=0}=r\frac{\partial u}{\partial t}\big|_{t=0}=r\psi(r), \\ v|_{r=0}=(ru)|_{r=0}=0\cdot u|_{r=0}=0. \quad \text{I 类齐次边界条件} \end{cases}$$

$$u(t,r)=\begin{cases} \frac{1}{2r}[(r+at)\varphi(r+at)+(r-at)\varphi(r-at)]+\frac{1}{2ar}\int_{r-at}^{r+at}\xi\psi(\xi)\mathrm{d}\xi, \quad t\leqslant\frac{r}{a}, \\ \frac{1}{2r}[(r+at)\varphi(r+at)-(at-r)\varphi(at-r)]+\frac{1}{2ar}\int_{at-r}^{r+at}\xi\psi(\xi)\mathrm{d}\xi, \quad t>\frac{r}{a} \end{cases}$$

其中, $f(r-at)$ 为发散波, $g(r+at)$ 为会聚波, $\frac{1}{r}$ 为波形的衰减 $(r>1)$ 或扩张 $(r<1)$ 因子

5 二阶 PDE 方程

$$a_{11}\frac{\partial^2 u}{\partial x^2}+2a_{12}\frac{\partial^2 u}{\partial x\partial y}+a_{22}\frac{\partial^2 u}{\partial y^2}+b_1\frac{\partial u}{\partial x}+b_2\frac{\partial u}{\partial y}+b_0u=0$$

或者可以写做

$$A_{11}\frac{\partial^2 u}{\partial \xi^2}+2A_{12}\frac{\partial^2 u}{\partial\xi\partial\eta}+A_{22}\frac{\partial^2 u}{\partial\eta^2}+B_1\frac{\partial u}{\partial\xi}+B_2\frac{\partial u}{\partial\eta}+B_0u=0$$

$$A_{11}=a_{11}\varphi_x^2+2a_{12}\varphi_x\varphi_y+a_{22}\varphi_y^2=0, \quad A_{22}=a_{11}\psi_x^2+2a_{12}\psi_x\psi_y+a_{22}\psi_y^2$$

$$A_{12}=a_{11}\varphi_x\psi_x+a_{12}\left(\varphi_x\psi_y+\varphi_y\psi_x\right)+a_{22}\varphi_y\psi_y,$$

$$B_1=a_{11}\varphi_{xx}+2a_{12}\varphi_{xy}+a_{22}\varphi_{yy}+b_1\varphi_x+b_2\varphi_y,$$

$$B_2=a_{11}\psi_{xx}+2a_{12}\psi_{xy}+a_{22}\psi_{yy}+b_1\psi_x+b_2\psi_y, \quad B_0=b_0(x(\xi,\eta),y(\xi,\eta))$$

▶得到特征方程:

$$a_{11}(\mathrm{d}y)^2-2a_{12}\,\mathrm{d}x\,\mathrm{d}y+a_{22}(\mathrm{d}x)^2=0$$

解的个数取决于:

$$\text{判别式}\Delta=a_{12}^2-a_{11}a_{22}$$

(1) $\Delta=a_{12}^2-a_{11}a_{22}>0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a_{12}+\sqrt{\Delta}}{a_{11}} \text{ 和 } \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{a_{12}-\sqrt{\Delta}}{a_{11}}, \text{ 解之得两族特征曲线: } \varphi(x,y)=h_1\psi(x,y)=h_2$$

$$\Rightarrow A_{11} = P\varphi = 0, \quad A_{22} = P\psi = 0$$

得到双曲型方程标准型

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{1}{2A_{12}}\left(B_1\frac{\partial u}{\partial\xi} + B_2\frac{\partial u}{\partial\eta} + B_0 u\right) = 0$$

求解注意复杂的链式法则：

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial u}{\partial\eta}\frac{\partial\eta}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial\xi}\right) = \frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\xi}\right)\frac{\partial\xi}{\partial x} + \frac{\partial}{\partial\eta}\left(\frac{\partial u}{\partial\xi}\right)\frac{\partial\eta}{\partial x}$$

对于$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$，其通解一般形式是$u = \int g(\eta)\mathrm{d}\eta + F(\xi) \triangleq G(\eta) + F(\xi)$

注: 双曲型另一标准型

$$令\begin{cases} t = \frac{1}{2}(\xi+\eta) \\ S = \frac{1}{2}(\xi-\eta) \end{cases} \Rightarrow \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial s^2} + 4\left(\tilde{B}_1\frac{\partial u}{\partial t} + \tilde{B}_2\frac{\partial u}{\partial s} + \tilde{B}_0 u\right) = 0$$

(2) $\Delta = a_{12}^2 - a_{11}a_{22} = 0$

只有一组隐式解$\varphi(x,y) = h$，任选$\psi(x,y)$, 使$J(\varphi,\psi) \neq 0$

得到抛物线型方程

$$\frac{\partial^2 u}{\partial\eta^2} + \frac{1}{A_{22}}\left(B_1\frac{\partial u}{\partial\xi} + B_2\frac{\partial u}{\partial\eta} + B_0 u\right) = 0$$

(3) $\Delta = a_{12}^2 - a_{11}a_{22} < 0$ 令新函数：$\xi =$ 实部, $\eta =$ 虚部，得到椭圆型方程

$$\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} + \frac{1}{A_{11}}\left(B_1\frac{\partial u}{\partial\xi} + B_2\frac{\partial u}{\partial\eta} + B_0 u\right) = 0$$

注: 经典方程

调和方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$——椭圆型，
热传导 $\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = 0$——抛物型，
波动方程 $\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$——双曲型

6 冲量原理

以无限长弦在外力作用下微小横振动方程的初值问题为例（相当于在波动方程加入干扰项）

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(t,x), t > 0, -\infty < x < +\infty, \\ u|_{t=0} = 0, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0. \quad f(t,x): 单位质量弦所受外力. \end{cases}$$

命题 (冲量原理) 设 $w=w(t,x;\tau)$ 是下面齐次方程初值问题的解:

$$\begin{cases}\dfrac{\partial^2 w}{\partial t^2}=a^2\dfrac{\partial^2 w}{\partial x^2}, t>\tau>0\\ w|_{t=\tau}=0,\quad \left.\dfrac{\partial w}{\partial t}\right|_{t=\tau}=f(\tau,x)\end{cases}$$

则 $u(t,x)=\displaystyle\int_0^t w(t,x;\tau)\mathrm{d}\tau$ 是初值问题方程的解.

证明: 证明冲量原理

$$\frac{\partial u}{\partial t}=w(t,x;\tau)|_{\tau=t}+\int_0^t\frac{\partial w(t,x;\tau)}{\partial t}d\tau=\int_0^t\frac{\partial w(t,x;\tau)}{\partial t}d\tau$$

$$\frac{\partial^2 u}{\partial t^2}=\left.\frac{\partial w(t,x;\tau)}{\partial t}\right|_{\tau=t}+\int_0^t\frac{\partial^2 w(t,x;\tau)}{\partial t^2}\,\mathrm{d}\tau=f(t,x)+a^2\int_0^t\frac{\partial^2 w(t,x;\tau)}{\partial x^2}\,\mathrm{d}\tau.$$

$$=f(t,x)+a^2\frac{\partial^2 u}{\partial x^2}.\quad u(0,x)=\int_0^0 w(0,x;\tau)\mathrm{d}\tau=0.$$

注

有时令 $t'=t-\tau$ 解决问题更加方便

例 2.3.6: 线性叠加原理 + 冲量原理

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}=a^2\dfrac{\partial^2 u}{\partial x^2}+f(t,x),\quad t>0,x\in\mathbb{R}\\ u|_{t=0}=\varphi(x),\quad \frac{\partial u}{\partial t}|_{t=0}=\psi(x),\end{cases}$$

由线性方程叠加原理 $u=u_1+u_2$，满足

$$\begin{cases}\dfrac{\partial^2 u_1}{\partial t^2}=a^2\dfrac{\partial^2 u_1}{\partial x^2}+f(t,x), t>0,x\in\mathbb{R}\\ u_1|_{t=0}=0,\quad \frac{\partial u_1}{\partial t}|_{t=0}=0\end{cases}\qquad \begin{cases}\dfrac{\partial^2 u_2}{\partial t^2}=a^2\dfrac{\partial^2 u_2}{\partial x^2}, t>0,x\in\mathbb{R}\\ u_2|_{t=0}=\varphi(x),\ \frac{\partial u_2}{\partial t}|_{t=0}=\psi(x)\end{cases}$$

$$\boxed{u_1(x,t)=\frac{1}{2a}\int_0^t\left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau.\ \text{(冲量原理和 d'Alembert 公式)}}$$

$$u_2(t,x)=\frac{1}{2}\{\varphi(x+at)+\varphi(x-at)\}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi.$$

$$\bigstar\quad u(t,x)=\frac{1}{2}\{\varphi(x+at)+\varphi(x-at)\}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi+\frac{1}{2a}\int_0^t\left\{\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\tau,\xi)\mathrm{d}\xi\right\}\mathrm{d}\tau$$

一般冲量原理:

$$u\left(t,x_1,x_2,\cdots,x_n\right)=\int_0^t w\left(t-\tau,x_1,x_2,\cdots,x_n;\tau\right)\mathrm{d}\tau$$

第 4 节 分离变量法

分离变量法：用于至少有一个变量是有界的问题

1. 两端固定的弦在初始微扰动影响下的自由横振动 $u=u(t,x)$：

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, t>0, 0<x<l, \\ u|_{x=0}=u|_{x=l}=0, t>0, \end{cases}$$

(两端固定在平衡位) 设$u(x,t)=T(t)X(x)$. 解固有值问题：

$$\begin{cases} X''(x)+\lambda X(x)=0, \quad 0<x<l \\ X(0)=X(l)=0 \end{cases}$$

固有值: 所有使固有值问题有非零解的 λ 值.

固有函数: 对任一固有值 λ, 固有值问题的相应非零解.

全部固有值为

$$\lambda_n=\left(\frac{n\pi}{l}\right)^2, \quad n=1,2,\cdots.$$

与 λ_n 相应的固有函数为 $X_n(x)=\sin\frac{n\pi}{l}x$.

对每个 $\lambda_n=\left(\frac{n\pi}{l}\right)^2$, 代入原方程, 得

$$T''(t)+a^2\lambda_n T(t)=0$$

, 特征方程:

$$\mu^2+a^2\left(\frac{n\pi}{l}\right)^2=0. \quad \mu_{n1}=\frac{an\pi}{l}i, \mu_{n2}=-\frac{an\pi}{l}i$$

. 故通解

$$T_n(t)=C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t, n=1,2,\cdots$$

根据叠加原理, 全部 u_n 的线性叠加仍然满足方程，故解为：

$$u(x,t)=\sum_{n=1}^{+\infty}u_n(x,t)=\sum_{n=1}^{+\infty}\left(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x.$$

2. 如果初值条件：$u(0,x)=\varphi(x), \quad u_t|_{x=0}=\psi(x), \quad 0<x<l$ 求 C_n, D_n

广义 Fourier 展开公式后得 $\because u|_{t=0}=\sum_{n=1}^{+\infty}C_n\sin\frac{n\pi}{l}x=\varphi(x)$. 系数为：

$$C_n=\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{n\pi}{l}\xi d\xi. \quad u_t|_{t=0}=\psi(x)$$

$$\frac{an\pi}{l}D_n=\frac{2}{l}\int_0^l\psi(\xi)\sin\frac{n\pi}{l}\xi d\xi, D_n=\frac{2}{an\pi}\int_0^l\psi(\xi)\sin\frac{n\pi}{l}\xi d\xi.$$

第 5 节 线性叠加原理

设 L 是将一个函数映射成另一个函数的算子, 若满足：$L(a_1u_1+a_2u_2)=a_1Lu_1+a_2Lu_2, \forall u_1,u_2\in D(L), \forall a_1,a_2\in\mathbb{R}$, 则称 L 为 $D(L)$ 上的线性算子.

例 2.5.1: 非齐次边值

$$\begin{cases} \Delta_2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1, x^2 + y^2 < 1, \\ u|_{x^2+y^2=1} = x^2 \end{cases}$$

设特解 $u_0 = a\left(x^2 + y^2\right)$, 代入方程, $\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} = 2a + 2a = 4a = 1$, 故 $a = \frac{1}{4}, u_0 = \frac{1}{4}\left(x^2 + y^2\right)$
令 $v = u - u_0$. 因 Δ_2 及边界条件是线性的, 故

$$\begin{cases} \Delta_2 v = \Delta_2 u - \Delta_2 u_0 = 1 - 1 = 0, \quad 1, x, y, x^2 - y^2, xy \text{ 都是调和函数} \\ v|_{x^2+y^2=1} = u|_{x^2+y^2=1} - u_0|_{x^2+y^2=1} = x^2 - \frac{1}{4}. \end{cases}$$

比较系数得到: $v = \frac{1}{4} + \frac{1}{2}\left(x^2 - y^2\right)$

$$u = v + u_0 = \frac{1}{4} + \frac{3}{4}x^2 - \frac{1}{4}y^2$$

第三章 固有值问题

第 1 节 线性函数空间

定理 3.1.1: 线性函数空间、完备正交基

$$\text{设} -\infty < a < b < +\infty. \quad L^2[a,b] \triangleq \left\{ f(x) \middle| \int_a^b |f(x)|^2 \, \mathrm{d}x < +\infty \right\}$$

加权的函数空间, 称$\rho(x)$ 为权函数. $\rho(x) \in C[a,b]$, $\rho(x) > 0, \quad \forall x \in [a,b]$

$$L^2_\rho[a,b] \triangleq \left\{ f(x) \middle| \int_a^b \rho(x) \left|f(x)\right|^2 \mathrm{d}x < +\infty \right\}$$

$L^2_\rho[a,b]$ 是无限维线性函数空间. 因为 $\{1, x, x^2, \cdots, x^n, \cdots\} \subseteq L^2_\rho[a,b]$.

性质 3.1.1: 线性函数

1. 内积:

$$< f(x), g(x) > \triangleq \int_a^b \rho(x) f(x) \overline{g(x)} \mathrm{d}x$$

2. 模:

$$\|f(x)\| = \sqrt{< f(x), f(x) >} = \sqrt{\int_a^b \rho(x) |f(x)|^2 \, \mathrm{d}x}.$$

3. 正交:

$$< f(x), g(x) > = 0$$

4. 相等：

$$\|f(x)-g(x)\| = \sqrt{\int_a^b \rho(x)|f(x)-g(x)|^2 dx} = 0$$

性质 3.1.2: 正交基

1. 正交：

$$< X_j(x), X_k(x) > = \begin{cases} 0, j \neq k \\ \|X_j(x)\|^2 \neq 0, j = k \end{cases}$$

2. 完备：

$$f(x) \in L_\rho^2[a,b], f(x) = \sum_{j=1}^{+\infty} c_j X_j(x), \text{即} \lim_{N\to+\infty} \left\| f(x) - \sum_{j=1}^{N} c_j X_j(x) \right\| = 0,$$

$$c_k = \frac{\langle f(x), X_k(x)\rangle}{\langle X_k(x), X_k(x)\rangle} = \frac{\langle f(x), X_k(x)\rangle}{\|X_k(x)\|^2} = \frac{\int_a^b \rho(x) f(x) \overline{X_k(x)} \mathrm{d}x}{\int_a^b \rho(x) \left|X_k(x)\right|^2 \mathrm{d}x}, \quad k = 1, 2, \cdots$$

第 2 节 Strum-Liouville 方程

Strum-Liouville 方程建立了固有函数和线性函数空间正交基的对应关系！

$$[k(x)X'(x)]' - q(x)X(x) + \lambda\rho(x)X(x) = 0$$

$k(x) \in C^1[a,b], q(x), \quad \rho(x) \in C[a,b], \quad \lambda \in \mathbb{R}$ 例如：

- $k(x) = 1 - x^2, \quad q(x) = \dfrac{m^2}{1-x^2}, \quad \rho(x) \equiv 1$

$$\left[\left(1-x^2\right) y'(x)\right]' + \left(\lambda - \frac{m^2}{1-x^2}\right) y(x) = 0$$

- $X''(x) + \lambda X(x) = 0$, S-L 型方程, $k(x) \equiv 1, q(x) \equiv 0, \rho(x) \equiv 1,$
- $(rR'(r))' + \dfrac{\lambda}{r}R(r) = 0, S - L$ 型方程, $k(r) = r, q(r) \equiv 0, \rho(r) = \dfrac{1}{r}.$

例 3.2.1: 普通二阶线性微分方程化成 S-L 标准型

$$b_0(x)X''(x) + b_1(x)X'(x) + b_2(x)X(x) + \lambda b_3(x)X(x) = 0$$

$$X''(x) + \frac{b_1(x)}{b_0(x)}X'(x) + \frac{b_2(x)}{b_0(x)}X(x) + \lambda\frac{b_3(x)}{b_0(x)}X(x) = 0$$

两边乘以 $k(x)=\exp(\int\dfrac{b_1(x)}{b_0(x)}\mathrm{d}x)$ 得 $S-L$ 型方程

$$\left\{\mathrm{e}^{\int\frac{b_1(x)}{b_0(x)}\mathrm{d}x}X'(x)\right\}'+\frac{b_2(x)}{b_0(x)}\mathrm{e}^{\int\frac{b_1(x)}{b_0(x)}\mathrm{d}x}X(x)+\lambda\frac{b_3(x)}{b_0(x)}\mathrm{e}^{\int\frac{b_1(x)}{b_0(x)}\mathrm{d}x}X(x)=0.$$

$$k(x)=\mathrm{e}^{\int\frac{b_1(x)}{b_0(x)}\mathrm{d}x},q(x)=-\frac{b_2(x)}{b_0(x)}k(x),\quad \rho(x)=\frac{b_3(x)}{b_0(x)}k(x).$$

1 常点 S-L 方程固有值问题

$$\begin{cases}[k(x)X'(x)]'-q(x)X(x)+\lambda\rho(x)X(x)=0,-\infty<a<x<b<+\infty\\ \alpha_1X(a)-\beta_1X'(a)=0,\quad \alpha_2X(b)+\beta_2X'(b)=0.\end{cases}$$

♣ 固有值可数个

♣ 固有值非负 $\lambda_n\geqslant 0$，且$\lambda_0=0$ 时候只存在第 II 类边界条件

♣ 正交性：当 $n\neq m$ 时, $<X_n(x),X_m(x)>=\int_a^b\rho(x)X_n(x)\overline{X_m(x)}\mathrm{d}x=0.$

♣ 完备性： 全体固有函数 $\{X_n(x)\}$ 构成$L_\rho^2[a,b]$ 的一组正交基

证明: 相关性质证明

证明固有函数在 $L_\rho^2[a,b]$ 中正交” . 先证明 L 在 $\mathscr{M}$ 内是自共轭算子即

$$\forall f(x),g(x)\in\mathscr{M},<Lf(x),g(x)>=\langle f(x),Lg(x)>$$

$$\begin{aligned}0&=<LX_n(x),X_m(x)>-<X_n(x),LX_m(x)>\\&=<\lambda_nX_n(x),X_m(x)>-<X_n(x),\lambda_mX_m(x)>\\&=\lambda_n<X_n(x),X_m(x)>-\overline{\lambda_m}<X_n(x),X_m(x)>=\left(\lambda_n-\overline{\lambda_m}\right)<X_n(x),X_m(x)>.\end{aligned}$$

已证 $\lambda_n\geq 0$ 和 $\lambda_m\geq 0$, 故是实数.
故 $(\lambda_n-\lambda_m)<X_n(x),X_m(x)>=0$. $\lambda_n\neq\lambda_m$, 故 $<X_n(x),X_m(x)>=0$,
即 $X_n(x)$ 和 $X_m(x)$ 在 $L_\rho^2[a,b]$ 中正交.
利用自共轭紧算子的谱理论, 可证 (1) 固有值、固有函数可数, $\lim_{\to+\infty}\lambda_n=+\infty$;
(4) 全体固有函数在 $L_\rho^2[a,b]$ 中完备, 构成 $L_\rho^2[a,b]$ 的基, 且是 $L_\rho^2[a,b]$ 的正交基.

2 周期条件下 S-L 方程固有值问题

$$\begin{cases}[k(x)X'(x)]'-q(x)X(x)+\lambda\rho(x)X(x)=0,a<x<b\\ X(a)=X(b),X'(a)=X'(b)\end{cases}$$

例 3.2.2: 有界细杆的温度分布

$$\begin{cases} u_t = a^2 u_{xx}, & t>0, 0<x<l, \quad \text{(有界, 分离变量法)} \\ u_x|_{x=0} = 0, & (u_x+\gamma u)|_{x=l} = 0, \quad \gamma = \dfrac{h}{k} > 0. \\ u|_{t=0} = \varphi(x). \end{cases}$$

$$\frac{T'}{a^2T} = \frac{X''}{X} = -\lambda, X_x = 0 \Longrightarrow X(x) = A\cos\omega_n t$$

$$X'(l)+\gamma X(l) = -A\omega\sin\omega l + \gamma A\cos\omega l = 0, A\neq 0, \frac{\sin\omega l}{\cos\omega l} = \tan\omega l = \frac{\gamma}{\omega}$$

$$\text{对每个}\lambda_n = \omega_n^2, \quad T'(t)+a^2\omega_n^2 T(t) = 0, \quad T_n(t) = C_n e^{-a^2\omega_n^2 t},$$

$$u_n(t) = \sum_{n=1}^{+\infty} A_n e^{-a^2\omega_n^2 t}\cos\omega_n x, \quad A_n = \frac{<\cos\omega_n x, \varphi(x)>}{<\cos\omega_n x, \cos\omega_n x>}$$

$$<\cos\omega_n x, \cos\omega_n x> = \|\cos\omega_n x\|^2 = \int_0^l \cos^2\omega_n x\,\mathrm{d}x = \int_0^l \frac{1+\cos 2\omega_n x}{2}\mathrm{d}x$$

$$= \frac{l}{2}+\frac{\sin 2\omega_n l}{4\omega_n} = \frac{l}{2}+\frac{1}{4\omega_n}\cdot\frac{2\tan\omega_n l}{1+\tan^2\omega_n l} = \frac{l}{2}+\frac{1}{4\omega_n}\cdot\frac{2\cdot\frac{\gamma}{\omega_n}}{1+\left(\frac{\gamma}{\omega_n}\right)^2} = \frac{l}{2}+\frac{\gamma}{2\left(\omega_n^2+\gamma^2\right)}$$

$$u(x,t) = \sum_{n=1}^{+\infty}\frac{2}{l+\frac{\gamma}{\omega_n^2+\gamma^2}}\left(\int_0^l \varphi(\xi)\cos\omega_n\xi\mathrm{d}\xi\right)\mathrm{e}^{-a^2\omega_n^2 t}\cos\omega_n x$$

第 3 节 Laplace 方程

1 二维 Laplace 方程圆内 I 类边值问题 (Dirichlet 问题)

$$\begin{cases} \Delta_2 u = 0, x^2+y^2<a^2 \\ u|_{x^2+y^2=a^2} = F(x,y)\text{(一般的连续函数)} \end{cases}$$

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0, & 0<r<a, -\infty<\theta<+\infty \\ u|_{r=a} = F(a\cos\theta, a\sin\theta). \end{cases}$$

$$r\left(rR'(r)\right)' - \lambda R(r) = 0, \quad \begin{cases} \Theta''(\theta)+\lambda\Theta(\theta) = 0 \\ \Theta(\theta+2\pi) = \Theta(\theta) \end{cases}$$

$$\frac{\frac{1}{r}\left(rR'(r)\right)'}{\frac{1}{r^2}R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

(a) 当$\lambda=0$ 时, $\mu^2=0, \mu_1=\mu_2=0$. 通解$\Theta(\theta)=A+B\theta$ 故$\lambda_0=0$ 是固有值，相应固有函数$\Theta_0(\theta)\equiv 1$
(b) 当$\lambda=-\omega^2<0, \omega>0$ 时，$\mu^2-\omega^2=0$ 无非零解
(c). 当$\lambda=\omega^2>0, \omega>0$ 时, $\mu^2+\omega^2=0, \mu_1=\omega i, \mu_2=-\omega\mathrm{i}$. 故 $\lambda_n=n^2(n=1,2,\cdots)$ 是固有值, 与 n^2 相应固有函数 $\Theta_n(\theta)=A_n\cos n\theta+B_n\sin n\theta$.

固有值：$\lambda_0=0, \lambda_n=n^2, n=1,2,\cdots$ 相应固有函数: $\Theta_0(\theta)\equiv 1, \Theta_n(\theta)=A_n\cos n\theta+B_n\sin n\theta$.

$\lambda_0=0$ 解 $R_0=C_0+D_0\ln r$, $\lambda=n^2$, $R_n=C_nr^n+D_nr^{-n}$

(1) 圆内只含正次幂项

$$u(r,\theta)=\frac{A_0}{2}\cdot 1+\sum_{n=1}^{+\infty}\left(\frac{r}{a}\right)^n(A_n\cos n\theta+B_n\sin n\theta)$$

广义 Fourier 展开求系数 (根据 $\int_0^{2\pi}\cos^2 n\varphi d\varphi=4\cdot\frac{1}{2}\frac{\pi}{2}=\pi$):

$$A_0=\frac{1}{\pi}\int_0^{2\pi}F(a\cos\varphi,a\sin\varphi)\cdot 1\,\mathrm{d}\varphi,A_n=\frac{1}{\pi}\int_0^{2\pi}F(a\cos\varphi,a\sin\varphi)\cos n\varphi\mathrm{d}\varphi,$$
$$B_n=\frac{1}{\pi}\int_0^{2\pi}F(a\cos\varphi,a\sin\varphi)\sin n\varphi\mathrm{d}\varphi,n=1,2,\cdots.$$

(2) 环域解（通解公式）:

$$u(r,\theta)=\left(\frac{C_0}{2}+\frac{D_0}{2}\ln r\right)\cdot 1+\sum_{n=1}^{+\infty}\left(C_nr^n+D_nr^{-n}\right)(A_n\cos n\theta+B_n\sin n\theta)$$

剩余步骤都在进行比较系数

例 3.3.1: 侧面绝热无内热源但有耗散的细长杆温度分布问题, 左端温度恒为 0 度, 右端绝热.

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}-bu,t>0,\quad 0<x<l\\ u|_{x=0}=0,\quad \frac{\partial u}{\partial x}\Big|_{x=l}=0,t>0\\ u|_{t=0}=\varphi(x)\end{cases}$$

$$\frac{T'(t)}{a^2T(t)}+\frac{b}{a^2}=\frac{X''(x)}{X(x)}=-\lambda,$$

与λ_n 相应固有函数$X_n(x)=\sin\frac{(2n+1)\pi}{2l}x,\qquad \lambda=\omega^2=\left(\frac{(2n+1)\pi}{2l}\right)^2$

$$T_n(t)=C_n\mathrm{e}^{-\left(a^2\lambda_n+b\right)t},n=0,1,2,\cdots$$

根据 $u=\sum_{n=0}^{+\infty}C_n\mathrm{e}^{-\left(a^2\lambda_n+b\right)t}\sin\frac{(2n+1)\pi}{2l}x\Longrightarrow u|_{t=0}=\sum_{n=0}^{+\infty}C_n\sin\frac{(2n+1)\pi}{2l}x=\varphi(x)$

$$C_n=\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{(2n+1)\pi}{2l}\xi d\xi,\quad n=0,1,2,\cdots$$

$$u(x,t)=\sum_{n=0}^{+\infty}\left(\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{(2n+1)\pi}{2l}\xi\mathrm{d}\xi\right)\mathrm{e}^{-\left\{a^2\left(\frac{(2n+1)\pi}{2l}\right)^2+b\right\}t}\sin\frac{(2n+1)\pi}{2l}x$$

2 扇形区域 Laplace 问题

$$\begin{cases}\Delta_2u=0,\quad 1<r<\mathrm{e},\quad 0<\theta<\pi/3\\ u|_{r=1}=0,\quad u|_{r=\mathrm{e}}=0\\ \bar{u}|_{\theta=0}=0,\quad u|_{\theta=\frac{\pi}{3}}=r\end{cases}$$

$$\frac{\frac{1}{r}\left(rR'(r)\right)'}{\frac{1}{r^2}R(r)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=-\lambda$$

$r^2R''(r)+rR'(r)+\omega^2R(r)=0, 1<r<e$. 欧拉方程令 $r=\mathrm{e}^t$, 则 $s=\ln r$,

方程化为 $\frac{\mathrm{d}^2R}{\mathrm{d}S^2}+\omega^2R=0$. $\Rightarrow$ 通解

$$R=A\cos\omega s+B\sin\omega s=A\cos(\omega\ln r)+B(\sin\omega\ln r)$$

代入边界条件推得 $A=0,\lambda_n=\omega_n^2=(n\pi)^2, R_n(r)=\sin(n\pi\ln r)$.

$\{\sin(n\pi\ln r), n=1,2,\cdots\}$ 是 $L^2_{\frac{1}{r}}[1,\mathrm{e}]$ 的一组正交基.

对每个 $\lambda_n=(n\pi)^2,\quad \Theta''(\theta)-(n\pi)^2\Theta(\theta)=0$. 通解 $\Theta_n(\theta)=C_n\mathrm{e}^{n\pi\theta}+D_n\mathrm{e}^{-n\pi\theta},\quad n=1,2,\cdots$

$$u(r,\theta)=\sum_{n=1}^{+\infty}R_n(r)\Theta_n(\theta)=\sum_{n=1}^{+\infty}\left(C_n\mathrm{e}^{n\pi\theta}+D_n\mathrm{e}^{-n\pi\theta}\right)\sin(n\pi\ln r)$$

$$2C_n\sinh\left(\frac{n\pi^2}{3}\right)=\frac{\langle r,\sin(n\pi\ln r)\rangle}{\langle\sin(n\pi\ln r),\sin(n\pi\ln r)\rangle}=\frac{\int_1^{\mathrm{e}}\frac{1}{r}\cdot r\sin(n\pi\ln r)\mathrm{d}r}{\int_1^{\mathrm{e}}\frac{1}{r}\sin^2(n\pi\ln r)\mathrm{d}r}$$

$$=\frac{\int_1^{\mathrm{e}}r\sin(n\pi\ln r)\mathrm{d}\ln r}{\int_1^{\mathrm{e}}\sin^2(n\pi\ln r)\mathrm{d}\ln r}\overset{s=\ln r}{=\!=\!=}\frac{\int_0^1\mathrm{e}^S\sin(n\pi S)\mathrm{d}S}{\displaystyle\int_0^1\frac{1-\cos2(n\pi\mathrm{S})}{2}}=\frac{n\pi\left\{1-(-1)^n\mathrm{e}\right\}}{1+n^2\pi^2}\cdot2$$

$$\text{故}u(r,\theta)=\sum_{n=1}^{+\infty}\frac{n\pi\left\{1-(-1)^n\mathrm{e}\right\}^2}{(1+n^2\pi^2)\sinh\frac{n\pi^2}{3}}\left(\mathrm{e}^{n\pi\theta}-\mathrm{e}^{-n\pi\theta}\right)\sin(n\pi\ln r)$$

注: 证明 $\displaystyle\int_0^1\mathrm{e}^t\sin(n\pi t)\mathrm{d}t=\frac{n\pi\left\{1-(-1)^n\mathrm{e}\right\}}{1+n^2\pi^2}$

$$\int_0^1\mathrm{e}^t\sin(n\pi t)\mathrm{d}t\overset{\text{分部积分}}{=}\int_0^1\sin(n\pi t)\mathrm{d}\mathrm{e}^t=-n\pi\int_0^1\mathrm{e}^t\cos(n\pi t)\mathrm{d}t$$

$$\overset{\text{分部积分}}{=}-n\pi\int_0^1\cos(n\pi t)\mathrm{d}\mathrm{e}^t=-\,n\pi\mathrm{e}^t\cos(n\pi t)\Big|_0^1+n\pi\int_0^1\{-n\pi\sin(n\pi t)\}\mathrm{e}^t\,\mathrm{d}t$$

$$=-n\pi\left\{(-1)^n\mathrm{e}-1\right\}-n^2\pi^2\int_0^1\mathrm{e}^t\sin(n\pi t)\mathrm{d}t\Leftarrow\left(\cos n\pi=(-1)^n\right)$$

$$\text{解得}\int_0^1\mathrm{e}^t\sin(n\pi t)\mathrm{d}t=\frac{n\pi\left\{1-(-1)^n\mathrm{e}\right\}}{1+n^2\pi^2}$$

3　矩形区域 Laplace 问题

矩形区域考虑直角坐标:

$$\begin{cases}\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0, 0<x<a, 0<y<b\\ \frac{\partial u}{\partial y}\Big|_{y=0}=0,\quad \frac{\partial u}{\partial y}\Big|_{y=b}=0\\ u|_{x=0}=0,\ u|_{x=a}=f(y)\end{cases}$$

$$\frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}\triangleq\lambda,$$

$$X''(x)-\lambda X(x)=0,(\text{ 不是满足 S-L 定理要求的 S- L 型方程})$$

$$\begin{cases} Y''(y)+\lambda Y(y)=0 \ (\text{满足 S-L 定理要求的 S-L 型方程}) \\ Y'(0)=Y'(b)=0 \end{cases}$$

据 $S-L$ 定理, 所有固有值 $\lambda \geq 0$. 无 I、III 类边界条件, 故 $\lambda_0=0$ 是固有值, 相应固有函数 $Y_0(y)\equiv 1$.

$Y(y)=A\cos\omega y+B\sin\omega y$ 得 $\omega_n=\dfrac{n\pi}{b},\lambda_n=\omega_n^2,\quad Y_n(y)=\cos\omega_n y,(\text{取 } A=1,)\, n=1,2,,$

由 S- L 定理得, $\{1,\cos\omega_n y,n=1,2,\cdots\}$ 是 $L^2[0,b]$ 的正交基.

对 $\lambda_0=0,X''(x)=0$, 通解 $X_0(x)=C_0+D_0x$.

对 $\lambda_n=\omega_n^2,X''(x)-\omega_n^2X(x)=0$, 通解 $X_n(x)=C_n\mathrm{e}^{\omega_n x}+D_n\mathrm{e}^{-\omega_n x}$

$$u(x,t)=(C_0+D_0x)\cdot 1+\sum_{n=1}^{+\infty}\left(C_n\mathrm{e}^{\omega_n x}+D_n\mathrm{e}^{-\omega_n x}\right)\cos\omega_n y$$

代入方程边界条件，得到

$$D_0=\frac{1}{ab}\int_0^b f(\xi)\mathrm{d}\xi,C_n=\frac{2}{b\left(\mathrm{e}^{\omega_n a}-\mathrm{e}^{-\omega_n a}\right)}\int_0^b f(\eta)\cos\omega_n\eta\mathrm{d}\eta,D_n=-C_n,n\geq 1,C_0=0$$

$$\Rightarrow u(x,t)=\left(\frac{1}{ab}\int_0^b f(\xi)\mathrm{d}\xi\right)x+\sum_{n=1}^{+\infty}\frac{2\int_0^b f(\eta)\cos\omega_n\eta\mathrm{d}\eta}{b\left(\mathrm{e}^{\omega_n a}-\mathrm{e}^{-\omega_n a}\right)}\left(\mathrm{e}^{\omega_n x}-\mathrm{e}^{-\omega_n x}\right)\cos\omega_n y$$

4 环域 Poisson 方程边值问题

例如环域，无冲量原理

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=12x^2-12y^2, a^2<x^2+y^2<b^2 \\ u|_{x^2+y^2=a^2}=1, \quad \frac{\partial u}{\partial n}\Big|_{x^2+y^2=b^2}=0 \end{cases}$$

方法 1： Fourier 展开求系数， 设$u(r,\theta)=R(r)\Theta(\theta)$

$$\frac{\frac{1}{r}\left(rR'(r)\right)'}{\frac{1}{r^2}R(r)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}=\lambda,$$

$$\begin{cases} \Theta''(\theta)+\lambda\Theta(\theta)=0,\rho(\theta)\equiv 1, \\ \Theta(\theta+2\pi)=\Theta(\theta)(\text{ 周期条件}). \end{cases}$$

$$\lambda_0=0,\quad \Theta_0(\theta)=1,$$

$$\lambda_n=n^2,\Theta_n(\theta)=\begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix},n\geq 1$$

将$f=12x^2-12y^2,\varphi_1(x)=1,\varphi_2(x)=0$ 按固有函数基展开

$$f=12x^2-12y^2=12r^2\left(\cos^2\theta-\sin^2\theta\right)=12r^2\cos 2\theta.$$

将u 代入$\Delta_2 u\equiv\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}=12r^2\cos 2\theta$ 比较固有函数$1,\cos n\theta,\sin n\theta$ 的系数得

$$\begin{cases} \frac{1}{r}(rA_0'(r))'=0, \\ A_0(a)=1,A_0'(b)=0. \end{cases}\quad \begin{cases} \frac{1}{r}\left(rA_2'(r)\right)'-\frac{2^2}{r^2}A_2(r)=12r^2 \\ A_2(a)=0,A_2'(b)=0. \end{cases}$$

$$\left\{\begin{array}{l} A_2(a)=a^2C_2+\frac{1}{a^2}D_2+a^4=0, \\ A_2'(b)=2bC_2-\frac{2}{b^3}D_2+4b^3=0 \end{array}\right. \left\{\begin{array}{l} C_2=-\frac{a^6+2b^6}{a^4+b^4} \\ D_2=\frac{2a^4b^6-b^4a^6}{a^4+b^4} \end{array}\right.$$

方法 2: **特解法:** 积分两次, 取$F(x)=x^4, G(y)=-y^4$

$$\begin{aligned} v(x,y)=x^4-y^4 &= r^4\left(\cos^4\theta-\sin^4\theta\right) \\ &= r^4\left(\cos^2\theta+\sin^2\theta\right)\left(\cos^2\theta-\sin^2\theta\right)=r^4\cos 2\theta \end{aligned}$$

令$w=u-v=u-r^4\cos 2\theta$,

$$\left\{\begin{array}{l} \Delta_2 w=0, \quad a<r<b \\ w|_{r=a}=1-a^4\cos 2\theta, \left.\frac{\partial w}{\partial r}\right|_{r=b}=-4b^3\cos 2\theta \end{array}\right.$$

根据 $w(r,\theta)=(C_0+D_0\ln r)\cdot 1+\sum_{n=1}^{+\infty}\left(C_n r^n+D_n r^{-n}\right)\left(A_n\cos n\theta+B_n\sin n\theta\right)$

$$\{1,\cos n\theta,\sin n\theta, n=1,2,\cdots\} \text{ 是} L^2[0,2\pi] \text{ 的正交基}$$

接着根据三角函数内的倍数确定 $n=2$

$$w(r,\theta)=(C_0+D_0\ln r)\cdot 1+\left(C_2A_2r^2+D_2A_2r^{-2}\right)\cos 2\theta$$

$$w|_{r=a}=(C_0+D_0\ln a)\cdot 1+\left(C_2A_2a^2+D_2A_2a^{-2}\right)\cos 2\theta=1-a^4\cos 2\theta$$

$$\left.\frac{\partial w}{\partial r}\right|_{r=b}=\frac{D_0}{b}\cdot 1+\left(2bC_2A_2-2D_2A_2b^{-3}\right)\cos 2\theta=-4b^3\cos 2\theta$$

$$u(r,\theta)=v(r,\theta)+w(r,\theta)=r^4\cos 2\theta+1+\left(-\frac{2b^6+a^6}{a^4+b^4}r^2+\frac{2a^4b^6-b^4a^6}{(a^4+b^4)\,r^2}\right)\cos 2\theta$$

注: 与物理学的联系

在解决静电学和静磁学问题的时候，常常用到这样的确定边界的 Poisson 方程

5 矩形域 Poisson 方程边值问题

$$\left\{\begin{array}{l} \Delta_2 u=f(x,y), \quad a_1<x<a_2, a_3<y<a_4 \\ \left.\left(\alpha_1 u-\beta_1\frac{\partial u}{\partial x}\right)\right|_{x=a_1}=\varphi_1(y), \quad \left.\left(\alpha_2 u+\beta_2\frac{\partial u}{\partial x}\right)\right|_{x=a_2}=\varphi_2(y) \\ \left.\left(\alpha_3 u-\beta_3\frac{\partial u}{\partial y}\right)\right|_{y=a_3}=\varphi_3(x), \quad \left.\left(\alpha_4 u+\beta_4\frac{\partial u}{\partial y}\right)\right|_{y=a_4}=\varphi_4(x) \end{array}\right.$$

Step1 处理边界条件中非齐次边界项.

$$\text{令} v(x,y)=\left\{\begin{array}{ll} A(y)x+B(y), & \text{若} \alpha_1,\alpha_2 \text{ 不同时为} 0, \\ A(y)x^2+B(y)x, & \text{若} \alpha_1=\alpha_2=0. \end{array}\right.$$

Step2 令$w=u-v$,

$$\left\{\begin{array}{l} \Delta_2 w=\Delta_2 u-\Delta_2 v=f(x,y)-\Delta_2 v\triangleq F(x,y), a_1<x<a_2, a_3<y<a_4 \\ \left.\left(\alpha_1 w-\beta_1\frac{\partial w}{\partial x}\right)\right|_{x=a_1}=0, \quad \left.\left(\alpha_2 w+\beta_2\frac{\partial w}{\partial x}\right)\right|_{x=a_2}=0 \\ w \text{ 分别在} y=a_3 \text{ 和} y=a_4 \text{ 的边界条件} \end{array}\right.$$

Step3 用 Fourier 展开法求w，设 $w(x,y)=X(x)Y(y)$ 代入 $\Delta_2 w=0$.

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ \alpha_1 X(a_1)-\beta_1 X'(a_1)=0, \alpha_2 X(a_2)+\beta_2 X'(a_2)=0 \end{cases}$$

第 4 节 齐次边界条件、非齐次发展型方程的混合问题

核心思路还是解出特解，向 S-L 型方程转化，再利用叠加原理求解。

例 3.4.1: 矩形非齐次边界的发展方程

$$\begin{cases} u_{tt}=a^2u_{xx}, t>0, \quad 0<x<l \\ u|_{x=0}=0, \quad u|_{x=l}=\sin\omega t \\ u|_{t=0}=\varphi(x), \quad u_t|_{t=0}=\psi(x) \end{cases}$$

因 $(\sin\omega t)''=-\omega^2\sin\omega t$, 故设 $v(x,t)=X(x)\sin\omega t$

得 $\begin{cases} X''(x)+\frac{\omega^2}{a^2}X(x)=0, \\ X(0)=0, X(l)=1. \end{cases}$ 故 $v(x,t)=\dfrac{\sin\dfrac{\omega x}{a}}{\sin\dfrac{\omega l}{a}}\sin\omega t$

令 $w=u-v$, 则

$$\begin{cases} w_{tt}-a^2w_{xx}=0, \\ w|_{x=0}=0, \quad w|_{x=l}=0, \\ w|_{t=0}=\varphi(x), \quad w_t|_{t=0}=\psi(x)-\dfrac{\sin\frac{\omega l}{a}}{\sin\frac{\omega l}{a}} \end{cases}$$

$$w(t,x)=\sum_{n=1}^{+\infty}\left(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

$$C_n=\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{n\pi}{l}\xi\mathrm{d}\xi, \quad \sin\left(\frac{\omega l}{a}\pm n\pi\right)=(-1)^n\sin\left(\frac{\omega l}{a}\right)$$

$$\begin{aligned} D_n&=\frac{2}{an\pi}\left\{\int_0^l\psi(\xi)\sin\frac{n\pi}{l}\xi\mathrm{d}\xi-\frac{\omega}{\sin\frac{\omega l}{a}}\int_0^l\sin\frac{\omega\xi}{a}\sin\frac{n\pi}{l}\xi\mathrm{d}\xi\right\} \\ &=\frac{2}{an\pi}\int_0^l\psi(\xi)\sin\frac{n\pi}{l}\xi\mathrm{d}\xi-\frac{2\omega la}{\omega^2l^2-a^2n^2\pi^2}(-1)^n \end{aligned}$$

$$u=v+w=\frac{\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}}\sin\omega t+\sum_{n=1}^{+\infty}\left(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

►一般形式:

$$\begin{cases} L_tu-L_xu=f(x), \quad t>t_0, \quad a<x<b, \\ \left(\alpha_1u-\beta_1\dfrac{\partial u}{\partial x}\right)\Big|_{x=a}=0, \left(\alpha_2u+\beta_2\dfrac{\partial u}{\partial x}\right)\Big|_{x=b}=0, \\ \text{关于}t\text{ 的初值条件}. \end{cases}$$

方法 1 ：特解法:

例 3.4.2: 非齐次发展方程

$$\begin{cases} u_{tt}-a^2u_{xx}=Ax, \quad t>0, \quad 0<x<l, \\ u|_{x=0}=u|_{x=l}=0, \\ u|_{t=0}=\varphi(x), \quad u_t|_{t=0}=\psi(x). \end{cases}$$

$v(x)=-\dfrac{A}{6a^2}\left(x^3-l^2x\right)$ 设$w(x,t)=u(x,t)-v(x)$, 则由叠加原理, 得

$$\begin{cases} w_{tt}-a^2w_{xx}=0, \quad t>0, \quad 0<x<l \\ w|_{x=0}=w|_{x=l}=0, \\ w|_{t=0}=u|_{t=0}-v|_{t=0}=\varphi(x)-v(x), w_t|_{t=0}=u_t|_{t=0}-v_t|_{t=0}=\psi(x) \end{cases}$$

$$w(x,t)=\sum_{n=1}^{+\infty}\left(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x,$$

$$\text{其中}C_n=\frac{2}{l}\int_0^l\{\varphi(\xi)-v(\xi)\}\sin\frac{n\pi}{l}\xi\mathrm{d}\xi, \quad D_n=\frac{2}{an\pi}\int_0^l\psi(\xi)\sin\frac{n\pi}{l}\xi\mathrm{d}\xi$$

$$\therefore u(x,t)=v(x)+w(x,t)=-\frac{A}{6a^2}\left(x^3-l^2x\right)+\sum_{n=1}^{+\infty}\left(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

方法 2 ► 冲量原理：如对于 在外力作用下有限长弦横振动方程的混合问题. ：

$$\begin{cases} L_tu-L_xu=f(t,x), t>0, a<x<b, \text{ 其中}L_t=\dfrac{\partial^2}{\partial t^2}+a_1(t)\dfrac{\partial}{\partial t}, \\ \left(\alpha_1u-\beta_1\dfrac{\partial u}{\partial x}\right)\Big|_{x=a}=0, \left(\alpha_2u+\beta_2\dfrac{\partial u}{\partial x}\right)\Big|_{x=b}=0, \\ u|_{t=0}=0, \quad \frac{\partial u}{\partial t}\big|_{t=0}=0. \quad L_x\text{：只依赖}x\text{ 的二阶微分算子}. \end{cases}$$

化为：

$$\begin{cases} L_{t'}w-L_xw=0, t'>0, \quad a<x<b, \quad L_{t'}=\dfrac{\partial^2}{\partial t'^2}+a_1\left(t'\right)\dfrac{\partial}{\partial t'} \\ \left(\alpha_1w-\beta_1\dfrac{\partial w}{\partial x}\right)\Big|_{x=a}=0, \left(\alpha_2w+\beta_2\dfrac{\partial w}{\partial x}\right)\Big|_{x=b}=0 \\ w|_{t'}=0=0, \dfrac{\partial w}{\partial t'}\Big|_{t'=0}=f(\tau,x). \quad (\text{其中}t'=t-\tau) \end{cases}$$

用分离变量法求 $w\left(t',x;\tau\right)=w(t-\tau,x;\tau)$ 最后

$$\boxed{u(t,x)=\int_0^t w(t-\tau,x;\tau)\mathrm{d}\tau}$$

例 3.4.3: 冲量原理例题

$$\begin{cases} \dfrac{\partial^2u}{\partial t^2}=a^2u_{xx}+\sin\dfrac{\pi x}{l}\sin\beta t, \quad t>0, 0<x<l, \beta>0, \\ u|_{x=0}=0, u|_{x=l}=0, \\ u|_{t=0}=0, u_t|_{t=0}=0. \end{cases}$$

根据冲量原理:

$$\begin{cases} \dfrac{\partial^2 w}{\partial t'^2} = a^2 w_{xx}, t' > 0, 0 < x < l, \\ w|_{x=0} = 0, \quad w|_{x=l} = 0, \quad (w \text{ 和} u \text{ 满足相同的边界条件. }) \\ w|_{t'=0} = 0, \left.\dfrac{\partial w}{\partial t'}\right|_{t'=0} = \sin\dfrac{\pi}{l}x \sin\beta\tau, \quad t' = t - \tau. \end{cases}$$

解得:

$$w(t', x; \tau) = \sum_{n=1}^{+\infty} \left(C_n(\tau) \cos\frac{an\pi}{l}t' + D_n(\tau) \sin\frac{an\pi}{l}t' \right) \sin\frac{n\pi}{l}x$$

$$\left\{\sin\frac{n\pi}{l}x, n = 1, 2, \ldots\right\} \text{ 是} L^2[0, l] \text{ 的正交基. } \frac{an\pi}{l} \text{ 为弦的固有频率.}$$

计算系数过程

$$w|_{t'=0} = \sum_{n=1}^{+\infty} C_n(\tau) \sin\frac{n\pi}{l}x = 0, \quad \text{故} C_n(\tau) = 0, \quad n = 1, 2, \cdots.$$

$$\frac{an\pi}{l} D_n(\tau) = \frac{2}{l} \int_0^l \left\{\sin\frac{\pi}{l}\xi \sin\beta\tau\right\} \sin\frac{n\pi}{l}\xi \mathrm{d}\xi = \frac{2\sin\beta\tau}{l} \int_0^l \sin\frac{\pi}{l}\xi \sin\frac{n\pi}{l}\xi \mathrm{d}\xi$$

$$= \begin{cases} \dfrac{2\sin\beta\tau}{l} \displaystyle\int_0^l \sin^2\frac{\pi}{l}\xi \mathrm{d}\xi = \sin\beta\tau, \text{ 当} n = 1 \text{ 时}, D_1(\tau) = \dfrac{l}{a\pi} \sin\beta\tau. \\ 0, \text{ 当} n \geq 2 \text{ 时. 根据正交性} \quad n \geq 2 \text{ 时}, D_n(\tau) = 0. \end{cases}$$

得到解为

$$w(t', x; \tau) = \frac{l\sin\beta\tau}{a\pi} \sin\frac{a\pi}{l}t' \sin\frac{\pi}{l}x = \frac{l\sin\beta\tau}{a\pi} \sin\frac{a\pi}{l}(t - \tau) \sin\frac{\pi}{l}x$$

$$u(t, x) = \int_0^t w(t - \tau, x; \tau) \mathrm{d}\tau = \left(\frac{l}{a\pi} \sin\omega_1 x\right) \int_0^t \sin\beta\tau \sin a\omega_1(t - \tau) \mathrm{d}\tau$$

$$\beta \neq a\omega_1, \quad u(t, x) = \frac{l}{2a\pi} (\sin\omega_1 x) \left(\frac{\sin\beta t + \sin a\omega_1 t}{\beta + a\omega_1} - \frac{\sin\beta t - \sin a\omega_1 t}{\beta - a\omega_1} \right)$$

$$\text{当} \beta = a\omega_1 \text{ 时, 出现共振: } u(t, x) = \frac{l}{2a\pi} \left(\frac{\sin a\omega_1 t}{a\omega_1} - t\cos a\omega_1 t \right) \sin\omega_1 x$$

例 3.4.4: 非齐次发展方程一二阶混合问题

$$\text{求解} \begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} - du, \quad t > 0, 0 < x < l, \\ u|_{x=0} = u|_{x=l} = 0, \quad t > 0, \\ u|_{t=0} = \varphi(x), \quad 0 < x < l. \end{cases}$$

分量变量得到

$$\frac{T'(t)}{a^2 T(t)} + \frac{d}{a^2} = \frac{X''(x)}{X(x)} + \frac{b}{a^2} \cdot \frac{X'(x)}{X(x)} = -\lambda$$

关于分量变量的方程为：

$$\begin{cases} X''(x)+\frac{b}{a^2}X'(x)+\lambda X(x)=0, 0<x<l(2.a) \\ X(0)=X(l)=0, \quad (2.b) \end{cases}$$

$$T'(t)+\left(a^2\lambda+d\right)T(t)=0, \quad t>0.$$

在 (2.a) 两边乘以$\mathrm{e}^{\int \frac{b}{a^2}\,\mathrm{d}x}=\mathrm{e}^{\frac{b}{a^2}x}$

$$\left(e^{\frac{b}{a^2}x}X'(x)\right)'+\lambda e^{\frac{b}{a^2}x}X(x)=0, \quad S-L \text{ 型方程,}$$

$$k(x)=e^{\frac{b}{a^2}x}>0, \quad q(x)\equiv 0, \quad \rho(x)=e^{\frac{b}{a^2}x}>0.$$

全体固有函数构成$L^2_{e^{\frac{b}{a^2}x}}[0,l]$ 的正交基

▶ ？如何求出固有函数呢？此处除了借助 特征方程为$\mu^2+\dfrac{b}{a^2}\mu+\lambda=0$ 分类讨论，可以直接从 S-L 出发：设$X(x)=e^{\alpha x}Y(x), \quad X'(x)=\alpha e^{\alpha x}Y(x)+e^{\alpha x}Y'(x)$

$$\Rightarrow Y''(x)+\left(2\alpha+\frac{b}{a^2}\right)Y'(x)+\left(\alpha^2+\frac{b}{a^2}\alpha+\lambda\right)Y(x)=0.$$

$$\text{令}2\alpha+\frac{b}{a^2}=0, \quad \alpha=-\frac{b}{2a^2}, \quad \alpha^2+\frac{b}{a^2}\alpha+\lambda=\lambda-\frac{b^2}{4a^4}$$

固有值 $\lambda_n=\dfrac{b^2}{4a^4}+\left(\dfrac{n\pi}{l}\right)^2$, 固有函数 $X_n(x)=\mathrm{e}^{-\frac{b}{2a^2}x}Y_n(x)=\mathrm{e}^{-\frac{b}{2a^2}x}\sin\dfrac{n\pi}{l}x, n=1,2,\cdots$

$$T'(t)+\left(\lambda a^2+d\right)T(t)=0, \quad T_n(t)=C_n\mathrm{e}^{-\left(a^2\lambda_n+d\right)t}$$

$$u(x,t)=\sum_{n=1}^{+\infty}T_n(t)X_n(t)=\sum_{n=1}^{+\infty}C_n\mathrm{e}^{-\left(\frac{a^2n^2\pi^2}{l^2}+\frac{b^2}{4a^2}+d\right)t}X_n(x)$$

$$u|_{t=0}=\sum_{n=1}^{+\infty}C_nX_n(x)=\varphi(x), C_n=\frac{\langle\varphi(x),X_n(x)\rangle}{\langle X_n(x),X_n(x)\rangle}$$

$$C_n=\frac{\int_0^l e^{\frac{b}{a^2}x}\varphi(x)\overline{X_n(x)}\mathrm{d}x}{\int_0^l e^{\frac{b}{a^2}x}\left|X_n(x)\right|^2\,\mathrm{d}x}=\frac{\int_0^l e^{\frac{b}{2a^2}x}\varphi(x)\sin\frac{n\pi}{l}x\,\mathrm{d}x}{\int_0^l \sin^2\frac{n\pi}{l}x\,\mathrm{d}x}=\frac{4\left\{1-(-1)^n\right\}}{n\pi}+\frac{2Al^3(-1)^n}{n^3\pi^3a^2}$$

例 3.4.5: 有界区间上非齐次边界条件下发展型方程的混合问题

$$\begin{cases} L_tu-L_xu=f(t,x), \quad t>t_0, \quad a<x<b, \\ \left(\alpha_1u-\beta_1\frac{\partial u}{\partial x}\right)\big|_{x=a}=g_1(t), \left(\alpha_2u+\beta_2\frac{\partial u}{\partial x}\right)\big|_{x=b}=g_2(t), \\ u|_{t=t_0}=\varphi_1(x), (a_2(t)\neq 0 \text{ 时}) u_t|_{t=t_0}=\varphi_2(x). \end{cases}$$

$$L_t=a_2(t)\frac{\partial^2}{\partial t^2}+a_1(t)\frac{\partial}{\partial t}, a_2(t), a_1(t) \text{ 至少有一个不为}0.$$

$$L_x=b_2(x)\frac{\partial^2}{\partial x^2}+b_1(x)\frac{\partial}{\partial x}+b_0(x), b_2(x)>0.$$

$g_1(t), g_2(t)$ 至少有一个不为0.

(1) 特解法. (I) 设$f(t,x)\equiv 0, g_1(t)\equiv h, g_2(t)\equiv m, h,m:$ 常数, $h^2+m^2\neq 0$

设 $v=v(x)$, 得 $L_t v(x)\equiv 0$,

$$\begin{cases} -L_x\bar{v}(x)=0, \quad a<x<b, \\ \alpha_1 v(a)-\beta_1 v'(a)=h, \alpha_2 v(b)+\beta_2 v'(b)=m \end{cases}$$

令 $w=u-v(x)$, 则

$$\begin{cases} L_t w-L_x w=0 \\ (\alpha_1 w-\beta_1\frac{\partial w}{\partial x})\big|_{x=a}=0,\ (\alpha_2 w+\beta_2\frac{\partial w}{\partial x})\big|_{x=b}=0 \\ w|_{t=t_0}=\varphi_1(x)-v(x),\ w_t|_{t=t_0}=\varphi_2(x)-0=\varphi_2(x) \end{cases}$$

于是得到通解

$$\begin{cases} L_t u-L_x u=0, \quad t>t_0, \quad a<x<b, (1.a) \\ (\alpha_1 u-\beta_1\frac{\partial u}{\partial x})\big|_{x=a}=hg(t), \quad (\alpha_2 u+\beta_2\frac{\partial u}{\partial x})\big|_{x=b}=mg(t) \\ \bar{u}|_{t=t_0}=\varphi_1(x), (a_2(t)\neq 0\text{ 时})\ u_t|_{t=t_0}=\varphi_2(x) \end{cases}$$

(2) 考虑分离变量法：令 $w=u-v=u-X(x)g(t)$, 则

$$\begin{cases} L_t w-L_x w=0 \\ (\alpha_1 u-\beta_1\frac{\partial u}{\partial x})\big|_{x=a}=0,\ (\alpha_2 u+\beta_2\frac{\partial u}{\partial x})\big|_{x=b}=0 \\ w|_{t=t_n}=\varphi_1(x)-X(x)g(0),\ w_t|_{t=t_0}=\varphi_2(x)-X(x)g'(0) \end{cases}$$

例如：

$$\begin{cases} u_{tt}=a^2u_{xx}, \qquad t>0, \quad 0<x<l, \\ u|_{x=0}=0, \qquad u|_{x=l}=\sin\omega t \\ u|_{t=0}=\varphi(x), \quad u_t|_{t=0}=\psi(x) \end{cases}$$

设$v(x,t)=X(x)\sin\omega t\Rightarrow X(x)=A\cos\dfrac{\omega}{a}x+B\sin\dfrac{\omega}{a}x.=\dfrac{\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}}\sin\omega t$

$$\begin{cases} w_{tt}-a^2w_{xx}=0, \\ w|_{x=0}=0, \quad w|_{x=l}=0, \\ w|_{t=0}=\varphi(x), \quad w_{t=0}=\psi(x)-\frac{\omega\sin\frac{\omega x}{a}}{\sin\frac{\omega l}{a}} \end{cases}$$

回到自由弦横振动

$$w(t,x)=\sum_{n=1}^{+\infty}\left(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

$$C_n=\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{n\pi}{l}\xi d\xi, \quad \sin\left(\frac{\omega l}{a}\pm n\pi\right)=(-1)^n\sin\left(\frac{\omega l}{a}\right).$$

$$D_n=\frac{2}{an\pi}\int_0^l\psi(\xi)\sin\frac{n\pi}{l}\xi d\xi-\frac{2\omega la}{\omega^2l^2-a^2n^2\pi^2(-1)^n}$$

注: 此题若使用特解法:

设$v(x,t)=A(t)x+B(t)$, 代入 x 的边界条件 则 $v(x,t)=\dfrac{\sin\omega t}{l}x$

$$\begin{cases} w_{tt}-a^2w_{xx}=0-(v_{tt}-a^2v_{xx})=\frac{\omega^2}{l}x\sin\omega t, t>0, 0<x<l,\\ w|_{x=0}=0,\quad w|_{x=l}=0,\\ w|_{t=0}=\varphi(x)-v|_{t=0}=\varphi(x),\, w_t|_{t=0}=\psi(x)-v_t|_{t=0}=\psi(x)-\frac{\omega}{l}x \end{cases}$$

之后采取叠加原理

$$\begin{cases} w_{1tt}-a^2w_{1xx}=0,\\ w_1|_{x=0}=w_1|_{x=l}=0,\\ w_1|_{t=0}=\varphi(x),\, w_{1t}|_{t=0}=\psi(x)-\frac{\omega}{l}x, \end{cases}\qquad \begin{cases} w_{2tt}-a^2w_{2xx}=\dfrac{\omega^2\sin\omega t}{l},\quad t>0, 0<x<l,\\ w_2|_{x=0}=w_2|_{x=l}=0,\\ w_2|_{t=0}=0,\quad w_{2t}|_{t=0}=0 \end{cases}$$

$$w_1(t,x)=\sum_{n=1}^{+\infty}\left(C_{1n}\cos\frac{an\pi}{l}t+D_{1n}\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

$$\begin{cases} z'_{t'}-a^2z_{xx}=0, t'>0, 0<x<l,\\ z|_{x=0}=z|_{x=l}=0,\quad z\,|_{t'=0}=0,\quad z_{t'}|_{t'=0}=\dfrac{\omega^2\sin\omega\tau}{l}x. \end{cases}$$

$$z\left(t',x;\tau\right)=\sum_{n=1}^{+\infty}\left(C_{2n}\cos\frac{an\pi}{l}t'+D_{2n}\sin\frac{an\pi}{l}t'\right)\sin\frac{n\pi}{l}x=\sum_{n=1}^{+\infty}\left(\frac{\omega^2\alpha_n}{a\pi n}\sin\omega\tau\right)\sin\frac{an\pi}{l}(t-\tau)\sin\frac{n\pi}{l}x$$

$$\begin{aligned} w_2(t,x)&==\int_0^t z(t-\tau,x;\tau)\mathrm{d}\tau\sum_{n=1}^{+\infty}\frac{\omega^2\alpha_n}{a\pi n}\left\{\int_0^t\sin\omega\tau\sin\frac{an\pi}{l}(t-\tau)\mathrm{d}\tau\right\}\sin\frac{n\pi}{l}x\\ &=\sum_{n=1}^{+\infty}\frac{\omega^2 l\alpha_n}{an\pi\left(\omega^2l^2-a^2n^2\pi^2\right)}\left(l\omega\sin\frac{an\pi}{l}t-an\pi\sin\omega t\right)\sin\frac{n\pi}{l}x \end{aligned}$$

▶思考：一般的非齐次边界混合？（整体思路：齐次化边界）

$$\begin{cases} L_tu-L_xu=f(t,x),\quad t>t_0,\quad a<x<b,\\ \left(\alpha_1u-\beta_1\frac{\partial u}{\partial x}\right)\big|_{x=a}=g_1(t),\,\left(\alpha_2u+\beta_2\frac{\partial u}{\partial x}\right)\big|_{x=b}=g_2(t),\\ \bar{u}|_{t=t_0}=\varphi_1(x),\,(a_2(t)\neq 0\text{ 时})\,u_t|_{t=t_0}=\varphi_2(x). \end{cases}$$

$L_t=a_2(t)\dfrac{\partial^2}{\partial t^2}+a_1(t)\dfrac{\partial}{\partial t}$, $a_2(t),a_1(t)$ 至少有一个不为0.

$L_x=b_2(x)\dfrac{\partial^2}{\partial x^2}+b_1(x)\dfrac{\partial}{\partial x}+b_0(x)$, $b_2(x)>0$.

$g_1(t),g_2(t)$ 至少有一个不为0.

解答: 考虑思路

1. 若α_1,α_2 不同时为0 设$v(x,t)=A(t)x+B(t)$
   $\alpha_1=\alpha_2=0$ 时, 设$v(x,t)=A(t)x^2+B(t)x$.

2. 令 $w=u-v$

$$\begin{cases} L_t w - L_x w = f(t,x) - (L_t v - L_x v)\,, t>t_0, \quad a<x<b \\ \left(\alpha_1 w - \beta_1 \frac{\partial w}{\partial x}\right|_{x=a} = 0,\ \left(\alpha_2 w + \beta_2 \frac{\partial w}{\partial x}\right)\Big|_{x=b} = 0 \\ \bar{w}|_{t=t_0} = \varphi_1(x) - v|_{t=t_0}\,, (a_2(t) \neq 0 \text{ 时})\ \ w_t|_{t=t_0} = \varphi_2(x) - v_t|_{t=0} \end{cases}$$

3. 分解：用分离变量法和冲量原理加分离变量法分别求解, 合并得 $w(t,x)$.

4. $u(t,x) = v(t,x) + w(t,x) = \cdots$.

例 3.4.6

$$\begin{cases} u_t = 4u_{xx} + xt, \\ u|_{x=0} = 1,\ u|_{x=1} = t^2 \\ u|_{t=0} = 1+4x, \end{cases}$$

设$v(x,t) = A(t)x + B(t)$, 代入边界条件

$$v(x,t) = \left(t^2-1\right)x + 1$$

令$w=u-v$, 则

$$\begin{cases} w_t - 4w_{xx} = (u_t - 4u_{xx}) - (v_t - 4v_{xx}) = -xt, t>0, 0<x<1 \\ w|_{x=0} = 0, \quad w|_{x=1} = 0 \\ w|_{t=0} = u|_{t=0} - v|_{t=0} = 5x \end{cases}$$

由线性叠加原理, $w = w_1 + w_2$

$$\begin{cases} w_{1t} - 4w_{1xx} = 0, t>0, 0<x<1, \\ w_1|_{x=0} = w_1|_{x=1} = 0, \\ w_1|_{t=0} = 5x. \end{cases} \qquad \begin{cases} w_{2t} - 4w_{2xx} = -xt, t>0, 0<x<1 \\ w_2|_{x=0} = w_2|_{x=1} = 0 \\ w_2|_{t=0} = 0 \end{cases}$$

由分离变量法

$$w_1 = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}10}{n\pi} e^{-4n^2\pi^2 t} \sin n\pi x$$

$$w_2(t,x) = \int_0^t z(t-\tau, x; \tau)\mathrm{d}\tau = \sum_{n=1}^{+\infty} \frac{2(-1)^n}{n\pi}\left(\int_0^t \tau \mathrm{e}^{-4n^2\pi^2(t-\tau)}\mathrm{d}\tau\right)\sin n\pi x$$

$$\begin{aligned} & u = v + w = v + (w_1 + w_2) \\ & = \left(t^2-1\right)x + 1 + \sum_{n=1}^{+\infty}\left[\frac{(-1)^{n+1}10}{n\pi}\mathrm{e}^{-4n^2\pi^2 t} + \frac{2(-1)^n}{n\pi}\left(\frac{t}{4n^2\pi^2} - \frac{1-e^{-4n^2\pi^2 t}}{16n^4\pi^4}\right)\right]\sin n\pi x \end{aligned}$$

第 5 节　特解法

前面给出两个例题，此处补充：

$$\begin{cases} u_t = 4u_{xx} + x, \quad t > 0, \quad 0 < x < l, \\ u|_{x=0} = 1, \quad u|_{x=l} = \sin \omega t, \\ u|_{t=0} = x. \end{cases}$$

设 $v(x,t) = A(t)x + B(t)$, 代入边界条件 $v(x,t) = \dfrac{\sin \omega t - 1}{l}x + 1$

令 $w = u - v$, 则

$$\begin{cases} w_t - 4w_{xx} = x - (v_t - 4v_{xx}) = x - \frac{\omega}{l}x\cos\omega t, \\ w|_{x=0} = 0, \quad w|_{x=l} = 0, \\ w|_{t=0} = x - v|_{t=0} = \left(1 + \frac{1}{l}\right)x - 1. \end{cases}$$

由线性叠加原理, $w = w_1 + w_2$, 其中

$$\begin{cases} w_{1t} - 4w_{1xx} = 0, \quad t > 0, 0 < x < l \\ w_1|_{x=0} = w_1|_{x=l} = 0 \\ w_1|_{t=0} = \left(1 + \frac{1}{l}\right)x - 1 \end{cases}$$

$$\frac{T'(t)}{4T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \qquad \lambda_n = \left(\frac{n\pi}{l}\right)^2, X_n(x) = \sin\frac{n\pi}{l}x, n = 1, 2, \cdots$$

$$w_1|_{t=0} = \sum_{n=1}^{+\infty} C_{1n}\sin\frac{n\pi}{l}x = \left(1 + \frac{1}{l}\right)x - 1$$

$$C_{1n} = \frac{2}{l}\int_0^l \left\{\left(1 + \frac{1}{l}\right)\xi - 1\right\}\sin\frac{n\pi}{l}\xi \mathrm{d}\xi = -\frac{2}{n\pi}\{(-1)^n l + 1\}$$

$$\begin{cases} w_{2t} - 4w_{2xx} = x - \frac{\omega}{l}x\cos\omega t \\ w_2|_{x=0} = w_2|_{x=l} = 0 \\ w_2|_{t=0} = 0 \end{cases}$$

$$\begin{cases} z_{t'} - 4^2 z_{xx} = 0, t' > 0, 0 < x < l, t' = t - \tau \\ z|_{x=0} = z|_{x=l} = 0, \quad z|_{t'=0} = x - \dfrac{\omega}{l}x\cos\omega\tau \end{cases}$$

$$z(t', x; \tau) = \sum_{n=1}^{+\infty} C_{2n}\mathrm{e}^{-4\lambda_n t'}\sin\frac{n\pi}{l}x, \quad C_{2n} = \frac{2}{l}\left(1 - \frac{\omega}{l}\cos\omega\tau\right)\int_0^l \xi\sin\frac{n\pi}{l}\xi\mathrm{d}\xi = \left(1 - \frac{\omega}{l}\cos\omega\tau\right)\alpha_n$$

$$w_2(t,x) = \int_0^t z(t-\tau, x; \tau)\mathrm{d}\tau = \sum_{n=1}^{+\infty}\alpha_n\beta_n(t)\mathrm{e}^{-4\lambda_n t}\sin\frac{n\pi}{l}x$$

$$\begin{aligned} \beta_n(t) &= \int_0^t \left(1 - \frac{\omega}{l}\cos\omega\tau\right)\mathrm{e}^{4\lambda_n\tau}\mathrm{d}\tau = \int_0^t \mathrm{e}^{4\lambda_n\tau}\mathrm{d}\tau - \frac{\omega}{l}\int_0^t \cos\omega\tau\mathrm{e}^{4\lambda_n\tau}\mathrm{d}\tau \\ &= \frac{1}{4\lambda_n}\left(\mathrm{e}^{4\lambda_n t} - 1\right) - \frac{\omega}{l}\int_0^t \cos\omega\tau\mathrm{e}^{4\lambda_n\tau}\mathrm{d}\tau \end{aligned}$$

$$\begin{aligned} u &= v + w = v + (w_1 + w_2) \\ &= \frac{\sin\omega t - 1}{l}x + 1 + \sum_{n=1}^{+\infty}\{C_{1n} + \alpha_n\beta_n(t)\}\mathrm{e}^{-4\lambda_n t}\sin\frac{n\pi}{l}x \end{aligned}$$

注

本题较为综合，主要考察公式的熟练掌握，难点在于积分和级数计算

第四章 分离变量法深究

第 1 节 Helmholtz 方程

对于一个四维空间函数分离变量：$u(t,x,y,z)=T(t)V(x,y,z)$，满足方程 u_{tt} 或$u_t=a^2\Delta_3 u$，则有

$$\frac{T''}{a^2T}=\frac{\Delta_3 V}{V}=-k^2$$

T 的部分满足：$T''+a^2k^2T=0$

$$T=\begin{cases} C_k\cos akt+D_k\sin akt, & k>0 \\ C_0+D_0t, & k=0 \\ C_\beta\cosh a\beta t+D_\beta\sinh a\beta t, & k=i\beta \end{cases}$$

考虑体积函数部分 $V(x,y,z)$ 满足 Helmholtz 方程

$$\Delta_3 V+k^2V=0$$

1 直角坐标 Helmholtz 方程

$$\begin{cases} \frac{X''}{X}=-\lambda, \quad \frac{Y''}{Y}=-\mu, \quad \frac{Z''}{Z}=-\nu \\ \frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}+k^2=0, \frac{X''}{X}=-\left(\frac{Y''}{Y}+\frac{Z''}{Z}+k^2\right)=-\lambda \\ \lambda+\mu+\nu=k^2 \end{cases}$$

例 4.1.1: 求长方体内稳恒 (与t 无关) 温度分布. (长方体内$\Delta_3 v=0$ 边值问题.)

$$\begin{cases} \Delta_3 v=0, 0<x<a, \quad 0<y<b, \quad 0<z<c \\ v|_{x=0}=\frac{\partial v}{\partial x}\Big|_{x=a}=0, \frac{\partial v}{\partial y}\Big|_{y=0}=v|_{y=b}=0 \\ v|_{z=0}=0, v|_{z=c}=\varphi(x,y) \end{cases}$$

X,Y 固有值问题：

$$\begin{cases} X''+\lambda X=0, \\ X(0)=X'(a)=0, \end{cases} \qquad \begin{cases} Y''+\mu Y=0 \\ Y'(0)=Y(b)=0 \end{cases}$$

$$\lambda_n=\gamma_n^2, \quad \gamma_n=\frac{(2n+1)\pi}{2a}, X_n(x)=\sin\gamma_n x, \quad n=0,1,2,\cdots$$

$$\mu_m=\beta_m^2, \quad \beta_m=\frac{(2m+1)\pi}{2b}, Y_m(y)=\cos\beta_m y, m=0,1,2,\cdots$$

$$\text{通解}Z_{nm}(z)=C_{nm}e^{\omega_{nm}z}+D_{nm}e^{-\omega_{nm}z}.$$

设$v(x,y,z)=\sum_{n,m=0}^{+\infty} Z_{nm}(z)X_n(x)Y_m(y)=\sum_{n,m=0}^{+\infty}\left(C_{nm}\mathrm{e}^{\omega_{nm}z}+D_{nm}\mathrm{e}^{-\omega_{nm}z}\right)\sin\gamma_n x\cos\beta_m y.$

定理 4.1.1: 二元正交基

(1) $\{X_n(x)\}_{n=1}^{+\infty}$ 是 $L^2_{\rho(x)}[0,a]$ 上的一元正交基,
(2) 对任意固定 n, $\{Y_{nm}(y)\}_{m=1}^{+\infty}$ 是 $L^2_{\sigma(y)}[0,b]$ 上一元正交基,
则 $\{X_n(x)Y_{nm}(y)\}_{n,m=1}^{+\infty}$ 是 $L^2_{\rho(x)\sigma(y)}[[0,a\mid\times[0,b]]$ 上的二元正交基,

$$L^2_{\rho(x)\sigma(y)}[[0,a]\times[0,b]]\equiv\left\{f(x,y)\left|\int_0^b\int_0^a\rho(x)\sigma(y)\right|f(x,y)\right|^2\mathrm{d}x\,\mathrm{d}y<+\infty\right\}$$

$$\forall f(x,y)\in L^2_{\rho(x)\sigma(y)}[[0,a]\times[0,b]],\ f(x,y)=\sum_{n,m=1}^{+\infty}C_{nm}X_n(x)Y_{nm}(y)$$

$$C_{nm}=\frac{\langle f(x,y),X_n(x)Y_{nm}(y)\rangle}{\|X_n(x)Y_{nm}(y)\|^2}=\frac{\int_0^b\int_0^a\rho(x)\sigma(y)f(x,y)\overline{X_n(x)Y_{nm}(y)}dxdy}{\|X_n(x)\|^2\|Y_{nm}(y)\|^2}$$

$$2C_{nm}\sinh\omega_{nm}c=\frac{\int_0^b\int_0^a\varphi(\xi,\eta)\sin\gamma_n\xi\cos\beta_m\eta\mathrm{d}\xi\mathrm{d}\eta}{\|\sin\gamma_n x\|^2\|\cos\beta_m y\|^2}$$

得到

$$C_{nm}=\frac{\frac{4}{ab}\int_0^b\int_0^a\varphi(\xi,\eta)\sin\gamma_n\xi\cos\beta_m\eta\mathrm{d}\xi\mathrm{d}\eta}{2\sinh\omega_{nm}c},D_{nm}=-C_{nm}$$

$$\text{故}v(x,y,z)=\sum_{n,m=0}^{+\infty}\frac{4\int_0^b\int_0^a\varphi(\xi,\eta)\sin\gamma_n\xi\cos\beta_m\eta\mathrm{d}\xi\mathrm{d}\eta}{ab\sinh\omega_{nm}c}\sinh\omega_{nm}z\sin\gamma_n x\cos\beta_m y,$$

$$\text{其中}\omega_{nm}=\sqrt{\gamma_n^2+\beta_m^2}.$$

例 4.1.2: 二维固有值

$$\begin{cases}\Delta_2 w+\eta w=0,\quad 0<x<a,\quad 0<y<b\\ w|_{x=0}=\frac{\partial w}{\partial x}\Big|_{x=a}=0,\quad \frac{\partial w}{\partial y}\Big|_{y=0}=\frac{\partial w}{\partial y}\Big|_{y=b}=0\end{cases}$$

设$w(x,y,z)=X(x)Y(y)$, 代入方程, 除以$X(x)Y(y)$, 代入边界条件,

$$\begin{cases}X''+\lambda X=0,\\ X(0)=X'(a)=0,\end{cases}\begin{cases}Y''+\mu Y=0,\\ Y'(0)=Y'(b)=0.\end{cases}\quad-\lambda-\mu+\eta=0.$$

$$\lambda_n=\left\{\frac{(2n+1)\pi}{2a}\right\}^2,\quad X_n(x)=\sin\frac{(2n+1)\pi}{2a}x,n=0,1,2,\cdots.$$

$$\mu_0=0,Y_0(y)=1,\mu_m=\left(\frac{m\pi}{b}\right)^2,Y_m(y)=\cos\frac{m\pi}{b}y,m=1,2,\cdots.$$

$$\eta_{nm}=\lambda_n+\mu_m,\quad w_{nm}=X_n(x)Y_m(y),\quad n,m=0,1,2,\cdots.$$

2 柱坐标 Helmholtz 方程

柱坐标变换：

$$\begin{cases} x = r\cos\theta, \quad r = \sqrt{x^2+y^2}, \quad 0<r<a \\ y = r\sin\theta, \quad -\infty<\theta<+\infty \\ z = z, \quad 0<z<h \end{cases}$$

柱坐标系下的 Laplace 算子

$$\Delta_3 v = \frac{1}{H}\partial_\alpha\left(\frac{H}{h^2}\partial_\alpha v\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial z^2}$$

$$\begin{cases} \dfrac{Z''(z)}{Z(z)} = -\mu, \quad Z''(z)+\mu Z(z) = 0. \\ \dfrac{\Theta''(\theta)}{\Theta(\theta)} = -\sigma, \begin{cases} \Theta''(\theta)+\sigma\Theta(\theta)=0, \\ \Theta(\theta+2\pi)=\Theta(\theta). \end{cases} \quad \begin{cases} \sigma_0 = 0, \Theta_0(\theta)\equiv 1, \\ \sigma_n = n^2, \Theta_n(\theta) = \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix}. \end{cases} \\ \dfrac{\frac{1}{r}\left(rR'(r)\right)'}{R(r)} - \dfrac{1}{r^2}\sigma - \mu + k^2 = 0 \end{cases}$$

$$\text{S-L 型：} \left(rR'(r)\right)' - \frac{\nu^2}{r}R(r) + \lambda r R(r) = 0$$

注

(1) $\lambda = 0$，是 Euler 方程

(2) $\lambda > 0$ 时, 称为v 阶 Bessel 方程. $\left(rR'(r)\right)' + \left(\lambda r - \dfrac{v^2}{r}\right)R(r) = 0$

$$x^2y''(x) + xy'(x) + \left(x^2 - v^2\right)y(x) = 0$$

根据 $y(x) = R\left(\frac{x}{\sqrt{\lambda}}\right)$，解是 $R_\nu(r) \triangleq y_\nu(\sqrt{\lambda}r)$

3 球坐标 Helmholtz 方程

球坐标系的 Laplace 算子

$$\Delta_3 v \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2}$$

设 $v(r,\theta) = R(r)\Theta(\theta)$, 代入方程, 两边除以 $\frac{1}{r^2}R(r)\Theta(\theta)$, 得

$$\frac{\left(r^2R'(r)\right)'}{R(r)} = -\frac{1}{\sin\theta}\cdot\frac{\left\{(\sin\theta)\Theta'(\theta)\right\}'}{\Theta(\theta)} = \lambda,$$

得到

$$\left\{(\sin\theta)\Theta'(\theta)\right\}' + \lambda\sin\theta\Theta(\theta) = 0, \quad \text{解：} \Theta(\theta) = y(\cos\theta)$$

写成：

$$\text{Legendre 方程：} \left\{\left(1-x^2\right)y'(x)\right\}' + \lambda y(x) = 0, x\in[-1,1]$$

注: m 阶伴随 Legendre 方程

$$\left\{\left(1-x^{2}\right) y^{\prime}(x)\right\}^{\prime}+\left(\lambda-\frac{m^{2}}{1-x^{2}}\right) y(x)=0, \quad x \in[-1,1]$$

第 2 节 Legendre 方程

定理 4.2.1: (Cauchy)

若 z_0 是常点 ($p(z), q(z)$ 解析点), 则对于初值问题

$$\left\{\begin{array}{l}w^{\prime \prime}(z)+p(z) w^{\prime}(z)+q(z) w(z)=0 \\ w\left(z_{0}\right)=a_{0}, \quad w^{\prime}\left(z_{0}\right)=a_{1}\end{array}\right.$$

$\exists R>0$, 使得在 $|z-z_0|<R$ 内 (3.2.1E) 存在唯一解 $w(z)$, 在 z_0 解析.

$$\text{解} w(z) \text{ 在} z_0 \text{ 可展开为幂级数} w(z)=\sum_{n=0}^{+\infty} a_{n}\left(z-z_{0}\right)^{n} .$$

且方程在常点z_0 某个邻域 $|z-z_0|<R$ 内有两个线性无关的解析解 (幂级数解)

$$\left\{\left(1-x^{2}\right) y^{\prime}(x)\right\}^{\prime}+\lambda y(x)=0, x \in(-1,1)$$

幂级数 $y(x)=\sum_{n=0}^{+\infty} a_n x^n$ 是方程在 $|x|<R$ 内解, 代入方程求系数

$$\left\{\left(1-x^{2}\right) y^{\prime}(x)\right\}^{\prime}=\sum_{n=0}^{+\infty}\{(n+2)(n+1) a_{n+2}-n(n+1) a_{n}\} x^{n}$$

$$a_{n+2}=\frac{(n-l)(n+l+1)}{(n+2)(n+1)} a_{n}, n=0,1,2, \cdots$$

(1) $a_0 \neq 0, a_1=0$

$$\begin{aligned} a_{2k} &=\frac{(2k-2-l) \cdots(2-l)(-l)(l+1)(2+l+1) \cdots(2k-2+l+1)}{2 \cdot 1 \cdot 4 \cdot 3 \cdots(2k) \cdot(2k-1)} a_{0} \\ &=\frac{2^{2k}\left(k-1-\frac{l}{2}\right) \cdots\left(1-\frac{l}{2}\right)\left(-\frac{l}{2}\right) \frac{l+1}{2}\left(1+\frac{l+1}{2}\right) \cdots\left(k-1+\frac{l+1}{2}\right)}{(2k)!} a_{0} \\ &=\frac{2^{2k} \cdot \Gamma\left(k-\frac{l}{2}\right) \Gamma\left(k+\frac{l+1}{2}\right)}{(2k)!\Gamma\left(-\frac{l}{2}\right) \Gamma\left(\frac{l+1}{2}\right)} a_{0} \neq 0, \quad k \in \mathbb{Z}^{+} \end{aligned}$$

$$y_{1}(x)=\sum_{k=0}^{+\infty} a_{2k} x^{2k}=a_{0} \sum_{k=0}^{+\infty} \frac{2^{2k} \Gamma\left(k-\frac{l}{2}\right) \Gamma\left(k+\frac{l+1}{2}\right)}{(2k)!\Gamma\left(-\frac{l}{2}\right) \Gamma\left(\frac{l+1}{2}\right)} x^{2k}$$

(2) $a_0=0, a_1 \neq 0$

$$a_{2k+1}=\frac{2^{2k} \Gamma\left(k+\frac{1-l}{2}\right) \Gamma\left(k+\frac{l+2}{2}\right)}{(2k+1)!\Gamma\left(\frac{1-l}{2}\right) \Gamma\left(\frac{l+2}{2}\right)} a_{1} \neq 0, k \in \mathbb{Z}^{+}$$

$$y_2(x)=\sum_{k=0}^{+\infty}a_{2k+1}x^{2k+1}=a_1\sum_{k=0}^{+\infty}\frac{2^{2k}\Gamma\left(k+\frac{1-l}{2}\right)\Gamma\left(k+\frac{l+2}{2}\right)}{(2k+1)!\Gamma\left(\frac{1-l}{2}\right)\Gamma\left(\frac{l+2}{2}\right)}x^{2k+1}$$

(3) 当 $l=2m$ (非负偶数), 即 $\lambda=2m(2m+1)\geq 0$ 时, 取 $a_0\neq 0, a_1=0$.

$$\text{得解}y_1(x)=\sum_{k=0}^{m}a_{2k}x^{2k}, 2m\text{ 次多项式. 可选取}a_0\text{ 使得}a_{2m}=\frac{(2\cdot 2m)!}{2^{2m}\{(2m)!\}}$$

(4) 当$l=2m+1$ (正奇数) 即$\lambda=(2m+1)(2m+2)\geq 0$ 时, 取$a_0=0, a_1\neq 0$

$y_2(x)=\sum_{k=0}^{m}a_{2k+1}x^{2k+1}$, 是 $2m+1$ 次多项式，可选取 a_1 使得 $a_{2m+1}=\dfrac{\{2(2m+1)\}!}{2^{2m+1}\{(2m+1)!\}^2}\neq 0$

综上，当 $\lambda=n(n+1)$，必有一解：$y(x)=\sum_{k=0}^{[\frac{n}{2}]}a_{n-2k}x^{n-2k}$, $[\cdot]$ 表示取整,) $\quad a_n=\dfrac{(2n)!}{2^n(n!)^2}$，

$\lambda=n(n+1)$Legendre 方程解：

$$P_n(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!}{k!2^n(n-2k)!(n-k)!}x^{n-2k}$$

$P_n(x)$ 是 $\{(1-x^2)\,y'(x)\}'+n(n+1)y(x)=0$ 在 $|x|<1$ 内的一个非零解；$|P_n(\pm 1)|$ 有界

解答: 另一个解的求解

利用解 $P_n(x)$ 和 ODE 中 Liouville 公式, 可得在 $|x|<1$ 内的另一个与 $P_n(x)$ 线性无关的解:

$$\begin{aligned}Q_n(x)&=P_n(x)\int\frac{1}{P_n^2(x)}e^{-\int\left(-\frac{2x}{1-x^2}\right)dx}dx\\&=\frac{1}{2}P_n(x)\ln\frac{1+x}{1-x}-\sum_{k=1}^{[\frac{n+1}{2}]}\frac{2n-4k+3}{(2k-1)(n-k+1)}P_{n-2k+1}(x)\end{aligned}$$

$|x|<1$ 时, $Q_n(x)$ 连续可微.

$\lambda=n(n+1)$ 时 Legendre 方程 $\left\{\left(1-x^2\right)y'(x)\right\}'+n(n+1)y(x)=0$ 的通解为

$$y(x)=C_nP_n(x)+D_nQ_n(x), C_n, D_n\text{为任意常数}, n=0,1,2,\cdot$$

1 第一类 Legendre 函数性质

$$P_n(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2n-2k)!}{k!2^n(n-2k)!(n-k)!}x^{n-2k}$$

(1) $P_n(x)$ 是 n 次多项式. $P_{2m}(x)$ 是 $2m$ 次的只含偶次幂的多项式, 是偶函数. $P_{2m+1}(x)$ 是 $2m+1$ 次的只含奇次幂的多项式, 是奇函数.

(2) 0 处取值:

$$P_{2m-1}(0)=0, \text{当}m=1,2,\cdots\text{ 时}$$
$$P_{2m}(0)=\frac{(-1)^m(2m-1)!!}{(2m)!!}\Rightarrow |P_{2m}(0)|\leq 1$$

(3) 奇偶性：$P_{2m}(x)$ 是偶函数, $P_{2m}(-x) = P_{2m}(x)$; $P_{2m+1}(x)$ 是奇函数,

$$\text{故} \quad P_n(-x) = (-1)^n P_n(x)$$

(4) 正交性（微分形式证明）$\{P_n(x), n = 0, 1, 2, \cdots\}$ 在$L^2[-1,1]$ 正交: 当$n \neq m$ $\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = 0$

(5) 模平方公式（采取母函数乘积证明 $\int_{-1}^{1} \frac{1}{1+t^2-2tx}dx = \sum_{n=0}^{+\infty} \frac{2}{2n+1}t^{2n}$ ）：

$$\int_{-1}^{1} \left| P_n(x) \right|^2 dx = \frac{2}{2n+1}, n = 0, 1, 2, \cdots$$

2 第一类 Legendre 函数表示方法

(1)（微分形式）Rodrigues 公式：

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left\{ \left(x^2-1\right)^n \right\}, n = 0, 1, 2, \cdots$$

(2)（复积分形式）Schläfli 公式：

$$P_n(x) = \frac{1}{2^n 2\pi \mathrm{i}} \oint_C \frac{\left(z^2-1\right)^n}{(z-x)^{n+1}} \mathrm{d}z, n = 0, 1, 2, \cdots$$

(3)（实积分形式）：

$$P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left(x + \sqrt{1-x^2}\mathrm{i}\cos\theta\right)^n \mathrm{d}\theta, n = 0, 1, 2, \cdots$$

$$|P_n(x)| \leq 1, \forall x \in [-1,1]; P_n(1) = 1, P_n(-1) = (-1)^n$$

(4) 母函数表示：

$$-1 < x < 1, \frac{1}{\sqrt{1-2xt+t^2}} = \begin{cases} \sum_{n=0}^{+\infty} P_n(x)t^n, & |t| < 1 \\ \frac{1}{t}\sum_{n=0}^{+\infty} P_n(x)\left(\frac{1}{t}\right)^n, & t > 1 \end{cases}$$

(5) 递推公式 (根据微分表和母函数推出):

根据微分表示：$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \Longleftrightarrow xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$

$$nP_n(x) = xP_n'(x) - {P_{n-1}}'(x), \quad \Rightarrow \text{计算} x^m P_n(x) \text{ 的积分}$$

$$(n+1)P_n(x) - P_{n+1}'(x) + xP_n'(x) = 0$$

$$P_n(x) = \frac{1}{2n+1}\left\{{P_{n+1}}'(x) - {P_{n-1}}'(x)\right\} \Rightarrow \text{计算} P_n(x) \text{ 的积分}$$

注: 第一类 Legendre 取值

$$P_0(x)=1,\quad P_1(x)=x$$
$$P_2(x)=\frac{3}{2}x^2-\frac{1}{2},\qquad P_3(x)=\frac{5}{2}x^3-\frac{3}{2}x$$
$$P_4(x)=\frac{35}{8}x^4-\frac{15}{4}x^2+\frac{3}{8},\quad P_5(x)=\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x$$

例 4.2.1: 当 $m\geq 1, n\geq 1$ 时, 求 $L_{m,n}=\int_0^1 x^m P_n(x)dx$

$$\begin{aligned}
L_{m,n}&=\frac{1}{n}\cdot\int_0^1 x^m\left\{xP_n'(x)-P_{n-1}'(x)\right\}\mathrm{d}x\\
&=\frac{1}{n}\left\{\int_0^1 x^{m+1}P_n'(x)\mathrm{d}x-\int_0^1 x^m P_{n-1}'(x)\mathrm{d}x\right\}\\
&=\frac{1}{n}\left\{1\cdot P_n(1)-0\cdot P_n(0)-(m+1)\int_0^1 x^m P_n(x)\mathrm{d}x\right.\\
&\left.-1\cdot P_{n-1}(1)+0\cdot P_{n-1}(0)+m\int_0^1 x^{m-1}P_{n-1}(x)\mathrm{d}x\right\}\\
&=-\frac{m+1}{n}L_{m,n}+\frac{m}{n}\int_0^1 x^{m-1}P_{n-1}(x)\mathrm{d}x.
\end{aligned}$$

$$L_{m,n}=\frac{m}{m+n+1}\int_0^1 x^{m-1}P_{n-1}(x)\mathrm{d}x=\frac{m}{m+n+1}L_{m-1,n-1}$$

$$\begin{aligned}
m\geqslant n,\quad L_{m,n}&=\frac{m(m-1)(m-2)\cdots\{m-(n-1)\}}{(m+n+1)(m+n+1-2)\cdots\{m+n+1-2(n-1)\}}\cdot\int_0^1 x^{m-n}\,\mathrm{d}x\\
&=\frac{m!}{(m-n)!}\cdot\frac{(m-n+1)!!}{(m+n+1)!!}\cdot\frac{1}{(m-n+1)}=\frac{m!}{(m-n)!!(m+n+1)!!}
\end{aligned}$$

$$m<n,\quad L_{m,n}=\frac{m(m-1)(m-2)\cdots\{m-(m-1)\}}{(m+n+1)(m+n+1-2)\cdots\{m+n+1-2(m-1)\}}\cdot\int_0^1 P_{n-m}(x)\mathrm{d}x$$

$$=\frac{m!(n-m+1)!!}{(m+n+1)!!}\cdot\int_0^1 P_{n-m}(x)\mathrm{d}x=\begin{cases}0, & n-m=2k\\ \dfrac{m!(n-m+1)!!}{(m+n+1)!!}\cdot\dfrac{(-1)^k(n-m-2)!!}{(n-m+1)!!}, & n-m=2k+1\end{cases}$$

例 4.2.2: 当$n\geq 2$ 时, 求$I_n=\int_0^1 P_n(x)\mathrm{d}x$

$$\begin{aligned}
I_n&=\frac{1}{2n+1}\int_0^1\left\{P_{n+1}'(x)-P_{n-1}'(x)\right\}\mathrm{d}x\\
&=\frac{1}{2n+1}\left\{P_{n+1}(x)-P_{n-1}(x)\right\}\Big|_0^1=\frac{P_{n-1}(0)-P_{n+1}(0)}{2n+1}
\end{aligned}$$

$$(1)\quad n=2k,k\geq 1\text{ 时}, P_{2k-1}(0)=P_{2k+1}(0)=0, \text{故}I_{2k}=\int_0^1 P_{2k}(x)\mathrm{d}x=0.$$

(2) 当 $n=2k+1, k\geq 1$ 时,

$$I_{2k+1}=\frac{P_{2k}(0)-P_{2k+2}(0)}{2(2k+1)+1}=\frac{1}{4k+3}\cdot\left\{\frac{(-1)^k(2k-1)!!}{(2k)!!}-\frac{(-1)^{k+1}(2k+1)!!}{(2k+2)!!}\right\}=\frac{(-1)^k(2k-1)!!}{(2k+2)!!}$$

注: 重要结论

1) m,n 一个是奇数, 另一个是偶数时, $\int_{-1}^{1}x^m P_n(x)\mathrm{d}x=0.$

2) $0\leq m<n$ 时, $\int_{-1}^{1}x^m P_n(x)dx=0.$

3 Legendre 函数系下展开

$$\forall f(x)\in L^2[-1,1],\quad f(x)=\sum_{n=0}^{+\infty}C_nP_n(x)$$

$$\int_{-1}^{1}|P_n(x)|^2\,dx=\frac{2}{2n+1},n=0,1,2,\cdots,\quad C_n=\frac{\langle P_n,f_n\rangle}{\langle P_n,P_n\rangle}=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)dx.$$

例 4.2.3

$$\text{分段函数：}f(x)=\begin{cases}0, & -1\leq x<\alpha,\\ \frac{1}{2}, & x=\alpha,\\ 1, & \alpha<x\leq 1\end{cases}$$

设 $f(x)=\sum_{n=0}^{+\infty}C_nP_n(x),\quad C_n=\frac{2n+1}{2}\int_{-1}^{1}f(x)P_n(x)\mathrm{d}x=\frac{2n+1}{2}\int_{\alpha}^{1}P_n(x)\mathrm{d}x$

$$C_0=\frac{1}{2}\int_{\alpha}^{1}1\,\mathrm{d}x=\frac{1}{2}(1-\alpha)$$

$$C_n=\frac{2n+1}{2}\int_{\alpha}^{1}\frac{1}{2n+1}\left\{P'_{n+1}(x)-P_{n-1}(x)\right\}\mathrm{d}x=-\frac{1}{2}\left\{P_{n+1}(\alpha)-P_{n-1}(\alpha)\right\}$$

$$\text{故}f(x)=\frac{1}{2}(1-\alpha)P_0(x)+\frac{1}{2}\sum_{n=1}^{+\infty}\left\{P_{n-1}(\alpha)-P_{n+1}(\alpha)\right\}P_n(x)$$

注: 待定系数法

对于给形式的函数，用待定系数法计算系数

4 轴对称 Laplace 方程球面边值问题

$$\text{求解}\quad\begin{cases}\Delta_3 v=0, \quad r=\sqrt{x^2+y^2+z^2}, 0\leqslant r<a\\ v|_{r=a}=f(\theta).\end{cases}$$

$$\Delta_3 v=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0\Rightarrow\frac{(r^2R')'}{R}=-\frac{\frac{1}{\sin\theta}\{(\sin\theta)\Theta'\}'}{\Theta}=\lambda$$

得到

$$\begin{cases}\{(\sin\theta)\Theta'\}+\lambda(\sin\theta)\Theta=0, \quad 0\le\theta\le\pi\\ |\Theta(0)|<+\infty, |\Theta(\pi)|<+\infty.\end{cases}$$

令 $x=\cos\theta, y(x)=\Theta(\arccos x)$, 得

$$\begin{cases}\{(1-x^2)\,y'(x)\}'+\lambda y(x)=0, x\in[-1,1],\\ |y(\pm1)|\text{ 有界.}\qquad 1,-1\text{ 都是正则奇点.}\end{cases}$$

$$\text{固有值}\lambda_n=n(n+1), n=0,1,2,\cdots,\ \text{相应固有函数为}P_n(x)$$

$\{P_n(\cos\theta), n=0,1,2,\cdots\}$ 是$L^2_{\sin\theta}[0,\pi]$ 的正交基

$$\text{对每一个}\lambda_n=n(n+1), n=0,1,2,\cdots$$
$$r^2R''+2rR'-n(n+1)R=0.$$

注: Euler 方程解

Euler 方程 $r^2R''+prR'+qR=0$ 在变换 $r=\mathrm{e}^S$ 下化为

$$\frac{\mathrm{d}^2R}{\mathrm{d}s^2}+(p-1)\frac{\mathrm{d}R}{\mathrm{d}s}+qR=0,$$

$$\text{通解: } R_n(r)=C_nr^n+D_nr^{-(n+1)}, n=0,1,2,\cdots$$

$$\text{设：}v(r,\theta)=\sum_{n=0}^{+\infty}R_n(r)\Theta_n(\theta)=\sum_{n=0}^{+\infty}\left(C_nr^n+D_nr^{-(n+1)}\right)P_n(\cos\theta)$$

当 $0<r<a$, 即球内问题时, 须 $|v|_{r=0}|$ 有界, 故 $D_n=0, n=0,1,2,\cdots$, 故

$$v(r,\theta)=\sum_{n=0}^{+\infty}A_n\left(\frac{r}{a}\right)^nP_n(\cos\theta).\quad\text{球内轴对称}\Delta_3v=0\text{一般解.}$$

$$v|_{r=a}=\sum_{n=0}^{+\infty}A_nP_n(\cos\theta)=f(\theta)$$

$$A_n=\frac{\langle f(\theta),P_n(\cos\theta)\rangle}{\langle P_n(\cos\theta),P_n(\cos\theta)\rangle}=\frac{\int_0^\pi f(\theta)P_n(\cos\theta)\sin\theta\mathrm{d}\theta}{\|P_n\left(\cos\theta\right\|^2}=\frac{2n+1}{2}\int_{-1}^1f(\arccos x)P_n(x)dx$$

例 4.2.4: $0 \le r < 3, v|_{r=3} = 2\cos^3\theta$

$$V|_{r=3} = \sum_{n=0}^{+\infty} A_n P_n(\cos\theta) = 2\cos^3\theta, \quad x = \cos\theta$$

$$2x^3 = A_1P_1(x) + A_3P_3(x) = \frac{4}{5}P_1(x) + \frac{6}{5}P_3(x)$$

$$V(r,\theta) = \frac{6}{5}\cdot\frac{r}{3}P_1(\cos\theta) + \frac{4}{5}\cdot\left(\frac{r}{3}\right)^3 P_3(\cos\theta) = \frac{2}{5}rP_1(\cos\theta) + \frac{4}{135}r^3P_3(\cos\theta)$$

例 4.2.5: $2 < r < 3,\ u|_{r=2} = 2, \quad u|_{r=3} = \cos\theta + 3\cos^2\theta$

- $$u(r,\theta) = \sum_{n=0}^{+\infty}\left(C_nr^n + D_nr^{-(n+1)}\right)P_n(\cos\theta)$$

$$u|_{r=2} = \sum_{n=0}^{+\infty}\left(2^nC_n + 2^{-(n+1)}D_n\right)P_n(\cos\theta) = 2$$

$$u|_{r=3} = \sum_{n=0}^{+\infty}\left(3^nC_n + 3^{-(n+1)}D_n\right)P_n(\cos\theta) = \cos\theta + 3\cos^2\theta$$

$$\sum_{n=0}^{+\infty}\left(2^nC_n + 2^{-(n+1)}D_n\right)P_n(x) = 2, \sum_{n=0}^{+\infty}\left(3^nC_n + 3^{-(n+1)}D_n\right)P_n(x) = x + 3x^2$$

$$\left(3C_1 + 3^{-2}D_1\right)P_1(x) + \left(C_0 + 3^{-1}D_0\right)P_0(x) + \left(3^2C_2 + 3^{-3}D_2\right)P_2(x) = x + 3x^2$$

$$2^nC_n + 2^{-(n+1)}D_n = 0, \text{当} n \ge 1 \text{时}, \quad C_0 + 2^{-1}D_0 = 2$$

$$u(r,\theta) = \left(-1 + \frac{6}{r}\right) + \left(\frac{9r}{19} - \frac{72}{19r^2}\right)P_1(\cos\theta) + \left(\frac{54r^2}{211} - \frac{1728}{211r^3}\right)P_2(\cos\theta)$$

例 4.2.6: 接地金属球壳 $u|_{r=a} = -\frac{q}{\sqrt{a^2+b^2-2ab\cos\theta}} = -\frac{q}{a}\cdot\sum_{n=0}^{+\infty}P_n(\cos\theta)\left(\frac{b}{a}\right)^n$

$$A_n = -\frac{q}{a}\cdot\left(\frac{b}{a}\right)^n$$

$$u(r,\theta) = \sum_{n=0}^{+\infty}A_n\left(\frac{r}{a}\right)^nP_n(\cos\theta) = -\frac{q}{a}\sum_{n=0}^{+\infty}\left(\frac{br}{a^2}\right)^nP_n(\cos\theta)$$

$$\text{球内电位：} U = u_0 + u = \frac{q}{\sqrt{r^2+b^2-2rb\cos\theta}} - \frac{q}{a\sqrt{1+\left(\frac{br}{a^2}\right)^2 - 2\frac{br}{a^2}\cos\theta}}$$

等效于：在距离球心 $\frac{a^2}{b}$ 且与 P 点在球心同一侧的同一射线上的点 P^1 放置一个虚设电荷 $(-4\pi\varepsilon qa/b)$ 在点 M 产生的电位. P' 与 P 在球心 O 同一侧且与球心 O 共线, $\left|\vec{OP'}\right|\cdot|\vec{OP}| = \frac{a^2}{b}\cdot b = a^2$ (球半径的平方), 称点 P' 与点 P 关于球面对称. (镜像法)

例 4.2.7: 球外边界

$$\begin{cases} \Delta_3 u = 0, \quad r > a, \\ u|_{r=a} = E_0 a\cos\theta - c, c\text{ 是常数},\ u|_{r=+\infty} = 0 \end{cases}$$

$$u(r,\theta) = \sum_{n=0}^{+\infty} B_n \left(\frac{r}{a}\right)^{-(n+1)} P_n(\cos\theta)$$

$$\sum_{n=0}^{+\infty} B_n P_n(x) = E_0 a x - c,\ \text{比较系数得, 当}n \geq 2\text{ 时}, B_n = 0,$$

$$u(r,\theta) = -c\left(\frac{r}{a}\right)^{-1} P_0(\cos\theta) + E_0 a\left(\frac{r}{a}\right)^{-2} P_1(\cos\theta) = -\frac{ca}{r} + \frac{E_0 a^3}{r^2}\cos\theta$$

例 4.2.8: ★ 上半球边值

$$\begin{cases} \Delta_3 u = 0, \quad 0 < r < a, \quad 0 \leq \theta < \frac{\pi}{2}, \\ u|_{r=a}\,(\text{球面}) = u_0\,(\text{常数}), \quad u|_{\theta=\frac{\pi}{2}}\,(\text{半球底面}) = 0. \end{cases}$$

重新解固有值问题

$$\frac{(r^2R')'}{R} = -\frac{\frac{1}{\sin\theta}\{(\sin\theta)\Theta'\}'}{\Theta} = \lambda$$

$$\begin{cases} \{(\sin\theta)\Theta'\}' + \lambda(\sin\theta)\Theta = 0, 0 \leq \theta < \frac{\pi}{2}, \\ |\Theta(0)| < +\infty, \Theta\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

令 $x = \cos\theta_\theta\ y(x) = \Theta(\arccos x)$

$$\begin{cases} \{(1-x^2)\,y'(x)\}' + \lambda y(x) = 0, x \in (0,1) \\ |y(1)| < +\infty, y(0) = 0 \end{cases}$$

当且仅当 $\lambda = n(n+1), n = 0,1,2,\cdots$ 时，方程有满足 $|y(1)| < +\infty$ 的非零解 $P_n(x)$.
由于 $P_{2m}(0) \neq 0, P_{2m+1}(0) = 0, m = 0,1,2,\ldots$

$$\text{固有值为}\lambda_m = (2m+1)(2m+2), m = 0,1,2,\cdots$$

根据 S-L 定理：$\{P_{2m+1}(\cos\theta)\}_{m=0}^{+\infty}$ 是 $L^2_{\sin\theta}\left[0,\frac{\pi}{2}\right]$ 的正交基

$$R_m(r) = C_m r^{2m+1} + D_m r^{-(2m+2)}$$

$$A_m = \frac{\langle u_0, P_{2m+1}(\cos\theta)\rangle}{\|P_{2m+1}(\cos\theta)\|^2} =$$

$$\because \|P_{2m+1}(\cos\theta)\|^2 = \int_0^1 |P_{2m+1}(x)|^2\ \mathrm{d}x = \frac{1}{2}\int_{-1}^1 |P_{2m+1}(x)|^2\ \mathrm{d}x = \frac{1}{2}\cdot\frac{2}{2(2m+1)+1} = \frac{1}{4m+3}$$

$$\langle u_0, P_{2m+1}(\cos\theta)\rangle = \int_0^{\frac{\pi}{2}} u_0 P_{2m+1}(\cos\theta)\sin\theta \mathrm{d}\theta = u_0\int_0^1 P_{2m+1}(x)\mathrm{d}x = \frac{(-1)^m(2m-1)!!}{(2m+2)!!}u_0$$

$$u(r,\theta) = u_0\sum_{m=0}^{+\infty}\frac{(-1)^m(2m-1)!!(4m+3)}{(2m+2)!!}\left(\frac{r}{a}\right)^{2m+1} P_{2m+1}(\cos\theta)$$

例 4.2.9

一个半径为 R, 厚度为 $\frac{R}{2}$ 的半空心球, 外球面和内球面上的温度始终保持为

$$f(\theta)=A\sin^2\frac{\theta}{2},\quad 0\leqslant\theta\leqslant\frac{\pi}{2},$$

底面温度保持为 $\frac{A}{2}$, 求半空心球内部各点的定常温度.

$$\begin{cases}\Delta_3 u=0,\quad 0\leq\theta\leq\dfrac{\pi}{2},\dfrac{R}{2}\leq r\leq R,\quad r=\sqrt{x^2+y^2+z^2}\\ u|_{r=\frac{R}{2}}=A\sin^2\dfrac{\theta}{2},u|_{r=R}=A\sin^2\dfrac{\theta}{2},u|_{\theta=\frac{\pi}{2}}=\dfrac{A}{2}\end{cases}$$

特解法：取$v=C(r)\theta+D(r)$, $v|_{\theta=\frac{\pi}{2}}=\dfrac{\pi}{2}C(r)+D(r)=\dfrac{A}{2}$

取$C(r)=0,D(r)=\dfrac{A}{2}\cdot v\equiv\dfrac{A}{2}\cdot(\Delta_3 v=0,$ 特解.) 令$w=u-\dfrac{A}{2}$, 则

$$\begin{cases}\Delta_3 w=\Delta_3 u-\Delta_3\left(\frac{A}{2}\right)=0,\quad 0\leq\theta\leq\frac{\pi}{2},\frac{R}{2}\leq r\leq R\\ w|_{r=\frac{R}{2}}=w|_{r=R}=A\sin^2\frac{\theta}{2}-\frac{A}{2},\quad w|_{\theta=\frac{\pi}{2}}=0\end{cases}$$

$$\begin{cases}\{(\sin\theta)\Theta'\}'+\lambda(\sin\theta)\Theta=0,0<\theta<\frac{\pi}{2},\\ |\Theta(0)|<+\infty,\Theta\left(\frac{\pi}{2}\right)=0.\end{cases}$$

固有值为 $\lambda_m=(2m+1)(2m+2),m=0,1,2,\cdots$, 相应固有函数为 $\Theta_m(\theta)=P_{2m+1}(\cos\theta)$. $\{P_{2m+1}(\cos\theta)\}_{m=0}^{+\infty}$ 是 $L^2_{\sin\theta}\left[0,\frac{\pi}{2}\right]$ 中的正交基.

$$\text{设：}w(r,\theta)=\sum_{m=0}^{+\infty}R_m(r)\Theta_m(\theta)=\sum_{m=0}^{+\infty}\left(C_m r^{2m+1}+D_m r^{-(2m+2)}\right)P_{2m+1}(\cos\theta)$$

$m\geq 1$ 时, $C_m=D_m=0.\quad P_1(x)=x.$

$w(r,\theta)=\left(C_0 r+D_0 r^{-2}\right)P_1(\cos\theta)=\left(C_0 r+D_0 r^{-2}\right)\cos\theta.$

$$\left.\begin{aligned}w|_{r=\frac{R}{2}}&=\left\{C_0\cdot\tfrac{R}{2}+D_0\left(\tfrac{R}{2}\right)^{-2}\right\}\cos\theta=-\tfrac{A}{2}\cos\theta\\ w|_{r=R}&=(C_0R+D_0R^{-2})\cos\theta=-\tfrac{A}{2}\cos\theta\end{aligned}\right\}\text{比较系数得到，}D_0=-\frac{AR^2}{14},\quad C_0=-\frac{3A}{7R}$$

$$\begin{cases}C_0\cdot\frac{R}{2}+D_0\left(\frac{R}{2}\right)^{-2}=-\frac{A}{2}\\ C_0R+D_0R^{-2}=-\frac{A}{2}\end{cases}$$

$$\text{故}w(r,\theta)=-\left(\frac{3Ar}{7R}+\frac{AR^2}{14r^2}\right)\cos\theta.\quad u=w+\frac{A}{2}=\frac{A}{2}-\left(\frac{3r}{7R}+\frac{R^2}{14r^2}\right)A\cos\theta.$$

第 3 节 Bessel 方程

定理 4.3.1: (Fuchs)

$$w''(z)+p(z)w'(z)+q(z)w(z)=0,\quad (z=a+bi,a,b\in\mathbb{R},i^2=-1.)$$

若 z_0 是 $\begin{cases} p(z)\text{ 的至多一级极点,} \\ q(z)\text{ 的至多二级极点,}\end{cases}$ 则称 z_0 为正则奇点

在正则奇点 z_0 某个去心邻域 $0<|z-z_0|<R$ 内存在两个线性无关的解: (沿支割线 (若有) 割开, 保证函数单值性) (广义幂级数解)

$$w_1(z)=(z-z_0)^{\rho_1}\sum_{n=0}^{+\infty}a_n\left(z-z_0\right)^n$$

$$x^2y''+xy'+\left(x^2-\nu^2\right)y=0,\quad \nu\geq 0$$

S-L 标准型: $y''+\dfrac{1}{x}y'+\dfrac{x^2-\nu^2}{x^2}y=0$

由 Fuchs 定理, 该方程在 $0<|x|<+\infty$ 内有形如 $y(x)=x^\rho\sum\limits_{n=0}^{+\infty}a_nx^n$ 的解, $a_0\neq 0$.

设$y(x)=x^\rho\sum_{n=0}^{+\infty}a_nx^n=\sum_{n=0}^{+\infty}a_nx^{n+\rho}$

$$\begin{aligned}&a_0\left\{\rho+\rho(\rho-1)-\nu^2\right\}x^\rho+a_1\left\{(1+\rho)+(1+\rho)\rho-\nu^2\right\}x^{1+\rho}\\&+\sum_{n=2}^{+\infty}\left\{a_n\left((n+\rho)+(n+\rho)(n+\rho-1)-\nu^2\right)+a_{n-2}\right\}x^{n+\rho}=0\end{aligned}$$

$$\Longrightarrow \rho^2=\nu^2,\quad n(n+2\rho)a_n+a_{n-2}=0,n\geq 2$$

取$\rho_1=\nu\geq 0,1+2\nu>0,(1+2\nu)a_1=0,$ 故$a_1=0.$) 得,

$$n\geq 2\text{时}, n+2\nu>0,a_n=-\frac{a_{n-2}}{n(n+2\nu)}$$

$$a_{2k}=\frac{(-1)^k}{2^{2k+\nu}k!\Gamma(k+\nu+1)}$$

$$y_1(x)=\sum_{k=0}^{+\infty}a_{2k}x^{2k+\nu}=\boxed{\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\triangleq J_\nu(x)}$$

$$y_2(x)=J_{-\nu}(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k-\nu+1)}\left(\frac{x}{2}\right)^{2k-\nu}=(-1)^\nu J_\nu(x)$$

$y_1(x),y_2(x)$ 线性无关，但在 $\nu=m$ 时线性相关，所以当 $\nu=m$ 时还需求 (3.1.8) 的另一个与 $J_m(x)$ 线性无关的解

$$N_\nu(x)=\begin{cases}\dfrac{\cos\nu\pi}{\sin\nu\pi}J_\nu(x)-\dfrac{1}{\sin\nu\pi}J_{-\nu}(x), & \text{当}\nu\neq m(\text{ 非负整数) 时,}\\ \lim\limits_{\mu\to m}N_\mu(x)=\lim\limits_{\mu\to m}\dfrac{(\cos\mu\pi)J_\mu(x)-J_{-\mu}(x)}{\sin\mu\pi}, & \text{当}\nu=m\text{ 时.}\end{cases}$$

$$x \to 0\ \text{时},\ \begin{cases} m > 0\ \text{时},\ N_m(x) \sim -\frac{(m-1)!}{\pi}\left(\frac{x}{2}\right)^{-m}, \\ m = 0\ \text{时},\ N_0(x) \sim \frac{2}{\pi}J_0(x)\ln\frac{x}{2}, \quad \text{故}\ \lim_{x\to 0} N_\nu(x) = \infty \end{cases}$$

注: 解的形式

根据上面的极限性质，如果 $|y(0)| < +\infty$ 有相关要求，则解为 $J_\nu(x)$，
无 $|y(0)|$ 的有界性限制则为：$CJ_\nu(x) + DN_\nu(x)$

1 Bessel 函数性质

对于 ν 阶 Bessel 方程：

$$J_\nu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

(1) 微分关系：$(x^\nu J_\nu)' = x^\nu J_{\nu-1}$， $(x^{-\nu}J_\nu)' = -x^{-\nu}J_{\nu+1}$

(2) 递推公式：

$$xJ_\nu{}' + \nu J_\nu = xJ_{\nu-1}$$
$$xJ_\nu{}' - \nu J_\nu = -xJ_{\nu+1}$$
$$2J_\nu{}' = J_{\nu-1} - J_{\nu+1}$$
$$2\nu x^{-1}J_\nu = J_{\nu-1} + J_{\nu+1}$$

(3) 零点：

- $J_\nu(x)$ 振荡, 有无穷多可数实零点, $\nu > -1$ 时, $J_\nu(x)$ 所有零点均在实轴
- $J_\nu(x), J_\nu'(x), J_\nu(x) + hxJ_\nu'(x)$($h$ 任意实常数) 都有无穷多可数实零点
- $\nu > -1$ 时, $J_\nu(x)$ 的非零零点是一级零点, $x = 0$ 是 $J_n(x)$ 和 $J_{-n}(x)$ 的 n 级零点.
- $J_\nu(x)$ 与 $J_{\nu+1}(x)$ 在正实轴的零点两两相间分布, $0 < J_\nu(x)$ 第一个正零点 $< J_{\nu+1}(x)$ 第一个正零点, 即 ν 越大, $J_\nu(x)$ 第一个正零点越大, 即离原点越远.

例 4.3.1: 递推公式分部积分的应用

Q1. 求 $\int x^3 J_0(x)\mathrm{d}x$

$$H_1 = \int x^2\left(xJ_0(x)\right)\mathrm{d}x = \int x^2\left(xJ_1(x)\right)'\mathrm{d}x = x^3J_1(x) - \int 2x\left(xJ_1(x)\right)\mathrm{d}x = x^3J_1(x) - 2\int x^2J_1(x)\mathrm{d}x$$

$$\begin{aligned} H_2 &= \int x^2J_1(x)\mathrm{d}x = -\int x^2J_0'(x)\mathrm{d}x \\ &= -x^2J_0(x) + 2\int xJ_0(x)\mathrm{d}x = -x^2J_0(x) + 2\int\left(xJ_1(x)\right)'\mathrm{d}x \\ &= -x^2J_0(x) + 2xJ_1(x) + c. \end{aligned}$$

$$\begin{aligned} H_1 &= x^3J_1(x) - 2H_2 = x^3J_1(x) - 2\left(-x^2J_0(x) + 2xJ_1(x) + c\right) \\ &= \left(x^3 - 4x\right)J_1(x) + 2x^2J_0(x) + c \end{aligned}$$

Q2. 求$I_2=-\int x^2J_{-2}(x)\mathrm{d}x$，首先用 $J_{-m}(x)=(-1)^mJ_m(x)$

$$
\begin{aligned}
\text{解:}I_2&=-(-1)^2\int x^2J_2(x)\mathrm{d}x=-\int x^3\left(x^{-1}J_2(x)\right)\mathrm{d}x\\
&=\int x^3\left(x^{-1}J_1(x)\right)'\mathrm{d}x=x^3\cdot x^{-1}J_1(x)-3\int x^2\left(x^{-1}J_1(x)\right)\mathrm{d}x\\
&=x^2J_1(x)-3\int xJ_1(x)\mathrm{d}x\\
&=x^2J_1(x)+3\int xJ_0'(x)\mathrm{d}x\\
&=x^2J_1(x)+3xJ_0(x)-3\int J_0(x)\mathrm{d}x
\end{aligned}
$$

Q3. 再比如 $H=\int_{-3}(x)\mathrm{d}x$

$$
\begin{aligned}
H&=(-1)^3\int xJ_3(x)\mathrm{d}x=-\int x^3\left(x^{-2}J_3(x)\right)\mathrm{d}x=\int x^3\left(x^{-2}J_2(x)\right)'\mathrm{d}x\\
&=x^3\cdot x^{-2}J_2(x)-3\int x^2\left(x^{-2}J_2(x)\right)\mathrm{d}x=xJ_2(x)-3\int J_2(x)\mathrm{d}x\\
&=x\left(2\cdot 1\cdot x^{-1}J_1(x)-J_0(x)\right)-3\int\left(J_0(x)-2J_1'(x)\right)\mathrm{d}x\\
&=2J_1(x)-xJ_0(x)-3\int J_0(x)\mathrm{d}x+6J_1(x)\\
&=8J_1(x)-xJ_0(x)-3\int J_0(x)\mathrm{d}x
\end{aligned}
$$

Q4. 求$\int J_{-4}(x)\mathrm{d}x$

$$
\begin{aligned}
\int J_{-4}(x)\mathrm{d}x=(-1)^4\int J_4(x)\mathrm{d}x&=\int\left\{J_2(x)-2J_3'(x)\right\}\mathrm{d}x\\
&=\int J_2(x)\mathrm{d}x-2J_3(x)\\
&=\int\left\{J_0(x)-2J_1'(x)\right\}\mathrm{d}x-2\cdot\left\{2\cdot 2x^{-1}J_2(x)-J_1(x)\right\}\\
&=\int J_0(x)\mathrm{d}x-2J_1(x)-8x^{-1}J_2(x)+2J_1(x)\\
&=\int J_0(x)\mathrm{d}x-8x^{-1}\left\{2\cdot 1\cdot x^{-1}J_1(x)-J_0(x)\right\}\\
&=\int J_0(x)dx-\frac{16}{x^2}J_1(x)+\frac{8}{x}J_0(x)
\end{aligned}
$$

2 Bessel 方程固有值问题

柱坐标 Helmholtz 方程

$$\Delta_3 v + k^2 v \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} + k^2 v = 0$$

设 $v(r,\theta,z) = R(r)\Theta(\theta)Z(z)$, 代入方程, 除以 $v = R\Theta Z$

$$\frac{\frac{1}{r}\left(rR'(r)\right)'}{R(r)} + \frac{1}{r^2}\cdot\frac{\Theta''(\theta)}{\Theta(\theta)} + \frac{Z''(z)}{Z(z)} + k^2 = 0.$$

其中: $\dfrac{\Theta''(\theta)}{\Theta(\theta)} = -\sigma,\quad \dfrac{Z''(z)}{Z(z)} = -\mu$, 记 $\sigma = \nu^2$, 记 $\lambda = -\mu + k^2$

$$\left(rR'(r)\right)' + \left(\lambda r - \frac{\nu^2}{r}\right)R(r) = 0,\quad 0<r<a$$

$$\begin{cases} \left(rR'(r)\right)' + \left(\lambda r - \frac{\nu^2}{r}\right)R(r) = 0, & 0<r<a, \\ |R(0)| < +\infty, \alpha R(a) + \beta R'(a) = 0 \end{cases}$$

当 $\lambda > 0$ 时, 记 $\lambda = \omega^2, \omega = \sqrt{\lambda} > 0$. 根令 $x = \sqrt{\lambda} r = \omega r, r = \frac{x}{\omega}, y(x) = R\left(\frac{x}{\omega}\right)$,

$$x^2 y''(x) + xy'(x) + \left(x^2 - \nu^2\right)y(x) = 0$$

$$R(r) = CJ_\nu(\omega r) + DN_\nu(\omega r)$$

- 第 I 类边界条件 (I) 记 ω_{1n} 是 $J_\nu(\omega a) = 0$ 的第 n 个正根, $J_\nu\left(\omega_{1n}a\right) = 0$,
  则固有值$\lambda_{1n} = \omega_{1n}^2$, 固有函数$R_{1n}(r) = J_\nu\left(\omega_{1n}r\right), n = 1,2,\cdots$
- 第 II 类边界条件 记ω_{2n} 是$J_\nu'(\omega a) = 0$ 的第n 个正根, $J_\nu'\left(\omega_{2n}a\right) = 0$
  则固有值$\lambda_{2n} = \omega_{2n}^2$, 固有函数$R_{2n}(r) = J_\nu\left(\omega_{2n}r\right), n = 1,2,\cdots$
  (特殊情况，仅在 $k = 2$ 成立): 当 (且仅当) $\alpha = 0, \nu = 0$ 时, $\lambda_{20} = 0$ 是固有值, 固有函数 $R_{20}(r) = 1 = J_0(0)$ 此时记 $\omega_{20} = 0$.
- 第 III 类边界条件 记ω_{3n} 是$\alpha J_\nu(\omega a) + \beta\omega J_{\nu'}(\omega a) = 0$ 的第n 个正根,
  则固有值$\lambda_{3n} = \omega_{3n}^2$, 固有函数$R_{2n}(r) = J_\nu\left(\omega_{3n}r\right), n = 1,2,\cdots$

定义模平方:

$$N_{\nu kn}^2 = \left\|J_\nu\left(\omega_{kn}r\right)\right\|^2 = \int_0^a r\left|J_\nu\left(\omega_{kn}r\right)\right|^2 \mathrm{d}r$$

$$N_{\nu kn}^2 = \frac{a^2}{2}\left\{J_\nu'\left(\omega_{kn}a\right)\right\}^2 + \frac{1}{2}\left(a^2 - \frac{\nu^2}{\omega_{kn}^2}\right)J_\nu^2\left(\omega_{kn}a\right)$$

性质 4.3.1: Bessel 函数模平方

1、$k = 1$，$N_{\nu 1n^2}\dfrac{a^2}{2}J_{\nu+1}^2\left(\omega_{1n^a}\right)$

2、$k = 2$，$N_{\nu 2n}^2 = \dfrac{1}{2}\left(a^2 - \dfrac{\nu^2}{\omega_{2n}^2}\right)J_\nu^2\left(\omega_{2n}a\right)$

3、$k=3$，$N_{\nu 3n}^2=\frac{1}{2}\left(a^2-\frac{\nu^2}{\omega_{3n}^2}+\frac{\alpha^2 a^2}{\beta^2\omega_{3n}^2}\right)J_\nu^2\left(\omega_{3n}a\right)$

▶**函数展开:**

$$\forall f(r)\in L_r^2[0,a],f(r)=\sum_{n=1(k\neq 2\text{ 或}\nu>0)}^{+\infty}C_nJ_\nu\left(\omega_{kn}r\right)$$

$$C_n=\frac{\left\langle f(r),J_\nu\left(\omega_{kn}r\right)\right\rangle}{\left\|J_\nu\left(\omega_{kn}r\right)\right\|^2}=\frac{1}{N_{\nu kn}^2}\int_0^a rf(r)J_\nu\left(\omega_{kn}r\right)\mathrm{d}r$$

$$\int_0^a rf(r)J_\nu\left(\omega_{kn}r\right)\mathrm{d}r=\frac{1}{\omega_{kn}^2}\int_0^{\omega_{kn}a}sf\left(\frac{s}{\omega_{kn}}\right)J_\nu(s)\mathrm{d}s$$

例 4.3.2: 将$f(x)=1-x^2$ 在$0\leq x\leq 1$ 按$\{J_0\left(\omega_nx\right)\}_{n=1}^{+\infty}$ 作广义 Fourier 展开

$$\text{设}\omega_n(n=1,2,\cdots)\text{ 是}J_0(x)=0\text{ 的全体正根}$$

$$f(x)=1-x^2=\sum_{n=1}^{+\infty}C_{n_0}\left(\omega_nx\right)$$

$$N_{\nu 1n}^2=\frac{a^2}{2}J_{\nu+1}^2\left(\omega_{1n}a\right),n\geq 1$$

$$C_n=\frac{\langle 1-x^2,J_0\left(\omega_nx\right)>}{\|J_0\left(\omega_nx\|^2\right.}=\frac{1}{N_{01n}^2}\int_0^1x\left(1-x^2\right)J_0\left(\omega_nx\right)dx$$

$$\begin{aligned}
C_n&=\frac{2}{J_1^2\left(\omega_n\right)}\cdot\frac{1}{\omega_n^4}\int_0^{\omega_n}s\left(\omega_n^2-s^2\right)J_0(s)\mathrm{d}s,\quad sJ_0(s)=\left(sJ_1(s)\right)'\\
&=\frac{2}{\omega_n^4J_1^2\left(\omega_n\right)}\int_0^{\omega_n}\left(\omega_n^2-s^2\right)\left(sJ_1(s)\right)'ds=\frac{-2}{\omega_n^4J_1^2\left(\omega_n\right)}\int_0^{\omega_n}(-2s)\left(sJ_1(s)\right)\mathrm{d}s\\
&=\frac{-4}{\omega_n^4J_1^2\left(\omega_n\right)}\cdot\int_0^{\omega_n}s^2J_0'(s)\mathrm{d}s(\text{ 分部积分})\\
&=\frac{-4}{\omega_n^4J_1^2\left(\omega_n\right)}\left\{\omega_n^2J_0\left(\omega_n\right)-\int_0^{\omega_n}2sJ_0(s)\mathrm{d}s\right\}\\
&==\frac{8}{\omega_n^4J_1^2\left(\omega_n\right)}\int_0^{\omega_n}\left(sJ_1(s)\right)'\mathrm{d}s=\frac{8\omega_nJ_1\left(\omega_n\right)}{\omega_n^4J_1^2\left(\omega_n\right)}=\frac{8}{\omega_n^3J_1\left(\omega_n\right)}
\end{aligned}$$

$$\text{故}f(x)=1-x^2=\sum_{n=1}^{+\infty}\frac{8}{\omega_n^3J_1\left(\omega_n\right)}J_0\left(\omega_nx\right)$$

注: 十分重要的化简关系式

$$sJ_0(s)=\left(sJ_1(s)\right)',\quad J_1(s)=-J_0'(s)$$

令$s=\omega_{kn}r,\quad r=\frac{s}{\omega_{kn}},\quad \mathrm{d}r=\frac{1}{\omega_{kn}}\mathrm{d}s$

$$\int_0^a rf(r)J_v\left(\omega_{kn}r\right)\mathrm{d}r=\frac{1}{\omega_{kn}^2}\int_0^{\omega_{kn}a}sf\left(\frac{S}{\omega_{kn}}\right)J_v(s)\mathrm{d}s$$

例 4.3.3: 柱形 Bessel 混合问题

$$\begin{cases} \frac{\partial u}{\partial t}=a^2\Delta_3 u, t>0, 0\le r<r_0, -\infty<z<+\infty, r=\sqrt{x^2+y^2}, \\ u|_{r=r_0}=0, \text{(柱侧表面温度恒为0 度)} \\ u|_{t=0}=x=r\cos\theta. \end{cases}$$

考虑分离变量:

$$\begin{cases} \frac{\partial u}{\partial t}=a^2\Delta_3 u=a^2\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right\}, t>0, 0<r<r_0, \\ u|_{r=r_0}=0, \quad u|_{t=0}=r\cos\theta. \quad -\infty<\theta<+\infty, \quad r=\sqrt{x^2+y^2} \end{cases}$$

$$\frac{T'(t)}{a^2T(t)}=\frac{\frac{1}{r}\left(rR'(r)\right)'}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}$$

$$\frac{T'(t)}{a^2T(t)}=\frac{\frac{1}{r}\left(rR'(r)\right)'}{R(r)}-\frac{\mu}{r^2}=-\lambda$$

于是得到两个固有值问题和一个微分方程:

$$\begin{cases} \Theta''+\mu\Theta=0 \\ \Theta(\theta+2\pi)=\Theta(\theta) \end{cases} \qquad \begin{cases} \left(rR'(r)\right)'+\left(\lambda r-\frac{\mu}{r}\right)R=0, \quad 0<r<r_0 \\ |R(0)|<+\infty, \quad R(r_0)=0 \end{cases} \qquad T'+a^2\lambda T=0$$

通过 Θ 的固有值问题

$$\begin{cases} \mu_0=0, \\ \Theta_0(\theta)=1 \end{cases} \quad \begin{cases} \mu_m=m^2, m=1,2,\cdots \\ \Theta_m(\theta)=A_m\cos m\theta+B_m\sin m\theta \end{cases}$$

$$x^2y''(x)+xy'(x)+\left(x^2-m^2\right)y(x)=0$$

$$y(x)=C_mJ_m(x)+D_mN_m(x)$$

通解

$$R(r)=C_mJ_m(\omega r)+D_mN_m(\omega r)$$

由 $|R(0)|<+\infty$, 得 $D_m=0$.

$$T'+a^2\omega_{m1n}^2T=0. \quad T_{mn}(t)=C_{mn}\mathrm{e}^{-a^2\omega_{m1n}^2t}$$

设 $u(t,r,\theta)=\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}C_{mn}\mathrm{e}^{-\left(a^2\omega_{m1n}^2t\right.}J_m\left(\omega_{m1n}r\right)\left(A_m\cos m\theta+B_m\sin m\theta\right)$

$$=\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}\mathrm{e}^{-a^2\omega_{m1n}^2t}J_m\left(\omega_{m1n}r\right)\left(C_{mn}\cos m\theta+D_{mn}\sin m\theta\right)$$

$$\sum_{n=1}^{+\infty}C_{1n}J_1\left(\omega_{11n}r\right)=r, \quad C_{1n}=\frac{\langle r, J_1\left(\omega_{11n}r\right)\rangle}{|\ J_1\left(\omega_{11n}r\right)^2\ |}=\frac{1}{N_{11n}^2}\int_0^{r_0}r\cdot rJ_1\left(\omega_{11n}r\right)\mathrm{d}r$$

$$\int_0^{r_0}r\cdot rJ_1\left(\omega_{11n}r\right)\mathrm{d}r=\frac{1}{\omega_{11n}^3}\int_0^{\omega_{11n}r_0}s^2J_1(s)\mathrm{d}s=-\frac{1}{\omega_{1n}^3}\int_0^{\omega_{1n}r_0}s^2J_0'(s)\mathrm{d}s$$

$$=-\frac{1}{\omega_{11n}^3}\left\{\omega_{11n}^2r_0^2J_0\left(\omega_{11n}r_0\right)-2\int_0^{\omega_{11n}r_0}sJ_0(s)ds\right\}\cdot sJ_0(s)=\left(sJ_1(s)\right)'=\frac{-r_0^2J_0\left(\omega_{11n}r_0\right)}{\omega_{11n}}$$

$$\text{故}u(t,r,\theta)=-\sum_{n=1}^{+\infty}\frac{2}{\omega_{11n}J_0\left(\omega_{1nn}r_0\right)}e^{-a^2\omega_{11n}^2t}J_1\left(\omega_{11n}r\right)\cos\theta$$

例 4.3.4: 柱形温度稳态分布

$$\begin{cases}\Delta_3 u=0, \quad 0\le r<a, 0<z<h\\ \frac{\partial u}{\partial r}\Big|_{r=a}=0, \text{侧面绝热 (边界热流密度为0}\\ \bar{u}|_{z=0}=0, u|_{z=h}=1-\dfrac{r^2}{a^2}, \text{上下面温度恒定}\end{cases}$$

S-L 方程

$$\begin{cases}(rR')'+\lambda rR=0, 0<r<a\\ |R(0)|<+\infty, R'(a)=0\end{cases}$$

当 $\lambda>0$ 时, 设 $\lambda=\omega^2>0, \omega=\sqrt{\lambda}>0$，通解 $R(r)=AJ_0(\sqrt{\lambda}r)+BN_0(\sqrt{\lambda}r)$.
记 ω_{2n} 是 $J_0'(\omega a)=0$ 的第 n 个正根, $J_0'(\omega_{2n}a)=0, n=1,2,\cdot\cdot$
固有值 $\lambda_0=\omega_{20}^2=0, \lambda_n=\omega_{2n}^2, n=1,2,\cdots$.
固有函数: $R_0(r)=J_0(0)=1, R_n(r)=J_0(\omega_{2n}r), n=1,2,\cdots$

性质 4.3.2: Bessel 函数正交基

$k\neq 2$ 或 $\nu>0$ 时, $\{J_\nu(\omega_{kn}r), n=1,2,\cdots\}$ 是 $L_r^2[0,a]$ 的正交基.
$k=2$ 且 $\nu=0$ 时, $\{J_0(\omega_{2n}r), n=0,1,2,\cdots\}$ 是 $L_r^2[0,a]$ 的正交基

$\{1, J_0(\omega_{2n}r), n=1,2,\cdots\}$ 是 $L_r^2[0,a]$ 的正交基.

$$\|1\|^2=\int_0^a r\cdot 1^2\,\mathrm{d}r=\frac{1}{2}a^2.$$

$$\|J_0(\omega_{2n}r)\|^2=N_{02n}^2=\frac{1}{2}\left(a^2-\frac{0^2}{\omega_{2n}^2}\mid J_0^2(\omega_{2n}a)=\frac{a^2}{2}J_0^2(\omega_{2n}a)\right.$$

求 Z. 对 $\lambda_0=0, Z''=0, Z_0(z)=A_0+B_0z_0$.
对 $\lambda_n=\omega_{2n}^2, Z''-\omega_{2n}^2Z=0, Z_n(z)=A_n\mathrm{e}^{\omega_{2n}z+B_n}\mathrm{e}^{-\omega_{2n}z}$.

$$\text{设}u(r,z)=(A_0+B_0z)\cdot 1+\sum_{n=1}^{+\infty}\left(A_n\mathrm{e}^{\omega_{2n}z}+B_n\mathrm{e}^{-\omega_{2n}z}\right)J_0(\omega_{2n}r)$$

$$B_0h=\frac{1}{||1||^2}\int_0^a r\left(1-\frac{r^2}{a^2}\right)\cdot 1\,\mathrm{d}r=\frac{2}{a^2}\cdot\int_0^a\left(r-\frac{r^3}{a^2}\right)\mathrm{d}r=\frac{1}{2}, B_0=\frac{1}{2h}$$

$$A_n\left(\mathrm{e}^{\omega_{2n}h}-\mathrm{e}^{-\omega_{2n}h}\right)=\frac{1}{||J_0(\omega_{2n}r)||^2}\int_0^a r\cdot\left(1-\frac{r^2}{a^2}\right)J_0(\omega_{2n}r)\,\mathrm{d}r$$

$$=\frac{1}{N_{02n}^2}\cdot\frac{1}{a^2\omega_{2n}^4}\int_0^{\omega_{2n}a}s\left(a^2\omega_{2n}^2-S^2\right)J_0(S)\mathrm{d}s=-\frac{4}{a^2\omega_{2n}^2J_0(\omega_{2n}a)}$$

$$u(r,z)=\frac{1}{2h}z-\frac{4}{a^2}\sum_{n=1}^{+\infty}\frac{\mathrm{e}^{\omega_{2n}z}-\mathrm{e}^{-\omega_{2n}z}}{\omega_{2n}^2J_0(\omega_{2n}a)\left(\mathrm{e}^{\omega_{2n}h}-\mathrm{e}^{-\omega_{2n}h}\right)}J_0(\omega_{2n}r)$$

其中 ω_{2n} 是 $J_0'(\omega a)=0$ 的第 n 个正根

例 4.3.5: Bessel 方程补充题

$$\begin{cases} u_{tt}=a^2\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), t>0, 0<x<l, a>0 \\ \left.\frac{\partial u}{\partial x}\right|_{x=l}=0, \\ u|_{t=0}=\varphi(x), \quad u_t|_{t=0}=0. \end{cases}$$

固有值方程

$$\frac{T''}{a^2T}=\frac{\frac{1}{x}\left(xX'\right)'}{X}=-\lambda$$

$$\begin{cases} \left(xX'\right)'+\lambda xX=0, 0<x<l, \\ |X(0)|<+\infty, X'(l)=0 \end{cases}$$

$$\text{因}|X(0)|<+\infty, \text{故}X(x)=J_0(\sqrt{\lambda}x)=J_0(\omega x)$$

固有值 $\lambda_0=\omega_{20}^2=0, \lambda_n=\omega_{2n}^2, \omega_{2n}$ 是 $J_0'(\omega l)=0$ 的第 n 个正根,
固有函数 $X_0(x)\equiv 1, X_n(x)=J_0\left(\omega_{2n}x\right).J_0'\left(\omega_{2n}l\right)=0, n=1,2,\cdots$.

$$\text{对}\lambda_0=\omega_{20}^2=0, T''=0, T_0(t)=A_0+B_0t.$$

$$\text{对}\lambda_n=\omega_{2n}^2, \quad T''+a^2\omega_{2n}^2T=0,$$

$$T_n(t)=A_n\cos a\omega_{2n}t+B_n\sin a\omega_{2n}t, n=1,2,\cdots$$

$$\text{设}u(t,x)=\sum_{n=0}^{+\infty}T_n(t)X_n(x)$$

$$=\left(A_0+B_0t\right)\cdot 1+\sum_{n=1}^{+\infty}\left(A_n\cos a\omega_{2n}t+B_n\sin a\omega_{2n}t\right)J_0\left(\omega_{2n}x\right)$$

$$u|_{t=0}=B_0+\sum_{n=1}^{+\infty}B_na\omega_{2n}J_0\left(\omega_{2n}x\right)=0, \text{ 故}B_n=0, n=0,1,2,\cdots$$

$$u|_{t=0}=A_0\cdot 1+\sum_{n=1}^{+\infty}A_nJ_0\left(\omega_{2n}x\right)=\varphi(x),$$

$$A_0=\frac{\int_0^l x\cdot 1\cdot\varphi(x)\mathrm{d}x}{\int_0^l x\cdot 1^2\,\mathrm{d}x}=\frac{2}{l^2}\int_0^l\xi\varphi(\xi)\mathrm{d}\xi$$

$$n\geqslant 1, \quad A_n=\frac{\left\langle\varphi(x), J_0\left(\omega_{2n}x\right)\right\rangle}{N_{02n}^2}=\frac{2}{l^2J_0^2\left(\omega_{2n}l\right)^2}\int_0^l\xi(\xi)J_0\left(\omega_{2n}\xi\right)\mathrm{d}\xi$$

$$\text{故}u(t,x)=A_0+\sum_{n=1}^{+\infty}A_nJ_0\left(\omega_{2n}x\right)\cos a\omega_{2n}t$$

第五章 积分变换法

第 1 节 Fourier 变换

定理 5.1.1: Fourier 变换和反变换

$$F[f(x)]=f(\lambda)=\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x$$

$$f\text{ 在点}x\text{ 连续时,}\quad f(x)=F^{-1}[f(\lambda)]=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\lambda)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda$$

性质 5.1.1

(1) 线性性质:

$$F\left[c_1f_1(x)+c_2f_2(x)\right]=c_1F\left[f_1(x)\right]+c_2F\left[f_2(x)\right],c_1,c_2\text{ 与}x\text{ 无关}$$

$$F^{-1}\left[a_1g_1(\lambda)+a_2g_2(\lambda)\right]=a_1F^{-1}\left[g_1(\lambda)\right]+a_2F^{-1}\left[g_2(\lambda)\right],a_1,a_2\text{ 与}\lambda\text{ 无关}$$

(2) 微分性质:

$$F\left[f^{(n)}(x)\right]=(i\lambda)^nF[f(x)]=(i\lambda)^nf(\lambda)$$

$$\Rightarrow F^{-1}\left[(i\lambda)^nf(\lambda)\right]=f^{(n)}(x)$$

(3) 频移公式

$$F\left[f(x)\mathrm{e}^{\mathrm{i}\lambda_0x}\right]=f\left(\lambda-\lambda_0\right)$$

(4) 位移公式

$$F\left[f\left(x-x_0\right)\right]=f(\lambda)e^{-i\lambda x_0}$$

(5) 相似性质

$$F[f(\alpha x)]=\frac{1}{|\alpha|}f\left(\frac{\lambda}{\alpha}\right),\alpha\neq 0\text{ 是实数}$$

(6) 积分性质: 设 $F[f(x)],F\left[\int_{-\infty}^{x} f(\xi)\mathrm{d}\xi\right]$ 存在, $f(0)=0$, 则

$$F\left[\int_{-\infty}^{x} f(\xi)\mathrm{d}\xi\right]=\frac{1}{i\lambda}F[f(x)]$$

(7) 卷积:

$$f(x)*g(x)=\int_{-\infty}^{+\infty} f(\xi)g(x-\xi)\mathrm{d}\xi$$

♣

$$F[f(x)*g(x)]=F[f(x)]F[g(x)]=f(\lambda)g(\lambda)$$

$$F^{-1}[f(\lambda)g(\lambda)]=F^{-1}[f(\lambda)]*F^{-1}[g(\lambda)]=f(x)*g(x)$$

♣

$$\{f(x)*g(x)\}*h(x)=f(x)*\{g(x)*h(x)\}$$

♣

$$f(x)*g(x)=g(x)*f(x)$$

♣

$$\{f(x)+g(x)\}*h(x)=f(x)*h(x)+g(x)*h(x)$$

(8) 反射性质

$$F[f(x)]=2\pi f(-\lambda)$$

例 5.1.1: 无界区间定解问题

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}, t>0, \quad -\infty<x<+\infty, a>0\text{且为常数}\\ u|_{t=0}=\varphi(x).\end{cases}$$

假设$u, \dfrac{\partial u}{\partial t}, \dfrac{\partial^2 u}{\partial x^2}, \varphi(x)$ 可以关于x 作 Fourier 变换, 形式地求解

$$\text{记}u(t,\lambda)=F[u(t,x)]=\int_{-\infty}^{+\infty}u(t,x)e^{-i\lambda x}dx,$$

$$F\left[\frac{\partial u}{\partial t}\right]=\int_{-\infty}^{+\infty}\frac{\partial u}{\partial t}\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}u(t,x)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x=\frac{\mathrm{d}u}{\mathrm{d}t}(t,\lambda)$$

$$F\left[a^2\frac{\partial^2 u}{\partial x^2}\right]=a^2F\left[\frac{\partial^2 u}{\partial x^2}\right]=a^2(\mathrm{i}\lambda)^2F[u]=-a^2\lambda^2 u$$

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}(t,\lambda)=-a^2\lambda^2u(t,\lambda), t>0\\ u|_{t=0}=C(\lambda)\end{cases}$$

通解 $u(t,\lambda)=Ce^{-a^2\lambda^2t}$.

$$u|_{t=0}=C=\varphi(\lambda)$$
$$u(t,\lambda)=\varphi(\lambda)e^{-a^2\lambda^2t}$$

$$\begin{aligned}u(t,x)&=F^{-1}[u(t,\lambda)]=F^{-1}\left[\varphi(\lambda)\mathrm{e}^{-a^2\lambda^2t}\right]\\&=F^{-1}[\varphi(\lambda)]*F^{-1}\left[\mathrm{e}^{-a^2\lambda^2t}\right]=\varphi(x)*F^{-1}\left[\mathrm{e}^{-a^2\lambda^2t}\right]\end{aligned}$$

注: $F^{-1}\left[\mathrm{e}^{-k^2\lambda^2}\right]$

$$F^{-1}\left[\mathrm{e}^{-k^2\lambda^2}\right]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-k^2\lambda^2}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda=\frac{1}{\pi}\int_0^{+\infty}\mathrm{e}^{-k^2\lambda^2}\cos\lambda x\,\mathrm{d}\lambda=\frac{1}{2k\sqrt{\pi}}\exp\left(-\frac{x^2}{4k^2}\right)$$

$$F^{-1}\left[\mathrm{e}^{-a^2\lambda^2 t}\right]=\frac{2}{2\pi}\int_0^{+\infty}\mathrm{e}^{-(a^2t)\lambda^2}\cos x\lambda \mathrm{d}\lambda=\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}$$

$$u(t,x)=\varphi(x)*\left(\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\right)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{\xi^2}{4a^2t}}\varphi(x-\xi)d\xi$$

例 5.1.2: 非齐次发展方程

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+x\cos 2t,-\infty<x<+\infty, \quad t>0,a>0\\ u|_{t=0}=\cos x\end{cases}$$

三种解法：特解法 + Fourier 变换/冲量原理 +Fourier 变换/ 直接用 Fourier 变换

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+f(t,x)\\ u|_{t=0}=\varphi(x)\end{cases}$$

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}=-a^2\lambda^2u+f(t,\lambda),t>0,\\ u|_{t=0}=\varphi(\lambda)=F[\varphi(x)]\end{cases}$$

$$u(t,\lambda)=\mathrm{e}^{-a^2\lambda^2t}\left\{\int_0^t f(\tau,\lambda)\mathrm{e}^{a^2\lambda^2\tau}\mathrm{d}\tau+\varphi(\lambda)\right\}$$

$$\begin{aligned}u(t,x)&=F^{-1}\left[\mathrm{e}^{-a^2\lambda^2t}\left\{\int_0^t f(\tau,\lambda)\mathrm{e}^{a^2\lambda^2\tau}\mathrm{d}\tau+\varphi(\lambda)\right\}\right]\\&=F^{-1}\left[\int_0^t\mathrm{e}^{-a^2\lambda^2(t-\tau)}f(\tau,\lambda)\mathrm{d}\tau+\mathrm{e}^{-a^2\lambda^2t}\varphi(\lambda)\right]\\&=F^{-1}\left[\int_0^t\mathrm{e}^{-a^2\lambda^2(t-\tau)}f(\tau,\lambda)\mathrm{d}\tau\right]+F^{-1}\left[\mathrm{e}^{-a^2\lambda^2t}\varphi(\lambda)\right]\\&=\frac{1}{2}x\sin 2t+e^{-a^2t}\cos x\end{aligned}$$

例 5.1.3: 复杂一些

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+b\frac{\partial u}{\partial x}+cu,,\infty<x<+\infty,t>0,\\ u|_{t=0}=\varphi(x)\end{cases}$$

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}=\left(-a^2\lambda^2+b\mathrm{i}\lambda+c\right)u,t>0,\\ u|_{t=0}=\varphi(\lambda)=F[\varphi(x)], \quad \text{得}u(t,\lambda)=\mathrm{e}^{\left(-a^2\lambda^2+b\mathrm{i}\lambda+c\right)t}\varphi(\lambda).\end{cases}$$

$$u(t,x)=F^{-1}\left[\mathrm{e}^{\left(-a^2\lambda^2+bi\lambda+c\right)t}\varphi(\lambda)\right]$$

$$=F^{-1}\left[\mathrm{e}^{-a^2\lambda^2t}\mathrm{e}^{\mathrm{i}bt\lambda}\mathrm{e}^{ct}\varphi(\lambda)\right]=\mathrm{e}^{ct}F^{-1}\left[\mathrm{e}^{-a^2\lambda^2t}\left(\mathrm{e}^{\mathrm{i}bt\lambda}\varphi(\lambda)\right)\right]$$

$$=\mathrm{e}^{ct}F^{-1}\left[\mathrm{e}^{-a^2\lambda^2t}\right]*F^{-1}\left[\mathrm{e}^{\mathrm{i}bt\lambda}\varphi(\lambda)\right]=\mathrm{e}^{ct}\left\{\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\right\}*\varphi(x+bt)$$

$$= \frac{\mathrm{e}^{ct}}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2}{4a^2 t}} \varphi(x-\xi+bt)\mathrm{d}\xi$$

1 Fourier 正弦变换、余弦变换和半无界问题

设 $f(x)$ 定义在 $[0,+\infty)$,

将 $f(x)$ 奇延拓到 $(-\infty,+\infty)$ 后作 Fourier 变换得到正弦变换；
将 $f(x)$ 偶延拓到 $(-\infty,+\infty)$ 后作 Fourier 变换得到正弦变换；

注: 求解问题类型

- I 类边界条件半无界问题用 Fourier 正弦变换求解.
- II 类边界条件半无界问题用 Fourier 余弦变换求解.

(1) Fourier 正弦变换: $F_S[f(x)] = f_s(\lambda) \triangleq \int_0^{+\infty} f(x)\sin\lambda x dx.$

反演公式: $f(x) = F_S^{-1}[f_S(\lambda)] \triangleq \frac{2}{\pi}\int_0^{+\infty} f_S(\lambda)\sin\lambda x\,\mathrm{d}\lambda.$

(2) Fourier 余弦变换: $F_c[f(x)] = f_c(\lambda) \triangleq \int_0^{+\infty} f(x)\cos\lambda x dx.$

反演公式: $f(x) = F_c^{-1}[f_c(\lambda)] \triangleq \frac{2}{\pi}\int_0^{+\infty} f_c(\lambda)\cos\lambda x\,\mathrm{d}\lambda.$

例 5.1.4: I 类边值—正弦变换

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, t>0, \quad x>0, \quad a>0 \text{ 是常数}, \\ u|_{x=0} = u_0, \text{ I 类边界冬件} \\ u|_{t=0} = 0, \text{ 添加 } u|_{x=+\infty} = \frac{\partial u}{\partial x}\Big|_{x=+\infty} = 0. \end{cases}$$

在 $x=0$ 满足 I 类边界条件, 用 Fourier 正弦变换

$$u_S(t,\lambda) = F_S[u(t,x)] = \int_0^{+\infty} u(t,x)\sin\lambda x\,\mathrm{d}x,$$

$$F_S\left[\frac{\partial u}{\partial t}\right] = \frac{du_S}{dt} \cdot \quad F_S\left[u|_{t=0}\right] = u_S|_{t=0}$$

$$F_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^{+\infty}\frac{\partial^2 u}{\partial x^2}\sin\lambda x\,\mathrm{d}x = \left(\frac{\partial u}{\partial x}\sin\lambda x\right)\Big|_{x=0}^{x=+\infty} - \lambda\int_0^{+\infty}\frac{\partial u}{\partial x}\cos\lambda x\,\mathrm{d}x$$
$$= -\lambda\int_0^{+\infty}\frac{\partial u}{\partial x}\cos\lambda x\,\mathrm{d}x = -\lambda\left\{(u\cos\lambda x)|_{x=0}^{x=+\infty} - (-\lambda)\int_0^{+\infty} u\sin\lambda x\,\mathrm{d}x\right\}$$
$$= -\lambda\{-u_0+\lambda u_s\} = \lambda u_0 - \lambda^2 u_s.$$

$$\text{故}\frac{\mathrm{d}u_s}{\mathrm{d}t} = a^2\left(-\lambda^2 u_S + \lambda u_0\right) = -a^2\lambda^2 u_S + a^2\lambda u_0, \quad u_S|_{t=0} = 0$$

$$\mathrm{e}^{a^2\lambda^2 t}u_s = \frac{u_0}{\lambda}\mathrm{e}^{a^2\lambda^2 t}\Big|_{t=0}^{t=t} = \frac{u_0}{\lambda}\left(\mathrm{e}^{a^2\lambda^2 t}-1\right). \text{故} u_S = \frac{u_0}{\lambda}\left(1-\mathrm{e}^{-a^2\lambda^2 t}\right)$$

$$u(t,x) = F_S^{-1}\left[u_s(t,\lambda)\right] = \frac{2}{\pi}\int_0^{+\infty}\frac{u_0}{\lambda}\left(1-e^{-a^2\lambda^2 t}\right)\sin\lambda x\,\mathrm{d}\lambda$$

$$= \frac{2u_0}{\pi}\left(\int_0^{+\infty}\frac{\sin\lambda x}{\lambda}\mathrm{d}\lambda - \int_0^{+\infty}\mathrm{e}^{-a^2\lambda^2 t}\frac{1}{\lambda}\sin\lambda x\,\mathrm{d}\lambda\right).$$

$$\because \int_0^{+\infty}\mathrm{e}^{-a^2\lambda^2 t}\frac{\sin\lambda x}{\lambda}\mathrm{d}\lambda = \int_0^{+\infty}\mathrm{e}^{-a^2\lambda^2 t}\left(\int_0^x\cos\lambda\xi\mathrm{d}\xi\right)\mathrm{d}\lambda$$

$$= \int_0^x\left(\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda\xi\mathrm{d}\lambda\right)\mathrm{d}\xi = \int_0^x\frac{\pi}{2a\sqrt{\pi t}}e^{-\frac{\xi^2}{4a^2 t}}\,\mathrm{d}\xi = \sqrt{\pi}\int_0^{\frac{x}{2a\sqrt{t}}}e^{-y^2}\,\mathrm{d}y$$

$$u(t,x) = u_0\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2a\sqrt{t}}}\mathrm{e}^{-y^2}\,\mathrm{d}y\right)$$

例 5.1.5: II 类边值-余弦变换

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}, t>0, x>0, a>0 \text{ 是常数}, \\ \frac{\partial u}{\partial x}\big|_{x=0} = 0, \quad \text{II 类边界条件} \\ u|_{t=0} = \varphi(x), \quad \text{添加 } u|_{x=+\infty} = \frac{\partial u}{\partial x}\big|_{x=+\infty} = 0. \end{cases}$$

在 $x=0$ 满足 II 类边界条件，用 Fourier 余弦变换求解. 即记 $u_c(t,\lambda) = F_c[u(t,x)] = \int_0^{+\infty}u(t,x)\cos\lambda x\,\mathrm{d}x$ 求解.

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_0^{+\infty}\frac{\partial^2 u}{\partial x^2}\cos\lambda x dx = \left(\frac{\partial u}{\partial x}\cos\lambda x\right)\Bigg|_{x=0}^{x=+\infty} + \lambda\int_0^{+\infty}\frac{\partial u}{\partial x}\sin\lambda x dx$$

$$= \lambda(u\sin\lambda x)|_{x=0}^{x=+\infty} - \lambda^2\int_0^{+\infty}u\cos\lambda x\,\mathrm{d}x = -\lambda^2 u_c$$

$$\begin{cases} \frac{du_c}{dt} = -a^2\lambda^2 u_c, t>0, \\ u_c|_{t=0} = \varphi_c(\lambda) \end{cases} \quad \text{故} u_c = \mathrm{e}^{-a^2\lambda^2 t}\varphi_c(\lambda).$$

$$u(t,x) = F_c^{-1}\left[\mathrm{e}^{-a^2\lambda^2 t}\varphi_c(\lambda)\right] = \frac{2}{\pi}\int_0^{+\infty}\mathrm{e}^{-a^2\lambda^2 t}\varphi_c(\lambda)\cos\lambda x\,\mathrm{d}\lambda$$

$$= \frac{2}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}\left\{\int_0^{1\infty}\varphi(\xi)\cos\lambda\xi\mathrm{d}\xi\cos\lambda x\,\mathrm{d}\lambda\right.$$

$$= \frac{2}{\pi}\int_0^{+\infty}\varphi(\xi)\left\{\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda\xi\cos\lambda x\,\mathrm{d}\lambda\right\}\mathrm{d}\xi$$

$$= \frac{2}{\pi}\int_0^{+\infty}\varphi(\xi)\left\{\int_0^{+\infty}\frac{1}{2}\mathrm{e}^{-a^2\lambda^2}[\cos\lambda(\xi+x)+\cos\lambda(\xi-x)]d\lambda\right\}\mathrm{d}\xi$$

$$= \frac{1}{\pi}\int_0^{+\infty}\varphi(\xi)\left\{\int_0^{+\infty}\mathrm{e}^{-a^2\lambda^2 t}\cos\lambda(\xi+x)\mathrm{d}\lambda + \int_0^{+\infty}\mathrm{e}^{-a^2\lambda^2 t}\cos\lambda(\xi-x)\mathrm{d}\lambda\right\}\mathrm{d}\xi$$

$$= \frac{1}{2a\sqrt{\pi t}}\int_0^{+\infty}\varphi(\xi)\left(\mathrm{e}^{-\frac{(\xi+x)^2}{4a^2 t}} + \mathrm{e}^{-\frac{(\xi-x)^2}{4a^2 t}}\right)\mathrm{d}\xi$$

2 一些采取 Fourier 变换题目

例 5.1.6: 无限长弦自由横振动

Fourier 变换. 设$u, \dfrac{\partial u}{\partial t}, \dfrac{\partial^2 u}{\partial t^2}, \dfrac{\partial^2 u}{\partial x^2}$ 都可关于x 作 Fourier 变换,

$$\begin{cases} \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -a^2\lambda^2 u, & t > 0 \\ u|_{t=0} = \varphi(\lambda), & \frac{\mathrm{d}u}{\mathrm{d}t}\Big|_{t=0} = \psi(\lambda) \end{cases}$$

$$u(t,\lambda) = A(\lambda)e^{\mathrm{i}a\lambda t} + B(\lambda)\mathrm{e}^{-\mathrm{i}a\lambda t}$$

$$A(\lambda) = \frac{1}{2}\left\{\varphi(\lambda) + \frac{1}{ia\lambda}\psi(\lambda)\right\}, \quad B(\lambda) = \frac{1}{2}\left\{\varphi(\lambda) - \frac{1}{ia\lambda}\psi(\lambda)\right\}$$

$$u(t,x) = \frac{1}{2}F^{-1}\left[\left\{\varphi(\lambda) + \frac{1}{\mathrm{i}a\lambda}\psi(\lambda)\right\}\mathrm{e}^{\mathrm{i}a\lambda t}\right] + \frac{1}{2}F^{-1}\left[\left\{\varphi(\lambda) - \frac{1}{\mathrm{i}a\lambda}\psi(\lambda)\right\}\mathrm{e}^{-\mathrm{i}a\lambda t}\right]$$

$$F^{-1}\left[\varphi(\lambda) \pm \frac{1}{\mathrm{i}a\lambda}\psi(\lambda)\right] = F^{-1}[\varphi(\lambda)] \pm \frac{1}{a}F^{-1}\left[\frac{1}{i\lambda}\psi(\lambda)\right] = \varphi(x) \pm \frac{1}{a}\cdot\int_{-\infty}^{x}\psi(\xi)\mathrm{d}\xi$$

$$F^{-1}\left[\left\{\varphi(\lambda) \pm \frac{1}{\mathrm{i}a\lambda}\psi(\lambda)\right\}\mathrm{e}^{\pm\mathrm{i}a\lambda t}\right] = \varphi(x \pm at) \pm \frac{1}{a}\int_{-\infty}^{x\pm at}\psi(\xi)\mathrm{d}\xi$$

$$u(t,x) = \frac{1}{2}\{\varphi(x+at) + \varphi(x-at)\} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi$$

例 5.1.7

$$\begin{cases} \Delta_2 u = 0, -\infty < x < +\infty, & y > 0, \\ u|_{y=0} = f(x), u(x,y) \text{ 有界}. \end{cases}$$

对原问题关于 x 作 Fourier 变换

$$\begin{cases} \frac{d^2 u}{dy^2} - \lambda^2 u = 0, y > 0, (*) \\ u|_{y=0} = f(\lambda), \\ v \to +\infty \text{ 时} u(\lambda, y) \text{ 有界} \end{cases}$$

$$\text{通解为} \quad u(\lambda,y) = C(\lambda)e^{|\lambda|y} + D(\lambda)e^{-|\lambda|y}$$

$$u|_{y=0} = D(\lambda) = f(\lambda). \text{ 故} u(\lambda,y) = f(\lambda)e^{-|\lambda|y}$$

$$u(x,y) = F^{-1}\left[f(\lambda)e^{-|\lambda|y}\right] = f(x) * F^{-1}\left[e^{-|\lambda|y}\right]$$

$$F^{-1}\left[\mathrm{e}^{-|\lambda|y}\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-|\lambda|y}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda, \quad \mathrm{e}^{\mathrm{i}\lambda x} = \cos\lambda x + \mathrm{i}\sin\lambda x$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-|\lambda|y}\cos\lambda x\,\mathrm{d}\lambda + \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-|\lambda|y}\sin\lambda x\,\mathrm{d}\lambda$$

$$= \frac{1}{2\pi}\cdot 2\int_{0}^{+\infty}\mathrm{e}^{-|\lambda|y}\cos\lambda x\,\mathrm{d}\lambda = \frac{1}{\pi}\int_{0}^{+\infty}\mathrm{e}^{-\lambda y}\cos\lambda x\,\mathrm{d}\lambda = \frac{y}{\pi\left(y^2+x^2\right)}$$

$$u(x,y) = f(x) * \frac{y}{\pi\left(y^2+x^2\right)} = \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi)}{y^2+(x-\xi)^2}d\xi$$

例 5.1.8: 使用 Fourier 优势明显

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + \cos x \cos 2t, -\infty < x < +\infty, t > 0, a > 0 \\ u|_{t=0} = x. \end{cases}$$

由线性叠加原理得$u = u_1 + u_2$, 其中u_1, u_2 分别满足

$$\begin{cases} \frac{\partial u_1}{\partial t} = a^2 \frac{\partial^2 u_1}{\partial x^2} \\ u_1|_{t=0} = \varphi(x) \end{cases} \quad \begin{cases} \frac{\partial u_2}{\partial t} = a^2 \frac{\partial^2 u_2}{\partial x^2} + f(t, x) \\ u_2|_{t=0} = 0 \end{cases}$$

$$u_1(t, x) = F^{-1}\left[\varphi(\lambda) e^{-a^2 \lambda^2 t}\right] = \varphi(x) * F^{-1}\left[e^{-a^2 \lambda^2 t}\right] = x$$

考虑$\begin{cases} \frac{\partial w}{\partial t'} = a^2 \frac{\partial^2 w}{\partial x^2}, t' > 0, -\infty < x < +\infty, \\ w|_{t'=0} = f(\tau, x), \end{cases}$ 对x 用 Fourier 变换得

$$w\left(t', x\right) = f(\tau, x) * F^{-1}\left[e^{-a^2 \lambda^2 t'}\right], t' = t - \tau = e^{-a^2(t-\tau)} \cos 2\tau \cos x$$

$$u_2 = \int_0^t \mathrm{e}^{-a^2(t-\tau)} \cos 2\tau \cos x \, \mathrm{d}\tau = (\cos x) \mathrm{e}^{-a^2 t} \int_0^t \mathrm{e}^{a^2 \tau} \cos 2\tau \mathrm{d}\tau.$$

故 $$u_2 = \frac{\left(2 \sin 2t + a^2 \cos 2t\right) \mathrm{e}^{a^2 t} - a^2}{a^2 + 4} \mathrm{e}^{-a^2 t} \cos x.$$

$$u(t, x) = u_1 + u_2 = x + \frac{2 \sin 2t + a^2 \cos 2t - a^2 e^{-a^2 t}}{a^2 + 4} \cos x$$

3 高维 Fourier 变换

$$F[f(x, y, z)] = f(\lambda, \mu, \nu) = \iiint_{-\infty}^{+\infty} f(x, y, z) \mathrm{e}^{-\mathrm{i} \lambda x} \mathrm{e}^{-\mathrm{i} \mu y} \mathrm{e}^{-\mathrm{i} \nu z} \, \mathrm{d} x \, \mathrm{d} y \, \mathrm{d} z$$
$$= \iiint_{-\infty}^{+\infty} f(x, y, z) \mathrm{e}^{-\mathrm{i}(\lambda x + \mu y + \nu z)} \mathrm{d} x \, \mathrm{d} y \, \mathrm{d} z$$

$$F\left[\frac{\partial f}{\partial x}\right] = i\lambda F[f] = i\lambda f, \quad F\left[\frac{\partial^2 f}{\partial x^2}\right] = (i\lambda)^2 F[f] = -\lambda^2 f$$

$$F\left[\Delta_3 f\right] = -\left(\lambda^2 + \mu^2 + \nu^2\right) f$$

$$F[f(x, y, z) * g(x, y, z)] = F[f(x, y, z)] F[g(x, y, z)] = f(\lambda, \mu, v) g(\lambda, \mu, \nu)$$

$$F^{-1}[f(\lambda, \mu, v) g(\lambda, \mu, v)] = F^{-1}[f(\lambda, \mu, v)] * F^{-1}[g(\lambda, \mu, \nu)]$$

$$f(x, y, z) * g(x, y, z) = \iiint_{-\infty}^{+\infty} f(\xi, \eta, \zeta) g(x - \xi, y - \eta, z - \zeta) \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}\zeta$$

例 5.1.9

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \Delta_3 u, t > 0, -\infty < x, y, z < +\infty, a > 0 \text{ 是常数}. \\ u|_{t=0} = \varphi(x, y, z). \end{cases}$$

进行 Fourier 变换得

$$\begin{cases} \frac{du}{dt} = -a^2\left(\lambda^2+\mu^2+v^2\right)u, t>0, \\ u|_{t=0} = \varphi(\lambda) \end{cases}$$

$$\text{故} \quad u(t,x,y,z) = \left(\frac{1}{2a\sqrt{\pi t}}\right)^3 \mathrm{e}^{-\frac{x^2+y^2+z^2}{4a^2t}} * \varphi(x,y,z)$$

$$= \left(\frac{1}{2a\sqrt{\pi t}}\right)^3 \iiint_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2+\eta^2+\zeta^2}{4a^2t}} \varphi(x-\xi, y-\eta, z-\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

第 2 节　Laplace 变换

$$\text{规定: } f(t) = f(t)H(t) = \begin{cases} f(t), & t \geqslant 0, \\ 0, & t<0, \end{cases} \text{ 其中 } H(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t<0. \end{cases}$$

设t 充分大时, $|f(t)| \le M\mathrm{e}^{\sigma_0 t}$, 记$p=\sigma+\mathrm{i}\lambda$, 当 $\mathrm{Re}\, p = \sigma > \sigma_0$ 时, $\int_0^{+\infty} f(t)\mathrm{e}^{-pt}\,\mathrm{d}t$ 绝对可积,

$$\text{称} \bar{f}(p) = \int_0^{+\infty} f(t)\mathrm{e}^{-pt}\,\mathrm{d}t \text{ 为} f(t) \text{ 的 Laplace 变换像函数,}$$

简称为 $f(t)$ 的 Laplace 变换, 记作 $L[f(t)]$, 即

$$L[f(t)] = \bar{f}(p) = \int_0^{+\infty} f(t)\mathrm{e}^{-pt}\,\mathrm{d}t = F\left[f(t)\mathrm{e}^{-\sigma t}H(t)\right]$$

Laplace 变换反演公式: 其中 $p=\sigma+i\lambda$, $\mathrm{e}^{-pt} = \mathrm{e}^{-\sigma t}\mathrm{e}^{-\mathrm{i}\lambda t}$.

$$f(t)H(t) = L^{-1}[\bar{f}(p)] = \frac{1}{2\pi i}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \bar{f}(p)e^{pt}dp, \sigma>\sigma_0$$

性质 5.2.1

(1) 变换线性: $L\left[C_1 f(t) + C_2 g(t)\right] = C_1 L[f(t)] + C_2 L[g(t)]$, 其中 C_1, C_2 与 t 无关.
反变换线性: $L^{-1}[\alpha\bar{f}(p) + \beta\bar{g}(p)] = \alpha L^{-1}[\bar{f}(p)] + \beta L^{-1}[\bar{g}(p)]$, 其中 α, β 与 p 无关.

(2) 微分公式:

$$\left[f^{(n)}(t)\right] = p^n L[f(t)] - p^{n-1}f(+0) - p^{n-2}f'(+0) - \ldots - pf^{(n-2)}(+0) - f^{(n-1)}(+0)$$

$$L\left[f'(t)\right] = pL[f(t)] - f(+0), f(+0) = \lim_{t\to0^+} f(t)$$

$$L\left[f''(t)\right] = p^2L[f(t)] - pf(+0) - f'(+0)$$

(3) 积分公式:

$$L\left[\int_0^t f(\tau)\mathrm{d}\tau\right] = \frac{1}{p}\bar{f}(p). \Rightarrow L^{-1}\left[\frac{1}{p}\bar{f}(p)\right] = H(t)\int_0^t f(\tau)\mathrm{d}\tau$$

(4) 延迟定理:

$$L[f(t-\tau)H(t-\tau)] = e^{-p\tau}\bar{f}(p), \tau>0$$

(5) 卷积

$$f(t)*g(t)=\{f(t)H(t)\}*\{g(t)H(t)\}=\int_0^t f(\tau)g(t-\tau)d\tau$$

$$L[f(t)*g(t)]=L[f(t)]L[g(t)]=\bar{f}(p)\bar{g}(p)$$

例 5.2.1: 半无界弦强迫振动

$$\begin{cases}\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2}, & t>0,x>0,a>0\text{ 是常数,}\\ \frac{\partial u}{\partial x}\Big|_{x=0}=f(t), & f(t)\neq\text{ 常数时, 不适宜用 Fourier 余弦变换}\\ u|_{t=0}=0, \quad \frac{\partial u}{\partial t}\Big|_{t=0}=0, & 0\text{ 初值 }\Rightarrow\text{ 关于}t\text{ 作 Laplace 变换}\end{cases}$$

$x>0$, 不适宜用分离变量法或 Fourier 变换法

$$L\left[\frac{\partial^2 u}{\partial t^2}\right]=p^2\bar{u}-pu|_{t=+0}-\frac{\partial u}{\partial t}\Big|_{t=+0}=p^2\bar{u}$$

$$L\left[\frac{\partial^2 u}{\partial x^2}\right]=\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2},\quad L\left[\frac{\partial u}{\partial x}\Big|_{x=0}\right]=\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\Big|_{x=0}=L[f(t)]=\bar{f}(p)$$

$$\begin{cases}p^2\bar{u}=a^2\frac{d^2\bar{u}}{dx^2}, x>0, (*)\\ \frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\Big|_{x=0}=\bar{f}(p),\ \bar{u}|_{x=+\infty}=0(\text{这是添加的自然条件})\end{cases}$$

通解 $\bar{u}=Ce^{\frac{p}{a}x}+De^{-\frac{p}{a}x}$，因 $\operatorname{Re}p=\sigma>0$, $\bar{u}|_{x=+\infty}=0$, 故 $C=0.\bar{u}=De^{-\frac{p}{a}x}$

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\Big|_{x=0}=-\frac{Dp}{a}=\bar{f}(p), D=-\frac{a}{p}\bar{f}(p).\text{ 故}\bar{u}=-\frac{a}{p}\bar{f}(p)\mathrm{e}^{-\frac{p}{a}x}$$

$$u(t,x)=L^{-1}\left[-\frac{a}{p}\bar{f}(p)e^{-\frac{p}{a}x}\right]=-aL^{-1}\left[\left(\frac{1}{p}\bar{f}(p)\right)e^{-\frac{x}{a}p}\right]$$

由积分公式得, $L^{-1}\left[\frac{1}{p}\bar{f}(p)\right]=H(t)\int_0^t f(\tau)\mathrm{d}\tau$. 由延迟定理,

$$u(t,x)=-aH\left(t-\frac{x}{a}\right)\int_0^{t-\frac{x}{a}}f(\tau)\mathrm{d}\tau$$

例 5.2.2: 半无界温度分布

$$\begin{cases}\frac{\partial v}{\partial t}=a^2\frac{\partial^2 v}{\partial x^2}+g(t), t>0, x>0\\ v|_{x=0}=f(t)\\ v|_{t=0}=u_0, u_0\text{ 是常数.}\end{cases}$$

通过 Lapalce 变换, 得到

$$\begin{cases}\frac{\mathrm{d}^2\bar{v}}{\mathrm{d}x^2}-\frac{p}{a^2}\bar{v}=-\frac{u_0}{a^2}-\frac{1}{a^2}\bar{g}(p), x>0\\ \bar{v}|_{x=0}=\bar{f}(p), |\bar{v}|_{x=+\infty}\text{ 有界}\end{cases}$$

先求齐次方程$\frac{\mathrm{d}^2\bar{v}}{\mathrm{d}x^2}-\frac{p}{a^2}\bar{v}=0$ 通解 : $\bar{v}_h(p,x)=C\mathrm{e}^{\frac{\sqrt{p}}{a}x}+D\mathrm{e}^{-\frac{\sqrt{p}}{a}x}$, 因$|\bar{v}|_{x=+\infty}|$ 有界, 故$C=0$

$$\text{右端非齐项与}x\text{ 无关, 故 有特解}\bar{v}_*(p,x)=A(p)=\frac{u_0}{p}+\frac{1}{p}\bar{g}(p), \text{与}x\text{ 无关}$$

$$\bar{v}|_{x=0}=D+\frac{u_0}{p}+\frac{1}{p}\bar{g}(p)=\bar{f}(p), D=\bar{f}(p)-\frac{u_0}{p}-\frac{1}{p}\bar{g}(p)$$

$$\bar{v}(p,x)=\left\{\bar{f}(p)-\frac{u_0}{p}-\frac{1}{p}\bar{g}(p)\right\}\mathrm{e}^{-\frac{\sqrt{p}}{a}x}+\frac{u_0}{p}+\frac{1}{p}\bar{g}(p)$$

$$\begin{aligned}v(t,x)&=L^{-1}\left[\left\{\bar{f}(p)-\frac{u_0}{p}-\frac{1}{p}\bar{g}(p)\right\}e^{-\frac{x}{a}\sqrt{p}}\right]+L^{-1}\left[\frac{u_0}{p}\right]+L^{-1}\left[\frac{1}{p}\bar{g}(p)\right]\\&=L^{-1}\left[\bar{f}(p)-\frac{u_0}{p}-\frac{1}{p}\bar{g}(p)\right]*L^{-1}\left[e^{-\frac{x}{a}\sqrt{p}}\right]+u_0L^{-1}\left[\frac{1}{p}\right]+L^{-1}\left[\frac{1}{p}\bar{g}(p)\right].\end{aligned}$$

$$\because L^{-1}\left[\bar{f}(p)-\frac{u_0}{p}-\frac{1}{p}\bar{g}(p)\right]=H(t)\left(f(t)-u_0-\int_0^t g(\eta)\mathrm{d}\eta\right),\quad L^{-1}\left[e^{-\alpha\sqrt{p}}\right]=H(t)\frac{\alpha e^{-\frac{\alpha^2}{4t}}}{2\sqrt{\pi}t^{\frac{3}{2}}}$$

$$\begin{aligned}v(t,x)&=\left\{H(t)\left(f(t)-u_0-\int_0^t g(\eta)\mathrm{d}\eta\right)\right\}*\left(\frac{x}{2a\sqrt{\pi}}\cdot\frac{H(t)}{t^{\frac{3}{2}}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\right)+u_0H(t)+H(t)\int_0^t g(\eta)\mathrm{d}\eta\\&=\frac{x}{2a\sqrt{\pi}}\int_0^t\left(f(t-\tau)-u_0-\int_0^{t-\tau}g(\eta)\mathrm{d}\eta\right)\cdot\frac{1}{\tau^{\frac{3}{2}}}e^{-\frac{x^2}{4a^2\tau}}\mathrm{d}\tau+H(t)\left(u_0+\int_0^t g(\eta)\mathrm{d}\eta\right)\end{aligned}$$

★复杂混合边界的求解

$$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2}+f(t), t>0, x>0, a>0\text{ 是常数},\\ u|_{x=0}=g(t),\ u|_{t=0}=\varphi(x),\ \frac{\partial u}{\partial t}\Big|_{t=0}=\psi(x).\end{cases}$$

根据线性叠加原理, 分解: $u=u_1+u_2$,

$$\begin{cases}\frac{\partial^2u_1}{\partial t^2}=a^2\frac{\partial^2u_1}{\partial x^2}, t>0,\quad x>0,\\ u_1|_{x=0}=0,\ u_1|_{t=0}=\varphi(x),\ \frac{\partial u_1}{\partial t}\Big|_{t=0}=\psi(x)\end{cases}$$

u_1 : 初值 x 奇延拓后, 用达朗贝尔公式或 Fourier 变换

$$\begin{cases}\frac{\partial^2u_2}{\partial t^2}=a^2\frac{\partial^2u_2}{\partial x^2}+f(t), t>0, x>0\\ u_2|_{x=0}=g(t),\quad u_2|_{t=0}=0,\ \frac{\partial u_2}{\partial t}\Big|_{t=0}=0\end{cases}$$

u_2 : 用 Laplace 变换求

例 5.2.3: 有限长弦受迫振动

$$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\frac{\partial^2u}{\partial x^2},\quad t>0, 0<x<l,\quad a>0, A\text{ 是常数},\\ u|_{x=0}=0,\ \frac{\partial u}{\partial x}\Big|_{x=l}=A\sin\omega t, \omega\neq\frac{2k-1}{2l}\pi a, k=1,2,\cdots,\\ u|_{t=0}=0,\ \frac{\partial u}{\partial t}\Big|_{t=0}=0\end{cases}$$

▶ 可以采取特解 + 分离变量法，此处采取 Laplace 变换法。

$$\text{设}L[u(t,x)]=\bar{u}(p,x)=\int_0^{+\infty}u(t,x)\mathrm{e}^{-pt}\,\mathrm{d}t, p=\sigma+\mathrm{i}\lambda, \sigma>0.$$

$$L\left[\frac{\partial^2 u}{\partial t^2}\right]=p^2\bar{u}-pu|_{t=0}-\left.\frac{\partial u}{\partial t}\right|_{t=0}=p^2\bar{u}$$

$$\begin{cases} p^2\bar{u}=a^2\frac{d^2\bar{u}}{\mathrm{d}x^2}, 0<x<l \\ \bar{u}|_{x=0}=0,\ \left.\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right|_{x=l}=AL[\sin\omega t]=\frac{A\omega}{p^2+\omega^2} \end{cases}$$

通解 $\bar{u}=C\cosh\frac{p}{a}x+D\sinh\frac{p}{a}x$，$\bar{u}|_{x=0}=C=0$. 故 $\bar{u}=D\sinh\frac{p}{a}x$

$$\left.\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right|_{x=l}=D\frac{p}{a}\cosh\frac{p}{a}l=\frac{A\omega}{p^2+\omega^2}. \text{ 故} D=\frac{A\omega a}{(p^2+\omega^2)\,p\cosh\frac{p}{a}l}$$

$$\text{故}\bar{u}(p,x)=\frac{A\omega a\sinh\frac{p}{a}x}{p\,(p^2+\omega^2)\cosh\frac{p}{a}l}$$

$$u(t,x)=L^{-1}[\bar{u}(p,x)]=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\bar{u}(p,x)e^{pt}\,\mathrm{d}p$$

其中 $p_0=0$ (可去奇点), $p_{01}=\omega\mathrm{i}$ (一级极点), $p_{02}=-\omega\mathrm{i}$ (一级极点), $p_k=\pm\mathrm{i}\varpi_k$ (一级极点), 其中 $\varpi_k=\dfrac{a(2k-1)\pi}{2I}, k\in\mathbb{Z}^+$, 所有奇点都在虚轴上, 任取$\sigma>0$. 作闭路包含虚轴奇点和 $\mathrm{Re}\,p=\sigma$

$$\begin{aligned}
u&=\left(\operatorname*{Res}_{p=\mathrm{i}\omega}+\operatorname*{Res}_{p=-\mathrm{i}\omega}\right)\left[\bar{u}(p,x)\mathrm{e}^{pt}\right]+\sum_{k=1}^{+\infty}\left(\operatorname*{Res}_{p=\mathrm{i}\varpi_k}+\operatorname*{Res}_{p=-\mathrm{i}\varpi_k}\right)\left[\bar{u}(p,x)\mathrm{e}^{pt}\right]\\
&=2\,\mathrm{Re}\left(\mathrm{Res}_{p=\mathrm{i}\omega}\left[\bar{u}(p,x)\mathrm{e}^{pt}\right]\right)+\sum_{k=1}^{+\infty}2\,\mathrm{Re}\left(\mathrm{Res}_{p=i\omega}\left[\bar{u}(p,x)\mathrm{e}^{pt}\right]\right)\\
&=2\,\mathrm{Re}\left\{\lim_{p\to\mathrm{i}\omega}(p-\mathrm{i}\omega)\bar{u}(p,x)e^{pt}\right\}+2\sum_{k=1}^{+\infty}\mathrm{Re}\left\{\left(\left.\frac{A\omega a\sinh\frac{p}{a}xe^{pt}}{p\,(p^2+\omega^2)}\right|_{p=i\varpi_k}\right)\cdot\left(\frac{1}{\left.\left(\cosh\frac{p}{a}l\right)'\right|_{p=i\varpi_k}}\right)\right\}\\
&=\frac{Aa\sin\omega t}{\omega\cos\frac{\omega l}{a}}\sin\frac{\omega x}{a}+\frac{16l^2A\omega a}{\pi}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}\sin\frac{\varpi_k}{a}x\sin\varpi_k t}{(2k-1)\left\{4l^2\omega^2-(2k-1)^2a^2\pi^2\right\}}.
\end{aligned}$$

注: 一些常用 Laplace 变换

$$L\left[e^{at}\right]=\frac{1}{p-a}, p\neq a. \Rightarrow a=0\text{ 时}, L[1]=L[H(t)]=\frac{1}{p}, p\neq 0$$

$$L[\cos\omega t]=\frac{p}{p^2+\omega^2}\qquad L[\sin\omega t]=\frac{\omega}{p^2+\omega^2}$$

$$L[\cosh\omega t]=\frac{p}{p^2-\omega^2}\qquad L[\sinh\omega t]=\frac{\omega}{p^2-\omega^2}$$

$$L^{-1}\left[\frac{1}{p-a}\right]=H(t)\mathrm{e}^{at},\qquad L^{-1}\left[\frac{1}{p}\right]=H(t)$$

$$L^{-1}\left[\frac{p}{p^2+\omega^2}\right]=H(t)\cos\omega t,\qquad L^{-1}\left[\frac{\omega}{p^2+\omega^2}\right]=H(t)\sin\omega t,$$

$$L^{-1}\left[\frac{p}{p^2-\omega^2}\right]=H(t)\cosh\omega t,\qquad L^{-1}\left[\frac{\omega}{p^2-\omega^2}\right]=H(t)\sinh\omega t.$$

性质 5.2.2: Laplace 变换的推论

(1) 多项式:

$$L\left[t^{n}\right]=\frac{n!}{p^{n+1}}, n \in \mathbb{N}. \Rightarrow L^{-1}\left[\frac{1}{p^{n}}\right]=\frac{t^{n-1}}{(n-1)!} H(t), \quad \forall n \in \mathbb{N}$$

(2) 复频移公式

$$L\left[f(t) \mathrm{e}^{p_{0} t} H(t)\right]=\bar{f}\left(p-p_{0}\right), \quad p_{0}: \text{任意复常数}$$

(3) 相似定理:

$$\forall \alpha>0, \quad L[f(\alpha t)]=\frac{1}{\alpha} \bar{f}\left(\frac{p}{\alpha}\right)$$

(4) 像函数相关公式:

$$\text{像函数微分公式}: L\left[(-t)^{n} f(t)\right]=\bar{f}^{(n)}(p).$$

$$\text{像函数积分公式:(P146)}: L\left[\frac{f(t)}{t}\right]=\int_{P}^{+\infty} \bar{f}(p) \mathrm{d} p,$$

右端积分路径取在 $\operatorname{Re} p>\sigma_{0}$ 内, σ_{0} 是 $f(t)$ 的增长指数

例 5.2.4: 初值问题

$$\left\{\begin{array}{l}\frac{\mathrm{d}^{2} y}{\mathrm{d} t^{2}}+3 y=t \\ \left.y\right|_{t=0}=\left.y^{\prime}\right|_{t=0}=0\end{array}\right.$$

解：设 $L[y(t)]=\bar{y}(p)$, 则

$$L\left[y^{\prime \prime}(t)\right]=p^{2} \bar{y}(p)-p y(+0)-y^{\prime}(+0)=p^{2} \bar{y}(p)$$

令 $g(t)=t$, 对方程作 Laplace 变换得, $p^{2} \bar{y}(p)+3 \bar{y}(p)=\bar{g}(p)$. 故 $\bar{y}(p)=\frac{\bar{g}(p)}{p^{2}+3}$. (2)

$$\begin{aligned} y(t)&=L^{-1}\left[\frac{\bar{g}(p)}{p^{2}+3}\right]=L^{-1}\left[\frac{1}{p^{2}+3}\right] * L^{-1}[\bar{g}(p)]=\frac{1}{\sqrt{3}} L^{-1}\left[\frac{\sqrt{3}}{p^{2}+(\sqrt{3})^{2}}\right] *(H(t) g(t)) \\ &=\frac{1}{\sqrt{3}}\{H(t) \sin \sqrt{3} t\} *\{H(t) g(t)\}=\frac{1}{\sqrt{3}} \int_{0}^{t} g(t-\tau) \sin \sqrt{3} \tau d \tau. \end{aligned}$$

$$\text{代入} g(t)=t \text{ 得} y(t)=\frac{1}{\sqrt{3}} \int_{0}^{t}(t-\tau) \sin \sqrt{3} \tau \mathrm{d} \tau=\frac{1}{3} t-\frac{\sqrt{3}}{9} \sin \sqrt{3} t.$$

第六章 基本解的方法

第 1 节 δ 函数，广义函数

在原点置单位点电荷，单位电荷均匀分布在 $[-\varepsilon, \varepsilon]$

电荷总量

$$Q=\int_{-\infty}^{+\infty}\rho_{\varepsilon}(x)\mathrm{d}x=\int_{-\varepsilon}^{\varepsilon}\frac{1}{2\varepsilon}\mathrm{d}x=1$$

定理 6.1.1: δ 函数定义

称集中在 $x=0$ 点单位物理量引起的密度函数为 δ 函数, 记为 $\delta(x)$

(1) $\delta(x)=\begin{cases}+\infty, & x=0\text{ 时},\\ 0, & x\neq 0\text{ 时},\end{cases}$ (2) $\displaystyle\int_{-\infty}^{+\infty}\delta(x)\mathrm{d}x=1.$

1 δ 函数性质

(1) 集中在 $x=\xi$ 点单位物理量引起的密度函数用 $\delta(x-\xi)$ 表示:

$\delta(x-\xi)=\begin{cases}+\infty, & x=\xi\text{ 时},\\ 0, & x\neq\xi\text{ 时},\end{cases}$ $\int_{-\infty}^{+\infty}\delta(x-\xi)\mathrm{d}x=1$

(2) 筛选性: $\forall\varphi(x)\in C(\mathbb{R})$($\mathbb{R}$ 上连续函数), $\mathbb{R}=(-\infty,+\infty), a<b$,

$$\int_a^b\delta(x)\varphi(x)\mathrm{d}x=\begin{cases}\varphi(0), & 0\in[a,b]\\ 0, & 0\notin[a,b]\end{cases}$$

注: 级数展开

设 $\{\varphi_n(x),n=0,1,2,\cdots\}$ 是 $L^2_{\rho(x)}[a,b]$ 的正交基,

$a<x,\xi<b$, 则 $\delta(x-\xi)=\sum\limits_{n=0}^{+\infty}A_n\varphi_n(x)$,

$$\text{其中}A_k=\frac{1}{\|\varphi_k(x)\|^2}\int_a^b\rho(x)\delta(x-\xi)\varphi_k(x)dx$$

根据δ 函数筛选性得

$$A_n=\frac{\varphi_k(\xi)\rho(\xi)}{\int_a^b\rho(x)\left|\varphi_k(x)\right|^2\,\mathrm{d}x}$$

(3) 对称性 $\delta(x-\xi)=\delta(\xi-x)$

设$f_1(x),f_2(x)$ 是广义函数, 则

$f_1(x)=f_2(x)\Leftrightarrow<f_1(x),\varphi(x)>=<f_2(x),\varphi(x)>,\forall\varphi(x)\in$ 基本函数空间: $C(\mathbb{R})$

定理 6.1.2

定义: $C_0^{\infty}(\mathbb{R})=\{\varphi(x)\mid$ 无穷次连续可导, 且只在有限区间内不为 0.$\}$, $C_0^{\infty}(\mathbb{R})$ 上线性泛函全体记为 $(C_0^{\infty}(\mathbb{R}))'$, $C_0^{\infty}(\mathbb{R})\subset C(\mathbb{R})$, 对偶空间$C'(\mathbb{R})\subset(C_0^{\infty}(\mathbb{R}))$

$$\left\langle f^{(n)}(x),\varphi(x)\right\rangle=(-1)^n\left\langle f(x),\varphi^{(n)}(x)\right\rangle$$

$$\text{即} \int_{-\infty}^{+\infty} f^{(n)}(x)\varphi(x)\mathrm{d}x = (-1)^n \int_{-\infty}^{+\infty} f(x)\varphi^{(n)}(x)\mathrm{d}x, \forall\varphi(x)\in C_0^\infty(\mathbb{R})$$

$$\therefore \delta^{(n)}(x) : \varphi(x) \to (-1)^n\varphi^{(n)}(0)$$

(4) 导数 $\delta^{(n)}$(x)

$$\int_{-\infty}^{+\infty} \delta^{(n)}(x)\varphi(x)\mathrm{d}x = (-1)^n \int_{-\infty}^{+\infty} \delta(x)\varphi^{(n)}(x)\mathrm{d}x = (-1)^n\varphi^{(n)}(0)$$

(5) δ 函数原函数——阶跃函数：$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$$\int_{-\infty}^{+\infty} H'(x)\varphi(x)\mathrm{d}x = -\int_{-\infty}^{+\infty} H(x)\varphi'(x)\mathrm{d}x = -\int_{0}^{+\infty} \varphi'(x)\mathrm{d}x = \varphi(0)$$

$$\text{故} H'(x) = \delta(x). \text{称} H(x) \text{ 是} \delta(x) \text{ 的一个原函数}$$

$y'(x) = \delta(x)$ 的通解是 $y(x) = H(x) + c$, 其中 c 是任意实常数

(6) δ 函数与连续函数 (基本函数空间中函数) 的乘积

$$\forall\alpha(x)\in C(\mathbb{R}), \alpha(x)\delta(x-\xi) = \alpha(\xi)\delta(x-\xi).$$

$$\text{例:} \mathrm{e}^x\delta(x+1) = \mathrm{e}^x\delta(x-(-1)) = \mathrm{e}^{-1}\delta(x+1), \delta(x)\cos x = \cos 0\delta(x) = \delta(x)$$

(7) δ 与连续可导函数的复合

$\forall u(x)\in C^1(\mathbb{R})$, 若 $u(x)$ 在实轴只有单零点 $x_k(k=1,2,\cdots,N)$, 即 $u(x_k)=0, u'(x_k)\neq 0, 1\leq k\leq N$, 则

$$\delta(u(x)) = \sum_{k=1}^{N} \frac{\delta(x-x_k)}{|u'(x_k)|}$$

例 6.1.1

$$\delta(\sin x) = \sum_{n=-\infty}^{+\infty} \frac{\delta(x-n\pi)}{|\cos n\pi|} = \sum_{n=-\infty}^{+\infty} \delta(x-n\pi)$$

$\delta(x^2+1) = 0$, 因 $x^2+1\geq 1$ 在 $\mathbb{R}$ 上无零点

$$\delta(ax) = \frac{\delta(x-0)}{|a|} = \frac{\delta(x)}{|a|}$$

$$\delta(x^2-a^2) = \frac{\delta(x+a)}{\left(2x|_{x=-a}\right)|} + \frac{\delta(x-a)}{|(2x|_{x=a})|} = \frac{\delta(x+a)}{2|a|} + \frac{\delta(x-a)}{2|a|}$$

$$|x|\delta(x^2) = \lim_{a\to 0}|x|\delta(x^2-a^2) \overset{(2)}{=} \lim_{a\to 0}\left(\frac{|x|\delta(x+a)}{2|a|} + \frac{|x|\delta(x-a)}{2|a|}\right)$$

$$= \lim_{a\to 0}\left(\frac{|-a|\delta(x+a)}{2|a|} + \frac{|a|\delta(x-a)}{2|a|}\right) = \lim_{a\to 0}\left(\frac{\delta(x+a)}{2} + \frac{\delta(x-a)}{2}\right) = \delta(x)$$

(8) 卷积：$\forall f(x) \in (C^{\infty}(\mathbb{R}))', \forall g(x) \in (C_0^{\infty}(\mathbb{R}))', (C^{\infty}(\mathbb{R}))' \subset (C_0^{\infty}(\mathbb{R}))'$ 则定义卷积 $f(x) * g(x) \in (C_0^{\infty}(\mathbb{R}))'$ 如下:

$$< f(x) * g(x), \varphi(x) >=< f(x), < g(y), \varphi(x+y) >>, \forall \varphi(x) \in C_0^{\infty}(\mathbb{R})$$

$$\text{写出：} f(x) * g(x) = \int_{-\infty}^{+\infty} f(\xi) g(x-\xi) \mathrm{d}\xi$$

注

(1) $\delta(x) * f(x) = f(x)$, (2) $\delta'(x) * f(x) = \delta(x) * f'(x) = f'(x)$

(3) $\delta^{(n)}(x) * f(x) = \delta(x) * f^{(n)}(x) = f^{(n)}(x)$

(9) Fourier 变换:

$$F[\delta] = \delta(\lambda) = \int_{-\infty}^{+\infty} \delta(x) \mathrm{e}^{-\mathrm{i}\lambda x} \, \mathrm{d}x = \left.\mathrm{e}^{-\mathrm{i}\lambda x}\right|_{x=0} = 1$$

$$\Rightarrow F^{-1}[1] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{i\lambda x} \mathrm{d}\lambda = \delta(x)$$

$$F^{-1}[\delta(\lambda)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta(\lambda) \mathrm{e}^{\mathrm{i}\lambda x} \, \mathrm{d}\lambda = \left.\frac{1}{2\pi} \mathrm{e}^{\mathrm{i}\lambda x}\right|_{\lambda=0} = \frac{1}{2\pi}$$

$$\Rightarrow F[1] = 2\pi\delta(\lambda), \text{即} \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\lambda x} \, \mathrm{d}x = \int_{-\infty}^{+\infty} \cos \lambda x \, \mathrm{d}x = 2\pi\delta(\lambda)$$

根据频移公式

$$F\left[\mathrm{e}^{\mathrm{i}ax}\right] = F\left[1 \cdot \mathrm{e}^{iax}\right] = 2\pi\delta(\lambda - a)$$

$$F[\cos ax] = F\left[\frac{\mathrm{e}^{\mathrm{i}ax} + \mathrm{e}^{-\mathrm{i}ax}}{2}\right] = \pi\delta(\lambda - a) + \pi\delta(\lambda + a)$$

$$F[\sin ax] = F\left[\frac{\mathrm{e}^{\mathrm{i}ax} - \mathrm{e}^{-\mathrm{i}ax}}{2\mathrm{i}}\right] = \mathrm{i}\pi\{\delta(\lambda + a) - \delta(\lambda - a)\}$$

对 $\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\lambda x} \, \mathrm{d}x = 2\pi\delta(\lambda)$ 两边关于 λ 求导得 $-\mathrm{i} \int_{-\infty}^{+\infty} x \mathrm{e}^{-\mathrm{i}\lambda x} \, \mathrm{d}x = 2\pi\delta'(\lambda)$, 再两边乘以 i 得

$$F[x] = 2\pi \mathrm{i}\delta'(\lambda). \Rightarrow F^{-1}\left[\delta'(\lambda)\right] = \frac{x}{2\pi\mathrm{i}} = -\mathrm{i}\frac{x}{2\pi}$$

例 6.1.2

$$\begin{cases} u_t = a^2 u_{xx}, t > 0, \quad 0 < x < l, a > 0, \\ u|_{x=0} = 0, \, u_x|_{x=l} = 0, (b) \text{ 分离变量法} \\ |_{t=0} = \varphi(x)\delta(x-\xi),, 0 < x, \xi < l. \end{cases}$$

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0, 0 < x < l, \\ X(0) = X'(l) = 0 \end{cases}$$

$$X(x)=B\sin\omega x.X'(l)=B\omega\cos\omega l=0,\cos\omega l=0$$
$$\omega_n=\frac{1}{l}\cdot\left(n+\frac{1}{2}\right)\pi=\frac{(2n+1)\pi}{2l}>0,n>-\frac{1}{2},n=0,1,2,\cdots$$

固有值 $\lambda_n=\omega_n^2$, 固有函数 $X_n(x)=\sin\omega_n x=\sin\dfrac{(2n+1)\pi}{2l}x$ 对每个

$$\lambda_n=\omega_n^2>0,T'+a^2\omega_n^2T=0,T_n=C_n\mathrm{e}^{-a^2\omega_n^2t}$$
$$设u(x,t)=\sum_{n=0}^{+\infty}T_n(t)X_n(x)=\sum_{n=0}^{+\infty}C_n\mathrm{e}^{-a^2\omega_n^2t}\sin\omega_n x$$
$$\rho(x)\equiv1,故\left\{\sin\omega_n x,n=0,1,2,\cdots\right\}是L^2[0,l]的正交基$$
$$u(x,0)=\sum_{n=0}^{+\infty}C_n\sin\omega_n x=\varphi(x)\delta(x-\xi)$$
$$C_n=\frac{\int_0^l\rho(x)\varphi(x)\delta(x-\xi)\sin\omega_n x\,\mathrm{d}x}{\int_0^l\rho(x)\sin^2\omega_n x\,\mathrm{d}x}=\frac{\rho(\xi)\varphi(\xi)\sin\omega_n\xi}{\int_0^l\rho(x)\frac{1-\cos2\omega_n x}{2}\,\mathrm{d}x}=\frac{2}{l}\varphi(\xi)\sin\omega_n\xi$$
$$故u(x,t)=\frac{2}{l}\varphi(\xi)\sum_{n=0}^{+\infty}\sin\frac{(2n+1)\pi\xi}{2l}\mathrm{e}^{-\frac{a^2(2n+1)^2\pi^2t}{4l^2}}\sin\frac{(2n+1)\pi x}{2l}$$

2　高维 δ 函数

定理 6.1.3

$$\delta\left(M-M_0\right)=\begin{cases}+\infty,M=M_0\text{ 时,}\\0,M\neq M_0\text{ 时,}\end{cases}$$
$$\iiint_{\mathbb{R}^3}\delta\left(M-M_0\right)\mathrm{d}M=1,\ 即\iiint_{\mathbb{R}^3}\delta(x-\xi,y-\eta,z-\zeta)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=1$$

$$\forall f(x,y,z)\in\left(C_0^\infty\left(\mathbb{R}^3\right)\right)',\frac{\partial^m f(x,y,z)}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}}\in\left(C_0^\infty\left(\mathbb{R}^3\right)\right)',\forall\varphi(x,y,z)\in C_0^\infty\left(\mathbb{R}^3\right)$$
$$\left\langle\frac{\partial^m f(x,y,z)}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}},\varphi(x,y,z)\right\rangle=(-1)^m\left\langle f(x,y,z),\frac{\partial^m\varphi(x,y,z)}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}}\right\rangle$$
$$\left\langle\frac{\partial^m\delta(x,y,z)}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}},\varphi(x,y,z)\right\rangle=(-1)^m\left\langle\delta(x,y,z),\frac{\partial^m\varphi(x,y,z)}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}}\right\rangle=(-1)^m\frac{\partial^m\varphi}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}}(0,0,0)$$

定义: $\forall f\in\mathscr{S}'\left(\mathbb{R}^3\right)$. **定义** $F[f],F^{-1}[f]\in\mathscr{S}'\left(\mathbb{R}^3\right)$:

$$\forall\varphi\in\mathscr{S}\left(\mathbb{R}^3\right),\quad\langle F[f],\varphi\rangle\overset{定义}{=}\langle f,F[\varphi]\rangle,\left\langle F^{-1}[f],\varphi\right\rangle\overset{定义}{=}\left\langle f,F^{-1}[\varphi]\right\rangle.$$

根据:
$$F[f(x,y,z)]=f(\lambda,\mu,\nu)=\iiint_{\mathbb{R}^3}f(x,y,z)\mathrm{e}^{-\mathrm{i}(\lambda x+\mu y+\nu z)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$F^{-1}[f(\lambda,\mu,\nu)]=\frac{1}{(2\pi)^3}\iint_{\mathbb{R}^3}f(\lambda,\mu,\nu)\mathrm{e}^{\mathrm{i}(\lambda x+\mu y+\nu z)}\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu$$

$$F[1]=\iint_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}(\lambda x+\mu y+\nu z)}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x\right)\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\mu y}\,\mathrm{d}y\right)\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\nu z}\,\mathrm{d}z\right)$$
$$=2\pi\delta(\lambda)\cdot 2\pi\delta(\mu)\cdot 2\pi\delta(\nu)=(2\pi)^3\delta(\lambda,\mu,\nu)$$

$F\left[\mathrm{e}^{\mathrm{i}ax}\right]=F\left[1\cdot\mathrm{e}^{\mathrm{i}ax}\right]=(2\pi)^3\delta(\lambda-a,\mu,\nu), F\left[\mathrm{e}^{\mathrm{i}by}\right]=(2\pi)^3\delta(\lambda,\mu-b,\nu), F\left[\mathrm{e}^{\mathrm{i}cz}\right]=(2\pi)^3\delta(\lambda,\mu,\nu-c)$

得到反演公式

$$F^{-1}[\delta(\lambda,\mu,\nu)]=\frac{1}{(2\pi)^3},\quad F^{-1}[1]=\delta(x,y,z)$$

进行一些升次

$$F[x]=(2\pi)^3 i\frac{\partial\delta(\lambda,\mu,\nu)}{\partial\lambda}\Longrightarrow F\left[x^2\right]=-(2\pi)^3\frac{\partial^2\delta(\lambda,\mu,\nu)}{\partial\lambda^2}$$
$$F\left[x^2+y^2+z^2\right]=-(2\pi)^3\Delta_3\delta(\lambda,\mu,\nu)\cdot\Rightarrow\cdot F^{-1}\left[\Delta_3\delta(\lambda,\mu,\nu)\right]=\frac{-1}{(2\pi)^3}\left(x^2+y^2+z^2\right)$$

▶**卷积:**

$$\forall f(x,y,z)\in\left(C^\infty\left(\mathbb{R}^3\right)\right)',\forall g(x,y,z)\in\left(C_0^\infty\left(\mathbb{R}^3\right)\right)',f*g\in\left(C_0^\infty\left(\mathbb{R}^3\right)\right)',\forall\varphi(x)\in C_0^\infty\left(\mathbb{R}^3\right),$$
$$\langle f(x,y,z)*g(x,y,z),\varphi(x,y,z)\rangle\overset{\text{定义}}{=}\langle f(\xi,\eta,\zeta),\langle g(u,v,w),\varphi(u+\xi,v+\eta,w+\zeta)\rangle\rangle$$

$$f(x,y,z)*g(x,y,z)=\iiint_{\mathbb{R}^3}f(\xi,\eta,\zeta)g(x-\xi,y-\eta,z-\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

满足： $\delta(x,y,z)*f(x,y,z)=f(x,y,z),\quad \dfrac{\partial^n\delta}{\partial x^n}*f=\delta*\dfrac{\partial^n f}{\partial x^n}=\dfrac{\partial^n f}{\partial x^n}$

$$F[f(x,y,z)*g(x,y,z)]=F[f(x,y,z)]F[g(x,y,z)]=f(\lambda,\mu,\nu)g(\lambda,\mu,\nu)$$

第 2 节 $Lu=0$ 方程基本解

$M=(x,y,z)\in\mathbb{R}^3,(-\infty<x,y,z<+\infty),L$ 是常系数线性偏微分算子,

$$\text{即}L=\sum_{m_1,m_2,m_3}a_{m_1m_2m_3}\frac{\partial^m}{\partial x^{m_1}\partial y^{m_2}\partial z^{m_3}},a_{m_1m_2m_3}\text{ 常数, 如, }L=\Delta_3$$

L 满足 $L(f*g)=(Lf)*g=f*(Lg)$. 例如, $\dfrac{\partial}{\partial x}(f*g)=\left(\dfrac{\partial f}{\partial x}\right)*g=f*\left(\dfrac{\partial g}{\partial x}\right)$

定理 6.2.1

称方程 $LU(M)=\delta(M)$ 的任一解 $U(M)$ 为齐次方程 $Lu(M)=0$ 的基本解，则

$$u(M)\triangleq U(M)*f(M)=\iint_{\mathbb{R}^3}U\left(M-M_0\right)f\left(M_0\right)\mathrm{d}M_0$$

是非齐次方程 $Lu(M)=f(M)$ 的一个解

$$Lu(M)=L(U(M)*f(M))=(LU(M))*f(M)$$
$$=\delta(M)*f(M)=f(M)$$

$$u(M)=\iiint_{\mathbb{R}^3}U\left(M-M_0\right)f\left(M_0\right)dM_0=U(M)*f(M)$$

例 6.2.1: $\Delta_3 u = 0$

求 $\Delta_3 u = 0$ 基本解, 即求解 $\Delta_3 U = \delta(M), M = (x,y,z) \in \mathbb{R}^3$

因 $-\infty < x,y,z < +\infty$, 故可用 Fourier 变换法. 记 $U(\lambda,\mu,\nu) = F[U(x,y,z)]$. 由 Fourier 变换微分公式得

$$F[\Delta_3 U] = (\mathrm{i}\lambda)^2 U + (\mathrm{i}\mu)^2 U + (\mathrm{i}\nu)^2 U = -\left(\lambda^2+\mu^2+\nu^2\right) U$$

$$F[\delta(M)] = 1. \text{ 记}\rho = \sqrt{\lambda^2+\mu^2+\nu^2}$$

$$-\rho^2 U = 1. \Rightarrow U = -\frac{1}{\rho^2} \Rightarrow U(x,y,z) = -F^{-1}\left[\frac{1}{\rho^2}\right]$$

$$U(x,y,z) = -F^{-1}\left[\frac{1}{\rho^2}\right] = -\frac{1}{(2\pi)^3}\iint_{\mathbb{R}^3}\frac{1}{\rho^2}\mathrm{e}^{\mathrm{i}(\lambda x+\mu y+\nu z)}\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu$$

$$\lambda x + \mu y + \nu z = \vec{\rho}\cdot\vec{r} = \rho r\cos\theta$$

$$\begin{aligned}
U(x,y,z) &= -\frac{1}{(2\pi)^3}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\frac{1}{Q^2}e^{i\rho r\cos\theta}\mathrm{R}^2\sin\theta d\varphi d\theta d\rho \\
&= -\frac{1}{(2\pi)^3}2\pi\int_0^{+\infty}\int_0^{\pi}\mathrm{e}^{\mathrm{i}\rho r\cos\theta}\sin\theta\mathrm{d}\theta\mathrm{d}\rho = -\frac{2\pi}{(2\pi)^3}\int_0^{+\infty}\left.\frac{-\mathrm{e}^{\mathrm{i}\rho r\cos\theta}}{\mathrm{i}\rho r}\right|_{\theta=0}^{\theta=\pi}\mathrm{d}\rho \\
&= -\frac{1}{(2\pi)^2}\int_0^{+\infty}\frac{-\mathrm{e}^{-\mathrm{i}}\rho r_+ \mathrm{e}\rho r}{\mathrm{i}\rho r}\mathrm{d}\rho = -\frac{1}{(2\pi)^2}\int_0^{+\infty}\frac{2\sin\rho r}{\rho r}\mathrm{d}\rho \text{ 令} y = \rho r \\
&= -\frac{2}{(2\pi)^2 r}\int_0^{+\infty}\frac{\sin y}{y}\mathrm{d}y = -\frac{2}{(2\pi)^2 r}\cdot\frac{1}{2}\pi = -\frac{1}{4\pi r}
\end{aligned}$$

定理 6.2.2: Green 公式

设 S 是 V 的边界, S 分片光滑, u,v 在 V 内有连续二阶偏导数, 在 $V\cup S$ 内有连续一阶偏导数, 则

注

$$\nabla u = \left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

$$\vec{n} = (n_x,n_y,n_z): S \text{ 的外法向量}$$

$$\frac{\partial u}{\partial n} = \nabla u\cdot\vec{n} = n_x\frac{\partial u}{\partial x} + n_y\frac{\partial u}{\partial y} + n_z\frac{\partial u}{\partial z}$$

$$\text{Green I}: \quad \oiint_S u\frac{\partial v}{\partial n}\,\mathrm{d}S = \iiint_V u\Delta_3 v\,\mathrm{d}V + \iiint_V \nabla u\cdot\nabla v\,\mathrm{d}V$$

$$\text{Green II}: \quad \oiint_S \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S = \iiint_V (u\Delta_3 v - v\Delta_3 u)\,\mathrm{d}V$$

例 6.2.2: $\Delta_2 u = 0$

$$\Delta_2 U = \delta(x,y)$$

$$\Delta_2 U = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) = 0. \quad r\frac{\mathrm{d}U}{\mathrm{d}r} = A$$

$$U(r) = A\ln r + B, r > 0$$

$$\varphi(0,0) = \langle \delta(x,y), \varphi\rangle = \langle \Delta_2 U, \varphi\rangle = (-1)^2 \langle U, \Delta_2\varphi\rangle = \iint_{\mathbb{R}^2} U\left(\Delta_2\varphi\right)\mathrm{d}x\,\mathrm{d}y.$$

$$\text{取}\varepsilon > 0\text{ 充分小, 使得}D_\varepsilon = \left\{(x,y)\in\mathbb{R}^2 \mid \sqrt{x^2+y^2} < \varepsilon\right\} \subset \Omega \subset D.$$

$$\begin{aligned}
\varphi(0,0) &= \iint_D U\Delta_2\varphi \mathrm{d}x\,\mathrm{d}y = \lim_{\varepsilon\to 0^+}\iint_{D-D_\varepsilon} U\Delta_2\varphi \mathrm{d}x\,\mathrm{d}y \text{ 在}D-D_\varepsilon\text{ 内用二维 Green 第二式} \\
&= \lim_{\varepsilon\to 0^+}\iint_{D-D_\varepsilon}\{U\Delta_2\varphi - \varphi\Delta_2 U\}\,\mathrm{d}x\,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0^+}\left\{\oint_{\partial D}\left(U\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial U}{\partial n}\right)\mathrm{d}l - \oint_{\partial D_\varepsilon}\left(U\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial U}{\partial r}\right)\mathrm{d}l\right\} \text{ 用积分中值定理} \\
&= \lim_{\varepsilon\to 0^+}(-2\pi\varepsilon)\left(U\frac{\partial\varphi}{\partial r} - \varphi\frac{\partial U}{\partial r}\right)\Bigg|_{(x^*,y^*)\in\partial D_\varepsilon} \text{ 代入}U(r) = A\ln r \\
&= \lim_{\varepsilon\to 0^+}\left(-2\pi\varepsilon(A\ln\varepsilon)\frac{\partial\varphi}{\partial r}(x^*,y^*) + 2\pi\delta\varphi(x^*,y^*)\frac{A}{\varepsilon}\right) = 2\pi A\varphi(0,0),
\end{aligned}$$

$$\text{故}U(r) = \frac{1}{2\pi}\ln r = -\frac{1}{2\pi}\ln\frac{1}{r}\text{ 是二维 Laplace 方程}\Delta_2 u = 0\text{ 的一个基本解}$$

例 6.2.3: 二维 Helmholtz 方程$\Delta_2 u + k^2 u = 0(k>0)$ 的一个基本解

$$\text{求解: } \Delta_2 U + k^2 U = \delta(x,y)$$

$$\Delta_2 U + k^2 U = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}U}{\mathrm{d}r}\right) + k^2 U = 0, \text{ 即 } (rU')' + k^2 rU = 0,$$

$$\text{S-L 标准型}, k(r) = \rho(r) = r, q(r) = 0, \text{零阶 Bessel 方程.}$$

$$U(r) = AJ_0(kr) + BN_0(kr)$$

$$\text{为了解在}r = 0\text{ 时有奇性}, B \neq 0, \text{取}A = 0, U(r) = BN_0(kr)$$

$$\begin{aligned}
\varphi(0,0) &= \langle\delta(x,y),\varphi\rangle = \langle\Delta_2 U,\varphi\rangle + k^2\langle U,\varphi\rangle = (-1)^2\langle U,\Delta_2\varphi\rangle + k^2\langle U,\varphi\rangle \\
&= \langle U, \Delta_2\varphi + k^2\varphi\rangle = \iint_{\mathbb{R}^2} U\left(\Delta_2\varphi + k^2\varphi\right)\mathrm{d}x\,\mathrm{d}y. \text{ (有限积分)}
\end{aligned}$$

利用 Green 公式

$$\begin{aligned}
\varphi(0,0) &= \lim_{\varepsilon\to 0^+}\iint_{D-D_\varepsilon} U\left(\Delta_2\varphi+k^2\varphi\right)\mathrm{d}x\,\mathrm{d}y\\
&= \lim_{\varepsilon\to 0^+}\iint_{D-D_\varepsilon}\left\{U\left(\Delta_2\varphi+k^2\varphi\right)-\left(\Delta_2 U+k^2U\right)\varphi\right\}\mathrm{d}x\,\mathrm{d}y\\
&= \lim_{\varepsilon\to 0^+}\left\{\oint_{\partial D}\left(U\frac{\partial\varphi}{\partial n}-\varphi\frac{\partial U}{\partial n}\right)\mathrm{d}l-\oint_{\partial D_\varepsilon}\left(U\frac{\partial\varphi}{\partial r}-\varphi\frac{\partial U}{\partial r}\right)\mathrm{d}l\right\}\\
&= \lim_{\varepsilon\to 0^+}(-2\pi\varepsilon)\left(U\frac{\partial\varphi}{\partial r}-\varphi\frac{\partial U}{\partial r}\right)(x^*,y^*)\in\partial D_\varepsilon\\
&= \lim_{\varepsilon\to 0^+}(-2\pi\varepsilon)\left(\frac{2B}{\pi}\ln(k\varepsilon)\frac{\partial\varphi}{\partial r}(x^*,y^*)-\varphi(x^*,y^*)\frac{2B}{\pi}\cdot\frac{1}{\varepsilon}\right)\\
&= \lim_{\varepsilon\to 0^+}\left\{-4B\varepsilon\ln(k\varepsilon)\frac{\partial\varphi}{\partial r}(x^*,y^*)+4B\varphi(x^*,y^*)\right\}=4B\varphi(0,0).\ \text{故}B=\frac{1}{4}.
\end{aligned}$$

故$U(r)=\frac{1}{4}N_0(kr)$, 是二维 Helmholtz 方程$\Delta_2u+k^2u=0(k>0)$ 的一个基本解

例 6.2.4: 变量代换 求$a^2u_{xx}+b^2u_{yy}+c^2u_{zz}=0$ 型方程的一个基本解

令$x=a\xi, y=b\eta, z=c\zeta$

注: Jacobi 行列式与 δ 函数

$$\left\{\begin{aligned}x&=x(\xi,\eta,\zeta),\\ y&=y(\xi,\eta,\zeta),\\ z&=z(\xi,\eta,\zeta),\end{aligned}\right.\quad \text{则}J=\begin{vmatrix}x_\xi & x_\eta & x_\zeta\\ y_\xi & y_\eta & y_\zeta\\ z_\xi & z_\eta & z_\zeta\end{vmatrix}\neq 0\quad \left\{\begin{aligned}x_0&=x(\xi_0,\eta_0,\zeta_0)\\ y_0&=y(\xi_0,\eta_0,\zeta_0)\\ z_0&=z(\xi_0,\eta_0,\zeta_0)\end{aligned}\right.$$

$$\delta(x-x_0,y-y_0,z-z_0)=\frac{1}{J}\delta(\xi-\xi_0,\eta-\eta_0,\zeta-\zeta_0)$$

则 $J=\begin{vmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{vmatrix}=\alpha\beta\gamma\neq 0.$

记 $\xi_0=\frac{x_0}{\alpha}, \eta_0=\frac{y_0}{\beta}, \zeta_0=\frac{z_0}{\gamma}$,

$$\delta(x-x_0,y-y_0,z-z_0)=\frac{1}{|J|}\delta(\xi-\xi_0,\eta-\eta_0,\zeta-\zeta_0)=\frac{1}{\alpha\beta\gamma\,|}\delta\left(\xi-\frac{x_0}{\alpha},\eta-\frac{y_0}{\beta},\zeta-\frac{z_0}{\gamma}\right)$$

$$\delta(x,y,z)=\frac{1}{abc}\delta(\xi,\eta,\zeta)$$

$$\text{求解}a^2U_{xx}+b^2U_{yy}+c^2U_{zz}=\delta(x,y,z)$$

$$\text{化为}U_{\xi\xi}+U_{\eta\eta}+U_{\zeta\zeta}=\frac{1}{abc}\delta(\xi,\eta,\zeta)$$

$$U=\frac{1}{abc}\left(-\frac{1}{4\pi r}\right)=-\frac{1}{4abc\pi\sqrt{\xi^2+\eta^2+\zeta^2}}=-\frac{1}{4abc\pi\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}$$

例 6.2.5: 求$\Delta_2\Delta_2 u=0$ 型方程的一个基本解

$$\text{即求}\Delta_2\left(\Delta_2\tilde{U}\right)=\delta(x,y)\ \text{的一个解}$$

$$\Delta_2\tilde{U}=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\tilde{U}}{\mathrm{d}r}\right)=\frac{1}{2\pi}\ln r$$

$$r\frac{\mathrm{d}\tilde{U}}{\mathrm{d}r}=\frac{1}{2\pi}\int r\ln r\,\mathrm{d}r=\frac{1}{4\pi}r^2\ln r-\frac{1}{8\pi}r^2$$

$$\text{积分后取}\tilde{U}=\int\left(\frac{1}{4\pi}r\ln r-\frac{1}{8\pi}r\right)\mathrm{d}r=\frac{1}{8\pi}r^2(\ln r-1).$$

1 Green 函数

(1) 球内 Laplace 的 I 类边值

$$\begin{cases}\Delta_3 G(M;M_0)=-\delta\left(M-M_0\right)\\ G\left(M;M_0\right)|_{r=R}=0,\end{cases}$$

设 $\Delta_3 U_0=-\delta\left(M-M_0\right)$, 有解 $U_0=\dfrac{1}{4\pi r\left(M,M_0\right)}$,

注: 镜像电荷法

通过虚设电荷以求解边界条件，电荷在讨论空间边界外，且距离和电荷量有要求

为了与 U_0 在球面 $r=R$ 上完全抵消, 在球外与 M_0 点关于球面 $r=R$ 对称的点 $M_1=(\rho_1,\theta_0,\varphi_0)$ 处虚设一点电荷, 电荷量为 $-Q,Q$ 待定

$$\rho_1=\frac{R^2}{\rho_0}.\ \text{此虚设电荷产生电场在}M\ \text{电位为}$$

$$U_1=-\frac{\mathrm{Q}/\varepsilon}{4\pi r\left(M,M_1\right)},\ r\left(M,M_1\right)=\sqrt{r^2+\rho_1^2-2r\rho_1\cos\psi}$$

$$\left(\frac{1}{r\left(M,M_0\right)}-\frac{\mathrm{Q}/\varepsilon}{r\left(M,M_1\right)}\right)\bigg|_{r=R}=0,\ \text{只须}\frac{Q}{\varepsilon}=\frac{r\left(M,M_1\right)}{r\left(M,M_0\right)}\bigg|_{r=R}$$

$$\text{故取}\frac{Q}{\varepsilon}=\frac{R}{\rho_0},\ \text{则}G=U_0+U_1\ \text{满足}\ G|_{r=R}=0.$$

$$G=U_0+U_1=\frac{1}{4\pi}\left(\frac{1}{r\left(M,M_0\right)}-\frac{R}{\rho_0 r\left(M,M_1\right)}\right)$$

$r\left(M,M_0\right)=\sqrt{r^2+\rho_0^2-2r\rho_0\cos\psi},\quad \psi:\vec{OM}$ 和$\vec{OM_0}$ 的夹角

$$r\left(M,M_1\right)=\sqrt{r^2+\rho_1^2-2r\rho_1\cos\psi}=\sqrt{r^2+\left(\frac{R^2}{\rho_0}\right)^2-2r\frac{R^2}{\rho_0}\cos\psi}=\frac{1}{\rho_0}\sqrt{r^2\rho_0^2+R^4-2rR^2\rho_0\cos\psi}$$

其中

$$\begin{aligned}\vec{OM}\cdot\vec{OM_0}&=r\rho_0\cos\psi=x\xi+y\eta+z\zeta\quad x\\&=r\rho_0\left\{\sin\theta\sin\theta_0\left(\cos\varphi\cos\varphi_0+\sin\varphi\sin\varphi_0\right)+\cos\theta\cos\theta_0\right\}\end{aligned}$$

$$\cos\psi = \cos\theta_0\cos\theta + \sin\theta_0\sin\theta\cos(\varphi-\varphi_0)$$

(2) 上半球 Laplace 问题 I 类边值 $\begin{cases}\Delta_3 G(M;M_0) = -\delta(M-M_0), & M = \\ G(M;M_0)|_{r=R} = 0, G(M;M_0)|_{\theta=\frac{\pi}{2}} = 0\end{cases}$

根据电荷 $M_0 = (\rho_0,\theta_0,\varphi_0), \varepsilon$ 构造镜像电荷（不仅关于球对称，还关于平面对称四个点），其位置和电量为：

$$M_1(\rho_1,\theta_0,\varphi_0), -\frac{R\varepsilon}{\rho_0}, \quad \rho_1 = \frac{R^2}{\rho_0}$$
$$M_2(\rho_0,\pi-\theta_0,\varphi_0), -\varepsilon$$
$$M_3(\rho_1,\pi-\theta_0,\varphi_0), \frac{R\varepsilon}{\rho_0}$$

$$U_0 = \frac{1}{4\pi r(M,M_0)}, \quad U_1 = -\frac{R}{4\pi\rho_0 r(M,M_1)}, \quad U_2 = \frac{-1}{4\pi r(M,M_2)}, \quad U_3 = \frac{R}{4\pi\rho_0 r(M,M_3)}$$
$$G = U_0+U_1+U_2+U_3$$

(3) 圆内 Laplace 问题 I 类边值

$$\begin{cases}\Delta_2 G(M;M_0) = -\delta(M-M_0) \\ G(M;M_0)|_{r=R} = 0,\end{cases}$$

设$\Delta_2 U_0 = -\delta(M-M_0), U_0 = \frac{1}{2\pi}\ln\frac{1}{r(M,M_0)}$, $\begin{cases} r(M,M_0) = \sqrt{r^2+\rho_0^2-2r\rho_0\cos\psi}, \\ \psi: O\vec{M} \text{ 和} O\vec{M}_0 \text{ 的夹角}.\end{cases}$

在圆外与 M_0 点关于圆周对称的点 $M_1 = (\rho_1,\theta_0)$ 处虚设一线电荷, 电荷量为 $-\varepsilon, \rho_1 = \dfrac{R^2}{\rho_0}$. 虚设电荷产生电场在 M 点电位为

$$U_1 = -\frac{1}{2\pi}\ln\frac{1}{r(M,M_1)} - \frac{C}{2\pi}, C: \text{常数}.$$

根据边界：

$$G(M;M_0) = \frac{1}{2\pi}\left\{\ln\frac{1}{r(M,M_0)} - \ln\frac{R}{\rho_0 r(M,M_1)}\right\}$$
$$r(M,M_0) = \sqrt{r^2+\rho_0^2-2r\rho_0\cos(\theta-\theta_0)}$$
$$r(M,M_1) = \sqrt{r^2+\rho_1^2-2r\rho_1\cos(\theta-\theta_0)} = \frac{1}{\rho_0}\sqrt{r^2\rho_0^2+R^4-2r\rho_0R^2\cos(\theta-\theta_0)}$$

注: 各种空间的 Green 函数

(A) 上半空间 I 类边值问题 $\begin{cases}\Delta_3 G(M;M_0) = -\delta(M-M_0), M = (x,y,z), M_0 = (\xi,\eta,\zeta), \\ G(M;M_0)|_{z=0} = 0. \quad z>0,\zeta>0,\Delta_3: \text{关于}M \text{ 的算子},\end{cases}$

$$G = U_0+U_1 = \frac{1}{4\pi r(M,M_0)} - \frac{1}{4\pi r(M,M_1)}$$

$$r(M,M_0) = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}\cdot r(M,M_1) = \sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}$$

(B) 二维左半平面 I 类边值问题

$$\begin{cases}\Delta_2 G(M;M_0) = -\delta(M-M_0), M = (x,y), M_0 = (\xi,\eta) \\ G(M;M_0)|_{x=0} = 0. \quad x<0,\xi<0,\Delta_2: \text{关于}M \text{ 算子}.\end{cases}$$

$$G=U_0+U_1=\frac{1}{2\pi}\left\{\ln\frac{1}{r\left(M,M_0\right)}-\ln\frac{1}{r\left(M,M_1\right)}\right\}=\frac{1}{4\pi}\ln\frac{r^2\left(M,M_1\right)}{r^2\left(M,M_0\right)}$$
$$=\frac{1}{4\pi}\ln\frac{(x+\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y-\eta)^2},\quad x<0,\xi<0.$$

(C) **四分之一空间** $(y>0,z>0)$ **I 类边值问题**

$$\begin{cases}\Delta_3 G\left(M;M_0\right)=-\delta\left(M-M_0\right), & M=(x,y,z),M_0=(\xi,\eta,\zeta)\\ G\left(M;M_0\right)\big|_{y=0}=0,\ G\left(M;M_0\right)\big|_{z=0}=0\end{cases}$$

$M_0=(\xi,\eta,\zeta),(+\varepsilon)$ M_0 **关于** $z=0$ **对称点** $M_1=(\xi,\eta,-\zeta)$, **虚设** $-\varepsilon$

M_0 **关于** $y=0$ **对称点** $M_2=(\xi,-\eta,\zeta)$, **虚设** $-\varepsilon$;

M_1 **关于左侧面** $y=0$ **对称点** $M_3=(\xi,-\eta,-\zeta)$, **虚设** ε;

$$G=U_0+U_1+U_2+U_3=\frac{1}{4\pi}\left(\frac{1}{r\left(M,M_0\right)}-\frac{1}{r\left(M,M_1\right)}-\frac{1}{r\left(M,M_2\right)}+\frac{1}{r\left(M,M_3\right)}\right)$$
$$r\left(M,M_0\right)=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$
$$r\left(M,M_1\right)=\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}$$
$$r\left(M,M_2\right)=\sqrt{(x-\xi)^2+(y+\eta)^2+(z-\zeta)^2}$$
$$r\left(M,M_3\right)=\sqrt{(x-\xi)^2+(y+\eta)^2+(z+\zeta)^2}$$

(D)

(4) **求 Green 函数的基本方法——Fourier 方法，用于复杂区域、复杂边界**

例 6.2.6: 求半条形域$D:0<x<a,y>0$ **内 Poisson 方程第 I 类边问题的 Green 函数**

$$\begin{cases}\Delta_2 G=-\delta(x-\xi,y-\eta),0<x,\xi<a,y,\eta>0,\\ G|_{x=0}=G|_{x=a}=G|_{y=0}=0,\quad G|_{y=+\infty}\ \text{有界}\end{cases}$$

第 I 步　考虑固有值问题

$$\begin{cases}\Delta_2 v+\lambda v=0,0<x<a,y>0,\\ v|_{x=0}=v|_{x=a}=v|_{y=0}=0,\ v|_{y=+\infty}\ \text{有界.}\end{cases}$$

$$\frac{X''}{X}+Y''+\lambda=0$$

$$\begin{cases}X''+\mu X=0\\ X(0)=X(a)=0\end{cases}\quad\begin{cases}Y''+\nu Y=0,\quad -\mu\\ Y(0)=0,Y(+\infty)\ \text{有界}\end{cases}\quad\begin{cases}\mu_n=\left(\frac{n\pi}{a}\right)^2,n=1,2,\cdots\\ X_n(x)=\sin\frac{n\pi}{a}x\end{cases}$$

$\nu>0$. 设$v=\omega^2,\omega>0,Y''+\omega^2 Y=0$

通解 $Y(y)=A\cos\omega y+B\sin\omega y.Y(0)=A=0$. 取 $B=1,Y(y)=\sin\omega y$, 满足 $Y(+\infty)$ 有界.

$$\text{固有值}\lambda_{n\omega}=\mu_n+\nu_\omega=\left(\frac{n\pi}{a}\right)^2+\omega^2,n=1,2,\cdots,\omega>0$$

$$\text{相应固有函数}v_{n\omega}(x,y)=X_n(x)Y_\omega(y)=\sin\frac{n\pi}{a}x\sin\omega y$$

第 II 步 II. 然后将 G 设成按固有函数列展开, 将 $\delta(x-\xi, y-\eta)$ 按固有函数列展开, 代入原来关于 G 的方程, 比较系数, 求出 G 展开式中所设的所有系数, 从而得解 G

$$\text{设} G=\sum_{n=1}^{+\infty} \int_0^{+\infty} C_n(\omega) v_{n\omega}(x, y) \mathrm{d}\omega,$$

$$\begin{aligned}
\Delta_2 G &= \sum_{n=1}^{+\infty} \int_0^{+\infty} C_n(\omega) \Delta_2 v_{n\omega}(x, y) \mathrm{d}\omega=-\sum_{n=1}^{+\infty} \int_0^{+\infty} C_n(\omega) \lambda_{n\omega} v_{n\omega}(x, y) \mathrm{d}\omega \\
&= -\sum_{n=1}^{+\infty}\left(\int_0^{+\infty} C_n(\omega) \lambda_{n\omega} \sin \omega y \mathrm{~d}\omega\right) \sin \frac{n\pi}{a} x=-\delta(x-\xi, y-\eta)
\end{aligned}$$

$$\begin{aligned}
\int_0^{+\infty} C_n(\omega) \lambda_{n\omega} \sin \omega y \mathrm{~d}\omega &= \frac{2}{a} \cdot \int_0^a \delta(x-\xi) \delta(y-\eta) \sin \frac{n\pi}{a} x \mathrm{~d}x \\
&= \frac{2}{a} \delta(y-\eta) \int_0^a \delta(x-\xi) \sin \frac{n\pi}{a} x \mathrm{~d}x=\frac{2}{a} \delta(y-\eta) \sin \frac{n\pi}{a} \xi
\end{aligned}$$

$$\text{故，} F_{s,\omega}\left[C_n(\omega) \lambda_{n\omega}\right]=\frac{2}{a} \delta(y-\eta) \sin \frac{n\pi}{a} \xi,$$

$$\begin{aligned}
C_n(\omega) \lambda_{n\omega} &= F_{s,y}^{-1}\left[\frac{2}{a} \delta(y-\eta) \sin \frac{n\pi}{a} \xi\right]=\frac{2}{\pi} \int_0^{+\infty} \frac{2}{a} \delta(y-\eta) \sin \frac{n\pi}{a} \xi \sin \omega y \mathrm{~d}y \\
&= \frac{4}{a\pi} \sin \frac{n\pi}{a} \xi \sin \omega\eta. \text{ 故} C_n(\omega)=\frac{4}{a\pi\lambda_{n\omega}} \sin \frac{n\pi}{a} \xi \sin \omega\eta.
\end{aligned}$$

$$G(x, y ; \xi, \eta)=\sum_{n=1}^{+\infty} \int_0^{+\infty} C_n(\omega) v_{n\omega}(x, y) \mathrm{d}\omega=\frac{4}{a\pi} \sum_{n=1}^{+\infty}\left(\int_0^{+\infty} \frac{\sin \omega\eta \sin \omega y}{\left(\frac{n\pi}{a}\right)^2+\omega^2} d\omega\right) \sin \frac{n\pi}{a} \xi \sin \frac{n\pi}{a} x$$

2 两类重点问题：$u_t / u_{tt}=Lu$

(1) ▶ $u_t=Lu$ 型

$$\left\{\begin{array}{ll}
u_t=Lu+f(t, M), & t>0 \\
\left.u\right|_{t=0}=\varphi(M)
\end{array}\right. \text{ 对应的基本解问题是：}\left\{\begin{array}{l}
\frac{\partial U}{\partial t}=LU, t>0, M \in \mathbb{R}^n \\
\left.U\right|_{t=0}=\delta(M) .
\end{array}\right.$$

由线性叠加原理，解$u=u_1+u_2, u_1, u_2$ 分别满足

$$\left\{\begin{array}{l}
u_{1t}=Lu_1, \quad t>0, M \in \mathbb{R}^n \\
\left.u_1\right|_{t=0}=\varphi(M)=\delta(M) * \varphi(M)=\int_{\mathbb{R}^n} \delta\left(M-M_0\right) \varphi\left(M_0\right) \mathrm{d}M_0
\end{array}\right.$$

$$\left\{\begin{array}{l}
u_{2t}=Lu_2+f(t, M), t>0, M \in \mathbb{R}^n, \\
\left.u_2\right|_{t=0}=0 .
\end{array}\right.$$

$$u_1(t, M)=\int_{\mathbb{R}n} U\left(t, M-M_0\right) \varphi\left(M_0\right) dM_0=U(t, M) * \varphi(M)$$

对 u_2 用冲量原理, 考虑 $\left\{\begin{array}{l}wt'=Lw, t'>0, M \in \mathbb{R}^n, \\ \left.w\right|_{t'=0}=f(\tau, M) .\end{array}\right.$

解：$w(t',M;\tau)=U(t',M)*f(\tau,M)=U(t-\tau,M)*f(\tau,M),\quad t'=t-\tau$

$$u_2(t,M)=\int_0^t U(t-\tau,M)*f(\tau,M)\mathrm{d}\tau$$

例 6.2.7: $u_t=a^2\Delta_n u$ 型方程初值问题基本解

$$\begin{cases}\dfrac{\partial U}{\partial t}=a^2\Delta_n U,\quad t>0,M=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n\\ U|_{t=0}=\delta(M)\end{cases}$$

$$\text{记}U(t,\lambda_1,\cdots,\lambda_n)=F[U(t,M)]=\int_{\mathbb{R}^n}U(t,M)\mathrm{e}^{-\mathrm{i}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)}\mathrm{d}M$$

做 Fourier 变换，得到：

$$\begin{cases}\dfrac{\mathrm{d}\hat{u}}{\mathrm{d}t}=-a^2\left(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2\right)\hat{u},t>0,\\ \hat{u}|_{t=0}=F[\delta(M)]=1\end{cases}$$

$$\Rightarrow\hat{u}=\mathrm{e}^{-a^2\left(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2\right)t}$$

$$\begin{aligned}U(t,M)&=F^{-1}\left[\mathrm{e}^{-a^2\left(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2\right)t}\right]\\&=\left(\frac{1}{2\pi}\right)^n\int_{\mathbb{R}^n}\mathrm{e}^{-a^2\left(\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2\right)t}\mathrm{e}^{\mathrm{i}(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)}\mathrm{d}\lambda_1\,\mathrm{d}\lambda_2\cdots\mathrm{d}\lambda_n\\&=\prod_{j=1}^n\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-a^2\lambda_j^2t}\mathrm{e}^{\mathrm{i}\lambda_jx_j}\,\mathrm{d}\lambda_j\right)=\prod_{j=1}^n\left(\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x_j^2}{4a2t}}\right)\\&=\left(\frac{1}{2a\sqrt{\pi t}}\right)^n\exp\left(-\sum_{i=1}^{+\infty}\frac{x_i^2}{4at}\right)\end{aligned}$$

? 如果方程改为：

$$\begin{cases}u_t=a^2\Delta_n u+f(t,M),t>0,M=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n,\\ u|_{t=0}=\varphi(M),\end{cases}$$

$$\begin{aligned}u(t,M)&=U(t,M)*\varphi(M)+\int_0^t U(t-\tau,M)*f(\tau,M)\mathrm{d}\tau\\&=\left(\frac{1}{2a\sqrt{\pi t}}\right)^n\int_{\mathbb{R}^n}\varphi(M-M_0)\,e^{-\frac{|M_0|^2}{4a^2t}}\,\mathrm{d}M_0\\&+\left(\frac{1}{2a}\right)^n\int_0^t\left\{\left(\frac{1}{\sqrt{\pi(t-\tau)}}\right)^n\int_{\mathbb{R}^n}f(\tau,M-M_0)\,e^{-\frac{|M_0|^2}{4a^2(t-\tau)}}\mathrm{d}M_0\right\}\mathrm{d}\tau,\end{aligned}$$

其中$M_0=(\xi_1,\xi_2,\cdots,\xi_n),M-M_0=(x_1-\xi_1,x_2-\xi_2,\cdots,x_n-\xi_n)$.

(1) ▶ $u_{tt}=a^2\Delta_n u$ 型方程初值问题基本解

基本解为：

$$\begin{cases}\frac{\partial^2U}{\partial t^2}=LU,t>0,M\in\mathbb{R}^n\\ U|_{t=0}=0,\,U_t|_{t=0}=\delta(M)\end{cases}$$

$$\begin{cases} u_{tt} = Lu + f(t,M), t>0, \quad M=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n, \quad n\geq 1 \\ u|_{t=0}=\varphi(M), \quad u_t|_{t=0}=\psi(M) \end{cases}$$

根据线性叠加原理, $u=u_1+u_2+u_3$,

$$\begin{cases} u_{1tt}=Lu_1, \quad t>0, M\in\mathbb{R}^n, \\ u_1|_{t=0}=0,\ u_{1t}|_{t=0}=\psi(M)=\delta(M)*\psi(M)=\int_{\mathbb{R}^n}\delta\left(M-M_0\right)\psi\left(M_0\right)\mathrm{d}M_0 \end{cases}$$

$$\begin{cases} u_{2tt}=Lu_2, \quad t>0, \quad M\in\mathbb{R}^n, \\ u_2|_{t=0}=\varphi(M),\ u_{2t}|_{t=0}=0 \end{cases}$$

$$\begin{cases} u_{3tt}=Lu_3+f(t,M), t>0, M\in\mathbb{R}^n, \\ u_3|_{t=0}=0, \quad u_{3t}|_{t=0}=0. \end{cases}$$

$$u_1(t,M)=\int_{\mathbb{R}^n}U\left(t,M-M_0\right)\psi\left(M_0\right)\mathrm{d}M_0=U(t,M)*\psi(M)$$

对于 u_3 考虑冲量原理:

$$\begin{cases} w_{t't'}=Lw, t'>0, M\in\mathbb{R}^n, \\ w|_{t'=0}=0,\ w_{t'}|_{t'=0}=f(\tau,M). \end{cases}$$

$$w\left(t',M;\tau\right)=U\left(t',M\right)*f(\tau,M)=U(t-\tau,M)*f(\tau,M)$$

$$u_3(t,M)=\int_0^t U(t-\tau,M)*f(\tau,M)\mathrm{d}\tau=\int_0^t\left\{\int_{\mathbb{R}^n}U\left(t-\tau,M-M_0\right)f\left(\tau,M_0\right)\mathrm{d}M_0\right\}\mathrm{d}\tau.$$

对于 u_2 考虑 $v=\int_0^t u_2 d\tau$

$$\begin{cases} v_{tt}=Lv, \quad t>0, M\in\mathbb{R}^n \\ v_{t=0}=0, \quad v_t|_{t=0}=\varphi(M) \end{cases}$$

$$u_2(t,M)=v_t(t,M)=\frac{\partial}{\partial t}\{U(t,M)*\varphi(M)\}$$

$$\therefore u(t,M)=U(t,M)*\psi(M)+\frac{\partial}{\partial t}\{U(t,M)*\varphi(M)\}+\int_0^t U(t-\tau,M)*f(\tau,M)d\tau$$

例 6.2.8: 三维波动方程 $u_{tt}=a^2\Delta_3 u$ 初值问题的基本解

$$\begin{cases} \frac{\partial^2 U}{\partial t^2}=a^2\Delta_3 U, \quad t>0, -\infty<x,y,z<+\infty \\ U|_{t=0}=0, \quad U_t|_{t=0}=\delta(x,y,z). \end{cases}$$

做 Fourier 变换，可得:

$$\begin{cases} \frac{d^2\hat{u}}{dt^2}=-a^2\left(\lambda^2+\mu^2+\nu^2\right)\hat{u}, \quad t>0, \\ \hat{u}|_{t=0}=0,\ \frac{\mathrm{d}\hat{u}}{\mathrm{d}t}\Big|_{t=0}=F[\delta(x,y,z)]=1. \end{cases}$$

记 $\rho=\sqrt{\lambda^2+\mu^2+\nu^2}$, 则 $\frac{d^2\hat{u}}{dt^2}+a^2p^2\hat{u}=0.$ $\hat{u}=C\cos a\rho t+D\sin a\rho t$

$$\text{由 } \hat{u}|_{t=0}=C=0, \text{得 } \frac{\mathrm{d}\hat{u}}{\mathrm{d}t}\Big|_{t=0}=Da\rho=1, D=\frac{1}{a\rho}, \quad \hat{u}(t,\lambda,\mu,\nu)=\frac{\sin a\rho t}{a\rho}.$$

$$U(t,x,y,z)=F^{-1}\left[\frac{\sin a\rho t}{a\rho}\right]=\frac{1}{(2\pi)^3 a}\iiint_{\mathbb{R}^3}\frac{\sin a\rho t}{\rho}\mathrm{e}^{\mathrm{i}(\lambda x+\mu y+\nu z)}\mathrm{d}\lambda\mathrm{d}\mu\mathrm{d}\nu$$

$$U(t,x,y,z)=\frac{1}{(2\pi)^3a}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi}\frac{\sin a\rho t}{\rho}\mathrm{e}^{\mathrm{i}\rho r\cos\theta}\rho^2\sin\theta\mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}\rho$$
$$=\frac{1}{(2\pi)^3a}\int_0^{+\infty}\int_0^{\pi}2\pi(\sin a\rho t)\mathrm{e}^{\mathrm{i}\rho r\cos\theta}\rho\sin\theta\mathrm{d}\theta\mathrm{d}\rho$$
$$=\frac{1}{(2\pi)^2a}\int_0^{+\infty}\left(-\frac{\mathrm{e}^{\mathrm{i}}\rho r\cos\theta}{\mathrm{i}r}\bigg|_{\theta=0}^{\theta=\pi}\right)\sin a\rho t\,\mathrm{d}\rho=\frac{1}{(2\pi)^2ar}\int_0^{+\infty}\frac{-\mathrm{e}^{-\mathrm{i}\rho r}+\mathrm{e}^{\mathrm{i}}\rho r}{i}\sin a\rho t\,\mathrm{d}\rho$$
$$=\frac{1}{(2\pi)^2ar}\int_0^{+\infty}\frac{2\sin\rho r\sin a\rho t}{(\rho\text{ 的偶函数})}\mathrm{d}\rho=\frac{1}{2(2\pi)^2ar}\int_{-\infty}^{+\infty}\frac{2\sin r\rho\sin at\rho\mathrm{d}\rho}{(\text{三数积化和差公式})}$$
$$=\frac{1}{8\pi^2ar}\int_{-\infty}^{+\infty}\{\cos(r-at)\rho-\cos(r+at)\rho\}\}\,d\rho$$
$$=\frac{1}{8\pi^2ar}\int_{-\infty}^{+\infty}\left\{\frac{\mathrm{e}^{-\mathrm{i}(r-at)\rho}-\mathrm{e}^{-\mathrm{i}(r+at)\rho}\}}{\text{虚部是}\rho\text{ 的奇函数, 虚部积分为 0.}}\right\}\mathrm{d}\rho$$
$$=\frac{1}{8\pi^2ar}\{2\pi\delta(r-at)-2\pi\delta(r+at)\}=\frac{\delta(r-at)}{4\pi ar},$$

即三维波动方程初值问题的基本解 $U(t,x,y,z)=\dfrac{\delta(r-at)}{4\pi ar}$，代入

$$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\Delta_3u+f(t,x,y,z), & t>0,-\infty<x,y,z<+\infty\\ u|_{t=0}=\varphi(x,y,z), & u_{t=0}=\psi(x,y,z)\end{cases}$$

$$u(t,x,y,z)=u_1(t,x,y,z)+u_2(t,x,y,z)+u_3(t,x,y,z)$$
$$=\frac{\delta(r-at)}{4\pi ar}*\psi(x,y,z)+\frac{\partial}{\partial t}\left\{\frac{\delta(r-at)}{4\pi ar}*\varphi(x,y,z)\right\}+\int_0^t\frac{\delta(r-a(t-\tau))}{4\pi ar}*f(\tau,x,y,z)\mathrm{d}\tau$$

将基本解代入到

$$\begin{cases}\frac{\partial^2u}{\partial t^2}=a^2\Delta_3u+f(t,x,y,z), & t>0,-\infty<x,y,z<+\infty\\ u|_{t=0}=\varphi(x,y,z), & u_t|_{t=0}=\psi(x,y,z)\end{cases}$$

$$u(t,x,y,z)=u_1(t,x,y,z)+u_2(t,x,y,z)+u_3(t,x,y,z)$$
$$=\frac{\delta(r-at)}{4\pi ar}*\psi(x,y,z)+\frac{\partial}{\partial t}\left\{\frac{\delta(r-at)}{4\pi ar}*\varphi(x,y,z)\right\}+\int_0^t\frac{\delta(r-a(t-\tau))}{4\pi ar}*f(\tau,x,y,z)\mathrm{d}\tau$$

记 $\tilde{r}=\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$, 则

$$u_1(t,x,y,z)=\frac{\delta(r-at)}{4\pi ar}*\psi(x,y,z)=\frac{1}{4\pi a}\iiint_{-\infty}^{+\infty}\frac{\delta(\tilde{r}-at)}{\tilde{r}}\psi(\xi,\eta,\zeta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta=tM_{at}(\psi)$$

其中，$M_{at}(\psi)=\dfrac{1}{4\pi(at)^2}\displaystyle\oiint_{S_{at}}\psi(\xi,\eta,\zeta)\mathrm{d}S$

$$u_2(t,x,y,z)=\frac{\partial}{\partial t}\{tM_{at}(\varphi)\}$$

$$u_3(t,x,y,z)=\int_0^t(t-\tau)M_{a(t-\tau)}(f(\tau,x,y,z))d\tau,\quad \text{令}\beta=a(t-\tau)$$

$$\frac{1}{4\pi a^2}\int_0^{at}\beta\left\{\int_0^{\pi}\left(\int_0^{2\pi}f\left(t-\frac{\beta}{a},x+\beta\sin\theta\cos\varphi,y+\beta\sin\theta\sin\varphi,z+\beta\cos\theta\right)\mathrm{d}\varphi\right)\sin\theta\mathrm{d}\theta\right\}\mathrm{d}\beta$$

例 6.2.9: 降维法：先把二维问题看成三维问题

6) $n=2$ 时, 求解 $\begin{cases} \frac{\partial^2 u}{\partial t^2}=a^2\Delta_2 u, t>0,-\infty<x,y<+\infty,a>0, \\ u|_{t=0}=\varphi(x,y), \quad u_t|_{t=0}=\psi(x,y). \quad (f\equiv 0). \end{cases}$

$$u(t,x,y)=tM_{at}(\psi(x,y))+\frac{\partial}{\partial t}\{tM_{at}(\varphi(x,y))\}$$

下面只须计算 $M_{at}(\psi(x,y)),M_{at}(\varphi(x,y))$

$$M_{at}(\psi)=\frac{1}{4\pi(at)^2}\oiint_{S_{at}}\psi(\xi,\eta)\mathrm{d}S.$$

球面$S_{at}=\left\{(\xi,\eta,\zeta)\mid\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}=at\right\}$ 分为上半球面$S_{at\,\text{上}}$, 下半球面$S_{at\,\text{下}}$

$$\text{因为}\psi\text{ 与}\zeta\text{ 无关, 故}M_{at}(\psi)=\frac{2}{4\pi(at)^2}\oiint_{S_{at\,\text{上}}}\psi(\xi,\eta)\mathrm{d}S$$

$$\text{在}S_{at\,\text{上}}\text{ 上}, \zeta=z+\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2},$$

记 $S_{at\,\text{上}}$ 在 xy 平面投影 $D_{at}=\left\{(\xi,\eta)\sqrt{(\xi-x)^2+(\eta-y)^2}\leq at\right\}$
记 dA 为在 $S_{at\,\text{上}}$ 上面元 $\mathrm{d}s$ 在 xy 平面投影. 则

$$\mathrm{d}s=\sqrt{1+\left(\frac{\partial\zeta}{\partial\xi}\right)^2+\left(\frac{\partial\zeta}{\partial\eta}\right)^2}\,\mathrm{d}A=\frac{at}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}\mathrm{d}A$$

$$\begin{aligned} tM_{at}(\psi(x,y))&=\frac{2t}{4\pi(at)^2}\oiint_{Sat\,\text{上}}\psi(\xi,\eta)\mathrm{d}s \\ &=\frac{2t\cdot at}{4\pi(at)^2}\iint_{D_{at}}\frac{\psi(\xi,\eta)}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}\mathrm{d}A \end{aligned}$$

$$u(t,x,y)=tM_{at}(\psi(x,y))+\frac{\partial}{\partial t}\{tM_{at}(\varphi(x,y))\}$$
$$=\frac{1}{2\pi a}\iint_{D_{at}}\frac{\psi(\xi,\eta)}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}\mathrm{d}\xi\mathrm{d}\eta+\frac{1}{2\pi a}\frac{\partial}{\partial t}\left(\iint_{D_{at}}\frac{\varphi(\xi,\eta)}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}\xi\mathrm{d}\eta\right)$$

$$\text{其中}D_{at}=\left\{(\xi,\eta)\mid\sqrt{(x-\xi)^2+(y-\eta)^2}\leq at\right\}.$$

▶ $\frac{\partial^2 u}{\partial t^2}=a^2\Delta_2 u$ 型方程初值问题基本解为

$$U(t,x,y)=\frac{H(at-r)}{2\pi a\sqrt{a^2t^2-r^2}}.$$

例 6.2.10: 非齐次基本解

$$\begin{cases} u_{tt}+2u_t=a^2u_{xx}-2u,t>0,x\in\mathbb{R} \\ u|_{t=0}=0, \quad u_t|_{t=0}=\psi(x) \end{cases}$$

$$\left\{\begin{array}{l} U_{tt}+2U_{t}=a^{2}U_{xx}-2U, t>0, x\in\mathbb{R} \\ U|_{t=0}=0, \quad U_{t}|_{t=0}=\delta(x) \end{array}\right.$$

$$U(t,\lambda)=F[U(t,x)]=\int_{-\infty}^{+\infty}U(t,x)\mathrm{e}^{-\mathrm{i}\lambda x}\,\mathrm{d}x$$

$$\left\{\begin{array}{l} Utt+2Ut=-a^{2}\lambda^{2}U-2U=-\left(a^{2}\lambda^{2}+2\right)U, t>0, x\in\mathbb{R} \\ U|_{t=0}=0, \quad Ut|_{t=0}=F[\delta(x)]=1 \end{array}\right.$$

$$\mu^{2}+2\mu+\left(a^{2}\lambda^{2}+2\right)=0, \quad \mu_{1,2}=-1\pm i\sqrt{a^{2}\lambda^{2}+1}$$

$$U=\mathrm{e}^{-t}\left(C\cos t\sqrt{a^{2}\lambda^{2}+1}+D\sin t\sqrt{a^{2}\lambda^{2}+1}\right)$$

$U|_{t=0}=C=0, Ut|_{t=0}=D\sqrt{a^{2}\lambda^{2}+1}=1, D=\dfrac{1}{\sqrt{a^{2}\lambda^{2}+1}}$. 故$U=\mathrm{e}^{-t}\dfrac{\sin t\sqrt{a^{2}\lambda^{2}+1}}{\sqrt{a^{2}\lambda^{2}+1}}$.

$$U=F^{-1}\left[\mathrm{e}^{-t}\frac{\sin ta\sqrt{\lambda^{2}+\frac{1}{a^{2}}}}{a\sqrt{\lambda^{2}+\frac{1}{a^{2}}}}\right]=\frac{\mathrm{e}^{-t}}{2a}J_{0}\left(\frac{1}{a}\sqrt{a^{2}t^{2}-x^{2}}\right)H(at-|x|)$$

$$u=\frac{\mathrm{e}^{-t}}{2a}J_{0}\left(\frac{1}{a}\sqrt{a^{2}t^{2}-x^{2}}\right)H(at-|x|)*\psi(x)=\frac{\mathrm{e}^{-t}}{2a}\int_{-at}^{at}J_{0}\left(\frac{1}{a}\sqrt{a^{2}t^{2}-\xi^{2}}\right)\psi(x-\xi)\mathrm{d}\xi$$

3 Poisson 公式

定理 6.2.3: Poisson 公式

设 $G(M;M_0)$ 是场位方程第 I 边值问题 $\left\{\begin{array}{l} \Delta u=-f(M), \quad M=(x,y,z)\in V \\ u|_{\partial V}=\varphi(M) \end{array}\right.$ 式 Green 函数，则式的解

$$u(M)=\int_{V}f\left(M_{0}\right)G\left(M;M_{0}\right)\mathrm{d}M_{0}-\oint_{\partial V}\varphi\left(M_{0}\right)\frac{\partial G}{\partial n}\,\mathrm{d}S_{0}$$

例如

$$\left\{\begin{array}{l} \Delta_{3}u=0, \quad 0\leq r<R \\ u|_{r=R}=\varphi(\theta,\varphi) \end{array}\right.$$

$$\begin{aligned} &u(r,\theta,\varphi) \\ &=-\frac{1}{4\pi}\oiint_{S_{R}}\varphi\left(\theta_{0},\varphi_{0}\right)\left\{\frac{\partial}{\partial\rho_{0}}\left(\frac{1}{\sqrt{r^{2}+\rho_{0}^{2}-2r\rho_{0}\cos\psi}}-\frac{R}{\sqrt{r^{2}\rho_{0}^{2}+R^{4}-2r\rho_{0}R^{2}\cos\psi}}\right)\right\}\Bigg|_{\rho_{0}=R}\mathrm{d}s_{0} \end{aligned}$$

其中 S_R 是以原点 O 为球心、R 为半径的球面.

例 6.2.11

用 Green 函数求解 $\left\{\begin{array}{l} 16u_{xx}+9u_{yy}=0, x>0, y>2, \\ u|_{x=0}=g(y), \quad u|_{y=2}=f(x). \end{array}\right.$

方程特征方程为 $16(\,\mathrm{d}y)^2+9(\,\mathrm{d}x)^2=0$,

$$(4\,\mathrm{d}y+3\mathrm{id}x)(4\,\mathrm{d}y-3\mathrm{id}x)=0, 4y+3\mathrm{i}x=h_1, \text{或} 4y-3\mathrm{i}x=h_2$$

故令 $\begin{cases}\alpha=3x,\\ \beta=4y,\end{cases}$ 则 $J\neq 0$,

$$u_x=3u_\alpha,\quad u_{xx}=9u_{\alpha\alpha}, u_y=4u_\beta,\quad u_{yy}=16u_{\beta\beta}$$

$$\begin{cases}u_{\alpha\alpha}+u_{\beta\beta}=0,\quad \alpha>0,\quad \beta>8,\\ u|_{\alpha=0}=g\left(\frac{\beta}{4}\right),\quad u|_{\beta=8}=f\left(\frac{\alpha}{3}\right).\end{cases}$$

先用镜像法求 Green 函数, 即求

$$\begin{cases}G_{\alpha\alpha}+G_{\beta\beta}=-\delta(\alpha-\xi,\beta-\eta), \underline{\alpha,\xi>0,\beta,\eta>8},\\ G|_{\alpha=0}=0, G|_{\beta=8}=0.\end{cases}$$

在 $M_0=(\xi,\eta)$ 放置 $+\varepsilon$ 点电荷, 在 $M=(\alpha,\beta)$ 产生电位 $U_0=\frac{1}{2\pi}\ln\frac{1}{r(M,M_0)}$.
在 M_0 关于 $\alpha=0$ 对称点 $M_1=(-\xi,\eta)$ 虚设电荷 $-\varepsilon$, 在 M 产生电位 $U_1=-\frac{1}{2\pi}\ln\frac{1}{r(M,M_1)}$
在 M_0 关于 $\beta=8$ 对称点 $M_2=(\xi,16-\eta)$ 虚设电荷 $-\varepsilon$, 在 M 产生电位 $U_2=-\frac{1}{2\pi}\ln\frac{1}{r(M,M_2)}$
在 M_1 关于 $\beta=8$ 对称点 $M_3=(-\xi,16-\eta)$ 虚设电荷 ε, 在 M 产生电位 $U_3=\frac{1}{2\pi}\ln\frac{1}{r(M,M_3)}$.

$$G=U_0+U_1+U_2+U_3=-\frac{1}{4\pi}\ln\left\{(\alpha-\xi)^2+(\beta-\eta)^2\right\}+\frac{1}{4\pi}\ln\left\{(\alpha+\xi)^2+(\beta-\eta)^2\right\}$$
$$+\frac{1}{4\pi}\ln\left\{(\alpha-\xi)^2+(\beta+\eta-16)^2\right\}-\frac{1}{4\pi}\ln\left\{(\alpha+\xi)^2+(\beta+\eta-16)^2\right\}$$

$$\because -\left.\frac{\partial G}{\partial \xi}\right|_{\xi=0}=\frac{2\alpha\cdot(-1)}{4\pi\left\{\alpha^2+(\beta-\eta)^2\right\}}-\frac{2\alpha}{4\pi\left\{\alpha^2+(\beta-\eta)^2\right\}}-\frac{2\alpha\cdot(-1)}{4\pi\left\{\alpha^2+(\beta+\eta-16)^2\right\}}+\frac{2\alpha}{4\pi\left\{\alpha^2+(\beta+\eta-16)^2\right\}}$$

$$\because -\left.\frac{\partial G}{\partial \eta}\right|_{\eta=8}=\frac{2(\beta-8)\cdot(-1)}{4\pi\left\{(\alpha-\xi)^2+(\beta-8)^2\right\}}-\frac{2(\beta-8)\cdot(-1)}{4\pi\left\{(\alpha+\xi)^2+(\beta-8)^2\right\}}-\frac{2(\beta-8)}{4\pi\left\{(\alpha-\xi)^2+(\beta-8)^2\right\}}+\frac{2(\beta-8)}{4\pi\left\{(\alpha+\xi)^2+(\beta-8)^2\right\}}$$

利用 Poisson 公式

$$u=-\int_8^{+\infty}\left(-\left.\frac{\partial G}{\partial \xi}\right|_{\xi=0}\right)g\left(\frac{\eta}{4}\right)d\eta-\int_0^{+\infty}\left(-\left.\frac{\partial G}{\partial \eta}\right|_{\eta=8}\right)f\left(\frac{\xi}{3}\right)d\xi$$

$$=\frac{1}{\pi}\int_8^{+\infty}\left(\frac{\alpha}{\alpha^2+(\beta-\eta)^2}-\frac{\alpha}{\alpha^2+(\beta+\eta-16)^2}\right)g\left(\frac{\eta}{4}\right)\mathrm{d}\eta+\frac{1}{\pi}\int_0^{+\infty}\left(\frac{\beta-8}{(\alpha-\xi)^2+(\beta-8)^2}-\frac{\beta-8}{(\alpha+\xi)^2+(\beta-8)^2}\right)f\left(\frac{\xi}{3}\right)\mathrm{d}\xi$$

代回 $\alpha=3x,\beta=4y$ 得原问题解:

$$u=\frac{3}{\pi}\int_8^{+\infty}\left(\frac{x}{9x^2+(4y-\eta)^2}-\frac{x}{9x^2+(4y+\eta-16)^2}\right)g\left(\frac{\eta}{4}\right)d\eta$$
$$+\frac{4}{\pi}\int_0^{+\infty}\left(\frac{y-2}{(3x-\xi)^2+(4y-8)^2}-\frac{y-2}{(3x+\xi)^2+(4y-8)^2}\right)f\left(\frac{\xi}{3}\right)d\xi$$

4 复杂卷积举例

(I) **常常把复杂的那项换成 ξ，比如下式**

$$\left\{\frac{1}{2a\sqrt{\pi(t-\tau)}}e^{-\frac{x^2}{4a^2(t-\tau)}}\right\}*(x\cos 2\tau)=\frac{\cos 2\tau}{2a\sqrt{\pi(t-\tau)}}\left\{\int_{-\infty}^{+\infty}(x-\xi)e^{-\frac{\xi^2}{4a^2(t-\tau)}}d\xi\right\}$$

$$=\frac{\cos 2\tau}{2a\sqrt{\pi(t-\tau)}}\left\{x\int_{-\infty}^{+\infty}e^{-\frac{\xi^2}{4a^2(t-\tau)}}\mathrm{d}\xi-\int_{-\infty}^{+\infty}\xi\mathrm{e}^{-\frac{\xi^2}{4a^2(t-\tau)}}\mathrm{d}\xi\right\}\text{(后一项是奇函数)}$$

$$=\frac{\cos 2\tau}{\sqrt{\pi}}\left\{x\int_{-\infty}^{+\infty}e^{-\lambda^2}\mathrm{d}\lambda-0\right\}=\frac{\cos 2\tau}{\sqrt{\pi}}(x\sqrt{\pi})=x\cos 2\tau$$

再比如

$$\left\{\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\right\}*\cos x=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{\xi^2}{4a^2t}}\cos(x-\xi)d\xi$$

$$=\frac{1}{2a\sqrt{\pi t}}\left\{\cos x\int_{-\infty}^{+\infty}e^{-\frac{\xi^2}{4a^2t}}\cos\xi\mathrm{d}\xi+\sin x\int_{-\infty}^{+\infty}e^{-\frac{\xi^2}{4a^2t}}\sin\xi\mathrm{d}\xi\right\}$$

$$=\frac{\cos x}{\sqrt{\pi}}\cdot 2\int_0^{+\infty}\mathrm{e}^{-\lambda^2}\cos 2a\sqrt{t}\lambda\mathrm{d}\lambda+0$$

$$=\frac{2\cos x}{\sqrt{\pi}}\cdot\frac{\pi}{2\sqrt{\pi}}\mathrm{e}^{-\frac{(2a\sqrt{t})^2}{4}}=\mathrm{e}^{-a^2t}\cos x$$

(II) **考虑从指数入手：**

$$\forall b\in\mathbb{R},\quad I\triangleq\left\{\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\right\}*\left(e^{-x^2}\cos bx\right)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{\xi^2}{4a^2t}}\mathrm{e}^{-(x-\xi)^2}\cos b(x-\xi)\mathrm{d}\xi$$

$$=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{1+4a^2t}{4a^2t}\left(\xi-\frac{4a^2t}{1+4a^2t}x\right)^2-\frac{x^2}{1+4a^2t}}\cos b(\xi-x)d\xi$$

$$\text{令}\eta=\xi-\frac{4a^2t}{1+4a^2t}x,\gamma=\gamma(t)\triangleq-\frac{1}{1+4a^2t}=\frac{4a^2t}{1+4a^2t}-1,\text{则}\xi-x=\eta+\gamma x$$

$$I=\frac{1}{2a\sqrt{\pi t}}e^{\gamma x^2}\int_{-\infty}^{+\infty}e^{-\frac{1+4a^2t}{4a^2t}\eta^2}\cos b(\eta+\gamma x)d\eta$$

$$=\frac{e^{\gamma x^2}}{2a\sqrt{\pi t}}\left\{\cos b\gamma x\int_{-\infty}^{+\infty}e^{-\frac{1+4a^2t}{4a^2t}\eta^2}\cos b\eta d\eta-\sin b\gamma x\int_{-\infty}^{+\infty}e^{-\frac{1+4a^2t}{4a^2t}\eta^2}\sin b\eta d\eta\right\}$$

$$=\frac{\cos b\gamma x}{2a\sqrt{\pi t}}e^{\gamma x^2}\cdot\frac{2\pi}{2\sqrt{\pi\cdot\frac{1+4a^2t}{4a^2t}}}e^{-\frac{b^2}{4\cdot\frac{1+4a^2t}{4a^2t}}}-0=\frac{\cos\frac{bx}{1+4a^2t}}{\sqrt{1+4a^2t}}e^{-\frac{x^2+b^2a^2t}{1+4a^2t}}$$

再比如

$$\left\{\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2t}}\right\}*\left(\mathrm{e}^{-bx}\right)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\xi^2}{4a^2t}}\mathrm{e}^{-b(x-\xi)}\mathrm{d}\xi$$

$$=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}e^{-\frac{\xi^2-4a^2bt\xi}{4a^2t}-bx}\mathrm{d}\xi=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\left(\xi-2a^2bt\right)^2-4a^4b^2t^2}{4a^2t}-bx}\mathrm{d}\xi$$

$$=\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{a^2b^2t-bx}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\left(\xi-2a^2bt\right)^2}{4a^2t}}\mathrm{d}\xi\quad\left\{\begin{array}{l}\text{令}\eta=\dfrac{1}{2a\sqrt{t}}(\xi-2a^2bt)\\ \mathrm{d}\xi=2a\sqrt{t}d\eta\end{array}\right.$$

$$=\frac{1}{\sqrt{\pi}}\mathrm{e}^{a^2b^2t-bx}\int_{-\infty}^{+\infty}\mathrm{e}^{-\eta^2}\mathrm{d}\eta=\mathrm{e}^{a^2b^2t-bx}$$

(III) 高维卷积（或者高维 Fourier 反变换）：先想办法分离变量

$$
\begin{aligned}
F^{-1}\left[\mathrm{e}^{-a\lambda^2 t+\mathrm{i}b\mu t}\right] &= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-a\lambda^2 t+\mathrm{i}b\mu t}\mathrm{e}^{\mathrm{i}(\lambda x+\mu y)}\mathrm{d}\lambda\mathrm{d}\mu \\
&= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-a\lambda^2 t}}{\mathrm{e}^{\mathrm{i}b\mu t}}\mathrm{e}^{\mathrm{i}\lambda x}\mathrm{e}^{\mathrm{i}\mu y}\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-a\lambda^2 t}\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}b\mu t}\mathrm{e}^{\mathrm{i}\mu y}\,\mathrm{d}\mu\right) \\
&= \left(\frac{1}{\pi}\int_{0}^{+\infty} \mathrm{e}^{-a\lambda^2 t}\cos\lambda x\,\mathrm{d}\lambda\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(bt+y)\mu}\mathrm{d}\mu\right) \\
&= \frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2 t}}\delta(y+bt)
\end{aligned}
$$

再比如

$$
\begin{aligned}
&\left(\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-\frac{x^2}{4a^2 t}}\delta(y+bt)\right) * \left(x+\mathrm{e}^{-x^2-y^2}\right) \\
=&\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2}{4a^2 t}}\delta(\eta+bt)\left\{(x-\xi)+\mathrm{e}^{-(x-\xi)^2-(y-\eta)^2}\right\}\mathrm{d}\xi\mathrm{d}\eta \\
=&\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2}{4a^2 t}}\left\{\int_{-\infty}^{+\infty}\delta(\eta+bt)\left\{(x-\xi)+\mathrm{e}^{-(x-\xi)^2-(y-\eta)^2}\right\}\mathrm{d}\eta\right\}\mathrm{d}\xi \\
=&\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2}{4a^2 t}}\left\{(x-\xi)+\mathrm{e}^{-(x-\xi)^2-(y+bt)^2}\right\}\mathrm{d}\xi \\
=&\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{\xi^2}{4a^2 t}}(x-\xi)\mathrm{d}\xi+\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-(y+bt)^2}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{\xi^2}{4a^2 t}-(x-\xi)^2}\,\mathrm{d}\xi \\
=&\ \frac{x}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{\xi^2}{4a^2 t}}\,\mathrm{d}\xi+\frac{1}{2a\sqrt{\pi t}}e^{-(y+bt)^2}\int_{-\infty}^{+\infty} e^{-\frac{4a^2t+1}{4a^2 t}\xi^2+2x\xi-x^2}\,\mathrm{d}\xi \\
=&\ \frac{x}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-u^2}\,\mathrm{d}u+\frac{1}{2a\sqrt{\pi t}}\mathrm{e}^{-(y+bt)^2}\mathrm{e}^{\frac{4a^2+x^2t+1}{4}-x^2}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{4a^2t+1}{4a^2t}\left(\xi-\frac{4a^2tx}{4a^2t+1}\right)^2}\,\mathrm{d}\xi \\
=&\ x+\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t+1}-(y+bt)^2}\frac{2a\sqrt{t}}{\sqrt{4a^2t+1}}\int_{-\infty}^{+\infty} e^{-v^2}\,\mathrm{d}v \\
=&\ x+\frac{1}{\sqrt{4a^2t+1}}e^{-\frac{x^2}{4a^2t+1}-(y+bt)^2}
\end{aligned}
$$

第七章 补充资料

第 1 节 2021 数理方程 A 型真题

一（12 分）求以下固有值问题的固有值, 固有函数

$$\begin{cases} y'' + \lambda y = 0, (0 < x < 21) \\ y(0) = 0, \quad y(21) = 0 \end{cases}$$

解：由 S-L 定理, 可设 $\lambda = \omega^2 > 0$, 这样解得

$$y = A\cos\omega x + B\sin\omega x$$

由 $y(0) = 0 \Longrightarrow A = 0, y(21) = B\sin 21\omega = 0$, 因此 $21\omega = n\pi \Longrightarrow \omega = \dfrac{n\pi}{21}, n = 1, 2, 3.$

固有值: $\lambda_n = \left(\dfrac{n\pi}{21}\right)^2$, , 固有函数: $y_n(x) = \sin\dfrac{n\pi x}{21}$

二 (16 分) 考虑如下形式定解问题:

$$\begin{cases} u_{tt} = 4u_{xx} + f(t,x), \quad (t > 0, -\infty < x < +\infty) \\ u(0,x) = 2x^2, u_t(0,x) = 3\cos x \end{cases}$$

(1) 在 $f(t,x) = 0$ 时, 求此问题的解。(2) 在 $f(t,x) = t^2x + 5\sin 2t\sin x$ 时, 求此问题的解。

解: (1) 在 $f(t,x) = 0$ 时, 由达朗贝尔公式:

$$u = \frac{2}{2}\left[(x+2t)^2 + (x-2t)^2\right] + \frac{1}{2\times 2}\int_{x-2t}^{x+2t} 3\cos\xi d\xi = 2x^2 + 8t^2 + \frac{3}{2}\sin 2t\cos x$$

(2) 在 $f(t,x) = t^2x + 5\sin 2t\sin x$ 时, 由叠加原理: $u = u_1 + u_2 + u_3$, 其中

$$\begin{cases} u_{1tt} = 4u_{1xx}, \quad (t > 0, -\infty < x < +\infty) \\ u_1(0,x) = 2x^2, \quad u_{1t}(0,x) = 3\cos x \end{cases}$$

$$\begin{cases} u_{2tt} = 4u_{2xx} + t^2x, \quad (t > 0, -\infty < x < +\infty) \\ u_2(0,x) = 0, u_{2t}(0,x) = 0 \end{cases}$$

$$\begin{cases} u_{3tt} = 4u_{3xx} + 5\sin 2t\sin x, \quad (t > 0, -\infty < x < +\infty) \\ u_3(0,x) = 0, u_{3t}(0,x) = 0 \end{cases}$$

由第一问的结论: $u_1 = 2x^2 + 8t^2 + \frac{3}{2}\sin 2t\cos x$, 直接观察可得: $u_2 = \frac{t^4}{12}x$, 使用冲量原理, 求得:

$$\begin{aligned} u_3 &= \frac{5}{2\times 2}\int_0^t d\tau\int_{x-2(t-\tau)}^{x+2(t-\tau)}\sin 2\tau\sin\xi d\xi = \frac{5}{2}\int_0^t \sin x\sin 2(t-\tau)\sin 2\tau d\tau \\ &= \frac{5}{4}\sin x\int_0^t (\cos(2t-4\tau) - \cos 2t)d\tau = \frac{5}{8}\sin 2t\sin x - \frac{5}{4}t\cos 2t\sin x \end{aligned}$$

综上, 这时

$$u = 2x^2 + 8t^2 + \frac{3}{2}\sin 2t\cos x + \frac{t^4}{12}x + \frac{5}{8}\sin 2t\sin x - \frac{5}{4}t\cos 2t\sin x$$

三 (10 分) 求解定解问题

$$\begin{cases} x\dfrac{\partial u}{\partial x} - 3y\dfrac{\partial u}{\partial y} + u = 0 \\ u(1,y) = y + 3y^2 \end{cases}$$

解：特征方程为：

$$\frac{dx}{x} = \frac{dy}{-3y}$$

得到首次积分: $x^3y = c$, 因此作变换:

$$\xi = x^3y, \quad \eta = y,$$

方程化为：

$$-3\eta\frac{\partial u}{\partial \eta} + u = 0. \Longrightarrow u = f(\xi)(\eta)^{\frac{1}{3}} = f\left(x^3y\right)y^{\frac{1}{3}}$$

再利用定解条件 $u(1,y) = y + 3y^2$, 得到 $f(\xi) = \xi^{\frac{5}{3}} + \xi^{\frac{2}{3}}$, 最后得到

$$u(x,y) = 3x^5y^2 + x^2y$$

四. (14 分) 求解混合问题:

$$\begin{cases} u_{tt} = 4u_{xx} + \delta(t-3, x-2), (t > 0, \quad 0 < x < 5) \\ u(t,0) = u_x(t,5) = 0 \\ u(0,x) = \sin\dfrac{1}{2}\pi x + 3\sin\dfrac{3}{10}\pi x, \quad u_t(0,x) = 0 \end{cases}$$

四.. 解: 使用叠加原理: $u = u_1 + u_2$ 其中

$$\begin{cases} u_{1tt} = 4u_{1xx}, \quad (t > 0, \quad 0 < x < 5) \\ u_1(t,0) = u_{1x}(t,5) = 0 \\ u_1(0,x) = \sin\frac{1}{2}\pi x + 3\sin\frac{3}{10}\pi x, \quad u_{1t}(0,x) = 0 \end{cases}$$

以及

$$\begin{cases} u_{2tt} = 4u_{2xx} + \delta(t-3, x-2), (t > 0, 0 < x < 5) \\ u_2(t,0) = u_{2x}(t,5) = 0 \\ u_2(0,x) = 0, \quad u_{2t}(0,x) = 0 \end{cases}$$

为了求 u_1, 作分离变量 $u_1 = T(t)X(x)$, 得到固有值问题:

$$\begin{cases} X'' + \lambda X = 0, (0 < x < 21) \\ X(0) = 0, \quad X'(5) = 0 \end{cases}$$

和常微: $T'' + 4\lambda T = 0$. 其中固有值问题的固有值和固有函数为:

$$\lambda_n = \left(\frac{2n\pi + \pi}{10}\right)^2, X_n(x) = \sin\frac{2n\pi + \pi}{10}x$$

相应地 $T_n(t) = C_n\cos\dfrac{2n\pi + \pi}{5}t + D_n\sin\dfrac{2n\pi + \pi}{5}t$, 这样, 由叠加原理可设:

$$u_1(t,x) = \sum_{n=0}^{+\infty}\left(C_n\cos\frac{2n\pi + \pi}{5}t + D_n\sin\frac{2n\pi + \pi}{5}t\right)\sin\frac{2n\pi + \pi}{10}x$$

再由 u_1 的初值条件, 定出:

$$u_1 = 3\cos\frac{3\pi}{5}t\sin\frac{3}{10}\pi x + \cos\pi t\sin\frac{1}{2}\pi x$$

为了求 u_2, 利用冲量原理

$$u_2 = \int_0^t W(t,x,\tau)d\tau$$

$W(t,x,\tau)$ 满足:

$$\begin{cases} W_{tt} = 4W_{xx}, \quad (t > \tau, \quad 0 < x < 5) \\ W(t,0) = W_x(t,5) = 0 \\ W|_{t=\tau} = 0, \quad W_t|_{t=\tau} = \delta(\tau - 3, x - 2) \end{cases}$$

利用变换 $t_1 = t - \tau$, 并利用分离变量法, 类似求得:

$$W(t,x,\tau) = \sum_{n=0}^{+\infty} \left(F_n \cos \frac{2n\pi + \pi}{5}(t-\tau) + G_n \sin \frac{2n\pi + \pi}{5}(t-\tau) \right) \sin \frac{2n\pi + \pi}{10} x$$

再利用 W 在 $t = \tau$ 的条件，得到 $F_n = 0$, 而 G_n 满足:

$$\sum_{n=0}^{+\infty} G_n \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{10} x = \delta(\tau - 3, x - 2) = \delta(\tau - 3)\delta(x - 2)$$

得出:

$$G_n = \frac{2}{2n\pi + \pi} \sin \frac{2n\pi + \pi}{5} \delta(\tau - 3)$$

即

$$W(t,x,\tau) = \sum_{n=0}^{+\infty} \left(\frac{2}{2n\pi + \pi} \delta(\tau - 3) \sin \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{5}(t-\tau) \right) \sin \frac{2n\pi + \pi}{10} x$$

因此

$$u_2(t,x) = \begin{cases} \displaystyle\sum_{n=0}^{+\infty} \left(\frac{2}{2n\pi + \pi} \sin \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{5}(t-3) \right) \sin \frac{2n\pi + \pi}{10} x, & t \geq 3 \\ 0, & t < 3 \end{cases}$$

综上, 此定解问题的解:

$$u(t,x) = \begin{cases} 3\cos \frac{3\pi}{5} t \sin \frac{3}{10}\pi x + \cos \pi t \sin \frac{1}{2}\pi x + \displaystyle\sum_{n=0}^{+\infty} \left[\frac{2}{2n\pi + \pi} \sin \frac{2n\pi + \pi}{5} \sin \frac{2n\pi + \pi}{5}(t-3) \right] \sin \frac{2n\pi + \pi}{10} x, & t \geqslant 3 \\ 3\cos \frac{3\pi}{5} t \sin \frac{3}{10}\pi x + \cos \pi t \sin \frac{1}{2}\pi x, & t < 3 \end{cases}$$

五 (12 分) 利用分离变量求解以下边值问题:

$$\begin{cases} \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial u}{\partial r} \right) + \dfrac{\partial^2 u}{\partial z^2} = 0, \quad \left(r = \sqrt{x^2 + y^2} < 2, 0 < z < 3 \right) \\ u|_{r=2} = 0 \\ u|_{z=0} = 8 - 2r^2, \quad u|_{z=3} = 0 \end{cases}$$

解: 作分离变量, 令 $u = R(r)Z(z)$, 则有

$$\frac{\frac{1}{r}\frac{d}{dr}\left(r \frac{dR(r)}{dr} \right)}{R} + \frac{\frac{d^2 Z(z)}{dz^2}}{Z} = 0$$

考虑到齐次边界条件并在 $r=0$ 附加自然边界条件, 得到 Bessel 方程固有值问题:

$$\begin{cases} r^2R''+rR'+\lambda r^2R=0 \\ |R(0)|<+\infty, R(2)=0 \end{cases}$$

和微分方程 $Z''-\lambda Z=0$. 进一步求得一系列分离变量形式的解:

$$u_n(r,z)=\left(A_n\cosh\omega_nz+B_n\sinh\omega_nz\right)J_0\left(\omega_nr\right)$$

其中 ω_n 是代数方程 $J_0(2\omega)=0$ 的第 n 个正根。由叠加原理,

$$u(r,z)=\sum_{n=1}^{+\infty}\left(A_n\cosh\omega_nz+B_n\sinh\omega_nz\right)J_0\left(\omega_nr\right)$$

利用边界条件:

$$u(r,3)=\sum_{n=1}^{+\infty}\left(A_n\cosh 3\omega_n+B_n\sinh 3\omega_n\right)J_0\left(\omega_nr\right)=0\Longrightarrow B_n=-\frac{\cosh 3\omega_n}{\sinh 3\omega_n}A_n=-\coth 3\omega_nA_n$$

$$u(r,0)=\sum_{n=1}^{+\infty}A_nJ_0\left(\omega_nr\right)=8-r^2$$

结合递推公式, 可算出广义 Fourier 系数:

$$\begin{aligned} A_n&=\frac{\int_0^2 r\left(8-2r^2\right)J_0\left(\omega_nr\right)dr}{N_{0n}^2}=2\frac{\int_0^2 r\left(4-r^2\right)J_0\left(\omega_nr\right)dr}{N_{0n}^2}=\frac{2}{N_{0n}^2}\frac{1}{\omega_n^2}\int_0^{2\omega_n}t\left(4-\frac{t^2}{\omega_n^2}\right)J_0(t)dt \\ &=\frac{2}{N_{0n}^2\omega_n^2}\left[\left(4-\frac{t^2}{\omega_n^2}\right)tJ_1(t)\Big|_0^{2\omega_n}+\frac{2}{\omega_n^2}\int_0^{2\omega_n}t^2J_1(t)dt\right]=\frac{8J_2\left(2\omega_n\right)}{\omega_n^2J_1^2\left(2\omega_n\right)}=\frac{16}{\omega_n^3J_1\left(2\omega_n\right)} \end{aligned}$$

进一步

$$B_n=-\frac{16\coth 3\omega_n}{\omega_n^3J_1\left(2\omega_n\right)}$$

最后

$$u(r,z)=\sum_{n=1}^{+\infty}\left(\frac{16}{\omega_n^3J_1\left(2\omega_n\right)}\cosh\omega_nz-\frac{16\coth 3\omega_n}{\omega_n^3J_1\left(2\omega_n\right)}\sinh\omega_nz\right)J_0\left(\omega_nr\right)$$

六. (16 分) 已知以下形式初值问题

$$\begin{cases} u_t=4u_{xx}+a^2u_{yy}+bu_x+cu, (t>0,-\infty<x,y<+\infty) \\ u|_{t=0}=\varphi(x)+g(x,y). \end{cases}$$

(1) 当 $a=b=c=0, g(x,y)=0$ 时, 设 u 不依赖于 y, 求这时 $u=u(t,x)$ 的解的表达式。

(2) 当 $a=1,b=3,c=2,\varphi(x)=x,g(x,y)=5e^{-x^2-y^2}$ 时, 求这时解 $u(t,x,y)$.

解: (1) 这时定解问题对应于:

$$\begin{cases} u_t=4u_{xx}(t>0,-\infty<x<+\infty) \\ u|_{t=0}=\varphi(x) \end{cases}$$

作 Fourier 变换, 令 $\bar{u}(t,\lambda)=\int_{-\infty}^{+\infty}u(t,x)e^{-i\lambda x}dx$, 则经过 Fourier 变换, 像函数 $\bar{u}$ 满足:

$$\begin{cases} \dfrac{d\bar{u}}{dt}=-4\lambda^2\bar{u} \\ \bar{u}|_{t=0}=\bar{\varphi}(\lambda) \end{cases}$$

于是解得:

$$\bar{u}=\bar{\varphi}(\lambda)e^{-4\lambda^2 t}$$

作反变换:

$$u(t,x)=\frac{1}{4\sqrt{\pi t}}\exp\left(-\frac{x^2}{16t}\right)*\varphi(x)=\frac{1}{4\sqrt{\pi t}}\int_{-\infty}^{+\infty}\exp\left(-\frac{(x-\xi)^2}{16t}\right)\varphi(\xi)d\xi$$

(2) 当 $a=1,b=3,c=2,\varphi(x)=x,g(x,y)=5e^{-x^2-y^2}$ 时,

$$\begin{cases} u_t=4u_{xx}+u_{yy}+3u_x+2u, \quad (t>0,-\infty<x,y<+\infty) \\ u|_{t=0}=x+5e^{-x^2-y^2} \end{cases}$$

令 $\bar{u}(t,\lambda,\mu)=\int_{-\infty}^{+\infty}u(t,x,y)e^{-i(\lambda x+\mu y)}dxdy$, 则经过 Fourier 变换, 像函数 $\bar{u}$ 满足:

$$\begin{cases} \dfrac{d\bar{u}}{dt}=\left(-4\lambda^2-\mu^2+3i\lambda+2\right)\bar{u} \\ \bar{u}|_{t=0}=2\pi h(\lambda)\delta\mu+G(\lambda,\mu) \end{cases}$$

其中 $h(\lambda)=\int_{-\infty}^{+\infty}xe^{-i\lambda x}dx, G(\lambda,\mu)=\int_{-\infty}^{+\infty}5e^{-x^2-y^2}e^{-i(\lambda x+\mu y)}dxdy$, 于是解得:

$$\bar{u}(t,\lambda,\mu)=2\pi h(\lambda)\delta\mu e^{\left(-4\lambda^2-\mu^2+3i\lambda+2\right)t}+G(\lambda,\mu)e^{\left(-4\lambda^2-\mu^2+3i\lambda+2\right)t}$$

作反变换:

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}2\pi h(\lambda)\delta\mu e^{\left(-4\lambda^2-\mu^2+3i\lambda+2\right)t}e^{i(\lambda x+\mu y)}d\lambda d\mu=\frac{e^{2t}}{2\pi}\int_{-\infty}^{+\infty}h(\lambda)e^{-4\lambda^2 t}e^{i\lambda(x+3t)}d\lambda$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}h(\lambda)e^{-4\lambda^2 t}e^{i\lambda x}d\lambda=x*\frac{1}{4\sqrt{\pi t}}\exp\left(-\frac{x^2}{16t}\right)=\frac{1}{4\sqrt{\pi t}}\int_{-\infty}^{+\infty}(x-\xi)\exp\left(-\frac{\xi^2}{16t}\right)d\xi$$

$$=\frac{x}{4\sqrt{\pi t}}\int_{-\infty}^{+\infty}\exp\left(-\frac{\xi^2}{16t}\right)d\xi=\frac{x}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\tau^2}d\tau=x$$

这样

$$F^{-1}\left[2\pi h(\lambda)\delta\mu e^{\left(-4\lambda^2-\mu^2+3i\lambda+2\right)t}\right]=e^{2t}(x+3t)$$

同样

$$F^{-1}\left[G(\lambda,\mu)e^{\left(-4\lambda^2-\mu^2\right)t}\right]=5e^{-x^2-y^2}*\left(\frac{1}{2\sqrt{\pi t}}\right)\left(\frac{1}{4\sqrt{\pi t}}\right)\exp\left\{-\frac{x^2}{16t}-\frac{y^2}{4t}\right\}$$

$$e^{-x^2}*\frac{1}{4\sqrt{\pi t}}\exp\left\{-\frac{x^2}{16t}\right\}=\frac{1}{\sqrt{1+16t}}e^{-\frac{1}{1+16t}x^2}, e^{-y^2}*\frac{1}{2\sqrt{\pi t}}\exp\left\{-\frac{y^2}{4t}\right\}=\frac{1}{\sqrt{1+4t}}e^{-\frac{1}{1+4t}y^2}$$

这样

$$F^{-1}\left[G(\lambda,\mu)e^{\left(-4\lambda^2-\mu^2\right)t}\right]=\frac{5}{8\pi t}\frac{1}{\sqrt{1+16t}}\frac{1}{\sqrt{1+4t}}e^{-\frac{1}{1+16t}x^2-\frac{1}{1+4t}y^2}$$

进一步

$$F^{-1}\left[G(\lambda,\mu)e^{\left(-4\lambda^2-\mu^2+3i\lambda+2\right)t}\right]=\frac{5}{8\pi t}\frac{e^{2t}}{\sqrt{1+16t}}\frac{1}{\sqrt{1+4t}}e^{-\frac{1}{1+16t}(x+3t)^2-\frac{1}{1+4t}y^2}$$

综上, 求得 $u(t,x,y)$ 为

$$u(t,x,y)=e^{2t}(x+3t)+\frac{5}{8\pi t}\frac{e^{2t}}{\sqrt{1+16t}}\frac{1}{\sqrt{1+4t}}e^{-\frac{1}{1+16t}(x+3t)^2-\frac{1}{1+4t}y^2}$$

七. (10 分) 求解以下定解问题, 其中 (r,θ,φ) 为球坐标.

$$\begin{cases} \Delta_3 u = 0, \quad 1<r<3 \\ u|_{r=1} = \cos 2\theta - 1 \\ u|_{r=3} = 21 + \cos\theta \end{cases}$$

解: 使用球坐标下 Laplace 方程轴对称情形下的求解公式:

$$u = \sum_{n=0}^{+\infty} \left(A_n r^n + B_n r^{-(n+1)}\right) P_n(\cos\theta)$$

利用边界条件得到:

$$u|_{r=1} = \sum_{n=0}^{+\infty} (A_n + B_n) P_n(\cos\theta) = 2\cos^2\theta - 2$$

$$u|_{r=2} = \sum_{n=0}^{+\infty} \left(A_n 3^n + B_n 3^{-(n+1)}\right) P_n(\cos\theta) = 21 + \cos\theta$$

因此

$$(A_0 + B_0) P_0(x) + (A_1 + B_1) P_1(x) + (A_2 + B_2) P_2(x) = -\frac{4}{3} P_0(x) + \frac{4}{3} P_2(x)$$

$$\left(A_0 + \frac{B_0}{3}\right) P_0(x) + \left(3A_1 + \frac{B_1}{9}\right) P_1(x) + \left(9A_2 + \frac{B_2}{27}\right) P_2(x) = 21 P_0(x) + P_1(x)$$

于是得到:

$$A_0 + B_0 = -\frac{4}{3}, \quad A_0 + \frac{B_0}{3} = 21, A_1 + B_1 = 0, 3A_1 + \frac{B_1}{9} = 1, A_2 + B_2 = \frac{4}{3}, 9A_2 + \frac{B_2}{27} = 0$$

解得:

$$A_0 = \frac{193}{6}, B_0 = -\frac{67}{2}, A_1 = \frac{9}{26}, B_1 = -\frac{9}{26}, A_2 = -\frac{2}{363}, B_2 = \frac{162}{121}$$

所以

$$u(r,\theta) = \frac{193}{6} - \frac{67}{2} r^{-1} + \left(\frac{9}{26} r - \frac{9}{26} r^{-2}\right) P_1(\cos\theta) + \left(-\frac{2}{363} r^2 + \frac{162}{121} r^{-3}\right) P_2(\cos\theta)$$

八 (10 分) 求解边值问题:

$$\begin{cases} u_{xx} + 4u_{yy} = 0, \quad (x>3, y>0) \\ u = \begin{cases} g(y), \text{ 当} x = 3, y > 0 \\ f(x), \text{ 当} x \geqslant 3, y = 0 \end{cases} \end{cases}$$

解: 作坐标变换 $\bar{x} = x - 3, \bar{y} = \frac{y}{2}$, 这样原问题变为

$$\begin{cases} u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}} = 0, \bar{x} > 0, \bar{y} > 0, \\ u = \begin{cases} g(2\bar{y}) \text{ 当} \bar{x} = 0, \bar{y} > 0 \\ f(\bar{x} + 3) \text{ 当} \bar{x} > 0, \bar{y} = 0 \end{cases} \end{cases}$$

对应的基本解问题是:

$$\begin{cases} \bar{\Delta} \bar{G} = \delta(\bar{M}), \bar{x} > 0, \bar{y} > 0, \text{相当于第一象限} \\ u = \begin{cases} g(2\bar{y}) \text{ 当} \bar{x} = 0, \bar{y} > 0 \\ f(\bar{x} + 3) \text{ 当} \bar{x} > 0, \bar{y} = 0 \end{cases} \end{cases}$$

利用镜像法, 在 $M_0=(\bar\xi,\bar\eta), M_1=(\bar\xi,-\bar\eta), M_2=(-\bar\xi,\bar\eta), M_3=(-\bar\xi,-\bar\eta)$ 依次放 $+\varepsilon,-\varepsilon,-\varepsilon,+\varepsilon$ 的线电荷, 产生电场电式叠加就是区域 $(\bar x>0,\bar y>0)$ 的格林函数 (对应的基本解 $G=-U=\frac{1}{2\pi}\ln r$):

$$\begin{aligned}\bar G=&\frac{1}{2\pi}\ln\sqrt{(\bar x+\bar\xi)^2+(\bar y-\bar\eta)^2}+\frac{1}{2\pi}\ln\sqrt{(\bar x-\bar\xi)^2+(\bar y+\bar\eta)^2}\\&-\frac{1}{2\pi}\ln\sqrt{(\bar x-\bar\xi)^2+(\bar y-\bar\eta)^2}-\frac{1}{2\pi}\ln\sqrt{(\bar x+\bar\xi)^2+(\bar y+\bar\eta)^2}\end{aligned}$$

对于区域在 $\bar y=0$ 的边界, 其外法方向 $\vec{n_0}=(0,-1)$, 则 $\left.\frac{\partial G}{\partial\vec{n_0}}\right|_{\bar\eta=0}=\vec{n_0}\cdot\left(\frac{\partial G}{\partial\xi},\frac{\partial G}{\partial\eta}\right)=-\left.\frac{\partial G}{\partial\bar\eta}\right|_{\bar\eta=0}$,

$$\begin{aligned}\left.\frac{\partial G}{\partial\bar\eta}\right|_{\bar y=0}&=\frac{1}{4\pi}\left.\left(\frac{2(\bar\eta-\bar y)}{(\bar x+\bar\xi)^2+(\bar\eta-\bar y)^2}+\frac{2(\bar\eta+\bar y)}{(\bar x-\bar\xi)^2+(\bar\eta+\bar y)^2}\right)\right|_{\bar\eta=0}\\&\quad-\frac{1}{4\pi}\left.\left(\frac{2(\bar\eta-\bar y)}{(\bar x-\bar\xi)^2+(\bar\eta-\bar y)^2}+\frac{2(\bar\eta+\bar y)}{(\bar x+\bar\xi)^2+(\bar\eta+\bar y)^2}\right)\right|_{\bar\eta=0}\\&=\frac{1}{\pi}\left(\frac{\bar y}{(\bar x-\bar\xi)^2+\bar y^2}-\frac{\bar y}{(\bar x+\bar\xi)^2+\bar y^2}\right).\\\left.\frac{\partial G}{\partial\bar\xi}\right|_{\bar x=0}&=\frac{1}{\pi}\left(\frac{\bar x}{\bar x^2+(\bar y-\bar\eta)^2}-\frac{\bar x}{\bar x^2+(\bar y+\bar\eta)^2}\right)\end{aligned}$$

因此:

$$\begin{aligned}u(\bar x,\bar y)&=-\int_l\varphi(M_0)\frac{\partial G}{\partial\vec n}(M;M_0)\,dl+\iint_D f(M_0)\,G(M;M_0)\,dM_0\\&=\int_{\substack{\bar\eta=0\\(\bar\xi>0)}}f(\bar\xi+3)\frac{\partial G}{\partial\bar\eta}d\bar\xi+\int_{\substack{\bar\xi=0\\(\bar\eta>0)}}g(2\bar\eta)\frac{\partial G}{\partial\bar\xi}d\bar\eta\\&=\frac{1}{\pi}\int_0^{+\infty}f(\bar\xi+3)\left(\frac{\bar y}{(\bar x-\bar\xi)^2+\bar y^2}-\frac{\bar y}{(\bar x+\bar\xi)^2+\bar y^2}\right)d\bar\xi\\&\quad+\frac{1}{\pi}\int_0^{+\infty}g(2\bar\eta)\left(\frac{\bar x}{\bar x^2+(\bar y-\bar\eta)^2}-\frac{\bar x}{\bar x^2+(\bar y+\bar\eta)^2}\right)d\bar\eta\end{aligned}$$

最后, 利用 $\bar x=x-3,\bar y=\frac{y}{2}$, 得到:

$$\begin{aligned}u(x,y)=&\frac{2}{\pi}\int_0^{+\infty}f(\xi+3)\left(\frac{y}{4(x-3-\xi)^2+y^2}-\frac{y}{4(x-3+\xi)^2+y^2}\right)d\xi\\&+\frac{4}{\pi}\int_0^{+\infty}g(2\eta)\left(\frac{x-3}{4(x-3)^2+(y-2\eta)^2}-\frac{x-3}{4(x-3)^2+(y+2\eta)^2}\right)d\eta\end{aligned}$$

第 2 节 2020 数理方程 A 型真题

一 (20 分) 求解定解问题: $\begin{cases} u_{xx}+3u_{xy}-4u_{yy}=0 \\ u(x,0)=\sin 3x, u_y(x,0)=2x \end{cases}$

解：写出特征方程：

$$(dy)^2-3dxdy-4(dx)^2=0$$

于是

$$(dy-4dx)(dy+dx)=0 \Rightarrow \begin{cases} \xi=y-4x \\ \eta=y+x \end{cases}$$

雅克比行列式不为 0：$J=\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}=\begin{vmatrix} -4 & 1 \\ 1 & 1 \end{vmatrix}\neq 0$, 给出系数：

$$\begin{cases} A_{11}=1\cdot(-4^2)+3\cdot(-4)-4\cdot 1^2=0, B_1=0 \\ A_{22}=1\cdot(-4)+3\cdot(1-4)-4\cdot 1=-17\neq 0 \\ A_{22}=1\cdot 1^2+3\cdot 1^2-4\cdot 1^2=0, B_2=0 \end{cases}$$

于是得到

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=0.$$

解的形式为：$u=f(\xi)+g(\eta)=f(y-4x)+g(y+x)$ 代入初值条件：

$$\Rightarrow \begin{cases} f(-4x)+g(x)=\sin 3x \Rightarrow f'(-4x)\cdot(-4)+g'(x)=3\cos 3x \\ f'(-4x)+g'(x)=2x \end{cases}$$

得到：

$$\Rightarrow \begin{cases} f'(-4x)=\frac{1}{5}(2x-3\cos 3x) \\ g'(x)=\frac{1}{5}(8x+\cos 3x) \end{cases}, \quad \Rightarrow \begin{cases} f(x)=-\frac{x^2}{20}-\frac{4}{5}\sin\frac{3}{4}x-\frac{4}{5}C \\ g(x)=\frac{4}{5}x^2+\frac{1}{5}\sin 3x+\frac{4}{5}C \end{cases}$$

故定解为：

$$\begin{aligned} \Rightarrow u &= f(y-4x)+g(y+x) \\ &= \frac{8}{5}x^2+\frac{6}{5}xy+\frac{17}{20}y^2+\frac{1}{5}\sin(3x+3y)+\frac{4}{5}\sin\left(3x-\frac{3}{4}y\right) \end{aligned}$$

二（12 分）考虑如下形式的一阶线性方程:

$$\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=f(x,y)$$

1) 在 $f(x,y)=0$ 时, 求方程通解。2) 在 $f(x,y)=xy$ 时, 求满足 $u(0,y)=y$ 的特解。

解：特征线方程：

$$\frac{dx}{1}=\frac{dy}{-y}, \quad x=-\ln y$$

$$\Rightarrow ye^x=\xi, \quad 令\eta=x.$$

雅克比行列式：

$$J=\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}=\begin{vmatrix} ye^x & e^x \\ 1 & 0 \end{vmatrix}=-e^x\neq 0$$

根据求导链式法则:

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x}+\dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x}=ye^x\dfrac{\partial u}{\partial \xi}+\dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial y}+\dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial y}=e^x\dfrac{\partial u}{\partial \xi} \end{cases}$$

(1) 根据 $f(x,y)=0$, 代入原方程得到

$$ye^x\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}-ye^x\frac{\partial u}{\partial \xi}=0, \frac{\partial u}{\partial \eta}=0$$

$$\Rightarrow u=F(\xi)=F\left(ye^x\right)$$

(2) 根据 $f(x,y)=xy$, 有 $\dfrac{\partial u}{\partial \eta}=xy\Rightarrow \dfrac{\partial u}{\partial \eta}=\dfrac{\eta\xi}{e^\eta}$

$$\Rightarrow u=\xi\left(G(\xi)-\frac{\eta+1}{e^\eta}\right)$$

$x=0,y=y$ 时候，$\xi=y,\eta=0$,

$$u(y,x)=yG(y)-\frac{y(x+1)}{e^x}=yG(y)-y=y$$

$$\Rightarrow G(y)=2,\quad u=ye^x\left(2-\frac{x+1}{e^x}\right)=2ye^x-(x+1)y$$

三（12 分）求以下固有值问题的固有值和固有函数:

$$\begin{cases} y''+\lambda y=0, & (0<x<20) \\ y(0)=0, & y'(20)=0 \end{cases}$$

解：根据 S-L 定理，因为不仅仅存在第 II 类边值条件，有 $\lambda_n=\omega_n^2>0$

$$y_n=A_n\cos\omega_n x+B_n\sin\omega_n x.$$

$$\begin{cases} y(0)=0, B_n=0, y(0)=A_n\cdot 1=0, A_n=0 \\ y'(20)=\omega_n B_n\cos\left(20\omega_n\right)=0, \end{cases}$$

$$\Rightarrow w_n=\frac{(2n+1)\pi}{2\times 20},\quad y_n=\sin\frac{2n+1}{40}\pi x,\quad \lambda_n^2=\frac{(2n+1)^2\pi^2}{1600}, n=0,1,2,\cdots$$

方法 2：分类讨论, 当 $\lambda=0$ 时, $y=ax+b$, 由边界条件可得 $y=0$, 舍去

当 $\lambda<0$ 时, $y=Ae^{20\sqrt{-\lambda}x}+Be^{-20\sqrt{-\lambda}x}$, 由边界条件可得 $y=0$, 舍去

当 $\lambda>0$ 时，$y=A\sin 20\sqrt{\lambda}x+B\cos 20\sqrt{\lambda}x$ ，由边界条件可得 $B=0, A\cos 20\sqrt{\lambda}=0$ 。

四（14 分) 求解定解问题:

$$\begin{cases} u_{tt}=4u_{xx}-3u, & (t>0, 0<x<5) \\ u(t,0)=u(t,5)=0 \\ u(0,x)=\varphi(x), & u_t(0,x)=0 \end{cases}$$

解：采取分离变量法，设 $u(t,x)=T(t)X(x)$

$$\Rightarrow T''X=4X''T-3TX$$

$$\Rightarrow \frac{T''}{4T}+\frac{3}{4}=\frac{X''}{X}=-\lambda$$

得到

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X(5)=0 \end{cases} \Rightarrow \begin{cases} \lambda_n=\omega_n^2=\left(\frac{n\pi}{5}\right)^2 \\ X_n(x)=\sin\frac{n\pi x}{5} \end{cases}$$

$$\Rightarrow T''+(4\lambda+3)T=0,\quad \omega_n'=\sqrt{3+4\left(\frac{n\pi}{5}\right)^2}$$
$$T_n(t)=A_n\sin\omega_n' t+B_n\cos\omega_n' t$$

于是:

$$u=\sum_{n=1}^{+\infty}(A_n\sin\omega_n' t+B_n\cos\omega_n' t)\sin\frac{n\pi x}{5}$$

代入边值条件，得到

$$\begin{cases} u|_{t=0}=\sum\limits_{n=1}^{+\infty}B_n\cos(\omega_n'\cdot 0)\sin\frac{n\pi x}{5}=\varphi(x) \\ u_t|_{t=0}=\sum\limits_{n=1}^{+\infty}\omega_n' A_n\cos(\omega_n'\cdot 0)\sin\frac{n\pi x}{5}=0 \end{cases}$$

$$A_n=0,\quad B_n=\frac{\langle\varphi(x),\sin\frac{n\pi x}{5}\rangle}{\langle\sin\frac{n\pi x}{5},\sin\frac{n\pi x}{5}\rangle}=\frac{2}{5}\int_0^5\varphi(\xi)\sin\frac{n\pi\xi}{5}d\xi$$

$$\Rightarrow u=\sum_{n=1}^{+\infty}\left(\frac{2}{5}\int_0^5\varphi(\xi)\sin\frac{n\pi\xi}{5}d\xi\right)\cos\left(\sqrt{3+4\left(\frac{n\pi}{5}\right)^2}t\right)\sin\frac{n\pi x}{5}$$

五 (14 分) 考虑如下定解问题:

$$\begin{cases} u_t=a^2\Delta_3 u, & (t>0,x^2+y^2<R^2,-\infty<z<+\infty) \\ u|_{x^2+y^2=R^2}=u_1 \\ u|_{t=0}=u_1+R^2-x^2-y^2 \end{cases}$$

其中 a,R,u_1 均为给定的常数且 $a>0,R>0$.

(1) 当 $u_1=0$ 时, 求解上述定解问题。(2) 当 $u_1\neq 0$ 时, 求解上述定解问题。

在柱坐标下讨论:

$$\begin{cases} u_t=a^2\Delta_3 u=a^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right], t>0, 0<r<R \\ u|_{r=R}=u_1,\quad u|_{t=0}=u_1+R^2-r^2 \end{cases}$$

考虑分离变量:

$$\frac{T'(t)}{a^2T(t)}=\frac{\frac{1}{r}(rR'(r))'}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)},\quad \frac{\Theta''(\theta)}{\Theta(\theta)}=-\mu$$

$$\frac{T'(t)}{a^2T(t)}=\frac{\frac{1}{r}(rR'(r))'}{R(r)}-\frac{\mu}{r^2}=-\lambda$$

于是得到两个固有值问题和一个微分方程:

$$\begin{cases} \Theta''+\mu\Theta=0 \\ \Theta(\theta+2\pi)=\Theta(\theta) \end{cases} \quad \begin{cases} (rR'(r))'+\left(\lambda r-\frac{\mu}{r}\right)R=0, \\ |R(0)|<+\infty,\text{此条件为自行添加} \end{cases} \quad T'+a^2\lambda T=0$$

通过 Θ 的固有值问题

$$\begin{cases}\mu_0=0,\\ \Theta_0(\theta)=1\end{cases}\qquad\begin{cases}\mu_m=m^2, m=1,2,\cdots\\ \Theta_m(\theta)=A_m\cos m\theta+B_m\sin m\theta\end{cases}$$

$$(rR')'+\left(\lambda r-\frac{m^2}{r}\right)R=0$$

通解 $R(r)=C_mJ_m(\omega r)+D_mN_m(\omega r), \omega=\sqrt{\lambda}$ 由 $|R(0)|<+\infty$, 得 $D_m=0$.

$$T'+a^2\omega_{m1n}^2T=0.\quad T_{mn}(t)=C_{mn}\mathrm{e}^{-a^2\omega_{m1n}^2t}$$

$$\Rightarrow\quad u(t,r,\theta)=\sum_{m=0}^{+\infty}\sum_{n=1}^{+\infty}\mathrm{e}^{-a^2\omega_{m1n}^2t}J_m\left(\omega_{m1n}r\right)\left(C_{mn}\cos m\theta+D_{mn}\sin m\theta\right)$$

(1) 当 $u_1=0, u|_{r=R}=0, J(\omega_{mn})$ 为 $J(\omega_{mn}R)=0$ 的第 n 个正根, $\lambda_n=\omega_n^2$

$$\begin{aligned}&m=0,\quad \sum_{n=1}^{+\infty}J_0\left(\omega_{on}r\right)A_0=R^2-r^2\\ &A_0=\frac{1}{{N_{0n}}^2}\int_0^R r\left(R^2-r^2\right)J_0\left(\omega_{0n}r\right)dr\end{aligned}\quad ,\quad \because N_{0n}^2=\|J_{0n}\|^2=\frac{R^2J_1^2\left(\omega_{0n}R\right)}{2}$$

$$\begin{aligned}\because&\int_0^R\left(R^2-r^2\right)rJ_0(\omega r)dr=\frac{1}{\omega^4}\int_0^{\omega R}\left(\omega^2R^2-x^2\right)xJ_0(x)dx\\
=&\frac{1}{\omega^4}\int_0^{\omega R}\left(\omega^2R^2-x^2\right)\left(xJ_1(x)\right)'dx=\frac{1}{\omega^4}\int_0^{\omega R}2xJ_1(x)dx^2=\frac{1}{\omega^4}\int_0^{\omega R}2x^2J_1(x)dx\\
=&\frac{1}{\omega^4}\int_0^{\omega R}2x^2\left(-J_0(x)\right)'dx=\left.\left(\frac{-1}{\omega^4}2x^2J_0(x)\right)\right|_0^{\omega R}+\frac{4}{\omega}\int_0^{\omega R}xJ_0(x)dx\\
=&-\frac{2}{\omega^4}(\omega R)^2J_0(\omega R)+\frac{4}{\omega^4}\left.\left(xJ_1(x)\right)\right|_0^{\omega R}=\frac{1}{\omega^4}\omega RJ_1(\omega R)=\frac{4R}{\omega^3}J_1(\omega R)\end{aligned}$$

$$\therefore A_0=\frac{2}{R^2J_1^2\left(\omega_{0n}R\right)}\frac{4R}{\omega_{0n}^3}J_1(\omega_{0n}R)=\frac{8}{RJ_1(\omega_{0n}R)\omega_{0n}^3}$$

$$u=\sum_{n=1}^{+\infty}A_0e^{-a^2\omega_{0n}^2t}J_0\left(\omega_{0n}r\right)=\sum_{n=1}^{+\infty}\frac{8}{RJ_1(\omega_{0n}R)\omega_{0n}^3}e^{-a^2\omega_{0n}^2t}J_0\left(\omega_{0n}r\right)$$

(2) 当 $u_1\neq0$ 时，根据叠加原理做变换 $u=u_1+v$，v 满足 $u_1=0$ 的情形，

$$u=u_1+\sum_{n=1}^{+\infty}A_0e^{-a^2\omega_{0n}^2t}J_0\left(\omega_{0n}r\right)=u_1+\sum_{n=1}^{+\infty}\frac{8}{RJ_1(\omega_{0n}R)\omega_{0n}^3}e^{-a^2\omega_{0n}^2t}J_0\left(\omega_{0n}r\right)$$

六 (12 分) (r,θ,φ) 为球坐标, 考虑下列定解问题:

$$\begin{cases}\Delta_3u=0,\quad (r<2)\\ u|_{r=2}=f(\theta)\end{cases}$$

(1) 当 $f(\theta)=1+\cos^2\theta$ 时, 求解上述定解问题。

(2) 当 $f(\theta)=\begin{cases}4,\quad (0\le\theta\le\alpha)\\ 0,\quad (\alpha<\theta\le\pi)\end{cases}$ 时, 求解上述定解问题。

解：球坐标系的 Laplace 方程的解的形式：

$$u(r,\theta,\varphi)=\sum_{n=0}^{+\infty}\left(c_nr^n+D_nr^{-(n+1)}\right)P_n(\cos\theta)$$

在轴对称情形下：

$$u(r,\theta)=\sum_{n=0}^{+\infty}A_n\left(\frac{r}{a}\right)^nP_n(\cos\theta)$$

(1) $f(\theta)=1+\cos^2\theta,\quad x=\cos\theta,$

$$\begin{aligned}&A_0P_0(x)+A_2P_2(x)=1+x^2,A_0=\frac{1}{3},A_2=\frac{2}{3}\\&\Rightarrow u(r,\theta,\varphi)=\frac{1}{3}\left(\frac{r}{a}\right)^2\left(\frac{3\cos^2\theta-1}{2}\right)\end{aligned}$$

或写作

$$u=\frac{4}{3}P_0(\cos\theta)+\frac{1}{6}r^2P_2(\cos\theta)$$

(2)

$$u(r,\theta)=\sum_{n=0}^{+\infty}A_n\left(\frac{r}{a}\right)^nP_n(\cos\theta)=\sum_{n=0}^{+\infty}A_n\left(\frac{r}{2}\right)^nP_n(\cos\theta)$$

$$f(\theta)=\left\{\begin{array}{ll}4, & 0\leqslant\theta\leqslant\alpha\\0, & \alpha<\theta\leqslant\pi\end{array}\right.\Rightarrow\sum_{n=1}^{+\infty}u|_{r=2}=A_nP_n(\cos\theta)=\left\{\begin{array}{ll}4, & 0\leqslant\theta\leqslant\alpha\\0, & \alpha<\theta<\pi\end{array}\right.$$

$$\begin{aligned}A_n&=\frac{\langle f(\theta),P_n(\cos\theta)\rangle}{\langle P_n(\cos\theta),P_n(\cos\theta)\rangle}=\frac{2n+1}{2}\int_0^\pi f(\theta)P_n(\cos\theta)\sin\theta d\theta\\&=\frac{2n+1}{2}\int_0^\pi f(\theta)P_n(\cos\theta)(-d\cos\theta)=\frac{2n+1}{2}\int_{-1}^1 f(\arccos x)P_n(x)dx\\&=\frac{2n+1}{2}\int_{\cos\alpha}^1 4P_n(x)dx=2\left[P_{n+1}(x)-P_{n-1}(x)\right]|_{\cos\alpha}^0\\&=2\left[-P_{n+1}(\cos\alpha)+P_{n-1}(\cos\alpha)\right]\end{aligned}$$

$$A_0=\frac{1}{2}\int_{\cos\alpha}^1 4dx=2(1-\cos\alpha)$$

$$u=2(1-\cos\alpha)+\sum_{n=1}^{+\infty}\frac{r^n}{2^{n-1}}\left[P_{n-1}(\cos\alpha)-P_{n+1}(\cos\alpha)\right]P_n(\cos\theta)$$

七 (12 分) 求解初值问题：

$$\left\{\begin{array}{l}u_t=u_{xx}+20u_x+u,\quad(t>0,-\infty<x<+\infty)\\u|_{t=0}=\delta(x-2)+3e^{-x^2}\end{array}\right.$$

解：进行 Fourier 变换：$\hat{u}(t,\lambda)=\int_{-\infty}^{+\infty}u(x)e^{-i\lambda x}dx$

其中，$F\left[3e^{-x^2}\right]=\int_{-\infty}^{+\infty}3e^{-x^2-i\lambda x}dx=3\int_{-\infty}^{+\infty}e^{-\left(x+\frac{i\lambda}{2}\right)^2-\frac{\lambda^2}{4}}dx\xlongequal{\text{围道积分}}3\sqrt{\pi}e^{-\frac{\lambda^2}{4}},F[\delta(x-2)]=e^{-2i\lambda}$

$$\Rightarrow \begin{cases} \dfrac{d\hat{u}}{dt} = -\lambda^2\hat{u} + 20(i\lambda)\hat{u} + \hat{u} \\ \hat{u}|_{t=0} = e^{-2i\lambda} + 3\sqrt{\pi}e^{-\frac{\lambda^2}{4}} \end{cases}$$

$$\widehat{u} = \left(e^{-2i\lambda} + 3\sqrt{\pi}e^{-\frac{\lambda^2}{4}}\right)\exp\left[\left(-\lambda^2 + 20i\lambda + 1\right)t\right]$$

由于位移公式和已知积分公式:

$$F^{-1}\left[e^{-\lambda^2 t}\right] = \frac{1}{2\sqrt{\pi t}}\exp\left(-\frac{x^2}{4t}\right)$$
$$F\left[f\left(x - x_0\right)\right] = f(\lambda)e^{-i\lambda x_0} \Rightarrow F^{-1}\left[f(\lambda)e^{-i\lambda x_0}\right] = f\left(x + x_0\right)$$

解得

$$\begin{aligned} u = F^{-1}[\hat{u}] &= e^t \cdot F^{-1}\left[e^{-\lambda^2 t}e^{(20t-2)i\lambda} + 3\sqrt{\pi}e^{-\lambda^2\left(t+\frac{1}{4}\right)}e^{20ti\lambda}\right] \\ &= e^t\left\{\frac{1}{2\sqrt{\pi t}}\exp\left[-\frac{(x+20t-2)^2}{4t}\right] + \frac{3}{2\sqrt{t+\frac{1}{4}}}\exp\left[-\frac{(x+20t)^2}{4t+1}\right]\right\} \end{aligned}$$

八 (14 分) 已知空间区域 $V = \{(x,y,z) \mid x > 0, y > 0, -\infty < z < +\infty\}$, 求 V 内泊松方程第一边值问题的格林函数并求解边值问题。

$$\begin{cases} \Delta_3 u = 0, & (x,y,z) \in V \\ u = \begin{cases} \varphi(x,z), & 当 y = 0, x \geq 0 \\ g(y,z), & 当 x = 0, y > 0 \end{cases} \end{cases}$$

解：先由镜像法求 Green 函数，Green 函数满足：$\begin{cases} G_{xx} + G_{yy} + G_{zz} = -\delta(x-\xi, y-\eta, z-y) \\ G|_{y=0,x>0} = 0,\ G|_{x=0,y>0} = 0 \end{cases}$

在 $M_0 = (\xi,\eta,\zeta)$ 放置点电荷 $+\varepsilon$, 在 $M = (x,y)$ 产生电位 $U_0 = \dfrac{1}{4\pi r_0(M,M_0)}$

在 M_0 关于 $x = 0$ 对称点 $M_1 = (-\xi,\eta,\zeta)$ 虚设点电荷 $-\varepsilon$, 在 M 产生电位 $U_1 = -\dfrac{1}{4\pi r_1(M,M_1)}$

在 M_0 关于 $y = 0$ 对称点 $M_2 = (\xi,-\eta,\zeta)$ 虚设点电荷 $-\varepsilon$, 在 M 产生电位 $U_2 = -\dfrac{1}{4\pi r_2(M,M_2)}$

在 M_0 关于 $x + y = 0$ 对称点 $M_3 = (-\xi,-\eta,\zeta)$ 虚设点电荷 ε, 在 M 产生电位 $U_3 = \dfrac{1}{4\pi r_3(M,M_3)}$

$$r_0(M,M_0) = \sqrt{(x-\xi)^2 + (y-\eta)^2},\quad r_1(M,M_1) = \sqrt{(x+\xi)^2 + (y-\eta)^2}$$
$$r_2(M,M_2) = \sqrt{(x-\xi)^2 + (y+\eta)^2},\quad r_3(M,M_3) = \sqrt{(x+\xi)^2 + (y+\eta)^2}$$

$$\begin{aligned} G = U_0 + U_1 + U_2 + U_3 &= \frac{1}{4\pi r(M,M_0)} - \frac{1}{4\pi r(M,M_1)} - \frac{1}{4\pi r(M,M_2)} + \frac{1}{4\pi r(M,M_3)} \\ &= \frac{1}{4\pi}\left\{\frac{1}{[(x-\xi)^2 + (y-\eta)^2]^{\frac{1}{2}}} - \frac{1}{[(x+\xi)^2 + (y-\eta)^2]^{\frac{1}{2}}} - \frac{1}{[(x-\xi)^2 + (y+\eta)^2]^{\frac{1}{2}}} - \frac{1}{[(x+\xi)^2 + (y+\eta)^2]^{\frac{1}{2}}}\right\} \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial G}{\partial \xi}\right|_{\xi=0} &= \frac{1}{4\pi}\left\{\frac{x-\xi}{[(x-\xi)^2 + (y-\eta)^2]^{\frac{3}{2}}} - \frac{x-\xi}{[(x+\xi)^2 + (y-\eta)^2]^{\frac{3}{2}}} - \frac{x+\xi}{[(x-\xi)^2 + (y+\eta)^2]^{\frac{3}{2}}} - \frac{x+\xi}{[(x+\xi)^2 + (y+\eta)^2]^{\frac{3}{2}}}\right\} \\ &= \frac{1}{4\pi}\left\{\frac{x}{[x^2 + (y-\eta)^2]^{\frac{3}{2}}} - \frac{x}{[x^2 + (y-\eta)^2]^{\frac{3}{2}}} - \frac{x}{[x^2 + (y+\eta)^2]^{\frac{3}{2}}} - \frac{x}{[x^2 + (y+\eta)^2]^{\frac{3}{2}}}\right\} \end{aligned}$$

$$\left.\frac{\partial G}{\partial \eta}\right|_{\eta=0}=\frac{1}{4\pi}\left\{\frac{y-\eta}{\left[(x-\xi)^2+(y-\eta)^2\right]^{\frac{3}{2}}}-\frac{y-\eta}{\left[(x+\xi)^2+(y-\eta)^2\right]^{\frac{3}{2}}}-\frac{y+\eta}{\left[(x-\xi)^2+(y+\eta)^2\right]^{\frac{3}{2}}}-\frac{y+\eta}{\left[(x+\xi)^2+(y+\eta)^2\right]^{\frac{3}{2}}}\right\}$$

$$=\frac{1}{4\pi}\left\{\frac{y}{\left[(x-\xi)^2+y^2\right]^{\frac{3}{2}}}-\frac{y}{\left[(x+\xi)^2+y^2\right]^{\frac{3}{2}}}-\frac{y}{\left[(x-\xi)^2+y^2\right]^{\frac{3}{2}}}-\frac{y}{\left[(x+\xi)^2+y^2\right]^{\frac{3}{2}}}\right\}$$

由于 $\frac{\partial G}{\partial n_1}=-\frac{\partial G}{\partial \xi},\quad \frac{\partial G}{\partial n_2}=-\frac{\partial G}{\partial \eta}$，根据 Poisson 公式

$$u=-\iint_S \varphi\left(M_0\right)\frac{\partial G}{\partial \vec{n}}\left(M;M_0\right)dS+\iiint_V f\left(M_0\right)G\left(M;M_0\right)dM_0$$

$$=\int_0^{+\infty}\left(\left.\frac{\partial G}{\partial \xi}\right|_{\xi=0}\right)g(\eta,z)d\eta+\int_0^{+\infty}\left(\left.\frac{\partial G}{\partial \eta}\right|_{\eta=0}\right)\varphi(\xi,z)d\xi$$

$$=\int_0^{+\infty}\frac{1}{4\pi}\left\{\frac{x}{\left[x^2+(y-\eta)^2\right]^{\frac{3}{2}}}-\frac{x}{\left[x^2+(y-\eta)^2\right]^{\frac{3}{2}}}-\frac{x}{\left[x^2+(y+\eta)^2\right]^{\frac{3}{2}}}-\frac{x}{\left[x^2+(y+\eta)^2\right]^{\frac{3}{2}}}\right\}g(\eta,z)d\eta$$

$$+\int_0^{+\infty}\frac{1}{4\pi}\left\{\frac{y}{\left[(x-\xi)^2+y^2\right]^{\frac{3}{2}}}-\frac{y}{\left[(x+\xi)^2+y^2\right]^{\frac{3}{2}}}-\frac{y}{\left[(x-\xi)^2+y^2\right]^{\frac{3}{2}}}-\frac{y}{\left[(x+\xi)^2+y^2\right]^{\frac{3}{2}}}\right\}\varphi(\xi,z)d\xi$$

第 3 节 考试一般给出的参考公式

1) 直角坐标系: $\Delta_3 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$,

柱坐标系: $\Delta_3 u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}$,

球坐标系: $\Delta_3 u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2}$.

2) Bessel 函数性质

若 ω 是 $J_\nu(\omega a)=0$ 的一个正根, 则有模平方 $N_{\nu 1}^2=\left\|J_\nu(\omega x)\right\|_1^2=\frac{a^2}{2}J_{\nu+1}^2(\omega a)$.

若 ω 是 $J_\nu'(\omega a)=0$ 的一个正根, 则有模平方 $N_{\nu 2}^2=\left\|J_\nu(\omega x)\right\|_2^2=\frac{1}{2}\left[a^2-\frac{\nu^2}{\omega^2}\right]J_\nu^2(\omega a)$.

若 ω 是 $\alpha J_\nu(\omega a)+\beta J_\nu'(\omega a)=0$ 的一个正根, 则有模平方 $N_{\nu 3n}^2=\frac{1}{2}\left[a^2-\frac{\nu^2}{\omega_{3n}^2}+\frac{a^2\alpha^2}{\beta^2\omega_{3n}^2}\right]J_\nu^2\left(\omega_{3n}a\right)$

Bessel 函数递推公式: $\frac{d}{dx}\left(x^\nu J_\nu(x)\right)=x^\nu J_{\nu-1}(x),\quad \frac{d}{dx}\left(\frac{J_\nu(x)}{x^\nu}\right)=-\frac{J_{\nu+1}(x)}{x^\nu}$

$$2J_\nu'(x)=J_{\nu-1}(x)-J_{\nu+1}(x),\quad \frac{2\nu}{x}J_\nu(x)=J_{\nu-1}(x)+J_{\nu+1}(x).$$

3) n 阶 Legendre 多项式: $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n,n=0,1,2,3,\ldots$;

递推公式:

1. $(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0$, 2. $nP_n(x)-xP_n'(x)+P_{n-1}'(x)=0$,

3. $nP_{n-1}(x)-P_n'(x)+xP_{n-1}'(x)=0$, 4. $P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$.

4) 卷积计算用到的公式: $\frac{1}{\pi}\int_0^{+\infty}e^{-a^2\lambda^2 t}\cos\lambda x d\lambda=\frac{1}{2a\sqrt{\pi t}}\exp\left(-\frac{x^2}{4a^2 t}\right)$

5) 由 Poisson 方程第一边值问题的格林函数 $G\left(M;M_0\right)$, 求得第一边值问题解 $u(M)$ 的公式:

空间区域: $u(M)=-\iint_S\varphi\left(M_0\right)\frac{\partial G}{\partial \vec{n}}\left(M;M_0\right)dS+\iiint_V f\left(M_0\right)G\left(M;M_0\right)dM_0$,

平面区域: $u(M)=-\int_l\varphi\left(M_0\right)\frac{\partial G}{\partial \vec{n}}\left(M;M_0\right)dl+\iint_D f\left(M_0\right)G\left(M;M_0\right)dM_0$.