中国科学技术大学
University of Science and Technology of China

理論力學學習參考

理论力学学习参考

程嘉杰　整理

USTC, 96 JinZhai Road, Hefei,
Anhui, P.R.China, 230026
中国安徽省合肥市金寨路 96 号
中国科学技术大学

文件网址：home.ustc.edu.cn/∼jiajie
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中国科大 2019-2021 学期理论力学 A 考试试题解析

程嘉杰整理
文件网址：home.ustc.edu.cn/∼jiajie

一、自聚焦光纤具有轴对称的折射率分布为,

$$n(r)=n_0\sqrt{1-\alpha^2r^2}$$

其中 $n_0>0,\alpha$ 是较小的正常数, r 是到光纤中轴线的距离。建立柱坐标系 r,θ,z,θ 表示绕中轴线的转角, z 轴沿中轴线方向。光线的轨迹记为:

$$r=r(\lambda),\quad \theta=\theta(\lambda),\quad z=z(\lambda)$$

根据费马原理, 几何光学的拉氏量可以写成

$$L=\frac{1}{2}n_0^2\left(1-\alpha^2r^2\right)\left(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2\right)$$

(1) 找出尽可能多的首次积分, 并说明理由;

(2) 写出这个系统的哈密顿量以及正则方程。

解： 想要找“运动积分”（守恒量），考虑从能量和动量角度出发，首先是 Euler-Lagrange 方程:

$$P_r=\frac{\partial L}{\partial \dot{r}}={n_0}^2\left(1-\alpha^2r^2\right)\dot{r}$$
$$P_\theta=\frac{\partial L}{\partial \dot{\theta}}={n_0}^2\left(1-\alpha^2r^2\right)r^2\dot{\theta}$$
$$P_z=\frac{\partial L}{\partial \dot{z}}={n_0}^2\left(1-\alpha^2r^2\right)\dot{z}$$

以及通过 Legendre 变换求解 Hamilton 量:

$$H=\dot{r}P_r+\dot{\theta}P_\theta+\dot{z}P_z-L=L=H(r,\theta,z,P_r,P_\theta,P_z)$$

因为 L 中不显含广义坐标 θ,z, 故这两个认为是循环坐标，对应的正则动量守恒，找到两个运动积分 P_θ,P_z，守恒量哈密顿量 H（系统的广义能量）也是一个运动积分。

在（2）问写哈密顿量，

$$H=\frac{P_r^2+P_\theta^2/r^2+P_z^2}{2n_0^2\left(1-\alpha^2r^2\right)}$$

使用 P_r,P_θ,P_z 表示根据 Hamilton 正则方程:

$$\left\{\begin{aligned}\dot{q_\alpha}&=\frac{\partial H}{\partial p_\alpha}\\ \dot{p_\alpha}&=-\frac{\partial H}{\partial q_\alpha}\end{aligned}\right.\Rightarrow\left\{\begin{aligned}&\dot{r}=\frac{p_r}{n_0^2\left(1-\alpha^2r^2\right)}, && \dot{p}_r=-\frac{\alpha^2r\left(p_r^2+p_z^2\right)+p_\theta^2\left(1+2\alpha^2r^2\right)/r^3}{n_0^2\left(1-\alpha^2r^2\right)}\\ &\dot{\theta}=\frac{p_\theta}{n_0^2r^2\left(1-\alpha^2r^2\right)}, && \dot{p}_\theta=0\\ &\dot{z}=\frac{p_z}{n_0^2\left(1-\alpha^2r^2\right)}, && \dot{p}_z=0\end{aligned}\right.$$

二、已知 n 个质点组成的 Hamilton 体系的 Hamilton 函数为

$$H=\sum_i \frac{p_i^2}{2m_i}+V\left(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N,t\right)$$

力学量 K 满足

$$\mathbf{K}=\mathbf{P}t-M\mathbf{r}_c$$

使用泊松括号找出势能 V 所需要满足的条件, 使得 K 为守恒量

解:

$$\begin{aligned}
0=\frac{d\boldsymbol{K}}{dt}&=\frac{\partial\boldsymbol{K}}{\partial t}+[\boldsymbol{K},H]\quad(\text{定义加性质})\\
&=\boldsymbol{p}+[\boldsymbol{p}t,H]-[M\boldsymbol{r}_c,H]\\
&=\boldsymbol{p}+\left[\sum_j\overrightarrow{p_j}t,\sum_i\frac{p_i^2}{2m_i}+V\right]-\left[\sum_i m_i\boldsymbol{r}_i,\sum_j\frac{p_j^2}{2m_j}+V\right]\\
&=\boldsymbol{p}+\sum_j[p_jt,V]-\left[\sum_i m_i\boldsymbol{r}_i,\sum_j\frac{p_j^2}{2m_j}\right]\\
&=\boldsymbol{p}+t\sum_j[p_j,V]-\left[\sum_i m_i\boldsymbol{r}_i,\sum_j\frac{p_j^2}{2m_j}\right]\\
&=\boldsymbol{p}+t\sum_j\left(0-\frac{\partial V}{\partial x_j}\right)-\left(\sum_i m_i\cdot\sum_j\frac{\overrightarrow{p_j}}{2m_j}-0\right)=-t\sum_j\frac{\partial V}{\partial x_j}
\end{aligned}$$

故势能 V 满足等势能面条件。

注 0.1. $\frac{d\boldsymbol{K}}{dt}$ 中的 K 是相空间 $(q_\alpha(t),p_\alpha(t),t)$ 的函数，而 $\frac{\partial\boldsymbol{K}}{\partial t}$ 中的 K 仅和 t 有关。

注 0.2. 还可以说明：

$$\left[x_i,f\left(x_i,\overrightarrow{p_i}\right)\right]=\sum_j\left(\frac{\partial x_i}{\partial x_j}\frac{\partial f_i}{\partial p_j}-\frac{\partial x_i}{\partial p_i}\frac{\partial f_i}{\partial x_j}\right)=\frac{\partial f}{\partial p_j}$$

$$\left[p_i,f\left(x_i,\overrightarrow{p_i}\right)\right]=\sum_j\left(\frac{\partial_i}{\partial x_j}\frac{\partial f}{\partial p_j}-\frac{\partial p_i}{\partial p_i}\frac{\partial f_i}{\partial x_j}\right)=-\frac{\partial f}{\partial x_j}$$

三、带电粒子在电磁场中的 Hamilton 函数为

$$H=\frac{(\mathbf{p}-q\mathbf{A})^2}{2m}+q\phi,\quad \mathbf{A}=\mathbf{A}(\mathbf{r},t),\quad \phi=\phi(\mathbf{r},t)$$

而规范变换引入了一组新的势场, 使得原来的电磁场在新的势场下描述不变, 引入 $\psi=\psi(\mathbf{r},t)$

$$\mathbf{A}'=\mathbf{A}+\nabla\psi,\quad \phi'=\phi-\partial_t\psi$$

解决下列问题:

(1) 通过 Legendre 变换求出 Lagrange 函数 $L(\mathbf{r},\dot{\mathbf{r}})$

(2) 证明变换

$$\mathbf{r}'=\mathbf{r},\quad \mathbf{p}'=\mathbf{p}+\nabla\psi$$

为正则变换, 并求出生成函数 $F_2\left(\mathbf{r},\mathbf{p}',t\right)$, 说明该正则变换为什么不存在第一类生成函数 $F_1\left(\mathbf{r},\mathbf{r}',t\right)$

(3) 求新 Hamilton 函数 $H'(\mathbf{r}',\mathbf{p}')$, 说明其和原来的 Hamilton 函数等价

解： (1)

$$
\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p} = \frac{p}{m} - \frac{qA}{m}, \quad p = m\dot{r} + qA \\
L(r,\dot{r}) &= \dot{r}p - H = \dot{r}(m\dot{r} + qA) - \frac{m\dot{r}^2}{2} - q\phi \\
&= \frac{m\dot{r}^2}{2} + qA\dot{r} - q\phi
\end{aligned}
$$

(2)

$$
\begin{aligned}
\frac{\partial(\boldsymbol{r},\boldsymbol{p}')}{\partial(\boldsymbol{r},\boldsymbol{p})} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \\
\delta F_2(\boldsymbol{r},\boldsymbol{p}') &= \boldsymbol{p}\delta\boldsymbol{r} + \boldsymbol{r}'\delta\boldsymbol{p}' \\
&= (\boldsymbol{p}' - q\nabla\psi)\,\delta\boldsymbol{r} + \boldsymbol{r}\delta\boldsymbol{p}' \\
&= \delta(\boldsymbol{p}'\cdot\boldsymbol{r} - q\psi) \\
\Rightarrow F_2(\boldsymbol{r},\boldsymbol{p}') &= \boldsymbol{p}'\cdot\boldsymbol{r} - q\psi
\end{aligned}
$$

不存在第一类生成函数是因为$\boldsymbol{r}=\boldsymbol{r}'$

(3)

$$
\begin{aligned}
H' &= H + \frac{\partial F_2}{\partial t} \\
&= \frac{(\boldsymbol{p} - q\boldsymbol{A})^2}{2m} + q\left(\phi - \frac{\partial\psi}{\partial t}\right) \\
&= \frac{(\boldsymbol{p}' - q\nabla\psi - q\boldsymbol{A})^2}{2m} + q\left(\phi - \frac{\partial\psi}{\partial t}\right) \\
&= \frac{(\boldsymbol{p}' - q\boldsymbol{A}')^2}{2m} + q\phi'
\end{aligned}
$$

四、讨论行星的运动。一质量为 m 的质点在引力势 $V(r) = -\frac{k}{r}(k>0)$ 中做束缚态平面轨道运动 $(E<0)$, 根据对称性, 采用极坐标 (r,θ) 。

1. 作用量 I_r, I_θ 分别定义为: $I_r \equiv \oint p_r(r)dr; \quad I_\theta \equiv \oint p_\theta(\theta)d\theta$, 环路积分沿着轨道积分。

   试根据位力 (Virial) 定理证明: $I_r + I_\theta = \oint \frac{k}{r}dt$;

2. 试根据求解哈密顿 - 雅克比方程, 求解哈密度主函数 $W(r,\theta) = W_r(r) + W_\theta(\theta)$;

3. 根据哈密顿主函数 $W_r(r), W_\theta(\theta)$, 计算相应的作用量 I_r, I_θ

   (你可能用到的公式: $\int_{x_{\min}}^{x_{\max}} \frac{\sqrt{-a+2bx-x^2}}{x}dx = \pi(b-\sqrt{a})$, 其中积分上下限为被积函数的两个根)。

解： 第一小问是考察动能和势能的关系，使用 Virial 定理：$\bar{T}=-\frac{1}{2}\overline{\sum_i \boldsymbol{F}_i\cdot\boldsymbol{r}_i}$

$$\begin{aligned}
I_r+I_\theta &= \oint\left(\dot{r}P_r+\dot{\theta}P_\theta\right)dt(\text{采取积分变换的方法})\\
&=\oint 2Tdt=2T_0\left(\frac{1}{T_0}\oint Tdt\right),(T_0\text{为一个运动周期})\\
&=2T_0\langle T\rangle_t=T_0(-\langle V\rangle_t)(\text{Virial 定理})\\
&=T_0\int_0^{T_0}\frac{k/r}{T_0}dt=\oint\frac{k}{r}dt
\end{aligned}$$

对于二小问，首先写出拉格朗日量 Lagrangian，在写出 Hamilton-Jacobi 方程主函数从而求解

$$\begin{aligned}
L&=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{k}{r}\\
H&=\frac{P_r^2}{2m}+\frac{P_\theta^2}{2mr^2}-\frac{k}{r}
\end{aligned}$$

$$S=S(r,\theta,t)=-Et+A+W_r(r)+W_\theta(\theta)$$

$$E=\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2+\frac{1}{2mr^2}\left(\frac{dW_\theta}{d\theta}\right)^2-\frac{K}{r}$$

$$\Rightarrow\left\{\begin{array}{l}\frac{dW_\theta}{d\theta}=L\\ E=\frac{1}{2m}\left(\frac{dW_r}{dr}\right)^2+\frac{L^2}{2mr^2}-\frac{K}{r}\end{array}\right.\Rightarrow\left\{\begin{array}{l}W_\theta=L\theta+A\\ W_r=\int^r\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{L^2}{r^2}}dr\end{array}\right.$$

$$I_\theta=\oint p_\theta d\theta=\oint\frac{dW_\theta}{d\theta}d\theta=2\pi L$$

$$I_r=\oint p_r dr=\oint\frac{dW_r}{dr}dr=2\int_{r_{\min}}^{r_{\max}}\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{L^2}{r^2}}dr=2\pi\left(k\sqrt{\frac{m}{2|E|}}-L\right)$$

五、某粒子在有心力作用下运动时的轨道方程为 $r=r_0\exp(a\theta)$, 当 $\theta=0$ 时, 粒子的速度为 v_0 求: (1) 粒子受到的力和势能表达式 (2) 该粒子运动时的能量 (3) 无穷远处所吸收的粒子的截面 σ

解:

$$u=\frac{1}{r}=\frac{1}{r_0}\mathrm{e}^{-a\theta}$$

对上式求导得

$$\frac{\mathrm{d}u}{\mathrm{d}\theta}=-\frac{a}{r_0}\mathrm{e}^{-a\theta},\quad\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}=\frac{a^2}{r_0}\mathrm{e}^{-a\theta}=a^2u$$

应用 Binet 公式, 有

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u=-\frac{F(r)}{mh^2u^2}$$

其中 $r^2\frac{\mathrm{d}\theta}{\mathrm{d}t}=h$。所以得到:

$$F=-mh^2u^2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\right)=-mh^2\left(1+a^2\right)u^3=-\frac{mh^2\left(1+a^2\right)}{r^3}.$$

这是立方反比的有心吸引力.

$$F=-\frac{\partial V}{\partial r},V=\int_r^r\frac{m\dot{\theta}\left(1+a^2\right)}{r}dr=\int\frac{m\left(1+a^2\right)}{ar^2}dr=-\frac{m\left(1+a^2\right)}{ar}$$

六、已知 Hamilton 量描述的体系为

$$H=\frac{1}{2m}\left(p^2+m^2\omega^2q^2\right)\cos(\Omega t)$$

用 Hamilton-Jacobi 方法求解 $p(t),q(t)$, 其中 $p(0)=p_0,q(0)=0$

解: 这里哈密顿函数显含时间. 为了计算化简方便，采取虚数变形:

$$H=\mathrm{Re}\frac{1}{2m}\left(p^2+m^2\omega^2q^2\right)\exp(i\Omega t)$$

技巧性比较强地，设第二类生成函数: $S_2(q,P,t)=\mathrm{e}^{\frac{i\Omega t}{2}}qP$ 由生成函数可得正则变换为

$$p=\frac{\partial S_2}{\partial q}=\mathrm{e}^{\frac{i\Omega t}{2}}P,\quad Q=\frac{\partial S_2}{\partial P}=\mathrm{e}^{\frac{i\Omega t}{2}}q.$$

变换后的系统的哈密顿函数（复数域）为

$$\tilde{H}(Q,P,t)=H(q,p,t)+\frac{\partial S_2}{\partial t}=\frac{P^2}{2m}+\frac{1}{2}m\omega_0^2Q^2+\frac{\Omega}{2}QP.$$

可见, 变换后哈密顿函数不再显含时间. 与新哈密顿函数相关联的哈密顿 - 雅可比方程为

$$\frac{1}{2m}\left(\frac{\partial S}{\partial Q}\right)^2+\frac{1}{2}m\omega_0^2Q^2+\frac{\Omega}{2}Q\frac{\partial S}{\partial Q}+\frac{\partial S}{\partial t}=0.$$

$$S(Q,E,t)=W_Q+A-Et$$

$$\Rightarrow\frac{1}{2m}\left(\frac{\partial W}{\partial Q}\right)^2+\frac{m\omega_0^2}{2}Q^2+\frac{\Omega}{2}Q\frac{\partial W}{\partial Q}=E$$

令 $x=\sqrt{m\omega_0}Q,a=\dfrac{\Omega}{\omega_0},b=\dfrac{2E}{\omega_0}$, 则方程写为

$$\left(\frac{\partial W}{\partial x}\right)^2+ax\frac{\partial W}{\partial x}+\left(x^2-b\right)=0.$$

因为初始条件为 $p(0)=p_0,q_(0)=0$, 此时有 $a>0,b>0$. 方程 (6) 的解为

$$\frac{\partial W}{\partial x}=-\frac{ax}{2}\pm\sqrt{b-\left(1-\frac{a^2}{4}\right)x^2},$$

其中第二项符号任意

$$W=-\frac{ax^2}{4}+\int\sqrt{b-\left(1-\frac{a^2}{4}\right)x^2}\,\mathrm{d}x$$

讨论:

$$(I)a<2\text{ 或}\frac{\Omega}{2\omega_0}<1,\quad q(t)=\cos(\frac{\Omega t}{2})Q=\cos(\frac{\Omega t}{2})\frac{x}{\sqrt{m\omega_0}}=A\cos(\frac{\Omega t}{2})\sin(\omega t+\delta)$$

$$\dot{q}=\frac{p}{m},\quad p=p_0+-\frac{\cos\left(\delta+t\left(\omega-\frac{\Omega}{2}\right)\right)}{2\omega-\Omega}-\frac{\cos\left(\delta+t\omega+\frac{t\Omega}{2}\right)}{2\omega+\Omega}$$

$$(II)a=2\text{ 或}\frac{\Omega}{2\omega_0}=1,\quad q(t)=(At+B)\cos(\frac{\Omega t}{2})$$

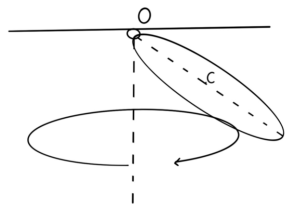
$$p=p_0+\frac{2\Omega(At+B)\sin\left(\frac{t\Omega}{2}\right)+4A\cos\left(\frac{t\Omega}{2}\right)}{\Omega^2}$$

$$(III)a > 2 \text{ 或} \frac{\Omega}{2\omega_0} > 1, \quad q(t) = A\cos(\frac{\Omega t}{2})\sinh(\omega_1 t + \delta_1)$$

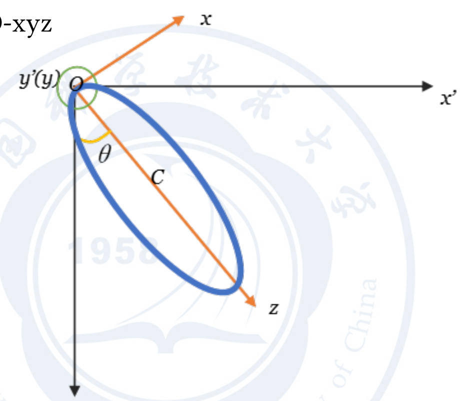
$$p = p_0 + \frac{A\left(2\Omega\sinh(t\omega(1))\sin\left(\frac{t\Omega}{2}\right) + 4\omega(1)\cosh(t\omega(1))\cos\left(\frac{t\Omega}{2}\right)\right)}{4\omega(1)^2 + \Omega^2}$$

七、如图, 质量为 m 半径为 a 的圆环绕圆环最上面 O 点旋转, O 点铰链连接固定, C 为圆环中心 OC 与竖直轴的夹角为 $60°$. 且圆环面始终垂直于 OC 和竖直轴形成的平面。

(1) 圆环绕 O 点旋转的角速度 Ω (2) 圆环旋转的角动量 (3) O 点作用于圆环的力



解: 如下图所示建立随动坐标系 O-xyz



将竖直向下的角速度分解到随动 (本体) 参考系方向, $\omega = \omega \boldsymbol{e_{z'}} = \omega\left(\frac{1}{2}\boldsymbol{e_z} - \frac{\sqrt{3}}{2}\boldsymbol{e_x}\right)$

随动参考系恰好是惯量主轴, 接着利用平行轴定理给出惯量张量:

$$I_{xx} = ma^2 + ma^2 = 2ma^2, \quad I_{yy} = ma^2 + \frac{3}{2}ma^2 = \frac{3}{2}ma^2, \quad I_{zz} = \frac{1}{2}ma^2,$$

$$I_{xy} = I_{yz} = I_{zx} = 0 \Rightarrow I = \begin{pmatrix} 2ma^2 & 0 & 0 \\ 0 & \frac{3}{2}ma^2 & 0 \\ 0 & 0 & \frac{1}{2}ma^2 \end{pmatrix}$$

$$\text{联立}\begin{cases} \text{Euler 动力学方程:} \ I_{yy}\omega_y - (I_{zz} - I_{xx})\,\omega_x\omega_y = M_y \\ \text{重力的力矩方程:} \ M_y = \overrightarrow{OC} \times m\boldsymbol{g} = -mga\sin\theta = -\frac{\sqrt{3}}{2}mga \end{cases}$$

$$\therefore \omega = \sqrt{\frac{4g}{3a}}, \qquad \boldsymbol{L} = I_{xx}\omega_x\boldsymbol{e_x} + I_{yy}\omega_y\boldsymbol{e_y} + I_{zz}\omega_z\boldsymbol{e_z} = ma^2\omega\left(-\sqrt{3}\boldsymbol{e_x} + \frac{1}{4}\boldsymbol{e_z}\right)$$

$$|\boldsymbol{L}| = ma^2\sqrt{\frac{4g}{3a}} \cdot \sqrt{3 + \frac{1}{4^2}} = \frac{7ma}{2}\sqrt{\frac{2g}{3}}$$

$$F_O = \sqrt{(mg)^2 + (mw^2 a\sin\theta)^2} = \sqrt{1^2 + \left(\frac{2\sqrt{3}}{3}\right)^2}\,mg = \frac{\sqrt{21}}{3}mg$$

八、用广义坐标 (θ,ϕ) 描述摆长为 l, 摆球质量为 m 的球面摆 (θ 为摆线与铅锤方向的夹角, ϕ 为摆球相对于悬挂点在水平面投影的方位角).

(1) 以 (θ,ϕ) 作为广义坐标写出球面摆的拉氏量, 用 Legendre 变换求出 Hamilton 量并写出相应的 Hamilton 方程

(2) 使用 Routh 变换得出等效拉氏量 $\widetilde{L}(\theta,\dot{\theta})$

(3) 对等效拉氏量 $\tilde{L}(\theta,\dot{\theta})$ 用 Legendre 变换求出等效 Hamilton 量 $\tilde{H}\left(\theta,p_\theta\right)$

(4) 用 Hamilton-Jacobi 方法求出等效 Hamilton 量 $\widetilde{H}\left(\theta,p_\theta\right)$ 的主函数 $S(t,\theta)$

解:

$$L=\frac{ml^2}{2}\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right)+mgl\cos\theta$$

$$p_\theta=\frac{\partial L}{\partial\dot{\theta}}=ml^2\dot{\theta},p_\phi=\frac{\partial L}{\partial\dot{\phi}}=ml^2\dot{\phi}\sin^2\theta(p_\phi,H\text{是运动积分})$$

$$H=p_\theta\dot{\theta}+p_\phi\dot{\phi}-L=\frac{1}{2ml^2}\left(p_\theta^2+\frac{p_\phi^2}{\sin^2\theta}\right)-mgl\cos\theta$$

$$\begin{aligned}R&=\dot{\phi}P_\phi-L\\&=ml^2\dot{\phi}^2\sin\theta-\frac{ml^2}{2}\left(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta\right)-mgl\cos\theta\\&=\frac{ml^2}{2}\left(\dot{\phi}^2\sin^2\theta-\dot{\theta}^2\right)-mgl\cos\theta=\widetilde{L}(\theta,\dot{\theta})\\&\frac{d}{dt}\frac{\partial R}{\partial\dot{\theta}}-\frac{\partial R}{\partial\theta}=0(\text{这是动力学方程})\\&\Rightarrow\frac{d}{dt}\left(-ml^2\dot{\theta}\right)=mgl\sin\theta\\&\Rightarrow\quad l\ddot{\theta}+g\sin\theta=0\end{aligned}$$

下面计算新变换下的动量和 Hamilton 量

$$\begin{aligned}&\frac{\partial\tilde{L}}{\partial\dot{\theta}}\Rightarrow\widetilde{P_\theta}=-ml\dot{\theta}\\&\widetilde{H}=\left(\theta,\tilde{P}_\theta\right)=\dot{\theta}\widetilde{P_\theta}-\widetilde{L}(\theta,\dot{\theta})\\&=\frac{ml^2}{2}\left(\dot{\theta}^2-\dot{\phi}^2\sin^2\theta\right)+mgl\cos\theta\\&=\frac{P_\theta^2}{2m}-\frac{ml^2\dot{\phi}^2\sin^2\theta}{2}+mgl\cos\theta\end{aligned}$$

$$\frac{1}{2m}\left(\frac{dW}{d\theta}\right)^2=\frac{ml^2\dot{\phi}^2}{2}\sin^2\theta+mgl\cos\theta.$$

$$W_\theta=\int^\theta\sqrt{m^2l^2\dot{\phi}^2\sin^2\theta+2m^2gl\cos\theta d\theta}$$

$$S(t,\theta)=-Et+A+\int^\theta\sqrt{m^2l^2\dot{\phi}^2\sin^2\theta+2m^2gl\cos\theta d\theta}$$

九、某粒子在一无限深方势阱中运动, 初始动量为 T_0, 势阱宽度为 a 。现将势阱宽度由 a 缓慢变化为 $2a$, 求其末态动能 T 。其势阱势能分布如下:

$$V=\begin{cases}0&,0<x<a\\\infty&,x<0,x>a\end{cases}$$

十、(1)(I) 已知势能函数 $U(x)=U_0\dfrac{x^2}{a^2}e^{-\frac{x}{a}}$, 求势能函数的极值点, (II) 在速度相空间 $(x,\dot{x})$ 中画出 $E=U_0, E=\dfrac{U_0}{3}, E=4e^{-2}U_0$ 的相轨迹, 并且标注箭头

(2) 粒子在竖直向下的重力场中做一维运动证明初始时刻四个相空间的点 (0, 0), (x, 0), (0, p), (x, p) 在经演化后面积不变

解:

$$\frac{\partial}{\partial x}\left(\frac{x^2e^{-x/a}}{a^2}\right)=\frac{xe^{-x/a}(2a-x)}{a^3}$$

故极值点在 $x=2a$ 处

十一、质量为 m 的拉格朗日陀螺, 即重力场中绕定点转动的对称陀螺的三个主转动惯量为: $I_1=I_2\neq I_3$, 其中质心到固定点的距离为 l 。

1. 写出选取三个欧拉角作为广义坐标时的刚体的哈密顿函数 $H(\varphi,\theta,\psi,P_\varphi,P_\theta,P_\psi)$;

2. 通过哈密顿 – 雅可比方程求积分形式的哈密度主函数 $S(\varphi,\theta,\psi,t;C_1,C_2,C_3)$, 其中 C_1,C_2,C_3 为运动积分。

3. 根据主函数 $S(\varphi,\theta,\psi,t;C_1,C_2,C_3)$, 试导出刚体积分形式的运动方程。

解:

$$L=\frac{I_1}{2}\left(\dot{\theta}^2+\dot{\varphi}^2\sin^2\theta\right)+\frac{I_3}{2}(\dot{\psi}+\dot{\varphi}\cos\theta)^2-mgl\cos\theta$$

广义动量为:

$$\begin{aligned}P_\theta&=I_1\dot{\theta}\\P_\varphi&=\left(I_1\sin^2\theta+I_3\cos^2\theta\right)\dot{\varphi}+I_3\cos\theta\dot{\psi}\\P_\psi&=I_3(\dot{\psi}+\cos\theta\dot{\varphi})\end{aligned}$$

系统存在三个运动积分: E,P_φ,P_ψ 。通过勒让德变换, 得到系统的哈密顿量:

$$H=\frac{1}{2I_1}\left[\frac{(P_\varphi-P_\psi\cos\theta)^2}{\sin^2\theta}+P_\theta^2\right]+\frac{P_\psi^2}{2I_3}+mgl\cos\theta$$

$$S=-Et+P_\varphi\varphi+P_\psi\psi+W_\theta(\theta)$$

代入哈密顿-雅可比方程, 得:

$$E=\frac{1}{2I_1}\left[\frac{(P_\varphi-P_\psi\cos\theta)^2}{\sin^2\theta}+\left(\frac{\partial W_\theta}{\partial\theta}\right)^2\right]+\frac{P_\psi^2}{2I_3}+mgl\cos\theta$$

求得:

$$\frac{dW_\theta}{d\theta}=\pm\sqrt{2I_1\left(E-\frac{P_\psi^2}{2I_3}-mgl\cos\theta\right)-\frac{(P_\varphi-P_\psi\cos\theta)^2}{\sin^2\theta}}\equiv\pm\sqrt{\Delta(\theta;E,P_\varphi,P_\psi)}$$

最终得到:

$$S=-Et+P_\varphi\varphi+P_\psi\psi\pm\int^\theta\sqrt{\Delta(\theta';E,P_\varphi,P_\psi)}d\theta'$$

由 $Q_\alpha=\dfrac{\partial S}{\partial P_\alpha}=\text{const.}$, P_α 分别取 (E,P_φ,P_ψ), 得到如下的运动方程:

$$\begin{aligned}
0=\frac{\partial S}{\partial E}&=-t\pm\int^\theta\frac{I_1}{\sqrt{\Delta\left(\theta';E,P_\varphi,P_\psi\right)}}d\theta'\\
0=\frac{\partial S}{\partial P_\varphi}&=\varphi\pm\int^\theta\frac{-\left(P_\varphi-P_\psi\cos\theta'\right)/\sin^2\theta'}{\sqrt{\Delta\left(\theta';E,P_\varphi,P_\psi\right)}}d\theta'\\
0=\frac{\partial S}{\partial P_\psi}&=\psi\pm\int^\theta\frac{-\frac{I_1}{I_3}P_\psi+\frac{\left(P_\varphi-P_\psi\cos\theta'\right)}{\sin^2\theta'}\cos\theta'}{\sqrt{\Delta\left(\theta';E,P_\varphi,P_\psi\right)}}d\theta'
\end{aligned}$$

十二、以向下为 x-轴的正方向, 讨论落体运动。

1. 设初始时刻的位移为 $x(t=0)=x_0$, 动量为 $p(t=0)=p_0$, 证明:

$$x(t),p(t)\to x_0,p_0$$

是正则变换。

2. 对此变换判断第三、四种生成函数是否存在; 若存在, 则求出各生成函数表达式和哈密顿函数 $K(t,x_0,p_0)$ 。

解:

$$\begin{cases}x=x_0+\dfrac{p_0}{m}t+\frac{1}{2}gt^2\\ p=p_0+mgt\end{cases}$$

$$\begin{cases}x_0=x-\dfrac{p}{m}t+\dfrac{1}{2}gt^2\\ p_0=p-mgt\end{cases}$$

判断可积性

$$\begin{aligned}
\delta F&=p\delta x-p_0\delta x_0=p\delta x-(p-mgt)\delta\left(x-\frac{p}{m}t+\frac{1}{2}gt^2\right)\\
&=mgt\delta x+\left(\frac{p}{m}t-gt^2\right)\delta p
\end{aligned}$$

即:

$$F=mgxt+\frac{p^2}{2m}t-pgt^2$$

所以是正则变换。

或者:

$$\frac{\partial\left(x_0,p_0\right)}{\partial(x,p)}=1$$

因此是正则变换。第三类生成函数 $F_3\left(p,x_0,t\right)$ 变元独立且

$$\frac{\partial x_0}{\partial x}=1\neq 0$$

所以 $F_3\left(p,x_0,t\right)$ 存在, 生成函数为

$$\left.\begin{aligned}F_3\left(t,p,x_0\right)=F-xp=mgxt\quad&+\tfrac{p^2}{2m}t-pgt^2-xp\\ x\quad&=x_0+\tfrac{p}{m}t-\tfrac{1}{2}gt^2\end{aligned}\right\}\Rightarrow$$

$$F_3\left(t,p,x_0\right)=-\frac{p^2}{2m}t+\frac{1}{2}gt^2p-px_0+mgtx_0-\frac{1}{2}mg^2t^3$$

旧的哈密顿函数为:

$$H=\frac{p^2}{2m}-mgx$$

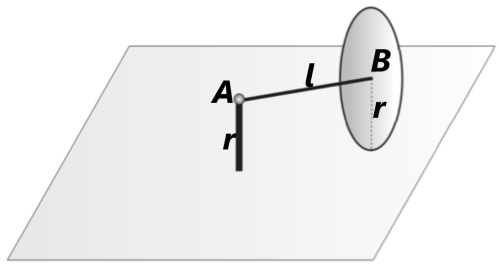
新的哈密顿函数为:

$$\widetilde{H}=H+\frac{\partial F_3}{\partial t}=-mg^2t^2$$

第四类生成函数 $F_4(p,p_0,t)$

$$\frac{\partial p_0}{\partial x}=0$$

所以第四类生成函数



十三、如图所示, 质量为 m, 半径为 r 的匀质圆盘竖直放置于水平面上, 圆盘中心 B 通过长为 l 并垂直于盘面的轻杆光滑连接到固定点 A, A 点高出水平面的距离也为 r, 这样 AB 始终平行于水平面, 盘面始终保持竖直。当圆盘在水平面上作无滑动滚动时, 其中心 B 绕着 A 点作圆周运动。如果圆盘中心 B 围绕 A 点以大小为 ω_0 的角速度作匀速圆周运动, 求圆盘对水平面施加的力 (只需求出力的大小即可)。

解: 先建立坐标系。在某时刻实验室坐标系与随体惯性系重合。选 A 为坐标原点, $\overrightarrow{AB}$ 轴为 z 轴, y 轴垂直向上, $A-xyz$ 形成右手坐标系。根据无滑滚动条件, 考虑 B 点的速度, 得到自转角速度 ω 为:

$$\omega_0 l=\omega r\Rightarrow\omega=\frac{l}{r}\omega_0$$

刚体的三个惯量主轴分别为:

$$I_3=\frac{1}{2}mr^2,\quad I_1=I_2=\frac{1}{4}mr^2+ml^2$$

刚体的角速度的角加速度分别为:

$$\omega_1=0,\quad\omega_2=\omega_0,\quad\omega_3=-\omega=-\frac{l}{r}\omega_0$$

方法一: 在实验室坐标系中讨论

$$\boldsymbol{N}=\frac{d\boldsymbol{L}}{dt}=\frac{d}{dt}\left[I_2\omega_2\boldsymbol{e}_{y'}+I_3\omega_3\boldsymbol{e}_{z'}\right]=I_3\omega_3\frac{d}{dt}\boldsymbol{e}_{z'}=I_3\omega_3\omega_2\boldsymbol{e}_{x'}=-\frac{1}{2}mrl\omega_0^2\boldsymbol{e}_{x'}$$

由 $N_1=-F_2l-rF_3,\quad F_2=N-mg,\quad F_3=0$, 得: $N=\frac{1}{2}mr\omega_0^2+mg$

方法二: 在随动惯性系中讨论

$$\text{由角速度在随动惯性系分量,}\begin{cases}\omega_1=\dot{\theta}\cos\psi+\dot{\varphi}\sin\theta\sin\psi\\ \omega_2=-\dot{\theta}\sin\psi+\dot{\varphi}\sin\theta\cos\psi\\ \omega_3=\dot{\psi}+\dot{\varphi}\cos\theta\end{cases}$$

$$\Rightarrow\dot{\omega}_1=\omega_2\omega_3,\quad\dot{\omega}_2=\dot{\omega}_3=0$$

代入欧拉方程:

$$
\begin{aligned}
&N_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\,\omega_2\omega_3 = I_3\omega_2\omega_3 = -\frac{1}{2}mrl\omega_0^2 \\
&N_2 = N_3 = 0
\end{aligned}
$$

由 $N_1 = -F_2 l - rF_3, \quad F_2 = N - mg, \quad F_3 = 0$, 得:

$$
N = \frac{1}{2}mr\omega_0^2 + mg
$$

十四、位于坐标原点具有磁荷 b 的磁单极子产生磁场, 其矢势在球坐标系 (r, θ, φ) 中可以表示为 $\boldsymbol{A} = \dfrac{b(1-\cos\theta)}{r\sin\theta}\boldsymbol{e}_\varphi$, 其中 $\boldsymbol{e}_\varphi$ 为方位角的单位矢量。考虑一个电荷为 e 、质量为 m 的粒子在该磁场中的运动。

1. 写出球坐标系中该带电粒子的拉格朗日函数的具体表达式, 并找出至少两个独立的运动积分;
2. 求直角坐标系 (x, y, z) 中相应的哈密顿函数 H 和关于第三个自由度 (z, p_z) 的哈密顿正则方程

解:

$$
\begin{aligned}
L &= \frac{1}{2}mv^2 - e(\phi - \boldsymbol{v}\cdot\boldsymbol{A}) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) + ev_\varphi A_\varphi \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) + eb(1-\cos\theta)\dot{\varphi}
\end{aligned}
$$

运动积分:

$$
p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mr^2\sin^2\theta\dot{\varphi} + eb(1-\cos\theta)
$$

$$
H = p_r\dot{r} + p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right)
$$

第二小问比较考验数学工具的应用: 直角坐标系与球坐标系的关系有 $\dfrac{y}{x} = \tan\varphi$,
所以有 $\dot{\varphi} = \dfrac{x\dot{y} - \dot{x}y}{x^2+y^2}$ 。因此直角坐标系下拉格朗日函数为:

$$
L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + eb\frac{r-z}{r}\frac{x\dot{y} - \dot{x}y}{x^2+y^2} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{eb}{r}\frac{x\dot{y} - \dot{x}y}{r+z}
$$

其中 $r = \sqrt{x^2+y^2+z^2}$ 。可得

$$
p_x = m\dot{x} - \frac{eb}{r(r+z)}y, p_y = m\dot{y} + \frac{eb}{r(r+z)}x, p_z = m\dot{z}
$$

$$
H = \frac{1}{2m}\left[\left(p_x + \frac{eb}{r(r+z)}y\right)^2 + \left(p_y - \frac{eb}{r(r+z)}x\right)^2 + p_z^2\right]
$$

得到正则方程:

$$
\begin{aligned}
\dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \\
\dot{p}_z &= -\frac{\partial H}{\partial z} = -\frac{eb}{m}\left[\left(p_x + \frac{eb}{r(r+z)}y\right)y - \left(p_y - \frac{eb}{r(r+z)}x\right)x\right]\frac{\partial}{\partial z}\frac{1}{r(r+z)} \\
&= \frac{eb}{m}\left[yp_x - xp_y + \frac{eb(r-z)}{r}\right]\frac{z(2r+z)+r^2}{r^3(r+z)^2} \\
&= \frac{eb}{mr^3}\left[yp_x - xp_y + \frac{eb(r-z)}{r}\right]
\end{aligned}
$$

十五、讨论质量为 m 的人造卫星围绕地球 (质量为 M) 作束缚态轨道运动。假设地球非球对称质量分布所引起的引力势为高阶小量 (一般作球谐函数展开), 显然卫星的轨道近似为椭圆。设卫星的轨道半长轴为 a, 引入无量纲化的卫星的角动量和 Runge-Lenz 矢量 (偏心率矢量):

$$\boldsymbol{j}=\frac{1}{\sqrt{GMa}}\boldsymbol{r}\times\boldsymbol{p},\quad \boldsymbol{e}=\frac{1}{GM}\boldsymbol{p}\times(\boldsymbol{r}\times\boldsymbol{p})-\frac{\boldsymbol{r}}{r}$$

试证明:

1. $[x_i,L_j]=\varepsilon_{ijk}x_k;\quad [p_i,L_j]=\varepsilon_{ijk}p_k;\quad [L_i,L_j]=\varepsilon_{ijk}L_k$; 其中 $\boldsymbol{L}=\boldsymbol{r}\times\boldsymbol{p}$;
2. $[j_i,j_j]=\frac{1}{\sqrt{GMa}}\varepsilon_{ijk}j_k;\quad [j_i,e_j]=\frac{1}{\sqrt{GMa}}\varepsilon_{ijk}e_k;\quad [e_i,e_j]=\frac{1}{\sqrt{GMa}}\varepsilon_{ijk}j_k$ 。

解: 第一种方法:

$$[x_1,L_1]=[x_1,x_2p_3-x_3p_2]=0$$

$$[x_1,L_2]=[x_1,x_3p_1-x_1p_3]=[x_1,x_3p_1]=x_3$$

$$[x_1,L_3]=[x_1,x_1p_2-x_2p_1]=-[x_1,x_2p_1]=-x_2$$

同理可证:

$$[x_2,L_1]=-x_3,\quad [x_2,L_2]=0,\quad [x_2,L_3]=x_1$$

$$[x_3,L_1]=x_2,\quad [x_3,L_2]=-x_1,\quad [x_3,L_3]=0$$

即:

$$[x_i,L_j]=\varepsilon_{ijk}x_k$$

$$[p_1,L_1]=[p_1,x_2p_3-x_3p_2]=0$$

$$[p_1,L_2]=[p_1,x_3p_1-x_1p_3]=-p_3\,[p_1,x_1]=p_3$$

$$[p_1,L_3]=[p_1,x_1p_2-x_2p_1]=p_2\,[p_1,x_1]=-p_2$$

同理可证:

$$[p_2,L_1]=-p_3,\quad [p_2,L_2]=0,\quad [p_2,L_3]=p_1$$

$$[p_3,L_1]=p_2,\quad [p_3,L_2]=-p_1,\quad [p_3,L_3]=0$$

即:

$$[p_i,L_j]=\varepsilon_{ijk}p_k$$

$$[L_1,L_1]=[x_2p_3-x_3p_2,x_2p_3-x_3p_2]$$

$$=[x_2p_3,-x_3p_2]-[x_3p_2,x_2p_3]=0$$

$$[L_1,L_2]=[x_2p_3-x_3p_2,x_3p_1-x_1p_3]$$

$$=[x_2p_3,x_3p_1]+[x_2p_3,-x_1p_3]+[-x_3p_2,x_3p_1]+[x_3p_2,x_1p_3]$$

$$=x_2p_1\,[p_3,x_3]+0+0+x_1p_2\,[x_3,p_3]$$

$$=-x_2p_1+x_1p_2=L_3$$

同理可证:

$$[L_1,L_3]=-L_2,\quad [L_2,L_3]=L_1$$

即:

$$[L_i,L_j]=\varepsilon_{ijk}L_k$$

$$
\begin{aligned}
[j_i, j_j] &= \frac{1}{\sqrt{GMa}}\frac{1}{\sqrt{GMa}}[L_i, L_j] \\
&= \frac{1}{\sqrt{GMa}} \cdot \frac{1}{\sqrt{GMa}}\varepsilon_{ijk}L_k \\
&= \frac{1}{\sqrt{GMa}}\varepsilon_{ijk}j_k
\end{aligned}
$$

$$
[j_1, e_1] = \frac{1}{\sqrt{GMa}}\left[L_1, \frac{1}{GM}(p_2L_3 - p_3L_2) - \frac{x_1}{r}\right]
$$

$$
\begin{aligned}
[L_1, p_2L_3 - p_3L_2] &= [L_1, p_2L_3] - [L_1, p_3L_2] \\
&= [L_1, p_2]L_3 + p_2[L_1, L_3] - [L_1, p_3]L_2 - p_3[L_1, L_2] \\
&= p_3L_3 - p_2L_2 + p_2L_2 - p_3L_3 = 0
\end{aligned}
$$

$$
\left[L_1, \frac{x_1}{r}\right] = \frac{1}{r}[L_1, x_1] + x_1\left[L_1, \frac{1}{r}\right] = x_1\left[L_1, \frac{1}{r}\right]
$$

$$
\begin{aligned}
\left[L_1, \frac{1}{r}\right] =& \frac{\partial}{\partial x_1}\left(\frac{1}{r}\right)[L_1, x_1] + \frac{\partial}{\partial x_2}\left(\frac{1}{r}\right)[L_1, x_2] + \frac{\partial}{\partial x_3}\left(\frac{1}{r}\right)[L_1, x_3] \\
=& 0 + \left(-\frac{1}{r^3}\right)x_2x_3 + \left(-\frac{1}{r^3}\right)x_3(-x_2) = 0
\end{aligned}
$$

同理可证:

$$
\left[L_2, \frac{1}{r}\right] = \left[L_3, \frac{1}{r}\right] = 0
$$

因此:

$$
[j_1, e_1] = 0
$$

$$
[j_1, e_2] = \frac{1}{\sqrt{GMa}}\left[L_1, \frac{1}{GM}(p_3L_1 - p_1L_3) - \frac{x_2}{r}\right]
$$

先计算:

$$
[L_1, p_3L_1 - p_1L_3] = [L_1, p_3]L_1 - p_1[L_1, L_3] = -p_2L_1 + p_1L_2 = (\boldsymbol{p} \times \boldsymbol{L})_3
$$

$$
\left[L_1, -\frac{x_2}{r}\right] = -x_2\left[L_1, \frac{1}{r}\right] + \frac{1}{r}[L_1, -x_2] = -\frac{x_3}{r}
$$

即:

$$
[j_1, e_2] = e_3
$$

同理可证:

$$
[j_1, e_3] = -e_2, \quad [j_2, e_3] = e_1
$$

即:

$$
[j_i, e_j] = \varepsilon_{ijk}e_k
$$

$$
[e_1, e_2] = \left[\frac{1}{GM}(p_2L_3 - p_3L_2) - \frac{x_1}{r}, \frac{1}{GM}(p_3L_1 - p_1L_3) - \frac{x_2}{r}\right]
$$

先计算:

$$
\begin{aligned}
&[p_2L_3-p_3L_2,p_3L_1-p_1L_3]\\
=&[p_2L_3,p_3L_1]-[p_2L_3,p_1L_3]-[p_3L_2,p_3L_1]+[p_3L_2,p_1L_3]\\
=&p_2\left[L_3,p_3L_1\right]+\left[p_2,p_3L_1\right]L_3-p_2\left[L_3,p_1L_3\right]-\left[p_2,p_1L_3\right]L_3\\
&-p_3\left[L_2,p_3L_1\right]-\left[p_3,p_3L_1\right]L_2+p_3\left[L_2,p_1L_3\right]+\left[p_3,p_1L_3\right]L_2\\
=&p_2p_3L_2+\left(-p_3\right)p_3L_3-p_2^2L_3-p_1^2L_3\\
&-p_3\left[L_2,p_3\right]L_1-p_3^2\left[L_2,L_1\right]-p_2p_3L_2+p_3\left[L_2,p_1\right]L_3+p_3p_1\left[L_2,L_3\right]\\
=&p_2p_3L_2-p_3^2L_3-p_2^2L_3-p_1^2L_3\\
&-p_1p_3L_1+p_3^2L_3-p_2p_3L_2+p_3\left(-p_3\right)L_3+p_1p_3L_1\\
=&-p^2L_3
\end{aligned}
$$

继续计算:

$$
\begin{aligned}
&\left[p_2L_3-p_3L_2,\frac{x_2}{r}\right]\\
=&\left[p_2L_3-p_3L_2,x_2\right]\frac{1}{r}+\left[p_2L_3-p_3L_2,\frac{1}{r}\right]x_2\\
=&\left\{-L_3-p_2x_1\right\}\frac{1}{r}+\left\{L_3\left[p_2,\frac{1}{r}\right]-L_2\left[p_3,\frac{1}{r}\right]\right\}x_2\\
=&\left\{-L_3-x_1p_2\right\}\frac{1}{r}+\frac{1}{r^3}\left\{x_2L_3-x_3L_2\right\}x_2
\end{aligned}
$$

$$
\begin{aligned}
&\left[\frac{x_1}{r},p_3L_1-p_1L_3\right]\\
=&\frac{1}{r}\left[x_1,p_3L_1-p_1L_3\right]+x_1\left[\frac{1}{r},p_3L_1-p_1L_3\right]\\
=&\frac{1}{r}\left\{-L_3+x_2p_1\right\}+x_1\left\{L_1\left[\frac{1}{r},p_3\right]-L_3\left[\frac{1}{r_1},p_1\right]\right\}\\
=&\frac{1}{r}\left\{-L_3+x_2p_1\right\}+x_1\frac{1}{r^3}\left\{-L_1x_3+L_3x_1\right\}
\end{aligned}
$$

两式相加:

$$
\begin{aligned}
&\frac{1}{r}\left\{-2L_3+p_1x_2-p_2x_1\right\}+\frac{1}{r^3}\left\{\left(x_1^2+x_2^2\right)L_3-x_1x_3L_1-x_2x_3L_2\right\}\\
=&\frac{1}{r}\left\{-3L_3\right\}+\frac{1}{r^3}\left\{\left(x_1^2+x_2^2\right)L_3-x_1x_3L_1-x_2x_3L_2\right\}\\
=&-\frac{2}{r}L_3-\frac{x_3}{r^3}\left\{x_3L_3+x_1L_1+x_2L_2\right\}\\
=&-\frac{2}{r}L_3-\frac{x_3}{r^3}\boldsymbol{r}\cdot\boldsymbol{L}
\end{aligned}
$$

因此,

$$
\begin{aligned}
[e_1,e_2]&=\frac{1}{(GM)^2}\left(-p^2L_3\right)+\frac{1}{GM}\left\{\frac{2}{r}L_3+\frac{x_3}{r^3}\boldsymbol{r}\cdot\boldsymbol{L}\right\}\\
&=\frac{L_3}{(GM)^2}\left(-p^2+\frac{2GM}{r}\right)+\frac{1}{GM}\cdot\frac{x_3}{r^3}(\boldsymbol{r}\cdot\boldsymbol{L})\\
&=\frac{L_3}{(GM)^2}(-2E)+\frac{1}{GM}\frac{x_3}{r^3}(\boldsymbol{r}\cdot\boldsymbol{L})
\end{aligned}
$$

利用:

$$E=-\frac{GM}{2a},\quad \boldsymbol{r}\cdot\boldsymbol{L}=0$$

可得:

$$[e_1,e_2]=\frac{L_3}{GMa}=\frac{1}{\sqrt{GMa}}j_3$$

同理可证:

$$[e_1,e_3]=-\frac{1}{\sqrt{GMa}}j_2,\quad [e_2,e_3]=\frac{1}{\sqrt{GMa}}j_1$$

即:

$$[e_i,e_j]=\frac{1}{\sqrt{GMa}}\varepsilon_{ijk}j_k$$

第二种方法:

$$\begin{aligned}
[x_i,L_j]&=[x_i,\varepsilon_{jlm}x_lp_m]\\
&=\frac{\partial}{\partial p_i}[\varepsilon_{jlm}x_lp_m]\\
&=\varepsilon_{jlm}x_l\delta_{im}\\
&=\varepsilon_{jli}x_l=\varepsilon_{ijl}x_l\\
[p_i,L_i]&=-\frac{\partial}{\partial x_i}L_j=-\frac{\partial}{\partial x_i}\varepsilon_{jlm}x_lp_m\\
&=-\varepsilon_{jlm}p_m\delta_{il}\\
&=-\varepsilon_{jim}p_m=\varepsilon_{ijm}p_m
\end{aligned}$$

$$\begin{aligned}
[L_i,L_j]&=[\varepsilon_{ilm}x_lp_m,L_j]=\varepsilon_{ilm}x_l[p_m,L_j]+\varepsilon_{ilm}[x_l,L_j]p_m\\
&=\varepsilon_{ilm}\varepsilon_{mjn}x_lp_n+\varepsilon_{ilm}\varepsilon_{ljn}x_np_m\\
&=\varepsilon_{ilm}\varepsilon_{mjn}x_lp_n+\varepsilon_{imn}\varepsilon_{mjl}x_lp_n\\
&=(\varepsilon_{mil}\varepsilon_{mjn}+\varepsilon_{mni}\varepsilon_{mjl})x_lp_n\\
&=(\delta_{ij}\delta_{ln}-\delta_{in}\delta_{lj}+\delta_{nj}\delta_{il}-\delta_{nl}\delta_{ij})x_lp_n\\
&=(\delta_{il}\delta_{nj}-\delta_{in}\delta_{lj})x_lp_n\\
&=\varepsilon_{ijk}\varepsilon_{lnk}x_lp_n\\
&=\varepsilon_{ijk}L_k
\end{aligned}$$

$$\begin{aligned}
[j_i,j_j]&=\frac{1}{\sqrt{GMa}}\frac{1}{\sqrt{GMa}}[L_i,L_j]\\
&=\frac{1}{\sqrt{GMa}}\cdot\frac{1}{\sqrt{GMa}}\varepsilon_{ijk}L_k\\
&=\frac{1}{\sqrt{GMa}}\varepsilon_{ijk}j_k\\
[j_i,e_j]&=\frac{1}{\sqrt{GMa}}\left[L_i,\frac{1}{GM}\varepsilon_{jlm}p_lL_m-\frac{x_j}{r}\right]\\
&=\frac{1}{\sqrt{GMa}}\frac{1}{GM}[L_i,\varepsilon_{jlm}p_lL_m]-\frac{1}{\sqrt{Gma}}\left[L_i,\frac{x_j}{r}\right]
\end{aligned}$$

分别计算第一项:

$$
\begin{aligned}
&[L_i, \varepsilon_{jlm} p_l L_m] = \varepsilon_{jlm}\left[L_i, p_l\right] L_m + \varepsilon_{jlm} p_l \left[L_i, L_m\right] \\
&= \varepsilon_{jlm}\varepsilon_{iln} p_n L_m + \varepsilon_{jlm}\varepsilon_{imn} p_l L_n \\
&= \left(\varepsilon_{jmn}\varepsilon_{iml} + \varepsilon_{jlm}\varepsilon_{imn}\right) p_k L_n \\
&= \left(\delta_{ji}\delta_{nl} - \delta_{jl}\delta_{ni} + \delta_{jn}\delta_{li} - \delta_{ji}\delta_{ln}\right) p_l L_n \\
&= \left(\delta_{jn}\delta_{li} - \delta_{jl}\delta_{ni}\right) p_l L_n \\
&= \varepsilon_{ijk}\varepsilon_{kln} p_l L_n = \varepsilon_{ijk}(\boldsymbol{p} \times \boldsymbol{L})_k
\end{aligned}
$$

第二项:

$$
\begin{aligned}
&\left[L_i, \frac{x_i}{r}\right] = \frac{1}{r}\left[L_i, x_j\right] + x_j \left[L_i, \frac{1}{r}\right] \\
=&\frac{1}{r}\varepsilon_{ijk} x_k + x_j \left(-\frac{\partial L_i}{\partial p_k}\right)\left(\frac{\partial}{\partial x^k}\frac{1}{r}\right) \\
=&\varepsilon_{ijk}\frac{x_k}{r} + x_j \cdot \frac{1}{r^3} \cdot x_k \frac{\partial L_i}{\partial p_k} \\
=&\varepsilon_{ijk}\frac{x_k}{r} + x_j \frac{1}{r^3} \cdot x_k \left[x_k, L_i\right] \\
=&\varepsilon_{ijk}\frac{x_k}{r} + x_j \cdot \frac{1}{r^3}\varepsilon_{ilk} x_l x_k \\
=&\varepsilon_{ijk}\frac{x_k}{r}
\end{aligned}
$$

因此,

$$
\begin{aligned}
\left[j_i, e_j\right] &= \frac{1}{\sqrt{GMa}}\frac{1}{GM}\left[L_i, \varepsilon_{jlm} p_l L_m\right] - \frac{1}{\sqrt{Gma}}\left[L_i, \frac{x_j}{r}\right] \\
&= \frac{1}{\sqrt{GMa}}\left(\frac{1}{GM}\varepsilon_{ijk}(\boldsymbol{p} \times \boldsymbol{L})_k - \varepsilon_{ijk}\frac{x_k}{r}\right) \\
&= \varepsilon_{ijk} e_k
\end{aligned}
$$

理论力学：原理公式题型

程嘉杰整理

文件网址：home.ustc.edu.cn/∼jiajie

目录

知识点 1 Lagrange 力学

第 1 节 位形空间

1. 自由度：系统独立变量数
2. 约束：分为完整约束 $f\left(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N; t\right)=0$ 只与位移有关
   不完整约束：$f(\vec{r}, \dot{\vec{r}}, t)=0$, 不可积分成完整约束，可以使用 Lagrange 乘子法
3. 广义坐标：
   $$\left\{q^\alpha\right\}, \quad \alpha=1,2, \cdots, s \tag{1.1}$$
   张成了一个 s 维空间，即系统位形空间，广义速度为 $v^\alpha=\dfrac{\mathrm{d} q^\alpha(t)}{\mathrm{d} t} \equiv \dot{q}^\alpha$
   可以选为长度、角度、能量和角动量量纲。位形空间是广义坐标张成的空间（s 维）
   注 1.1. 若 N 个质点组成的质点系只存在 k 个完整约束, 则系统的自由度为 $s=3N-k$.
4. 度规（矩阵）$g_{\mu\nu}$ 表示流形上线元
   $$\mathrm{d} s^2=g_{\mu \nu} \mathrm{d} x^\mu \mathrm{d} x^\nu \tag{1.2}$$

第 2 节 变分法

泛函：函数到数的映射，绝大多数情况下，物理中遇到的泛函都可写出积分形式

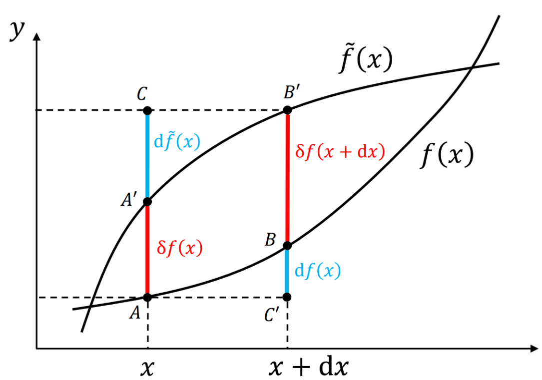
$$S[f]=\int_{x_1}^{x_2} \mathrm{d} x L\left(x, f(x), f^{\prime}(x), f^{\prime \prime}(x), \cdots, f^{(n)}(x), \cdots\right) \tag{1.3}$$

性质 1.1. 变分：

1. 类似于微分的求导性质

$$\begin{aligned}
\delta f &= f[y(x)+\delta y(x)]-f[y(x)] \\
\delta\left(f_1+f_2\right) &= \delta f_1+\delta f_2 \\
\delta\left(f_1 f_2\right) &= f_1 \delta f_2+\delta f_1 f_2 \\
\delta\left(\frac{f_1}{f_2}\right) &= \frac{f_2 \delta f_1-f_1 \delta f_2}{f_2^2}
\end{aligned} \tag{1.4}$$

变分、微分区别



2. 等时变分, 变分和微分可交换次序

$$\delta(d y)=d(\delta y) \tag{1.5}$$

$$\text{宗量（自变量）的变分引起泛函变分:}\quad f(x) \to \tilde{f}(x) = f(x) + \varepsilon\delta f(x) \tag{1.6}$$

于是泛函的变分为：

$$S[\tilde{f}] = S[f] + \varepsilon\delta S[f] + \frac{\varepsilon^2}{2}\delta^2 S[f] + \frac{\varepsilon^3}{3!}\delta^3 S[f] + \cdots \tag{1.7}$$

注 1.2. 一阶泛函导数的作用：

将函数的变分 $\delta f(x)$（一个无穷小的函数）映射到泛函的变分 δS（一个无穷小的数）。

例 1.1. 变分的计算

$$\begin{aligned}\delta S[f] &= \int_{x_1}^{x_2} \mathrm{d}x\delta L\left(x, f(x), f'(x), \cdots, f^{(n)}(x), \cdots\right). \\ &= \int_{x_1}^{x_2} \mathrm{d}x\left(\frac{\partial L}{\partial f}\delta f + \frac{\partial L}{\partial f'}\delta f' + \cdots + \frac{\partial L}{\partial f^{(n)}}\delta f^{(n)} + \cdots\right).\end{aligned} \tag{1.8}$$

一阶泛函导数

$$\frac{\partial L}{\partial f'}\delta f' \equiv \frac{\partial L}{\partial f'}\delta\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) = \frac{\partial L}{\partial f'}\frac{\mathrm{d}(\delta f)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\delta f\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right)\delta f. \tag{1.9}$$

所以代入原积分式子第一项

$$\begin{aligned}\int_{x_1}^{x_2} \mathrm{d}x\frac{\partial L}{\partial f'}\delta f' &= \int_{x_1}^{x_2} \mathrm{d}x\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\delta f\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right)\delta f\right] \\ &= \left(\frac{\partial L}{\partial f'}\delta f\right)\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \mathrm{d}x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right)\delta f\end{aligned} \tag{1.10}$$

边界项若为零，则

$$\frac{\partial L}{\partial f'}\delta f' \simeq -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right)\delta f \tag{1.11}$$

$$\frac{\partial L}{\partial f^{(n)}}\delta f^{(n)} \simeq (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\partial L}{\partial f^{(n)}}\right)\delta f \tag{1.12}$$

最终得到：

$$\frac{\delta S[f]}{\delta f} = \sum_{n=0}(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\partial L}{\partial f^{(n)}}\right) = \frac{\partial L}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial f'}\right) + \cdots + (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\partial L}{\partial f^{(n)}}\right) + \cdots \tag{1.13}$$

例 1.2. 泛函导数的操作定义

$$\delta S[f] = \int \mathrm{d}x\frac{\delta S}{\delta f(x)}\delta f(x) \equiv \frac{\mathrm{d}}{\mathrm{d}\varepsilon}S[\phi + \varepsilon\delta f]\bigg|_{\varepsilon=0} \tag{1.14}$$

第 3 节　最小作用量原理

对于所有自然现象，作用量趋于最小值

$$S = \int_{t_0}^{t_1}(T - V)dt \tag{1.15}$$

定义一般拉氏量 Lagrangian： $\mathcal{L} = T - V$

1.3.1 Euler-Lagrange 方程

对 Lagrangian 拉氏量 $\mathcal{L}$ 的作用量进行泛函取极值，由例1.1

$$
\begin{aligned}
S[y(x)] &= \int_{x_1}^{x_2} f(y(x),\dot{y}(x),x)dx \\
\delta S &= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial \dot{y}}+Q_\alpha\right)\delta y dx = 0 \\
&\Rightarrow \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}-\frac{\partial f}{\partial y} = Q_\alpha
\end{aligned} \tag{1.16}
$$

Q_α 代表非保守力 (惯性力）的部分，当非保守力为 0 时

$$
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}-\frac{\partial \mathcal{L}}{\partial q_\alpha}=0 \tag{1.17}
$$

也可以用 d'Alembert 原理推导给定

$$
\begin{aligned}
\delta S &= 0 \\
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}-\frac{\partial \mathcal{L}}{\partial q_\alpha}&=0
\end{aligned} \tag{1.18}
$$

例 1.3. 最速降线、*Fermat* 原理

$$
\begin{aligned}
t_{12} &= \int_{x_1}^{x_2}\frac{ds}{v} = \int_{x_1}^{x_2}\sqrt{\frac{1+\dot{y}^2}{2gy}}dx \\
t &= \int_0^L \frac{ds}{v} = \int_0^L \frac{n(x)}{c}\sqrt{1+\dot{y}^2}dx
\end{aligned}
$$

例 1.4. 悬链线势能、短程线和最小回转面积

$$
\begin{aligned}
V &= \int_{x_A}^{x_B}\rho g y ds = \rho g\int_{x_A}^{x_B} y\sqrt{1+\dot{y}^2}dx \\
l &= \int_A^B \sqrt{\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2} \\
S &= 2\pi\int_{x_0}^{x_1} y ds = 2\pi\int_{x_0}^{x_1} y\sqrt{1+(\frac{dy}{dx})^2}dx
\end{aligned}
$$

例 1.5. *Atwood* 机、单双摆问题

$$
\begin{aligned}
&T=\frac{1}{2}(m_1+m_2)\dot{x}^2, \quad V=-mgx-m_2g(l-x) \\
&\frac{\partial L}{\partial x}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}=0 \quad \Rightarrow \quad (m_1+m_2)\ddot{x}=(m_1-m_2)g
\end{aligned}
$$

非保守力下 Euler-Lagrange 方程

例 1.6. 有心力场下受阻尼 $f=-kv$

$$
\begin{aligned}
&\mathcal{L}=T-V=\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\frac{GMm}{r} \\
&\because v=\dot{r}\hat{e_r}+r\dot{\theta}\hat{e_\theta} \\
&\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}-\frac{\partial \mathcal{L}}{\partial r}=m\ddot{r}-mr\dot{\theta}^2+\frac{GMm}{r^2}=-kv_r=-k\dot{r} \\
&\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}}-\frac{\partial \mathcal{L}}{\partial \theta}=mr^2\ddot{\theta}+2mr\dot{r}\dot{\theta}=-kv_\theta=-kr\dot{\theta}
\end{aligned}
$$

第 4 节 d'Alembert 原理

理想约束系统所受主动力和惯性力产生的总虚功为零

$$\sum_{\alpha}\left(\vec{F}_{\alpha}-m_{\alpha}\ddot{\vec{x}}_{\alpha}\right)\cdot\delta\vec{x}_{\alpha}=0 \tag{1.19}$$

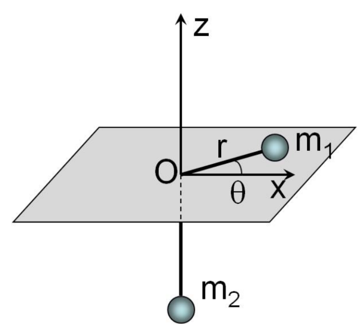
1. **虚功原理**

$$\delta W=\sum\left(\vec{F}_i+\vec{R}_i\right)\cdot\delta\vec{r}_i \tag{1.20}$$

2. **理想约束满足**

$$\sum\vec{R}_i\cdot\delta\vec{r}_i=0,\quad \delta W=\sum\left(\vec{F}_i\right)\cdot\delta\vec{r}_i \tag{1.21}$$

例 1.7. *连线穿孔两小球的运动自由度为 2，广义坐标选为：r; θ*



$$F_1=-m_1g\hat{e}_z,\quad F_2=-m_2g\hat{e}_z$$

$$r_1=r\hat{e}_r,\quad r_2=z\hat{e}_z=-(l-r)\hat{e}_z$$
$$\delta r_1=\delta r\hat{e}_r+r\delta\theta\hat{e}_\theta,\quad \delta r_2=\delta z\hat{e}_z=\delta r\hat{e}_z$$
$$\ddot{r_1}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+(r\ddot{\theta}+2\dot{r}\dot{\theta})\hat{e}_\theta,\quad \ddot{r_2}=\ddot{r}\hat{e}_z$$

由 d'Alembert 原理, 主动力和惯性力虚功和为 0

$$(-m_2g\hat{e}_z-m_2\ddot{r}\hat{e}_z)\cdot\delta r\hat{e}_z+\left\{-m_1g\hat{e}_r-m_1\left[\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+(r\ddot{\theta}+2\dot{r}\dot{\theta})\hat{e}_\theta\right]\right\}\cdot(\delta r\hat{e}_r+r\delta\theta\hat{e}_\theta)=0$$

$$\Rightarrow(m_1+m_2)\ddot{r}-m_1r\dot{\theta}^2+m_2g=0$$
$$\Rightarrow r\ddot{\theta}+2\dot{r}\dot{\theta}=0$$

例 1.8. *在无摩擦的光滑水平面内有一半径为 R、圆心固定的圆盘，以角速度 ω 绕圆心转动，在转盘边缘挂了一长度为 l、质量可以忽略的轻杆，杆的末端挂了一质量为 m 的小球（像个单摆）*

$$x=R\cos\omega t+l\cos(\omega t+\theta),\quad y=R\sin\omega t+l\sin(\omega t+\theta)$$

$$\dot{x}=-\mathrm{R}\omega\sin\omega t-\ell(\omega+\dot{\theta})\sin(\omega t+\theta),\quad \dot{y}=R\omega\cos\omega t+\ell(\omega+\dot{\theta})\cos(\omega t+\theta)$$
$$\ddot{x}=-\omega^2R\cos\omega t-\omega^2l\cos(\theta+\omega t)-\dot{\theta}\omega l\cos(\theta-\omega t)-\ddot{\theta}l\sin(\theta+\omega t)-\dot{\theta}\omega l\cos(\theta+\omega t)$$
$$\ddot{y}=-\omega^2R\sin\omega t-\omega^2l\sin(\theta+\omega t)-\dot{\theta}\omega l\sin(\theta-\omega t)+\ddot{\theta}l\cos(\theta+\omega t)-\dot{\theta}\omega l\sin(\theta+\omega t)$$

$$\delta x=-l\sin(\omega t+\theta)\delta\theta,\quad \delta y=l\cos(\omega t+\theta)\delta\theta$$
$$(\vec{F}-m\ddot{r})\cdot\delta\vec{r}=0\Rightarrow\quad l\ddot{\theta}+\omega^2R\sin\theta=0$$

或者考虑 *Euler-Lagrange* 方程方法

$$\mathrm{T}=\frac{1}{2}\,\mathrm{m}\left(\dot{\mathrm{x}}^2+\dot{\mathrm{y}}^2\right)=\frac{1}{2}\,\mathrm{m}\left[\mathrm{R}^2\omega^2+2R\ell\omega(\omega+\dot{\theta})\cos\theta+\ell^2(\omega+\dot{\theta})^2\right]$$
$$\frac{\partial T}{\partial\dot{\theta}}=\mathrm{mR}\ell\omega\cos\theta+\mathrm{m}\ell^2(\omega+\dot{\theta}),\quad \frac{\partial T}{\partial\theta}=-\mathrm{mR}\ell\omega(\omega+\dot{\theta})\sin\theta$$
$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial\mathrm{T}}{\partial\dot{\theta}}\right)=-\mathrm{mR}\ell\omega\dot{\theta}\sin\theta+\mathrm{m}\ell^2\ddot{\theta}$$

第 5 节　Maupertuis 原理

Maupertuis 定义简约作用量

$$S=\int_A^B mvds=\int\sum_i p_i dq_i \tag{1.22}$$

用能量 E 为参数代入变分方程求解轨道

$$E=\frac{1}{2}\sum_{i,k}m_{ik}(q)\dot{q}_i\dot{q}_k+V(q)$$

注 1.3. *Einstein* 求和规则：只求和重复指标 i（哑指标不用求和）

$$\mathrm{d}t=\sqrt{\frac{\sum m_{ik}\,\mathrm{d}q_i\,\mathrm{d}q_k}{2(E-V)}}$$
$$\sum_i p_i\,\mathrm{d}q_i=\sum_{i,k}m_{ik}\frac{\mathrm{d}q_k}{\mathrm{d}t}\,\mathrm{d}q_i$$
$$S_0=\int\sqrt{2(E-V)\sum_{i,k}m_{ik}\,\mathrm{d}q_i\,\mathrm{d}q_k}.$$
$$\int\sqrt{\frac{\sum m_{ik}\,\mathrm{d}q_i\,\mathrm{d}q_k}{2(E-V)}}=t-t_0.$$

这个方程和二体问题的轨道方程是一致的

注 1.4. Мещерский 变质量物体运动方程：$m\dfrac{dv}{dt}=(u-v)\dfrac{dm}{dt}+F$

第 6 节　Noether 定理

Einstein 指出空间的首要特性：均匀性对应冲量分量守恒；而时间的均匀性对应能量守恒

Noether 定理: 假设 F 是拉格朗日系统 $L(L:TQ\to\mathbb{R})$ 的单参数对称群, 于是 Noether 荷守恒

$$p_\alpha\delta q_\alpha-\ell=\text{ const.} \tag{1.23}$$

其中，l 与 Lagrangian 的变分有关

$$\delta\mathcal{L}\equiv\frac{d}{dt}l \tag{1.24}$$

注 1.5. 空间平移操作 $q_\varepsilon(t)=q(t)+\varepsilon v$，时间平移 $q_\varepsilon(t)=q(t+\varepsilon)$，转动操作 $q_\varepsilon(t)=e^{\varepsilon X}q(t)$

动量守恒	空间平移不变性
能量守恒	时间平移不变性
角动量守恒	空间转动不变性

例 1.9. 质量为 m，带电量为 e 的电荷在均匀磁场中运动，磁场方向为 z 方向，磁场强度为 B

$$\mathcal{L}=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+\frac{eB}{2c}(x\dot{y}-y\dot{x})$$

说明：该系统存在沿 z 轴转动对称性，平移对称性，找出运动积分（仅由初始条件决定的恒定变量），难点在于找到恰当的“操作”（与 ε 相关函数）

$$x_\varepsilon=x+\alpha\varepsilon,\quad y_\varepsilon=y+\beta\varepsilon,\quad z_\varepsilon=z+\gamma\varepsilon$$

$$\delta x=\frac{d(x+\alpha\varepsilon)}{d\varepsilon}=\alpha,\quad \delta y=\beta,\quad \delta z=\gamma,\quad \delta\dot{x}=\frac{d\delta x}{dt}=\delta\dot{y}=\delta\dot{z}=0$$

$$\delta\mathcal{L}=\frac{d}{dt}\frac{eB}{2c}(\alpha y-\beta x)\equiv\frac{d}{dt}l$$

平移下的 *Noether* 荷

$$p_x\delta x+p_y\delta y+p_z\delta z-l=\left(m\dot{x}-\frac{eB}{c}y\right)\alpha+\left(m\dot{y}+\frac{eB}{c}x\right)\beta+m\dot{z}\gamma=const.$$

找一个转动角度

$$\begin{pmatrix}x_\varepsilon\\ y_\varepsilon\\ z_\varepsilon\end{pmatrix}=\begin{pmatrix}\cos\varepsilon & \sin\varepsilon & 0\\ -\sin\varepsilon & \cos\varepsilon & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\delta x=y,\quad \delta y=-x,\quad \delta z=0;\quad \delta\dot{x}=\dot{y},\quad \delta\dot{y}=-\dot{x},\quad \delta\dot{z}=0$$

$$\because \delta\mathcal{L}=m(\dot{x}\delta\dot{x}+\dot{y}\delta\dot{y}+\dot{z}\delta\dot{z})+\frac{eB}{2c}(\delta x\dot{y}+x\delta\dot{y}-\delta y\dot{x}-y\delta\dot{x})=0\Rightarrow l=0$$

转动下的 *Noether* 荷

$$p_x\delta x+p_y\delta y+p_z\delta z=m(x\dot{y}-x\dot{y})-\frac{eB}{2c}\left(x^2+y^2\right)=const.$$

知识点 2　Hamilton 力学

通过 Legendre 变换求 Hamilton 量

$$H=H\left(t,q^\alpha,p_\alpha\right)=p_\alpha\dot{q}^\alpha-\mathcal{L}\tag{2.1}$$

Routh 函数

用循环坐标的变换减去 Lagrangian, 得到非循环坐标的函数考虑 Euler-Lagrange 方程

$$R=\sum_{\alpha=m+1}^{s}p_\alpha\dot{q}^\alpha-\mathcal{L}\tag{2.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial\dot{q}^\alpha}-\frac{\partial R}{\partial q^\alpha}=0,\quad \alpha=1,\cdots,m\tag{2.3}$$

第 1 节 Hamilton 正则方程 Canonical Equations

$$\begin{cases} \dot{q}_\alpha = \dfrac{\partial H}{\partial p_\alpha} \\ \dot{p}_\alpha = -\dfrac{\partial H}{\partial q_\alpha} \end{cases} \tag{2.4}$$

第 2 节 Poisson 括号

$$\begin{aligned} \text{相空间的函数：} \frac{df(q,p,t)}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial f}{\partial p_\alpha}\dot{p}_\alpha \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial H}{\partial q_\alpha} \\ &\equiv \frac{\partial f}{\partial t} + [f, H] \end{aligned} \tag{2.5}$$

Poisson 括号定义

$$[u, v] \equiv \sum_{\alpha=1}^{s} \frac{\partial(u, v)}{\partial(q_\alpha, p_\alpha)} = \sum_{\alpha=1}^{s}\left(\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right) \tag{2.6}$$

性质 2.1.

- 反对称

$$[u, v] = -[v, u] \tag{2.7}$$

- 微商性质

$$\frac{\partial}{\partial x}[u, v] = \left[\frac{\partial u}{\partial x}, v\right] + \left[u, \frac{\partial v}{\partial x}\right] \tag{2.8}$$

证明：

$$\begin{aligned} \frac{\partial}{\partial x}[u, v] &= \frac{\partial}{\partial x}\sum_{\alpha=1}^{s}\left(\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha} - \frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha}\right) \\ &= \sum_{\alpha=1}^{s}\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial q_\alpha}\frac{\partial v}{\partial p_\alpha} + \frac{\partial u}{\partial q_\alpha}\frac{\partial}{\partial x}\frac{\partial v}{\partial p_\alpha} - \frac{\partial}{\partial x}\frac{\partial u}{\partial p_\alpha}\frac{\partial v}{\partial q_\alpha} - \frac{\partial u}{\partial p_\alpha}\frac{\partial}{\partial x}\frac{\partial v}{\partial q_\alpha}\right) \\ &= \sum_{\alpha=1}^{s}\left(\frac{\partial}{\partial q_\alpha}\frac{\partial u}{\partial x}\frac{\partial v}{\partial p_\alpha} - \frac{\partial}{\partial p_\alpha}\frac{\partial u}{\partial x}\frac{\partial v}{\partial q_\alpha} + \frac{\partial u}{\partial q_\alpha}\frac{\partial}{\partial p_\alpha}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial p_\alpha}\frac{\partial}{\partial q_\alpha}\frac{\partial v}{\partial x}\right) \\ &= \left[\frac{\partial u}{\partial x}, v\right] + \left[u, \frac{\partial v}{\partial x}\right] \end{aligned}$$

□

- 分配律和结合律

$$\begin{aligned} [u, v + w] &= [u, v] + [u, w] \\ [u, vw] &= [u, v]w + v[u, w] \end{aligned} \tag{2.9}$$

- *Jacobi* 恒等式：

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0 \tag{2.10}$$

- 广义坐标的 *Poisson* 括号性质

$$
\begin{gathered}
[q_\alpha, q_\beta] = 0, \quad [p_\alpha, p_\beta] = 0, \quad [q_\alpha, p_\beta] = \delta_{\alpha\beta} \\
[q_\alpha, f] = \frac{\partial f}{\partial p_\alpha} = \frac{\partial (q_\alpha, f)}{\partial (q_\alpha, p_\alpha)}, \quad [p_\alpha, f] = -\frac{\partial f}{\partial q_\alpha} = \frac{\partial (p_\alpha, f)}{\partial (q_\alpha, p_\alpha)}
\end{gathered} \tag{2.11}
$$

例 2.1. 证明: 以 r^2, p^2 和 $\boldsymbol{r}\cdot\boldsymbol{p}$ 为自变量的任意函数 f 与角动量分量 J_z 的泊松括号为 *0*

$$\because f = f\left(r^2, p^2, \boldsymbol{r}\cdot\boldsymbol{p}\right), \text{其中} \boldsymbol{r}\cdot\boldsymbol{p} = xp_x + yp_y + zp_z, J_z = \vec{r}\times\vec{p} = \begin{pmatrix} x & y \\ p_x & p_y \end{pmatrix} = xp_y - yp_x.$$

$$
\begin{aligned}
[f, J_z] = & \sum_\alpha \left(\frac{\partial f}{\partial q_\alpha}\frac{\partial J_z}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial J_z}{\partial q_\alpha}\right) \\
= & \frac{\partial f}{\partial x}\frac{\partial J_z}{\partial p_x} - \frac{\partial f}{\partial p_x}\frac{\partial J_z}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial J_z}{\partial p_y} - \frac{\partial f}{\partial p_y}\frac{\partial J_z}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial J_z}{\partial p_z} - \frac{\partial f}{\partial p_z}\frac{\partial J_z}{\partial x} \\
= & -y\frac{\partial f}{\partial x} - p_y\frac{\partial f}{\partial p_x} + x\frac{\partial f}{\partial y} + p_x\frac{\partial f}{\partial p_y} \\
= & -y\left(\frac{\partial f}{\partial r^2}2x + \frac{\partial f}{\partial(\boldsymbol{r}\cdot p)}p_x\right) - p_y\left(\frac{\partial f}{\partial p^2}2p_x + \frac{\partial f}{\partial(\boldsymbol{r}\cdot p)}x\right) \\
& + x\left(\frac{\partial f}{\partial r^2}2y + \frac{\partial f}{\partial(\boldsymbol{r}\cdot p)}p_y\right) + p_x\left(\frac{\partial f}{\partial p^2}2p_y + \frac{\partial f}{\partial(\boldsymbol{r}\cdot\boldsymbol{p})}y\right) \\
= & 0
\end{aligned}
$$

第 3 节 正则变换 Canonical Transformation

判断正则变换

找一个原函数（即第一类生成函数）或考虑 Jacobi 行列式为 0

$$p\delta q - P\delta Q = \delta f \text{或者} \frac{\partial(Q,P)}{\partial(q,p)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \end{vmatrix} = 1 \tag{2.12}$$

2.3.1 生成函数

共有四类生产函数

$$
\begin{aligned}
p_\alpha dq_\alpha - P_\alpha dQ_\alpha + (\widetilde{H} - H)dt &= dF_1(q_\alpha, Q_\alpha, t) \\
p_\alpha dq_\alpha + Q_\alpha dP_\alpha + (\widetilde{H} - H)dt &= dF_2(q_\alpha, P_\alpha, t) = d(F_1 + Q_\alpha P_\alpha) \\
-q_\alpha dp_\alpha - P_\alpha dQ_\alpha + (\tilde{H} - H)dt &= dF_3(p_\alpha, Q_\alpha, t) = d(F_1 - p_\alpha q_\alpha) \\
-q_\alpha dp_\alpha + Q_\alpha dP_\alpha + (\tilde{H} - H)dt &= dF_4(p_\alpha, P_\alpha, t) = d(F_1 - p_\alpha q_\alpha + Q_\alpha P_\alpha)
\end{aligned} \tag{2.13}
$$

故经过 Hamilton 正则变换，新的 Hamiltonian 为

$$\widetilde{H} = H + \frac{\partial F}{\partial t} \tag{2.14}$$

2.3.2 Hamilton-Jacobi 方程

$$\frac{\partial S}{\partial t}+H(q,p,t)=0 \tag{2.15}$$

$$\frac{\partial S}{\partial t}+H\left(q_1,\cdots,q_s;\frac{\partial S}{\partial q_1},\cdots,\frac{\partial S}{\partial q_s};t\right)=0 \tag{2.16}$$

Hamilton 主函数 S、特征函数 W

$$S=\int Ldt=\int\left(p_\alpha\dot{q}_\alpha-H\right)dt=\int p_\alpha dq_\alpha-Et+A \tag{2.17}$$

$$S=F_2\left(q_1,\ldots,q_s;P_1,\ldots,P_s;t\right)+A \tag{2.18}$$

$$W\left(q_1,\ldots,q_s\right)=\int p_\alpha dq_\alpha \tag{2.19}$$

故可以改写 Hamiltonian 中的广义坐标

$$p_\alpha=\frac{\partial S}{\partial q_\alpha}=\frac{\partial W}{\partial q_\alpha} \tag{2.20}$$

$$Q_\alpha=\frac{\partial F_2}{\partial P_\alpha}=\frac{\partial S}{\partial P_\alpha}=\begin{cases}-t+\dfrac{\partial W}{\partial E} & \alpha=1(\text{第一个坐标方向})\\ \dfrac{\partial W}{\partial P_\alpha} & \alpha=2,3,\ldots,s(\text{第 2-s 个坐标方向})\end{cases} \tag{2.21}$$

注 2.1. *一般得出 H-J 方程的方法：*

第 1 步 p_α *用* $\dfrac{\partial W}{\partial q_\alpha}$ *代入 Hamiltonian, 右侧常数令为能量* E

第 2 步 运动积分动量为常数, 分离变量如 $W(r,\theta,\varphi)=W_r(r)+W_\theta(\theta)+W_\varphi(\varphi)$

第 3 步 得到 W *的不同参数的积分式子，代回主函数：*$S=-Et+A+W(q_\alpha)$

例 2.2. *用 H-J 方程讨论质点* m *做斜抛运动轨道方程, 初速度为* v_0*, 水平仰角为* α*.*

解： *Hamilton 函数* $H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+mgy$*, H-J 方程*

$$E=\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right]+mgy$$

由于 x 是循环坐标, $p_x=\dfrac{\partial W}{\partial x}$ 为常数, 记为 C_1. 设 $W=W_1(x)+W_2(y)$, 则有

$$\left(\frac{\partial W_1}{\partial x}\right)^2=2mE-2m^2gy-\left(\frac{\partial W_2}{\partial y}\right)^2=C_1^2$$

解得

$$W_1=C_1x+C',\quad W_2=\pm\int\sqrt{2mE-C_1^2-2m^2gy}dy$$

$$\Rightarrow W = C_1 x \pm \int \sqrt{2mE - C_1^2 - 2m^2 gy}\,\mathrm{d}y + C'$$

$$\frac{\partial W}{\partial C_1} = x \pm \int \frac{C_1}{\sqrt{2mE - C_1^2 - 2m^2 gy}}\,\mathrm{d}y = x \pm \frac{C_1}{m^2 g}\sqrt{2mE - C_1^2 - 2m^2 gy} = x_0$$

代入 $C_1 = p_x = mv_0 \cos\alpha, E = \frac{1}{2}mv_0^2$, 解得

$$y = \frac{2mE - C_1^2 - \frac{m^4 g^2 (x-x_0)^2}{C_1^2}}{2m^2 g} = \frac{v_0^2 \sin^2\alpha}{2g} - \frac{g\left(x - x_0\right)^2}{2v_0^2 \cos^2\alpha}$$

设初始时刻质点在原点, 可得 $x_0 = \dfrac{v_0^2 \cos\alpha \sin\alpha}{g}$, 代入上式化简得轨道方程

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2 \cos^2\alpha}$$

第 4 节　Virial 定理

Virial 定理反映了系统的某种统计性质

$$\bar{T} = -\frac{1}{2}\overline{\sum_i \boldsymbol{F}_i \cdot \boldsymbol{r}_i} \tag{2.22}$$

若各 F_i 均为保守力, 则 $\bar{T} = \frac{1}{2}\overline{\sum_i \nabla V_i \cdot r_i}$ 若 $V = kr^{n+1} \Rightarrow \bar{T} = \dfrac{n+1}{2}\bar{V}$.

对于引力或电磁力, $n = -2 \Rightarrow \bar{T} = -\frac{1}{2}\bar{V}$, 而对谐振子, $n = 1 \Rightarrow \bar{T} = \bar{V}$.

知识点 3　动力学

第 1 节　两体问题

3.1.1　轨道方程

对两体的 Lagrangian（μ 为约化质量）:

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{3.1}$$

由 Euler-Lagrange 方程得到:

$$\mu\ddot{r} - \mu r\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{3.2}$$

于是轨道方程只需求解:

$$\begin{gathered} \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{L^2}{\mu r^2} + V(r) = E = \text{ const.} \\ \text{其中} L = \mu r^2\dot{\theta}, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \end{gathered} \tag{3.3}$$

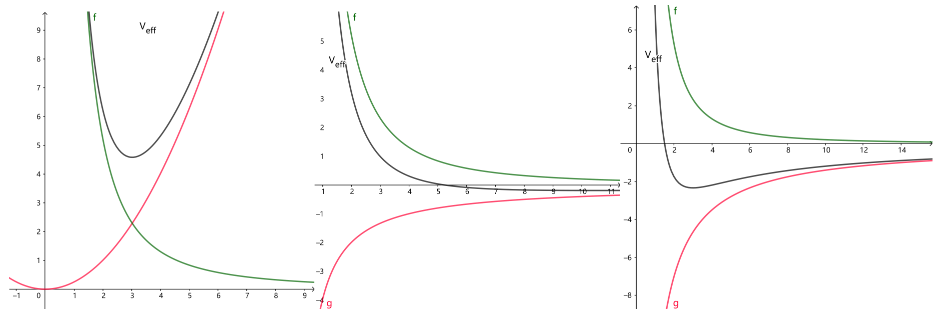
得到 r, θ 与 t 的表达式

$$\begin{gathered} \dot{r} = \sqrt{\frac{2}{\mu}\left(E - V - \frac{L^2}{2\mu r^2}\right)} = \sqrt{\frac{2}{\mu}\left(E - V_{eff}\right)} \Rightarrow t = \int^r \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - V_{eff}\right)}} \\ \dot{\theta} = \frac{L}{\mu r^2} \Rightarrow \quad \theta = L\int^t \frac{dt}{\mu r^2} = \int^r \frac{Ldr}{r^2\sqrt{2\mu\left(E - V_{eff}\right)}} \end{gathered} \tag{3.4}$$

注 3.1. 有效势能的问题：

$$V_{eff} = V + \frac{L^2}{2mr^2} \tag{3.5}$$

不同情况下的等效势能



微分 (Binet) 和积分形式轨道方程

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{\mu}{L^2}\frac{d}{du}V(\frac{1}{u}) = -\frac{\mu F(r)}{L^2 u^2} = -\frac{F(r)}{\mu h^2 u^2}, \quad h = r^2\dot{\theta} \tag{3.6}$$

$$\theta = \int^r \frac{L \cdot dr}{\mu r^2 \sqrt{\frac{2}{\mu}(E - \mathrm{V}_{eff})}}, \quad \Delta\theta = 2\int_{r_{\min}}^{r_{\max}} \frac{L \cdot dr}{\mu r^2 \sqrt{\frac{2}{\mu}(E - V - \frac{L^2}{2mr^2})}} \tag{3.7}$$

注 3.2. 考虑存在微扰的情况，

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{\mu}{L^2}\frac{d}{du}(V(\frac{1}{u}) + \delta V(\frac{1}{u})) \tag{3.8}$$

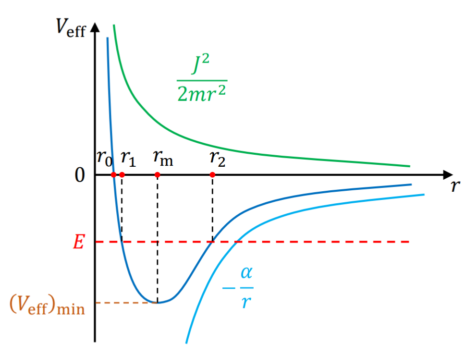
$$\delta\theta = -2\frac{\partial}{\partial L}\int_{r_{\min}}^{r_{\max}} \sqrt{2m(E-V) - \frac{L^2}{r^2}}\,\mathrm{d}r = \frac{\partial}{\partial L}\int_{r_{\min}}^{r_{\max}} \frac{2m\delta V\,\mathrm{d}r}{\sqrt{2m\left(E + \frac{k}{r}\right) - \frac{L^2}{r^2}}} = \frac{\partial}{\partial L}\left(\frac{2m}{L}\int_0^{\pi} r^2 \delta V\,\mathrm{d}\varphi\right) \tag{3.9}$$

Kepler 问题

首先写出 Kelper 问题下有效势能

$$V_{\mathrm{eff}} = -\frac{GMm}{r} + \frac{1}{2}\frac{L^2}{\mu r^2} = -\frac{k}{r} + \frac{1}{2}\frac{L^2}{\mu r^2} \Rightarrow \frac{d^2 u}{d\theta^2} + u = -\frac{\mu V(\frac{1}{u})}{L^2 u^2} = \frac{\mu k}{L^2}. \tag{3.10}$$

$$\frac{1}{r} = \frac{\mu k}{L^2}\left[1 + \sqrt[2]{1 + \frac{2EL^2}{\mu k^2}}\cos(\theta - \theta_0)\right] \tag{3.11}$$



偏心率和能量

$$
\text{偏心率：}\quad e=\sqrt{1+\frac{2EL^2}{\mu k^2}} \tag{3.12}
$$

$$
\begin{gathered}
e>1,\quad E>0:\text{双曲线}\\
e=1,\quad E=0:\text{抛物线}\\
e<1,\quad E<0:\text{椭圆}\\
e=0\quad E=-\frac{\mu k^2}{2L^2}:\text{圆周}
\end{gathered}
$$

$$
\text{引入：}\ p\equiv\frac{L^2}{\mu k} \tag{3.13}
$$

轨道方程改为：

$$
\theta=\int\frac{p}{r^2}\frac{\mathrm{d}r}{\sqrt{e^2-\left(1-\frac{p}{r}\right)^2}} \tag{3.14}
$$

通过积分得到

$$
r=\frac{p}{1+e\cos\theta} \tag{3.15}
$$

轨道的长轴只与能量有关

$$
\begin{aligned}
2a&=r_{\min}+r_{\min}=\frac{p}{1+e}+\frac{p}{1-e}=\frac{p}{1-e^2}\\
&=\frac{L^2}{\mu k}\frac{1}{1-\left(1+\dfrac{2EL^2}{\mu k^2}\right)}=-\frac{k}{E}.
\end{aligned} \tag{3.16}
$$

轨道半短轴和 E、L 有关

$$
b=a\sqrt{1-e^2}=\frac{L}{\sqrt{-2\mu E}} \tag{3.17}
$$

在角动量为常数的情况下，可以导出 Kepler 第三定律

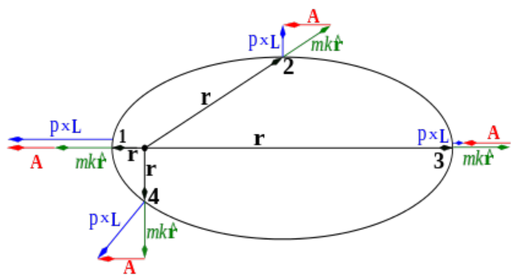
$$
\begin{aligned}
&L=\mu r^2\dot{\theta}=\ \text{const.}\\
\Rightarrow T&=\int_0^{2\pi}\frac{\mu r^2}{L}d\theta=\frac{\mu\cdot 2\pi ab}{L}\\
&=\pi k\sqrt{\frac{\mu}{2|E|^3}}=2\pi\sqrt{\frac{\mu a^3}{k}}
\end{aligned} \tag{3.18}
$$

3.1.2 Laplace-Runge-Lenz 矢量：

$$
\vec{A}=\vec{p}\times\vec{L}-\mu k\hat{\vec{r}} \tag{3.19}
$$

$$
\vec{A}\cdot\vec{r}=Ar\cos\left(\theta-\theta_0\right)=\vec{r}\cdot(\vec{P}\times\vec{L}-\mu k\hat{\vec{r}})=L^2-\mu kr \tag{3.20}
$$

$$
\begin{aligned}
\vec{A}\cdot\vec{A}&=(\vec{P}\times\vec{L})^2-2\mu k\hat{\vec{r}}\cdot(\vec{P}\times\vec{L})+\mu^2k^2=P^2L^2-2\mu L^2\frac{k}{r}+\mu^2k^2\\
&=2\mu L^2\left(\frac{P^2}{2\mu}-\frac{k}{r}+\frac{\mu k^2}{2L^2}\right)=2\mu L^2\left(E+\frac{\mu k^2}{2L^2}\right)=\mu^2k^2e^2
\end{aligned} \tag{3.21}
$$



3.1.3 质点碰撞

势能空间

质量为 m 的质点以速度 v_1 从一个势能为常数 V_1 的半空间运动到另一个势能为常数 V_2 的半空间. 求质点运动方向的改变.

$$\frac{v_2}{v_1}=\frac{\sin\theta_1}{\sin\theta_2}=\sqrt{1+\frac{2}{mv_1^2}(V_1-V_2)}. \tag{3.22}$$

要求根号下式子大于 0 才可以进入另一个势能空间

质点分裂

把实验室参考系的速度放到质心系考虑:

$$\boldsymbol{v}_{10}=\frac{m_2}{m_1+m_2}\boldsymbol{v},\quad \boldsymbol{v}_{20}=-\frac{m_1}{m_1+m_2}\boldsymbol{v} \tag{3.23}$$

$$\tan\theta_0=\frac{m_2\sin\theta}{m_1+m_2\cos\theta} \tag{3.24}$$

故得到进入某一微元立体角 $d\Omega$ 的粒子比例

$$\frac{d\Omega}{4\pi}=\frac{2\pi\sin\theta_0 d\theta_0}{4\pi}=\frac{\sin\theta_0 d\theta_0}{2} \tag{3.25}$$

下面计算动能的分布

$$\mathrm{d}\cos\theta_0=\frac{\mathrm{d}\left(v^2\right)}{2v_0V}\Rightarrow\frac{\mathrm{d}T}{2mv_0V} \tag{3.26}$$

$$\frac{1}{2}\left(v_0-V\right)^2\leqslant T\leqslant\frac{1}{2}\left(v_0+V\right)^2 \tag{3.27}$$

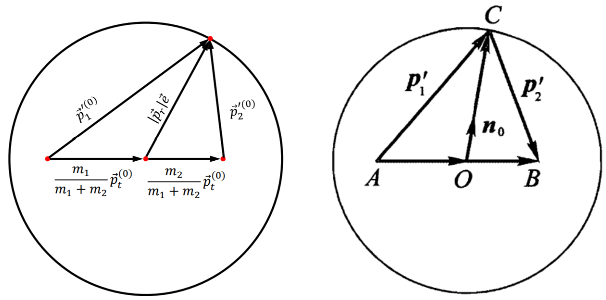
弹性碰撞

在两质点质心系中，碰撞后速度

$$\boldsymbol{v}_{10}=\frac{m_2}{m_1+m_2}\boldsymbol{v},\quad \boldsymbol{v}_{20}=-\frac{m_1}{m_1+m_2}\boldsymbol{v} \tag{3.28}$$

还原到实验室参考系

$$\boldsymbol{v}_1'=\frac{m_2}{m_1+m_2}v\boldsymbol{n}_0+\frac{m_1\boldsymbol{v}_1+m_2\boldsymbol{v}_2}{m_1+m_2},\boldsymbol{v}_2'=-\frac{m_1}{m_1+m_2}v\boldsymbol{n}_0+\frac{m_1\boldsymbol{v}_1+m_2\boldsymbol{v}_2}{m_1+m_2}. \tag{3.29}$$



于是可以画出如图的参考矢量圆模型

$$\begin{cases} \overrightarrow{OC} = m\boldsymbol{v} \\ \overrightarrow{AO} = \dfrac{m_1\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right)}{m_1 + m_2} \\ \overrightarrow{OB} = \dfrac{m_2\left(\boldsymbol{p}_1 + \boldsymbol{p}_2\right)}{m_1 + m_2} \end{cases} \tag{3.30}$$

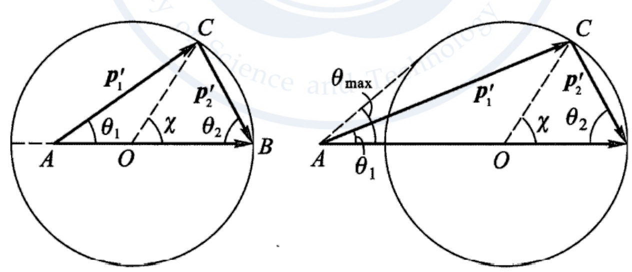
$$\tan\theta_0 = \frac{v_0 \sin\theta_0}{v_0\cos\theta_0 + V} \tag{3.31}$$

接着用角度关系互相表示偏转角 χ 和速度

$$\tan\theta_1 = \frac{m_2\sin\chi}{m_1 + m_2\cos\chi}, \quad \theta_2 = \frac{\pi - \chi}{2} \tag{3.32}$$

$$v_1' = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2\cos\chi}}{m_1 + m_2} v, \quad v_2' = \frac{2m_1 v}{m_1 + m_2}\sin\frac{\chi}{2}. \tag{3.33}$$

可以导出碰撞后的第二个质点获得的最大能量



(a) $m_1 < m_2$　　(b) $m_1 > m_2$

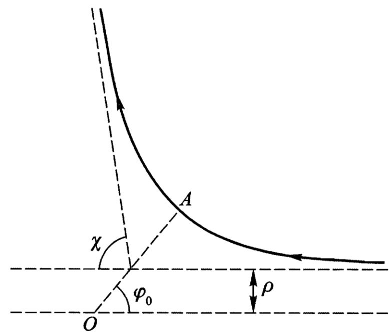
$$E'_{2\,\max} = \frac{m_2 v'^2_{2\,\max}}{2} = \frac{4m_1m_2}{(m_1+m_2)^2} E_1 \tag{3.34}$$

质点散射

根据公式3.4, 质点散射中的轨道渐近线夹角, 积分积到无穷

$$\varphi_0 = \int_{r_{\min}}^{\infty} \frac{L dr}{r^2\sqrt{2m[E - V(r) - \dfrac{L^2}{2mr^2}]}} \tag{3.35}$$

其中，$r_{\min}$ 计算方法：$m\rho v_\infty = m v_{\min} r_{\min}$



有效截面

通过瞄准距离和 v_∞ 计算：

$$
\begin{aligned}
E &= \frac{mv_\infty^2}{2}, \quad L = m\rho v_\infty \\
\varphi_0 &= \int_{r_{\min}}^{\infty} \frac{\rho dr}{r^2\sqrt{1-\frac{\rho^2}{r^2}-\frac{2V}{mv_\infty^2}}}
\end{aligned}
\tag{3.36}
$$

$$
\begin{cases}
\frac{1}{2}mv_\infty^2 = \frac{1}{2}mv_{\min}^2 + V\left(r_{\min}\right) \\
m\rho v_\infty = mr_{\min}v_{\min}
\end{cases}
\tag{3.37}
$$

$$
\begin{aligned}
\mathrm{d}\sigma &= 2\pi\rho\mathrm{d}\rho \\
&= 2\pi\rho(\chi)\left|\frac{\mathrm{d}\rho(\chi)}{\mathrm{d}\chi}\right|\mathrm{d}\chi \\
&= 2\pi\rho(\chi)\left|\frac{\mathrm{d}\rho(\chi)}{\mathrm{d}\chi}\right|\frac{\mathrm{do}}{2\pi\sin\chi d\chi} \\
&= \frac{\rho(\chi)}{\sin\chi}\left|\frac{\mathrm{d}\rho}{\mathrm{d}\chi}\right|\mathrm{do}
\end{aligned}
\tag{3.38}
$$

计算总的截面积即对立体角积分：

$$
\sigma = \int_0^{4\pi} \frac{\rho(\chi)}{\sin\chi}\left|\frac{\mathrm{d}\rho}{\mathrm{d}\chi}\right| do \tag{3.39}
$$

第 2 节　微振动

3.2.1　n 维振动

写出 Lagrangian 和 E-L 方程

$$
\mathcal{L} = T - V = \frac{1}{2}\sum_{\alpha,\beta}\left(m_{\alpha\beta}\dot{x}_\alpha\dot{x}_\beta - k_{\alpha\beta}x_\alpha x_\beta\right) \tag{3.40}
$$

$$
\sum_{\alpha,\beta}\left(m_{\alpha\beta}\ddot{x}_\beta + k_{ij}x_\beta\right) = 0. \tag{3.41}
$$

矩阵形式为

$$\mathcal{L}=\frac{1}{2}\boldsymbol{M}\dot{X}^2+\frac{1}{2}\boldsymbol{K}X^2 \qquad \Rightarrow \qquad \boldsymbol{M}\ddot{X}+\boldsymbol{K}X=0 \tag{3.42}$$

其中，定义

$$\boldsymbol{M}=\begin{pmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \vdots & & & \vdots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{pmatrix} \qquad \boldsymbol{K} = \begin{pmatrix} k_{11} & k_{12} & \dots & k_{1n} \\ k_{21} & k_{22} & \dots & k_{2n} \\ \vdots & & & \vdots \\ k_{n1} & k_{n2} & \dots & k_{nn} \end{pmatrix} \tag{3.43}$$

解的一般形式为

$$\ddot{X}=\frac{d^2}{dt^2}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad X=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{3.44}$$

$$\Rightarrow x_\beta = A_\beta \exp(iwt),$$

欲求振动频率 ω 则求解行列式，称之为久期方程

$$|-\omega^2\boldsymbol{M}+\boldsymbol{K}|=0 \tag{3.45}$$

或者求解特征值，

$$|-\lambda\boldsymbol{M}+\boldsymbol{K}|=0 \tag{3.46}$$

求简正坐标: 由线性代数，两个正定的实对称矩阵，一定存线性变换，使其同时对角化。

$$\tilde{\boldsymbol{K}}=\boldsymbol{R}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{R}, \quad \tilde{\boldsymbol{M}}=\boldsymbol{R}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{R}, \quad \text{同时对角化} \tag{3.47}$$

将 ω 的解代回原矩阵方程可以得到 X 的通解，

$$(-\omega^2\boldsymbol{M}+\boldsymbol{K})X=\begin{pmatrix} -\omega^2 m_{11}+k_{11} & -\omega^2 m_{12}+k_{12} & \dots & -\omega^2 m_{1n}+k_{1n} \\ -\omega^2 m_{21}+k_{21} & -\omega^2 m_{22}+k_{22} & \dots & -\omega^2 m_{2n}+k_{2n} \\ & \vdots & & \vdots \\ -\omega^2 m_{n1}+k_{n1} & -\omega^2 m_{n2}+k_{n2} & \dots & -\omega^2 m_{nn}+k_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=0 \tag{3.48}$$

再对通解进行组合即为振动方程，可以单位化基矢得到正交模列

3.2.2　阻尼振动

考虑摩擦力的状态下，如果速度足够小, 可以将摩擦力按速度的赛次展开. 因为在静止物体上没有任何摩擦力作用，所以零次项为零. 不为零的第一项与速度成正比

$$f_{\mathrm{fr}}=-\alpha\dot{x} \tag{3.49}$$

$$m\ddot{x}=-kx-\alpha\dot{x}. \tag{3.50}$$

特征方程为

$$r^2+2\lambda r+\omega_0^2=0,\qquad \lambda=\frac{\alpha}{2m},\omega_0^2=\frac{k}{m} \tag{3.51}$$

再结合力学微分方程讨论解的形式

对多自由度系统而言，广义摩擦力可以写作一个矩阵的导数

$$\begin{aligned} f&=-\textstyle\sum_k \alpha_{ik}\dot{x}_k=-\frac{\partial F}{\partial \dot{x}_i}\\ F&=\tfrac{1}{2}\textstyle\sum_{i,k}\alpha_{ik}\dot{x}_i\dot{x}_k \end{aligned} \tag{3.52}$$

而且这个矩阵和能量有关

$$\frac{\mathrm{d}E}{\mathrm{d}t}=-2F \tag{3.53}$$

故可以改写 Euler-Lagrange 方程

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i}-\frac{\partial L}{\partial x_i}=-\frac{\partial F}{\partial \dot{x}_i} \tag{3.54}$$

下面参考上面的思路可以求解久期方程

$$\boldsymbol{K}\ddot{X}+\boldsymbol{F}\dot{X}+\boldsymbol{M}X=0 \tag{3.55}$$

3.2.3 强迫振动

$$\ddot{x}+2\lambda\dot{x}+\omega_0^2x=\frac{f}{m}\cos\gamma t \tag{3.56}$$

求解这个方程用复数形式更方便, 为此在右端用 $\mathrm{e}^{\mathrm{i}\gamma t}$ 代替 $\cos\gamma t$:

$$\ddot{x}+2\lambda\dot{x}+\omega_0^2x=\frac{f}{m}\mathrm{e}^{\mathrm{i}\gamma t}.$$

求 $x=B\mathrm{e}^{\mathrm{i}\gamma t}$ 形式的特解, 对于 B 有

$$B=\frac{f}{m\left(\omega_0^2-\gamma^2+2i\lambda\gamma\right)}.$$

将 B 写成 $b\mathrm{e}^{\mathrm{i}\delta}$ 的形式, 对于 b 和 δ 有

$$b=\frac{f}{m\sqrt{\left(\omega_0^2-\gamma^2\right)^2+4\lambda^2\gamma^2}},\qquad \tan\delta=\frac{2\lambda\gamma}{\gamma^2-\omega_0^2}.$$

$$x=a\mathrm{e}^{-\lambda t}\cos(\omega t+\alpha)+b\cos(\gamma t+\delta) \tag{3.57}$$

第 3 节 刚体问题

刚体满足：任意两点保持平动状态；对任意参考系，角速度相同

$$T=\frac{1}{2}I_{ij}\omega^i\omega^j \tag{3.58}$$

$$L=I_{ij}\omega_j \tag{3.59}$$

3.3.1　惯量张量

$$
\begin{aligned}
I_{ij} &\equiv \sum_{\alpha=1}^{N} m_\alpha \left(\delta_{ij}\left(\boldsymbol{r}_\alpha\right)^2-\left(\boldsymbol{r}_\alpha\right)_i\left(\boldsymbol{r}_\alpha\right)_j\right) \\
(3-dim) &= \sum_{\alpha=1}^{N} m_\alpha \begin{pmatrix} y_\alpha^2+z_\alpha^2 & -x_\alpha y_\alpha & -x_\alpha z_\alpha \\ -y_\alpha x_\alpha & x_\alpha^2+z_\alpha^2 & -y_\alpha z_\alpha \\ -z_\alpha x_\alpha & -z_\alpha y_\alpha & x_\alpha^2+y_\alpha^2 \end{pmatrix}
\end{aligned} \tag{3.60}
$$

惯量椭球

$$
I_{11}x^2+I_{22}y^2+I_{33}z^2+2I_{12}xy+2I_{23}yz+2I_{31}zx=1 \tag{3.61}
$$

平行轴定理证明

$$
\begin{aligned}
I_{ij}^{Q} &= \Sigma_\alpha m_\alpha\left(\delta_{ij}\Sigma_k R_{\alpha,k}^2-R_{\alpha,i}R_{\alpha,j}\right) \\
&= \Sigma_\alpha m_\alpha\left[\delta_{ij}\Sigma_k\left(r_{\alpha,k}+a_k\right)^2-\left(r_{\alpha,i}+a_i\right)\left(r_{\alpha,j}+a_j\right)\right] \\
&= \Sigma_\alpha m_\alpha\left[\left(\delta_{ij}\Sigma_k r_{\alpha,k}^2-r_{\alpha,i}r_{\alpha,j}\right)+\left(\delta_{ij}\Sigma_k a_k^2-a_ia_j\right)+\left(\delta_{ij}\Sigma_k 2r_{\alpha,k}a_k-a_ir_{\alpha,j}-r_{\alpha,i}a_j\right)\right] \\
&= I_{ij}^{C}+M\left(\delta_{ij}\Sigma_k a_k^2-a_ia_j\right)=I_{ij}^{C}+M\left(\delta_{ij}a^2-a_ia_j\right)
\end{aligned} \tag{3.62}
$$

主转动惯量、惯量主轴

选用恰当的坐标使得转动惯量三个分量两两正交

$$
\boldsymbol{R}\boldsymbol{I}\boldsymbol{R}^{\mathrm{T}}=\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{3.63}
$$

例 3.1. 均质正方体，质量和边长分别为 $m; a$，取参考点为其一个顶点 A，试求匀质正方体相对其顶点 A 的转动惯量以及主转动惯量

$$
\begin{aligned}
I_{11} &= \int_0^a\int_0^a\int_0^a \rho\left(y^2+z^2\right)dxdydz=\frac{2}{3}\rho a^5=\frac{2}{3}ma^2 \\
I_{12} &= -\int_0^a\int_0^a\int_0^a \rho xydxdydz=-\frac{1}{4}\rho a^5=-\frac{1}{4}ma^2
\end{aligned}
$$

$$
\mathbf{I}=\begin{pmatrix} \frac{2}{3}ma^2 & -\frac{1}{4}ma^2 & -\frac{1}{4}ma^2 \\ -\frac{1}{4}ma^2 & \frac{2}{3}ma^2 & -\frac{1}{4}ma^2 \\ -\frac{1}{4}ma^2 & -\frac{1}{4}ma^2 & \frac{2}{3}ma^2 \end{pmatrix}
$$

取正方体体对角线为 z 轴，则得到主转动惯量（这是角度旋转法）

先绕 z 轴逆时针转 45°，再绕 y'' 轴逆时针转 θ 角，其中 $\cos\theta=\frac{1}{\sqrt{3}}, \sin\theta=\frac{2}{\sqrt{6}}$

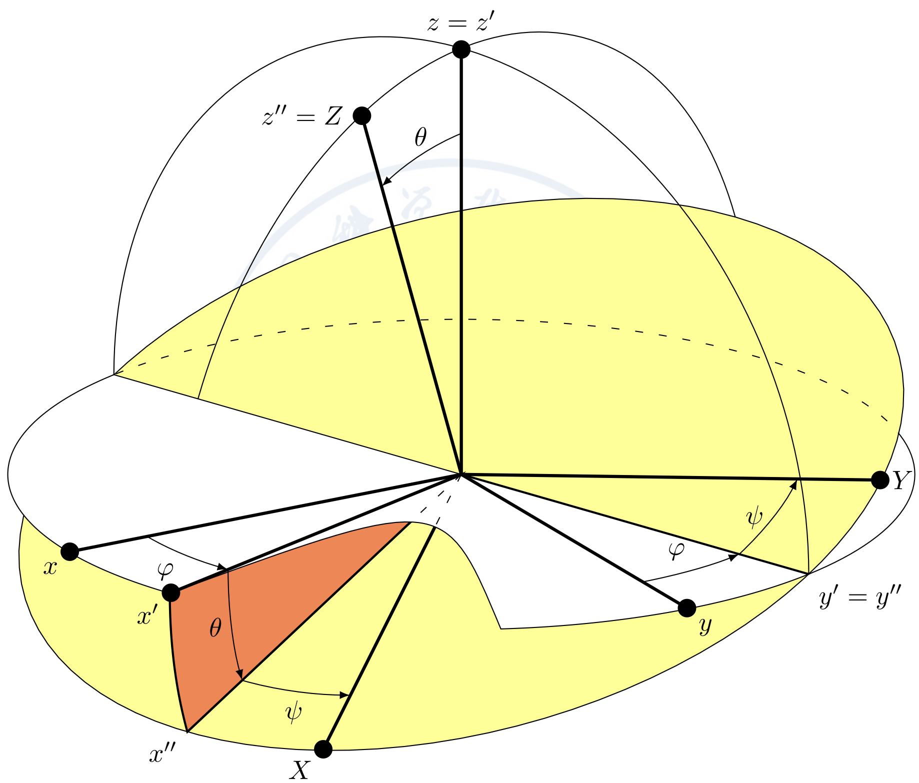
$$
X'=\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}X=\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}X=\boldsymbol{R}X
$$

$$\mathbf{I}' = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{2}{3}ma^2 & -\frac{1}{4}ma^2 & -\frac{1}{4}ma^2 \\ -\frac{1}{4}ma^2 & \frac{2}{3}ma^2 & -\frac{1}{4}ma^2 \\ -\frac{1}{4}ma^2 & -\frac{1}{4}ma^2 & \frac{2}{3}ma^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \boldsymbol{R I R^T} = \begin{pmatrix} \frac{11}{12}ma^2 & 0 & 0 \\ 0 & \frac{11}{12}ma^2 & 0 \\ 0 & 0 & \frac{1}{6}ma^2 \end{pmatrix} \quad \text{或者用特征值方法求解}$$

3.3.2 Euler 角

Euler 角：进动角 φ，章动角 θ，自转角 ψ



3.3.3 Euler 动力学方程

$$\begin{aligned} I_1\dot{\omega}_x - (I_2 - I_3)\,\omega_y\omega_z &= N_x \\ I_2\dot{\omega}_y - (I_3 - I_1)\,\omega_z\omega_x &= N_y \\ I_3\dot{\omega}_z - (I_1 - I_2)\,\omega_x\omega_y &= N_z \end{aligned} \tag{3.64}$$

其中，各个方向的力矩为

$$N_x \equiv -\frac{\partial V}{\partial \varphi}, \quad N_y \equiv -\frac{\partial V}{\partial \theta}, \quad N_z \equiv -\frac{\partial V}{\partial \psi} \tag{3.65}$$

3.3.4 对称 Euler 陀螺

$$I_1 = I_2 \neq I_3$$

$$\left\{\begin{array}{c} I_1\dot{\omega}_x - (I_1 - I_3)\,\omega_y\omega_z = 0 \\ I_1\dot{\omega}_x - (I_3 - I_1)\,\omega_z\omega_x = 0 \\ I_3\dot{\omega}_z = 0 \end{array}\right.$$

$$\omega_z = \text{ const } = \omega_{z0}$$

$$\left\{\begin{array}{l} \dot{\omega}_x = -\left(\frac{I_3-I_1}{I_1}\omega_{z0}\right)\omega_y \\ \dot{\omega}_y = \left(\frac{I_3-I_1}{I_1}\omega_{z0}\right)\omega_x \end{array}\right.$$

若令 $I_3 > I_1$, 定义: $\Omega \equiv \dfrac{I_3 - I_1}{I_1}\omega_{z0}$

$$\left\{\begin{array}{l} \dot{\omega}_x + \Omega\omega_y = 0 \\ \dot{\omega}_y - \Omega\omega_x = 0 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} \omega_x = A\cos\left(\Omega t + \phi_0\right) \\ \omega_y = A\sin\left(\Omega t + \phi_0\right) \end{array}\right.$$

积分常数 $A = \omega_0 \sin\alpha$(**自转轴和进动轴夹角**), ϕ_0 **由初始条件给出**!

$$\omega = |\vec{\omega}| = \sqrt{\omega_x^2 + \omega_y^2\omega_z^2} = \sqrt{A^2 + \omega_{z0}^2} = \text{const}$$

例 3.2. 一回转仪绕其质心作自由转动, $I_1 = I_2 = nI_3$, 初始时对称轴与进动轴间的夹角为 θ_0. 证明:

(1) 刚体作规则进动;

(2) 自转角速度与进动角速度以 $\dot{\psi} = (n-1)\dot{\varphi}\cos\theta_0$ 相联系.

证明: *(1)* $I_1 = I_2 = nI_3$, 为对称欧拉陀螺. 由欧拉动力学方程可解得

$$\left\{\begin{array}{l} \omega_x = A\cos(\Omega t + \phi_0) \\ \omega_y = A\sin(\Omega t + \phi_0) \\ \omega_z = \omega_{z0} \end{array}\right. \Rightarrow \left\{\begin{array}{l} L_x = L\sin\theta\sin\psi = I_1\omega_x = I_1 A\cos\left(\Omega t + \phi_0\right) \\ L_y = L\sin\theta\cos\psi = I_1\omega_y = I_1 A\sin\left(\Omega t + \phi_0\right) \\ L_z = L\cos\theta = I_3\omega_{z0} \end{array}\right.$$

所以

$$\cos\theta = \frac{L_z}{L} = \frac{I_3\omega_{z0}}{L} = \textit{Const.} = \cos\theta_0$$

故陀螺作规则进动.

$$(2) \qquad \Omega = \frac{I_3 - I_1}{I_1}\omega_{z0} = \frac{1-n}{n}\omega_{z0} \Rightarrow \omega_{z0} = -\frac{n}{n-1}\Omega = \frac{n}{n-1}\dot{\psi}$$

$$\dot{\varphi} = (\omega_{z0} + \Omega)\sec\theta_0 = \frac{I_3}{I_1}\omega_{z0}\sec\theta_0 = \frac{1}{n}\frac{n}{n-1}\dot{\psi}\sec\theta_0 = \frac{1}{n-1}\dot{\psi}\sec\theta_0$$

$$\dot{\psi} = (n-1)\dot{\varphi}\cos\theta_0$$

□

3.3.5 Lagrange 陀螺

$$
\begin{aligned}
T &= \frac{I_1}{2}\left(\dot\theta^2+\dot\varphi^2\sin^2\theta\right)+\frac{I_3}{2}(\dot\psi+\dot\varphi\cos\theta)^2\\
\mathcal{L} &= \frac{I_1}{2}\left(\dot\theta^2+\dot\varphi^2\sin^2\theta\right)+\frac{I_3}{2}(\dot\psi+\dot\varphi\cos\theta)^2-mgl\cos\theta
\end{aligned}
\tag{3.66}
$$

对于循环坐标列出等式

$$
\begin{cases}
P_\varphi & =\left(I_1\sin^2\theta+I_3\cos^2\theta\right)\dot\varphi+I_3\cos\theta\dot\psi=\text{ const }=L_{z'}\\
P_\psi & =I_3(\dot\psi+\cos\theta\dot\varphi)=\text{ const }=L_z=I_3\omega_z\\
\omega_3 & =\text{ const}
\end{cases}
\tag{3.67}
$$

$$
E=\frac{I_1}{2}\left(\dot\theta^2+\dot\varphi^2\sin^2\theta\right)+\frac{I_z^2}{2I_3}+mgl\cos\theta=\text{const}
\tag{3.68}
$$

能量守恒

$$
E=\frac{I_1}{2}\left(\dot\theta^2+\dot\varphi^2\sin^2\theta\right)+\frac{I_z^2}{2I_3}+mgl\cos\theta=\text{const}
\tag{3.69}
$$

得到

$$
\begin{cases}
\dot\varphi=\dfrac{L_{z'}-L_z\cos\theta}{I_1\sin^2\theta}\\
\dot\psi=\dfrac{L_z}{I_3}-\dfrac{L_{z'}-L_z\cos\theta}{I_1\sin^2\theta}
\end{cases}
\tag{3.70}
$$

引入

$$
E'=\frac{I_1'}{2}\dot\theta^2+V_{\text{eff}}(\theta)
\tag{3.71}
$$

其中

$$
E'=E-\frac{L_3^2}{2I_3}-mgl,\quad V_{\text{eff}}(\theta)=\frac{(L_2-L_3\cos\theta)^2}{2I_1'\sin^2\theta}-mgl(1-\cos\theta).
\tag{3.72}
$$

运动方程写为：

$$
t=\int\frac{\mathrm{d}\theta}{\sqrt{\frac{2}{I_1'}\left[E'-V_{\text{eff}}(\theta)\right]}}
\tag{3.73}
$$

注 3.3. 对于一般陀螺，都是考虑两个守恒：

$$
\begin{aligned}
I_1\Omega_1^2+I_2\Omega_2^2+I_3\Omega_3^2&=2E\\
I_1^2\Omega_1^2+I_2^2\Omega_2^2+I_3^2\Omega_3^2&=L^2
\end{aligned}
\tag{3.74}
$$