Bode Plot of Filter Circuits

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1 RC Low-pass Filter

We start to consider a RC low-pass filter circuit as shown in fig. 1.

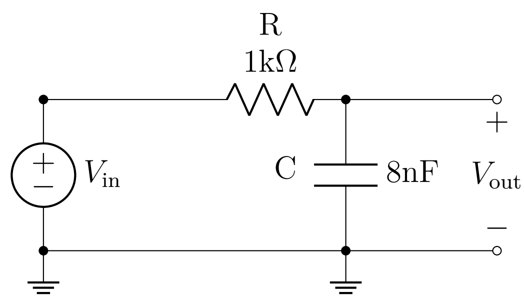


Fig. 1: RC low-pass filter circuit

The transfer function (or gain) of the circuit is given by

$$H(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_{\text{C}}}{Z_{\text{R}} + Z_{\text{C}}} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega RC} \tag{1.1}$$

1 bel (B) is equal to the logarithm of the ratio of two power levels, i.e., or a bel is a factor of 10 change in power. Also, a decibel (dB) is one-tenth of a bel, which is widely used.

$$1 \text{ B} = \log_{10}\left(\frac{P_1}{P_2}\right), \quad 1 \text{ dB} = 0.1 \text{ B} \tag{1.2}$$

Then, the magnitude (in bels) of the transfer function is given by, could be understood as attenuation or gain of the input signal,

$$A_{\text{bel}} = \log_{10}\left(\frac{P_{\text{out}}}{P_{\text{in}}}\right) = \log_{10}\left(\frac{V_{\text{out}}}{V_{\text{in}}}\right)^2 \tag{1.3}$$

Transform into the magnitude in decibels, we have the **magnitude of the transfer function in decibels** is given by

$$A_{\text{dB}} = 10\log_{10}\left(\frac{V_{\text{out}}}{V_{\text{in}}}\right)^2 = 20\log_{10}|H(j\omega)| \tag{1.4}$$

We are interested in when the power reduce to half, i.e., $P_{\text{out}} = P_{\text{in}}/2$, then we have

$$A_{\text{dB}} = 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) \approx -3.01 \text{ dB} \tag{1.5}$$

For the low-pass filter, the attenuation is

$$A_{\text{dB}} = -20\log_{10}\left(\sqrt{1+\omega^2R^2C^2}\right) \tag{1.6}$$

One could find the frequency at which the power is reduced to half, by solving the equation

$$|H(f)| = \frac{1}{\sqrt{2}} \cdot |H(f)|_{\max} \tag{1.7}$$

Therefore, we have

$$f_{-3 \text{ dB}} = \frac{1}{2\pi RC} \tag{1.8}$$

As we continue to pass -3 dB, the attenuation continues to fall off, eventually reach an approximately straight line on the Bode plot as shown in fig. 2. The fall-off rate or called roll-off rate of this line is in unit of -20 dB/decade if the gain falls like $1/f$ as in fig. 3. Therefore, this frequency is called as **cut-off frequency**, which divides passband and stopband.
The **phase of the transfer function** is given by

$$\phi(\omega) = \arctan\frac{\text{Im}H(j\omega)}{\text{Re}H(j\omega)} = \arg(H(j\omega)) \tag{1.9}$$

which means the phase difference between the output signal and the input signal. In RC low-pass filter, the phase is given by

$$\phi(\omega) = \arctan\left(\frac{-\omega RC}{1}\right) = -\arctan(\omega RC) \tag{1.10}$$

The signal remains nearly constant under certain frequency range, we define the length of the signal platform as **Band Width** (BW). The bandwidth (BW) is the difference between the upper -3 dB frequency (f_H) and the lower -3 dB frequency (f_L) :

$$BW = f_H - f_L \tag{1.11}$$

The **center** frequency is the geometric mean of the lower and upper cutoff frequencies (f_L and f_H) of a bandpass system. Mathematically, in many systems, it means the peak of the filter's response. The center frequency is given by

$$f_0 = \sqrt{f_L \cdot f_H} \tag{1.12}$$

Also, we could use **quality factor** Q to describe the selectivity of the filter, or to determine how sharp the peak is. The quality factor is defined as the ratio of the center frequency to the bandwidth.

$$Q = \frac{f_0}{BW} \tag{1.13}$$

A more modern way is to write the magnitude and phase of the transfer function in the **Laplace domain**, i.e., let $s = j\omega = j2\pi f$ is the complex frequency variable. $\omega = 2\pi f$ is the angular frequency, we have

$$H(s) = \frac{1}{1+sRC} \tag{1.14}$$

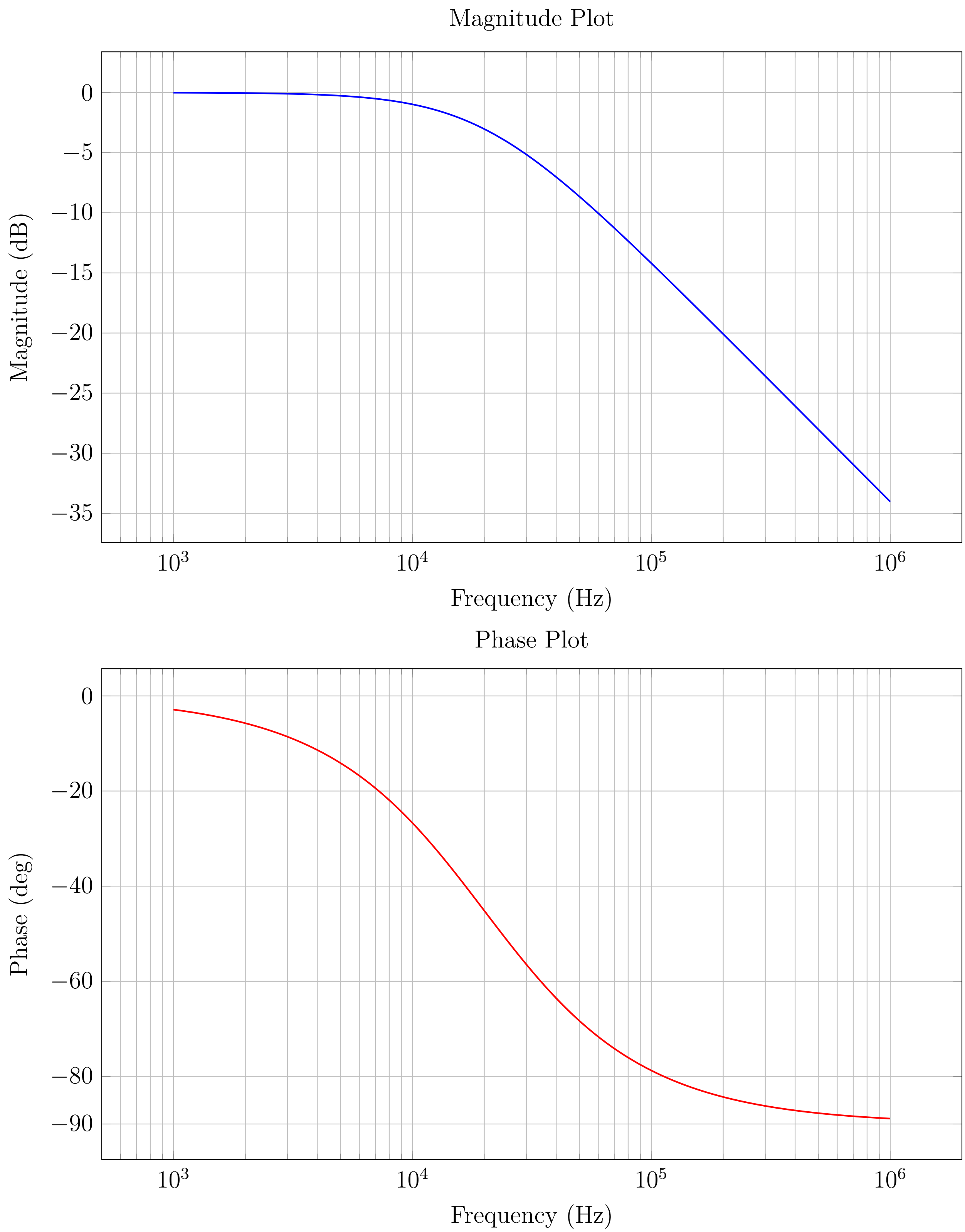


Fig. 2: Bode plot of RC low-pass filter (in logarithmic scale)

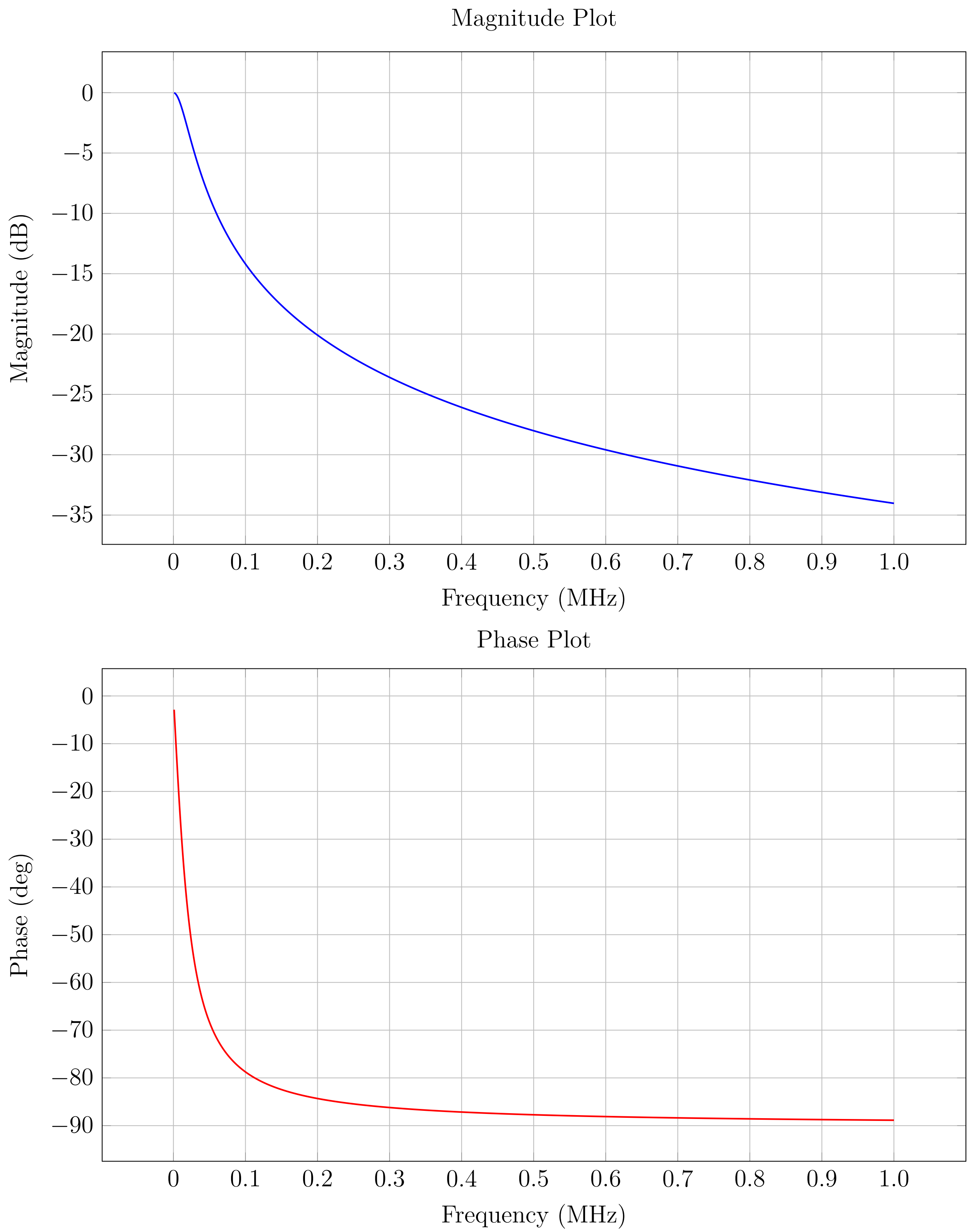


Fig. 3: Bode plot of RC low-pass filter (in normal scale)

2 RLC Resonant

Consider a **series RLC resonant circuit** as shown in fig. 4. Here the R is the Thévenin equivalent resistance of the circuit looking from the output, it contains the resistance of the inductor and the power source (function generator).

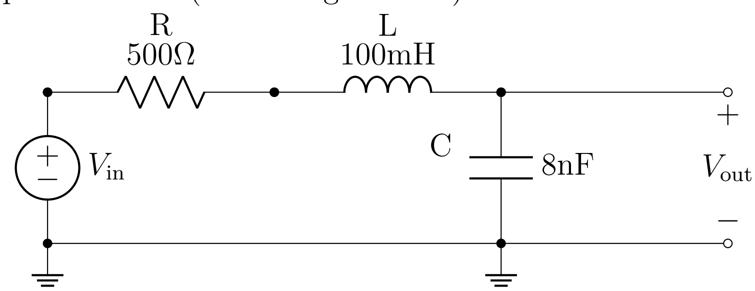


Fig. 4: Series RLC resonant circuit

The transfer function of the circuit is given by

$$H(s) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_{\text{C}}}{Z_{\text{R}} + Z_{\text{L}} + Z_{\text{C}}} = \frac{1/j\omega C}{R + j\omega L + 1/j\omega C} = \frac{1}{1 + j\omega RC - \omega^2 LC} \tag{2.1}$$

The magnitude of the transfer function is given by

$$|H(j\omega)| = \frac{1}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \tag{2.2}$$

The attenuation is given by

$$A_{\text{dB}} = 20 \log_{10} \frac{1}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}} \tag{2.3}$$

The phase of the transfer function is given by

$$\phi(\omega) = \arg(H(s)) = \arctan\left(\frac{-\omega RC}{1-\omega^2 LC}\right) \tag{2.4}$$

By analyzing the Bode plot in this case, we observe a peak in gain around the resonant frequency. This corresponds to the frequency at which the inductive and capacitive reactances are equal in magnitude, meaning their imaginary components cancel out ($\omega L = 1/(\omega C)$). The **self resonant frequency** is given by

$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{\omega_{\text{LC}}}{2\pi} \tag{2.5}$$

In RLC resonant in fig. 4, the resonant frequency is calculated as 5626.98 Hz. Comparing with fig. 5, we observe that the magnitude peak (17 dB) occurs around 5570 Hz, slightly lower than the theoretical value due to the presence of R as shown in eq. (2.9). At resonance, the phase shift is expected to be -90 degrees or $-\pi/2$ radians as shown in eq. (2.4).
For this case, the band width is the width of the resonant signal curve at half maximum:

$$\text{BW} = f_H - f_L = \Delta f = \frac{1}{2\pi}\frac{R}{L} \tag{2.6}$$

the quality factor Q (aka damping factor, calculated as 7.07) in series RLC circuit is defined:

$$Q = \frac{f_0}{\text{BW}} = \frac{\omega_{\text{LC}}}{\omega_{\text{LR}}} = \frac{L}{R\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{2.7}$$

Ony may ask how to derive the band width of RLC resonant circuit, the key idea is to find high/low cut-off frequency using the eq. (2.2) to find frequency at its half maximum. Simplify its numerator, we have a quartic equation in ω:

$$C^2L^2\omega^4 + (C^2R^2 - 2CL)\omega^2 + 1 \tag{2.8}$$

To maximize the signal, we need to minimize this formula, which requires ω to be:

$$\omega = \sqrt{\frac{2CL - C^2R^2}{2C^2L^2}} \tag{2.9}$$

which is smaller than ω_{LC}, when R is negligible, $\omega = \frac{1}{\sqrt{LC}}$, going back to eq. (2.5).
Also, the maximum value of the signal implies that the impedance has only real part, which corresponds to the resistance R. Analytically, t the half-maximum frequency points, the total impedance satisfies:

$$Z = R \pm jR = \sqrt{2}R\angle \pm 45^\circ \tag{2.10}$$

which causes the signal amplitude to be reduced by half.
Applying the impedance formula, we have

$$Z = R + \left(j\omega L - \frac{1}{j\omega C}\right) \tag{2.11}$$

By solving the equation that imaginary part satisfying,

$$\pm R = j\omega L - \frac{1}{j\omega C} \tag{2.12}$$

We obatin the lower and upper cut-off frequency.

$$\omega_L = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{2.13a}$$

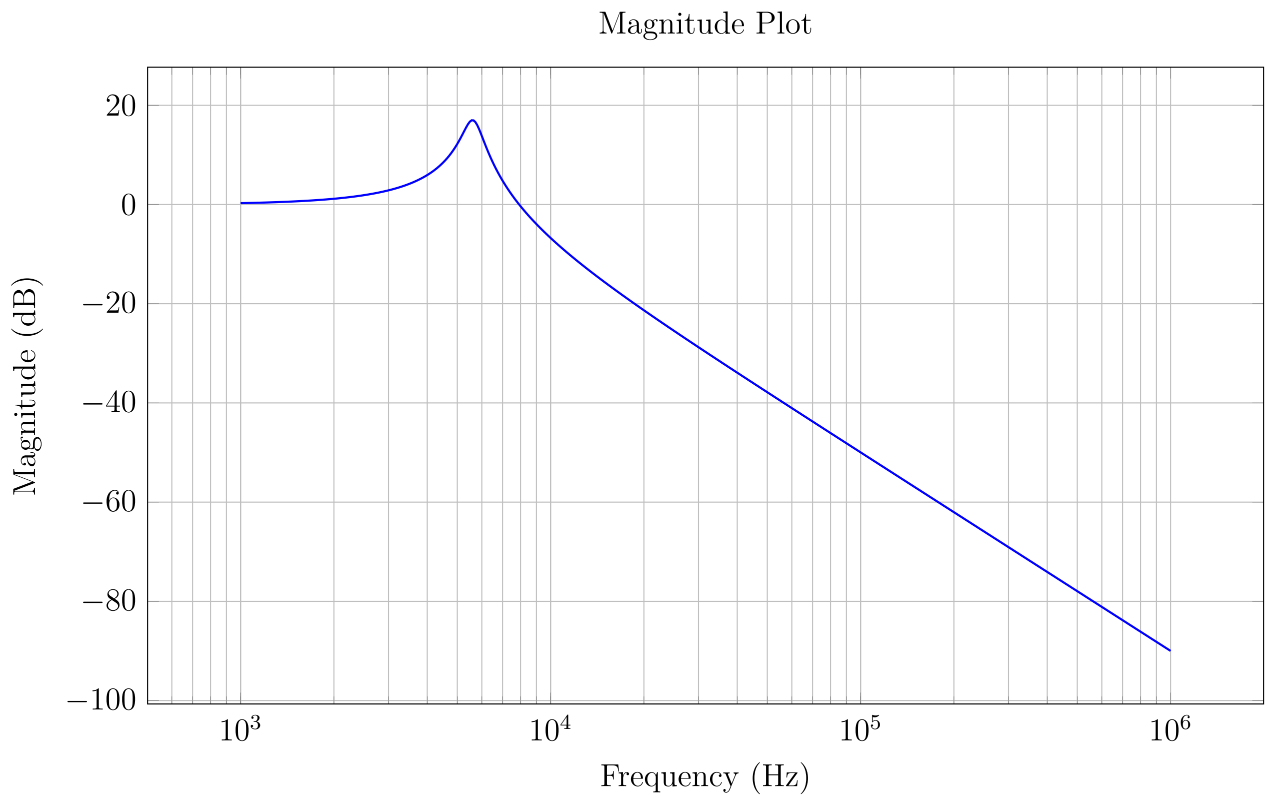
$$\omega_H = +\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \tag{2.13b}$$

So the band width is $\omega_H - \omega_L = \Delta\omega = \frac{R}{L}$ so proved the eq. (2.6).
More generally, we could express the cut-off frequency in terms of the quality factor $Q = \frac{\omega_0}{\Delta\omega}$:

$$\omega_L \approx \omega_0 - \frac{\Delta\omega}{2} = \omega_0\left(1 - \frac{1}{2Q}\right) \tag{2.14a}$$

$$\omega_H \approx \omega_0 + \frac{\Delta\omega}{2} = \omega_0\left(1 + \frac{1}{2Q}\right) \tag{2.14b}$$

and finally use $f = \frac{\omega}{2\pi}$ to get the frequency in Hz. Therefore, the height of the peak is proportional to Q, and the width of the peak is inversely proportional to Q, as shown in fig. 7. For parallel RLC circuit, its quality factor is inverse of quality factor of the series RLC circuit.



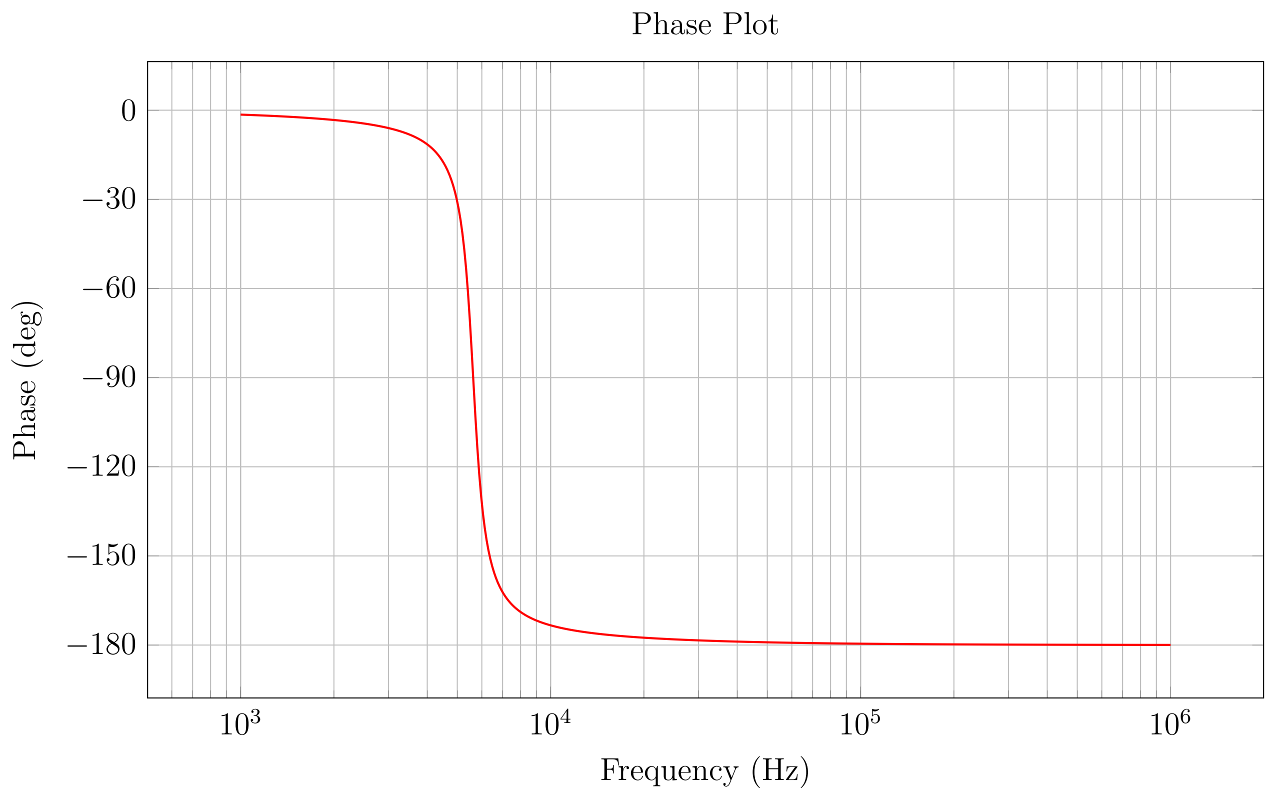


Fig. 5: Bode plot of RLC resonant (in logarithmic scale)

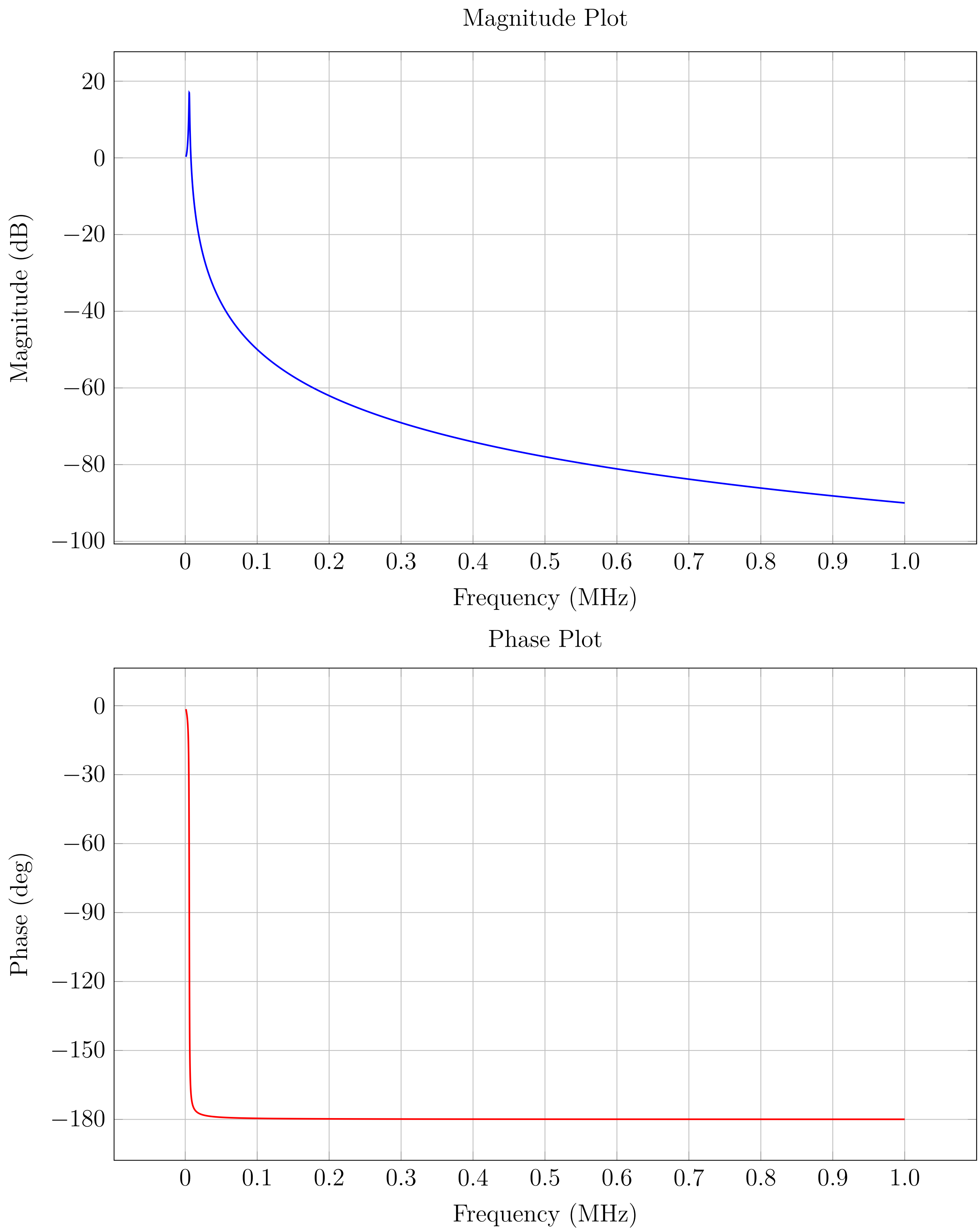


Fig. 6: Bode plot of RLC resonant (in normal scale)

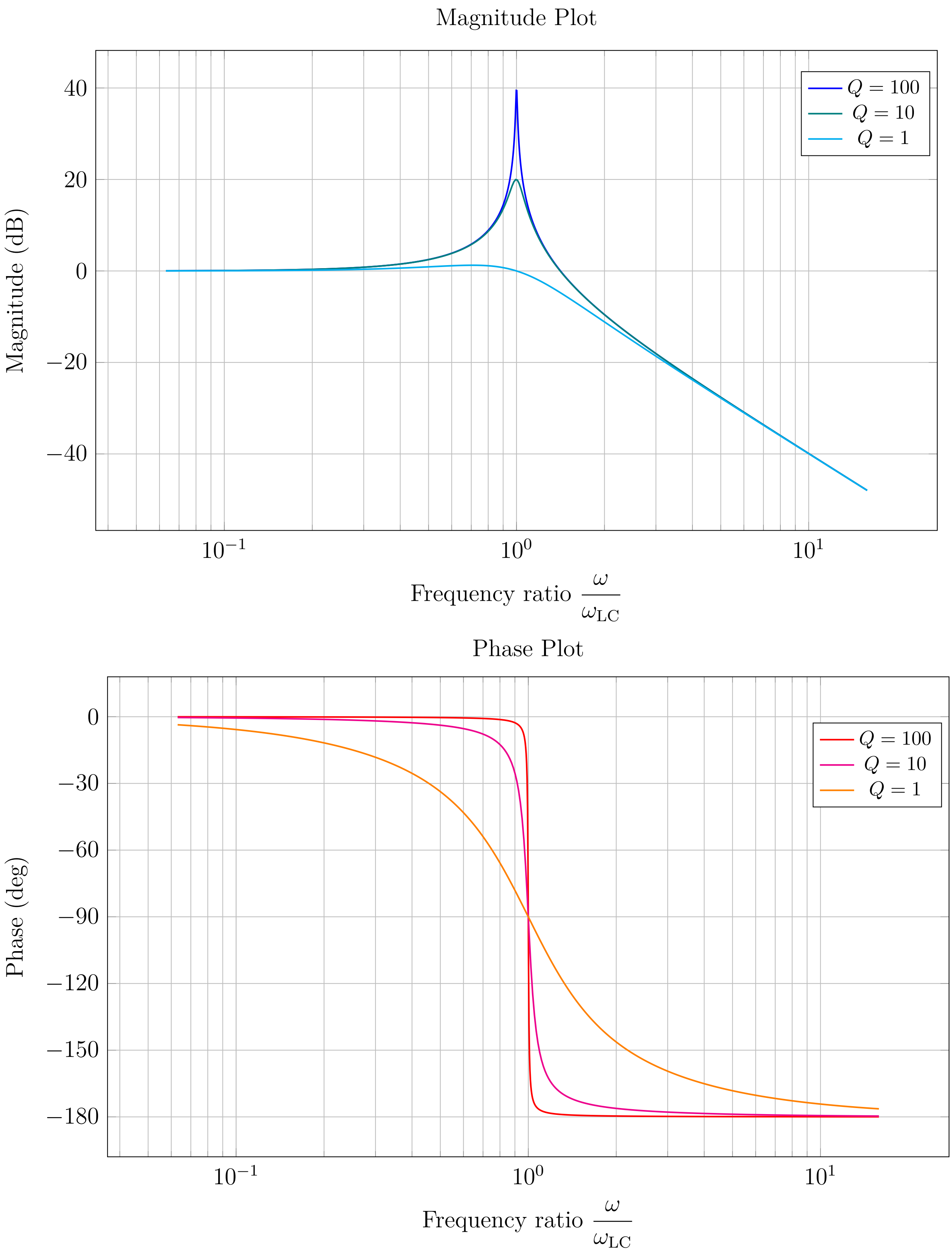


Fig. 7: Bode plot of RLC resonant with different quality factors $Q = \dfrac{\omega_{\mathrm{LC}}}{\omega_{\mathrm{LR}}} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

Here we rewrite the transfer function from eq. (2.1) as

$$H\left(s' = \frac{\omega}{\omega_{\mathrm{LC}}}\right) = \frac{1}{1 + s'/Q - s'^2} \tag{2.15}$$

3 RL Low-pass Filter

Opposite to capacitor, the inductor has a positive reactance, which means the impedance of the inductor increases with frequency. We can say the **capacitors favor high frequency current**, while the **inductors favor low frequency current**. Draw a RL low-pass filter in fig. 8.

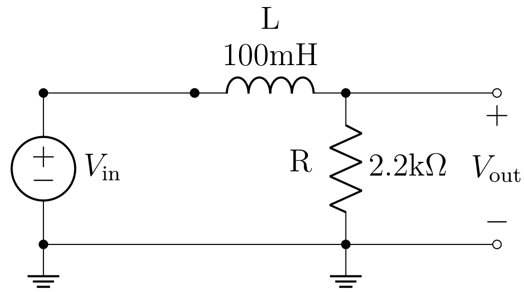


Fig. 8: RL low-pass filter circuit

Similarly, we could derive the gain of the RL low-pass filter circuit by using the voltage divider rule. Assume $R_L = 500\ \Omega$, where we already included the internal resistance of inductor and function generator. The transfer function of the RL low-pass filter is given by

$$H(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_{\text{R}}}{Z_{\text{R}} + Z_{\text{L}}} = \frac{R}{R + R_{\text{L}} + j\omega L}, \tag{3.1}$$

But in reality, the inductor contains a **parasitic capacitance** C_{p} which may have a significant effect, especially at high frequencies, typically around microwave frequencies.

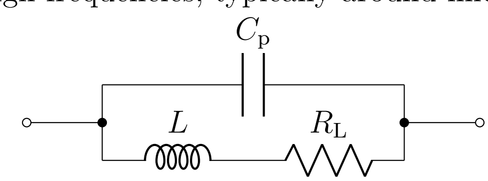


Fig. 9: Equivalent impedance of inductor with parasitic capacitance

Using equivalent circuit like in fig. 9, the transfer function can be rewritten as

$$H(\omega) = \frac{R}{R + \frac{(R_{\text{L}} + j\omega L)/(j\omega C_{\text{p}})}{R_{\text{L}} + j\omega L + \frac{1}{j\omega C_{\text{p}}}}} = \frac{R}{R + D(\omega)} \tag{3.2}$$

It is often challenging to determine the exact value of parasitic capacitance beforehand. A common method to identify when the gain of the RL low-pass filter is minimized at a certain frequency is by finding the self-resonant frequency of the inductor. To determine this frequency, we set the numerator in eq. (3.2) to be purely real number (i.e., with no imaginary component)

$$D(\omega) = \frac{Z_{\text{L}} Z_{\text{C}}}{Z_{\text{L}} + Z_{\text{C}}} = \operatorname{Re} D(\omega) = R_{\text{L}} \tag{3.3}$$

By determining this frequency experimentally, we could get the value of parasitic capacitance using the self-resonant frequency in eq. (2.5). I measured $f_0 = 208$ kHz, then $C_{\text{p}} = 5.85$ pF.

$$C_{\text{p}} = \frac{1}{\omega_0^2 L} = \frac{1}{(2\pi f_0)^2 L} \tag{3.4}$$

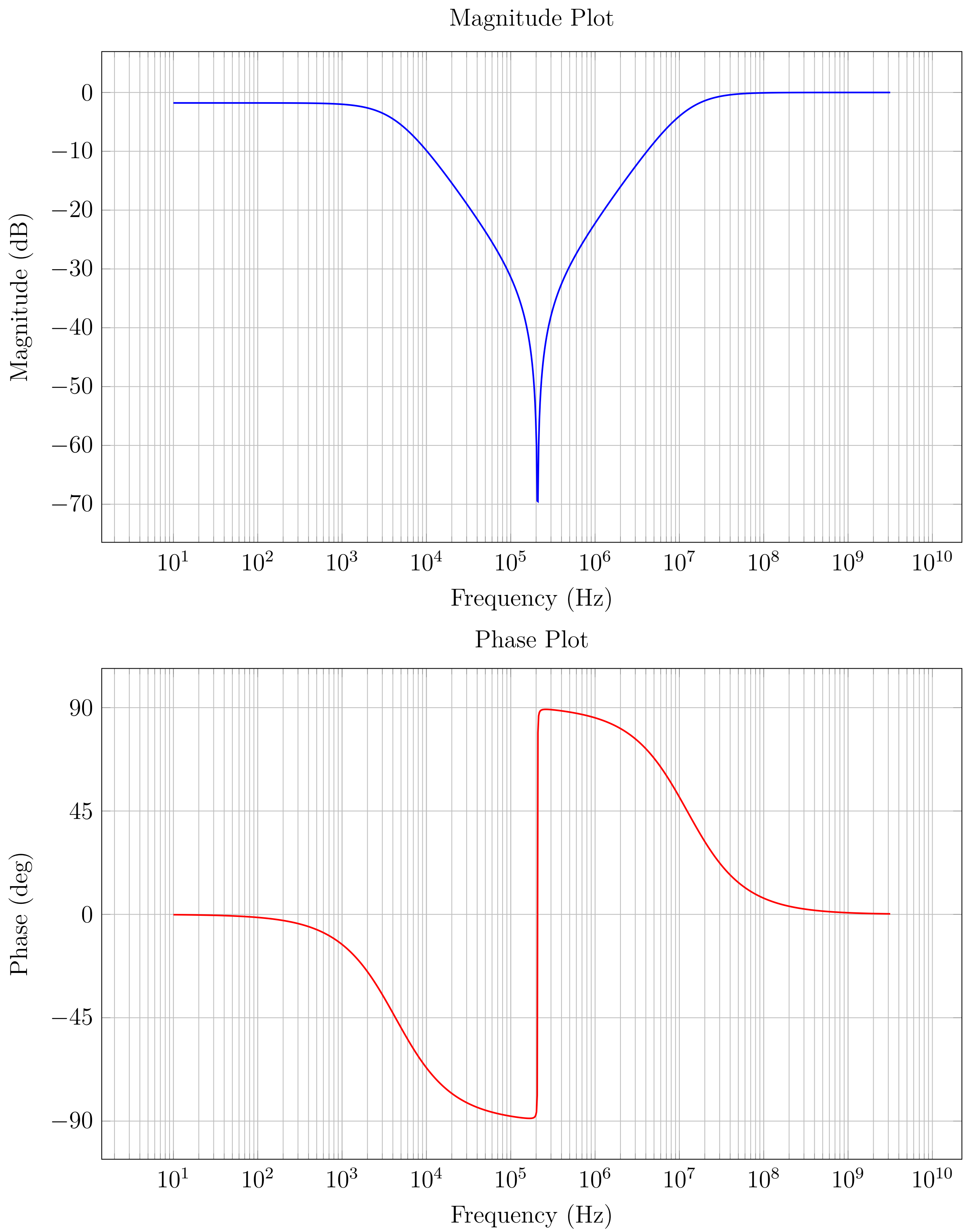


Fig. 10: Bode plot of RL low-pass filter (in logarithmic scale)
Analogy: RL low-pass filter + RC high-pass filter + RLC resonant circuit.
But for RC low-pass filter, the resonant frequency is above 10^8 Hz, not shown in fig. 2

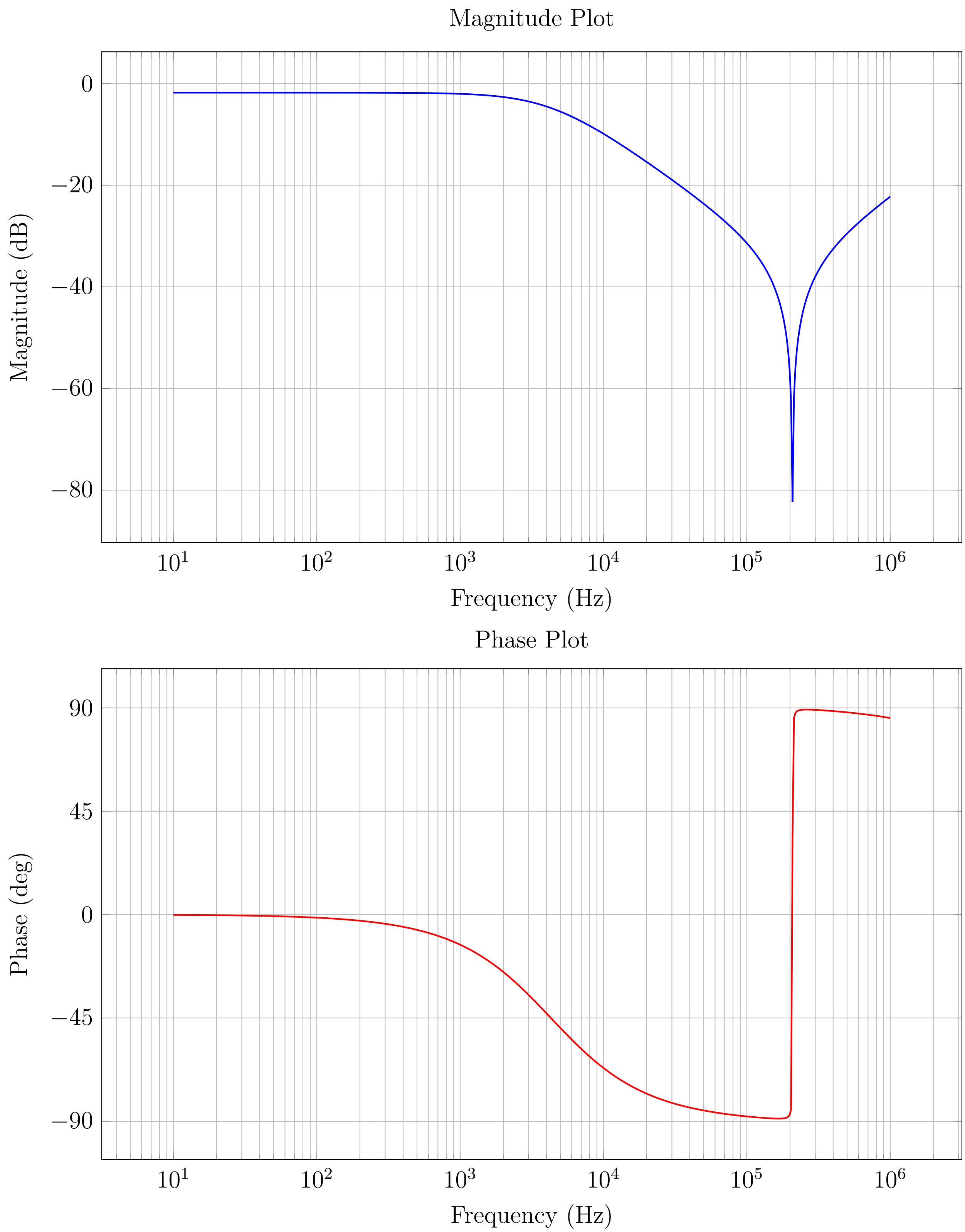


Fig. 11: Bode plot of RL low-pass filter (in logarithmic scale)

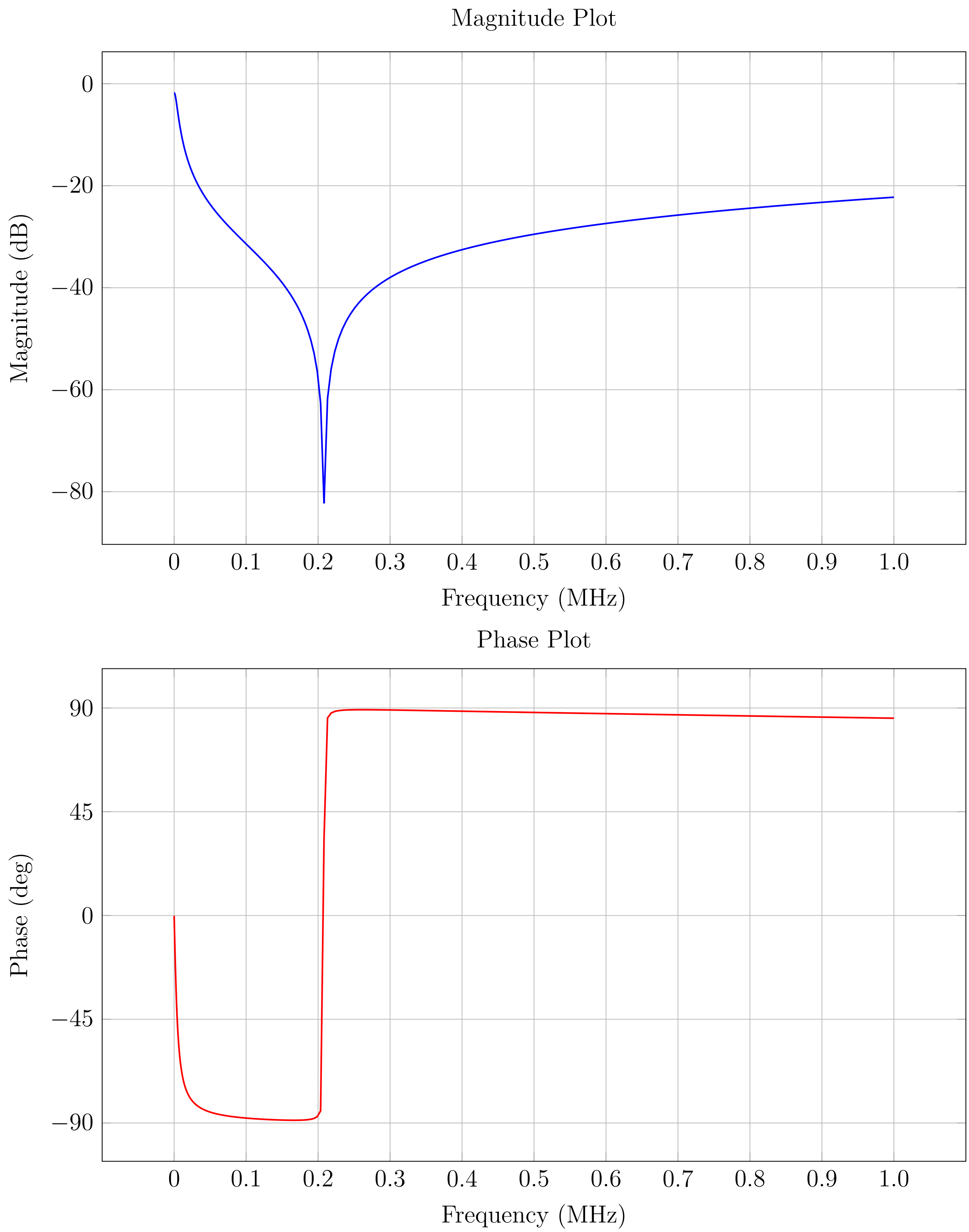


Fig. 12: Bode plot of RL low-pass filter (in normal scale)

4 Band-pass Filter

Draw a band-pass filter circuit as shown in fig. 20.

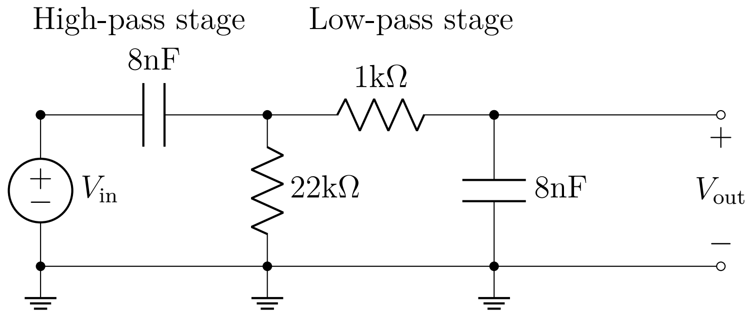


Fig. 13: RC band-pass filter circuit (cascaded high-pass and low-pass stages)

With $f_H = f_{\text{low-pass}} = 1/(2\pi R_2 C_2) \geqslant f_L = f_{\text{high-pass}} = 1/(2\pi R_1 C_1)$, we get $R_1 C_1 \geqslant R_2 C_2$.

The transfer function of the band-pass filter is given by (as a thrid order band-pass filter)

$$H(s) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_{R_1}//(Z_{R_2}+Z_{C_2})}{[Z_{R_1}//(Z_{R_2}+Z_{C_2})]+Z_{C_1}} \cdot \frac{Z_{C_2}}{Z_{C_2}+Z_{R_2}} = \frac{R_1\left(R_2+\frac{1}{sC_2}\right)}{R_1\left(R_2+\frac{1}{sC_2}\right)+\frac{1}{sC_1}\left(R_1+R_2+\frac{1}{sC_2}\right)} \cdot \frac{\frac{1}{sC_2}}{R_2+\frac{1}{sC_2}} \tag{4.1}$$

The magnitude of the transfer function is given by (behaves like third order band-pass filter)

$$|H(s)| = \left| \frac{R_1\left(R_2+\frac{1}{sC_2}\right)}{R_1\left(R_2+\frac{1}{sC_2}\right)+\frac{1}{sC_1}\left(R_1+R_2+\frac{1}{sC_2}\right)} \cdot \frac{\frac{1}{sC_2}}{R_2+\frac{1}{sC_2}} \right| \tag{4.2}$$

The attenuation is given by

$$A_{\text{dB}} = 20\log_{10}|H(s)| \tag{4.3}$$

The phase of the transfer function is given by

$$\phi(\omega) = \arg(H(s)) \tag{4.4}$$

Center Frequency (f_0) (calculated as 4241.49 Hz):

$$f_0 = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}} \tag{4.5}$$

Bandwidth (BW) (calculated as 18990.78 Hz):

$$\text{BW} = f_H - f_L \approx \frac{1}{2\pi}\left(\frac{1}{R_2 C_2} - \frac{1}{R_1 C_1}\right) \tag{4.6}$$

Quality Factor (Q) (calculated as 0.2233):

$$Q = \frac{f_0}{\text{BW}} = \frac{\sqrt{R_1 R_2 C_1 C_2}}{R_1 C_1 - R_2 C_2} \tag{4.7}$$

A very low quality factor (close to 0) indicates that the system is heavily overdamped. This means it does not behave as a narrow band-pass filter, but rather as a wide band-pass filter. Note that when the frequency is near the center frequency, the phase difference is expected to be 0 (at $f_0 \approx 4241.49$ Hz) and the gain will peak, as shown in fig. 14.

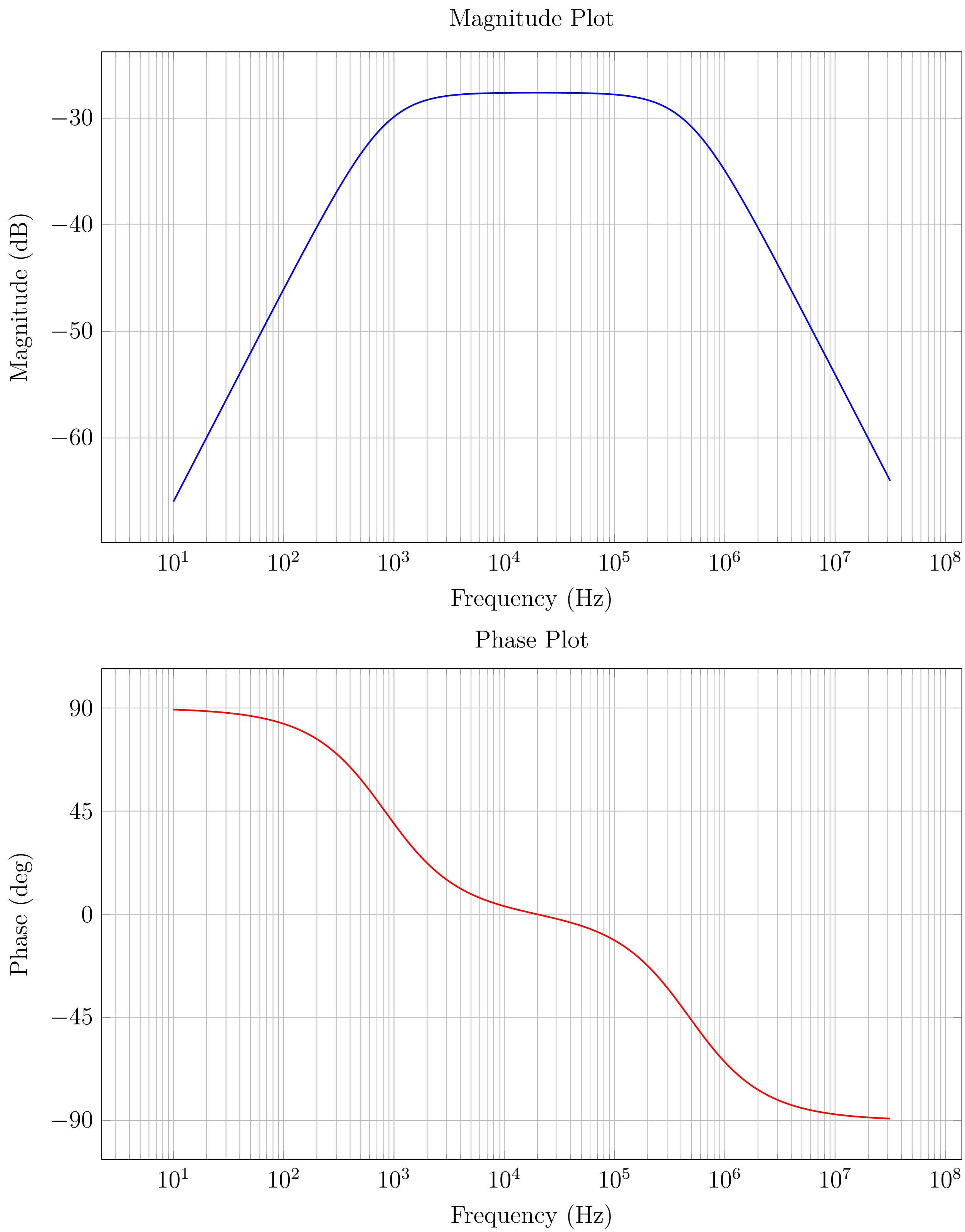


Fig. 14: Bode plot of band-pass filter (in logarithmic scale)

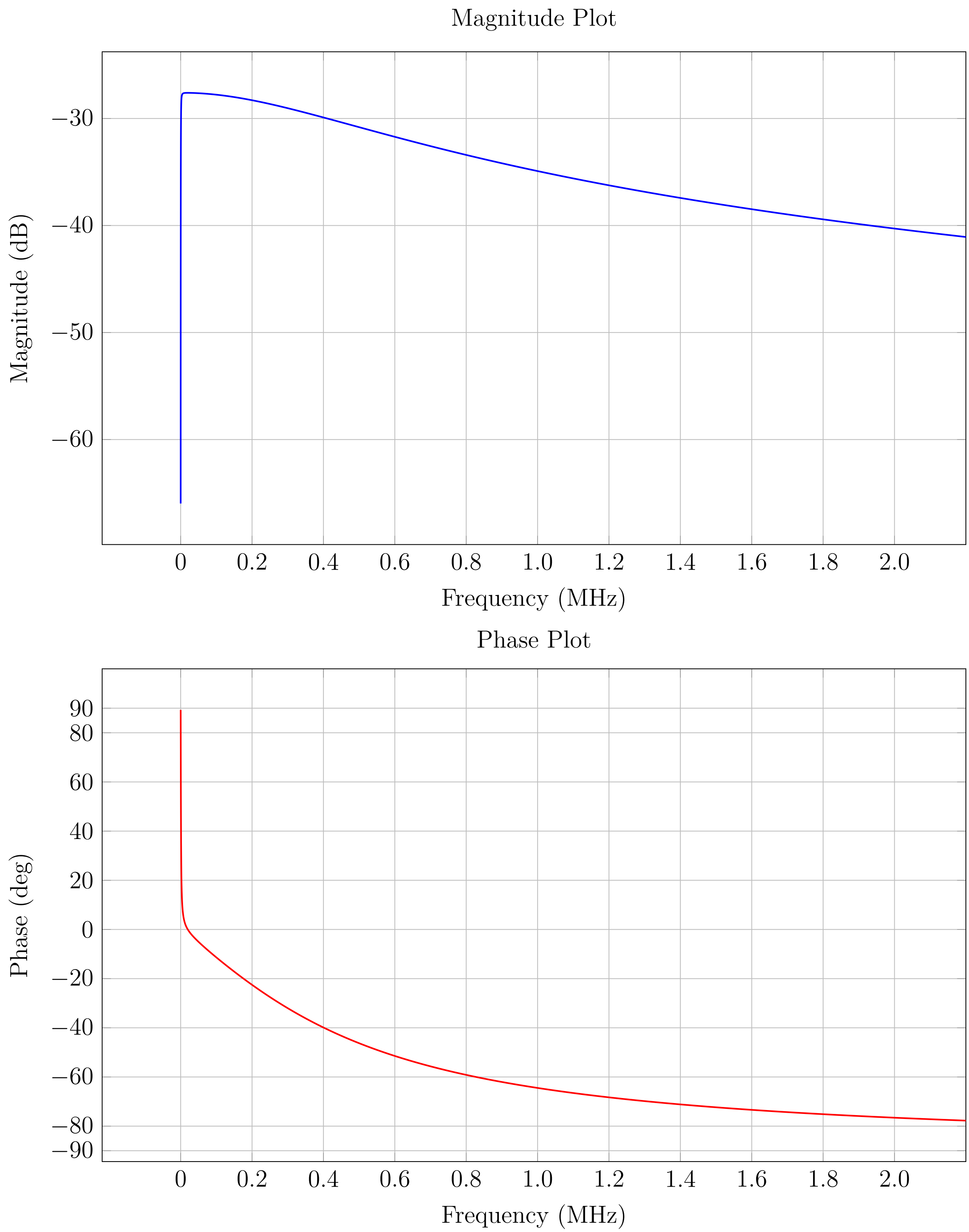


Fig. 15: Bode plot of band-pass filter (in normal scale)

In the case of $R_1C_1 = R_2C_2$, with $|Z_{R_1}| \ll |Z_{R_2} + Z_{C_2}|$, or generally for impedance of stages:

$$|Z_{\text{high-pass}}| \ll |Z_{\text{low-pass}}| \ll |Z_{\text{out}}|, \tag{4.8}$$

with this **inequality of impedance** satisfied, we could neglect $|Z_{R_2} + Z_{C_2}|$ in eq. (4.1), and simplify the transfer function to a product of a RC low-pass and a RC high-pass filter:

$$H(s) = \frac{V_{\text{out}}}{V_{\text{in}}} \approx \frac{Z_{R_1}}{Z_{R_1} + Z_{C_1}} \cdot \frac{Z_{C_2}}{Z_{C_2} + Z_{R_2}} = \frac{\frac{1}{sC_1}}{R_1 + \frac{1}{sC_1}} \times \frac{R_2}{R_2 + \frac{1}{sC_2}} \tag{4.9}$$

We could interpret the band-pass filter as a combination of a low-pass and a high-pass filter, which is shown in fig. 16. The Bode plot of the band-pass filter is deviated from the theoretical curve (depending on the difference of $Z_{\text{high-pass}}$ and $Z_{\text{low-pass}}$), but the peak still occurs at the center frequency f_0. We use dashed line to represent RC low pass and RC high pass in the plot for better clarity.

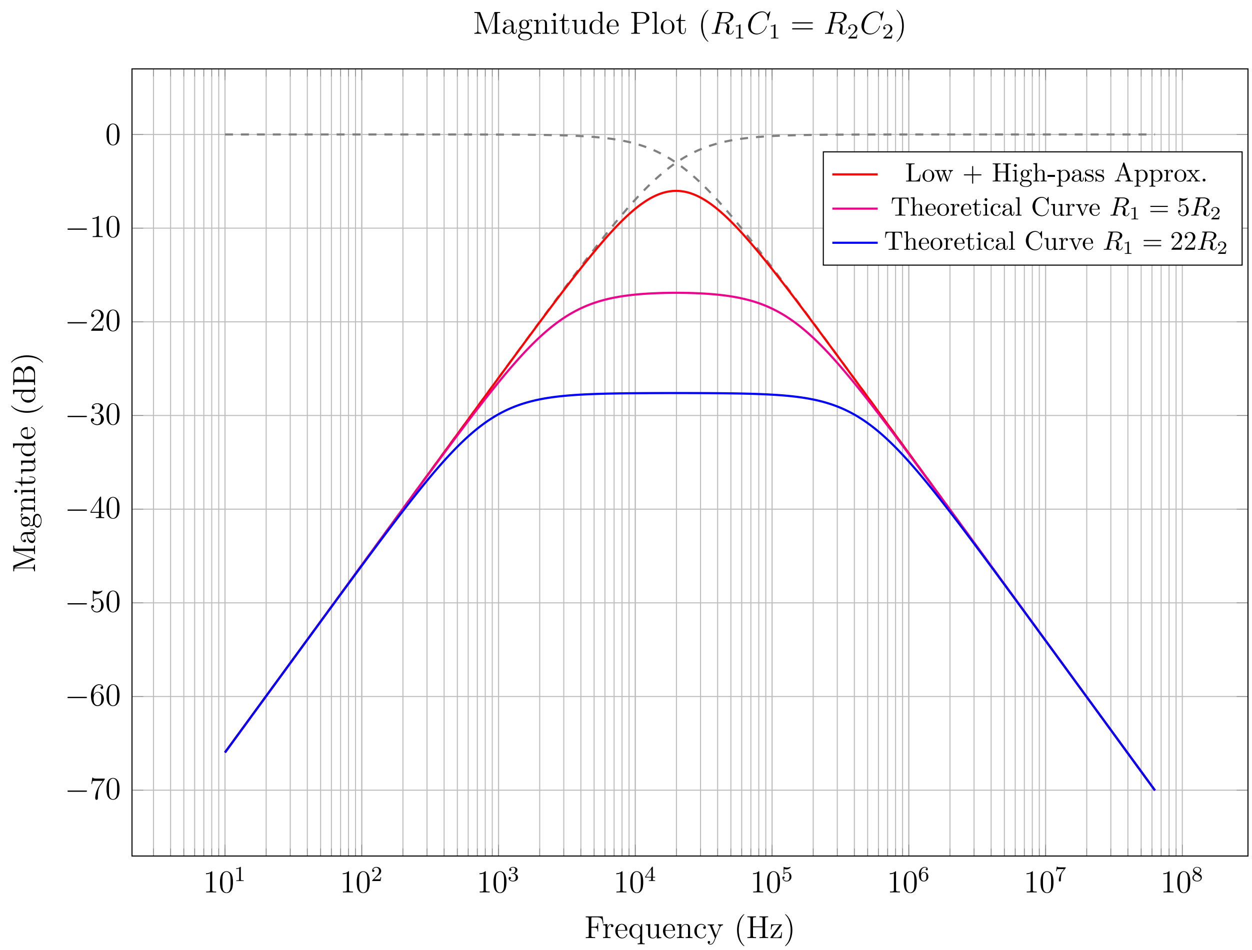


Fig. 16: Comparison between theoretical and approximation of Bode plots of band-pass filter

From this plot, we know that for for $R_1 \gg R_2$ we should use eq. (4.1) to calculate the actual transfer function; otherwise, we won't find the platform area of the band-pass filter. But for $R_2 \gg R1$ theoretical curve closely matches the approxmation red curve in fig. 17.

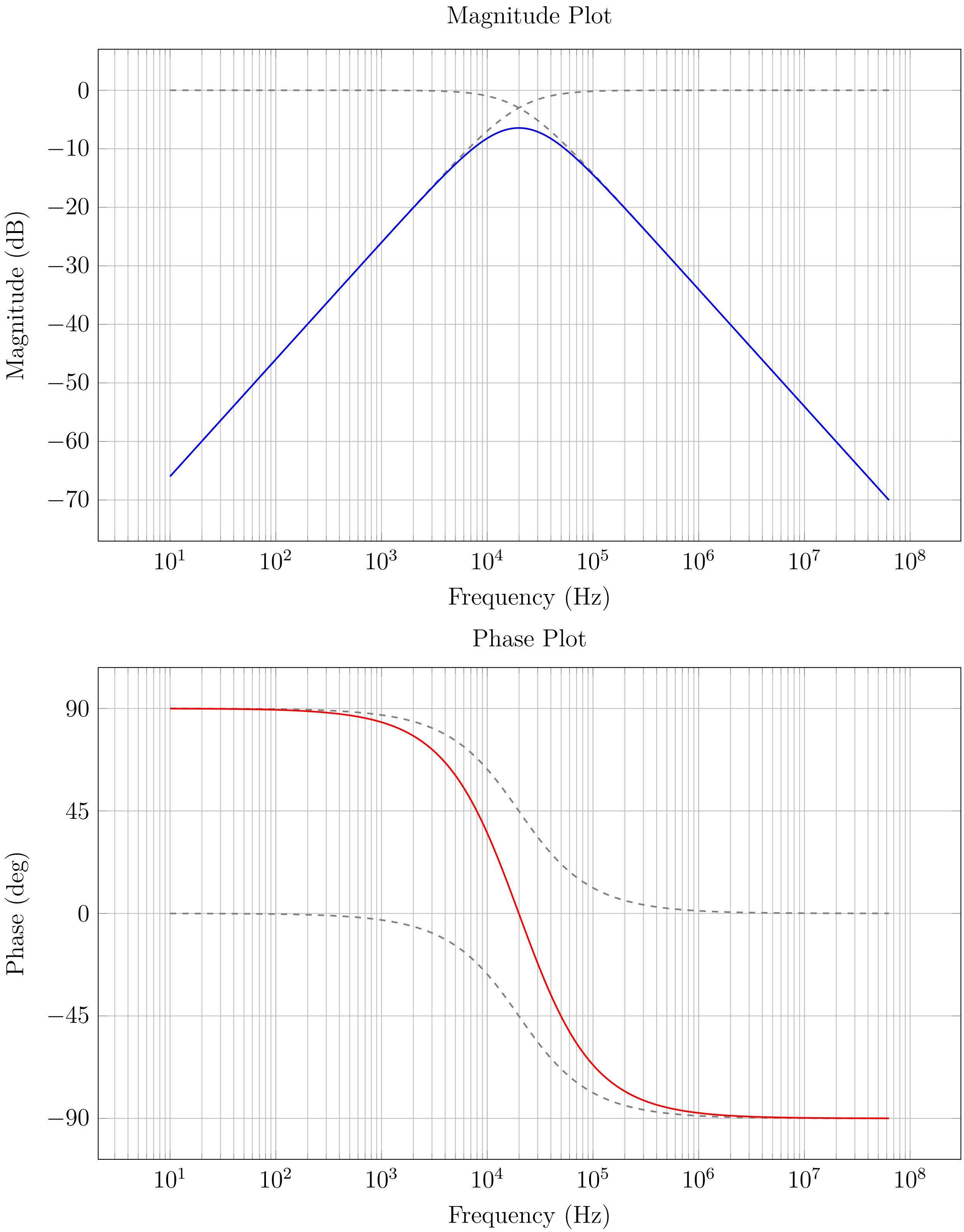


Fig. 17: Bode plot of band-pass filter when $R_1C_1 = R_2C_2$ (in logarithmic scale)
Grey lines represent the RC low-pass and RC high-pass filters' lines.

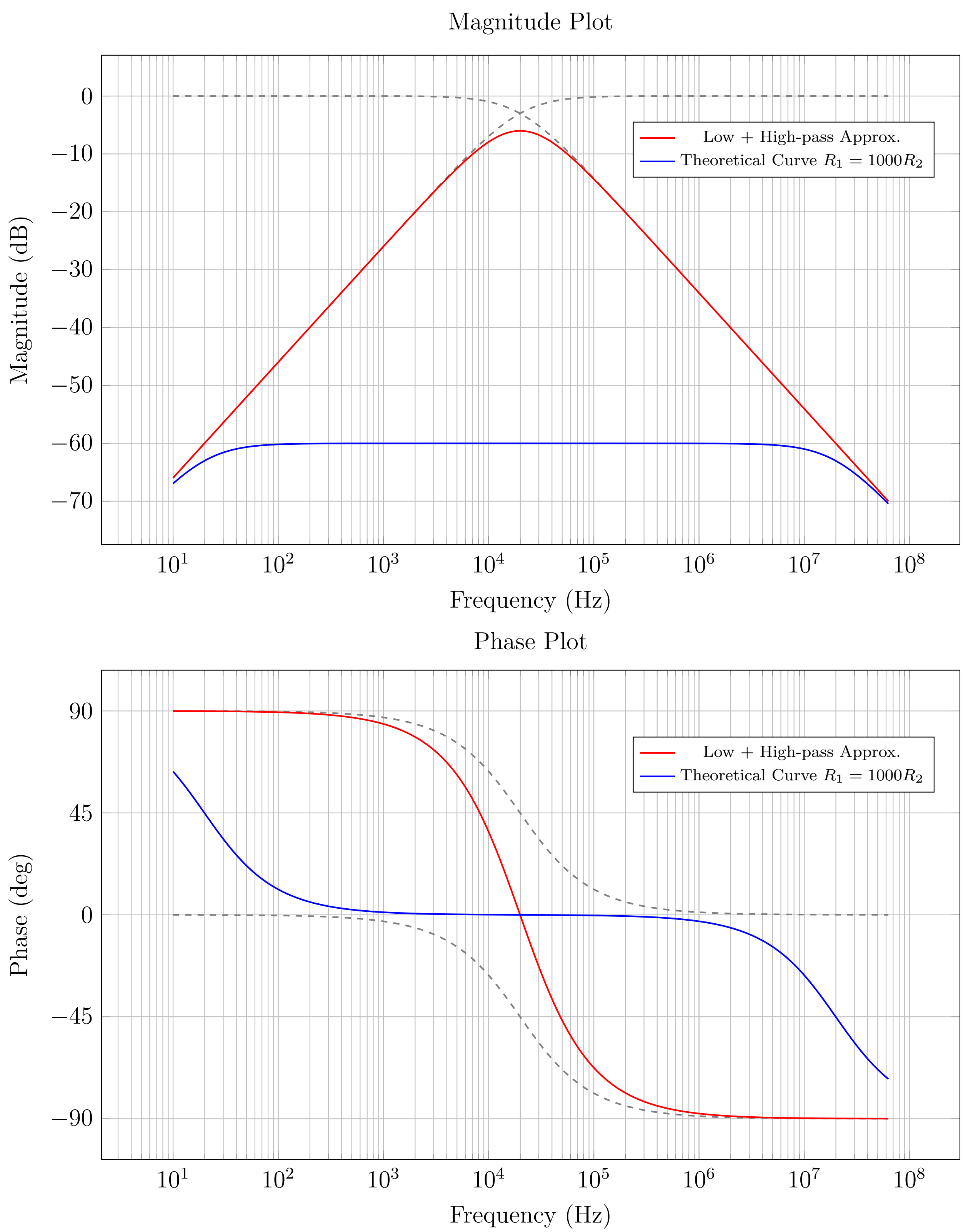


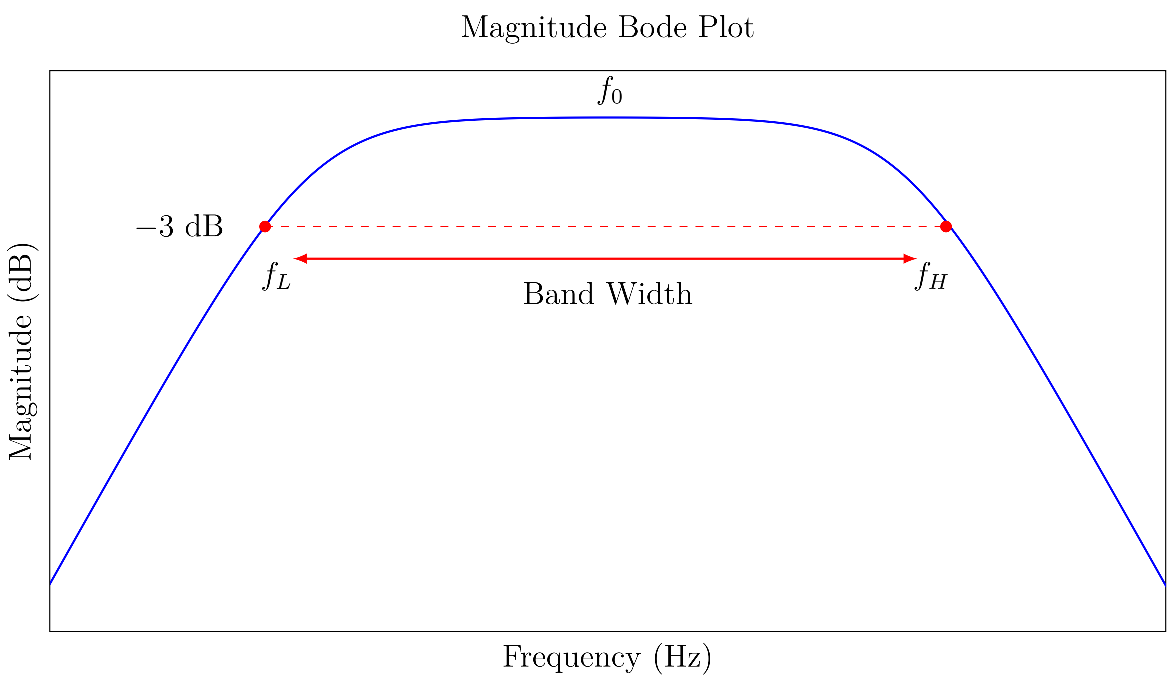
Fig. 18: Bode plot of band-pass filter when $R_1C_1 = R_2C_2$ (in logarithmic scale): Change the ratio of R_1/R_2 to high enough to create a wide band-pass filter

Supplementary Materials

Table 1: Filter components characteristics and parameters

Parameter	**RC Low-pass**	**RLC resonant**	**Band-pass**
Impedance $Z(s)$, $s = j\omega$	$R = R$ $C = \frac{1}{sC}$	$R = R$ $L = sL$ $C = \frac{1}{sC}$	R_1, R_2 C_1, C_2
Magnitude of Gain $\lvert H(s)\rvert$	$\frac{1}{\sqrt{1+(sRC)^2}}$	$\frac{1}{\sqrt{(1-s^2LC)^2+(sRC)^2}}$	$\left\lvert \frac{R_1\left(R_2+\frac{1}{sC_2}\right)}{R_1\left(R_2+\frac{1}{sC_2}\right)+\frac{1}{sC_1}\left(R_1+R_2+\frac{1}{sC_2}\right)} \cdot \frac{\frac{1}{sC_2}}{R_2+\frac{1}{sC_2}} \right\rvert$
Phase difference $\phi(s)$	$\arctan(jsRC)$	$\arctan\left(\frac{jsRC}{1+s^2LC}\right)$	$\arg(H(s))$
Center frequency f_0	-	$\frac{1}{2\pi\sqrt{LC}}$	$\frac{1}{2\pi\sqrt{R_1R_2C_1C_2}}$
Cutoff frequency $f_{-3\mathrm{dB}}$	$\frac{1}{2\pi RC}$	$\mp\frac{R}{4\pi L}+\frac{1}{2\pi}\sqrt{\left(\frac{R}{2L}\right)^2+\frac{1}{LC}}$	$\approx \frac{1}{2\pi R_1C_1}, \frac{1}{2\pi R_2C_2}$
Band Width (BW)	-	$\frac{1}{2\pi}\frac{R}{L}$	$\frac{1}{2\pi}\left(\frac{1}{R_2C_2}-\frac{1}{R_1C_1}\right)$
Quality fatcor	-	$\frac{1}{R}\sqrt{\frac{L}{C}}$	$\frac{\sqrt{R_1R_2C_1C_2}}{R_1C_1-R_2C_2}$

Fig. 19: Definition of the Band Width, Cutoff frequencies, and Center frequency



All the Bode plots above are drawn with raw data calculated by Mathematica. You may find the code to draw them in the attached Mathematica notebook "Lab4_Bode_Plot.nb" file. The data is exported to ".dat" file and plotted in logarithmic scale and normal scale.

5 N-stage RC Low-pass

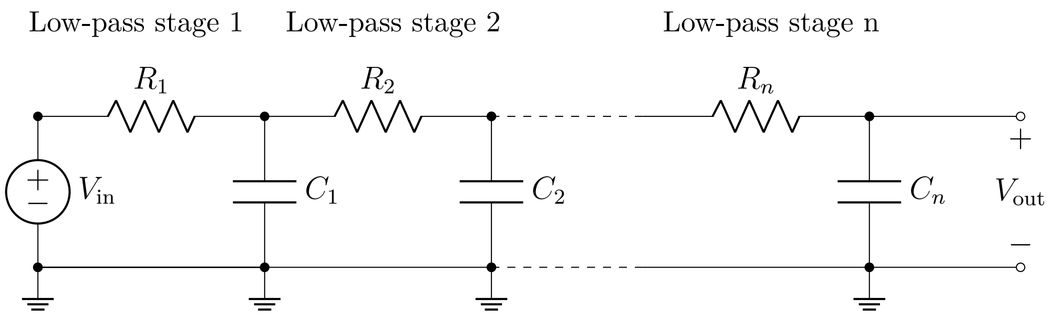


Fig. 20: n RC low-pass filter circuit, to achieve better low-pass filtering

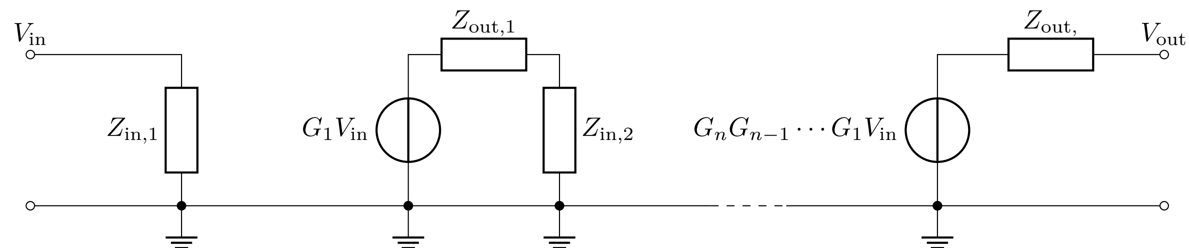


Fig. 21: Thévenin equivalent circuit of n-stage RC low-pass fine filter

After one RC low-pass filter stage, the output voltage is given by

$$V_{\text{out},2} = G_1 V_{\text{in}} \frac{Z_{\text{in},2}}{Z_{\text{out},1} + Z_{\text{in},2}} \tag{5.1}$$

After n stages, the output voltage is given by

$$V_{\text{out,n}} = V_{\text{in}} \prod_{i=n-1}^{1} G_{i+1} \frac{Z_{\text{in,i+1}}}{Z_{\text{out,i}} + Z_{\text{in,i+1}}} \tag{5.2}$$

If assume $|Z_{\text{in,i+1}}| \gg |Z_{\text{out,i}}|$ to make the fraction nearly 1, like in eq. (4.8), then

$$V_{\text{out,n}} = G_n G_{n-1} \cdots G_1 V_{\text{in}} = \prod_{i=n}^{1} G_i V_{\text{in}} \tag{5.3}$$

Recall that G_n is the gain of the n-th stage, we could use the transfer function of the RC low-pass filter to find gain of each stage, from eq. (1.1) we have

$$G_i = \frac{1}{1 + j\omega R_i C_i} \tag{5.4}$$

where ω is the angular frequency of the input signal. If we set the $R_i C_i$ of each stage to be the same, the overall gain is n-th power of the gain of each stage, which is

$$G = \prod_{i=n}^{1} G_i = \left(\frac{1}{1 + j\omega R_i C_i} \right)^n \tag{5.5}$$

The low-pass filtering of n-stage RC low-pass filter is very good, as shown in the Bode plots.

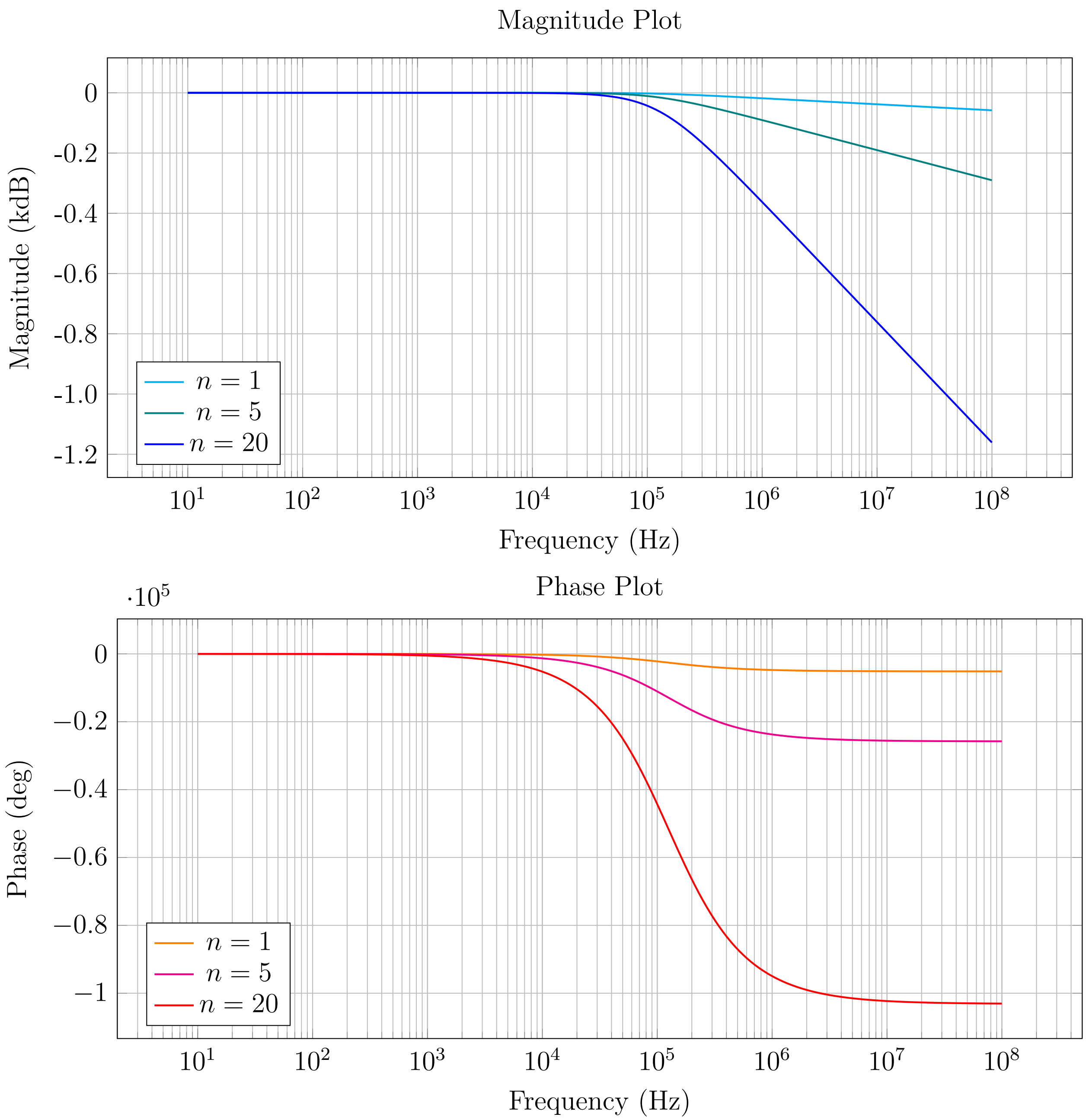


Fig. 22: Bode plots of n-stage RC low-pass filter (logarithmic scale)

The magnitude is

$$20\log_{10}\left(\frac{1}{\sqrt{1+(\omega RC)^2}}\right)^n = -10n\log_{10}\left(1+(\omega RC)^2\right) \tag{5.6}$$

and the phase is

$$-\arctan((\omega RC)^n) = -n\arctan(\omega RC) \tag{5.7}$$

In electronics, n-stage RC low-pass filters are commonly used for better low-pass filtering, offering enhanced high-frequency attenuation which help reduce noise, a sharper transition between the passband and stopband, and a more gradual phase shift.