Enhanced Aerial Reorientation Performance Using a 3-DoF Morphable Inertial Tail Inspired by Kangaroo Rats

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I. MODELING

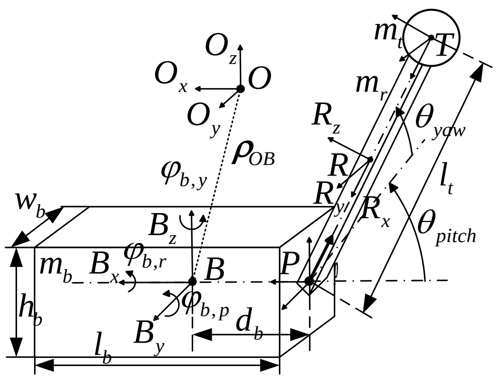

Fig. 1. Analytical models of the kangaroo rat and our tailed robot with a 3-DoF tail.

We start with an angular momentum equation expressed in the inertial frame $\{O\}$:

$$H^O = I_b^O \boldsymbol{\omega}_b^O + I_R^O \boldsymbol{\omega}_r^O + m_b \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O, \tag{1}$$

where I_b^O and I_r^O are the inertias of the robot body and tail link in the inertial frame $\{O\}$. m_t, m_r, m_b denote the mass of the body, the tail link, and the tail end mass. $\rho_{OT}^O, \rho_{OR}^O, \rho_{OB}^O$ are the vectors from the origin of the frame $\{O\}$ to the origins of the tail end frame $\{T\}$, the tail link frame $\{R\}$, and the body frame $\{B\}$. It can be also expressed in the body frame as:

$$H^B = I_b^B \boldsymbol{\omega}_b^B + I_R^B \boldsymbol{\omega}_r^B + m_b (R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + m_r (R_B^O)^T \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t (R_B^O)^T \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O, \tag{2}$$

where R_B^O is the rotation matrix from frame $\{B\}$ to frame $\{O\}$. Based on $\boldsymbol{\rho}_{OB}^O = R_B^O \boldsymbol{\rho}_{OB}^B$, we have $\dot{\boldsymbol{\rho}}_{OB}^O = \dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B$. Then, we have:

$$\begin{aligned}
&(R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O \\
&= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B\right) + (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left((R_B^O)^T \dot{R}_B^O \boldsymbol{\rho}_{OB}^B\right) + \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \\
&\left((R_B^O)^T R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\left[\boldsymbol{\omega}_b^O \times\right] R_B^O\right) \boldsymbol{\rho}_{OB}^B\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\boldsymbol{\omega}_b^O \times \boldsymbol{\rho}_{OB}^O\right)\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \boldsymbol{\rho}_{OB}^B \times \left(\boldsymbol{\omega}_b^B \times \boldsymbol{\rho}_{OB}^B\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \left(\boldsymbol{\rho}_{OB}^B\right)^2 \boldsymbol{\omega}_b^B - (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OB}^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= -\left[\boldsymbol{\rho}_{OB}^B \times\right]\left[\boldsymbol{\rho}_{OB}^B \times\right] \boldsymbol{\omega}_b^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B,
\end{aligned} \tag{3}$$

where $R(x \times y) = (Rx) \times (Ry)$ and $\Omega_b^O = \left[\boldsymbol{\omega}_b^O \times\right] = \dot{R}_B^O (R_B^O)^T$ are used. Here $\left(\boldsymbol{\rho}_{OB}^B\right)^2 = \left(\boldsymbol{\rho}_{OB}^B\right) \cdot \left(\boldsymbol{\rho}_{OB}^B\right)$. Then Eq. 2 becomes:

$$\begin{aligned}
H^B = {} & I_b^B \boldsymbol{\omega}_b^B + I_R^B \boldsymbol{\omega}_r^B + m_b \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B + m_r \boldsymbol{\rho}_{OR}^B \times \dot{\boldsymbol{\rho}}_{OR}^B + \\
& m_t \boldsymbol{\rho}_{OT}^B \times \dot{\boldsymbol{\rho}}_{OT}^B + m_b \left(\boldsymbol{\rho}_{OB}^B\right)^2 \boldsymbol{\omega}_b^B + m_r \left(\boldsymbol{\rho}_{OR}^B\right)^2 \boldsymbol{\omega}_b^B + m_t \\
& \left(\boldsymbol{\rho}_{OT}^B\right)^2 \boldsymbol{\omega}_b^B - m_b (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OB}^B - m_r (\boldsymbol{\rho}_{OR}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OR}^B - \\
& m_t (\boldsymbol{\rho}_{OT}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OT}^B.
\end{aligned} \tag{4}$$

At the robot center of mass (CoM), we have:

$$m_b \boldsymbol{\rho}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O = \boldsymbol{0}, \tag{5}$$

here

$$\begin{aligned}
&\boldsymbol{\rho}_{OR}^O = \boldsymbol{\rho}_{OT}^O - \boldsymbol{\rho}_{RT}^O = \boldsymbol{\rho}_{OT}^O - R_B^O \boldsymbol{\rho}_{RT}^B = \\
&\boldsymbol{\rho}_{OT}^O - \frac{1}{2} l_t R_B^O \left[-c_{pitch} c_{yaw} \quad -s_{yaw} \quad s_{pitch} c_{yaw}\right]^T,
\end{aligned} \tag{6}$$

where $c_{pitch} = \cos\theta_{pitch}$, $s_{pitch} = \sin\theta_{pitch}$, $c_{yaw} = \cos\theta_{yaw}$, $s_{yaw} = \sin\theta_{yaw}$. θ_{pitch} is the tail swing angle in the body pitch direction and θ_{yaw} is the tail swing angle in the body yaw direction. $\boldsymbol{p}^B = \left[-c_{pitch} c_{yaw} \quad -s_{yaw} \quad s_{pitch} c_{yaw}\right]^T$ is unit tail reorientation vector in frame $\{B\}$. l_t denotes the tail length. Substitute Eq. 6 into Eq. 5, we get:

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$$
\begin{aligned}
\boldsymbol{\rho}_{OT}^{B} &= -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t \\
&\left[-c_{pitch}c_{yaw} \quad -s_{yaw} \quad s_{pitch}c_{yaw}\right]^T, \\
\boldsymbol{\rho}_{OR}^{B} &= -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} - \frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t \\
&\left[-c_{pitch}c_{yaw} \quad -s_{yaw} \quad s_{pitch}c_{yaw}\right]^T.
\end{aligned}
\tag{7}
$$

We also have:

$$
\begin{aligned}
\dot{\boldsymbol{\rho}}_{OT}^{B} &= -\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)}\dot{l}_t \\
&\left[-c_{pitch}c_{yaw} \quad -s_{yaw} \quad s_{pitch}c_{yaw}\right]^T + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t [s_{pitch}c_{yaw}\dot{\theta}_{pitch} \\
&+c_{pitch}s_{yaw}\dot{\theta}_{yaw} \quad -c_{yaw}\dot{\theta}_{yaw} \quad c_{pitch}c_{yaw}\dot{\theta}_{pitch} - s_{pitch}s_{yaw}\dot{\theta}_{yaw}]^T,
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{\rho}}_{OR}^{B} &= -\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^{B} - \frac{1}{2}\frac{m_t}{(m_r+m_t)}\dot{l}_t \\
&\left[-c_{pitch}c_{yaw} \quad -s_{yaw} \quad s_{pitch}c_{yaw}\right]^T - \frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t [s_{pitch}c_{yaw}\dot{\theta}_{pitch} \\
&+c_{pitch}s_{yaw}\dot{\theta}_{yaw} \quad -c_{yaw}\dot{\theta}_{yaw} \quad c_{pitch}c_{yaw}\dot{\theta}_{pitch} - s_{pitch}s_{yaw}\dot{\theta}_{yaw}]^T.
\end{aligned}
$$

Eq. 4 can be updated as:

$$
\begin{aligned}
H^B =\; & I_b^B\boldsymbol{\omega}_b^B + I_r^B\boldsymbol{\omega}_r^B + m_b\boldsymbol{\rho}_{OB}^{B}\times\dot{\boldsymbol{\rho}}_{OB}^{B} + m_r\boldsymbol{\rho}_{OR}^{B}\times\dot{\boldsymbol{\rho}}_{OR}^{B} + \\
& m_t\boldsymbol{\rho}_{OT}^{B}\times\dot{\boldsymbol{\rho}}_{OT}^{B} + m_b\left(\boldsymbol{\rho}_{OB}^{B}\right)^2\boldsymbol{\omega}_b^B + m_r\left(\boldsymbol{\rho}_{OR}^{B}\right)^2\boldsymbol{\omega}_b^B + m_t \\
& \left(\boldsymbol{\rho}_{OT}^{B}\right)^2\boldsymbol{\omega}_b^B - m_b(\boldsymbol{\rho}_{OB}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^{B} - m_r(\boldsymbol{\rho}_{OR}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OR}^{B} - \\
& m_t(\boldsymbol{\rho}_{OT}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OT}^{B} \\
=\; & I_b^B\boldsymbol{\omega}_b^B + m_b\boldsymbol{\rho}_{OB}^{B}\times\dot{\boldsymbol{\rho}}_{OB}^{B} + m_r\begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} - \\ \frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t\boldsymbol{p}^B \end{pmatrix}\times \\
& \begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^{B} - \frac{1}{2}\frac{m_t}{(m_r+m_t)}\dot{l}_t\boldsymbol{p}^B \\ -\frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t\dot{\boldsymbol{p}}^B \end{pmatrix} \\
& +m_t\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t\boldsymbol{p}^B\right)\times \\
& \begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)}\dot{l}_t\boldsymbol{p}^B \\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t\dot{\boldsymbol{p}}^B \end{pmatrix} + m_b\left(\boldsymbol{\rho}_{OB}^{B}\right)^2\boldsymbol{\omega}_b^B \\
& +m_r\left(\begin{pmatrix}\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} - \frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t\boldsymbol{p}^B\right)\cdot \\ \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} - \frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t\boldsymbol{p}^B\right)\end{pmatrix}\right)\boldsymbol{\omega}_b^B + \\
& m_t\left(\begin{pmatrix}\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t\boldsymbol{p}^B\right)\cdot \\ \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t\boldsymbol{p}^B\right)\end{pmatrix}\right)\boldsymbol{\omega}_b^B - \\
& m_b(\boldsymbol{\rho}_{OB}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^{B} - m_r\left(\begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} - \\ \frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t\boldsymbol{p}^B \end{pmatrix}\cdot\boldsymbol{\omega}_b^B\right) \\
& \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} - \frac{1}{2}\frac{m_t}{(m_r+m_t)} l_t\boldsymbol{p}^B\right) - \\
& m_t\left(\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t\boldsymbol{p}^B\right)\cdot\boldsymbol{\omega}_b^B\right) \\
& \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t\boldsymbol{p}^B\right) \\
=\; & I_b^B\boldsymbol{\omega}_b^B + I_r^B\boldsymbol{\omega}_r^B + \frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(\boldsymbol{\rho}_{OB}^{B}\times\dot{\boldsymbol{\rho}}_{OB}^{B} + \\
& \left(\boldsymbol{\rho}_{OB}^{B}\right)^2\boldsymbol{\omega}_b^B - (\boldsymbol{\rho}_{OB}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^{B}) + \frac{1}{4}\left(\frac{m_t m_r}{(m_r+m_t)}\right) l_t^{2}(\boldsymbol{p}^B \\
& \times\dot{\boldsymbol{p}}^B + \boldsymbol{\omega}_b^B - (\boldsymbol{p}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{p}^B) \\
=\; & I_b^B\boldsymbol{\omega}_b^B + I_r^B\boldsymbol{\omega}_r^B + \frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(-\left[\boldsymbol{\rho}_{OB}^{B}\times\right] \\
& \left[\boldsymbol{\rho}_{OB}^{B}\times\right]\boldsymbol{\omega}_b^B + \boldsymbol{\rho}_{OB}^{B}\times\dot{\boldsymbol{\rho}}_{OB}^{B}) + \frac{1}{4}\left(\frac{m_t m_r}{(m_r+m_t)}\right) l_t^{2}(- \\
& \left[\boldsymbol{p}^B\times\right]\left[\boldsymbol{p}^B\times\right]\boldsymbol{\omega}_b^B + \boldsymbol{p}^B\times\dot{\boldsymbol{p}}^B).
\end{aligned}
\tag{8}
$$

A closed path starting from the origin of frame $\{O\}$ passing through body CoM B, tail base P, and tail end mass CoM P can be expressed as:

$$
\boldsymbol{\rho}_{OB}^{O} + \boldsymbol{\rho}_{BP}^{O} + \boldsymbol{\rho}_{PT}^{O} - \boldsymbol{\rho}_{OT}^{O} = \boldsymbol{0},
$$

where

$$
\begin{aligned}
\boldsymbol{\rho}_{BP}^{O} &= R_B^O\boldsymbol{\rho}_{BP}^{B} = R_B^O\boldsymbol{\rho}_{BP}^{B} = R_B^O\left[-d_b \quad 0 \quad 0\right]^T, \\
\boldsymbol{\rho}_{PT}^{O} &= R_B^O\boldsymbol{\rho}_{PT}^{B} = R_B^O\left(l_t\boldsymbol{p}^B\right).
\end{aligned}
\tag{9}
$$

d_b denotes the distance between the body CoM and the tail pivot. Combining Eq. 7 and Eq. 9 gives:

$$
\begin{aligned}
\boldsymbol{\rho}_{OT}^{B} &= -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t \boldsymbol{p}^{B} \\
&= R_O^B\left(\boldsymbol{\rho}_{OB}^{O} + \boldsymbol{\rho}_{BP}^{O} + \boldsymbol{\rho}_{PT}^{O}\right) \\
&= \boldsymbol{\rho}_{OB}^{B} + \begin{bmatrix} -d_b & 0 & 0 \end{bmatrix}^T + l_t \boldsymbol{p}^{B}.
\end{aligned}
$$

Then:

$$
\begin{aligned}
\boldsymbol{\rho}_{OB}^{B} &= \frac{m_r+m_t}{m_b+m_r+m_t}\begin{pmatrix} \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t \boldsymbol{p}^{B} + \begin{bmatrix} d_b & 0 & 0 \end{bmatrix}^T \\ -l_t \boldsymbol{p}^{B} \end{pmatrix} \\
&= \frac{m_r+m_t}{m_b+m_r+m_t}\left(\frac{-\frac{1}{2}m_r - m_t}{(m_r+m_t)} l_t \boldsymbol{p}^{B} + \begin{bmatrix} d_b & 0 & 0 \end{bmatrix}^T\right)
\end{aligned}
$$

$$
\dot{\boldsymbol{\rho}}_{OB}^{B} = -\frac{\frac{1}{2}m_r + m_t}{m_b+m_r+m_t}(\dot{l}_t \boldsymbol{p}^{B} + l_t \dot{\boldsymbol{p}}^{B})
$$

Besides, we have:

$$
I_r^B \boldsymbol{\omega}_r^B = R_R^B I_r^R \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,
$$

$$
R_R^B = \begin{bmatrix} c_{pitch}c_{yaw} & -c_{pitch}s_{yaw} & s_{pitch} \\ s_{yaw} & c_{yaw} & 0 \\ -s_{pitch}c_{yaw} & s_{pitch}s_{yaw} & c_{pitch} \end{bmatrix}
$$

$$
\boldsymbol{\omega}_r^B = \begin{bmatrix} s_{pitch}\dot{\theta}_{yaw} & \dot{\theta}_{pitch} & c_{pitch}\dot{\theta}_{yaw} \end{bmatrix}^T
$$

$$
I_r^R = \frac{m_r}{12}\begin{bmatrix} w_t^2 + h_t^2 & 0 & 0 \\ 0 & l_t^2 + h_t^2 & 0 \\ 0 & 0 & l_t^2 + w_t^2 \end{bmatrix}
$$

Eq. 8 can be expressed as:

$$
H^B = A\boldsymbol{\omega}_B^B + F\boldsymbol{u}, \tag{10}
$$

$$
\begin{aligned}
A = I_b^B &- \frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)} S(\boldsymbol{\rho}_{OB}^{B})S(\boldsymbol{\rho}_{OB}^{B}) \\
&- \frac{m_t m_r l_t^{\,2}}{4(m_r+m_t)} S(\boldsymbol{p}^{B})S(\boldsymbol{p}^{B}),
\end{aligned}
$$

$$
\begin{aligned}
F\boldsymbol{u} = &\frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} \times \dot{\boldsymbol{\rho}}_{OB}^{B} + \frac{m_t m_r l_t^{\,2}}{4(m_r+m_t)} \boldsymbol{p}^{B} \times \dot{\boldsymbol{p}}^{B} \\
&+ R_R^B I_r^R \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,
\end{aligned}
$$

And

$$
F = \begin{bmatrix}
-\dfrac{(c_{yaw}c_{pitch}s_{yaw}(l_t^2 m_r^2 - m_r^2 w_t^2 + 4l_t^2 m_b m_r + 12 l_t^2 m_b m_t + 4l_t^2 m_r m_t - m_b m_r w_t^2 - m_r m_t w_t^2))}{12(m_b+m_r+m_t)}, \\
\dfrac{m_r s_{yaw}{}^2 (h_t^2 + w_t^2) + m_r c_{yaw}{}^2 (h_t^2 + l_t^2)}{12} + \dfrac{l_t m_b c_{yaw}(m_r + 2m_t)\begin{pmatrix} 2d_b m_r c_{pitch} + 2d_b m_t c_{pitch} + l_t m_r c_{yaw} + \\ 2 l_t m_t c_{yaw} \end{pmatrix}}{4(m_b+m_r+m_t)(m_r+m_t)} + \dfrac{l_t^2 m_r m_t c_{yaw}{}^2}{4(m_r+m_t)}, \\
\dfrac{(c_{yaw}s_{yaw}s_{pitch}(l_t^2 m_r^2 - m_r^2 w_t^2 + 4l_t^2 m_b m_r + 12 l_t^2 m_b m_t + 4l_t^2 m_r m_t - m_b m_r w_t^2 - m_r m_t w_t^2))}{12(m_b+m_r+m_t)}, \\
\dfrac{(s_{pitch}(m_r^2 w_t^2 + l_t^2 m_r^2 + 4l_t^2 m_b m_r + 12 l_t^2 m_b m_t + 4l_t^2 m_r m_t + m_b m_r w_t^2 + m_r m_t w_t^2))}{12(m_b+m_r+m_t)}, \\
-\dfrac{(d_b l_t m_b s_{yaw} s_{pitch}(m_r + 2m_t))}{2(m_b+m_r+m_t)}, \\
\dfrac{(l_t^2 m_r^2 c_{pitch} + m_r^2 w_t^2 c_{pitch} + 4l_t^2 m_b m_r c_{pitch} + 12 l_t^2 m_b m_t c_{pitch} + 4l_t^2 m_r m_t c_{pitch} + m_b m_r w_t^2 c_{pitch} + m_r m_t w_t^2 c_{pitch} + 6 d_b l_t m_b m_r c_{yaw} + 12 d_b l_t m_b m_t c_{yaw})}{12(m_b+m_r+m_t)}, \\
0 \\
\dfrac{(d_b m_b c_{yaw} s_{pitch}(m_r + 2m_t))}{2(m_b+m_r+m_t)} \\
\dfrac{(d_b m_b s_{yaw}(m_r + 2m_t))}{2(m_b+m_r+m_t)}
\end{bmatrix}
$$

$$
\boldsymbol{u} = \begin{bmatrix} \dot{\theta}_{pitch} & \dot{\theta}_{yaw} & \dot{l}_t \end{bmatrix}.
$$

II. Tail Motion Analysis

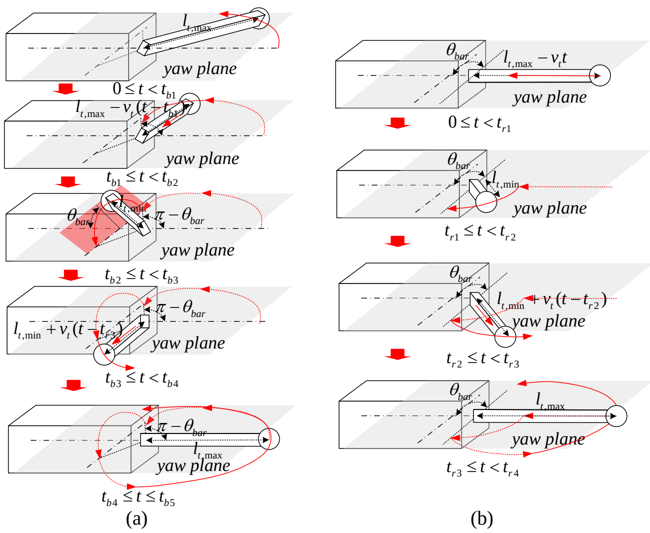

Fig. 2. Tail motion trajectories in the analysis. (a) During TBBMP. (b) During TRMP.

In the analysis, we assume $H^B = 0$ and is subjected to no external forces during aerial reorientation. The center of the whole robot keeps still:

$$m_b \boldsymbol{\rho}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O = \mathbf{0},$$

Then:

$$\begin{aligned} & m_b \boldsymbol{\rho}_{OB}^B + m_r(\boldsymbol{\rho}_{OB}^B + \boldsymbol{\rho}_{BR}^B) + m_t(\boldsymbol{\rho}_{OB}^B + \boldsymbol{\rho}_{BT}^B) = \mathbf{0}, \\ & \boldsymbol{\rho}_{OB}^B = -\frac{m_r \boldsymbol{\rho}_{BR}^B + m_t \boldsymbol{\rho}_{BT}^B}{m_b + m_r + m_t} \end{aligned} \tag{11}$$

A. Tail Bypass Barrier Motion Pattern (TBBMP)

As shown in Fig. 2(a), the tail motion trajectory in frame $\{B\}$ can be expressed as:

$$\boldsymbol{\rho}_{PT}^B =$$

$$\begin{cases} \begin{bmatrix} -l_{t,\max}\cos(\omega_t t), & -l_{t,\max}\sin(\omega_t t), & 0, \end{bmatrix}^T, 0 \le t < t_{b1} \\ \begin{bmatrix} -(l_{t,\min} + v_t(t_{b2} & -(l_{t,\min} + v_t(t_{b2} & 0, \\ -t))\cos(\omega_t t), & -t))\sin(\omega_t t), & \end{bmatrix}^T, t_{b1} \le t < t_{b2} \\ \begin{bmatrix} l_{t,\min} & -l_{t,\min}\sin\theta_{bar}\cos & l_{t,\min}\sin\theta_{bar}\sin \\ \cos\theta_{bar}, & (\omega_t(t - t_{b2})), & (\omega_t(t - t_{b2})), \end{bmatrix}^T, \\ t_{b2} \le t < t_{b3} \\ \begin{bmatrix} -(l_{t,\min} + v_t(t - t_{b3})) & -(l_{t,\min} + v_t(t - t_{b3})) & \\ \cos(\omega_t(t - t_{b3}) + & \sin(\omega_t(t - t_{b3}) + & 0, \\ \pi + \theta_{bar}), & \pi + \theta_{bar}), & \end{bmatrix}^T, t_{b3} \le t < t_{b4} \\ \begin{bmatrix} -l_{t,\max}\cos(\omega_t(t - & -l_{t,\max}\sin(\omega_t(t - & 0, \\ t_{b3}) + \pi + \theta_{bar}), & t_{b3}) + \pi + \theta_{bar}), & \end{bmatrix}^T, t_{b4} \le t \le t_{b5} \end{cases}$$

where

$$t_{b1} = \frac{\pi - \theta_{bar}}{\omega_t} - \frac{l_{t,\max} - l_{t,\min}}{v_t}, \; t_{b2} = \frac{\pi - \theta_{bar}}{\omega_t}, \; t_{b3} = \frac{2\pi - \theta_{bar}}{\omega_t},$$

$$t_{b4} = \frac{2\pi - \theta_{bar}}{\omega_t} + \frac{l_{t,\max} - l_{t,\min}}{v_t}, \text{ and } t_{b5} = \frac{4\pi - 3\theta_{bar}}{\omega_t}.$$

At stage 1, during $t \in [0, t_{b1})$,

$$\begin{aligned} \boldsymbol{\rho}_{BR}^B &= \frac{1}{2} l_{t,\max} \boldsymbol{p}^B + [-d_b \quad 0 \quad 0]^T \\ &= \begin{bmatrix} -d_b - \frac{1}{2} l_{t,\max}\cos(\omega_t t) & -\frac{1}{2} l_{t,\max}\sin(\omega_t t) & 0 \end{bmatrix}^T, \\ \boldsymbol{\rho}_{BT}^B &= \begin{bmatrix} -d_b - l_{t,\max}\cos(\omega_t t) & -l_{t,\max}\sin(\omega_t t) & 0 \end{bmatrix}^T. \end{aligned}$$

$$\begin{aligned} \boldsymbol{\rho}_{OB}^B &= -\frac{m_r \boldsymbol{\rho}_{BR}^B + m_t \boldsymbol{\rho}_{BT}^B}{m_b + m_r + m_t} \\ &= \frac{1}{m_b + m_r + m_t} \begin{bmatrix} m_r\left(d_b + \frac{l_{t,\max}\cos(\omega_t t)}{2}\right) + m_t(d_b + l_{t,\max} \\ \cos(\omega_t t)) \\ l_{t,\max}\sin(\omega_t t)\left(\frac{m_r}{2} + m_t\right) \\ 0 \end{bmatrix} \end{aligned}$$

Based on Eq. 10, we can get the expression of $\boldsymbol{\omega}_b^B$. Here to get simple expression in the analysis, we assume $m_r = 0$. We have:

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{l_{t,\max} m_b m_t \omega_t (l_{t,\max} + d_b cos(\omega_t t))}{\substack{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\max} \\ + m_b m_t l_{t,\max}{}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)}} \end{bmatrix}$$

The changed angle in yaw direction at $t = t_{b1}$ is:

$$\Delta\varphi_{b,y,1} = \omega_t \int_0^{t_{b1}} -\frac{l_{t,\max} m_b m_t (l_{t,\max} + d_b cos(\omega_t t))}{\substack{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\max} \\ + m_b m_t l_{t,\max}{}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)}} dt$$

At stage2, during $t \in [t_{b1}, t_{b2})$,

$$\boldsymbol{\rho}_{OB}^{B} = -\frac{m_r \boldsymbol{\rho}_{BR}^{B} + m_t \boldsymbol{\rho}_{BT}^{B}}{m_b + m_r + m_t}$$

$$= \frac{1}{m_b + m_r + m_t} \begin{bmatrix} m_r (d_b + \frac{l_t \cos(\omega_t t)}{2}) + m_t (d_b + l_t \\ \cos(\omega_t t)) \\ l_t \sin(\omega_t t)(\frac{m_r}{2} + m_t) \\ 0 \end{bmatrix}$$

We can get:

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} 0 \\ 0 \\ -\frac{(m_b m_t (\omega_t l_t^2 + d_b \omega_t cos(\omega_t t) l_t - d_b v_t sin(\omega_t t)))}{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_t + m_b m_t l_t^2 + I_{b,y}^B (m_b + m_t))} \end{bmatrix},$$

The changed angle in yaw direction at $t = t_{b2}$ is:

$$\varphi_{b,y,2}(t) = \int_{t_{b1}}^{t_{b2}} -\frac{(m_b m_t (\omega_t l_t^2 + d_b \omega_t cos(\omega_t t) l_t - d_b v_t sin(\omega_t t)))}{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_t + m_b m_t l_t^2 + I_{b,y}^B (m_b + m_t))} dt$$
$$+ \omega_t \int_0^{t_{b1}} -\frac{l_{t,\max} m_b m_t (l_{t,\max} + d_b cos(\omega_t t))}{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\max} + m_b m_t l_{t,\max}{}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)} dt$$

$l_t = l_{t,\min} + v_t (t_{b2} - t).$

At stage 3, during $t \in \left[t_{b2}, t_{b3}\right)$

$$\boldsymbol{\rho}_{OB}^{B} = -\frac{m_r \boldsymbol{\rho}_{BR}^{B} + m_t \boldsymbol{\rho}_{BT}^{B}}{m_b + m_r + m_t}$$

$$= -\frac{1}{m_b + m_r + m_t} \begin{bmatrix} (\frac{m_r}{2} + m_t) l_t c_{bar} - (m_r + m_t) d_b \\ -(\frac{m_r}{2} + m_t) l_t s_{bar} \cos(\omega_t (t - t_{r2})) \\ (\frac{m_r}{2} + m_t) l_t s_{bar} \sin(\omega_t (t - t_{r2})) \end{bmatrix}$$

Here to get simple expression for the analysis, we assume $m_r = 0$. Especially, in specific case $d_b = l_t \cos\theta_{bar}$ for creating pure body roll rotation. We can get:

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} \frac{l_t^2 m_b m_t \omega_t sin^2\theta_{bar}}{m_b m_t l_t^2 sin^2\theta_{bar} + I_{b,r}^B (m_b + m_r)} \\ 0 \\ 0 \end{bmatrix}$$

When $t = t_{r3}$, the changed angle in the body roll direction is:

$$\Delta\varphi_{b,r} = \frac{\pi l_t^2 m_b m_t sin^2\theta_{bar}}{m_b m_t l_t^2 sin^2\theta_{bar} + I_{b,r}^B (m_b + m_r)}$$

Similarly, at stage 4, during $t \in \left[t_{b3}, t_{b4}\right)$,

$$\boldsymbol{\rho}_{OB}^{B} = -\frac{m_r \boldsymbol{\rho}_{BR}^{B} + m_t \boldsymbol{\rho}_{BT}^{B}}{m_b + m_r + m_t}$$

$$= \frac{1}{m_b + m_r + m_t} \begin{bmatrix} (m_r + m_t) d_b - (\frac{m_r}{2} + m_t) l_t \\ \cos(\omega_t (t - t_{b3}) + \theta_{bar}) \\ -(\frac{m_r}{2} + m_t) l_t \sin(\omega_t (t - t_{b3}) + \theta_{bar}) \\ 0 \end{bmatrix},$$

$l_t = l_{t,\min} + v_t (t - t_{b3}).$

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} 0 \\ 0 \\ -\frac{(m_b m_t (\omega_t l_t^2 - d_b \omega_t cos(\theta_{bar} + \omega_t t - \omega_t t_{b3}) l_t - d_b v_t sin(\theta_{bar} + \omega_t t - \omega_t t_{b3})))}{(m_b m_t d_b^2 - 2 m_b m_t cos(\theta_{bar} + \omega_t t - \omega_t t_{b3}) d_b l_t + m_b m_t l_t^2 + I_{b,y}^B (m_b + m_t))} \end{bmatrix},$$

Similarly, at stage 5, during $t \in \left[t_{b4}, t_{b5}\right]$,

$$\boldsymbol{\rho}_{OB}^{B} = -\frac{m_r \boldsymbol{\rho}_{BR}^{B} + m_t \boldsymbol{\rho}_{BT}^{B}}{m_b + m_r + m_t} = \frac{1}{m_b + m_r + m_t} \begin{bmatrix} (m_r + m_t) d_b - (\frac{m_r}{2} + m_t) l_{t,\max} \\ \cos(\omega_t (t - t_{b3}) + \theta_{bar}) \\ -(\frac{m_r}{2} + m_t) l_{t,\max} \sin(\omega_t (t - t_{b3}) + \theta_{bar}) \\ 0 \end{bmatrix}$$

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} 0 \\ 0 \\ -\frac{(l_{t,\max} m_b m_t \omega_t (l_{t,\max} - d_b cos(\theta_{bar} + \omega_t t - \omega_t t_{b3})))}{(m_b m_t d_b^2 - 2 m_b m_t cos(\theta_{bar} + \omega_t t - \omega_t t_{b3}) d_b l_{t,\max} + m_b m_t l_{t,\max}{}^2 + I_{b,y}^B (m_b + m_t))} \end{bmatrix},$$

The final changed angle after TBBMP is presented in the paper.

B. Tail Bypass Barrier Motion Pattern (TBBMP)

As shown in Fig. 2(b), the tail motion trajectory in frame $\{B\}$ can be expressed as:

$$\boldsymbol{\rho}_{PT,r}^{B} = \begin{cases} [-l_t, \quad 0, \quad 0,]^T, 0 \le t < t_{r1}, l_t = l_{t,\max} - v_t t, \\ [-l_{t,\max} \cos(-\omega_t (t - t_{r1})), \quad -l_{t,\max} \sin(-\omega_t (t - t_{r1})), \quad 0,]^T, \\ t_{r1} \le t < t_{r2}, \\ [-l_t \cos(\omega_t (t - t_{r2})), \quad -l_t \sin(\omega_t (t - t_{r2})), \quad 0,]^T, \\ t_{r2} \le t < t_{r3}, \ l_t = l_{t,\min} + v_t (t - t_{r1}), \\ [-l_{t,\max} \cos(\omega_t (t - t_{r2})), \quad -l_{t,\max} \sin(\omega_t (t - t_{r2})), \quad 0,]^T, \\ t_{r3} \le t \le t_{r4}, \end{cases}$$

At stage 1, during $t \in [0, t_{r1})$, there is no yaw angle change.

At stage 2, during $t \in [t_{r1}, t_{r2})$. Similar to the method during Section A, the changed angle in yaw direction is:

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} 0 \\ 0 \\ \frac{l_{t,\min} m_b m_t \omega_t (l_{t,\min} + d_b cos(\omega_t t))}{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\min} + m_b m_t l_{t,\min}{}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)} \end{bmatrix}$$

The changed angle in yaw direction at $t = t_{r2}$ is:

$$\Delta\varphi_{b,y} = \omega_t \int_{t_{r1}}^{t_{r2}} \frac{l_{t,\min} m_b m_t \omega_t (l_{t,\min} + d_b cos(\omega_t t))}{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\min} + m_b m_t l_{t,\min}{}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)} dt$$

At stage 3, during $t \in [t_{r2}, t_{r3})$. Similar to the method during Section A, the changed speed in yaw direction is same to the derivations in stage 4 of Section A.

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} 0 \\ 0 \\ -\frac{(m_b m_t (\omega_t l_t^2 - d_b \omega_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) l_t - d_b v_t sin(\theta_{bar} + \omega_t t - \omega_t t_{r2})))}{(m_b m_t d_b^2 - 2 m_b m_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) d_b l_t + m_b m_t l_t^2 + I_{b,y}^B (m_b + m_t))} \end{bmatrix},$$

The changed angle in yaw direction at $t = t_{r3}$ is:

$$\Delta\varphi_{b,y} = \int_{t_{r1}}^{t_{r2}} -\frac{(m_b m_t (\omega_t l_t^2 - d_b \omega_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) l_t - d_b v_t sin(\theta_{bar} + \omega_t t - \omega_t t_{r2})))}{(m_b m_t d_b^2 - 2 m_b m_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) d_b l_t + m_b m_t l_t^2 + I_{b,y}^B (m_b + m_t))} dt + \omega_t \int_{t_{r1}}^{t_{r2}} \frac{l_{t,\min} m_b m_t \omega_t (l_{t,\min} + d_b cos(\omega_t t))}{(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\min} + m_b m_t l_{t,\min}{}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)} dt$$

At stage 4, during $t \in [t_{r3}, t_{r4}]$

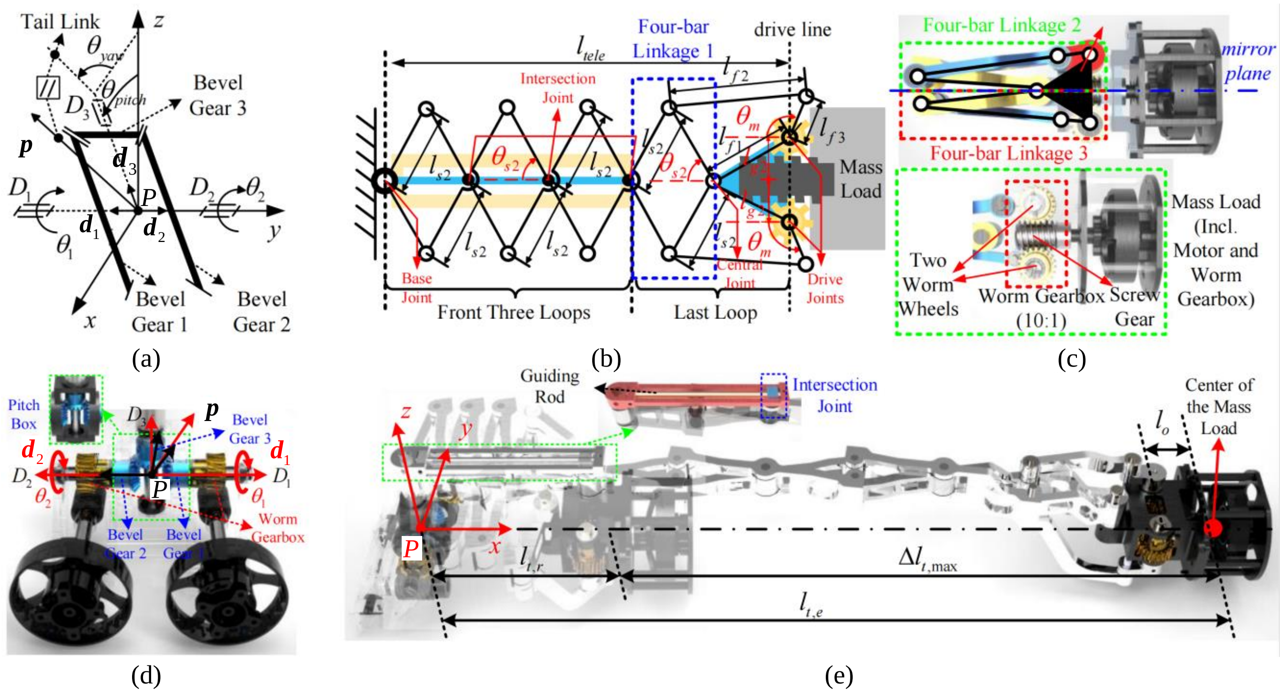

Fig. 6. Overview of a 3-DoF morphable inertial tail design. Tail orientation structure (TOS) based on the differential bevel gear mechanism: (a) Kinematics; (b) Mechanical design. Tail morphable inertial structure (TMIS) based on the scissor lift parallel mechanism: (c) Kinematics; (d) The last linkage loop kinematics and components of the tail mass load; (e) Mechanical design.

$$\boldsymbol{\omega}_b^B = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{(m_b m_t(\omega_t l_{t,\max}^2 - d_b \omega_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) l_{t,\max}}{\begin{array}{l}(m_b m_t d_b^2 - 2 m_b m_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) d_b l_{t,\max} \\ + m_b m_t l_{t,\max}^2 + I_{b,y}^B (m_b + m_t))\end{array}} \end{bmatrix},$$

The changed angle in yaw direction at $t = t_{r4}$ is:

$$\begin{aligned}\Delta\varphi_{b,y} = &\int_{t_{r1}}^{t_{r2}} -\frac{\begin{array}{c}(m_b m_t(\omega_t l_t^2 - d_b \omega_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) l_t \\ - d_b v_t sin(\theta_{bar} + \omega_t t - \omega_t t_{r2})))\end{array}}{\begin{array}{c}(m_b m_t d_b^2 - 2 m_b m_t cos(\theta_{bar} + \omega_t t - \omega_t t_{r2}) d_b l_t \\ + m_b m_t l_t^2 + I_{b,y}^B (m_b + m_t))\end{array}} dt \\ &+ \omega_t \int_{t_{r1}}^{t_{r2}} \frac{l_{t,\min} m_b m_t \omega_t (l_{t,\min} + d_b cos(\omega_t t))}{\begin{array}{c}(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\min} \\ + m_b m_t {l_{t,\min}}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)\end{array}} dt \\ &- \omega_t \int_{t_{r1}}^{t_{r2}} \frac{l_{t,\max} m_b m_t \omega_t (l_{t,\max} + d_b cos(\omega_t t))}{\begin{array}{c}(m_b m_t d_b^2 + 2 m_b m_t cos(\omega_t t) d_b l_{t,\max} \\ + m_b m_t {l_{t,\min}}^2 + I_{b,y}^B m_b + I_{b,y}^B m_t)\end{array}} dt\end{aligned}$$

The final changed angle after multiple TRMP is presented in the paper.

III. Robotic Tail Kinematics

The tail length kinematics is:

$$l_t = 8 l_{s2} \cos\theta_{s2} + \sqrt{l_{f1}^2 - l_g^2} + l_o,$$

$$\theta_{s2} = \pi - \sin^{-1}\left(\frac{l_{g2}}{l_{f1}}\right) -$$

$$\cos^{-1}\left(\frac{l_{s2}^2 + l_{f1}^2 + l_{f3}^2 - 2 l_{f1} l_{f3} \cos\left(\theta_m + \sin^{-1}\left(\frac{l_{g2}}{l_{f1}}\right)\right) - l_{f2}^2}{2 l_{s2} \sqrt{l_{f1}^2 + l_{f3}^2 - 2 l_{f1} l_{f3} \cos\left(\theta_m + \sin^{-1}\left(\frac{l_{g2}}{l_{f1}}\right)\right)}}\right)$$

$$-\varepsilon_m \cos^{-1}\left(\frac{l_{f1} - l_{f3} \cos\left(\theta_m + \sin^{-1}\left(\frac{l_{g2}}{l_{f1}}\right)\right)}{\sqrt{l_{f1}^2 + l_{f3}^2 - 2 l_{f1} l_{f3} \cos\left(\theta_m + \sin^{-1}\left(\frac{l_{g2}}{l_{f1}}\right)\right)}}\right),$$

$$\theta_m \in [0, 2\pi],$$

where $\varepsilon_m = 1$ when $\theta_m \in \left[0, \pi - \sin^{-1}\left(\frac{l_{g2}}{l_{f1}}\right)\right] \cup \left(2\pi - \tan^{-1}\right.$ $\left.\left(\frac{l_{g2}}{l_{f1}}\right), 2\pi\right]$ and $\varepsilon_m = -1$ when $\theta_m \in \left[\pi - \tan^{-1}\left(\frac{l_{g2}}{l_{f1}}\right), 2\pi -\right.$

$\tan^{-1}\left(\frac{l_{g2}}{l_{f1}}\right)\Bigg]$. θ_{s2} is the angle from the $+x$ axis to the bars of the front three loops and the four-bar linkage 1 in the last loop. l_o is the distance between the center of mass load and central joint.

IV. ROBOT HARDWARE

The robotic tail was equipped with a T-motor Antigravity 5008 KV170 (135g Incl. Cable) motor and gear systems (gear ratio 20:1) to provide high output torque and rapid system response. The motor was driven by an open-source VESC motor driver capable of a continuous current up to 50 A and a peak current up to 240 A. Positions of the tail actuators and platform joints were measured using AS5047D magnetic encoders. The orientation angle and acceleration of the robot body were obtained from an IMU module LPMS-BE2. The robot's release and landing moments were detected by sudden changes in acceleration along the vertical direction.The entire embedded system was controlled by the Raspberry Pi computing platform operating at a 500 Hz sampling rate.