Bio-inspired Soft Variable-Stiffness Prehensile Tail Enabling Versatile Grasping and Enhancing Dynamic Mobility

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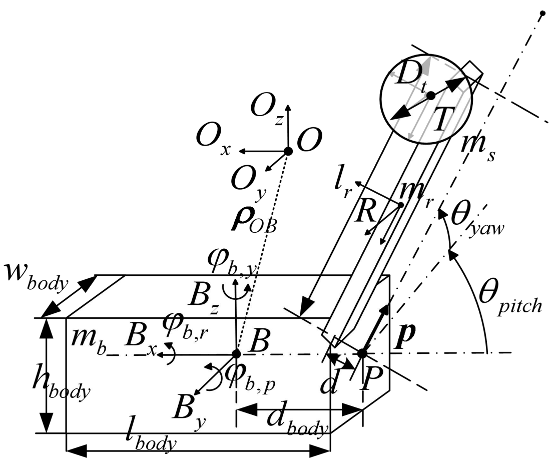

Fig. 1. Analytical models of the kangaroo rat and our tailed robot with a 3-DoF tail.

We start with an angular momentum equation expressed in the inertial frame $\{O\}$:

$$\begin{aligned}H^O =& I_b^O \boldsymbol{\omega}_b^O + I_r^O \boldsymbol{\omega}_r^O + I_t^O \boldsymbol{\omega}_t^O + m_b \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + \\ & m_r \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O ,\end{aligned} \tag{2}$$

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where I_b^O and I_r^O are the inertias of the robot body and tail link in the inertial frame $\{O\}$. m_t, m_r, m_b denote the mass of the body, the tail link, and the tail end mass. $\rho_{OT}^O, \rho_{OR}^O, \rho_{OB}^O$ are the vectors from the origin of the frame $\{O\}$ to the origins of the tail end frame $\{T\}$, the tail link frame $\{R\}$, and the body frame $\{B\}$. It can be also expressed in the body frame as:

$$\begin{aligned}H^B =& I_b^B \boldsymbol{\omega}_b^B + I_r^B \boldsymbol{\omega}_r^B + I_t^O \boldsymbol{\omega}_t^O + m_b (R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + \\ & m_r (R_B^O)^T \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t (R_B^O)^T \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O ,\end{aligned} \tag{2}$$

where R_B^O is the rotation matrix from frame $\{B\}$ to frame $\{O\}$. Based on $\boldsymbol{\rho}_{OB}^O = R_B^O \boldsymbol{\rho}_{OB}^B$, we have $\dot{\boldsymbol{\rho}}_{OB}^O = \dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B$. Then, we have:

$$\begin{aligned}& (R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O \\ &= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B \right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B \right) \\ &= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B \right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B \right) + (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B \right) \times \left(R_B^O \dot{\boldsymbol{\rho}}_{OB}^B \right) \\ &= \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B \right) \times \left((R_B^O)^T \dot{R}_B^O \boldsymbol{\rho}_{OB}^B \right) + \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B \right) \times \\ & \left((R_B^O)^T R_B^O \dot{\boldsymbol{\rho}}_{OB}^B \right) \\ &= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\left[\boldsymbol{\omega}_b^O \times \right] R_B^O \right) \boldsymbol{\rho}_{OB}^B \right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\ &= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\boldsymbol{\omega}_b^O \times \boldsymbol{\rho}_{OB}^O \right) \right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\ &= \boldsymbol{\rho}_{OB}^B \times \left(\boldsymbol{\omega}_b^B \times \boldsymbol{\rho}_{OB}^B \right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\ &= \left(\boldsymbol{\rho}_{OB}^B \right)^2 \boldsymbol{\omega}_b^B - (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OB}^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\ &= - \left[\boldsymbol{\rho}_{OB}^B \times \right] \left[\boldsymbol{\rho}_{OB}^B \times \right] \boldsymbol{\omega}_b^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B ,\end{aligned} \tag{3}$$

where $R(x \times y) = (Rx) \times (Ry)$ and $\Omega_b^O = \left[\boldsymbol{\omega}_b^O \times \right] = \dot{R}_B^O (R_B^O)^T$ are used. Here $\left(\boldsymbol{\rho}_{OB}^B \right)^2 = \left(\boldsymbol{\rho}_{OB}^B \right) \cdot \left(\boldsymbol{\rho}_{OB}^B \right)$ .Then Eq. 2 becomes:

$$\begin{aligned}H^B =& I_b^B \boldsymbol{\omega}_b^B + I_r^B \boldsymbol{\omega}_r^B + I_r^B \boldsymbol{\omega}_r^B + m_b \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B + \\ & m_r \boldsymbol{\rho}_{OR}^B \times \dot{\boldsymbol{\rho}}_{OR}^B + m_t \boldsymbol{\rho}_{OT}^B \times \dot{\boldsymbol{\rho}}_{OT}^B + m_b \left(\boldsymbol{\rho}_{OB}^B \right)^2 \boldsymbol{\omega}_b^B \\ & + m_r \left(\boldsymbol{\rho}_{OR}^B \right)^2 \boldsymbol{\omega}_b^B + m_t \left(\boldsymbol{\rho}_{OT}^B \right)^2 \boldsymbol{\omega}_b^B - m_b (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \\ & \boldsymbol{\rho}_{OB}^B - m_r (\boldsymbol{\rho}_{OR}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OR}^B - m_t (\boldsymbol{\rho}_{OT}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OT}^B .\end{aligned} \tag{4}$$

At the robot center of mass (CoM), we have:

$$m_b \boldsymbol{\rho}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O = \boldsymbol{0}, \tag{5}$$

here

$$\boldsymbol{\rho}_{OR}^{O} = \boldsymbol{\rho}_{OT}^{O} - \boldsymbol{\rho}_{RT}^{O} = \boldsymbol{\rho}_{OT}^{O} - R_B^O \boldsymbol{\rho}_{RT}^{B} =$$

$$\boldsymbol{\rho}_{OT}^{O} - R_B^O \left(\begin{bmatrix} -\frac{1}{2} l_r c_{pitch} c_{yaw} & -\frac{1}{2} l_r s_{yaw} & \frac{1}{2} l_r s_{pitch} c_{yaw} \end{bmatrix}^T \right. \tag{6}$$

$$\left. + \begin{bmatrix} -\frac{1}{2} D_t c_{pitch} s_{yaw} & \frac{1}{2} D_t c_{yaw} & \frac{1}{2} D_t s_{pitch} s_{yaw} \end{bmatrix}^T \right),$$

where $c_{pitch} = \cos\theta_{pitch}$, $s_{pitch} = \sin\theta_{pitch}$, $c_{yaw} = \cos\theta_{yaw}$, $s_{yaw} = \sin\theta_{yaw}$. θ_{pitch} is the tail swing angle in the body pitch direction and θ_{yaw} is the tail swing angle in the body yaw direction. $\boldsymbol{p}^B = \begin{bmatrix} -c_{pitch} c_{yaw} & -s_{yaw} & s_{pitch} c_{yaw} \end{bmatrix}^T$ is unit tail reorientation vector in frame $\{B\}$. l_r denotes the tail length. Substitute Eq. 6 into Eq. 5, we get:

$$\left(m_r + m_t\right) \boldsymbol{\rho}_{OT}^{O} = -m_b \boldsymbol{\rho}_{OB}^{O} + m_r R_B^O$$

$$\left(\begin{bmatrix} -\frac{1}{2} l_r c_{pitch} c_{yaw} & -\frac{1}{2} l_r s_{yaw} & \frac{1}{2} l_r s_{pitch} c_{yaw} \end{bmatrix}^T \right.$$

$$\left. + \begin{bmatrix} -\frac{1}{2} D_t c_{pitch} s_{yaw} & \frac{1}{2} D_t c_{yaw} & \frac{1}{2} D_t s_{pitch} s_{yaw} \end{bmatrix}^T \right),$$

$$\boldsymbol{\rho}_{OT}^{B} = -\frac{m_b}{\left(m_r + m_t\right)} \boldsymbol{\rho}_{OB}^{B} + \frac{m_r}{\left(m_r + m_t\right)}$$

$$\left(\begin{bmatrix} -\frac{1}{2} l_r c_{pitch} c_{yaw} & -\frac{1}{2} l_r s_{yaw} & \frac{1}{2} l_r s_{pitch} c_{yaw} \end{bmatrix}^T \right.$$

$$\left. + \begin{bmatrix} -\frac{1}{2} D_t c_{pitch} s_{yaw} & \frac{1}{2} D_t c_{yaw} & \frac{1}{2} D_t s_{pitch} s_{yaw} \end{bmatrix}^T \right),$$

$$\boldsymbol{\rho}_{OR}^{B} = \boldsymbol{\rho}_{OT}^{B} - \boldsymbol{\rho}_{RT}^{B} =$$

$$-\frac{m_b}{\left(m_r + m_t\right)} \boldsymbol{\rho}_{OB}^{B} - \frac{m_t}{\left(m_r + m_t\right)}$$

$$\left(\begin{bmatrix} -\frac{1}{2} l_r c_{pitch} c_{yaw} & -\frac{1}{2} l_r s_{yaw} & \frac{1}{2} l_r s_{pitch} c_{yaw} \end{bmatrix}^T \right.$$

$$\left. + \begin{bmatrix} -\frac{1}{2} D_t c_{pitch} s_{yaw} & \frac{1}{2} D_t c_{yaw} & \frac{1}{2} D_t s_{pitch} s_{yaw} \end{bmatrix}^T \right) \tag{7}$$

We also have:

$$\dot{\boldsymbol{\rho}}_{OT}^{B} = -\frac{m_b}{\left(m_r + m_t\right)} \dot{\boldsymbol{\rho}}_{OB}^{B} +$$

$$+\frac{1}{2} l_r \frac{m_r}{\left(m_r + m_t\right)} [s_{pitch} c_{yaw} \dot{\theta}_{pitch}$$

$$+c_{pitch} s_{yaw} \dot{\theta}_{yaw} \quad -c_{yaw} \dot{\theta}_{yaw} \quad c_{pitch} c_{yaw} \dot{\theta}_{pitch} - s_{pitch} s_{yaw} \dot{\theta}_{ya\,w}]^T$$

$$+\frac{1}{2} D_t \frac{m_r}{\left(m_r + m_t\right)} [s_{pitch} s_{yaw} \dot{\theta}_{pitch} - c_{pitch} c_{yaw} \dot{\theta}_{yaw}$$

$$-s_{yaw} \dot{\theta}_{yaw} \quad c_{pitch} s_{yaw} \dot{\theta}_{pitch} + s_{pitch} c_{yaw} \dot{\theta}_{yaw}]^T,$$

$$\dot{\boldsymbol{\rho}}_{OR}^{B} = -\frac{m_b}{\left(m_r + m_t\right)} \dot{\boldsymbol{\rho}}_{OB}^{B}$$

$$-\frac{1}{2} l_r \frac{m_t}{\left(m_r + m_t\right)} [s_{pitch} c_{yaw} \dot{\theta}_{pitch}$$

$$+c_{pitch} s_{yaw} \dot{\theta}_{yaw} \quad -c_{yaw} \dot{\theta}_{yaw} \quad c_{pitch} c_{yaw} \dot{\theta}_{pitch} - s_{pitch} s_{yaw} \dot{\theta}_{ya\,w}]^T$$

$$-\frac{1}{2} D_t \frac{m_t}{\left(m_r + m_t\right)} [s_{pitch} s_{yaw} \dot{\theta}_{pitch} - c_{pitch} c_{yaw} \dot{\theta}_{yaw}$$

$$-s_{yaw} \dot{\theta}_{yaw} \quad c_{pitch} s_{yaw} \dot{\theta}_{pitch} + s_{pitch} c_{yaw} \dot{\theta}_{yaw}]^T,.$$

Eq. 4 can be updated as:

$$
\begin{aligned}
H^{B} =& I_b^B\boldsymbol{\omega}_b^B + I_r^B\boldsymbol{\omega}_r^B + I_t^B\boldsymbol{\omega}_t^B + m_b\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B + m_r\boldsymbol{\rho}_{OR}^B\times\dot{\boldsymbol{\rho}}_{OR}^B + \\
& m_t\boldsymbol{\rho}_{OT}^B\times\dot{\boldsymbol{\rho}}_{OT}^B + m_b\left(\boldsymbol{\rho}_{OB}^B\right)^2\boldsymbol{\omega}_b^B + m_r\left(\boldsymbol{\rho}_{OR}^B\right)^2\boldsymbol{\omega}_b^B + m_t \\
& \left(\boldsymbol{\rho}_{OT}^B\right)^2\boldsymbol{\omega}_b^B - m_b(\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B - m_r(\boldsymbol{\rho}_{OR}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OR}^B - \\
& m_t(\boldsymbol{\rho}_{OT}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OT}^B \\
=& I_b^B\boldsymbol{\omega}_b^B + \left(I_r^B + I_t^B\right)\boldsymbol{\omega}_t^B + m_b\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B + m_r \\
& \begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B - \\ \frac{1}{2}l_r\frac{m_t}{(m_r+m_t)}\boldsymbol{p}^B \\ -\frac{1}{2}D_t\frac{m_t}{(m_r+m_t)}\boldsymbol{q}^B \end{pmatrix} \times \begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^B \\ -\frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t\dot{\boldsymbol{p}}^B \\ -\frac{1}{2}D_t\frac{m_t}{(m_r+m_t)}\dot{\boldsymbol{q}}^B \end{pmatrix} \\
& +m_t\begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B + \frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\boldsymbol{p}^B \\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B \end{pmatrix}\times \\
& \begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^B + \frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\dot{\boldsymbol{p}}^B \\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\dot{\boldsymbol{q}}^B \end{pmatrix} + m_b\left(\boldsymbol{\rho}_{OB}^B\right)^2\boldsymbol{\omega}_b^B \\
& +m_r\left(\begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B - \frac{1}{2}\frac{m_t}{(m_r+m_t)}l_r\boldsymbol{p}^B \\ -\frac{1}{2}\frac{m_t}{(m_r+m_t)}D_t\boldsymbol{q}^B \end{pmatrix}\cdot \begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B - \frac{1}{2}\frac{m_t}{(m_r+m_t)}l_r\boldsymbol{p}^B \\ -\frac{1}{2}\frac{m_t}{(m_r+m_t)}D_t\boldsymbol{q}^B \end{pmatrix}\right)\boldsymbol{\omega}_b^B + \\
& m_t\left(\begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B + \frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\boldsymbol{p}^B \\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B \end{pmatrix}\cdot \begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B + \frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\boldsymbol{p}^B \\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B \end{pmatrix}\right)\boldsymbol{\omega}_b^B - \\
& m_b(\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B - m_r\left(\begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B - \\ \frac{1}{2}\frac{m_t}{(m_r+m_t)}l_r\boldsymbol{p}^B - \\ \frac{1}{2}\frac{m_t}{(m_r+m_t)}D_t\boldsymbol{q}^B \end{pmatrix}\cdot\boldsymbol{\omega}_b^B\right) \\
& \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B - \frac{1}{2}\frac{m_t}{(m_r+m_t)}l_r\boldsymbol{p}^B - \frac{1}{2}\frac{m_t}{(m_r+m_t)}D_t\boldsymbol{q}^B\right) \\
& -m_t\left(\begin{pmatrix} -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B + \frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\boldsymbol{p}^B \\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B \end{pmatrix}\cdot\boldsymbol{\omega}_b^B\right) \\
& \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B + \frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B + \frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B\right) \\
=& I_b^B\boldsymbol{\omega}_b^B + I_r^B\boldsymbol{\omega}_r^B + I_t^B\boldsymbol{\omega}_t^B + \frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B + \\
& \left(\boldsymbol{\rho}_{OB}^B\right)^2\boldsymbol{\omega}_b^B - (\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B) + \frac{m_t m_r}{(m_r+m_t)}(\begin{pmatrix} (\frac{1}{2}l_r\boldsymbol{p}^B + \frac{1}{2}D_t\boldsymbol{q}^B) \\ \times(\frac{1}{2}l_t\dot{\boldsymbol{p}}^B + \frac{1}{2}D_t\dot{\boldsymbol{q}}^B) \end{pmatrix} + \\
& (\frac{1}{4}l_r^2 + \frac{1}{4}D_t^2)\boldsymbol{\omega}_b^B - \left((\frac{1}{2}l_t\boldsymbol{p}^B + \frac{1}{2}D_t\boldsymbol{q}^B)\cdot\boldsymbol{\omega}_b^B\right)(\frac{1}{2}l_t\boldsymbol{p}^B + \frac{1}{2}D_t\boldsymbol{q}^B)) \\
=& I_b^B\boldsymbol{\omega}_b^B + I_r^B\boldsymbol{\omega}_r^B + \frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(-\left[\boldsymbol{\rho}_{OB}^B\times\right] \\
& \left[\boldsymbol{\rho}_{OB}^B\times\right]\boldsymbol{\omega}_b^B + \boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B) + \frac{m_t m_r}{(m_r+m_t)}(- \\
& \left[(\frac{1}{2}l_t\boldsymbol{p}^B + \frac{1}{2}D_t\boldsymbol{q}^B)\times\right]\left[(\frac{1}{2}l_t\boldsymbol{p}^B + \frac{1}{2}D_t\boldsymbol{q}^B)\times\right]\boldsymbol{\omega}_b^B \\
& +\begin{pmatrix} (\frac{1}{2}l_r\boldsymbol{p}^B + \frac{1}{2}D_t\boldsymbol{q}^B) \\ \times(\frac{1}{2}l_t\dot{\boldsymbol{p}}^B + \frac{1}{2}D_t\dot{\boldsymbol{q}}^B) \end{pmatrix}).
\end{aligned}
\tag{8}
$$

Here $\boldsymbol{q}^B = \begin{bmatrix} -c_{pitch}s_{yaw} & c_{yaw} & s_{pitch}s_{yaw} \end{bmatrix}^T$ is unit reorientation vector of R_y in frame $\{B\}$. A closed path starting from the origin of frame $\{O\}$ passing through body CoM B, tail base P, and tail end mass CoM P can be expressed as:

$$
\boldsymbol{\rho}_{OB}^O + \boldsymbol{\rho}_{BP}^O + \boldsymbol{\rho}_{PT}^O - \boldsymbol{\rho}_{OT}^O = \boldsymbol{0},
$$

where

$$
\begin{aligned}
\boldsymbol{\rho}_{BP}^O &= R_B^O\boldsymbol{\rho}_{BP}^B = R_B^O\boldsymbol{\rho}_{BP}^B = R_B^O\begin{bmatrix} -d_b & 0 & 0 \end{bmatrix}^T, \\
\boldsymbol{\rho}_{PT}^O &= R_B^O\boldsymbol{\rho}_{PT}^B = R_B^O\left(l_r\boldsymbol{p}^B + d\boldsymbol{r}^B + \frac{D_t}{2}\boldsymbol{q}^B\right).
\end{aligned}
\tag{9}
$$

d_b denotes the distance between the body CoM and the tail pivot. Combining Eq. 7 and Eq. 9 gives:

$$\begin{aligned}
\boldsymbol{\rho}_{OT}^{B} &= -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{m_r}{(m_r+m_t)} \\
&\left(\begin{bmatrix} -\frac{1}{2}l_r c_{pitch} c_{yaw} & -\frac{1}{2}l_r s_{yaw} & \frac{1}{2}l_r s_{pitch} c_{yaw} \end{bmatrix}^T \right. \\
&\left. + \begin{bmatrix} -\frac{1}{2}D_t c_{pitch} s_{yaw} & \frac{1}{2}D_t c_{yaw} & \frac{1}{2}D_t s_{pitch} s_{yaw} \end{bmatrix}^T \right) \\
&= -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^{B} + \frac{m_r}{(m_r+m_t)}\left(\frac{1}{2}l_r \boldsymbol{p}^B + \frac{1}{2}D_t \boldsymbol{q}^B\right) \\
&= R_O^B\left(\boldsymbol{\rho}_{OB}^{O} + \boldsymbol{\rho}_{BP}^{O} + \boldsymbol{\rho}_{PT}^{O}\right) \\
&= \boldsymbol{\rho}_{OB}^{B} + \begin{bmatrix} -d_b & 0 & 0 \end{bmatrix}^T + l_r \boldsymbol{p}^B + d\boldsymbol{r}^B + \frac{D_t}{2}\boldsymbol{q}^B.
\end{aligned}$$

Here $\boldsymbol{r}^B = \begin{bmatrix} s_{pitch} & 0 & c_{pitch} \end{bmatrix}^T$ is unit reorientation vector of R_z in frame $\{B\}$.

Then:

$$\boldsymbol{\rho}_{OB}^{B} = \frac{m_r+m_t}{m_b+m_r+m_t}\begin{pmatrix} \frac{m_r}{(m_r+m_t)}\left(\frac{1}{2}l_r \boldsymbol{p}^B + \frac{1}{2}D_t \boldsymbol{q}^B\right) + \\ \begin{bmatrix} d_b & 0 & 0 \end{bmatrix}^T \\ -l_r \boldsymbol{p}^B - d\boldsymbol{r}^B - \frac{D_t}{2}\boldsymbol{q}^B \end{pmatrix}$$

$$= \frac{m_r+m_t}{m_b+m_r+m_t}\left(\frac{-\frac{1}{2}m_r - m_t}{(m_r+m_t)} l_r \boldsymbol{p}^B + \begin{bmatrix} d_b & 0 & 0 \end{bmatrix}^T - d\boldsymbol{r}^B - \frac{m_t}{(m_r+m_t)}\frac{1}{2}D_t \boldsymbol{q}^B\right)$$

$$\dot{\boldsymbol{\rho}}_{OB}^{B} = \frac{m_r+m_t}{m_b+m_r+m_t}\left(\frac{-\frac{1}{2}m_r - m_t}{(m_r+m_t)} l_r \dot{\boldsymbol{p}}^B - d\dot{\boldsymbol{r}}^B - \frac{m_t}{(m_r+m_t)}\frac{1}{2}D_t \dot{\boldsymbol{q}}^B\right)$$

Besides, we have:

$$I_r^B \boldsymbol{\omega}_r^B = R_R^B I_r^R \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,$$

$$I_t^B \boldsymbol{\omega}_t^B = R_R^B I_t^T \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,$$

$$R_R^B = \begin{bmatrix} c_{pitch}c_{yaw} & -c_{pitch}s_{yaw} & s_{pitch} \\ s_{yaw} & c_{yaw} & 0 \\ -s_{pitch}c_{yaw} & s_{pitch}s_{yaw} & c_{pitch} \end{bmatrix}$$

$$\boldsymbol{\omega}_r^B = \boldsymbol{\omega}_t^B = \begin{bmatrix} s_{pitch}\dot{\theta}_{yaw} & \dot{\theta}_{pitch} & c_{pitch}\dot{\theta}_{yaw} \end{bmatrix}^T$$

$$I_r^R = \frac{m_r}{12}\begin{bmatrix} w_t^2+h_t^2 & 0 & 0 \\ 0 & l_t^2+h_t^2 & 0 \\ 0 & 0 & l_t^2+w_t^2 \end{bmatrix}$$

$$I_t^R = \frac{m_r}{10}\begin{bmatrix} D_t^2 & 0 & 0 \\ 0 & D_t^2 & 0 \\ 0 & 0 & D_t^2 \end{bmatrix}$$

Eq. 8 can be expressed as:

$$H^B = A\boldsymbol{\omega}_b^B + F\boldsymbol{u}, \tag{10}$$

$$\begin{aligned}
A = I_b^B &- \frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)} S(\boldsymbol{\rho}_{OB}^B) S(\boldsymbol{\rho}_{OB}^B) \\
&- \frac{m_t m_r}{(m_r+m_t)} S\left(\frac{1}{2}l_r \boldsymbol{p}^B + \frac{1}{2}D_t \boldsymbol{q}^B\right) S\left(\frac{1}{2}l_r \boldsymbol{p}^B + \frac{1}{2}D_t \boldsymbol{q}^B\right),
\end{aligned}$$

$$\begin{aligned}
F\boldsymbol{u} &= \frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)} \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B + \frac{m_t m_r}{(m_r+m_t)}\left(\frac{1}{2}l_r \boldsymbol{p}^B + \right. \\
&\left. \frac{1}{2}D_t \boldsymbol{q}^B\right) \times \left(\frac{1}{2}l_r \dot{\boldsymbol{p}}^B + \frac{1}{2}D_t \dot{\boldsymbol{q}}^B\right) + R_R^B (I_r^R + I_t^T)\left(R_R^B\right)^T \boldsymbol{\omega}_r^B,
\end{aligned}$$

And

$$\boldsymbol{u} = \begin{bmatrix} \dot{\theta}_{pitch} & \dot{\theta}_{yaw} \end{bmatrix}.$$