Enhanced Aerial Reorientation Performance Using a 3-DoF Morphable Inertial Tail Inspired by Kangaroo Rats

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I. Modeling

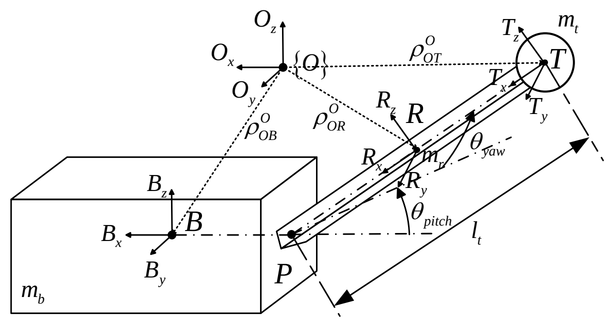

Fig. 1. Analytical models of the kangaroo rat and our tailed robot with a 3-DoF tail.

We start with an angular momentum equation expressed in the inertial frame $\{O\}$:

$$H^O = I_b^O \boldsymbol{\omega}_b^O + I_R^O \boldsymbol{\omega}_r^O + m_b \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O, \tag{1}$$

where I_b^O and I_r^O are the inertias of the robot body and tail link in the inertial frame $\{O\}$. m_t, m_r, m_b denote the mass of the body, the tail link, and the tail end mass. $\rho_{OT}^O, \rho_{OR}^O, \rho_{OB}^O$ are the vectors from the origin of the frame $\{O\}$ to the origins of the tail end frame $\{T\}$, the tail link frame $\{R\}$, and the body frame $\{B\}$. It can be also expressed in the body frame as:

$$H^B = I_b^B \boldsymbol{\omega}_b^B + I_R^B \boldsymbol{\omega}_r^B + m_b (R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + m_r (R_B^O)^T \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t (R_B^O)^T \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O, \tag{2}$$

where R_B^O is the rotation matrix from frame $\{B\}$ to frame $\{O\}$. Based on $\boldsymbol{\rho}_{OB}^O = R_B^O \boldsymbol{\rho}_{OB}^B$, we have $\dot{\boldsymbol{\rho}}_{OB}^O = \dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B$. Then, we have:

$$\begin{aligned}
&(R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O \\
&= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B\right) + (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left((R_B^O)^T \dot{R}_B^O \boldsymbol{\rho}_{OB}^B\right) + \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \\
&\left((R_B^O)^T R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\left[\boldsymbol{\omega}_b^O \times\right] R_B^O\right) \boldsymbol{\rho}_{OB}^B\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\boldsymbol{\omega}_b^O \times \boldsymbol{\rho}_{OB}^O\right)\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \boldsymbol{\rho}_{OB}^B \times \left(\boldsymbol{\omega}_b^B \times \boldsymbol{\rho}_{OB}^B\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \left(\boldsymbol{\rho}_{OB}^B\right)^2 \boldsymbol{\omega}_b^B - (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OB}^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= -\left[\boldsymbol{\rho}_{OB}^B \times\right]\left[\boldsymbol{\rho}_{OB}^B \times\right] \boldsymbol{\omega}_b^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B,
\end{aligned} \tag{3}$$

where $R(x \times y) = (Rx) \times (Ry)$ and $\Omega_b^O = \left[\boldsymbol{\omega}_b^O \times\right] = \dot{R}_B^O (R_B^O)^T$ are used. Here $\left(\boldsymbol{\rho}_{OB}^B\right)^2 = \left(\boldsymbol{\rho}_{OB}^B\right) \cdot \left(\boldsymbol{\rho}_{OB}^B\right)$. Then Eq. 2 becomes:

$$\begin{aligned}
H^B =& I_b^B \boldsymbol{\omega}_b^B + I_R^B \boldsymbol{\omega}_r^B + m_b \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B + m_r \boldsymbol{\rho}_{OR}^B \times \dot{\boldsymbol{\rho}}_{OR}^B + \\
& m_t \boldsymbol{\rho}_{OT}^B \times \dot{\boldsymbol{\rho}}_{OT}^B + m_b \left(\boldsymbol{\rho}_{OB}^B\right)^2 \boldsymbol{\omega}_b^B + m_r \left(\boldsymbol{\rho}_{OR}^B\right)^2 \boldsymbol{\omega}_b^B + m_t \\
& \left(\boldsymbol{\rho}_{OT}^B\right)^2 \boldsymbol{\omega}_b^B - m_b (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OB}^B - m_r (\boldsymbol{\rho}_{OR}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OR}^B - \\
& m_t (\boldsymbol{\rho}_{OT}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OT}^B.
\end{aligned} \tag{4}$$

At the robot center of mass (CoM), we have:

$$m_b \boldsymbol{\rho}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O = \boldsymbol{0}, \tag{5}$$

here

$$\begin{aligned}
&\boldsymbol{\rho}_{OR}^O = \boldsymbol{\rho}_{OT}^O - \boldsymbol{\rho}_{RT}^O = \boldsymbol{\rho}_{OT}^O - R_B^O \boldsymbol{\rho}_{RT}^B = \\
&\boldsymbol{\rho}_{OT}^O - \frac{1}{2} l_t R_B^O \begin{bmatrix} -c_{pitch} c_{yaw} & -s_{yaw} & s_{pitch} c_{yaw} \end{bmatrix}^T,
\end{aligned} \tag{6}$$

where $c_{pitch} = \cos\theta_{pitch}$, $s_{pitch} = \sin\theta_{pitch}$, $c_{yaw} = \cos\theta_{yaw}$, $s_{yaw} = \sin\theta_{yaw}$. θ_{pitch} is the tail swing angle in the body pitch direction and θ_{yaw} is the tail swing angle in the body yaw direction. $\boldsymbol{p}^B = \begin{bmatrix} -c_{pitch} c_{yaw} & -s_{yaw} & s_{pitch} c_{yaw} \end{bmatrix}^T$ is unit tail reorientation vector in frame $\{B\}$. l_t denotes the tail length. Substitute Eq. 6 into Eq. 5, we get:

$$\begin{aligned}
&\boldsymbol{\rho}_{OT}^B = -\frac{m_b}{(m_r + m_t)} \boldsymbol{\rho}_{OB}^B + \frac{1}{2} \frac{m_r}{(m_r + m_t)} l_t \\
&\begin{bmatrix} -c_{pitch} c_{yaw} & -s_{yaw} & s_{pitch} c_{yaw} \end{bmatrix}^T, \\
&\boldsymbol{\rho}_{OR}^B = -\frac{m_b}{(m_r + m_t)} \boldsymbol{\rho}_{OB}^B - \frac{1}{2} \frac{m_t}{(m_r + m_t)} l_t \\
&\begin{bmatrix} -c_{pitch} c_{yaw} & -s_{yaw} & s_{pitch} c_{yaw} \end{bmatrix}^T.
\end{aligned} \tag{7}$$

We also have:

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$$\dot{\boldsymbol{\rho}}_{OT}^{B}=-\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^{B}+\frac{1}{2}\frac{m_r}{(m_r+m_t)}\dot{l}_t$$
$$\begin{bmatrix}-c_{pitch}c_{yaw} & -s_{yaw} & s_{pitch}c_{yaw}\end{bmatrix}^T+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t[s_{pitch}c_{yaw}\dot{\theta}_{pitch}$$
$$+c_{pitch}s_{yaw}\dot{\theta}_{yaw}\quad -c_{yaw}\dot{\theta}_{yaw}\quad c_{pitch}c_{yaw}\dot{\theta}_{pitch}-s_{pitch}s_{yaw}\dot{\theta}_{yaw}]^T,$$

$$\dot{\boldsymbol{\rho}}_{OR}^{B}=-\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^{B}-\frac{1}{2}\frac{m_t}{(m_r+m_t)}\dot{l}_t$$
$$\begin{bmatrix}-c_{pitch}c_{yaw} & -s_{yaw} & s_{pitch}c_{yaw}\end{bmatrix}^T-\frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t[s_{pitch}c_{yaw}\dot{\theta}_{pitch}$$
$$+c_{pitch}s_{yaw}\dot{\theta}_{yaw}\quad -c_{yaw}\dot{\theta}_{yaw}\quad c_{pitch}c_{yaw}\dot{\theta}_{pitch}-s_{pitch}s_{yaw}\dot{\theta}_{yaw}]^T.$$

Eq. 4 can be updated as:

$$H^B=I_b^B\boldsymbol{\omega}_b^B+I_r^B\boldsymbol{\omega}_r^B+m_b\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B+m_r\boldsymbol{\rho}_{OR}^B\times\dot{\boldsymbol{\rho}}_{OR}^B+$$
$$m_t\boldsymbol{\rho}_{OT}^B\times\dot{\boldsymbol{\rho}}_{OT}^B+m_b\left(\boldsymbol{\rho}_{OB}^B\right)^2\boldsymbol{\omega}_b^B+m_r\left(\boldsymbol{\rho}_{OR}^B\right)^2\boldsymbol{\omega}_b^B+m_t$$
$$\left(\boldsymbol{\rho}_{OT}^B\right)^2\boldsymbol{\omega}_b^B-m_b(\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B-m_r(\boldsymbol{\rho}_{OR}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OR}^B-$$
$$m_t(\boldsymbol{\rho}_{OT}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OT}^B$$
$$=I_b^B\boldsymbol{\omega}_b^B+m_b\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B+m_r\begin{pmatrix}-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B-\\ \frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t\boldsymbol{p}^B\end{pmatrix}\times$$
$$\begin{pmatrix}-\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^B-\frac{1}{2}\frac{m_t}{(m_r+m_t)}\dot{l}_t\boldsymbol{p}^B\\ -\frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t\dot{\boldsymbol{p}}^B\end{pmatrix}$$
$$+m_t\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)\times$$
$$\begin{pmatrix}-\frac{m_b}{(m_r+m_t)}\dot{\boldsymbol{\rho}}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}\dot{l}_t\boldsymbol{p}^B\\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\dot{\boldsymbol{p}}^B\end{pmatrix}+m_b\left(\boldsymbol{\rho}_{OB}^B\right)^2\boldsymbol{\omega}_b^B$$
$$+m_r\begin{pmatrix}\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B-\frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)\cdot\\ \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B-\frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)\end{pmatrix}\boldsymbol{\omega}_b^B+$$
$$m_t\begin{pmatrix}\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)\cdot\\ \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)\end{pmatrix}\boldsymbol{\omega}_b^B-$$
$$m_b(\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B-m_r\left(\begin{pmatrix}-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B-\\ \frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t\boldsymbol{p}^B\end{pmatrix}\cdot\boldsymbol{\omega}_b^B\right)$$
$$\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B-\frac{1}{2}\frac{m_t}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)-$$
$$m_t\left(\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)\cdot\boldsymbol{\omega}_b^B\right)$$
$$\left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B\right)$$
$$=I_b^B\boldsymbol{\omega}_b^B+I_r^B\boldsymbol{\omega}_r^B+\frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B+$$
$$\left(\boldsymbol{\rho}_{OB}^B\right)^2\boldsymbol{\omega}_b^B-(\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B)+\frac{1}{4}\left(\frac{m_tm_r}{(m_r+m_t)}\right)l_t^2(\boldsymbol{p}^B$$
$$\times\dot{\boldsymbol{p}}^B+\boldsymbol{\omega}_b^B-(\boldsymbol{p}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{p}^B)$$
$$=I_b^B\boldsymbol{\omega}_b^B+I_r^B\boldsymbol{\omega}_r^B+\frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(-\left[\boldsymbol{\rho}_{OB}^B\times\right]$$
$$\left[\boldsymbol{\rho}_{OB}^B\times\right]\boldsymbol{\omega}_b^B+\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B)+\frac{1}{4}\left(\frac{m_tm_r}{(m_r+m_t)}\right)l_t^2(- \tag{8}$$
$$\left[\boldsymbol{p}^B\times\right]\left[\boldsymbol{p}^B\times\right]\boldsymbol{\omega}_b^B+\boldsymbol{p}^B\times\dot{\boldsymbol{p}}^B).$$

A closed path starting from the origin of frame $\{O\}$ passing through body CoM B, tail base P, and tail end mass CoM P can be expressed as:

$$\boldsymbol{\rho}_{OB}^O+\boldsymbol{\rho}_{BP}^O+\boldsymbol{\rho}_{PT}^O-\boldsymbol{\rho}_{OT}^O=\boldsymbol{0},$$

where

$$\begin{aligned}\boldsymbol{\rho}_{BP}^O&=R_B^O\boldsymbol{\rho}_{BP}^B=R_B^O\boldsymbol{\rho}_{BP}^B=R_B^O\begin{bmatrix}-l_b/2 & 0 & 0\end{bmatrix}^T,\\ \boldsymbol{\rho}_{PT}^O&=R_B^O\boldsymbol{\rho}_{PT}^B=R_B^O\left(l_t\boldsymbol{p}^B\right).\end{aligned} \tag{9}$$

l_b denotes the body length. Combining Eq. 7 and Eq. 9 gives:

$$\begin{aligned}\boldsymbol{\rho}_{OT}^B&=-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B\\ &=R_O^B\left(\boldsymbol{\rho}_{OB}^O+\boldsymbol{\rho}_{BP}^O+\boldsymbol{\rho}_{PT}^O\right)\\ &=\boldsymbol{\rho}_{OB}^B+\begin{bmatrix}-l_b/2 & 0 & 0\end{bmatrix}^T+l_t\boldsymbol{p}^B.\end{aligned}$$

Then:

$$\boldsymbol{\rho}_{OB}^{B} = \frac{m_r + m_t}{m_b + m_r + m_t}\begin{pmatrix} \frac{1}{2}\frac{m_r}{(m_r+m_t)} l_t \boldsymbol{p}^B + \left[l_b/2 \quad 0 \quad 0 \right]^T \\ -l_t \boldsymbol{p}^B \end{pmatrix}$$

$$= \frac{m_r + m_t}{m_b + m_r + m_t}\left(\frac{-\frac{1}{2}m_r - m_t}{(m_r+m_t)} l_t \boldsymbol{p}^B + \left[l_b/2 \quad 0 \quad 0 \right]^T\right)$$

$$\dot{\boldsymbol{\rho}}_{OB}^{B} = -\frac{\frac{1}{2}m_r + m_t}{m_b + m_r + m_t}(\dot{l}_t \boldsymbol{p}^B + l_t \dot{\boldsymbol{p}}^B)$$

Besides, we have:

$$I_R^B \boldsymbol{\omega}_r^B = R_R^B I_r^R \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,$$

$$R_R^B = \begin{bmatrix} c_{pitch}c_{yaw} & -c_{pitch}s_{yaw} & s_{pitch} \\ s_{yaw} & c_{yaw} & 0 \\ -s_{pitch}c_{yaw} & s_{pitch}s_{yaw} & c_{pitch} \end{bmatrix}$$

$$\boldsymbol{\omega}_r^B = \left[s_{pitch}\dot{\theta}_{yaw} \quad \dot{\theta}_{pitch} \quad c_{pitch}\dot{\theta}_{yaw} \right]^T$$

$$I_r^R = \frac{m_r}{12}\begin{bmatrix} w_t^2 + h_t^2 & 0 & 0 \\ 0 & l_t^2 + h_t^2 & 0 \\ 0 & 0 & l_t^2 + w_t^2 \end{bmatrix}$$

Eq. 8 can be expressed as:

$$H^B = A\boldsymbol{\omega}_B^B + F\boldsymbol{u},$$

$$A = I_b^B - \frac{m_b(m_r + m_t + m_b)}{(m_r + m_t)} S(\boldsymbol{\rho}_{OB}^B) S(\boldsymbol{\rho}_{OB}^B) - \frac{m_t m_r l_t^2}{4(m_r + m_t)} S(\boldsymbol{p}^B) S(\boldsymbol{p}^B),$$

$$F\boldsymbol{u} = \frac{m_b(m_r + m_t + m_b)}{(m_r + m_t)} \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B + \frac{m_t m_r l_t^2}{4(m_r + m_t)} \boldsymbol{p}^B \times \dot{\boldsymbol{p}}^B + R_R^B I_r^R \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,$$

And

$$F = \begin{bmatrix} -\frac{(c_{yaw}c_{pitch}s_{yaw}(l_t^2m_r^2 - m_r^2w_t^2 + 4l_t^2m_bm_r + 12l_t^2m_bm_t + 4l_t^2m_rm_t - m_bm_rw_t^2 - m_rm_tw_t^2))}{12(m_b + m_r + m_t)}, & \frac{m_r s_{yaw}^{\ 2}(h_t^2 + w_t^2) + m_r c_{yaw}^{\ 2}(h_t^2 + l_t^2)}{12} + \frac{l_t m_b c_{yaw}(m_r + 2m_t)\begin{pmatrix} l_b m_r c_{pitch} + l_b m_t c_{pitch} + l_t m_r c_{yaw} + \\ 2 l_t m_t c_{yaw} \end{pmatrix}}{4(m_b + m_r + m_t)(m_r + m_t)} + \frac{l_t^2 m_r m_t c_{yaw}^{\ 2}}{4(m_r + m_t)}, & \frac{(c_{yaw}s_{yaw}s_{pitch}(l_t^2m_r^2 - m_r^2w_t^2 + 4l_t^2m_bm_r + 12l_t^2m_bm_t + 4l_t^2m_rm_t - m_bm_rw_t^2 - m_rm_tw_t^2))}{12(m_b + m_r + m_t)}, \\ \frac{(s_{pitch}(m_r^2w_t^2 + l_t^2m_r^2 + 4l_t^2m_bm_r + 12l_t^2m_bm_t + 4l_t^2m_rm_t + m_bm_rw_t^2 + m_rm_tw_t^2))}{12(m_b + m_r + m_t)}, & -\frac{(l_b l_t m_b s_{yaw} s_{pitch}(m_r + 2m_t))}{4(m_b + m_r + m_t)}, & \frac{(l_t^2m_r^2c_{pitch} + m_r^2w_t^2c_{pitch} + 4l_t^2m_bm_rc_{pitch} + 12l_t^2m_bm_tc_{pitch} + 4l_t^2m_rm_tc_{pitch} + m_bm_rw_t^2c_{pitch} + m_rm_tw_t^2c_{pitch} + 3l_bl_tm_bm_rc_{yaw} + 6l_bl_tm_bm_tc_{yaw})}{12(m_b + m_r + m_t)}, \\ 0 & \frac{(l_b m_b c_{yaw} s_{pitch}(m_r + 2m_t))}{4(m_b + m_r + m_t)} & \frac{(l_b m_b s_{yaw}(m_r + 2m_t))}{4(m_b + m_r + m_t)} \end{bmatrix}$$

$$\boldsymbol{u} = \left[\dot{\theta}_{pitch} \quad \dot{\theta}_{yaw} \quad \dot{l}_t \right].$$