Quadrupedal Robot with a Prehensile Tail: A Bioinspired Soft Variable Stiffness Robotic Tail for Versatile Grasping and Enhanced Mobility

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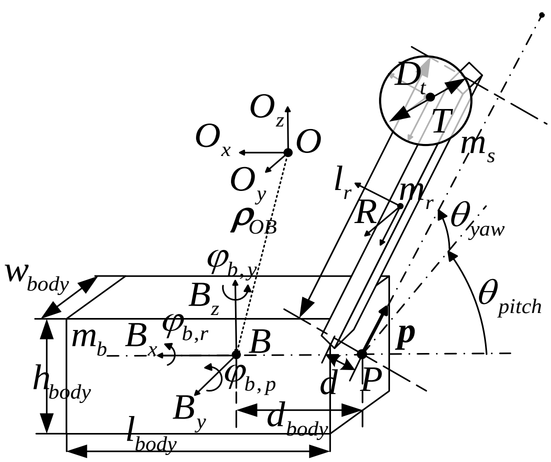

(a)

Fig. 1. Analytical models of the kangaroo rat and our tailed robot with a 3-DoF tail.

We start with an angular momentum equation expressed in the inertial frame $\{O\}$:

$$H^O = I_b^O \boldsymbol{\omega}_b^O + I_r^O \boldsymbol{\omega}_r^O + I_t^O \boldsymbol{\omega}_t^O + m_b \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O, \tag{2}$$

where I_b^O and I_r^O are the inertias of the robot body and tail link in the inertial frame $\{O\}$. m_t, m_r, m_b denote the mass of the body, the tail link, and the tail end mass. $\rho_{OT}^O, \rho_{OR}^O, \rho_{OB}^O$ are the vectors from the origin of the frame $\{O\}$ to the origins of the tail end frame $\{T\}$, the tail link frame $\{R\}$, and the body frame $\{B\}$. It can be also expressed in the body frame as:

$$H^B = I_b^B \boldsymbol{\omega}_b^B + I_r^B \boldsymbol{\omega}_r^B + I_t^O \boldsymbol{\omega}_t^O + m_b (R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O + m_r (R_B^O)^T \boldsymbol{\rho}_{OR}^O \times \dot{\boldsymbol{\rho}}_{OR}^O + m_t (R_B^O)^T \boldsymbol{\rho}_{OT}^O \times \dot{\boldsymbol{\rho}}_{OT}^O, \tag{2}$$

where R_B^O is the rotation matrix from frame $\{B\}$ to frame $\{O\}$. Based on $\boldsymbol{\rho}_{OB}^O = R_B^O \boldsymbol{\rho}_{OB}^B$, we have $\dot{\boldsymbol{\rho}}_{OB}^O = \dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B$. Then, we have:

$$\begin{aligned}
&(R_B^O)^T \boldsymbol{\rho}_{OB}^O \times \dot{\boldsymbol{\rho}}_{OB}^O \\
&= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B + R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(\dot{R}_B^O \boldsymbol{\rho}_{OB}^B\right) + (R_B^O)^T \left(R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left(R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \left((R_B^O)^T \dot{R}_B^O \boldsymbol{\rho}_{OB}^B\right) + \left((R_B^O)^T R_B^O \boldsymbol{\rho}_{OB}^B\right) \times \\
&\left((R_B^O)^T R_B^O \dot{\boldsymbol{\rho}}_{OB}^B\right) \\
&= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\left[\boldsymbol{\omega}_b^O \times\right] R_B^O\right) \boldsymbol{\rho}_{OB}^B\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \boldsymbol{\rho}_{OB}^B \times \left((R_B^O)^T \left(\boldsymbol{\omega}_b^O \times \boldsymbol{\rho}_{OB}^O\right)\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \boldsymbol{\rho}_{OB}^B \times \left(\boldsymbol{\omega}_b^B \times \boldsymbol{\rho}_{OB}^B\right) + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= \left(\boldsymbol{\rho}_{OB}^B\right)^2 \boldsymbol{\omega}_b^B - (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OB}^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B \\
&= -\left[\boldsymbol{\rho}_{OB}^B \times\right]\left[\boldsymbol{\rho}_{OB}^B \times\right] \boldsymbol{\omega}_b^B + \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B,
\end{aligned} \tag{3}$$

where $R(x \times y) = (Rx) \times (Ry)$ and $\Omega_b^O = \left[\boldsymbol{\omega}_b^O \times\right] = \dot{R}_B^O (R_B^O)^T$ are used. Here $\left(\boldsymbol{\rho}_{OB}^B\right)^2 = \left(\boldsymbol{\rho}_{OB}^B\right) \cdot \left(\boldsymbol{\rho}_{OB}^B\right)$. Then Eq. 2 becomes:

$$\begin{aligned}
H^B = {} & I_b^B \boldsymbol{\omega}_b^B + I_r^B \boldsymbol{\omega}_r^B + I_r^B \boldsymbol{\omega}_r^B + m_b \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B + \\
& m_r \boldsymbol{\rho}_{OR}^B \times \dot{\boldsymbol{\rho}}_{OR}^B + m_t \boldsymbol{\rho}_{OT}^B \times \dot{\boldsymbol{\rho}}_{OT}^B + m_b \left(\boldsymbol{\rho}_{OB}^B\right)^2 \boldsymbol{\omega}_b^B \\
& + m_r \left(\boldsymbol{\rho}_{OR}^B\right)^2 \boldsymbol{\omega}_b^B + m_t \left(\boldsymbol{\rho}_{OT}^B\right)^2 \boldsymbol{\omega}_b^B - m_b (\boldsymbol{\rho}_{OB}^B \cdot \boldsymbol{\omega}_b^B) \\
& \boldsymbol{\rho}_{OB}^B - m_r (\boldsymbol{\rho}_{OR}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OR}^B - m_t (\boldsymbol{\rho}_{OT}^B \cdot \boldsymbol{\omega}_b^B) \boldsymbol{\rho}_{OT}^B.
\end{aligned} \tag{4}$$

At the robot center of mass (CoM), we have:

$$m_b \boldsymbol{\rho}_{OB}^O + m_r \boldsymbol{\rho}_{OR}^O + m_t \boldsymbol{\rho}_{OT}^O = \boldsymbol{0}, \tag{5}$$

here

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$$\boldsymbol{\rho}_{OR}^{O}=\boldsymbol{\rho}_{OT}^{O}-\boldsymbol{\rho}_{RT}^{O}=\boldsymbol{\rho}_{OT}^{O}-R_B^O\boldsymbol{\rho}_{RT}^{B}=$$
$$\boldsymbol{\rho}_{OT}^{O}-R_B^O\left(\left[-\frac{1}{2}l_r c_{pitch}c_{yaw}\quad -\frac{1}{2}l_r s_{yaw}\quad \frac{1}{2}l_r s_{pitch}c_{yaw}\right]^T \right. \tag{6}$$
$$\left.+\left[-\frac{1}{2}D_t c_{pitch}s_{yaw}\quad \frac{1}{2}D_t c_{yaw}\quad \frac{1}{2}D_t s_{pitch}s_{yaw}\right]^T\right),$$

where $c_{pitch}=\cos\theta_{pitch}$, $s_{pitch}=\sin\theta_{pitch}$, $c_{yaw}=\cos\theta_{yaw}$, $s_{yaw}=\sin\theta_{yaw}$. θ_{pitch} is the tail swing angle in the body pitch direction and θ_{yaw} is the tail swing angle in the body yaw direction. $\boldsymbol{p}^B=\left[-c_{pitch}c_{yaw}\quad -s_{yaw}\quad s_{pitch}c_{yaw}\right]^T$ is unit tail reorientation vector in frame $\{B\}$. l_r denotes the tail length. Substitute Eq. 6 into Eq. 5, we get:

$$\left(m_r+m_t\right)\boldsymbol{\rho}_{OT}^{O}=-m_b\boldsymbol{\rho}_{OB}^{O}+m_r R_B^O$$
$$\left(\left[-\frac{1}{2}l_r c_{pitch}c_{yaw}\quad -\frac{1}{2}l_r s_{yaw}\quad \frac{1}{2}l_r s_{pitch}c_{yaw}\right]^T\right.$$
$$\left.+\left[-\frac{1}{2}D_t c_{pitch}s_{yaw}\quad \frac{1}{2}D_t c_{yaw}\quad \frac{1}{2}D_t s_{pitch}s_{yaw}\right]^T\right),$$

$$\boldsymbol{\rho}_{OT}^{B}=-\frac{m_b}{\left(m_r+m_t\right)}\boldsymbol{\rho}_{OB}^{B}+\frac{m_r}{\left(m_r+m_t\right)}$$
$$\left(\left[-\frac{1}{2}l_r c_{pitch}c_{yaw}\quad -\frac{1}{2}l_r s_{yaw}\quad \frac{1}{2}l_r s_{pitch}c_{yaw}\right]^T\right.$$
$$\left.+\left[-\frac{1}{2}D_t c_{pitch}s_{yaw}\quad \frac{1}{2}D_t c_{yaw}\quad \frac{1}{2}D_t s_{pitch}s_{yaw}\right]^T\right),$$

$$\boldsymbol{\rho}_{OR}^{B}=\boldsymbol{\rho}_{OT}^{B}-\boldsymbol{\rho}_{RT}^{B}=$$
$$-\frac{m_b}{\left(m_r+m_t\right)}\boldsymbol{\rho}_{OB}^{B}-\frac{m_t}{\left(m_r+m_t\right)}$$
$$\left(\left[-\frac{1}{2}l_r c_{pitch}c_{yaw}\quad -\frac{1}{2}l_r s_{yaw}\quad \frac{1}{2}l_r s_{pitch}c_{yaw}\right]^T\right.$$
$$\left.+\left[-\frac{1}{2}D_t c_{pitch}s_{yaw}\quad \frac{1}{2}D_t c_{yaw}\quad \frac{1}{2}D_t s_{pitch}s_{yaw}\right]^T\right) \tag{7}$$

We also have:

$$\dot{\boldsymbol{\rho}}_{OT}^{B}=-\frac{m_b}{\left(m_r+m_t\right)}\dot{\boldsymbol{\rho}}_{OB}^{B}+$$
$$+\frac{1}{2}l_r\frac{m_r}{\left(m_r+m_t\right)}[s_{pitch}c_{yaw}\dot{\theta}_{pitch}$$
$$+c_{pitch}s_{yaw}\dot{\theta}_{yaw}\quad -c_{yaw}\dot{\theta}_{yaw}\quad c_{pitch}c_{yaw}\dot{\theta}_{pitch}-s_{pitch}s_{yaw}\dot{\theta}_{yaw}]^T$$
$$+\frac{1}{2}D_t\frac{m_r}{\left(m_r+m_t\right)}[s_{pitch}s_{yaw}\dot{\theta}_{pitch}-c_{pitch}c_{yaw}\dot{\theta}_{yaw}$$
$$-s_{yaw}\dot{\theta}_{yaw}\quad c_{pitch}s_{yaw}\dot{\theta}_{pitch}+s_{pitch}c_{yaw}\dot{\theta}_{yaw}]^T,$$

$$\dot{\boldsymbol{\rho}}_{OR}^{B}=-\frac{m_b}{\left(m_r+m_t\right)}\dot{\boldsymbol{\rho}}_{OB}^{B}$$
$$-\frac{1}{2}l_r\frac{m_t}{\left(m_r+m_t\right)}[s_{pitch}c_{yaw}\dot{\theta}_{pitch}$$
$$+c_{pitch}s_{yaw}\dot{\theta}_{yaw}\quad -c_{yaw}\dot{\theta}_{yaw}\quad c_{pitch}c_{yaw}\dot{\theta}_{pitch}-s_{pitch}s_{yaw}\dot{\theta}_{yaw}]^T$$
$$-\frac{1}{2}D_t\frac{m_t}{\left(m_r+m_t\right)}[s_{pitch}s_{yaw}\dot{\theta}_{pitch}-c_{pitch}c_{yaw}\dot{\theta}_{yaw}$$
$$-s_{yaw}\dot{\theta}_{yaw}\quad c_{pitch}s_{yaw}\dot{\theta}_{pitch}+s_{pitch}c_{yaw}\dot{\theta}_{yaw}]^T,.$$

Eq. 4 can be updated as:

$$H^B=I_b^B\boldsymbol{\omega}_b^B+I_r^B\boldsymbol{\omega}_r^B+I_t^B\boldsymbol{\omega}_t^B+m_b\boldsymbol{\rho}_{OB}^{B}\times\dot{\boldsymbol{\rho}}_{OB}^{B}+m_r\boldsymbol{\rho}_{OR}^{B}\times\dot{\boldsymbol{\rho}}_{OR}^{B}+$$
$$m_t\boldsymbol{\rho}_{OT}^{B}\times\dot{\boldsymbol{\rho}}_{OT}^{B}+m_b\left(\boldsymbol{\rho}_{OB}^{B}\right)^2\boldsymbol{\omega}_b^B+m_r\left(\boldsymbol{\rho}_{OR}^{B}\right)^2\boldsymbol{\omega}_b^B+m_t$$
$$\left(\boldsymbol{\rho}_{OT}^{B}\right)^2\boldsymbol{\omega}_b^B-m_b(\boldsymbol{\rho}_{OB}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^{B}-m_r(\boldsymbol{\rho}_{OR}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OR}^{B}-$$
$$m_t(\boldsymbol{\rho}_{OT}^{B}\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OT}^{B}$$
$$=I_b^B\boldsymbol{\omega}_b^B+\left(I_r^B+I_t^B\right)\boldsymbol{\omega}_t^B+m_b\boldsymbol{\rho}_{OB}^{B}\times\dot{\boldsymbol{\rho}}_{OB}^{B}+m_r$$
$$\begin{pmatrix}-\frac{m_b}{\left(m_r+m_t\right)}\boldsymbol{\rho}_{OB}^{B}-\\ \frac{1}{2}l_r\frac{m_t}{\left(m_r+m_t\right)}\boldsymbol{p}^B\\ -\frac{1}{2}D_t\frac{m_t}{\left(m_r+m_t\right)}\boldsymbol{q}^B\end{pmatrix}\times\begin{pmatrix}-\frac{m_b}{\left(m_r+m_t\right)}\dot{\boldsymbol{\rho}}_{OB}^{B}\\ -\frac{1}{2}\frac{m_t}{\left(m_r+m_t\right)}l_t\dot{\boldsymbol{p}}^B\\ -\frac{1}{2}D_t\frac{m_t}{\left(m_r+m_t\right)}\dot{\boldsymbol{q}}^B\end{pmatrix}$$
$$+m_t\begin{pmatrix}-\frac{m_b}{\left(m_r+m_t\right)}\boldsymbol{\rho}_{OB}^{B}+\frac{1}{2}\frac{m_r}{\left(m_r+m_t\right)}l_r\boldsymbol{p}^B\\ +\frac{1}{2}\frac{m_r}{\left(m_r+m_t\right)}D_t\boldsymbol{q}^B\end{pmatrix}\times$$
$$\begin{pmatrix}-\frac{m_b}{\left(m_r+m_t\right)}\dot{\boldsymbol{\rho}}_{OB}^{B}+\frac{1}{2}\frac{m_r}{\left(m_r+m_t\right)}l_r\dot{\boldsymbol{p}}^B\\ +\frac{1}{2}\frac{m_r}{\left(m_r+m_t\right)}D_t\dot{\boldsymbol{q}}^B\end{pmatrix}+m_b\left(\boldsymbol{\rho}_{OB}^{B}\right)^2\boldsymbol{\omega}_b^B$$
$$+m_r\begin{pmatrix}\begin{pmatrix}-\frac{m_b}{\left(m_r+m_t\right)}\boldsymbol{\rho}_{OB}^{B}-\frac{1}{2}\frac{m_t}{\left(m_r+m_t\right)}l_r\boldsymbol{p}^B\\ -\frac{1}{2}\frac{m_t}{\left(m_r+m_t\right)}D_t\boldsymbol{q}^B\end{pmatrix}\cdot\\ \begin{pmatrix}-\frac{m_b}{\left(m_r+m_t\right)}\boldsymbol{\rho}_{OB}^{B}-\frac{1}{2}\frac{m_t}{\left(m_r+m_t\right)}l_r\boldsymbol{p}^B\\ -\frac{1}{2}\frac{m_t}{\left(m_r+m_t\right)}D_t\boldsymbol{q}^B\end{pmatrix}\end{pmatrix}\boldsymbol{\omega}_b^B+$$

$$
\begin{aligned}
& m_t\left(\left(\begin{array}{l}-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\boldsymbol{p}^B\\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B\end{array}\right)\cdot\left(\begin{array}{l}-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\boldsymbol{p}^B\\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B\end{array}\right)\right)\boldsymbol{\omega}_b^B-\\
& m_b(\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B-m_r\left(\left(\begin{array}{l}-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B-\\ \frac{1}{2}\frac{m_t}{(m_r+m_t)}l_r\boldsymbol{p}^B-\\ \frac{1}{2}\frac{m_t}{(m_r+m_t)}D_t\boldsymbol{q}^B\end{array}\right)\cdot\boldsymbol{\omega}_b^B\right)\\
& \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B-\frac{1}{2}\frac{m_t}{(m_r+m_t)}l_r\boldsymbol{p}^B-\frac{1}{2}\frac{m_t}{(m_r+m_t)}D_t\boldsymbol{q}^B\right)\\
& -m_t\left(\left(\begin{array}{l}-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_r\boldsymbol{p}^B\\ +\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B\end{array}\right)\cdot\boldsymbol{\omega}_b^B\right)\\
& \left(-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}l_t\boldsymbol{p}^B+\frac{1}{2}\frac{m_r}{(m_r+m_t)}D_t\boldsymbol{q}^B\right)\\
& =I_b^B\boldsymbol{\omega}_b^B+I_r^B\boldsymbol{\omega}_r^B+I_t^B\boldsymbol{\omega}_t^B+\frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B+\\
& \left(\boldsymbol{\rho}_{OB}^B\right)^2\boldsymbol{\omega}_b^B-(\boldsymbol{\rho}_{OB}^B\cdot\boldsymbol{\omega}_b^B)\boldsymbol{\rho}_{OB}^B)+\frac{m_t m_r}{(m_r+m_t)}(\left(\begin{array}{l}(\frac{1}{2}l_r\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B)\\ \times(\frac{1}{2}l_t\dot{\boldsymbol{p}}^B+\frac{1}{2}D_t\dot{\boldsymbol{q}}^B)\end{array}\right)+\\
& (\frac{1}{4}l_r^2+\frac{1}{4}D_t^2)\boldsymbol{\omega}_b^B-\left((\frac{1}{2}l_t\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B)\cdot\boldsymbol{\omega}_b^B\right)(\frac{1}{2}l_t\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B))\\
& =I_b^B\boldsymbol{\omega}_b^B+I_r^B\boldsymbol{\omega}_r^B+\frac{m_b(m_r+m_t+m_b)}{(m_r+m_t)}(-\left[\boldsymbol{\rho}_{OB}^B\times\right]\\
& \left[\boldsymbol{\rho}_{OB}^B\times\right]\boldsymbol{\omega}_b^B+\boldsymbol{\rho}_{OB}^B\times\dot{\boldsymbol{\rho}}_{OB}^B)+\frac{m_t m_r}{(m_r+m_t)}(-\\
& \left[(\frac{1}{2}l_t\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B)\times\right]\left[(\frac{1}{2}l_t\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B)\times\right]\boldsymbol{\omega}_b^B\\
& +\left(\begin{array}{l}(\frac{1}{2}l_r\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B)\\ \times(\frac{1}{2}l_t\dot{\boldsymbol{p}}^B+\frac{1}{2}D_t\dot{\boldsymbol{q}}^B)\end{array}\right)).
\end{aligned}
\tag{8}
$$

Here $\boldsymbol{q}^B=\left[-c_{pitch}s_{yaw}\quad c_{yaw}\quad s_{pitch}s_{yaw}\right]^T$ is unit reorientation vector of R_y in frame $\{B\}$. A closed path starting from the origin of frame $\{O\}$ passing through body CoM B, tail base P, and tail end mass CoM P can be expressed as:

$$
\boldsymbol{\rho}_{OB}^O+\boldsymbol{\rho}_{BP}^O+\boldsymbol{\rho}_{PT}^O-\boldsymbol{\rho}_{OT}^O=\boldsymbol{0},
$$

where

$$
\begin{aligned}
\boldsymbol{\rho}_{BP}^O &= R_B^O\boldsymbol{\rho}_{BP}^B=R_B^O\boldsymbol{\rho}_{BP}^B=R_B^O\left[-d_b\quad 0\quad 0\right]^T,\\
\boldsymbol{\rho}_{PT}^O &= R_B^O\boldsymbol{\rho}_{PT}^B=R_B^O\left(l_r\boldsymbol{p}^B+d\boldsymbol{r}^B+\frac{D_t}{2}\boldsymbol{q}^B\right).
\end{aligned}
\tag{9}
$$

d_b denotes the distance between the body CoM and the tail pivot. Combining Eq. 7 and Eq. 9 gives:

$$
\begin{aligned}
\boldsymbol{\rho}_{OT}^B &= -\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{m_r}{(m_r+m_t)}\\
&\left(\left[-\frac{1}{2}l_rc_{pitch}c_{yaw}\quad -\frac{1}{2}l_rs_{yaw}\quad \frac{1}{2}l_rs_{pitch}c_{yaw}\right]^T\right.\\
&\left.+\left[-\frac{1}{2}D_tc_{pitch}s_{yaw}\quad \frac{1}{2}D_tc_{yaw}\quad \frac{1}{2}D_ts_{pitch}s_{yaw}\right]^T\right)\\
&=-\frac{m_b}{(m_r+m_t)}\boldsymbol{\rho}_{OB}^B+\frac{m_r}{(m_r+m_t)}(\frac{1}{2}l_r\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B)\\
&=R_O^B\left(\boldsymbol{\rho}_{OB}^O+\boldsymbol{\rho}_{BP}^O+\boldsymbol{\rho}_{PT}^O\right)\\
&=\boldsymbol{\rho}_{OB}^B+\left[-d_b\quad 0\quad 0\right]^T+l_r\boldsymbol{p}^B+d\boldsymbol{r}^B+\frac{D_t}{2}\boldsymbol{q}^B.
\end{aligned}
$$

Here $\boldsymbol{r}^B=\left[s_{pitch}\quad 0\quad c_{pitch}\right]^T$ is unit reorientation vector of R_z in frame $\{B\}$.

Then:

$$
\begin{aligned}
\boldsymbol{\rho}_{OB}^B &= \frac{m_r+m_t}{m_b+m_r+m_t}\left(\begin{array}{l}\frac{m_r}{(m_r+m_t)}(\frac{1}{2}l_r\boldsymbol{p}^B+\frac{1}{2}D_t\boldsymbol{q}^B)+\\ \left[d_b\quad 0\quad 0\right]^T\\ -l_r\boldsymbol{p}^B-d\boldsymbol{r}^B-\frac{D_t}{2}\boldsymbol{q}^B\end{array}\right)\\
&=\frac{m_r+m_t}{m_b+m_r+m_t}(\frac{-\frac{1}{2}m_r-m_t}{(m_r+m_t)}l_r\boldsymbol{p}^B+\left[d_b\quad 0\quad 0\right]^T-d\boldsymbol{r}^B\\
&-\frac{m_t}{(m_r+m_t)}\frac{1}{2}D_t\boldsymbol{q}^B)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{\rho}}_{OB}^B &= \frac{m_r+m_t}{m_b+m_r+m_t}(\frac{-\frac{1}{2}m_r-m_t}{(m_r+m_t)}l_r\dot{\boldsymbol{p}}^B-d\dot{\boldsymbol{r}}^B\\
&-\frac{m_t}{(m_r+m_t)}\frac{1}{2}D_t\dot{\boldsymbol{q}}^B)
\end{aligned}
$$

Besides, we have:

$$I_r^B \boldsymbol{\omega}_r^B = R_R^B I_r^R \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,$$

$$I_t^B \boldsymbol{\omega}_t^B = R_R^B I_t^T \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,$$

$$R_R^B = \begin{bmatrix} c_{pitch}c_{yaw} & -c_{pitch}s_{yaw} & s_{pitch} \\ s_{yaw} & c_{yaw} & 0 \\ -s_{pitch}c_{yaw} & s_{pitch}s_{yaw} & c_{pitch} \end{bmatrix}$$

$$\boldsymbol{\omega}_r^B = \boldsymbol{\omega}_t^B = \left[s_{pitch}\dot{\theta}_{yaw} \quad \dot{\theta}_{pitch} \quad c_{pitch}\dot{\theta}_{yaw} \right]^T$$

$$I_r^R = \frac{m_r}{12}\begin{bmatrix} w_t^2 + h_t^2 & 0 & 0 \\ 0 & l_t^2 + h_t^2 & 0 \\ 0 & 0 & l_t^2 + w_t^2 \end{bmatrix}$$

$$I_t^R = \frac{m_r}{10}\begin{bmatrix} D_t^2 & 0 & 0 \\ 0 & D_t^2 & 0 \\ 0 & 0 & D_t^2 \end{bmatrix}$$

Eq. 8 can be expressed as:

$$H^B = A\boldsymbol{\omega}_b^B + F\boldsymbol{u}, \tag{10}$$

$$A = I_b^B - \frac{m_b(m_r + m_t + m_b)}{\left(m_r + m_t\right)} S(\boldsymbol{\rho}_{OB}^B) S(\boldsymbol{\rho}_{OB}^B)$$
$$- \frac{m_t m_r}{\left(m_r + m_t\right)} S(\frac{1}{2} l_r \boldsymbol{p}^B + \frac{1}{2} D_t \boldsymbol{q}^B) S(\frac{1}{2} l_r \boldsymbol{p}^B + \frac{1}{2} D_t \boldsymbol{q}^B),$$

$$F\boldsymbol{u} = \frac{m_b(m_r + m_t + m_b)}{\left(m_r + m_t\right)} \boldsymbol{\rho}_{OB}^B \times \dot{\boldsymbol{\rho}}_{OB}^B + \frac{m_t m_r}{\left(m_r + m_t\right)} (\frac{1}{2} l_r \boldsymbol{p}^B +$$
$$\frac{1}{2} D_t \boldsymbol{q}^B) \times (\frac{1}{2} l_r \dot{\boldsymbol{p}}^B + \frac{1}{2} D_t \dot{\boldsymbol{q}}^B) + R_R^B (I_r^R + I_t^T) \left(R_R^B\right)^T \boldsymbol{\omega}_r^B,$$

And

$$\boldsymbol{u} = \left[\dot{\theta}_{pitch} \quad \dot{\theta}_{yaw} \right].$$