

Special unipotent representations
of real classical groups
and theta correspondence

Ma, Jia-Jun

(joint with Dan Barbasch, Binyong Sun and Chengbo Zhu)

School of Mathematical Sciences
Shanghai Jiao Tong University

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(Zhejiang University)

Classical groups and special unipotent representations

	G	\mathbf{G}	\mathbf{G}^\vee	
D_n	$O(p, 2n - p)$	$O(2n, \mathbb{C})$	$O(2n, \mathbb{C})$	D_n
C_n	$Sp(2n, \mathbb{R})$	$Sp(2n, \mathbb{C})$	$SO(2n + 1, \mathbb{C})$	B_n
B_n	$O(p, 2n + 1 - p)$	$O(2n + 1, \mathbb{C})$	$Sp(2n, \mathbb{C})$	C_n
\tilde{C}_n	$Mp(2n, \mathbb{R})$	$Sp(2n, \mathbb{C})$	$Sp(2n, \mathbb{C})$	C_n
D_n	$O^*(n)$	$SO(2n, \mathbb{C})$	$SO(2n, \mathbb{C})$	D_n
C_n	$Sp(p, n - p)$	$Sp(2n, \mathbb{C})$	$SO(2n + 1, \mathbb{C})$	B_n
A_n	$U(p, n - p)$	$GL(n, \mathbb{C})$	$GL(n, \mathbb{C})$	A_n
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 All *special unipotent representations* of G are *unitarizable*.

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- *Atlas of Lie group:* \rightsquigarrow complete answer for exceptional groups.

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- **Lemma:**

$$\begin{aligned} & \# \{ \pi \in \text{Irr}_\mu(\mathfrak{g}, K)(G) \mid \text{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \} \\ &= \sum_{\substack{\tau \in \mathcal{D} \\ \mathcal{D} \rightsquigarrow \mathcal{O}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathcal{G}_\lambda(\mathfrak{g}, K)] \end{aligned}$$

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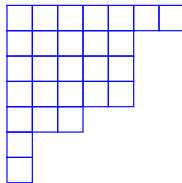
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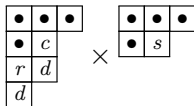
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- $[\tau : \mathcal{G}_\rho(G)]$ is counted by painted bi-partitions $\mathrm{PBP}(\check{\mathcal{O}}).$

Example of PBP

$$\check{O} = [7, 5, 5, 5, 3, 1, 1] =$$



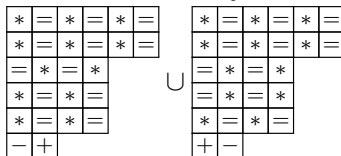
$\text{PBP}_{\check{O}}(\text{Sp}(2n, \mathbb{R}))$



⋮

\mapsto

Associated cycle



⋮

Nilpotent orbits with “good/bad parity”

- Bad parity (must occur with even multiplicity in $\check{\mathcal{O}}$):

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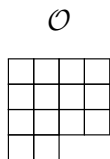
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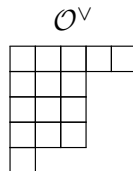
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- Example of good parity:



$\mathrm{Sp}(14, \mathbb{C})$



$\mathrm{SO}(15, \mathbb{C})$

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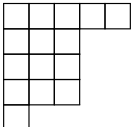
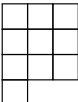


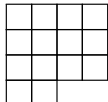
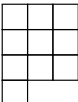


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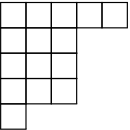
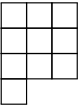


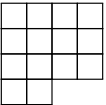
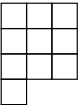


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- We assume $\check{\mathcal{O}}$ has **good parity** from now on.

Example of descent sequences

\mathbf{G}_i^\vee	$\mathrm{SO}(15, \mathbb{C})$	$\mathrm{O}(10, \mathbb{C})$	$\mathrm{SO}(7, \mathbb{C})$	$\mathrm{O}(4, \mathbb{C})$
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[Kraft-Procesi's](#) resolution of singularities of the closure of complex nilpotent orbits.

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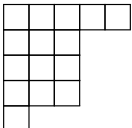
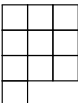


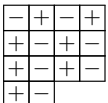
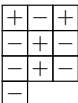


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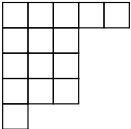
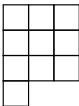


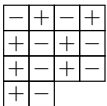
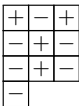


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Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

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$$\pi_{\chi} := (\omega_{G_{2a}, G_{2a-1}} \widehat{\otimes} \omega_{G_{2a-1}, G_{2a-2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1, G_0} \otimes \chi)_{G_{2a-1} \times G_{2a-2} \times \cdots \times G_0}$$

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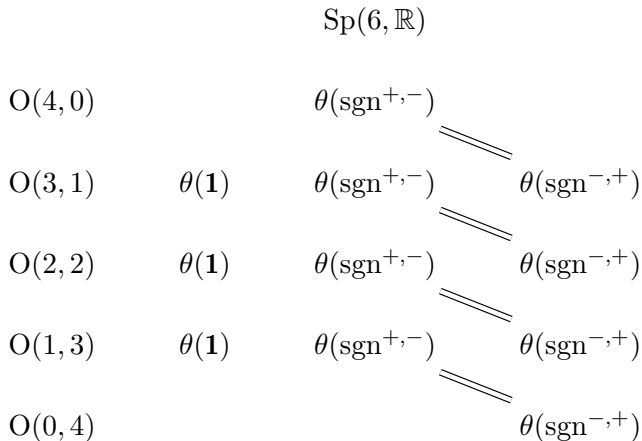
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Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.

$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.



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Whittaker cycle = Wavefront cycle.

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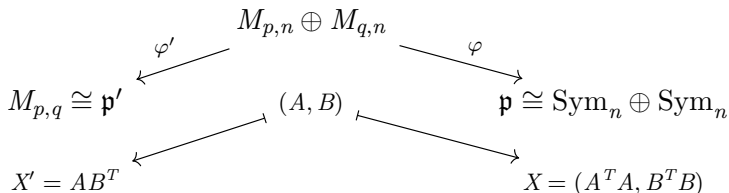
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such that

$$\mathrm{AC}(\Theta(\pi')) \preceq \vartheta^{\mathrm{geo}}(\mathrm{AC}(\pi')),$$

for any π' with $\mathrm{AV}(\pi') \subset \overline{\mathcal{O}'}$

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- Stable range lifting trick: Suppose $n > p + q$.

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- The **injectivity** of theta lifting is crucial!

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- **Question:** How to describe $\pi_{\psi, \eta}$ explicitly?

Dan Barabasch, M. , Binyong Sun and Chen-Bo Zhu
Special unipotent representations: orthogonal and symplectic groups
ArXiv e-prints: <https://arxiv.org/abs/1712.05552v2>

Thank you for your attention!