COUNTING SPECIAL UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS

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Contents

1. Counting unipotent representations

In this section, let $G_{\mathbb{C}}$ be a connected complex reductive group and $\mathfrak{g}$ is its Lie algebra. Fix a antiholomorphic involution σ on $G_{\mathbb{C}}$ and a corresponding Cartan involution θ of $G_{\mathbb{C}}$. Let G be a finite central extension of a open subgroup os $G_{\mathbb{C}}^{\sigma}$ and

$$\mathrm{pr}\colon G \to G_{\mathbb{C}}^{\sigma}$$

be the canonical projection. Let $K = \mathrm{pr}^{-1}(G_{\mathbb{C}}^{\sigma})$ and $W = W(G_{\mathbb{C}})$.

Let ${}^{a}\mathfrak{h}$ be the abstract Cartan subalgebra of $\mathfrak{g}$ and ${}^{a}X$ be the lattice of abstarct weight spaces. Let ${}^{a}R \subseteq {}^{a}X$, ${}^{a}R^{+}$ and ${}^{a}Q$ be the abstract root system, the set of positive roots and the root lattice. Let

$$\mathsf{C} = \left\{ \mu \in {}^{a}\mathfrak{h}^{*} \;\middle|\; \begin{array}{l} \text{either } \langle \mathrm{Re}(\mu), \check{\alpha} \rangle > 0 \text{ or} \\ \langle \mathrm{Re}(\mu), \check{\alpha} \rangle = 0 \text{ and } \sqrt{-1}\,\langle \mathrm{Im}(\mu), \check{\alpha} \rangle > 0 \end{array} \right\}$$

and $\overline{\mathsf{C}}$ be the closure of C in ${}^{a}\mathfrak{h}$.

We identify the set of infinitesimal characters with ${}^{a}\mathfrak{h}^{*}/W$.

1.1. **Coherent family.** For each finite dimensional $\mathfrak{g}$-module or $G_{\mathbb{C}}$-module F, let F^{*} be its contragredient representation and let $\Delta(F) \subseteq {}^{a}X$ denote the multi-set of weights in F.

Let $\Pi_{\Lambda_0}(G_{\mathbb{C}})$ be the set of irreducible finite dimensional representations of $G_{\mathbb{C}}$ with extermal weight in Λ_0 and $\mathcal{G}_{\Lambda}(G_{\mathbb{C}})$ be the subgroup generated by $\Pi_{\Lambda_0}(G_{\mathbb{C}})$. Let

$${}^{a}P := \{\, \mu \in {}^{a}X \mid \mu \text{ is a } {}^{a}\mathfrak{h}\text{-weight of an } F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}}) \,\}.$$

Via the highest weight theory, every W-orbit $W \cdot \mu$ in ${}^{a}P$ corresponds with the irreducible finite dimensional representation $F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}})$ with extermal weight μ.

Now the Grothendieck group $\mathcal{G}(G_{\mathbb{C}})$ of finite dimensional representation of $G_{\mathbb{C}}$ is identified with $\mathbb{Z}[{}^{a}P/W]$. In fact $\mathcal{G}(G_{\mathbb{C}})$ is a $\mathbb{Z}$-algebra under the tensor product and equiped with the involution $F \mapsto F^{*}$.

Fix a W-invariant sub-lattice $\Lambda_0 \subset {}^{a}X$ containing ${}^{a}Q$.

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Take a lattice $\Lambda = \lambda + \Lambda_0 \in \mathfrak{h}^*/\Lambda$ with $\lambda \in \overline{\mathsf{C}}$. Let

$$R(\lambda) := \{\, \alpha \in {}^aR \mid \langle \lambda, \check\alpha \rangle \in \mathbb{Z} \,\} \quad \text{and} \quad W(\lambda) := \langle s_\alpha | \alpha \in R(\lambda) \rangle .$$

Definition 1.1. *Suppose $\mathcal{M}$ is an abelian group with $\mathcal{G}_\Lambda(G_\mathbb{C})$-action*

$$\mathcal{G}_\Lambda(G_\mathbb{C}) \times \mathcal{M} \ni (F, m) \mapsto F \otimes m.$$

In addition, we fix a subgroup $\mathcal{M}_{[\mu]}$ of $\mathcal{M}$ for each for each W-coset $[\mu] \in \Lambda/W$.

A function $f \colon \Lambda \to \mathcal{M}$ is called a coherent family based on Λ if it satisfies $f(\mu) \in \mathcal{M}_\mu$ and

$$F \otimes f(\mu) = \sum_{\nu \in \Delta(F)} f(\mu + \nu) \qquad \forall \mu \in \Lambda, F \in \Pi_{\Lambda_0}(G_\mathbb{C}).$$

Let $\mathrm{Coh}_\Lambda(\mathcal{M})$ be the abelian group of all coherent families based on Λ and value in $\mathcal{M}$.

In this paper, we will consider the following cases.

Example 1.2. *Suppose $\mathcal{M} = \mathbb{Q}$ and $F \otimes m = \dim(F) \cdot m$ for $F \in \Pi_{\Lambda_0}(G_\mathbb{C})$ and $m \in \mathcal{M}$. We let $\mathcal{M}_\mu = \mathcal{M}$ for every $\mu \in \Lambda$. When $\Lambda = \Lambda_0$, the set of W-harmonic polynomials on ${}^a\mathfrak{h}^*$ is natrually identified with $\mathrm{Coh}_\Lambda(\mathcal{M})$ via restriction (Vogan's result)*

Example 1.3. *Let $\mathcal{G}(\mathfrak{g}, K)$ be the Grothendieck group of finite length $(\mathfrak{g}, K)$-modules and $\mathcal{G}_\mu(\mathfrak{g}, K)$ be the subgroup of $\mathcal{G}(\mathfrak{g}, K)$ generated by the set of irreducible $(\mathfrak{g}, K)$-modules with infinitesimal character μ.*

Then $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$ is the group of coherent families of Harish-Chandra modules.

Example 1.4. *Fixing a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, let $\mathcal{G}(\mathfrak{g}, \mathfrak{b})$ be the Gorthendieck group of the category $\mathscr{O}^\mathfrak{b}$. The space $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, \mathfrak{b}))$ is defined similarly.*

Note that the lattice Λ is stable under the $W(\lambda)$ action. We can define $W(\lambda)$ action on $\mathrm{Coh}_\Lambda(\mathcal{M})$ by

$$w \cdot f(\mu) = f(w^{-1}\mu) \qquad \forall \mu \in \Lambda, w \in W(\lambda).$$

Translation principal assumption. Recall that we have fixed a set of simple roots of W_Λ. We make the following assumption for $\mathrm{Coh}_\Lambda(\mathcal{M})$.

- There is a basis $\{\, \Theta_\gamma \mid \gamma \in \mathrm{Parm} \,\}$ of $\mathrm{Coh}_\Lambda(\mathcal{M})$ where Parm is a parameter set;
- For every $\mu \in \Lambda$, the evalutation at μ is surjective;
$$\mathrm{ev}_\mu \colon \mathrm{Coh}_\Lambda(\mathcal{M}) \to \mathcal{M}_{[\mu]} \qquad f \mapsto f(\mu)$$
is surjective;
- a subset $\tau(\gamma)$ of the simple roots of W_Λ is attached to each $\gamma \in \mathrm{Parm}$ such that $s \cdot \Theta_\gamma = -\Theta_\gamma$;
- $\Theta_\gamma(\mu) = 0$ if and only if $\tau(\gamma) \cap R_\mu \neq \varnothing$.
- $\{\, \Theta_\gamma(\mu) \mid \tau(\gamma) \cap R_\mu = \varnothing \,\}$ form a basis of $\mathcal{M}_\gamma$.

The translation principle assumption implies

$$\mathrm{Ker}\,\mathrm{ev}_\mu = \mathrm{span}\,\{\, \Theta_\gamma \mid \tau(\gamma) \cap R_\mu \neq \varnothing \,\} \tag{1.1}$$

Now we have the following counting lemma.

Lemma 1.5. *For each μ, we have*

$$\dim \mathcal{M}_{[\mu]} = \dim(\mathrm{Coh}_\Lambda(\mathcal{M}))_{W_\mu} = [\mathrm{Coh}_\Lambda(\mathcal{M}), 1_{W_\mu}].$$

Proof. Clearly

$$\mathrm{span}\,\{\, \Theta_\gamma - w\Theta_\gamma \mid \gamma \in \mathrm{Parm}, w \in W_\Lambda \,\} \subseteq \ker \mathrm{ev}_\mu$$

since $w \cdot \Theta_\gamma(\mu) = \Theta_\gamma(w^{-1} \cdot \mu) = \Theta_\gamma(\mu)$ for $w \in W_\mu$. Combine this with (1.1), we conclude that

$$\mathrm{span}\,\{\, \Theta_\gamma - w\Theta_\gamma \mid \gamma \in \mathrm{Parm}, w \in W_\Lambda \,\} = \ker \mathrm{ev}_\mu.$$

Therefore, ev_μ induces an isomorphism $(\mathrm{Coh}_\Lambda(\mathcal{M}))_{W_\mu} \to \mathcal{M}_{[\mu]}$. Now the dimension equality follows. □

Example 1.6. *For the case of category $\mathcal{O}$. We can take* Parm $= W_\Lambda$. *Let* $\Theta_w(\mu) = L(w\mu)$ *for each* $w \in W_\Lambda$ *with* $\tau(w) = \{ s_\alpha \mid \alpha \in \Delta^+, w\alpha \notin R^+ \}$.

Example 1.7. *In the Harish-Chandra module case,* Parm *consists of certain irreducible K-equavariant local systems on a K-orbit of the flag variety of G. $\tau(\gamma)$ is the τ-invariants of the parameter γ.*

2. Parameterize of Unipotent representations

We fix an abstract complex Cartan subgroup $\mathbf{H}_a$ and $\mathfrak{h}_a$ in $\mathbf{G}$ and a set of simple roots Π_a. Let $\mathcal{P}(\mathbf{G})$ be the set of all Langlands parameters of G-modules with character ρ (i.e. the infinitesimal character of the trivial representation). For $\gamma \in \mathcal{P}(\mathbf{G})$, let $\mathcal{L}(\gamma)$, $\mathcal{S}(\gamma)$ and Φ_γ be the corresponding Langlands quotient, standard module and coherent family such that $\Phi_\gamma(\rho) = \mathcal{L}(\gamma)$. Let $\mathcal{M}(\mathbf{G})$ be the span of $\mathcal{L}(\gamma)$. Let $\{ \mathbb{B} \}$ be the set of all blocks. Then $\mathcal{P}(\mathbf{G}) = \bigsqcup_{\mathcal{B}} \mathcal{B}$. The Weyl group $W = W(G)$ acts on $\mathcal{M}(\mathbf{G})$ by coherent continuation. Let $\mathcal{M}_{\mathcal{B}}$ be the submodule of $\mathcal{M}(\mathbf{G})$ spand by $\gamma \in \mathcal{B}$, then

$$\mathcal{M}(\mathbf{G}) = \bigoplus_{\mathcal{B}} \mathcal{M}_{\mathcal{B}}$$

Let $\tau(\gamma) \subset \Pi_a$ be the τ-invariant of γ.

Let $\check{\mathcal{O}}$ be even orbit. $\lambda = \frac{1}{2}\check{h}$. Define

$$S(\lambda) = \{ \alpha \in \Pi_a \mid \langle \alpha, \lambda \rangle = 0 \}.$$

Let $\mathcal{P}_\lambda(\mathbf{G})$ be the set of all Langlands parameters with infinitesimal character λ. Let $T_{\lambda,\rho}$ be the translation functor. Let

$$\mathcal{B}(S) = \{ \gamma \in \mathcal{B} \mid S \cap \tau(\gamma) = \varnothing \}$$

and

$$\mathcal{P}(\mathbf{G}, S) = \bigsqcup_{\mathcal{B}} \mathcal{B}(S)$$

Then

$$\begin{aligned} \mathcal{P}(\mathbf{G}, S) &\longrightarrow \mathcal{P}_\lambda(\mathbf{G}) \\ \gamma &\longmapsto T_{\lambda,\rho}(\gamma) \end{aligned}$$

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Let

$$\mathcal{B}(S, \mathcal{O}) = \{ \gamma \in \mathcal{B}(S) \mid \mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}} \}$$

Let

$$\begin{aligned} m_S(\sigma) &= [\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] \\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{M}_{\mathcal{B}}] \end{aligned}$$

Barbasch [10, Theorem 9.1] established the following theorem.

Theorem 2.1.

$$|\mathcal{B}(S, \mathcal{O})| = \sum_{\sigma} m_{\mathcal{B}}(\sigma) m_S(\sigma)$$

Here $\sigma \times \sigma$ running over the $W \times W$ appears in the double cell $\mathcal{C}(\mathcal{O})$.

Proof. We need to take the graded module of $\mathcal{M}(\mathbf{G})$ with respect to the $\overset{LR}{\leqslant}$. By abuse of notation, we identify the basis $\mathcal{P}(\mathbf{G})$ with its image in the graded module. Note that $S \cap \tau(\lambda) = \varnothing$ if and only if $W(S)$ acts on γ trivially by [70, Lemma 14.7]. On the other hand, by [70, Theorem 14.10, and page 58], $\mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}}$ only if γ generate a W-module in the double cell of $\mathcal{O}$. □

Now assume $S = S(\lambda)$. By [12, Cor 5.30 b) and c)], $[\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] = [\mathbf{1}|_{W(S)} : \sigma] \leqslant 1$.

3. Primitive ideals

In this section, we recall some results about the primitive ideals and cells developed by Barbasch Vogan, Joseph and Lusztig etc.

3.1. **Heighest weight module.** Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a fixed Borel. Let

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

and $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$.

In the rest of this section, we fix a lattice $\Lambda \in \mathfrak{h}^*/X^*$. Let R_Λ and R_Λ^+ be the set of integral root system and the set of positive integral roots. Write W_Λ for the integral Weyl group.

For $\mu \in \Lambda$,

$$\begin{array}{ccccccc} \mu \geq 0 & \Leftrightarrow & \mu \text{ is dominant} & \Leftrightarrow & \langle \lambda, \check{\alpha} \rangle \geqslant 0 & \forall \alpha \in R_\Lambda^+ \\ \mu \not> 0 & \Leftrightarrow & \mu \text{ is regular} & \Leftrightarrow & \langle \lambda, \check{\alpha} \rangle \neq 0 & \forall \alpha \in R_\Lambda^+ \end{array}$$

regular if $\langle \lambda, \check{\alpha} \rangle \neq 0$.

For each $w \in W_\Lambda$, it give a coherent family such that

$$M_w(\mu) = M(w\mu) \quad \forall \mu \in \Lambda \text{ dominant}$$

[Note that the Gorthendieck group $\mathcal{K}(\mathcal{O})$ of category $\mathcal{O}$ is naturally embedded in the space of formule character. M_w is a coherent family: the formal character $\mathrm{ch} M_w(\mu)$ of $M_w(\mu)$ is $e^{w\mu}/\prod_{\alpha \in R^+}(1 - e^{-\alpha})$. It is clear that $\mathrm{ch} M_w$ satisfies the condition for coherent continuation.]

For each $w \in W_\Lambda$, the function

$$\widetilde{p}_w(\mu) := \mathrm{rank}(\mathcal{U}(\mathfrak{g})/\mathrm{Ann}(L(w\mu))) \qquad \forall \mu \in \Lambda \text{ dominant}.$$

extends to a Harmonic polynomial on $\mathfrak{h}^*$. [Let Λ^+ be the set of dominant weights in Λ. Assume $\lambda > 0$, i.e. dominant regular. Then $\mathbb{R}_\lambda^0 =$, $w^{w\lambda} = w^{-1}$. The set

$$\hat{F}_{w\lambda} = \{ \mu \in \Lambda \mid w^{-1}\mu \geq 0 \} = w\Lambda^+.$$

Joseph's $p_w(\mu) = \mathrm{rank}\,\mathcal{U}(\mathfrak{g})/(\mathrm{Ann}\, L(w\mu))$ is defined on $\hat{F}_{w\lambda} = w\Lambda^+$. Now his $\widetilde{p}_w(\mu) = w^{-1}p_w$ is defined on Λ^+ and given by $\widetilde{p}_w(\mu) = \mathrm{rank}\,\mathcal{U}(\mathfrak{g})/(\mathrm{Ann}\, L(w\mu))$.]

Let $\mathrm{Coh}_{[\Lambda]}$

4. Counting in type A

The results in this section are well known to the experts.

Let YD be the set of Young diagrams viewed as a finite multiset of positive integers. The set of nilpotent orbits in $\mathrm{GL}_n(\mathbb{C})$ is identified with Young diagram of n boxs.

Let $\check{G}_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$. Fix an orbit $\check{\mathcal{O}} \in \mathrm{Nil}(ckG)$, let $\check{\mathcal{O}}_e$ (resp. $\check{\mathcal{O}}_o$) be the partition conssists of all even (resp. odd) rows in $\check{\mathcal{O}}$.

Let S_n deonte the Weyl group of $\mathrm{GL}_n(\mathbb{C})$. Let $W_n := S_n \ltimes \{ \pm 1 \}^n$ denote the Weyl group of type B_n or C_n. Let sgn denote the sign representation of the Weyl group. The group

W_n is naturally embeded in S_{2n}. For W_n, let ϵ denote the unique non-trivial character which is trivial on S_n. Note that ϵ is also the restriction of the sgn of S_{2n} on W_n.

4.1. **Special unipotent repesentations of $G = \mathrm{GL}_n(\mathbb{C})$.** By [12], the set of unipotent representations of $G = \mathrm{GL}_n(\mathbb{C})$ one-one corresponds to nilpotent orbits in $\mathrm{Nil}(\check{G}_\mathbb{C})$. Suppose $\check{\mathcal{O}}$ has rows

$$\mathbf{r}_1(\check{\mathcal{O}}) \geqslant \mathbf{r}_2(\check{\mathcal{O}}) \geqslant \cdots \geqslant \mathbf{r}_k(\check{\mathcal{O}}) > 0.$$

Then $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}})$ has columns $\mathbf{c}_i(\mathcal{O}) = \mathbf{r}_i(\check{\mathcal{O}})$ for all $i \in \mathbb{N}^+$. The map $\check{\mathcal{O}} \mapsto \mathcal{O}$ is a bijection.

We set $\mathsf{CP} = \mathsf{YD}$ be the set of Young diagrams. For $\tau \in \mathsf{CP}$ which has k columns, let 1_c be the trivial representations of $\mathrm{GL}_c(\mathbb{C})$.

$$\pi_\tau = 1_{\mathbf{c}_1(\tau)} \times 1_{\mathbf{c}_2(\tau)} \times \cdots \times 1_{\mathbf{c}_k(\tau)}.$$

The Vogan duality gives a duality between Harish-Chandra cells. In this case, Harish-Chandra cells is the double cell of Lusztig. Now we have a duality

$$\pi_\tau \leftrightarrow \pi_{\tau^t}.$$

Let $\tau' := \nabla(\tau)$ be the partition obtained by deleting the first column of τ. Let $\theta_{a,b}$ (resp. $\Theta_{a,b}$) be the theta lift (resp. big theta lift) from $\mathrm{GL}_a(\mathbb{C})$ to $\mathrm{GL}_b(\mathbb{C})$. Then we have

$$\pi_\tau = \theta_{|\tau'|,|\tau|}(\pi_\tau).$$

4.2. **Counting unipotent repesentations of $\mathrm{GL}_n(\mathbb{R})$.** Now let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G}_\mathbb{C})$. Recall the decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_e \cup \check{\mathcal{O}}_o$. Let $n_e = \left|\check{\mathcal{O}}_e\right|, n_o = \left|\check{\mathcal{O}}_o\right|$ and $\lambda_{\check{\mathcal{O}}} = \frac{1}{2}\check{h}$.

Then

$$W(\lambda_{\check{\mathcal{O}}}) \cong S_{\left|\check{\mathcal{O}}_e\right|} \times S_{\left|\check{\mathcal{O}}_o\right|}.$$

$$W_{\lambda_{\check{\mathcal{O}}}} = \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_e)} \times \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_o)}$$

By the formula of a-function, one can easily see that The cell in $W(\lambda_{\check{\mathcal{O}}})$ consists of the unique representatoin $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}}(1)$. Now the W-cell is $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W} 1$ corresponds to the orbit $\mathcal{O} = \check{\mathcal{O}}^t$ under the Springer correspondence. [WLOG, we assume $\check{\mathcal{O}} = \check{\mathcal{O}}_o$.

Let $\sigma \in \widehat{S_n}$. We identify σ with a Young diagram. Let $c_i = \mathbf{c}_i(\sigma)$. Then $\sigma = J_{W'}^{S_n}\epsilon_{W'}$ where $W' = \prod S_{c_i}$ (see Carter's book). This impiles Lusztig's a-function takes value

$$a(\sigma) = \sum_i c_i(c_i - 1)/2$$

Compairing the above with the dimension formula of nilpotent Orbits [17, Collary 6.1.4], we get (for the formula, see Bai ZQ-Xie Xun's paper on GK dimension of $SU(p,q)$)

$$\tfrac{1}{2}\dim(\sigma) = \dim(L(\lambda)) = n(n-1)/2 - a(\sigma).$$

Here $\dim(\sigma)$ is the dimension of nilpotent orbit attached to the Young diagram of σ (it is the Springer correspondence, regular orbit maps to trivial representation, note that $a(\mathrm{triv}) = 0$), $L(\lambda)$ is any highest weight module in the cell of σ.

Return to our question, let $S' = \prod_i S_{\mathbf{c}_i(\check{\mathcal{O}})}$. We want to find the component σ_0 in $\mathrm{Ind}_{S'}^{S_n} 1$ whose $a(\sigma_0)$ is maximal, i.e. the Young diagram of σ_0 is minimal.

By the branaching rule, $\sigma \subset \mathrm{Ind}_{S'}^{S_n} 1$ is given by adding rows of lenght $\mathbf{c}_i(\check{\mathcal{O}})$ repeatly (Each time add at most one box in each column). Now it is clear that $\sigma_0 = \check{\mathcal{O}}^t$ is desired.

This agrees with the Barbasch-Vogan duality d_{BV} given by

$$\check{\mathcal{O}} \xrightarrow{Springer} \check{\mathcal{O}} \xrightarrow{\otimes \mathrm{sgn}} \check{\mathcal{O}}^t \xrightarrow{Springer} \check{\mathcal{O}}^t.$$

]

The $W_{[\lambda_{\check{\mathcal{O}}}]}$-module $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}$ is given by the following formula:

$$\begin{aligned}\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]} &\cong \mathcal{C}_{n_e} \otimes \mathcal{C}_{n_o} \quad \text{with} \\ \mathcal{C}_n &:= \bigoplus_{\substack{s,a,b \\ 2s+a+b=n}} \mathrm{Ind}_{W_s \times S_a \times S_b}^{S_n} \epsilon \otimes 1 \otimes 1.\end{aligned}$$

According to Vogan duality, we can obtain the above formula by tensoring sgn on the forumla of the unitary groups in [8, Section 4].

By branching rules of the symmetric groups, $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ can be parameterized by painted partition.

$$\mathrm{PP}_{\check{\mathcal{O}}} = \left\{\, \tau\colon \mathrm{Box}(\check{\mathcal{O}}^t) \to \{\, \bullet, c, d \,\} \;\middle|\; \begin{array}{l}\text{“}\bullet\text{” occures with even} \\ \text{mulitplicity in each column}\end{array} \right\}.$$

[The typical diagram of all columns with even length $2c$ are

$$\begin{array}{|c|c|c|c|c|c|}\hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bullet & \cdots & \bullet & c & \cdots & c \\ \hline \bullet & \cdots & \bullet & d & \cdots & d \\ \hline \end{array}$$

The typical diagram of all columns with odd length $2c+1$ are

$$\begin{array}{|c|c|c|c|c|c|}\hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline c & \cdots & c & d & \cdots & d \\ \hline \end{array}$$

]

Let $\mathrm{sgn}_n\colon \mathrm{GL}_n(\mathbb{R}) \to \{\, \pm 1 \,\}$ be the sign of determinant. Let 1_n be the trivial representation of $\mathrm{GL}_n(\mathbb{R})$. For $\tau \in \mathrm{PP}_{\check{\mathcal{O}}}$, we attache the representation

$$\pi_\tau := \bigtimes_j \underbrace{1_j \times \cdots \times 1_j}_{c_j\text{-terms}} \times \underbrace{\mathrm{sgn}_j \times \cdots \times \mathrm{sgn}_j}_{d_j\text{-terms}}. \tag{4.1}$$

Here

- j running over all column lengths in $\check{\mathcal{O}}^t$,
- d_j is the number of columns of length j ending with the symbol “d”,
- c_j is the number of columns of length j ending with the symbol “$\bullet$” or “c”, and
- “$\times$” denote the parabolic induction.

4.3. **Special Unipotent representations of $G = \mathrm{GL}_m(\mathbb{H})$.** Suppose that $\mathcal{O}$ is the complexificiation of a rational nilpotent $\mathrm{GL}_m(\mathbb{H})$-orbit. Then $\mathcal{O}$ has only even length columns. Therefore, $\mathrm{Unip}_{\check{\mathcal{O}}}(G) \neq \emptyset$ only if $\check{\mathcal{O}} = \check{\mathcal{O}}_e$.

In this case the coherent continuation representation is given by

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) = \mathrm{Ind}_{W_m}^{S_{2m}} \epsilon$$

and $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is a singleton. For each partition τ only having even columns, we define

$$\pi_\tau := \bigtimes_i 1_{\mathbf{c}_i(\tau)/2}.$$

4.4. **Counting special unipotent repesentations of** $\mathrm{U}(p,q)$**.** We call the parity of $|\check{\mathcal{O}}|$ the "good pairity". The other pairity is called the "bad parity". We write $\check{\mathcal{O}} = \check{\mathcal{O}}_g \cup \check{\mathcal{O}}_b$ where $\check{\mathcal{O}}_g$ and $\check{\mathcal{O}}_b$ consist of good pairity length rows and bad pairity rows respectively.

Let $(n_g, n_b) = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$. Now as the $S_{n_g} \times S_{n_b}$

$$\bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=n}} \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathrm{U}(p,q)) = \mathcal{C}_{gd} \otimes \mathcal{C}_{bd}$$

where

$$\begin{aligned}
\mathcal{C}_{gd} &= \bigoplus_{\substack{s,a,b\in\mathbb{N}\\ 2s+a+b=n_g}} \mathrm{Ind}_{W_s\times S_a\times S_b}^{S_{n_g}} 1\otimes \mathrm{sgn}\otimes \mathrm{sgn}\\
\mathcal{C}_{bd} &= \begin{cases} \mathrm{Ind}_{W_{\frac{n_b}{2}}}^{S_{n_b}} 1 & \text{if } n_b \text{ is even}\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

By the above formula, we have

Lemma 4.1. *(i) The set* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,q)) \neq \emptyset$ *if and only if* $p=q$ *and each row lenght in* $\check{\mathcal{O}}$ *has even multiplicity.*

(ii) Suppose $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$*, let* $\check{\mathcal{O}}'$ *be the Young diagram such that* $\mathbf{r}_i(\check{\mathcal{O}}') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ *and* π' *be the unique special uinpotent representation in* $\mathrm{Unip}_{\check{\mathcal{O}}'}(\mathrm{GL}_p(\mathbb{C}))$*. Then the unique element in* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p))$ *is given by*

$$\pi := \mathrm{Ind}_P^{\mathrm{U}(p,p)} \pi'$$

where P *is a parabolic subgroup in* $\mathrm{U}(p,p)$ *with Levi factor equals to* $\mathrm{GL}_p(\mathbb{C})$*.*

(iii) In general, when $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$*, we have a natural bijection*

$$\begin{aligned}
\mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) &\longrightarrow \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}(n_1+p,n_2+p))\\
\pi_0 &\longmapsto \mathrm{Ind}_P^{\mathrm{U}(n_1+p,n_2+p)} \pi'\otimes\pi_0
\end{aligned}$$

where P *is a parabolic subgroup with Levi factor* $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{U}(n_1,n_2)$*.*

The above lemma ensure us to reduce the problem to the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Now assume $\check{\mathcal{O}} = \check{\mathcal{O}}_g$ and so $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ corresponds to the blocks of the infinitesimal character of the trivial representation.

By [8, Theorem 4.2], Harish-Chandra cells in $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ are in one-one correspondence to real nilpotent orbits in $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t$.

[From the branching rule, the cell is parametered by painted partition

$$\mathrm{PP}(\mathrm{U}) := \left\{\, \tau\in\mathrm{PP} \;\middle|\; \begin{array}{l}\mathrm{Im}(\tau)\subseteq\{\bullet, s, r\}\\ \text{“}\bullet\text{” occures even times in each row}\end{array} \right\}.$$

The bijection $\mathrm{PP}(\mathrm{U}) \to \mathsf{SYD}, \tau\mapsto\mathscr{O}$ is given by the following recipe: The shape of $\mathscr{O}$ is the same as that of τ. $\mathscr{O}$ is the unique (upto row switching) signed Young diagram such that

$$\mathscr{O}(i,\mathbf{r}_i(\tau)) := \begin{cases} +, & \text{when } \tau(i,\mathbf{r}_i(\tau)) = r;\\ -, & \text{otherwise, i.e. } \tau(i,\mathbf{r}_i(\tau))\in\{\bullet, s\}.\end{cases}$$

Example 4.2.

$$\begin{array}{|c|c|c|c|c|}\hline \bullet & \bullet & \bullet & \bullet & r \\ \hline \bullet & \bullet \\ \cline{1-2} s & r \\ \cline{1-2} s \\ \cline{1-1} r \\ \cline{1-1} \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|}\hline + & - & + & - & + \\ \hline + & - \\ \cline{1-2} - & + \\ \cline{1-2} - \\ \cline{1-1} + \\ \cline{1-1} \end{array}$$

]

Now the following lemma is clear.

Lemma 4.3. *When $\check{\mathcal{O}} = \check{\mathcal{O}}_g$, the associated varity of every special unipotent representations in* $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U})$ *is irreducible. Moreover, the following map is a bijection.*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1, n_2)) & \longrightarrow & \{\, \textit{rational forms of } \check{\mathcal{O}}^t \,\} \\ \pi_0 & \mapsto & \mathrm{AV}^{\mathrm{weak}}(\pi_0). \end{array}$$

□

Remark. Note that the parabolic induction of an rational nilpotent orbit can be reducible. Therefore, when $\check{\mathcal{O}}_b \neq \varnothing$, the special unipotent representations can have reducible associated variety. Meanwhile, it is easy to see that the map $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) \ni \pi \mapsto \mathrm{AV}^{\mathrm{weak}}(\pi)$ is still injective.

We will show that every elements in $\mathrm{Unip}_{\check{\mathcal{O}}_g}$ can be constructed by iterated theta lifting. For each τ, let $\mathscr{O}$ be the corresponding real nilpotent orbit. Let $\mathrm{Sign}(\mathscr{O})$ be the signature of $\mathscr{O}$, $\nabla(\mathscr{O})$ be the signed Young diagram obtained by deleteing the first column of $\mathscr{O}$. Suppose $\mathscr{O}$ has k-columns. Inductively we have a sequence of unitary groups $\mathrm{U}(p_i, q_i)$ with $(p_i, q_i) = \mathrm{Sign}(\nabla^i(\mathscr{O}))$ for $i = 0, \cdots, k$. Then

$$\pi_\tau = \theta_{\mathrm{U}(p_1,q_1)}^{\mathrm{U}(p_0,q_0)} \theta_{\mathrm{U}(p_2,q_2)}^{\mathrm{U}(p_1,q_1)} \cdots \theta_{\mathrm{U}(p_k,q_k)}^{\mathrm{U}(p_{k-1},q_{k-1})}(1) \tag{4.2}$$

where 1 is the trivial representation of $\mathrm{U}(p_k, q_k)$.

Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Form the duality between cells of $\mathrm{U}(p,q)$ and $\mathrm{GL}(n, \mathbb{R})$. We have an ad-hoc (bijective) duality between unipotent representations:

$$\begin{array}{rcl} d_{\mathrm{BV}} \colon \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) & \to & \mathrm{Unip}_{\check{\mathcal{O}}^t}(\mathrm{GL}(\mathbb{R})) \\ \pi_\tau & \mapsto & \pi_{d_{\mathrm{BV}}(\tau)} \end{array}$$

Here $\check{\mathcal{O}}^t = d_{\mathrm{BV}}(\check{\mathcal{O}})$ and $d_{\mathrm{BV}}(\tau)$ is the pained bipartition obtained by transposeing τ and replace s and r by c and d respectively. See (4.2) and (4.1) for the definition of special unipotent representations on the two sides.

5. Counting in type BCD

In this section, we count special unipotent representations of real orthogonal groups (type B,D), symplectic groups (type C) and metaplectic groups (type $\widetilde{C}$).

Recall that

$$\text{good pairity} = \begin{cases} \text{odd} & \text{in type } C \text{ and } D \\ \text{even} & \text{in type } C \text{ and } D \end{cases}$$

We decompose $\check{\mathcal{O}} = \check{\mathcal{O}}_g \cup \check{\mathcal{O}}_b$ as before.

Let $\widetilde{\mathrm{sgn}}$ be the character of W_n inflated from the sign character of S_n via the natural map $W_n \to S_n$. Recall the following formula

$$\mathrm{Ind}_{W_b \ltimes \{\pm 1\}^b}^{W_{2b}} 1 \otimes \mathrm{sgn} = \sum_{\sigma} (\sigma, \sigma)$$

where σ running over all Young diagrams of size b.

5.1. **Counting special unipotent representation of $G = \mathrm{Sp}(p,q)$.** . The dual group of $G_{\mathbb{C}} = \mathrm{Sp}(2n,\mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{SO}(2n+1,\mathbb{C})$. We set $(2m+1, 2m') = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_n|)$. By Vogan duality, the dual group for class $[\lambda_{\check{\mathcal{O}}}]$ is

$$\mathrm{SO}(2m+1,\mathbb{C}) \times \mathrm{O}(2m',\mathbb{C}).$$

Now we have

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) \otimes \mathrm{sgn} = \mathcal{C}_g \otimes \mathcal{C}_b$$

with

$$\begin{aligned}
\mathcal{C}_g &= \bigoplus_{p+q=m} \mathrm{Coh}_{[\rho]}(\mathrm{Sp}(p,q)) \\
&= \bigoplus_{\substack{b,s,r\in\mathbb{N}\\ 2b+s+r=m}} \mathrm{Ind}_{W_b \ltimes \{\pm 1\}^b \times W_s \times W_b}^{W_m} 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn} \otimes \mathrm{sgn} \\
&= \bigoplus_{\substack{b,s,r\in\mathbb{N}\\ 2b+s+r=m}} \mathrm{Ind}_{W_{2b} \times W_s \times W_b}^{W_m} (\sigma,\sigma) \otimes \mathrm{sgn} \otimes \mathrm{sgn} \\
\mathcal{C}_b &= \mathrm{Ind}_{W_{\frac{m'}{2}} \ltimes \{\pm 1\}^{\frac{m'}{2}}}^{W_{m'}} \mathrm{sgn}
\end{aligned}$$

In $\mathrm{SO}(2m',\mathbb{C})$, the dobule cell corresponds to $\check{\mathcal{O}}_b$ consists of a single representation

$$\check{\tau}_b = (\check{\sigma}_b, \check{\sigma}_b) \text{ such that } \mathbf{r}_i(\check{\sigma}_b) = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)/2$$

Let $\check{\mathcal{O}}_b'$ be the Young diagram such that $\mathbf{r}_i(\check{\mathcal{O}}_b') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ and π_b' be the unique special unipotent representation attached to $\mathrm{GL}_p(\mathbb{H})$

Lemma 5.1. *The set* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{Sp}(p,q)) \neq \emptyset$ *only if* $p = q = |\check{\sigma}_b|$ *and in this case it consists a single element*

$$\pi_b := \mathrm{Ind}_P^{\mathrm{Sp}(p,p)} \pi'_{\check{\sigma}_b}$$

where P has the Levi subgroup $\mathrm{GL}_p(\mathbb{H})$.

The special reprsentation corresponds to $\check{\mathcal{O}}_b$ is

$$aa$$

Let

$$\mathsf{PBP}_{\check{\mathcal{O}}_g}(C^*) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath,\jmath) = \check{\tau}_g \\ \mathrm{Im}\mathcal{P} \subset \{\bullet\}, \mathrm{Im}\mathcal{Q} \subset \{\bullet, s, r\} \end{array} \right\}.$$

Lemma 5.2. *The set of* $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q))$ *is paramterized by the set such that*

$$ss$$

[Let $W = W(G_{\mathbb{C}})$ where $G_{\mathbb{C}}$ is natrualy embeded in $\mathrm{GL}(n,\mathbb{C})$. Let $s_{\varepsilon_i i, \varepsilon_j j}$ be the permutation matrix of index i,j, where the (i,j)-th entry is ε_i, the (j,i)-th entry is ε_j and the other place is the identity matrix. Let $w^{\epsilon}_{i,\pm j}$ be the element such that

- it is in $G_{\mathbb{C}}$ and entries in $\{0\} \cup \mu_4$
- it lifts the element $e_i \leftrightarrow \pm e_j$ in the Weyl group.
- $(w^{\epsilon}_{i,\pm j})^2 = \epsilon 1 \in \{\pm 1\}$.

Let

$$h^{+}_{\pm i} = \mathrm{diag}(1, \cdots, 1, \pm 1, 1, \cdots, 1)$$

where ± 1 is the i-th place. Let

$$h^{-}_{\pm i} = \mathrm{diag}(1, \cdots, 1, \pm\sqrt{-1}, 1, \cdots, 1)$$

where $\pm\sqrt{-1}$ is the i-th place. Let $e_{\pm i} = s_{\pm i, \pm(n-i+1)}$.

Let $\check{w}^{\pm}_{i,\pm j}$ be the lift of $e_i \leftrightarrow \pm e_j$ such that

- it is in $G_{\mathbb{C}}$ and entries in $\{ 0 \} \cup \mu_4$
- it lifts the element $e_i \leftrightarrow \pm e_j$ in the Weyl group.
- $(w^{\epsilon}_{i,\pm j})^2|_{[i,j]} = \epsilon 1 \in \{ \pm 1 \}$ here "$|_{[i,j]}$" means restricts on the $e_i, e_j, -e_i, -e_j$-weights space.

Let

$$\begin{aligned}
x_{b,s,r} &= w^+_{1,2} \cdots w^+_{2b-1,2b}\, h^+_{2b+1} \cdots h^+_{2b+s} \\
y^+_{b,s,r} &= \breve{w}^+_{1,-2} \cdots \breve{w}^+_{2b-1,-2b}\, e_{2b+1} \cdots e_{2b+s+r} \\
y^-_{b,s,r} &= \breve{w}^-_{1,-2} \cdots \breve{w}^-_{2b-1,-2b}
\end{aligned}$$

We compute the parameter space

$$\begin{aligned}
\mathcal{Z}_g &= \bigcup_{2b+s+r=m} W_m \cdot (x^+_{b,s,r}, y^+_{b,s,r}) \\
\mathcal{Z}_b &= \bigcup_{2b=m} W_m \cdot (x^+_{b,0,0}, y^-_{b,0,0})
\end{aligned}$$

]

5.2. **Counting special unipotent representation of $G = \mathrm{Sp}(2n, \mathbb{R})$.** Now we have

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) \otimes \mathrm{sgn} = \mathcal{C}_g \otimes \mathcal{C}_b$$

with

$$\begin{aligned}
\mathcal{C}_g &= \bigoplus_{p+q=2m+1} \mathrm{Coh}_{[\rho]}(\mathrm{SO}(p,q)) \\
&= \bigoplus_{2b+s+r+c+d=m} \mathrm{Ind}^{W_m}_{W_b \ltimes \{ -1 \}^b \times W_s \times W_r \times W_c \times W_d} 1 \otimes \det \otimes \det \otimes 1 \otimes 1 \\
\mathcal{C}_b &= \mathrm{Coh}_{[\rho]}(\mathrm{SO}^*(2m)) = \bigoplus_{2s+b=m} \mathrm{Ind}^{W_m}_{W_s \ltimes \{ -1 \}^s \times S_b} 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn}
\end{aligned}$$

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