COMBINATORICS FOR UNIPOTENT REPRESENTATIONS

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Contents

1. Statement of the main result

1.1. **Nilpotent Orbits.** Let $\mathbf{G} = \mathrm{Sp}, \mathrm{O}$, and $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ is a complex nilpotent orbit. "Type M" means that $\widetilde{G}$ is a metaplectic group. its dual group $\check{\mathbf{G}}$ is a symplectic group with the same rank, and its duality means the metaplectic dual. Let $\mathrm{DRC}(\mathcal{O})$ be the set of dot-r-c diagrams parameterizing unipotent representations of the real forms of $\mathbf{G}$.

We say an orbit $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathbf{G}})$ is purely even (good parity in Moeglin-Renard's notation) if

all row lengths of $\check{\mathcal{O}}$ are even when $\check{\mathbf{G}}$ is a symplectic group.
all row lengths of $\check{\mathcal{O}}$ are odd when $\check{\mathbf{G}}$ is an orthogonal group.

Definition 1.1. *Now we describe the nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ who is the Barbasch-Vogan dual of a purely even orbit $\check{\mathcal{O}}$. If G is*

Type C *Then $\mathcal{O}$ is special and it has columns*

$$(C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is even for } i = 0, \cdots, k-1.$$

Note that, by the definition of special orbit, we have $C_{2i} \equiv C_{2i-1} \pmod 2$, $C_{2i} = C_{2i-1}$ if C_{2i} is odd for $i = 1, \cdots k$.

Type M *Then $\mathcal{O}$ is metaplecitic special and it has columns*

$$(C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is odd for } i = 0, \cdots, k-1.$$

Note that, by the definition of metaplectic special, we have $C_{2i} \equiv C_{2i-1} \pmod 2$, $C_{2i} = C_{2i-1}$ if C_{2i} is even for $i = 1, \cdots k$.

Type D *and $\mathcal{O}$ is special and it has columns*

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is even for } i = 0, \cdots, k.$$

Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is purely even of type C and C_{2k+1} is always even.

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Type B *and $\mathcal{O}$ is special and it has columns*

$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0)$ *such that $C_{2i+1} > C_{2i}$ if C_{2i} is odd for $i = 0, \cdots, k$.*

Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is purely even of type M and C_{2k+1} is always odd.

Let $\mathrm{Nil}^{\mathrm{dpe}}(\star)$ the set of duals of purely even orbits for type $\star =$ B,C,D,M respectively.

1.2. **Descent of orbits in the purely even case.** For $\{\star, \star'\} = \{C, D\}, \{B, M\}$, we define the notion of descent of orbits. The descent map is defined in the following way (under the above notation):

$$\overline{\nabla}: \qquad \mathrm{Nil}^{\mathrm{dpe}}(\star) \longrightarrow \mathrm{Nil}^{\mathrm{dpe}}(\star')$$

$$\mathcal{O} \longmapsto \mathcal{O}'$$

such that

- when $\star$ is B or D, $\mathcal{O}' := \nabla(\mathcal{O})$;
- when $\star$ is C or M,

$$\mathcal{O}' := \begin{cases} \nabla(\mathcal{O}) & \text{if } C_{2k} \text{ is even and } \star = B \\ & \text{or } C_{2k} \text{ is odd and } \star = M; \\ (C_{2k-1}+1, C_{2k-2}, C_{2k-3}, \cdots, C_0) & \text{otherwise.} \end{cases}$$

In the second case, $C_{2k} = C_{2k-1}$ and $\mathcal{O} \mapsto \overline{\nabla}(\mathcal{O})$ is a good generalized descent.

The following set of orbit are essential to us:

Definition 1.2. *A nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ is called* noticed *if G is*

Type D *and $\mathcal{O}$ is special and it has columns*

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k.$$

Type B *and $\mathcal{O}$ is special and it has columns*

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k.$$

Type C *and $\overline{\nabla}(\mathcal{O})$ is noticed of type D.*

Type M *and $\overline{\nabla}(\mathcal{O})$ is notice of type B.*

Furthermore, We say a noticed orbit $\mathcal{O}$ of type D or B is +noticed *if $C_{2k+1} - C_{2k} \geqslant 2$.*

The following inductive definition of noticed orbit is easy to verify.

Lemma 1.3. *An orbit $\mathcal{O}$ of type D (resp. type B) is noticed if and only if*

i) $\overline{\nabla}^2(\mathcal{O})$ is noticed, when C_{2k} is even (resp. odd), or

ii) $\overline{\nabla}^2(\mathcal{O})$ is +noticed, when C_{2k} is odd (resp. even).

1.3. **Relavent Weyl group representations and dot-r-c diagrams.** Retain the notation in Definition 1.2. Now we describe all relevent representations of the Weyl group appear in the counting algorithm.

[We use "s" and "d" to denote "r'" and "c'" in Dan's notes respectively.]

Type D We define c_i as the following

$$c_{2i} := \lfloor \frac{C_{2i}}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1}+1}{2} \rfloor \quad \forall i = 0, 1, \cdots, k,$$

$$c_{2k+1} := \frac{C_{2k+1}}{2}$$

The (special) Weyl group representation associated to the special orbit $\mathcal{O}$ has columns

$$\tau = \tau_L \times \tau_R = (c_{2k+1}, c_{2k-1}, \ldots, c_1) \times (c_{2k}, \ldots, c_0).$$

The other representations is obtained from the special one by interchanging following pairs when $c_{2i-1} \neq c_{2i} + 1$ and $i = 1, \cdots, k$ (these are pairs coming from even length columns)

$$(c_{2i-1}, c_{2i}) \longleftrightarrow (c_{2i} + 1, c_{2i-1} - 1).$$

[Note that we always have $c_{2i-1} - 1 \geqslant c_{2i-2}$ by our assumptions of the shape of the orbit.]

The dot-r-c diagrams of type D: For each representation $\tau = \tau_L \times \tau_R$, dots are filled in both sides forming the same shape of Young diagram, a column of "s" is added next to dots on the left side diagram, a column of "r" is added next to dots on the left side diagram, add a row of "c"s and finally add a row of "d"s on the left side of the diagram; Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step. [At most one r/s is added in each row of the existing dots diagram and at most one c/d is added to each column.]

In summary, $s/r/c/d$ are all added on the left diagram. Fixing the representation τ, the left diagram $\boldsymbol{\tau}_L$ determin $\boldsymbol{\tau}_R$ completely.

Type C Set $C_{2k+1} = 0$. The Weyl group representations are obtained by the same formula of Type D.

The dot-r-c diagrams of type C are obtained in almost the same way as that of type D, except that s are added on the right side of the diagram.

Type B We define c_i as the following

$$c_{2i} := \lfloor \frac{C_{2i} + 1}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1}}{2} \rfloor \quad \forall i = 1, 2, \cdots, k,$$
$$c_{2k+1} := \frac{C_{2k+1} - 1}{2}$$

Note that C_{2k+1} is always odd and $C_0 = 0$. The (special) Weyl group representation associated to the special orbit $\mathcal{O}$ has columns

$$\tau = \tau_L \times \tau_R = (c_{2k}, \ldots, c_2) \times (c_{2k+1}, c_{2k-1}, \ldots, c_1).$$

The other representations is obtained from the special one by interchanging following pairs when $c_{2i-1} \neq c_{2i} + 1$ and $i = 1, \cdots, k$ (these are pairs coming from odd length columns)

$$(c_{2i}, c_{2i-1}) \longleftrightarrow (c_{2i-1}, c_{2i}).$$

[Note that we always have $c_{2i-1} \geqslant c_{2i-2}$ by our assumptions of the shape of the orbit.]

The dot-r-c diagrams of type B: For each representation $\tau = \tau_L \times \tau_R$, dots are filled in both sides forming the same shape of Young diagram, a column of "s" is added next to dots on the right side of the diagram, a column of "r" is added next to dots on the right side diagram, add a row of "d"s on the right side of the diagram, and finally add a row of "c"s on the left side of the diagram. Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step. [At most one r/s is added in each row of the existing dots diagram and at most one c/d is added to each column.]

In summary, $s/r/d$ are added on the right and c is added on the left.

Each dot-r-c diagram of type B is extended into two diagrams by adding an extra label "a" or "b". This extension helps us to determine the signature of special orthogonal groups.

Fixing the representation τ, the right diagram $\boldsymbol{\tau}_R$ determin $\boldsymbol{\tau}_L$ completely.

Type M Set $C_{2k+1} = 0$. The Weyl group representations are obtained by the same formula of Type B.

The dot-r-c diagrams of type M are obtained in almost the same way as the type B except that s are added on the left side of the diagram and we do not add labels a/b.

We let $\mathrm{DRC}(\mathcal{O})$ denote the set of dot-r-c diagrams defined above. Note that for type B, there is an additional mark a/b attached to the diagram.

In addition, for $\star = B, C, D, M$, let

$$\mathrm{DRC}(\star) := \bigsqcup_{\mathcal{O}\in \mathrm{Nil}^{\mathrm{dpe}}(\star)} \mathrm{DRC}(\mathcal{O}).$$

1.4. **Main theorem.** Let $\mathrm{LS}(\mathcal{O})$ denote the Grothendieck group of $\widetilde{\mathbf{K}}$-equivariant local systems on $\mathcal{O}$ for various real forms of $\mathbf{G}$. The following is our main theorem on the counting of unipotent representations.

Let $\check{\mathcal{O}}$ be a purely even orbit and $\mathcal{O}$ be its dual orbit.

Theorem 1.4. *We can define an injective map using convergence range theta lifting:*

$$\begin{aligned} \pi: \quad \mathrm{DRC}(\mathcal{O}) &\longrightarrow \mathrm{Unip}(\mathcal{O}) \\ \boldsymbol{\tau} &\longmapsto \pi_{\boldsymbol{\tau}} \end{aligned}$$

i) When $\mathcal{O}$ is of type C or type M, π is a bijection.
ii) When $\mathcal{O}$ is of type D or type B, the following map is a bijection

$$\begin{aligned} \mathrm{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow \mathrm{Unip}(\mathcal{O}) \\ (\boldsymbol{\tau}, \varepsilon) &\longmapsto \pi_{\boldsymbol{\tau}} \otimes (\det)^{\varepsilon} \end{aligned}$$

iii) All unipotent representations attached to $\mathcal{O}$ are unitarizable.
iv) Suppose $\mathcal{O}$ is noticed. The associated character map is an injective map:

$$\mathrm{Ch}_{\mathcal{O}}: \quad \mathrm{Unip}(\mathcal{O}) \hookrightarrow \mathrm{LS}(\mathcal{O})$$

v) Suppose $\mathcal{O}$ is quasi-distinguished. The image of associated character map is exactly the set of admissible orbit data. In other words, we have a bijection

$$\mathrm{Ch}_{\mathcal{O}}: \quad \mathrm{Unip}(\mathcal{O}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{LS}^{\mathrm{aod}}(\mathcal{O})$$

According to Moeglin-Renard's work, Arthur's unipotent Arthur packet (defined by endoscopic transfer) is constructed by iterated theta lifting. The theorem implies that ABV's unipotent Arthur packet coincide with Arthur's.

Two unipotent representation π_1 and π_2 are called in the same LS packet if $\mathcal{L}_{\pi_1} = \mathcal{L}_{\pi_2}$. For a local system $\mathcal{L} \in \mathrm{LS}(\mathcal{O})$, let

$$[\mathcal{L}] = \{\, \pi \mid \mathrm{Ch}_{\mathcal{O}}(\pi) = \mathcal{L} \,\}$$

denote the set of all unipotent representations attached to $\mathcal{L}$. We call $[\mathcal{L}]$ a LS packet.

Remark. 1. We use convergence range theta lift to construct π inductively with respect to the number of columns in $\mathcal{O}$.
2. When $\mathcal{O}$ is not noticed, a LS packet may not be a singleton and we use the injectivity of theta lifting for a fixed dual pair in an essential way.

3. The size of LS packet $[\mathcal{L}]$ could be arbitrary large when the size of $\mathcal{O}$ groups. For example, $\mathcal{O}$ is the regular orbit of $\mathrm{Sp}(2n,\mathbb{C})$, there is only 5 possible local systems when $n \geqslant 3$. But there are $2n+1$ unipotent representations. See **??**.
4. One will see easily from the proof that the associated character $\mathrm{Ch}_{\mathcal{O}}(\pi)$ for every $\pi \in \mathrm{Unip}(\mathcal{O})$ is always multiplicity free. Even when $\mathrm{Ch}_{\mathcal{O}}(\pi)$ is not irreducible, the character of the local system attached the same row length in different irreducible components are always the same.

2. Matching doc-r-c diagrams, local systems, and unipotent representations

2.1. **Descent maps of dot-r-c diagram in the algorithm.** Suppose $\mathbf{G}$ is of type B or D, we define a map

$$\begin{aligned} \mathrm{Sign}\colon \quad \mathrm{DRC}(B) \sqcup \mathrm{DRC}(D) &\longrightarrow \mathbb{N}\times\mathbb{N} \\ \tau &\longmapsto (p,q) \end{aligned}$$

Here (p,q) is the signature of the orthogonal group $\mathrm{O}(p,q)$ corresponding to the dot-r-c diagram τ, which is calculated by the following formula

$$\begin{aligned} p &= \# \bullet + 2\#r + \#c + \#d + \#A \\ q &= \# \bullet + 2\#s + \#c + \#d + \#B \end{aligned}$$

Suppose $\mathcal{O}$ is the trivial orbit of $\mathrm{Sp}(2c_0,\mathbb{R})$ ($c_0 \geqslant 0$) or $\mathrm{Mp}(0,\mathbb{R})$ ($c_0 = 0$). Let

$$\tau_{\mathcal{O}} = \begin{array}{c} \varnothing \\ \\ \\ \end{array} \times \begin{array}{c} s \\ \vdots \\ s \end{array} \tag{2.1}$$

such that there are c_0-entries marked by "s" in the right diagram. Then $\mathrm{DRC}(\mathcal{O}) = \{\, \tau_{\mathcal{O}} \,\}$ is a singleton. Define $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$ the trivial representation of $\mathrm{Sp}(2c_0,\mathbb{R})$ or $\mathrm{Mp}(0,\mathbb{R}) = \mathbf{1}$. That by convention, $\mathrm{Sp}(0,\mathbb{R}) = \mathrm{Mp}(0,\mathbb{R})$ is the trivial group. Let

$$\mathrm{DRC}_0(C) = \{\, \tau_{\mathcal{O}} \mid \mathcal{O} = (2c_0), c_0 = 0,1,2,\cdots \,\} \text{ and } \mathrm{DRC}_0(M) = \{\, \tau_{(0)} = \varnothing \times \varnothing \,\}.$$

The algorithm of matching dot-r-c diagrams is encoded in the descent map $\overline{\nabla}$ of dot-r-c diagrams defined below. For $(\star,\star') = (D,C),(B,M)$, we define maps

$$\begin{aligned} \overline{\nabla}\colon \quad \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) &\longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star)) \times \mathbb{Z}/2\mathbb{Z} \\ \tau &\longmapsto (\tau',\epsilon). \end{aligned}$$

Here

i) τ is in $\mathrm{DRC}(\mathcal{O})$ where $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(\star) \sqcup \mathrm{Nil}^{\mathrm{dpe}}(\star')$;
ii) $\tau' \in \mathrm{DRC}(\mathcal{O}')$ where $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$;
iii) i$\epsilon = 0$ when $\mathcal{O} \mapsto \mathcal{O}'$ is a generalized descent from type C or M to type B or D.

To ease the notations, we define

$$\overline{\nabla}_1 \colon \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) \longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star))$$

such that $\overline{\nabla}_1(\tau)$ is the first component of $\overline{\nabla}(\tau)$.

We define $\mathcal{L}$ and π inductively with respect to the number of columns of $\mathcal{O}$.

i) Our reduction terminates when $\mathcal{O}$ is the trivial orbit of $\mathrm{Sp}(2c_0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R})$. In this case, $\mathrm{DRC}(\mathcal{O}) = \{\tau_{\mathcal{O}}\}$ which is attached to the trivial representation. We remark that the theta lift of the trivial representation $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$ of $\mathrm{Sp}(0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R})$ to $\mathrm{O}(p, q)$ for any signature p, q is the trivial representation $\mathbf{1}^{+,+}_{p,q}$ of $\mathrm{O}(p, q)$.[1]

ii) Suppose $\mathcal{O} \in \mathrm{Nil}(\star) \sqcup \mathrm{Nil}(\star')$ is a nontrivial orbit. Let $\tau \in \mathrm{DRC}(\mathcal{O})$ and

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \in \mathrm{DRC}(\mathcal{O}') \times \mathbb{Z}/2\mathbb{Z} \text{ where } \mathcal{O}' = \overline{\nabla}(\mathcal{O}).$$

a) Suppose $\tau \in \mathrm{DRC}(C)$ or $\mathrm{DRC}(M)$. Now $\widetilde{G} = \mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Mp}(2n, \mathbb{R})$ with $n = |\tau|$. By induction, τ' is attached to a unipotent representation $\pi_{\tau'}$ of $\widetilde{G}' = \mathrm{O}(p, q)$ where $(p, q) = \mathrm{Sign}(\tau')$. We define

$$\pi_\tau := \bar{\Theta}_{\widetilde{G}', \widetilde{G}}(\pi_{\tau'} \otimes (\det)^{\epsilon}).$$

Here det is the determinant character of $\mathrm{O}(p, q)$.

b) Suppose $\tau \in \mathrm{DRC}(D)$ or $\mathrm{DRC}(B)$. Now $\widetilde{G} = \mathrm{O}(p, q)$ with $(p, q) = \mathrm{Sign}(\tau)$. By induction, τ' is attached to a unipotent representation of $\widetilde{G}'$ where $\widetilde{G}' = \mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Mp}(2n, \mathbb{R})$ and $n = |\tau'|$. We define

$$\pi_\tau := \bar{\Theta}_{\widetilde{G}', \widetilde{G}}(\pi_{\tau'}) \otimes (\mathbf{1}^{+,-}_{p,q})^{\epsilon}.$$

We let

$$\mathscr{L}_\tau := \mathrm{Ch}_{\mathcal{O}}(\pi_\tau), \text{ and}$$
$${}^{\ell}\mathrm{LS}(\mathcal{O}) := \{\mathcal{L}_\tau \mid \tau \in \mathrm{DRC}(\mathcal{O})\}$$

to be the set of local systems obtained from our algorithm.

We summarize our definitions in the following diagram with $\star = \mathcal{O}/C/D/B/M$.

$$\begin{tikzcd}
\mathrm{DRC}(\star) \arrow[rrr, "\pi"] \arrow[ddd, "\mathcal{L}"'] & & & \mathrm{Unip}(\star) \arrow[ddd, "\mathrm{Ch}"] \\
& \tau \arrow[r, mapsto] \arrow[d, mapsto] & \pi_\tau \arrow[d, mapsto] & \\
& \mathcal{L}_\tau \arrow[r, leftrightarrow] & \mathscr{L}_\tau & \\
\mathcal{K}_{\mathsf{M}}(\star) \arrow[rrr, hook, two heads] & & & \mathrm{LS}(\star)
\end{tikzcd}$$

Here $\mathcal{K}_{\mathsf{M}}(\star)$ is a combinatiorially defined monoid parameterizing the local systems on the relevent nilpotent orbits, see.

3. The definition of $\overline{\nabla}$

From now on, we assume

$$\tau = (\tau_{L,0}, \tau_{L,1}, \cdots) \times (\tau_{R,0}, \tau_{R,1}, \cdots).$$

For any τ let $\tau^{\bullet}$ be the subdiagram consisting of $\bullet$ and s entries only.

[1]When $c_0 = 0$, this is an assignment by convention.

3.1. **dot-s switching algorithm.** We say a bipartition $\tau = \tau_L \times \tau_R$ is interlaced with lager left part if τ has columns $(\tau_{L,0} \geqslant \tau_{L,1}, \cdots,) \times (\tau_{R,0}, \tau_{R,1}, \cdots,)$ such that

$$\tau_{L,0} \geqslant \tau_{R,0} \geqslant \tau_{L,1} \geqslant \tau_{R,1} \geqslant \tau_{L,2} \geqslant \cdots .$$

$\mathrm{bi}\mathcal{P}_{\mathrm{L}}$ denote the set of all interlaced bipartitions with larger left parts. Similarly, define $\mathrm{bi}\mathcal{P}_{\mathrm{R}}$ be the set of bipartitoins with larger right parts:

$$\mathrm{bi}\mathcal{P}_{\mathrm{R}} = \{\, \tau = (\tau_{L,0} \geqslant \tau_{L,1}, \cdots,) \times (\tau_{R,0}, \tau_{R,1}, \cdots,) \mid \tau_{L,0} \geqslant \tau_{R,0} \geqslant \tau_{L,1} \geqslant \tau_{R,1} \geqslant \tau_{L,2} \geqslant \cdots \}$$

Let DS_{L} (resp. DS_{R}) be the set of of filled diagrams with only "s" entries on the left (resp. right) diagram and the rest are all "$\bullet$".

Note that for each $\boldsymbol{\tau} \in \mathrm{DS}_{\mathrm{L}}$, its shape τ is in $\mathrm{bi}\mathcal{P}_{\mathrm{L}}$. On the other hand, for each $\tau \in \mathrm{bi}\mathcal{P}_{\mathrm{L}}$, we can fill "$\bullet$" in all entries on the right diagram and corresponding entries on the left, and than fill "s" in the rest entries on the left. This procedure yields a valid dot-s diagram. In summary, $\mathrm{bi}\mathcal{P}_{\mathrm{L}}$ and DS_{L} are naturally bijective to each other.

Let $\overline{\nabla}_{\mathrm{L}}\colon \mathrm{bi}\mathcal{P}_{\mathrm{L}} \to \mathrm{bi}\mathcal{P}_{\mathrm{R}}$ (resp. $\overline{\nabla}_{\mathrm{R}}\colon \mathrm{bi}\mathcal{P}_{\mathrm{R}} \to \mathrm{bi}\mathcal{P}_{\mathrm{L}}$) be the operation of deleting the longest column on the left (resp. on the right). Then the operation gives a well defined maps between DS_{L} and DS_{R} making the following diagrams commute (the vertical maps are the natural identification discussed above):

$$\begin{tikzcd}
\mathrm{bi}\mathcal{P}_{\mathrm{L}} \arrow[r, "\overline{\nabla}_{\mathrm{L}}"] \arrow[d, equal] & \mathrm{bi}\mathcal{P}_{\mathrm{R}} \arrow[d, equal] \\
\mathrm{DS}_{\mathrm{L}} \arrow[r, dashed, "\overline{\nabla}_{\mathrm{L}}"] & \mathrm{DS}_{\mathrm{R}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{bi}\mathcal{P}_{\mathrm{R}} \arrow[r, "\overline{\nabla}_{\mathrm{R}}"] \arrow[d, equal] & \mathrm{bi}\mathcal{P}_{\mathrm{L}} \arrow[d, equal] \\
\mathrm{DS}_{\mathrm{R}} \arrow[r, dashed, "\overline{\nabla}_{\mathrm{R}}"] & \mathrm{DS}_{\mathrm{L}}
\end{tikzcd}$$

By abuse of notation, we also call the dashed arrow above $\overline{\nabla}_{\mathrm{L}}$ and $\overline{\nabla}_{\mathrm{R}}$.

3.2. **Bijection between "special" and "non-special" diagrams for type C.** We define a bijection between "special" diagram $\boldsymbol{\tau}^s$ and "non-special" diagrams $\boldsymbol{\tau}$.

Definition 3.1. *Let $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \in \mathrm{Nil}^{\mathrm{dpe}}(C)$ with $k \geqslant 0$. Let*

$$\tau = (\tau_{2k-1}, \cdots, \tau_1, \tau_{-1} = 0) \times (\tau_{2k}, \cdots, \tau_0).$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(C)$. We say τ has

- special shape *if* $\tau_{2k-1} \leqslant \tau_{2k} + 1$*;*
- non-special shape *if* $\tau_{2k-1} > \tau_{2k} + 1$.

Clearly, this gives a partition of the set of Weyl group representations attached to $\mathcal{O}$.

Suppose $k \geqslant 1$, $C_{2k} = 2c_{2k}$ and $C_{2k-1} = 2c_{2k-1}$. The special shape representation τ^s

$$\tau^s = \tau^s_L \times \tau^s_R = (c_{2k-1}, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \cdots, \tau_0)$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau^{ns}_L \times \tau^{ns}_R = (c_{2k} + 1, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \cdots, \tau_0)$$

In the paring $\tau^s \leftrightarrow \tau^{ns}$, τ_j is unchanged for $j \leqslant 2k-2$. We call τ^s the special shape *and τ^{ns} the corresponding* non-special shape

Definition 3.2. *Suppose C_{2k} is even. We retain the notation in Definition 3.1 where the shapes τ^s and τ^{ns} correspond with each other. We define a bijection between $\mathrm{DRC}(\tau^s)$ and $\mathrm{DRC}(\tau^{ns})$:*

$$\begin{aligned}
\mathrm{DRC}(\tau^s) &\longleftrightarrow \mathrm{DRC}(\tau^{ns}) \\
\boldsymbol{\tau}^s &\longleftrightarrow \boldsymbol{\tau}^{ns}
\end{aligned}$$

Without loss of generality, we can assume that $\boldsymbol{\tau}^s \in \mathrm{DRC}(\tau^s)$ *and* $\boldsymbol{\tau}^{ns} \in \mathrm{DRC}(\tau^{ns})$ *have the following shapes:*

$$
\boldsymbol{\tau}^s : \begin{array}{lll} \cdots & \cdots & \cdots \\ x_0 & * & \cdots \\ x_1 & x_2 & \cdots \end{array} \times \begin{array}{ll} \cdots & \cdots \\ v_0 & \cdots \\ x_3 & \\ s & \\ \vdots & \\ s & \end{array} \quad \longleftrightarrow \quad \boldsymbol{\tau}^{ns} : \begin{array}{lll} \cdots & \cdots & \cdots \\ y_0 & * & \cdots \\ r & y_2 & \cdots \\ \vdots & & \\ r & & \\ y_1 & & \\ y_3 & & \end{array} \times \begin{array}{ll} \cdots & \cdots \\ w_0 & \cdots \end{array} \tag{3.1}
$$

(Here the grey parts have length $(C_{2k} - C_{2k-1})/2$ *(could be zero length), the row contains* x_0 *and* y_0 *may be empty, and* $*/\cdots$ *are arbitrary entries. We remark that the value of* v_0 *and* w_0 *are completely determined by* x_0 *and* y_0*. In particular,* v_0 *and* w_0 *are non-empty if and only if* $x_0/y_0 \neq \emptyset$*, i.e.* $C_{2k-1} \geqslant 4$*.)*

The correspondence between (x_0, x_1, x_2, x_3)*, with* (y_0, y_1, y_2, y_3) *is given by Table 1. The rest part of the diagrams are unchanged.*

We define

$$
\mathrm{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \mathrm{DRC}(\tau^s) \quad \text{and} \quad \mathrm{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \mathrm{DRC}(\tau^{ns})
$$

where τ^s *(resp.* τ^{ns}*) runs over all specail (resp. non-special) shape representations attached to* $\mathcal{O}$*. Clearly* $\mathrm{DRC}(\mathcal{O}) = \mathrm{DRC}^s(\mathcal{O}) \sqcup \mathrm{DRC}^{ns}(\mathcal{O})$*.*

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

3.3. Special and non-special shapes of type D.

Now we define the notion of special and non-special shape of type D.

Definition 3.3. *Let*

$$
\tau = (\tau_{2k+1}, \tau_{2k-1}, \cdots, \tau_1) \times (\tau_{2k}, \cdots, \tau_0)
$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(D)$*. We say* τ *has*

- special shape *if* $\tau_{2k-1} \leqslant \tau_{2k} + 1$*;*
- non-special shape *if* $\tau_{2k-1} > \tau_{2k} + 1$.

Suppose $\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \cdots, C_0 \geqslant 0)$ *with* $k \geqslant 1$.[2] *A representation* τ^s *attached to* $\mathcal{O}$ *has the shape*

$$
\tau^s = \tau^s_L \times \tau^s_R = (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \cdots, \tau_0)
$$

is paired with

$$
\tau^{ns} = \tau^{ns}_L \times \tau^{ns}_R = (c_{2k+1}, c_{2k} + 1, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \cdots, \tau_0).
$$

In the paring $\tau^s \leftrightarrow \tau^{ns}$*,* τ_j *is unchanged for* $j \leqslant 2k-2$*.*

3.4. Definition of $\overline{\nabla}$ of Type D and C.

The induction starts with type C: For the trivial orbit $\mathcal{O}$ of $\widetilde{G} = \mathrm{Sp}(2c_0, \mathbb{R})$, $\mathrm{DRC}(\mathcal{O}) = \{ \boldsymbol{\tau}_{\mathcal{O}} \}$ and $\pi_{\boldsymbol{\tau}_{\mathcal{O}}} = \mathbf{1}$ is the trivial representation, see (2.1).

[2]Note that C_{2k} and C_{2k-1} are non-zero even integers.

3.4.1. *The defintion of* ϵ.

(1). When $\tau \in \mathrm{DRC}(D)$, ϵ is determined by the "basal disk" of τ (see (6.3) for the definition of x_τ):

$$\epsilon_\tau := \begin{cases} 0, & \text{if } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

(2). When $\tau \in \mathrm{DRC}(C)$, the ϵ is determined by the lengths of $\tau_{L,0}$ and $\tau_{R,0}$:

$$\epsilon_\tau := \begin{cases} 0, & \text{if } \tau \text{ has special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| \leqslant 1; \\ 1, & \text{if } \tau \text{ has non-special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| > 1. \end{cases}$$

3.4.2. *Initial cases.* Let $\mathcal{O}$ be a nilpotent orbit of type D with at most 2 columns.

(3). Suppose $\mathcal{O} = (2c_1, 2c_0)$. Then $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$ is the trivial orbit of $\mathrm{Sp}(2c_0, \mathbb{R})$. $\mathrm{DRC}(\mathcal{O})$ consists of diagrams of shape $\tau_L \times \tau_R = (c_1,) \times (c_0,)$. The set $\mathrm{DRC}(\mathcal{O}')$ is a singleton $\{\tau_{\mathcal{O}'}\}$, and every element $\tau \in \mathrm{DRC}(\mathcal{O})$ maps to $\tau_{\mathcal{O}'}$ (see (2.1)):

$$\mathrm{DRC}(\mathcal{O}) \ni \tau : \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \mapsto \tau_{\mathcal{O}'} : \begin{array}{c} \varnothing \end{array} \times \begin{array}{c} s \\ \vdots \\ s \end{array}$$

	τ'	τ^s	τ^{ns}	
	$\begin{array}{ll} x_0' & * \\ x_1' & x_2' \end{array} \times \begin{array}{l} * \end{array}$	$\begin{array}{ll} x_0 & * \\ x_1 & x_2 \end{array} \times \begin{array}{ll} * & * \\ x_3 & \\ s & \\ \vdots & \\ s & \end{array}$	$\begin{array}{ll} y_0 & * \\ r & y_2 \\ \vdots & \\ r & \\ y_1 & \\ y_3 & \end{array} \times \begin{array}{ll} * & * \end{array}$	
$x_1' = s$ then $z_0 = \varnothing /\ s$, $x_0 = \varnothing /\ \bullet$, $x_1 = \bullet$, $x_2 = x_2'$	$\begin{array}{ll} x_0' & * \\ s & x_2 \end{array} \times \begin{array}{l} * \end{array}$	$\begin{array}{ll} x_0 & * \\ \bullet & x_2 \end{array} \times \begin{array}{ll} * & * \\ \bullet & \\ s & \\ \vdots & \\ s & \end{array}$	$\begin{array}{ll} x_0 & * \\ r & x_2 \\ \vdots & \\ r & \\ c & \\ d & \end{array} \times \begin{array}{ll} * & * \end{array}$	$x_2 \neq r$
		$\begin{array}{ll} x_0 & * \\ \bullet & r \end{array} \times \begin{array}{ll} * & * \\ \bullet & \\ s & \\ \vdots & \\ s & \end{array}$	$\begin{array}{ll} x_0 & * \\ r & c \\ \vdots & \\ r & \\ r & \\ d & \end{array} \times \begin{array}{ll} * & * \end{array}$	$x_2 = r$
$x_1' \neq s$ then $x_1 = x_1'$, $x_2 = x_2'$	$\begin{array}{ll} x_0' & * \\ x_1' & x_2 \end{array} \times \begin{array}{l} * \end{array}$	$\begin{array}{ll} x_0 & * \\ x_1 & x_2 \end{array} \times \begin{array}{ll} * & * \\ s & \\ s & \\ \vdots & \\ s & \end{array}$	$\begin{array}{ll} x_0 & * \\ r & x_2 \\ \vdots & \\ r & \\ r & \\ x_1 & \end{array} \times \begin{array}{ll} * & * \end{array}$	$x_0' \neq c$ then $x_0' = \varnothing / s /\ r$, $x_0 = \varnothing / \bullet /\ r$
		$\begin{array}{ll} c & * \\ d & x_2 \end{array} \times \begin{array}{ll} * & * \\ s & \\ s & \\ \vdots & \\ s & \end{array}$	$\begin{array}{ll} r & * \\ r & x_2 \\ \vdots & \\ r & \\ c & \\ d & \end{array} \times \begin{array}{ll} * & * \end{array}$	$x_0' = c$ then $x_0 = x_0' = c$, $x_1 = x_1' = d$, $x_2 = x_2' = \varnothing/d$

TABLE 1. "special-non-special" switch

We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \text{ and } x_\tau := x_n$$

We call $\mathbf{p}_\tau = \mathbf{x}_\tau$ the "peduncle" part of τ and x_τ the "basal disk" of τ. Note that $\mathbf{x}_\tau$ consists of entries marked by $s/r/c/d$.

3.4.3. *The descent from C to D.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

(4). Suppose $|\tau_{L,0}| - |\tau_{R,0}| < 2$, i.e. τ has special shape. Keep r, c, d unchanged, delete $\tau_{R,0}$ and fill the remaining part with "$\bullet$" and "s" by $\overline{\nabla}_{\mathrm{R}}(\tau^\bullet)$ using the dot-s switching algorithm.

(5). Suppose $|\tau_{L,0}| - |\tau_{R,0}| \geqslant 2$. We define

$$\tau' = \overline{\nabla}_1(\tau) := \overline{\nabla}_1(\tau^s)$$

where τ^s is the special diagram corresponding to τ defined in Definition 3.2.

Lemma 3.4. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} even. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1}, \cdots, C_0)$. Then*

$$\mathrm{DRC}^s(\mathcal{O}) \xrightarrow{\quad\overline{\nabla}_1\quad} \mathrm{DRC}(\mathcal{O}')$$

$$\mathrm{DRC}^{ns}(\mathcal{O}) \xrightarrow{\quad\overline{\nabla}_1\quad} \mathrm{DRC}(\mathcal{O}')$$

are a bijections.

Proof. The claim for $\mathrm{DRC}^s(\mathcal{O})$ is clear by the definition of descent. The claim for $\mathrm{DRC}^{ns}(\mathcal{O})$ reduces to that of $\mathrm{DRC}^s(\mathcal{O})$ using Definition 3.2. ☐

Lemma 3.5. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} odd. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1}+1, \cdots, C_0)$. Then the following map is a bijection*

$$\mathrm{DRC}(\mathcal{O}) \xrightarrow{\quad\overline{\nabla}_1\quad} \{\, \tau' \in \mathrm{DRC}(\mathcal{O}') \mid x_\tau \neq s \,\}.$$

Proof. This is clear by the descent algorithm. ☐

3.4.4. *Descent from D to C.* Now we consider the general case of the descent from type D to type C. So we assume $\mathcal{O}$ has at least 3 columns. First note that the shape of $\tau' = \overline{\nabla}(\tau)$ is the shape of τ deleting the longest column on the left.

The definition splits in cases below. In all these cases we define

$$\mathbf{x}_\tau := x_1 \cdots x_n \text{ and } x_\tau := x_n \tag{3.2}$$

which is marked by $s/r/c/d$. For the part marked by $*/\cdots$, $\overline{\nabla}$ keeps r, c, d and maps the rest part consisting of $\bullet$ and s by dot-s switching algorithm.

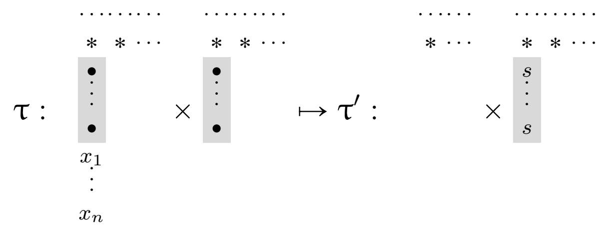
(6). When $C_{2k} = C_{2k-1}$ is odd, we could assume τ and τ' hase the following forms with $n = (C_{2k+1} - C_{2k} + 1)/2$.

$$\tau : \begin{array}{llll} \cdots\cdots\cdot & & \cdots\cdots\cdot & \\ * & * \cdots & w_0 * \cdots & \\ x_1 & y_1 \cdots \times & & \\ \vdots & & & \\ x_n & & & \end{array} \mapsto \tau' : \begin{array}{ll} \cdots\cdots & \cdots\cdots\cdot \\ * \cdots & w_0 * \cdots \\ z_1 \cdots \times & \end{array}$$

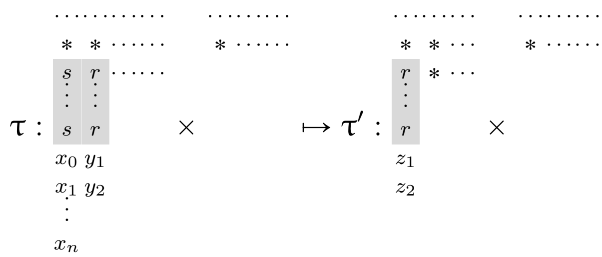
Suppose $y_1 = c$ and $(x_1, \cdots, x_n) = (r, \cdots, r)$ or $(r, \cdots, r, d)$, then let $z_1 = r$; otherwise, set $z_1 := y_1$. We define

$$\mathbf{p}_\tau := \begin{matrix} x_1 & y_1 \\ \vdots & \\ x_n & \end{matrix}.$$

(7). When C_{2k} is even and $\boldsymbol{\tau}$ has special shape, the descent is given by the following diagram



(8). When C_{2k} is even and $\boldsymbol{\tau}$ has non-special shape, then



where y_1, y_2 are non empty. In most of the case, we just delete the longest column on the left and set $(z_1, z_2) = (y_1, y_2)$. The exceptional cases are listed below:

- When $x_0 = r$, we have $(y_1, y_2) = (c, d)$. We let
$$\begin{array}{cc} x_0 & y_1\\ x_1 & y_2 \\ \vdots & \\ x_n & \end{array} = \begin{array}{cc} r & c\\ x_1 & d\\ \vdots & \\ x_n & \end{array} \mapsto \begin{array}{c} z_1 \\ z_2\end{array} := \begin{array}{c} r\\ r\end{array}$$
- When $x_0 = c$, we have $n = 1$ and $(x_0, x_1) = (y_1, y_2) = (c, d)$. We let
$$\begin{array}{cc} x_0 & y_1\\ x_1 & y_2\end{array} = \begin{array}{cc} c & c\\ d & d\end{array} \mapsto \begin{array}{c} z_1\\ z_2\end{array} := \begin{array}{c} r\\ c\end{array}.$$
- We remark that x_0 never equals to d.

3.4.5. *A key property in the descent case.* We retain the notation in Definition 3.3, where
$$\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \cdots, C_0) \text{ such that } k \geqslant 1.$$
Let $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$. The following lemma is the key property satisfied by our definition

Lemma 3.6. *Let τ^s and τ^{ns} are two representations attached to $\mathcal{O}$ as in Definition 3.3. Then*

i) For every $\boldsymbol{\tau} \in \mathrm{DRC}(\tau^s) \sqcup \mathrm{DRC}(\tau^{ns})$, the shape of $\overline{\nabla}^2(\boldsymbol{\tau})$ is
$$\tau'' = (c_{2k-1}, \tau_{2k-3}, \tau_1) \times (\tau_{2k-2}, \cdots, \tau_0)$$

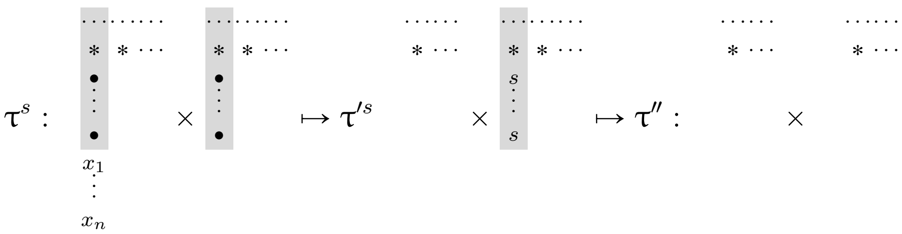
ii) Let $\mathcal{O}_1 = (2(c_{2k+1} - c_{2k}),) \in \mathrm{Nil}(D)$ be the trivial orbit of $\mathrm{O}(2(c_{2k-1} - c_{2k}), \mathbb{C})$. We define $\delta\colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ by $\delta(\boldsymbol{\tau}) = (\overline{\nabla}^2(\boldsymbol{\tau}), \mathbf{x}_{\boldsymbol{\tau}})$. The following maps are bijections
$$\begin{array}{ccccc} \mathrm{DRC}(\tau^s) & \xrightarrow{\delta} & \mathrm{DRC}(\tau'') \times \mathrm{DRC}(\mathcal{O}_1) & \xleftarrow{\delta} & \mathrm{DRC}(\tau^{ns}) \\ \boldsymbol{\tau}^s & \longmapsto & (\overline{\nabla}^2(\boldsymbol{\tau}^s), \mathbf{x}_{\boldsymbol{\tau}^s}) & & \\ & & (\overline{\nabla}^2(\boldsymbol{\tau}^{ns}), \mathbf{x}_{\boldsymbol{\tau}^{ns}}) & \longleftarrow & \boldsymbol{\tau}^{ns} \end{array}$$

In particular, we obtain an one-one correspondence $\boldsymbol{\tau}^s \leftrightarrow \boldsymbol{\tau}^{ns}$ such that $\delta(\boldsymbol{\tau}^s) = \delta(\boldsymbol{\tau}^{ns})$.

iii) Suppose τ^s *and* τ^{ns} *correspond as the above such that* $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$. *Then*

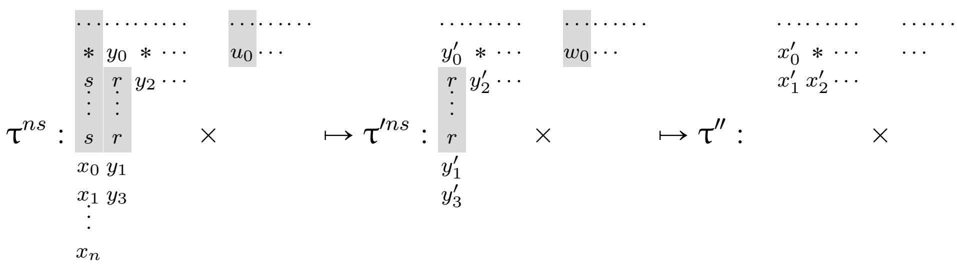
$$\mathrm{Sign}(\tau^s) = \mathrm{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \mathrm{Sign}(\tau'') + \mathrm{Sign}(\mathbf{x}). \tag{3.3}$$

Proof. Suppose τ^s has special shape. the the behavior of τ^s under the descent map $\overline{\nabla}$ is illostrated as the following:



Note that τ^s is obtained from τ'' by attaching the grey part consisting totally $2c_{2k}$ dots and $\mathbf{x}$. Now the claims for τ^s is clear.

Now consider the descent of a non-special diagram τ^{ns}:



The bijection follows from the observation that 1. τ^{ns} is obtained by attaching the grey part $x_0, \cdots, x_n$ and y_0, y_1, y_2, y_3 to the $*/\cdots$ part of τ''; 2. the value of (y_0, y_1, y_2, y_3) is completely determined by (x_0', x_1', x_2') and $\mathbf{x} = x_1 \cdots x_n$.

We leave it to the reader for checking case by case that the grey part of τ^{ns} has signature $(C_{2k} - 2, C_{2k} - 2)$ and

$$\mathrm{Sign}(x_0 y_0 y_1 y_2 y_3) - \mathrm{Sign}(x_0' x_1' x_2') = (2, 2).$$

Therefore (6.4) follows.

[First assume that $C_{2k} = 2$. Note that we can not/(or now?) assume $k = 1$, consider the orbit $\mathcal{O}'' = (2, 1, 1, 1, 1)$.

(i) $x_1' = s$, now $y_0' = \varnothing/\bullet$ when $x_0' = \varnothing/s$
  (a) Suppose $x_2' \neq r$. Then $(y_2', y_1', y_3') = (x_2', r, d)$, $(x_0, y_0, y_2, y_1, y_3) = (s, x_0', x_2', c, d)$. Therefore, the sign difference is $\mathrm{Sign}(s, c, d) - \mathrm{Sign}(s) = (2, 2)$.
  (b) Suppose $x_2' = r$. Then $(y_2', y_1', y_3') = (c, r, d)$. $(x_0, y_0, y_2, y_1, y_3) = (s, x_0', c, r, d)$. Therefore, the sign difference is $\mathrm{Sign}(s, c, r, d) - \mathrm{Sign}(s, r) = (2, 2)$.

(ii) $x_1' \neq s$.
  (a) Suppose $x_0' \neq c$. Then $x_0' = \varnothing/s/r$, $x_1' = r/c/d$, $(y_0', y_2', y_1', y_3') = (\varnothing/\bullet/r, x_2', r, x_1')$.
    1). $x_1' = r$. Then $(x_0, y_0, y_2, y_1, y_3) = (r, x_0', x_2', c, d)$. Therefore, the sign difference is $\mathrm{Sign}(r, c, d) - \mathrm{Sign}(r) = (2, 2)$.
    2). $x_1' = c$. Then $x_1 = d$. $(x_0, y_0, y_2, y_1, y_3) = (c, x_0', x_2', c, d)$. Therefore, the sign difference is $\mathrm{Sign}(c, c, d) - \mathrm{Sign}(c) = (2, 2)$.
    3). $x_1' = d$. Then $(x_0, y_0, y_2, y_1, y_3) = (s, x_0', y_2', y_1', y_3') = (s, x_0', x_2', r, d)$. Therefore, the sign difference is $\mathrm{Sign}(s, r, d) - \mathrm{Sign}(d) = (2, 2)$.
  $(x_0, y_0, y_2, y_1, y_3) = (s, r, x_2', c, d)$. Therefore, the sign difference is $\mathrm{Sign}(s, r, x_2', c, d) - \mathrm{Sign}(c, d, x_2') = (2, 2)$.

(b) Suppose $x_0' = c$. Then $x_1' = d$, $(y_0', y_2', y_1', y_3') = (r, x_2', c, d)$. $(x_0, y_0, y_2, y_1, y_3) = (s, r, x_2', c, d)$. Therefore, the sign difference is $\mathrm{Sign}(s, r, x_2', c, d) - \mathrm{Sign}(c, d, x_2') = (2, 2)$.

] $\square$

3.4.6. *A key proposition in the generalized descent case.* We now assume that $k \geqslant 1$ and C_{2k} is odd, i.e. $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a generalized descent.

Without loss of generality, we can assume the dot-r-c diagrams have the following shapes with $n = (C_{2k+1} - C_{2k} + 1)/2$:

$$\tau : \begin{array}{ll} \cdots\cdots\cdots & \cdots\cdots \\ * \; * \; * \; * & * \; \cdots \\ x_1\, y_1 \cdots\cdots \times & \\ \vdots & \\ x_n & \end{array} \xrightarrow{\nabla_1} \tau' : \begin{array}{ll} \cdots\cdots\cdots & \cdots\cdots \\ * \; * \; * & * \; \cdots \\ x_{\tau''} \cdots\cdots \times & \end{array} \xrightarrow{\nabla_1} \tau'' : \begin{array}{ll} \cdots\cdots\cdots & \cdots\cdots \\ * \; * \; * & * \; \cdots \\ x_{\tau''} \cdots\cdots \times & \end{array} \tag{3.4}$$

The diagram τ is obtained from the diagram of τ by adding $\bullet$ at the grey parts, changing $x_{\tau''}$ to y_1 and attaching $x_1 \cdots x_n$.

We define

$$\mathbf{p}_\tau := \begin{array}{ll} x_1 & y_1 \\ \vdots & \\ x_n & \end{array}$$

and call $\mathbf{p}_\tau$ the peduncle part of τ. Let $\mathcal{O}_1 = (C_{2k+1} - C_{2k} + 1,) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. We define $\mathbf{u}_\tau \in \mathrm{DRC}(\mathcal{O}_1)$ by the following formula:

$$\mathbf{u}_\tau := u_1 \cdots u_n = \begin{cases} r \cdots rc, & \text{when } \mathbf{p}_\tau = \begin{array}{ll} r & c \\ \vdots & \\ r & \end{array} \\ r \cdots cd, & \text{when } \mathbf{p}_\tau = \begin{array}{ll} r & c \\ \vdots & \\ d & \end{array} \\ x_1 \cdots , x_n & \text{otherwise.} \end{cases} \tag{3.5}$$

Lemma 3.7. *The map* $\delta \colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ *given by* $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathbf{u}_\tau)$ *is injective. Moreover,*

$$\mathrm{Sign}(\tau) = \mathrm{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \mathrm{Sign}(\mathbf{u}_\tau).$$

The map $\tilde{\delta} \colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective.*

Proof. The the injectivity of δ and the siginature formula follows directly from our algorithm. Now the injectivity of $\tilde{\delta}$ follows from $\mathbf{x}_\tau \mapsto (\mathrm{Sign}(\mathbf{x}_\tau), \epsilon)$ is injective by Claim 5.2. $\square$

4. Convention on the local system

An irreducible local system is represented by a *marked Young diagram*. A general local system is understand as the union of irreducible ones. By our computation later, it would be clear that the local systems appeared in our cases are always *multiplicity one*.

4.1. **Commutative free monoid.** Given a set A, the commutative free monoid $\mathsf{C}(\mathbb{A})$ on $\mathbf{A}$ consists all finite multisets with elements drawn from A. It is equipped with the binary operation, say $\bullet$ such that

$$\{ a_1, \cdots, a_{k_1} \} \bullet \{ b_1, \cdots, b_{k_2} \} := \{ a_1, \cdots, a_{k_1}, b_1, \cdots, b_{k_2} \}.$$

In the following, we always identify A with the subset of $\mathsf{C}(\mathsf{A})$ consists of singletons.

Suppose M is a commutative monoid, a map $f \colon \mathsf{A} \to \mathsf{M}$ is uniquelly extends to a morphism $\mathsf{C}(\mathsf{A}) \to \mathsf{M}$ which is still called f by abuse of notations.

In particular a map $f\colon \mathsf{A}\to\mathsf{B}$ between sets uniquely extends to an morphism $f\colon \mathsf{C}(\mathsf{A})\to\mathsf{C}(\mathsf{B})$ by

$$f(\{\, a_1,\cdots,a_k\,\}) = \{\, f(a_1),\cdots,f(a_k)\,\}\,.$$

4.2. **Marked Young diagram.** We now explain the combinatiorial objects parameterizing local systems. Basically ,we use $+/-$ (resp. $\ast/=$) to denote the associated character on the corresponding factor of the isotropic group. Let $\star = B, C, D, M$.

- **Young diagram** Let YD be the commutative free monoid $\mathsf{C}(\mathbb{N}^+)$ on the set of positive integers $\mathbb{Z}_{\geqslant 1}$. An element $\mathcal{O} = \{\, R_1,\cdots,R_k\,\}$ is identified with the Young diagram whose set of row lengths is $\mathcal{O}$. Therefore, we identify YD with the set of Young diagrams. Let
$$|\ \ |\colon \mathsf{YD}\longrightarrow\mathbb{N}$$
be the map sending $\{\, R_1,\cdots R_k\,\}$ to $\sum_{i=1}^k R_i$.
  Let $\mathsf{YD}(\star)$ to denote the Young diagrams corresponding to partitions of type $\star$. The set $\mathsf{YD}(\star)$ parameterizes the set of nilpotent orbits of type $\star$. Note that $\mathsf{YD}(C) = \mathsf{YD}(M)$.
- **Signed Young diagram** Let S be the set of non-empty finite length strings of alternating signs of the form $+-+\cdots$ or $-+-\cdots$. Let $\mathsf{SYD} := \mathsf{C}(\mathsf{S})$ be the monoid on S.
  The monoid SYD is identified with the set of all signed Young diagrams: $\mathscr{O} = \{\, \mathbf{r}_1,\cdots,\mathbf{r}_k\,\}\in\mathsf{SYD}$ is identified with the signed Young diagram whose rows are filled by $\mathbf{r}_j$. The map $\mathsf{S}\to\mathbb{Z}_{\geqslant 1}$ sending a string to its length extends to the morphism
$$\begin{aligned}\mathcal{Y}\colon \mathsf{SYD} &\longrightarrow \mathsf{YD}\\ \{\, \mathbf{r}_1,\cdots,\mathbf{r}_k\,\} &\longmapsto \{\, |\mathbf{r}_1|,\cdots,|\mathbf{r}_k|\,\}\end{aligned}$$
where $|\mathbf{r}_j|$ denote the length of the string $\mathbf{r}_j$. By abused of notation, we also denote $|\ \ |\circ\mathcal{Y}$ by $|\ \ |$.
  We define $\mathsf{SYD}(\star)$ to be a subset of the preimage of $\mathsf{YD}(\star)$ under the above map: When $\star = C, M$ (resp. $\star = D, B$),
$$\mathsf{SYD}(\star) = \left\{\ \mathscr{O}\in\mathsf{SYD}\ \middle|\ \begin{array}{l}\mathcal{Y}(\mathscr{O})\in\mathsf{YD}(\star)\\ \text{For each positive odd number (resp.}\\ \text{even number) } r\text{, the strings } +-\cdots\\ \text{and } -+\cdots \text{ of the length } r \text{ appears in } \mathscr{O} \text{ with the same multiplicity (could be 0).}\end{array}\right\}$$
  The set $\mathsf{SYD}(\star)$ parameterizes the real nilpotent orbits of type $\star$ and $\mathbf{K}$-orbits in $\mathfrak{p}$.
  We define two signature maps $\mathsf{S}\to\mathbb{N}\times\mathbb{N}$ as the following. For $\mathbf{r}\in\mathsf{S}$, define
$$\mathrm{Sign}(\mathbf{r}) := (\#+(\mathbf{r}), \#-(\mathbf{r}))\quad\text{and}$$
$${}^l\mathrm{Sign}(\mathbf{r}) := \begin{cases}(1,0) & \text{if } \mathbf{r} = +\cdots\\ (0,1) & \text{if } \mathbf{r} = -\cdots\end{cases}.$$
  The above defined signature maps naturally extends to signature maps $\mathrm{Sign}\colon\mathsf{SYD}\to\mathbb{N}\times\mathbb{N}$ and $\mathrm{Sign}\colon\mathsf{SYD}\to\mathbb{N}\times\mathbb{N}$.
- **Marked Young diagram** Let M be the set of finite length strings of alternating marks of the form $+-+\cdots$,$-+-\cdots$, $\ast{=}\ast\cdots$, or $={\ast}=\cdots$.

Let $\mathsf{MYD} := \mathsf{C}(\mathsf{M})$ which is identified with the set of Young diagrams marked with alternating marks $+/-$, $\ast/=$. Define $\mathscr{F}\colon \mathsf{M} \to \mathsf{S}$ by the formula

$$\mathscr{F}(\mathbf{r}) := \begin{cases} + - + \cdots & \text{if } \mathbf{r} = + - + \cdots \text{ or } \ast{=}\ast \cdots \\ - + - \cdots & \text{if } \mathbf{r} = - + - \cdots \text{ or } {=}\ast{=} \cdots \end{cases} \qquad \forall \mathbf{r} \in \mathsf{M}$$

such that $\mathscr{F}(\mathbf{r})$ and $\mathbf{r}$ have the same length. $\mathscr{F}$ extends to the map $\mathscr{F} : \mathsf{MYD} \to \mathsf{SYD}$.

By abused of notation, we also denote $|\ \ | \circ \mathcal{Y}$ by $|\ \ |$.

For $\mathcal{L} \in \mathsf{MYD}$, we define

$$\mathrm{Sign}(\mathcal{L}) = \mathrm{Sign}(\mathscr{F}(\mathcal{L})) \quad \text{and} \quad {}^{l}\mathrm{Sign}(\mathcal{L}) = {}^{l}\mathrm{Sign}(\mathscr{F}(\mathcal{L})).$$

We define $\mathsf{MYD}(\star)$ to be a subset of the preimage of the above map:

$$\mathsf{MYD}(\star) = \left\{ \mathcal{L} \in \mathsf{MYD}(\star) \;\middle|\; \begin{array}{l} \mathscr{F}(\mathcal{L}) \in \mathsf{SYD}(\star); \\ \mathbf{r}_1 = \mathbf{r}_2 \text{ if } \mathbf{r}_1, \mathbf{r}_2 \in \mathcal{L} \text{ and } \mathscr{F}(\mathbf{r}_1) = \mathscr{F}(\mathbf{r}_2); \\ \mathbf{r} = + - + \cdots \text{ or } - + - \cdots \text{ if the length} \\ \text{of } \mathbf{r} \text{ is odd (resp. even) when } \star \in \{ C, M \} \\ \text{(resp. } \star \in \{ B, D \}). \end{array} \right\}$$

We will use $\mathsf{MYD}(\star)$ to parameterize irreducible local systems attached to the nilpotent orbits of type $\star$.

For monoid YD, SYD or MYD, the identity element is the empty set $\varnothing$, and the monoid binary operator is denoted by "$\cdot$". The $\varnothing$ is always in $\mathsf{MYD}(\star)$ by convention. According to the definition of $\mathsf{MYD}(\star)$. it could happen that $\mathcal{L}_1 \cdot \mathcal{L}_2 \notin \mathsf{MYD}(\star)$ for $\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{MYD}(\star)$.

We define an involution

$$\overline{}\colon \mathsf{M} \to \mathsf{M} \tag{4.1}$$

on M by switching the symbols $+$ with $\ast$ and $-$ with $=$. For example $\overline{+ - +} = \ast{=}\ast$ and $\overline{=\ast} = -+$.

We define the map

$$\dagger\colon \mathsf{M} \to \mathsf{M} \tag{4.2}$$

by extending the length of the strings to the left by one. For example $\dagger(+-+) = -+-+$ and $\dagger(=\ast) = \ast{=}\ast$.

For $\mathcal{L}, \mathcal{C} \in \mathsf{MYD}$, the expression $\mathcal{L} \succ \mathcal{C}$ means that there is $\mathcal{B} \in \mathsf{MYD}$ such that the factorization $\mathcal{L} = \mathcal{B} \cdot \mathcal{C}$ holds.

In the following, we will represent an element $\mathcal{L} \in \mathsf{MYD}$ by the Young diagram whose rows are filled with the strings in $\mathcal{L}$. The terminology "l-row" means a row of length l. If we do not care whether the mark is $\ast$ or $+$ (resp. $=$ or $=$), we will use $\divideontimes$ (resp. $\dot{-}$) to mark the corresponding position.

Example 4.1. *The following two diagrams $\ddagger_{p,q}$ and $\dagger_{p,q}$ are in* $\mathsf{MYD}(B) \cup \mathsf{MYD}(D)$.

$$\ddagger_{p,q} := \begin{array}{c} + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array}$$

represents the local system on the trivial orbit of $\mathrm{O}(p,q)$ *which has the trivial character on* $\mathrm{O}(p)$ *and* det *on* $\mathrm{O}(q)$. *Similarly,*

$$\dagger_{p,q} := \begin{matrix} + \\ \vdots \\ + \\ - \\ \vdots \\ - \end{matrix}$$

represents the trivial local system on the trivial orbit.

4.2.1. *Local systems.* We define $\mathcal{K}_{\mathsf{M}}$ to be the free commutative monoid generated by MYD. The binary operation in $\mathcal{K}_{\mathsf{M}}$ is denoted by "+".

The operation "$\cdot$" naturally extends to $\mathcal{K}_{\mathsf{M}}$:

$$(\sum_i m_i \mathcal{L}_i) \cdot (\sum_j m_j \mathcal{L}'_j) := \sum_{i,j} m_i m_j (\mathcal{L}_i \cdot \mathcal{L}'_j).$$

where $\mathcal{L}_i, \mathcal{L}'_j \in \mathsf{MYD}$, $m_i, m_j \in \mathbb{Z}_{\geqslant 0}$.

Now $(\mathcal{K}_{\mathsf{M}}, +, \cdot)$ is in fact a commutative semiring. We let 0 be the additive identity element in $\mathcal{K}_{\mathsf{M}}$ and $\varnothing$ is the multiplicative identify. Note that $\mathcal{L} \cdot 0 = 0$ for any $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$ by definition.

For $\mathcal{L}, \mathcal{D} \in \mathcal{K}_{\mathsf{M}}$, $\mathcal{L} \succ \mathcal{D}$ means that there exists $\mathcal{B} \in \mathcal{K}_{\mathsf{M}}$ such that $\mathcal{L} = \mathcal{B} \cdot \mathcal{D}$. In addition, we write $\mathcal{L} \supset \mathcal{D}$ if there is $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{K}_{\mathsf{M}}$ such that $\mathcal{L}_1 \succ \mathcal{D}$ and $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}$.

Let $\mathcal{K}_{\mathsf{M}}(\star)$ be the sub-monoid of $\mathcal{K}_{\mathsf{M}}$ generated by $\mathsf{MYD}(\star)$. Note that $\mathcal{K}_{\mathsf{M}}(\star)$ is not a sub semi-ring of $\mathcal{K}_{\mathsf{M}}$.

For $\mathcal{L} = \sum_{i=1}^k \mathcal{L}_i \in \mathcal{K}_{\mathsf{M}}$ with $\mathcal{L}_i \in \mathsf{MYD}$, we let

$$\mathrm{Sign}(\mathcal{L}) = \{\, \mathrm{Sign}(\mathcal{L}_i) \mid i = 1, \cdots, k \,\} \text{ and } {}^l\mathrm{Sign}(\mathcal{L}) = \left\{\, {}^l\mathrm{Sign}(\mathcal{L}_i) \;\middle|\; i = 1, \cdots, k \,\right\}$$

4.3. **Operations on $\mathcal{K}_{\mathsf{M}}(\star)$.** The operation $\dagger$ defined in (4.2) extends to an operation

$$\dagger \colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$$

on $\mathcal{K}_{\mathsf{M}}$ which adding a column on the left of the original marked Young diagram.

4.3.1. *Character twists on* $\mathcal{K}_{\mathsf{M}}(D)$ *and* $\mathcal{K}_{\mathsf{M}}(B)$. Recall that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ paramterizes the characters of a indefinite orthogonal group $\mathrm{O}(p,q)$:

$$\chi = (\chi^+, \chi^-) \longleftrightarrow (\mathbf{1}^{-,+})^{\chi^+} \otimes (\mathbf{1}^{+,-})^{\chi^-}.$$

We define the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on M by:

$$\mathbf{r} \otimes \chi := \begin{cases} \bar{\mathbf{r}} & \text{if } p+q \equiv 1 \text{ and } \chi^+ p + \chi^- q \equiv 1 \pmod 2 \\ \mathbf{r} & \text{otherwise} \end{cases}$$

where $\mathbf{r} \in \mathsf{M}, \chi = (\chi^+, \chi^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $(p,q) = \mathrm{Sign}(\mathbf{r})$.

The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action extends to actions on $\mathcal{K}_{\mathsf{M}}(D)$ and $\mathcal{K}_{\mathsf{M}}(B)$.

4.3.2. *Twists on* $\mathcal{K}_{\mathsf{M}}(C)$ *and* $\mathcal{K}_{\mathsf{M}}(M)$. We define an involution $✠_C \colon \mathsf{M} \to \mathsf{M}$ on M by

$$✠(\mathbf{r}) := \begin{cases} \bar{\mathbf{r}} & \text{if } |\mathbf{r}| \equiv 2 \pmod 4 \\ \mathbf{r} & \text{otherwise.} \end{cases}$$

The involution ✠ extends to an involution on $\mathcal{K}_{\mathsf{M}}$.We will only apply ✠ on $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}(C)$ or $\mathcal{K}_{\mathsf{M}}(M)$.

4.3.3. *Truncations.* We define the Truncation maps

$$\mathsf{T}^{+}\colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$$

such that for $\mathcal{L} \in \mathsf{MYD}$

$$\mathsf{T}^{+}(\mathcal{L}) = \begin{cases} \mathcal{L}_1 & \text{if there is } \mathcal{L}_1 \in \mathsf{MYD} \text{ such that } \mathcal{L}_1 \cdot + = \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathsf{T}^{-}\colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$ similarly such that $\mathcal{L} = \mathsf{T}^{-}(\mathcal{L}) \cdot -$ if $\mathcal{L} \succ -$ and $\mathsf{T}^{-}(\mathcal{L}) = 0$ otherwise for $\mathcal{L} \in \mathsf{MYD}$.

To symplify the notation, we write $\mathscr{L}^{\pm} := \mathsf{T}^{\pm}(\mathscr{L})$ for $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$.

4.4. **Theta lifts of local systems.**

4.4.1. *Lift from type C to D.* For each signature $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that $p+q$ is even, we define a map

$$\vartheta_{CD,(p,q)}\colon \mathcal{K}_{\mathsf{M}}(C) \to \mathcal{K}_{\mathsf{M}}(D)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(C)$. Let $(p_1, q_1) = {}^{l}\mathrm{Sign}(\mathcal{L})$ and

$$(p_0, q_0) = (p,q) - \mathrm{Sign}(\mathcal{L}) - (q_1, p_1).$$

We define

$$\vartheta_{CD,(p,q)}(\mathcal{L}) = \begin{cases} (\dagger \maltese^{\frac{p-q}{2}}(\mathcal{L})) \cdot \dagger_{p_0,q_0} & \text{if } p_0 \geqslant 0 \text{ and } q_0 \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

4.4.2. *Lift from type D to type C.* For each non-zero integer $n \in \mathbb{N}$, we define a map

$$\vartheta_{DC,n}\colon \mathcal{K}_{\mathsf{M}}(D) \to \mathcal{K}_{\mathsf{M}}(C)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(D)$. Let $(p_1, q_1) = {}^{l}\mathrm{Sign}(\mathcal{L})$ and $(p,q) = \mathrm{Sign}(\mathcal{L})$. Let

$$n_0 = n - (p+q)/2 - (p_1+q_1)/2$$

We define

$$\vartheta_{DC,n}(\mathcal{L}) = \begin{cases} \maltese^{\frac{p-q}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geqslant 0} \\ \maltese^{\frac{p-q}{2}}(\dagger\mathcal{L}^{+} + \dagger\mathcal{L}^{-}) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that $\maltese^{\frac{p-q}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) = (\maltese^{\frac{p-q}{2}}(\dagger\mathcal{L})) \cdot \dagger_{n_0,n_0}$.

[We take a splitting $\mathrm{O}(p,q) \times \mathrm{Sp}(n,\mathbb{R})$ in to the big metaplectic group Mp such that $\mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C})$ acts on the Fock model linearly. The maximal compact $K_{\mathrm{Sp}(n,\mathbb{R})} = \mathrm{U}(2n)$ acts on the Fock model by the character $\zeta = \det^{(p-q)/2}$. For the component of the rational nilpotent orbit $\mathrm{Sp}(2n,\mathbb{R})$, the factor of the component group is Sp if the corresponding row has odd length. Otherwise the component group is $\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})$ where (p_{2k}, q_{2k}) is the signature corresponding to the $2k$-rows. $\zeta|_{\mathrm{O}(p_{2k})} = \det_{\mathrm{O}(p_{2k})}^{(p-q)k/2}$ which is nontrivial if and only if $p - q \equiv 2 \pmod 4$ and $2k \equiv 2 \pmod 4$. This gives the formula above.]

In the proof later, we don't care above the marks for rows with length longer than 1 in the most of the case. So we will omit $\maltese_{(p,q)}$. When there is no confusion, we will simply write ϑ for $\vartheta_{CD,(p,q)}$ and $\vartheta_{CD,(p,q)}$.

4.4.3. *Lift from type M to B.* For each signature $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that $p+q$ is odd, we define a map

$$\vartheta_{MB,(p,q)} \colon \mathcal{K}_{\mathsf{M}}(M) \to \mathcal{K}_{\mathsf{M}}(B)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(M)$. Let $(p_1,q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and

$$(p_0,q_0) = (p,q) - \mathrm{Sign}(\mathcal{L}) - (q_1,p_1).$$

We define

$$\vartheta_{MB,(p,q)}(\mathcal{L}) = \begin{cases} (\dagger \maltese^{\frac{p-q+1}{2}}(\mathcal{L})) \cdot \dagger_{p_0,q_0} & \text{if } p_0 \geqslant 0 \text{ and } q_0 \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

4.4.4. *Lift from type B to type M.* For each non-zero integer $n \in \mathbb{N}$, we define a map

$$\vartheta_{BM,n} \colon \mathcal{K}_{\mathsf{M}}(B) \to \mathcal{K}_{\mathsf{M}}(M)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(D)$. Let $(p_1,q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and $(p,q) = \mathrm{Sign}(\mathcal{L})$. Let

$$n_0 = n - (p+q)/2 - (p_1+q_1)/2$$

We define

$$\vartheta_{BM,n}(\mathcal{L}) = \begin{cases} \maltese^{\frac{p-q-1}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geqslant 0} \\ \maltese^{\frac{p-q-1}{2}}(\dagger\mathcal{L}^+ + \dagger\mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that $\maltese^{\frac{p-q-1}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) = (\maltese^{\frac{p-q-1}{2}}(\dagger\mathcal{L})) \cdot \dagger_{n_0,n_0}$.

[For odd orthogonal-metaplectic group case, we still take the splitting such that $\mathrm{O}(p,q)$ acts linearly.

The maximal compact $\widetilde{K}_{\mathrm{Sp}(n,\mathbb{R})} = \widetilde{\mathrm{U}(2n)}$ acts on the Fock model by the character $\zeta = \det^{(p-q)/2}$.

Then $\det^{(p-q)/2}|_{\mathrm{O}(p_{2k})\times \mathrm{O}(q_{2k})} = \det_{\mathrm{O}(p_{2k})}^{\frac{(p-q)k}{2}} \boxtimes \det_{\mathrm{O}(q_{2k})}^{\frac{(p-q)k}{2}}$. Note that $p-q$ is an odd number. When k is even the character on $\widetilde{\mathrm{O}(p_{2k})}/\widetilde{\mathrm{O}(q_{2k})}$ are $\mathbf{1}$ or $\det$. When k is odd the character would be $\det^{\frac{1}{2}}$ or $\det^{\frac{1}{2}+1}$.

We assume the default character on $2k$-rows is $\det^{\frac{k}{2}}$, and we use $+/-$ to mark the row. Otherwise, we use $*/=$ to mark the rows.]

4.5. **Examples and remarks.**

4.5.1. *Example.*

Example 4.2. *Let* $\mathcal{T} := \dagger\dagger(\dagger_{2,1}) + \dagger\dagger(\dagger_{1,2})$ *and* $\mathcal{P} = \ddagger_{1,3}$. *Then*

$$\mathcal{T} \cdot \mathcal{P} = \begin{array}{ccc} \divideontimes & \dotminus & \divideontimes \\ \divideontimes & \dotminus & \divideontimes \\ \dotminus & \divideontimes & \dotminus \\ + & & \\ = & & \\ = & & \\ = & & \end{array} \cup \begin{array}{ccc} \divideontimes & \dotminus & \divideontimes \\ \dotminus & \divideontimes & \dotminus \\ \dotminus & \divideontimes & \dotminus \\ + & & \\ = & & \\ = & & \\ = & & \end{array}.$$

4.5.2. *Signs of the local systems obtained by iterated lifting.* Suppose $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$ is obtained by iterated theta lifting and character twisting in Section 4.3.1. Then the set ${}^l\mathrm{Sign}(\mathcal{L})$ has restricted possibilities. First, $\mathrm{Sign}(\mathcal{L})$ is always a singleton.

When $\mathcal{O}$ is of type B/D, ${}^l\mathrm{Sign}(\mathcal{L})$ is a singleton $\{ (p_1, q_1) \}$ where

$$(p_1, q_1) = \mathrm{Sign}(\mathcal{L}) - (n_0, n_0) \quad \text{and} \quad 2n_0 = \left|\overline{\nabla}(\mathcal{O})\right|.$$

When $\mathcal{O}$ is of type C/M, ${}^l\mathrm{Sign}(\mathcal{L}) = \{ (p_1, q_1) \}$ is a singleton in the most of the case. The ${}^l\mathrm{Sign}(\mathcal{L})$ may have two elements if $\mathcal{L}$ is obtained by good generalized lifting, and it has the form ${}^l\mathrm{Sign}(\mathcal{L}) = \{ (p_1 + 1, q_1), (p_1, q_1 + 1) \}$.

5. Proof of main theorem in the type C/D case

We do induction on the number of columns of the nilpotent orbits.

5.1. **Notation.** In this section, $\mathcal{O}$ is always an nilpotent orbit of type D in $\mathrm{Nil}^{\mathrm{dpe}}(D)$. We always let

$$\begin{aligned}\mathcal{O} &= (C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0),\\ \mathcal{O}' &= \overline{\nabla}(\mathcal{O}) = (C_{2k}, C_{2k-1}, \cdots, C_1, C_0).\end{aligned}$$

When $k \geqslant 1$, we let

$$\mathcal{O}'' = \overline{\nabla}^2(\mathcal{O}) = \begin{cases} (C_{2k-1}, \cdots, C_1, C_0) & \text{when } C_{2k} \text{ is even,}\\ (C_{2k-1}+1, \cdots, C_1, C_0) & \text{when } C_{2k} \text{ is odd.}\end{cases}$$

For a $\tau \in \mathrm{DRC}(\mathcal{O})$, we let

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \quad \text{and} \quad (\tau'', \epsilon') = \overline{\nabla}(\tau') \text{ (when } k \geqslant 1).$$

5.1.1. *Main proposition.* The aim of this section is to prove the following main proposition holds for $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(D)$.

Proposition 5.1. *We have:*

i) $\mathcal{L}_\tau$ is non-zero. In particular, $\pi_\tau \neq 0$.

ii) Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$, then $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.

iii) Suppose $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$, then $\pi_{\tau_1} \neq \pi_{\tau_2}$.

iv) The local system $\mathcal{L}_\tau$ is disjoint with their determinant twist:

$$\{ \mathcal{L}_\tau \mid \tau \in \mathrm{DRC}(\mathcal{O}) \} \cap \{ \mathcal{L}_\tau \otimes \det \mid \tau \in \mathrm{DRC}(\mathcal{O}) \} = \varnothing. \tag{5.1}$$

v) x_τ determine the factorization of local system described in Table 2:

$$\mathcal{L}_\tau = \sum_{i=-k}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}.$$

The key properties of the factorization are

$$\begin{aligned}(\mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i})^+ &= \begin{cases} \mathcal{B}_{\tau,i} \cdot (\mathcal{D}_{\tau,i})^+ & \text{for } i \geqslant 0\\ 0 & \text{for } i < 0\end{cases},\\ (\mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i})^- &= \begin{cases} 0 & \text{for } i > 0\\ \mathcal{B}_{\tau,i} \cdot (\mathcal{D}_{\tau,i})^- & \text{for } i \leqslant 0\end{cases}\end{aligned} \tag{5.2}$$

$$\text{and} \quad {}^l\mathrm{Sign}(\mathcal{D}_{\tau,i}) = \mathrm{Sign}(\mathbf{x}_\tau).$$

vi) In particular, we have:

(i) $x_\tau = s$, then $\mathcal{L}_\tau^+ = \mathcal{L}_\tau^- = 0$.

(ii) $x_\tau = r/c$, then $\mathcal{L}_\tau^+ \neq 0$ and $\mathcal{L}_\tau^- = 0$.

(iii) $x_\tau = d$, *then* $\mathcal{L}_\tau^+ \neq 0$ *and* $\mathcal{L}_\tau^- \neq 0$.

vii) If $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ *is a usual descent, we have a simpler factorization:*

$$\mathcal{L}_\tau = \mathcal{B}_{\tau,0} \cdot \mathcal{D}_{\tau,0} \tag{5.4}$$

where $\mathcal{D}_{\tau,0}$ *is given by* (5.3).

In particular, $\mathcal{D}_{\tau,0} = \begin{smallmatrix} = \\ = \end{smallmatrix}$ *(resp.* $\mathcal{D}_{\tau,0} = \begin{smallmatrix} + \\ + \end{smallmatrix}$*) if and only if all 1-rows of* $\mathcal{L}_\tau$ *have "=" marks (resp. "+" marks), i.e.* $x_\tau = s$ *(resp.* $x_\tau = r$*).*

viii) When $\mathcal{O}'$ *is noticed, the map* $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}') \longrightarrow {}^\ell\mathrm{LS}(\mathcal{O}')$ *is a bijection.*

ix) When $\mathcal{O}$ *is noticed, the map* $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}) \longrightarrow {}^\ell\mathrm{LS}(\mathcal{O})$ *is a bijection.*

x) When $\mathcal{O}$ *is +noticed, the maps*

$$\begin{aligned} \Upsilon_{\mathcal{O}}^+\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau^+ \neq 0 \,\} &\longrightarrow \{\, \mathcal{L}_\tau^+ \,\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^+ \qquad \textit{and} \\ \Upsilon_{\mathcal{O}}^-\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau^- \neq 0 \,\} &\longrightarrow \{\, \mathcal{L}_\tau^- \,\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^- \end{aligned}$$

are injective.

xi) When $\mathcal{O}$ *is noticed, the map*

$$\begin{aligned} \Upsilon_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau^+ \neq 0 \,\} &\longrightarrow \{\, (\mathcal{L}_\tau^+, \mathcal{L}_\tau^-) \mid \mathcal{L}_\tau^+ \neq 0 \,\} \\ \mathcal{L}_\tau &\longmapsto (\mathcal{L}_\tau^+, \mathcal{L}_\tau^-) \end{aligned} \tag{5.5}$$

is injective.[3]

[3] In fact, we only need to know whether $\mathcal{L}_\tau^-$ is non-zero or not.

x_τ	$\mathcal{L}_\tau$	$\mathcal{D}_\tau$	
s	$\sum_{i=0}^{k} \mathcal{B}_{\tau,-i} \cdot \mathcal{D}_{\tau,-i}$	$\mathcal{D}_{\tau,i} = \begin{cases} 0 & \textit{when } i > 0 \\ \begin{smallmatrix} \dotminus \cdots \dotminus \\ = \end{smallmatrix} & \textit{when } i \leqslant 0 \end{cases}$	$\exists i \geqslant 0$ *s.t.* $\mathcal{B}_{\tau,-i} \neq 0$
r	$\sum_{i=0}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}$	$\mathcal{D}_{\tau,i} = \begin{cases} \begin{smallmatrix} \divideontimes \cdots \divideontimes \\ + \end{smallmatrix} & \textit{when } i \geqslant 0 \\ 0 & \textit{when } i < 0 \end{cases}$	$\exists i \geqslant 0$ *s.t.* $\mathcal{B}_{\tau,i} \neq 0$
c	$\sum_{i=-k}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}$	$\mathcal{D}_{\tau,i} = \begin{cases} \begin{smallmatrix} \dotminus \cdots \dotminus \\ + \end{smallmatrix} & \textit{when } i \geqslant 0, \\ \begin{smallmatrix} \divideontimes \cdots \divideontimes \\ = \end{smallmatrix} & \textit{when } i < 0 \end{cases}$	$\exists i \geqslant 0$ *s.t.* $\mathcal{B}_{\tau,i} \neq 0$.
d	$\sum_{i=-k}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}$	$\mathcal{D}_{\tau,i} = \begin{cases} \begin{smallmatrix} \dotminus \cdots \dotminus \\ + \end{smallmatrix} & \textit{when } i \geqslant 0, \\ \begin{smallmatrix} \divideontimes \cdots \divideontimes \\ - \end{smallmatrix} & \textit{when } i < 0 \end{cases}$	$\exists i, j \geqslant 0$ *s.t.* $\begin{cases} \mathcal{B}_{\tau,i} \neq 0, \\ \mathcal{B}_{\tau,-j} \neq 0. \end{cases}$

In the above table $\divideontimes \cdots \divideontimes$ *or* $\dotminus \cdots \dotminus$ *has length* $2i+1$ *where we do not specify the associated character of the local system on these rows when* $i \neq 0$.
(The associated character depends on τ *and can be specified inductively.)*
The local system of $\mathcal{D}_{\tau,0}$ *is specified in the following table:*

(5.3)

x_τ	s	r	c	d
$\mathcal{D}_{\tau,0}$	$\begin{smallmatrix} = \\ = \end{smallmatrix}$	$\begin{smallmatrix} + \\ + \end{smallmatrix}$	$\begin{smallmatrix} + \\ = \end{smallmatrix}$	$\begin{smallmatrix} + \\ - \end{smallmatrix}$

TABLE 2. Factorization of the local system

Note that, $\mathrm{DRC}(\mathcal{O})$ counts the unipotent representations of special orthogonal groups a priori. But Proposition 5.1 (iv)), implies that the representation π_τ and $\pi_\tau \otimes \det$ are two different representations of the orthogonal groups. Therefore $\mathrm{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z}$ counts the set of unipotent representations of orthogonal groups.

The factorization (5.3) also holds in more general cases, we indicate when it holds in the proof of Proposition 5.1.

The main theorem would be a consequence of Proposition 5.1.

5.1.2. *How to think about the factorization of local systems.* When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is the usual descent, the local system $\mathcal{L}_\tau$ could be compared to a water hydra, see in Figure 1.

The tentacles part $\mathcal{T}_\tau = \dagger\mathcal{L}_{\tau'}$ could be reducible. The peduncle part $\mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}$ and $\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau$.

On the other hand, $\mathcal{L}_\tau$ also could be factorized into the body part and the basal disk part:

$$\mathcal{L}_\tau = \mathcal{B}_{\tau,0} \cdot \mathcal{D}_{\tau,0}.$$

Here $\mathcal{D}_{\tau,0}$ is given in (5.3).

When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a generalized descent, the situation is more complicated as the local system $\mathcal{L}_\tau$ may consists of several "hydras".

To ease the notation, we will set $\mathcal{B}_\tau := \mathcal{B}_{\tau,0}$ and $\mathcal{D}_\tau := \mathcal{D}_{\tau,0}$ in the following proof.

5.2. **The initial case:** When $k = 0$, $\mathcal{O} = (C_1 = 2c_1, C_0 = 2c_0)$ has at most two columns. Now $\pi_{\tau'}$ is the trivial representation of $\mathrm{Sp}(2c_0, \mathbb{R})$. The lift $\pi_{\tau'} \mapsto \bar{\Theta}(\pi_{\tau'})$ is in fact a stable range theta lift.For $\tau \in \mathrm{DRC}(\mathcal{O})$, let $(p_1, q_1) = \mathrm{Sign}(\mathbf{x}_\tau)$ and x_τ be the foot of τ. It is easy to see that (using associated character formula)

$$\mathcal{L}_{\pi_\tau} = \mathcal{T}_\tau \cdot \mathcal{P}_\tau, \text{ where } \mathcal{T}_\tau := \dagger\dagger_{c_0,c_0}, \text{ and } \mathcal{P}_\tau := \begin{cases} \ddagger_{p_1,q_1} & x_\tau \neq d, \\ \dagger_{p_1,q_1} & x_\tau = d. \end{cases}$$

Note that $\mathcal{L}_{\pi_\tau}$ is irreducible.

Let $\mathcal{O}_1 = (2(c_1 - c_0)) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. Using the above formula, one can check that the following maps are bijections

$$\begin{array}{ccccc} \mathrm{DRC}(\mathcal{O}_1) & \longleftarrow & \mathrm{DRC}(\mathcal{O}) & \longrightarrow & {}^\ell\mathrm{LS}(\mathcal{O}) \\ \mathbf{x}_\tau & \longleftarrow\!\shortmid & \tau & \longmapsto & \mathcal{L}_\tau. \end{array}$$

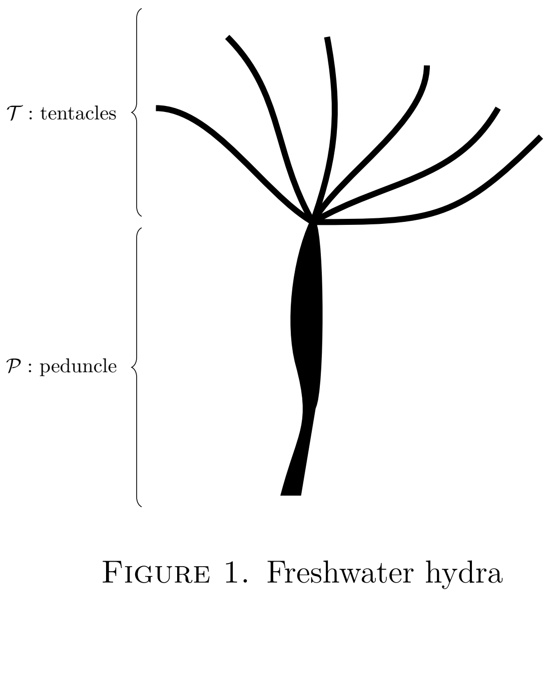


FIGURE 1. Freshwater hydra

To see that $\mathrm{DRC}(\mathcal{O}) \ni \tau \mapsto \mathcal{L}_\tau$ is an injection, one check the following claim holds:

Claim 5.2. *The map* $\mathrm{DRC}(\mathcal{O}_1) \longrightarrow \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective (see Section 6.2.1 for the definition of* ϵ_τ*). Moreover,* $\epsilon_\tau = 0$ *only if* $\mathrm{Sign}(\tau) \geqslant (0,1)$.

Moreover, $\mathcal{L}_\tau \otimes \det \notin {}^\ell\mathrm{LS}(\mathcal{O})$ for any $\tau \in \mathrm{DRC}(\mathcal{O})$. In fact, if $\mathrm{Sign}(\mathbf{x}_\tau) \succeq (1,0)$, $\mathcal{L}_\tau$ on the 1-rows with $+$-signs is trivial and $\mathcal{L}_\tau \otimes \det$ has non-trivial restriction. When $\mathrm{Sign}(\mathbf{x}_\tau) \not\succeq (1,0)$, $\mathbf{x}_\tau = s\cdots s$, all 1-rows of $\mathcal{L}_\tau$ are marked by "=" and all 1-rows of $\mathcal{L}_\tau \otimes \det$ are marked by "$-$".

Obviously we have a factoriation of the peduncle:

$$\mathcal{P}_\tau = \mathcal{L}_{\mathbf{x}_\tau} = \mathcal{B}_{\mathbf{x}_\tau} \cdot \mathcal{D}_{\mathbf{x}_\tau} \text{ with } \mathcal{B}_{\mathbf{x}_\tau} := \begin{cases} \ddagger_{\mathrm{Sign}(x_1\cdots x_{n-1})} & \text{when } x_\tau \neq d \\ \dagger_{\mathrm{Sign}(x_1\cdots x_{n-1})} & \text{when } x_\tau = d \end{cases} \tag{5.6}$$

and $\mathcal{D}_{\mathbf{x}_\tau}$ given in (5.3). Now

$$\mathcal{L}_\tau = \mathcal{B}_\tau \cdot \mathcal{D}_\tau \text{ with } \mathcal{B}_\tau := \mathcal{T}_\tau \cdot \mathcal{B}_{\mathbf{x}_\tau} \text{ and } \mathcal{D}_\tau := \mathcal{D}_{\mathbf{x}_\tau}. \tag{5.7}$$

and so the factorization (5.4) holds.

Therefore, our main theorem, Theorem 1.4, holds for $\mathcal{O}$.

5.3. **The descent case.** Now we assume $k \geqslant 1$ and C_{2k} is even. In this case $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a usual descent.

5.3.1. *Unipotent representations attached to* $\mathcal{O}'$*.* Therefore, $\mathrm{LS}(\mathcal{O}'') \longrightarrow \mathrm{LS}(\mathcal{O}')$ given by $\mathcal{L} \mapsto \vartheta(\mathcal{L})$ is an injection between abelian groups.

For $\mathcal{L}'' \in {}^\ell\mathrm{LS}(\mathcal{O}'') \sqcup {}^\ell\mathrm{LS}(\mathcal{O}'') \otimes \det$, let $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and $(p_2, q_2) = (C_{2k} - q_1, C_{2k} - p_1)$. Then

$$\vartheta(\mathcal{L}'') = \dagger\mathcal{L}'' \cdot \dagger_{p_2,q_2}.$$

By (5.1), we have a injection

$$\begin{aligned} {}^\ell\mathrm{LS}(\mathcal{O}'') \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow {}^\ell\mathrm{LS}(\mathcal{O}') \\ (\mathcal{L}'', \epsilon') &\longmapsto \vartheta(\mathcal{L}'' \otimes \det^{\epsilon'}). \end{aligned} \tag{5.8}$$

In particular, we have $\mathcal{L}_{\tau'} \neq 0$ and $\pi_{\tau'} \neq 0$ for every $\tau' \in \mathrm{DRC}(\mathcal{O}')$.

5.3.2. *Unipotent representations attached to* $\mathcal{O}$*.* Now suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (5.8), $\epsilon'_1 = \epsilon'_2$ and $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$. In particular, τ''_1 and τ''_2 have the same signature. By Lemma 3.4 and the induction hypothesis, $\tau''_1 \neq \tau''_2$ and so $\pi_{\tau''_1} \otimes \det^{\epsilon'_1} \neq \pi_{\tau''_2} \otimes \det^{\epsilon'_2}$. Now the injectivity of theta lifting yields

$$\pi_{\tau'_2} = \bar{\Theta}(\pi_{\tau''_1} \otimes \det^{\epsilon'_1}) \neq \bar{\Theta}(\pi_{\tau''_2} \otimes \det^{\epsilon'_2}) = \pi_{\tau'_2}$$

Suppose $\mathcal{O}'$ is noticed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is noticed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is an injection into $\mathrm{LS}(\mathcal{O}'')$. Now (5.8) implies $\tau' \mapsto \mathcal{L}_{\tau'}$ is also an injection.

Suppose $\mathcal{O}'$ is quasi-distingushed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is quasi-distingushed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is a bijection with $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}'')$. Since the component group of each K-nilpotent orbit in $\mathcal{O}'$ is naturally isomorphic to the component group of its descent. So we deduce that $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection onto $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}')$.

This proves the main proposition for $\mathcal{O}'$.

Now let $\tau \in \mathrm{DRC}(\mathcal{O})$. By (6.4), we have

$$\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau \neq 0, \text{ where } \mathcal{T}_\tau := \dagger\mathcal{L}_{\tau'}, \text{ and } \mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}. \tag{5.9}$$

In particular, $\pi_\tau \neq 0$ and we could determine $\mathbf{x}_\tau$ from the marks on the 1-rows of $\mathcal{L}_\tau$.

The factorization of $\mathcal{L}_\tau$ in the fashion of (2) is also given by (5.6) and (5.7).

Now suppose $\tau_1 \neq \tau_2$ and $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (5.9), the result in Section 5.2 and Lemma 3.6 (ii)), we have $\mathbf{x}_{\tau_1} = \mathbf{x}_{\tau_2}$, $\epsilon_1 = \epsilon_2$ and $\tau_1'' \neq \tau_2''$. By (5.9), we have $\mathcal{L}_{\tau_1'} = \mathcal{L}_{\tau_2'}$. Applying the main proposition of $\mathcal{O}'$, we have $\epsilon_1' = \epsilon_2'$ and $\tau_1' \neq \tau_2'$. Now by the injectivity of theta lifing, we conclude that

$$\pi_{\tau_2} = \bar{\Theta}(\pi_{\tau_1'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_1} \neq \bar{\Theta}(\pi_{\tau_2'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_2} = \pi_{\tau_2'}$$

5.3.3. Note that $\mathcal{L}_\tau$ is obtained from $\mathcal{L}_{\tau'}$ by attaching the peduncle determined by $\mathbf{x}_\tau$. The claims about noticed and quasi-distinguished orbit are easy to verify. We leave them to the reader.

5.4. **The general descent case.** We assume $k \geqslant 1$ and all the properties are satisfied by $\tau'' \in \mathrm{DRC}(\mathcal{O}'')$. We retain the notation in Section 3.4.6.

5.4.1. *Local systems.* We now describe the local systems $\mathcal{L}_{\tau'}$ and $\mathcal{L}_\tau$ more precisely using our formula of the associated character and the induction hypothesis. Note that these local systems are always non-zero which implies that $\pi_{\tau'}$ and π_τ are unipotent representations attached to $\mathcal{O}'$ and $\mathcal{O}$ respectively.

First note that in τ' and τ'', $x_{\tau''} \neq s$ by the definition of dot-r-c diagram. [4] By the induction hypothesis, $\mathcal{L}^+_{\tau''} \neq 0$.

Let $(p_1'', q_1'') := {}^l\mathrm{Sign}(\mathcal{L}_{\tau''})$. Then

$$ {}^l\mathrm{Sign}(\mathcal{L}^+_{\tau''}) = (p_0 - 1, q_0), \text{ and } {}^l\mathrm{Sign}(\mathcal{L}^-_{\tau''}) = (p_0, q_0 - 1) \text{ if } \mathcal{L}^-_{\tau''} \neq \varnothing. \tag{5.10}$$

$$\mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''}) = \dagger\mathcal{L}^+_{\tau''} + \dagger\mathcal{L}^-_{\tau''} \neq 0 \tag{5.11}$$

where ${}^l\mathrm{Sign}(\dagger\mathcal{L}^+_{\tau''}) = (q_1'', p_1'' - 1)$ and ${}^l\mathrm{Sign}(\dagger\mathcal{L}^-_{\tau''}) = (q_1'' - 1, p_1'')$ (if $\dagger\mathcal{L}^-_{\tau''} \neq 0$).

Let $(p_1, q_1) := {}^l\mathrm{Sign}(\mathcal{L}_\tau)$ and $(e, f) := (p_1 - p_1'' + 1, q_1 - q_1'' + 1)$. Using (5.2) and (3.5), one can show that

$$(e, f) = \mathrm{Sign}(\mathbf{u}_\tau).$$

[It suffice to consider the most left three columns of the peduncle part: this part has signature ${}^l\mathrm{Sign}(\mathcal{P}_\tau) + (1,1) = \mathrm{Sign}(\mathbf{u}_\tau x_{\tau''}) = \mathrm{Sign}(\mathbf{u}_\tau) + {}^l\mathrm{Sign}(\mathcal{D}_{\tau''})$. Therfore,

$$\mathrm{Sign}(\mathbf{u}_\tau) = {}^l\mathrm{Sign}(\mathcal{P}_\tau) - {}^l\mathrm{Sign}(\mathcal{D}_{\tau''}) + (1,1) = {}^l\mathrm{Sign}(\mathcal{L}_\tau) - {}^l\mathrm{Sign}(\mathcal{L}_{\tau''}) + (1,1).$$

]

Now

$$\mathcal{L}_\tau = \dagger\dagger\mathcal{L}^+_{\tau''} \cdot \mathcal{E}^+_\tau + \dagger\dagger\mathcal{L}^-_{\tau''} \cdot \mathcal{E}^-_\tau \tag{5.12}$$

where

$$\mathcal{E}^+_\tau = \begin{cases} \ddagger_{e,f-1} & \text{when } f \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e,f-1} & \text{when } f \geqslant 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{E}^-_\tau = \begin{cases} \ddagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau = d\,. \\ 0 & \text{otherwise} \end{cases}$$

By induction hypothesis, we have a factorization of $\mathcal{L}_{\tau''}$ as in (2)

$$\mathcal{L}_{\tau''} = \sum_{i=-(k-1)}^{k-1} \mathcal{B}_{\tau'',i} \cdot \mathcal{D}_{\tau'',i}.$$

[4] Suppose $\tau'' \in \mathrm{DRC}(\mathcal{O}'')$ has the shape in (3.4) and $x_{\tau''} = s$. By the induction hypothesis, $\mathcal{L}^+_{\tau''} = \mathcal{L}^-_{\tau''} = 0$ and so the theta lift of $\pi_{\tau''}$ vanishes.

Then we have the following factorization of $\mathcal{L}_\tau$:

$$\mathcal{L}_\tau = \sum_{i\neq 0} \mathcal{T}_{\tau,i}\cdot \mathcal{P}_{\tau,i}, \text{ with } \mathcal{T}_{\tau,i} := \begin{cases} \dagger\dagger\mathcal{B}_{\tau'',i-1} & i\geqslant 1 \\ \dagger\dagger\mathcal{B}_{\tau'',i+1} & i\leqslant -1. \end{cases}$$

Now we describe $\mathcal{P}_{\tau,i}$ according to $x_{\tau''}$ case by case. In the following discussion,

$$\divideontimes\cdots\divideontimes \qquad \text{and} \qquad \dot{-}\cdots\dot{-}$$

always denote a row has length $2i+1$ with unspecified associated character.

5.4.2. *Case* $x_{\tau''}=r$. We will see that the factorization (5.4) always holds when $x_{\tau''}=r$ and $n\geqslant 2$. We have three sub-cases:

(i) $\mathbf{p}_\tau = \begin{array}{l} r\ \ c\cdots \\ \vdots \\ r \end{array}$. Then $\mathcal{L}_\tau = \sum_{i=1}^k (\dagger\dagger\mathcal{B}_{\tau'',i-1})\cdot\mathcal{P}_{\tau,i}$, where

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ + \\ \vdots \\ + \end{array} \qquad \forall i\geqslant 1. \tag{5.13}$$

When $n\geqslant 2$, we have $\mathcal{P}_{\tau,i}\succeq \begin{array}{c} + \\ + \end{array}$ and the factorization (5.4) holds.

When $n=1$, we set

$$\mathcal{B}_{\tau,0}=0, \mathcal{B}_{\tau,i}=\mathcal{T}_{\tau,i} \text{ and } \mathcal{D}_{\tau,i}=\mathcal{P}_{\tau,i} \text{ for } i\geqslant 1. \tag{5.14}$$

Now $\mathcal{L}_\tau = \sum_{i=1}^k \mathcal{B}_{\tau,i}\cdot\mathcal{D}_{\tau,i}$.

(ii) $\mathbf{p}_\tau = \begin{array}{l} r\ \ c\cdots \\ \vdots \\ r \\ d \end{array}$. In this case, $n\geqslant 2$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ - \\ + \\ \vdots \\ + \end{array} \quad \succ \begin{array}{c} - \\ + \end{array} \qquad \forall i\geqslant 1 \tag{5.15}$$

In particular, (5.4) holds.

(iii) $\mathbf{p}_\tau = \begin{array}{l} s\ \ r\cdots \\ \vdots \\ s \\ x_j \\ \vdots \\ x_n \end{array}$. In this case $\#s(\mathbf{x}_\tau)\geqslant 1$.

When $n\geqslant 2$, the factorization (5.4) holds. More precisely, we have four cases according to the mark of x_n.

(a) $x_n = s$, i.e. $\mathbf{x}_\tau = s\cdots s$. Then

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ = \\ \vdots \\ = \end{array} \qquad \forall i\geqslant 1$$

When $n=1$, $\mathcal{L}_\tau = \sum_{i=-1}^{-k}\mathcal{B}_{\tau,i}\cdot\mathcal{D}_{\tau,i}$ with $\mathcal{B}_{\tau,i}=\mathcal{T}_{\tau,i}$ and $\mathcal{D}_{\tau,i}=\mathcal{P}_{\tau,i}$.

(b) $x_n = r$. Then $n\geqslant 2$ and

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array} \quad \succeq \begin{array}{c} + \\ + \\ = \end{array} \succ \begin{array}{c} + \\ + \end{array} \qquad \forall i\geqslant 1$$

(c) $x_n = c$. Then $n \geqslant 2$ and

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \divideontimes \cdots \divideontimes \\ + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array} \succeq \begin{array}{c} + \\ = \\ = \end{array} \succ \begin{array}{c} + \\ = \end{array} \quad \forall i \geqslant 1$$

(d) $x_n = d$. Then $n \geqslant 2$ and

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \divideontimes \cdots \divideontimes \\ + \\ \vdots \\ + \\ - \\ \vdots \\ - \end{array} \succeq \begin{array}{c} + \\ - \\ - \end{array} \succ \begin{array}{c} + \\ - \end{array} \quad \forall i \geqslant 1$$

5.4.3. *Case* $x_{\tau''} = c$. We always have $\mathcal{P}_{\tau,2} = \varnothing$.

(i) $x_n = s$. In this case $x_\tau = s \cdots s$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ = \\ \vdots \\ = \end{array} \qquad \forall i \geqslant 1. \tag{5.16}$$

When $n \geqslant 2$, the factorization (5.4) holds.

When $n = 1$, we have

$$\mathcal{L}_\tau = \sum_{i=-1}^{-k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i} \text{ where } \mathcal{B}_{\tau,i} = \mathcal{T}_{\tau,-i} \text{ and } \mathcal{D}_{\tau,i} = \mathcal{P}_{\tau,-i}. \tag{5.17}$$

(ii) $x_n = r$. In this case, $\mathbf{x}_\tau = s \cdots sr \cdots r$ where $\#s(\mathbf{x}_\tau) > 1$ and $n \geqslant 2$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ = \\ \vdots \\ = \\ + \\ \vdots \\ + \end{array} \succeq \begin{array}{c} = \\ + \\ + \end{array} \succ \begin{array}{c} + \\ + \end{array} \quad \forall i \geqslant 1.$$

In particular, (5.4) holds.

(iii) $x_n = c$.

(a) When $\#s(\mathbf{x}_\tau) > 1$ and so $n \geqslant 2$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array} \succ \begin{array}{c} + \\ = \end{array} \quad \forall i \geqslant 1.$$

The factorization (5.4) holds.

(b) When $\mathbf{x}_\tau = r \cdots rc$ with $n_r := \#r(\mathbf{x}_\tau) \geqslant 0$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \\ \vdots \\ + \end{array} \qquad \forall i \geqslant 1.$$

We have $\mathcal{L}_\tau = \sum_{i=1}^k \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}$ where $\mathcal{D}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \end{array}$ and $\mathcal{B}_{\tau,i} = \mathcal{T}_{\tau,i} \cdot \ddagger_{2n_r,0}$.

(iv) $x_n = d$. In this case, $n \geqslant 2$ and $\#s(\mathbf{x}_\tau) + \#c(\mathbf{x}_\tau) \geqslant 1$. Therefore,

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dotminus\ \divideontimes\ \dotminus\cdots \\ - \\ \vdots \\ - \\ + \\ \vdots \\ + \end{array} \succ \begin{array}{c} - \\ + \end{array}, \quad \forall i \geqslant 1 \tag{5.18}$$

and the factorization (5.4) holds.

5.4.4. *Case $x_{\tau''} = d$.* In this case, $\mathbf{x}_\tau$ could be anything.

(i) $\mathbf{x}_\tau = s \cdots s$. We have

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dotminus\cdots\dotminus \\ = \\ \vdots \\ = \end{array} \qquad \forall i \geqslant 1.$$

When $n \geqslant 2$, the factorization (5.4) holds. When $n = 1$, the factorizaton is given by (5.17).

(ii) $\mathbf{x}_\tau = r \cdots r$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ + \\ \vdots \\ + \end{array} \qquad \forall i \leqslant -1.$$

When $n \geqslant 2$, the factorization (5.4) holds.

When $n = 1$,

$$\mathcal{L}_\tau = \sum_{i=1}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i} \text{ where } \mathcal{B}_{\tau,i} = \mathcal{T}_{\tau,-i} \text{ and } \mathcal{D}_{\tau,i} = \mathcal{P}_{\tau,-i}.$$

(iii) $\mathbf{x}_\tau = s \cdots s r \cdots r$ with $n_s := \#s(\mathbf{x}_\tau) \geqslant 1$ and $n_r := \#r(\mathbf{x}_\tau) \geqslant 1$

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dotminus\cdots\dotminus \\ = \\ + \\ + \\ + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array} \qquad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ + \\ = \\ = \\ + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array} \qquad \forall i \geqslant 1$$

We have

$$\mathcal{L}_\tau = \sum_{i=0}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}.$$

Here

$$\begin{aligned} \mathcal{B}_{\tau,0} &= \sum_{i=1}^{k} \mathcal{T}_{\tau,i} \cdot \begin{array}{l} \dotminus\cdots\dotminus \\ = \end{array} \cdot \ddagger_{2n_r-2,2n_s-2}, & \mathcal{D}_{\tau,0} &= \begin{array}{c} + \\ + \end{array}, \\ \mathcal{B}_{\tau,i} &= \mathcal{T}_{\tau,-i} \cdot \ddagger_{2n_r-2,2n_s}, & \mathcal{D}_{\tau,i} &= \begin{array}{l} \divideontimes\cdots\divideontimes \\ + \end{array}, \quad \forall i \geqslant 1 \end{aligned}$$

(iv) $\mathbf{x}_\tau = s \cdots s c$ where $n_s := \#s(\mathbf{x}_\tau) \geqslant 0$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dotminus\cdots\dotminus \\ + \\ = \\ \vdots \\ = \end{array} \qquad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ = \\ = \\ \vdots \\ = \end{array} \qquad \forall i \geqslant 1 \tag{5.19}$$

We have

$$\mathcal{L}_\tau = \sum_{i=1}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i} + \sum_{i=1}^{k} \mathcal{B}_{\tau,-i} \cdot \mathcal{D}_{\tau,-i}.$$

For $i \geqslant 1$,

$$\mathcal{B}_{\tau,i} = \mathcal{T}_{\tau,i} \cdot \ddagger_{0,2n_s}, \qquad \mathcal{D}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \end{array},$$

$$\mathcal{B}_{\tau,-i} = \mathcal{T}_{\tau,-i} \cdot \ddagger_{0,2n_s}, \qquad \mathcal{D}_{\tau,-i} = \begin{array}{l} \divideontimes \cdots \divideontimes \\ = \end{array}.$$

(v) $\mathbf{x}_\tau = r \cdots rc$ where $n_r = \#r(\mathbf{x}_\tau) \geqslant 1$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \\ + \\ \vdots \\ + \end{array} \qquad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes \cdots \divideontimes \\ = \\ + \\ \vdots \\ + \end{array} \qquad \forall i \geqslant 1$$

We have

$$\mathcal{L}_\tau = \sum_{i=0}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}.$$

where

$$\mathcal{B}_{\tau,0} = \sum_{i=1}^{k} \mathcal{T}_{\tau,-i} \cdot \begin{array}{l} \divideontimes \cdots \divideontimes \\ + \end{array} \cdot \ddagger_{2n_r-2,0}, \qquad \mathcal{D}_{\tau,0} = \begin{array}{l} = \\ + \end{array},$$

$$\mathcal{B}_{\tau,i} = \mathcal{T}_{\tau,i} \cdot \ddagger_{2n_r,0}, \qquad \mathcal{D}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \end{array} \quad \forall i \geqslant 1.$$

(vi) $\mathbf{x}_\tau = s \cdots sr \cdots rc$ where $n_s := \#s(\mathbf{x}_\tau) \geqslant 1, n_r := \#r(\mathbf{x}_\tau) \geqslant 1$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \\ + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array} \succ \begin{array}{l} + \\ = \end{array}, \quad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes \cdots \divideontimes \\ = \\ + \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array} \succ \begin{array}{l} + \\ = \end{array} \quad \forall i \geqslant 1$$

Now the factorization (5.4) holds.

(vii) $\mathbf{x}_\tau = s \cdots sd$ where $\#s(\mathbf{x}_\tau) \geqslant 1$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-} \cdots \dot{-} \\ + \\ - \\ \vdots \\ - \end{array} \qquad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes \cdots \divideontimes \\ - \\ - \\ \vdots \\ - \end{array} \qquad \forall i \geqslant 1 \tag{5.20}$$

We have

$$\mathcal{L}_\tau = \sum_{i=0}^{k} \mathcal{B}_{\tau,-i} \cdot \mathcal{D}_{\tau,-i}.$$

where

$$\mathcal{B}_{\tau,0} = \sum_{i=1}^{k} \mathcal{T}_{\tau,i} \cdot \begin{array}{l} \dot{-} \cdots \dot{-} \\ - \end{array} \cdot \dagger_{0,2n_s-2}, \qquad \mathcal{D}_{\tau,0} = \begin{array}{l} + \\ - \end{array},$$

$$\mathcal{B}_{\tau,-i} = \mathcal{T}_{\tau,-i} \cdot \ddagger_{0,2n_s}, \qquad \mathcal{D}_{\tau,-i} = \begin{array}{l} \divideontimes \cdots \divideontimes \\ - \end{array} \quad \forall i \geqslant 1.$$

(viii) $\mathbf{x}_\tau = r\cdots rd$ where $n_r = \#r(\mathbf{x}_\tau) \geqslant 1$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-}\cdots\dot{-} \\ + \\ + \\ \vdots \\ + \end{array} \qquad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ - \\ + \\ \vdots \\ + \end{array} \qquad \forall i \geqslant 1$$

We have

$$\mathcal{L}_\tau = \sum_{i=0}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i}.$$

where

$$\begin{aligned} \mathcal{B}_{\tau,0} &= \sum_{i=1}^{k} \mathcal{T}_{\tau,-i} \cdot \begin{array}{l} \divideontimes\cdots\divideontimes \\ + \end{array} \cdot \dagger_{2n_r-2,0}, & \mathcal{D}_{\tau,0} &= \begin{array}{l} - \\ + \end{array}, \\ \mathcal{B}_{\tau,i} &= \mathcal{T}_{\tau,i} \cdot \dagger_{2n_r,0}, & \mathcal{D}_{\tau,i} &= \begin{array}{l} \dot{-}\cdots\dot{-} \\ + \end{array} \quad \forall i \geqslant 1. \end{aligned}$$

(ix) $\mathbf{x}_\tau = s\cdots sr\cdots rd$ where $\#s(\mathbf{x}_\tau) \geqslant 1, \#r(\mathbf{x}_\tau) \geqslant 1$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-}\cdots\dot{-} \\ + \\ + \\ \vdots \\ + \\ - \\ \vdots \\ - \end{array} \succ \begin{array}{l} + \\ - \end{array}, \quad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ - \\ + \\ \vdots \\ + \\ - \\ \vdots \\ - \end{array} \succ \begin{array}{l} + \\ - \end{array} \quad \forall i \geqslant 1.$$

Now the factorization (5.4) holds.

(x) $\mathbf{x}_\tau = s\cdots sr\cdots rcd$ where $\#s(\mathbf{x}_\tau) \geqslant 0, \#r(\mathbf{x}_\tau) \geqslant 0$.

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-}\cdots\dot{-} \\ + \\ + \\ \vdots \\ + \\ - \\ \vdots \\ - \end{array} \succ \begin{array}{l} + \\ - \end{array}, \quad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ - \\ + \\ \vdots \\ + \\ - \\ \vdots \\ - \end{array} \succ \begin{array}{l} + \\ - \end{array} \quad \forall i \geqslant 1.$$

Now the factorization (5.4) holds.

(xi) $\mathbf{x}_\tau = d$. In this case, $\mathbf{p}_\tau = dd\cdots$

$$\mathcal{P}_{\tau,i} = \begin{array}{l} \dot{-}\cdots\dot{-} \\ + \end{array} \qquad \mathcal{P}_{\tau,-i} = \begin{array}{l} \divideontimes\cdots\divideontimes \\ - \end{array} \qquad \forall i \geqslant 1$$

We have

$$\mathcal{L}_\tau = \sum_{i=0}^{k} \mathcal{B}_{\tau,i} \cdot \mathcal{D}_{\tau,i} + \sum_{i=0}^{k} \mathcal{B}_{\tau,-i} \cdot \mathcal{D}_{\tau,-i}.$$

where

$$\mathcal{B}_{\tau,i} = \mathcal{T}_{\tau,i}, \quad \mathcal{D}_{\tau,i} = \mathcal{P}_{\underline{\tau},i} \quad \forall i = -k, \cdots, -1, 1, \cdots, k.$$

5.4.5. *Summary of the factorization.* We have the non-vanishing of the lifted local system $\mathcal{L}_\tau$ and the factorization described in Table 2 holds.

5.4.6. *Unipotent representations attached to $\mathcal{O}'$.* In this section, we let $\tau' \in \mathrm{DRC}(\mathcal{O}')$ and $\tau'' = \overline{\nabla}_1(\tau') \in \mathrm{DRC}(\mathcal{O}'')$. First note that $x_{\tau''} \neq s$ and so $\mathcal{L}^+_{\tau''}$ always non-zero.

Recall (5.11). We claim that we can recover $\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''}$ from $\mathcal{L}_{\tau'}$:

Claim 5.3. *The map $\Omega_{\mathcal{O}'} \colon \mathcal{L}_{\tau'} \mapsto (\mathcal{L}^+_{\tau''}, \mathcal{L}^-_{\tau''})$ is a well defined map.*

Proof. When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has two elements, $x_{\tau''} = d$ and ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'}) = \{ (q''_1, p''_1 - 1), (q''_1 - 1, p''_1) \}$. In (5.11), $\dagger\mathcal{L}^+_{\tau''}$ (resp. $\dagger\mathcal{L}^-_{\tau''}$) consists of components whose first column has siginature $(q''_1, p''_1 - 1)$ (resp. $(q''_1 - 1, p''_1)$). When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has only one elements, $x_\tau = r/c$, ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'}) = \{ (q''_1, p''_1 - 1) \}$, $\mathcal{L}_{\tau'} = \dagger\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''} = 0$

In any case, we get

$$\mathrm{Sign}(\tau'') = (n_0, n_0) + (p_0, q_0) \quad \text{where } 2n_0 = |\nabla(\mathcal{O}'')| \,. \tag{5.21}$$

Using the signature $\mathrm{Sign}(\tau'')$, we could recover the twisting characters in the theta lifting of local system. Therefore $\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''}$ can be recovered from $\mathcal{L}_{\tau'}$. □

Claim 5.4. *Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$. Then $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.*

Proof. It suffice to consider the case when $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (5.21) in the proof of Claim 5.3, $\mathrm{Sign}(\tau''_1) = \mathrm{Sign}(\tau''_2)$. On the other hand, $\epsilon_1 = \epsilon_2 = 0$ and $\tau''_1 \neq \tau''_2$ by Lemma 3.5. So

$$\pi_{\tau'_1} = \bar{\Theta}(\pi_{\tau''_1}) \neq \bar{\Theta}(\pi_{\tau''_2}) = \pi_{\tau'_2}$$

by the injectivity of theta lift. □

5.4.7. *Unipotent representations attached to $\mathcal{O}$.*

Claim 5.5. *Suppose $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$. Then $\pi_{\tau_1} \neq \pi_{\tau_2}$.*

Proof. It suffice to consider the case that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. On the other hand, the twisting $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ since it is determined by the local system: $\epsilon_{\tau_i} = 0$ if and only if $\mathcal{L}_{\tau_i} \supset -$.

We conclude that $\tau''_1 \neq \tau''_2$ and $\tau'_1 \neq \tau'_2$ by Lemma 3.7 and Lemma 3.5. By Claim 5.4 and the injectivity of theta lift,

$$\pi_{\tau_1} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_1}} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_2}} = \pi_{\tau_2}$$

□

5.4.8. *Noticed orbits.* In this section, we prove the claims about noticed and +noticed orbits.

Claim 5.6. *Suppose $\mathcal{O}'$ is noticed. Then*

(i) The map $\{ \mathcal{L}_{\tau''} \mid x_{\tau''} \neq s \} \longrightarrow \{ \mathcal{L}_{\tau'} \} = {}^\ell\mathrm{LS}(\mathcal{O}')$ given by $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''})$ is a bijection.

(ii) The map $\mathrm{DRC}(\mathcal{O}') \to {}^\ell\mathrm{LS}(\mathcal{O}')$ given by $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection.

Proof. (i) Note that $\mathcal{O}''$ is noticed by definition. Hence $\Upsilon_{\mathcal{O}''}$ is invertible. Recall Claim 5.3, we see that

$$(\Upsilon_{\mathcal{O}''})^{-1} \circ \Omega_{\mathcal{O}'} \colon \mathcal{L}_{\tau'} \mapsto (\mathcal{L}^+_{\tau''}, \mathcal{L}^-_{\tau''}) \mapsto \mathcal{L}_{\tau''}$$

gives the inverse of the map in the claim.

(ii) Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$. By Lemma 3.5, $\tau''_1 \neq \tau''_2$. Hence $\mathcal{L}_{\tau''_1} \neq \mathcal{L}_{\tau''_2}$ by the induction hypothesis. Therefore, $\mathcal{L}_{\tau'_1} \neq \mathcal{L}_{\tau'_2}$ by (i) of the claim. □

Claim 5.7. *Suppose $\mathcal{O}$ is noticed, $\mathcal{L} \colon \mathrm{DRC}(\mathcal{O}) \mapsto {}^\ell\mathrm{LS}(\mathcal{O})$ given by $\tau \mapsto \mathcal{L}_\tau$ is bijective.*

Proof. We prove the claim by contradiction. We assume $\tau_1 \neq \tau_2$ such that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$.

Recall (5.12): there is an pair of non-negative integers $(e_i, f_i) = \mathrm{Sign}(\mathbf{u}_\tau)$ such that

$$\mathcal{L}_{\tau_i} = \dagger\dagger\mathcal{L}^+_{\tau''_i} \cdot \mathcal{E}^+_{\tau_i} + \dagger\dagger\mathcal{L}^-_{\tau''} \cdot \mathcal{E}^-_{\tau_i}.$$

(i) Suppose that $\mathcal{L}_{\tau_i}$ contains a 1-row marked by $-$ or $=$. Then we have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$, which can be read from the mark $-/=$. Moreover, the term $\dagger\dagger\mathcal{L}^+_{\tau''_i} \cdot \mathcal{E}^+_{\tau_i} \neq 0$ and we can recover it from $\mathcal{L}_{\tau_i}$. So $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$. Since $\mathcal{O}''$ is +noticed, $\tau''_1 = \tau''_2$ by (5.8) and the induction hypothesis. Note that $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by Lemma 3.7 which is contradict to our assumption.

(ii) Now we assume that $\mathcal{L}_{\tau_i}$ does not contains a 1-row marked by $-/=$. By checking the list in Sections 5.4.2 to 5.4.4, we conclude that $\mathbf{p}_{\tau_i}$ must have one of the following form:

$$\mathbf{p}_1 = \begin{smallmatrix} r & c \\ \vdots & \\ r & \\ r & \end{smallmatrix}, \quad \mathbf{p}_2 = \begin{smallmatrix} r & c \\ \vdots & \\ r & \\ c & \end{smallmatrix}, \quad \text{and} \quad \mathbf{p}_3 = \begin{smallmatrix} r & d \\ \vdots & \\ r & \\ r & \end{smallmatrix}$$

We consider case by case according to the set $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\}$:

(a) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_1 \,\}$ or $\{\, \mathbf{p}_2 \,\}$. In these cases, $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by Lemma 3.7. On the other hand, $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ by (5.12). Since $\mathcal{O}''$ is +noticed, we get $\tau''_1 = \tau''_2$ a contradiction.

(b) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_3 \,\}$. We still have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by Lemma 3.7. On the other hand, $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$ by (5.12). This again contradict to the injectivity of $\mathcal{L}_{\tau''} \mapsto \mathcal{L}^-_{\tau''}$.

(c) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_1, \mathbf{p}_2 \,\}$. The the argument above, we have $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ Now $\tau''_1 \neq \tau''_2$ since $\{\, x_{\tau''_1}, x_{\tau''_2} \,\} = \{\, r, c \,\}$. This contradict to that $\mathcal{O}''$ is +noticed again.

(d) Suppose $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ or $\mathbf{p}_2, \mathbf{p}_{\tau_2} = \mathbf{p}_3$. Then $\mathcal{L}_{\tau_1} = \dagger\dagger\mathcal{L}^+_{\tau''_1} \cdot \mathcal{E}^+_{\tau''_1}$ and $\mathcal{L}_{\tau_2} = \dagger\dagger\mathcal{L}^-_{\tau''_2} \cdot \mathcal{E}^-_{\tau''_2}$. This implies $\dagger\mathcal{L}^+_{\tau''_1} = \dagger\mathcal{L}^-_{\tau''_2}$ and $|\mathrm{Sign}(\tau''_1)| \equiv |\mathrm{Sign}(\tau''_2)| + 2 \pmod 4$.

By the list in Sections 5.4.2 to 5.4.4, we see that $\mathcal{L}_{\tau''_2} \supset \begin{smallmatrix} - \\ + \end{smallmatrix}$ and $\mathcal{L}^-_{\tau''_2} \supset +$.

This leads to a contradiction: Thanks to the character twist in the theta lifting formula of the local system, we see that the the associated character restricted on the 2-row $-\ +$ of $\mathcal{L}^+_{\tau_1}$ and $\mathcal{L}^-_{\tau_2}$ must be are different.

We finished the proof of the claim. $\square$

5.4.9. *The maps $\Upsilon^+_{\mathcal{O}}$, $\Upsilon^-_{\mathcal{O}}$ and $\Upsilon_{\mathcal{O}}$.*

Claim 5.8. *Suppose $\mathcal{O}$ is +noticed. The map $\Upsilon^+_{\mathcal{O}} \colon \{\, \mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0 \,\} \to \{\, \mathcal{L}^+_\tau \neq 0 \,\}$ is injective.*

Proof. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $\succ \begin{smallmatrix} + \\ + \end{smallmatrix}$. This is equivalent to $\mathcal{L}^+_\tau$ has an irreducible component $\succ +$. Now all irreducible components of $\mathcal{L}_\tau \succ +$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^+_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^+_\tau$.

(ii) Suppose that $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ do not contain a component $\succ \begin{smallmatrix} + \\ + \end{smallmatrix}$.

By Claim 5.7 $\tau_1 \neq \tau_2$.[5] We now show that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2} \neq 0$ leads to a contradiction. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$ By checking the list in Sections 5.4.2 to 5.4.4, for $i = 1, 2$,

[5] $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ clearly implies $\tau_1 \neq \tau_2$.

we have

$$\mathbf{p}_{\tau_i} = \begin{array}{ll} s & x_{\tau''} \\ \vdots & \\ s & \\ x_{\tau_i} & \end{array} \qquad \text{where } x_{\tau_i''} = c/d, x_{\tau_i} = c/d.$$

By looking at (5.19) and (5.20), we see that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$ implies

(5.22) $$\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}.$$

Moreover, we see that every components of $\mathcal{L}_{\tau_i}$ has a 1-row of mark "$-/=$" and we can determine ϵ_{τ_i} by the mark $-/=$ of the 1-row appeared in $\mathcal{L}^+_{\tau_i}$. In particular, we have $\epsilon_1 = \epsilon_2$. Now Lemma 3.7 implies $\tau_1'' \neq \tau_2''$. This is contradict to (5.22) and our induction hypothesis since $\mathcal{O}''$ is also +noticed.

□

Claim 5.9. *Suppose $\mathcal{O}$ is noticed. Then the map $\Upsilon_{\mathcal{O}}\colon \mathcal{L}_\tau \mapsto (\mathcal{L}^+_\tau, \mathcal{L}^-_\tau)$ is injective.*

Proof. It suffice to consider the case where $\mathcal{O}$ is noticed but not +noticed, i.e. $C_{2k+1} = C_{2k} + 1$. In this case, $n = 1$. The peduncle $\mathbf{p}_\tau$ have four possible cases:

$$r \ \ c\ , \qquad c \ \ c\ , \qquad c \ \ d\ , \qquad \text{or} \qquad d \ \ d\ .$$

By (5.12), $\mathcal{L}^+_\tau = \dagger\dagger\mathcal{L}^+_{\tau''}$.

Now suppose $\tau_1 \neq \tau_2$ such that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$. Since $\mathcal{O}''$ is +noticed, we have $\tau_1'' = \tau_2''$ by the induction hypothesis (see Claim 5.8). By Lemma 3.7, this only happens when $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\ c\ \ d\ ,\ d\ \ d\ \}$. Since one of $\mathcal{L}^-_{\tau_i}$ is zero and the other is non-zero, we conclude that $\Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_1}) \neq \Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_2})$. □

Claim 5.10. *Suppose $\mathcal{O}$ is +noticed. The map $\Upsilon^-_{\mathcal{O}}\colon \{\mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0\} \to \{\mathcal{L}^-_\tau \neq 0\}$ is injective.*

Proof. The proof is similar to that of Claim 5.8. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $\succ \begin{smallmatrix} - \\ - \end{smallmatrix}$. This is equivalent to $\mathcal{L}^-_\tau$ has an irreducible component $\succ -$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^-_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^-_\tau$.

(ii) Now we make a weaker assumption that $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$ such that $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ do not contain a component $\succ \begin{smallmatrix} - \\ - \end{smallmatrix}$.

By checking the list in Sections 5.4.2 to 5.4.4, for $i = 1, 2$, we have $\mathbf{p}_{\tau_i}$ must be one of the following

$$\mathbf{p}_1 = \begin{array}{ll} r & c \\ \vdots & \\ r & \\ d & \end{array} \qquad \text{or} \qquad \mathbf{p}_2 = \begin{array}{ll} r & d \\ \vdots & \\ r & \\ d & \end{array}.$$

Without of loss of generality, we have the following possibilities:

(a) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_1$. We have $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$ and obtain the contradiction.

(b) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_2$. We have $\mathcal{L}^-_{\tau_1''} = \mathcal{L}^-_{\tau_2''}$ and obtain the contradiction.

(c) $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ and $\mathbf{p}_{\tau_2} = \mathbf{p}_2$. We get $\dagger\mathcal{L}^+_{\tau_1''} = \dagger\mathcal{L}^-_{\tau_2''}$. Note that $x_{\tau_1''} = r$. We inspect the list Sections 5.4.2 to 5.4.4 for the generalized descent case and the factorization (5.3) for the descent case. We conclude that $\mathcal{L}^+_{\tau_1''}$ must contain a 1-row with $+$ mark. We obtain the contradiction to $\dagger\mathcal{L}^+_{\tau_1''} = \dagger\mathcal{L}^-_{\tau_2''}$ by looking at the associated character on the $-\ +$ row by the same argument in the case (d) of the proof of Claim 5.7.

This finished the proof. □

6. Type B/M case

6.1. "Special" and "non-special" diagrams.

6.1.1. *type M.*

Definition 6.1. *Let* $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0) \in \mathrm{Nil}^{\mathrm{dpe}}(M)$ *with* $k \geqslant 1$. *Let*

$$\tau = (\tau_{2k}, \cdots, \tau_2) \times (\tau_{2k-1}, \cdots, \tau_1).$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(M)$. *We say* τ *has*

- special shape *if* $\tau_{2k} \geqslant \tau_{2k-1}$*;*
- non-special shape *if* $\tau_{2k} < \tau_{2k-1}$.

Clearly, this gives a partition of the set of Weyl group representations attached to $\mathcal{O}$.

Suppose $k \geqslant 1$, $C_{2k} = 2c_{2k} - 1$ *and* $C_{2k-1} = 2c_{2k-1} + 1$. *The special shape representation* τ^s

$$\tau^s = \tau^s_L \times \tau^s_R = (c_{2k}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k-1}, \tau_{2k-3} \cdots, \tau_1).$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau^{ns}_L \times \tau^{ns}_R = (c_{2k-1}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k}, \tau_{2k-3}, \cdots, \tau_1)$$

In the paring $\tau^s \leftrightarrow \tau^{ns}$, τ_j *is unchanged for* $j \leqslant 2k-2$. *We call* τ^s *the* special shape *and* τ^{ns} *the corresponding* non-special shape

Definition 6.2. *Suppose* C_{2k} *is odd. We retain the notation in Definition 6.1 where the shapes* τ^s *and* τ^{ns} *correspond with each other. We define a bijection between* $\mathrm{DRC}(\tau^s)$ *and* $\mathrm{DRC}(\tau^{ns})$*:*

$$\begin{array}{ccc} \mathrm{DRC}(\tau^s) & \longleftrightarrow & \mathrm{DRC}(\tau^{ns}) \\ \boldsymbol{\tau}^s & \longleftrightarrow & \boldsymbol{\tau}^{ns} \end{array}$$

Without loss of generality, we can assume that $\boldsymbol{\tau}^s \in \mathrm{DRC}(\tau^s)$ *and* $\boldsymbol{\tau}^{ns} \in \mathrm{DRC}(\tau^{ns})$ *have the following shapes:*

$$\boldsymbol{\tau}^s : \begin{array}{l} \cdots\cdots\cdots \\ y_1 \cdots\cdots \\ s \\ \vdots \\ s \\ y_2 \end{array} \times \begin{array}{l} \cdots\cdots\cdots \\ x_0 \cdots\cdots \\ \\ \\ \\ \\ \end{array} \quad \longleftrightarrow \quad \boldsymbol{\tau}^{ns} : \begin{array}{l} \cdots\cdots\cdots \\ y_0 \cdots\cdots \\ \\ \\ \\ \\ \end{array} \times \begin{array}{l} \cdots\cdots\cdots \\ x_1 \cdots\cdots \\ r \\ \vdots \\ r \\ x_2 \end{array} \tag{6.1}$$

Here the grey parts have length $(C_{2k} - C_{2k-1})/2$ *(could be zero length). The entries* x_0, x_1, y_0 *and* y_1 *are either all empty or all non-empty. The correspondence between* (x_0, x_1, x_2)*, with* (y_0, y_1, y_2) *is given by switching* r *and* s, d *and* c. *i.e.*

$$\begin{cases} y_i = s \Leftrightarrow x_i = r \\ y_i = c \Leftrightarrow x_i = d \end{cases} \quad \text{for } i = 0, 1, 2.$$

We define

$$\mathrm{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \mathrm{DRC}(\tau^s) \quad \text{and} \quad \mathrm{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \mathrm{DRC}(\tau^{ns})$$

where τ^s *(resp.* τ^{ns}*)runs over all specail (resp. non-special) shape representations attached to* $\mathcal{O}$. *Clearly* $\mathrm{DRC}(\mathcal{O}) = \mathrm{DRC}^s(\mathcal{O}) \sqcup \mathrm{DRC}^{ns}(\mathcal{O})$.

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

6.1.2. *Special and non-special shapes of type B.* Now we define the notion of special and non-special shape of type B.

Definition 6.3. *Let*

$$\tau = (\tau_{2k}, \cdots, \tau_2) \times (\tau_{2k+1}, \tau_{2k-1}, \cdots, \tau_1)$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(B)$. *We say* τ *has*

- special shape *if* $\tau_{2k} \geqslant \tau_{2k-1}$*;*
- non-special shape *if* $\tau_{2k} < \tau_{2k-1}$.

Suppose $k \geqslant 1$ *and*

(6.2) $\mathcal{O} = (C_{2k+1} = 2c_{2k+1} + 1, C_{2k} = 2c_{2k} - 1, C_{2k-1} = 2c_{2k-1} + 1, \cdots, C_1 > 0, C_0 = 0).$

A representation τ^s *attached to* $\mathcal{O}$ *has the shape*

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \cdots, \tau_1)$$

is paired with

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k-1}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k+1}, c_{2k}, \tau_{2k-3}, \cdots, \tau_1).$$

In the paring $\tau^s \leftrightarrow \tau^{ns}$ *described as the above,* τ_j *are unchanged for* $j \leqslant 2k-2$.

6.2. **Definition of $\overline{\nabla}$ for type B and M.**

6.2.1. *The defintion of* ϵ.

(1). When $\tau \in \mathrm{DRC}(B)$, ϵ is determined by the "basal disk" of τ (see (6.3) for the definition of x_τ):

$$\epsilon_\tau := \begin{cases} 0, & \text{if } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

(2). When $\tau \in \mathrm{DRC}(M)$, the ϵ is determined by the lengths of $\tau_{L,0}$ and $\tau_{R,0}$:

$$\epsilon_\tau := \begin{cases} 0, & \text{if } |\tau_{L,0}| - |\tau_{R,0}| \geqslant 0; \\ 1, & \text{if } |\tau_{L,0}| - |\tau_{R,0}| < 0. \end{cases}$$

6.2.2. *Initial cases.*

(3). Suppose $\mathcal{O} = (2c_1 + 1)$ has only one column. Then $\mathcal{O}'_0 := \overline{\nabla}(\mathcal{O})$ is the trivial orbit of $\mathrm{Mp}(0, \mathbb{R})$. $\mathrm{DRC}(\mathcal{O})$ consists of diagrams of shape $\tau_L \times \tau_R = \varnothing \times (c_1,)$. The set $\mathrm{DRC}(\mathcal{O}'_0)$ is a singleton $\{\, \tau'_0 = \varnothing \times \varnothing \,\}$, and every element $\tau \in \mathrm{DRC}(\mathcal{O})$ maps to τ_0 (see (2.1)):

$$\mathrm{DRC}(\mathcal{O}) \ni \tau := \quad \begin{array}{c} \varnothing \\ \\ \\ \\ \end{array} \times \begin{array}{c} x_1 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau'_0 := \varnothing \times \varnothing$$

Here $m_\tau = a$ or b is the mark of τ. We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := x_1 \cdots x_n m_\tau.$$

We call $\mathbf{p}_\tau = \mathbf{x}_\tau$ the "peduncle" part of τ.

6.2.3. *The descent from M to B.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

(4). Suppose $|\tau_{L,0}| \geqslant |\tau_{R,0}|$, i.e. τ has special shape. Keep r, d unchanged, delete $\tau_{L,0}$ and fill the remaining part with "$\bullet$" and "s" by $\overline{\nabla}_{\mathrm{R}}(\tau^{\bullet})$ using the dot-s switching algorithm. The mark $m_{\tau'}$ of $\tau' := \overline{\nabla}_1(\tau)$ is given by the following formula:

$$m_{\tau'} := \begin{cases} a & \text{if } \tau_{L,0} \text{ ends with } s \text{ or } \bullet, \\ b & \text{if } \tau_{L,0} \text{ ends with } c. \end{cases}$$

(5). Suppose $|\tau_{L,0}| < |\tau_{R,0}|$, i.e. τ has non-special shape. We define

$$\overline{\nabla}_1(\tau) := \overline{\nabla}_1(\tau^s)$$

where τ^s is the special diagram corresponding to τ defined in Definition 6.2.

Lemma 6.4. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} is odd. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1}, \cdots, C_0)$. Then*

$$\mathrm{DRC}^s(\mathcal{O}) \xrightarrow{\overline{\nabla}_1} \mathrm{DRC}(\mathcal{O}')$$

$$\mathrm{DRC}^{ns}(\mathcal{O}) \xrightarrow{\overline{\nabla}_1} \mathrm{DRC}(\mathcal{O}')$$

are a bijections.

Proof. The claim for $\mathrm{DRC}^s(\mathcal{O})$ is clear by the definition of descent. The claim for $\mathrm{DRC}^{ns}(\mathcal{O})$ reduces to that of $\mathrm{DRC}^s(\mathcal{O})$ using Definition 6.2. $\square$

Lemma 6.5. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} even. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1} + 1, \cdots, C_0)$. Then the following map is a bijection*

$$\mathrm{DRC}(\mathcal{O}) \xrightarrow{\overline{\nabla}_1} \left\{ \tau' \in \mathrm{DRC}(\mathcal{O}') \,\middle|\, \begin{array}{l} m_{\tau'} \neq b \text{ and} \\ \tau'_{R,0} \text{ dose not ends with } s. \end{array} \right\}.$$

Proof. This is clear by the descent algorithm. $\square$

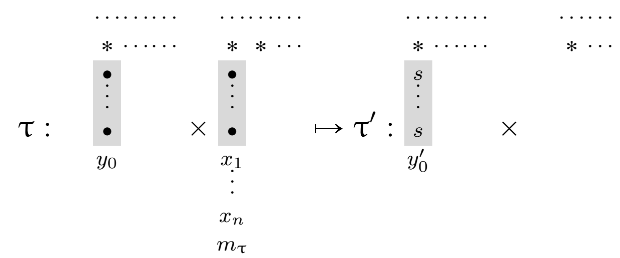
6.2.4. *Descent from B to M.* Now we define the general case of the descent from type B to type M. We assume that $k \geqslant 1$ i.e. $\mathcal{O}$ has at least 3 columns. First note that the shape of $\tau' = \overline{\nabla}(\tau)$ is the shape of τ deleting the most left column on the right diagram.

The definition splits in cases below. In all these cases we define

$$\tilde{\mathbf{x}}_\tau := x_1 \cdots x_n m_\tau \text{ and } \tilde{x}_\tau := x_n \tag{6.3}$$

which is marked by $s/r/c/d$. For the part marked by $*/\cdots$, $\overline{\nabla}$ keeps r, c, d and maps the rest part consisting of $\bullet$ and s by dot-s switching algorithm.

(6). When C_{2k} is odd and τ has special shape, the descent is given by the following diagram



Here the grey columns has length $(C_{2k} - C_{2k-1})/2$ and $n = (C_{2k+1} - C_{2k})/2$. The "$*/\cdots$" part of τ' is given by keeping the corresponding entries marked by $r/d/c$

in τ unchange and filling $s/\bullet$ accordingly in the rest of the entries. The entry y_0' of τ' is given by the following formula:

$$y_0' := \begin{cases} s & \text{if } x_1 = \bullet \text{ or } (x_1, m_\tau) = (r/d, a) \\ c & \text{if } x_1 = s \text{ or } (x_1, m_\tau) = (r/d, b). \end{cases}$$

(7). When C_{2k} is odd and τ has non-special shape, then

$$\tau : \begin{array}{l} \cdots\cdots\cdots \\ * \cdots\cdots \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \times \begin{array}{ll} \cdots\cdots & \cdots \\ * & * \cdots \\ s & r \\ \vdots & \vdots \\ s & r \\ x_1 & x_0 \\ \vdots & \\ x_n & \\ m_\tau & \end{array} \mapsto \tau' : \begin{array}{l} \cdots\cdots\cdots \\ * \cdots\cdots \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \times \begin{array}{l} \cdots\cdots \\ * \cdots \\ r \\ \vdots \\ r \\ x_0' \\ \\ \\ \\ \end{array}$$

Here the grey columns has length $(C_{2k} - C_{2k-1})/2$ and $n = (C_{2k+1} - C_{2k})/2$. The "$*/\cdots$" part of τ' is given by keeping the corresponding entries marked by $r/d/c$ in τ unchange and filling $s/\bullet$ accordingly in the rest of the entries. The entry x_0' of τ' is given by the following formula:

$$x_0' := \begin{cases} r & \text{if } (x_1, m_\tau) = (r/d, a), \\ d & \text{if } (x_1, m_\tau) = (r/d, b), \\ x_0 & \text{if } x_1 = s. \end{cases}$$

Note that

(8). When $C_{2k} = C_{2k-1}$ is even, we could assume τ and τ' hase the following forms with $n = (C_{2k+1} - C_{2k} + 1)/2$.

$$\tau : \begin{array}{l} \cdots\cdots\cdots \\ * \cdots\cdots \\ y_0 \cdots \\ \\ \\ \\ \end{array} \times \begin{array}{ll} \cdots\cdots & \cdots \\ * & * \cdots \\ x_1 & x_0 \\ \vdots & \\ x_n & \\ m_\tau & \end{array} \mapsto \tau' : \begin{array}{l} \cdots\cdots\cdots \\ * \cdots\cdots \\ y_0' \cdots \\ \\ \\ \\ \end{array} \times \begin{array}{l} \cdots\cdots \\ * \cdots \\ x_0' \\ \\ \\ \\ \end{array}$$

In the most of the case, τ' is obtained from τ by deleting the first column of τ_R, keeping $c/r/d$ and filling $s/\bullet$ accordingly. The exceptional cases are when $x_1 = r/d$. In the exceptional case we always have $x_0 = d$. We define

$$x_0' := d,$$
$$y_0' := \begin{cases} s & \text{if } m_\tau = a, \\ c & \text{if } m_\tau = b. \end{cases}$$

Note that this definition is similar to that of the special shape diagrams in the descent case. We define the "peduncle" of τ to be

$$\mathbf{p}_\tau = \begin{array}{ll} x_1 & x_0 \\ \vdots & \\ x_n & \\ m_\tau & \end{array}.$$

In all the cases, let $\mathcal{O}_1 = (2n,)$ is the trivial nilpotent of type D. For each $\tau \in \mathrm{DRC}(\mathcal{O})$, we define $\mathbf{x}_\tau \in \mathrm{DRC}(\mathcal{O}_1)$ by require the following identity of multisets:

$$\{\, \text{entry of } \mathbf{x}_\tau \,\} = \begin{cases} \{\, c, x_2, \cdots, x_n \,\} & \text{if } (x_1, m_\tau) = (\bullet/s, a), \\ \{\, s, x_2, \cdots, x_n \,\} & \text{if } (x_1, m_\tau) = (\bullet/s, b), \\ \{\, x_1, x_2, \cdots, x_n \,\} & \text{otherwise } (x_1 = r/d). \end{cases}$$

In the above definition we include the initial case where $\mathcal{O}$ is a trivial orbit of type B. We define $\mathbf{x}_\tau = \varnothing$ when $\mathcal{O}$ is the trivial orbit of $\mathrm{O}(1, \mathbb{C})$.

We define the "basal disk" of τ as the following:

$$x_\tau := \begin{cases} \mathbf{x}_\tau & \text{if } C_{2k+1} = C_{2k} + 1. \\ c & \text{if } x_n m_\tau = sa \text{ or } \bullet a \\ x_n & \text{otherwise.} \end{cases}$$

6.2.5. *A key proposition for descent case.* Now we assume $\mathcal{O}$ is given by (6.2) Let $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$.

The following lemma is the key property satisfied by our definition

Lemma 6.6. *Suppose $k \geqslant 1$ and τ^s and τ^{ns} are two representations attached to $\mathcal{O}$ as in Definition 6.3. Then*

i) For every $\tau \in \mathrm{DRC}(\tau^s) \sqcup \mathrm{DRC}(\tau^{ns})$, the shape of $\overline{\nabla}^2(\tau)$ is

$$\tau'' = (\tau_{2k-2}, \cdots, \tau_2) \times (c_{2k-1}, \tau_{2k-3}, \tau_1)$$

ii) We define $\delta\colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ by $\delta(\tau) = (\overline{\nabla}^2(\tau), \mathbf{x}_\tau)$. The following maps are bijective

$$\begin{array}{ccccc} \mathrm{DRC}(\tau^s) & \xrightarrow{\ \delta\ } & \mathrm{DRC}(\tau'') \times \mathrm{DRC}(\mathcal{O}_1) & \xleftarrow{\ \delta\ } & \mathrm{DRC}(\tau^{ns}) \\ \tau^s & \longmapsto & (\overline{\nabla}^2(\tau^s), \mathbf{x}_{\tau^s}) & & \\ & & (\overline{\nabla}^2(\tau^{ns}), \mathbf{x}_{\tau^{ns}}) & \longleftarrow & \tau^{ns} \end{array}$$

In particular, we obtain an one-one correspondence $\tau^s \leftrightarrow \tau^{ns}$ such that $\delta(\tau^s) = \delta(\tau^{ns})$.

iii) Suppose τ^s and τ^{ns} correspond as the above such that $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$. Then

$$\mathrm{Sign}(\tau^s) = \mathrm{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \mathrm{Sign}(\tau'') + \mathrm{Sign}(\mathbf{x}). \tag{6.4}$$

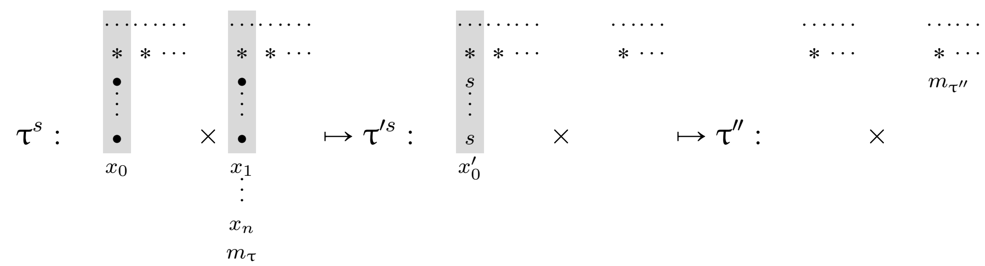
Note that $\mathrm{Sign}(\mathbf{x}) \in \mathbb{N} \times \mathbb{N}$.

Proof. By our assumption, we have $c_{2k+1} \geqslant c_{2k} > c_{2k-1}$ and

$$\tau^s = (c_{2k}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k+1}, c_{2k-1}).$$

The the behavior of τ^s under the descent map $\overline{\nabla}$ is illustrated as the following:

(6.5)



The grey part consists of totally $2(c_{2k} - 1) = C_{2k} - 1$ dots. Hence the total signature of the gray part is $(C_{2k} - 1, C_{2k} - 1)$. Note that τ^s is obtained from the "$* \cdots$" part of τ'' by attaching the grey part and $x_0, \cdots, x_n, m_\tau$.

The descent of a non-special diagram τ^{ns}:

$$
(6.6)\qquad \tau^{ns}: \begin{array}{l} \cdots\cdots\cdots \\ * \ * \cdots \\ \\ \\ \\ \\ \\ \\ \end{array} \times \begin{array}{lll} \cdots & \cdots\cdots & \\ * & * \cdots & \\ s & r & \\ \vdots & \vdots & \\ s & r & \\ x_1 & x_0 & \\ \vdots & & \\ x_n & & \\ m_\tau & & \end{array} \mapsto \tau'^{ns}: \begin{array}{l} \cdots\cdots\cdots \\ * \ * \cdots \\ \\ \\ \\ \\ \\ \\ \end{array} \times \begin{array}{l} \cdots\cdots \\ * \cdots \\ r \\ \vdots \\ r \\ x'_0 \\ \\ \\ \end{array} \mapsto \tau'': \begin{array}{l} \cdots\cdots \\ * \cdots \\ \\ \\ \\ \\ \\ \\ \end{array} \times \begin{array}{l} \cdots\cdots \\ * \cdots \\ m_{\tau''} \\ \\ \\ \\ \\ \\ \end{array}
$$

Here the gray parts in τ consists of $C_{2k}-1$ marks of $\bullet$, s, or r and s and r occur with the same multiplicity. Hence the total signature of the gray part is $(C_{2k}-1, C_{2k}-1)$.

To prove the lemma, it suffice to verify the following claims which we leave to the reader:

- In both the special shape and non-special shape cases, the following map is bijective:
$$x_0x_1\cdots x_n m_{\tau^{s/ns}} \mapsto (\mathbf{x}_{\tau^{s/ns}}, m_{\tau''}).$$
- The following equation of signatures holds
$$\mathrm{Sign}(x_0x_1\cdots x_n m_\tau) - \mathrm{Sign}(m_{\tau''}) = \mathrm{Sign}(\mathbf{x}_\tau).$$

In the verification, one can use the fact that $m_\tau = m_{\tau''}$ when $x_1 = r/d$.

[Suppose $x_1 = r/d$. Then $x_0 = c/d$ and
$$\mathrm{Sign}(x_0\cdots x_n m_\tau) - \mathrm{Sign}(m_{\tau''}) = (1,1) + \mathrm{Sign}(x_1\cdots x_n).$$

Now we consider the generic cases, i.e. $x_1 = \bullet/s$: In this case, we clain that $\mathrm{Sign}(x_0x_1) - \mathrm{Sign}(m_{\tau''}) = (1,2)$. Now
$$
\begin{aligned}
&\mathrm{Sign}(x_0x_1\cdots x_n m_\tau) - \mathrm{Sign}(m_{\tau''}) \\
=&(1,1) + \mathrm{Sign}(x_2\cdots x_n m_\tau) + (0,1) \\
=&\begin{cases} (1,1) + \mathrm{Sign}(x_2\cdots x_n) + \mathrm{Sign}(c) & \text{if } m_\tau = a \\ (1,1) + \mathrm{Sign}(x_2\cdots x_n) + \mathrm{Sign}(s) & \text{if } m_\tau = b \end{cases}
\end{aligned}
$$

First consider the special shape diagram $\tau = \tau^s$.

(i) Suppose $m_{\tau''} = a$. Then $x_0 \times x_1 = \bullet \times \bullet$.
(ii) Suppose $m_{\tau''} = b$. Then $x_0 \times x_1 = c \times s$.

Now consider the non-special shape diagram $\tau = \tau^{ns}$.

(i) Suppose $m_{\tau''} = a$. Then $\emptyset \times x_1x_0 = \emptyset \times sr$.
(ii) Suppose $m_{\tau''} = b$. Then $\emptyset \times x_1x_0 = \emptyset \times sd$.

The macthing between τ^s and τ^{ns} is also clear by the above listing of cases.] $\square$

6.2.6. *A key proposition for the generalized descent case.* Now assume C_{2k} is even. By our assumption, we have $c_{2k+1} \geqslant c_{2k} = c_{2k-1}$.

$$
(6.7)\qquad \tau: \begin{array}{l} \cdots\cdots\cdots \\ y_0 \cdots\cdots \\ \\ \\ \\ \\ \end{array} \times \begin{array}{ll} \cdots & \cdots\cdots \\ x_1 & x_0 \cdots \\ x_2 & \\ \vdots & \\ x_n & \\ m_\tau & \end{array} \mapsto \tau': \begin{array}{l} \cdots\cdots\cdots \\ y'_0 \cdots\cdots \\ \\ \\ \\ \\ \end{array} \times \begin{array}{l} \cdots\cdots \\ x'_0 \cdots \\ \\ \\ \\ \\ \end{array} \mapsto \tau'': \begin{array}{l} \cdots\cdots \\ \cdots\cdots \\ \\ \\ \\ \\ \end{array} \times \begin{array}{l} \cdots\cdots \\ x'' \cdots \\ m_{\tau''} \\ \\ \\ \\ \end{array}
$$

Here the gray parts in τ are two columns of $\bullet$ of length $C_{2k}/2-1$.

Lemma 6.7. *In* (6.7), $x'' = x_0$. *The map* $\delta\colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ *given by* $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathbf{x}_\tau)$ *is injective. Moreover,*

$$\mathrm{Sign}(\tau) = \mathrm{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \mathrm{Sign}(\mathbf{x}_\tau).$$

The map $\tilde{\delta}\colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective.*

Proof. The claim $x'' = x_0$, the injectivity of δ and the siginature formula follows directly from our algorithm.

Now the injectivity of $\tilde{\delta}$ follows from $\mathbf{x}_\tau \mapsto (\mathrm{Sign}(\mathbf{x}_\tau), \epsilon)$ is injective by Claim 5.2. [We can fully recover τ from $\mathbf{x}_\tau$ and τ''.

Suppose $\mathbf{x}_\tau$ dose not contains s or c. Then $x_1 = r/d$, $m_\tau = m_{\tau''}$, $y_0 = c$. Hence the claim holds.

$$\mathrm{Sign}(\tau) = \mathrm{Sign}(\tau'') + (C_{2k} - 2, C_{2k} - 2) + \mathrm{Sign}(c) + \mathrm{Sign}(\mathbf{x}_\tau)$$

Now assume $\mathbf{x}_\tau$ contain at least one s or c. Now

$$m_\tau = \begin{cases} a & \text{if } \mathbf{x}_\tau \text{ contains } c \\ b & \text{otherwise} \end{cases}$$

Therefore, $\mathrm{Sign}(x_2 \cdots x_n m_\tau) = \mathrm{Sign}(\mathbf{x}_\tau) - (0, 1)$.

Clearly, we can recover $x_2 \cdots x_n$ from $\mathbf{x}_\tau$ by deleting the c (if it exists) or a s. On the other hand, (y_0, x_1) is completely determined by $m_{\tau''}$:

$$(y_0, x_1) = \begin{cases} (\bullet, \bullet) & \text{if } m_{\tau''} = a \\ (c, s) & \text{if } m_{\tau''} = b \end{cases}$$

Hence $\mathrm{Sign}(y_0 x_1) = \mathrm{Sign}(m_{\tau''}) + (1, 2)$.

Now $\mathrm{Sign}(y_0 x_0 x_1 \cdots x_n m_\tau) = \mathrm{Sign}(\mathbf{x}_\tau) + \mathrm{Sign}(x'' m_{\tau''}) + (1, 2) - (0, 1)$. This yields the signature identity.] □

6.3. **Proof of the type BM case.** We will reduce the proof to the type CD case.

In this section, $k \geqslant 1$ and $\mathcal{O} = (C_{2k+1}, C_{2k}, \cdots, C_1, C_0 = 0) \in \mathrm{Nil}^{\mathrm{dpe}}(B)$. $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' := \overline{\nabla}(\mathcal{O}')$.

Take $\tau \in \mathrm{DRC}(\mathcal{O})$, we marks the entries in τ as in (6.5) to (6.7).

6.3.1. *Usual decent case.* Thanks to Lemma 6.6, the proof in the descent case is exactly the same as that in Section 5.3.

6.3.2. *Reduction to type CD case in the general descent case.* Suppose C_{2k} is even. We define $\tilde{\mathcal{O}} = (C_{2k+1} - C_{2k} + 1, 1, 1) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$ and $\tilde{\mathcal{O}}'' := \overline{\nabla}_1(\tilde{\mathcal{O}}) = (2)$.

The key proposition of reduction.

Proposition 6.8. *We have the following properties:*

(a) *When* $\tau'' = \overline{\nabla}_1^2(\tau)$ *for certain* $\tau \in \mathrm{DRC}(\mathcal{O})$, *we have* $x_{\tau''} \neq s$;
(b) *For each* τ'' *such that* $x_{\tau''} \neq s$, *we have a bijection*

$$\begin{array}{rcl} \bar{\delta}\colon \{\, \tau \in \mathrm{DRC}(\mathcal{O}) \mid \overline{\nabla}_1^2(\tau) = \tau'' \,\} & \longrightarrow & \{\, (x_{\tilde{\tau}}, \mathbf{u}_{\tilde{\tau}}) \mid \tilde{\tau} \in \mathrm{DRC}(\tilde{\mathcal{O}}) \text{ s.t. } \overline{\nabla}_1^2(\tilde{\tau}) = x_{\tau''} \,\} \\ \tau & \longmapsto & (x_{\tau''}, \mathbf{x}_\tau) \end{array}$$

where $\mathbf{u}_{\tilde{\tau}}$ *is defined in* (3.5).

The situation could be summarized in the following commutative diagram:

$$\begin{tikzcd}
 & \tau \arrow[r, mapsto, "\delta"] \arrow[d, mapsto] & (\tau'', \mathbf{x}_\tau) \arrow[r, mapsto] \arrow[d, mapsto] & \tau'' \arrow[d, mapsto] \\
x_\tau \arrow[d, dashed, leftrightarrow] & \mathbf{p}_\tau \arrow[l, mapsto] \arrow[r, mapsto] \arrow[d, leftrightarrow] & (x_{\tau''}, \mathbf{x}_\tau) \arrow[r, mapsto] \arrow[d, equal] & x_{\tau''} \arrow[d, equal] \\
x_{\tilde{\tau}} & \tilde{\tau} \arrow[l, mapsto] \arrow[r, mapsto, "\delta"] & (\tilde{\tau}'', \mathbf{u}_{\tilde{\tau}}) \arrow[r, mapsto] & \tilde{\tau}''
\end{tikzcd}$$

We remark that x_τ and $x_{\tilde{\tau}}$ are equal in the most of the case, especially when $n = 1$. But there are exceptional cases.

Proof. This follows from a case by case verification according to our algorithm. We list all possible cases below:

When $n = 1$, see Table 3.

Now assume $n \geqslant 1$. We let $\mathbf{x} = x_1 x_2 \cdots x_n$ and define

$$\mathbf{x}' = \begin{cases} \mathbf{x} \text{ deleteing “}c\text{”} & \text{if } c \in \mathbf{x} \\ \mathbf{x} \text{ deleteing “}s\text{”} & \text{if } c \notin \mathbf{x}, s \in \mathbf{x} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now all the cases are listed in Table 4.

□

Thanks to $\bar{\delta}$ defined in the above lemma, we can use (5.12). The rest of the proof is similar to that in Section 5.4. We leave the details to the reader.

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x_τ	$\tilde{x}_{\tau''}$ $m_{\tau''}$	$\mathbf{p}_\tau$	$(x_{\tau''}, \mathbf{x}_\tau)$	$\tilde{\tau}_L$
r	r a	$\bullet\ r$ b	(r, s)	$s\ r$
		$\bullet\ r$ a	(r, c)	$r\ c$
r	r b	$s\ r$ b	(r, s)	$s\ r$
		$s\ r$ a	(r, c)	$r\ c$
c	s a	$\bullet\ s$ b	(c, s)	$s\ c$
		$\bullet\ s$ a	(c, c)	$c\ c$
d	d a	$\bullet\ d$ b	(d, s)	$s\ d$
		$\bullet\ d$ a	(d, c)	$c\ d$
		$r\ d$ a	(d, r)	$r\ d$
		$d\ d$ a	(d, d)	$d\ d$
	d b	$s\ d$ b	(d, s)	$s\ d$
		$s\ d$ a	(d, c)	$c\ d$
		$r\ d$ b	(d, r)	$r\ d$
		$d\ d$ b	(d, d)	$d\ d$

TABLE 3. Reduction when $n = 1$

x_τ	$\tilde{x}_{\tau''}$ $m_{\tau''}$	$\mathbf{p}_\tau$	$x_{\tau''}, \mathbf{x}_\tau$	$\tilde{\tau}_L$	conditions	
r	r a	$\bullet\ r$ $\mathbf{x}'$ a	$r\ ,\ \mathbf{x}$	$\mathbf{x}\ r$	$x_1 \neq r$	$c \in \mathbf{x}$
				$r\ c$ $\mathbf{x}'$	$x_1 = r$	
		$\bullet\ r$ $\mathbf{x}'$ b	$r\ ,\ \mathbf{x}$	$\mathbf{x}\ r$	$c \notin \mathbf{x}, s \in \mathbf{x}$	
r	r b	$s\ r$ $\mathbf{x}'$ a	$r\ ,\ \mathbf{x}$	$\mathbf{x}\ r$	$x_1 \neq r$	$c \in \mathbf{x}$
				$r\ c$ $\mathbf{x}'$	$x_1 = r$	
		$s\ r$ $\mathbf{x}'$ b	$r\ ,\ \mathbf{x}$	$\mathbf{x}\ r$	$c \notin \mathbf{x}, s \in \mathbf{x}$	
c	s a	$\bullet\ s$ $\mathbf{x}'$ a	$c\ ,\ \mathbf{x}$	$\mathbf{x}\ c$	$c \in \mathbf{x}$	$s \in \mathbf{x}$ is automatic
		$\bullet\ s$ $\mathbf{x}'$ b			$c \notin \mathbf{x}$	
d	d a	$\bullet\ d$ $\mathbf{x}'$ a	$d\ ,\ \mathbf{x}$	$\mathbf{x}\ d$	$c \in \mathbf{x}$	
		$\bullet\ d$ $\mathbf{x}'$ b			$x_1 = s$	$c \notin \mathbf{x}$
		$\mathbf{x}\ d$ a			$x_1 \neq s$	$c \notin \mathbf{x}$
d	d b	$s\ d$ $\mathbf{x}'$ a	$d\ ,\ \mathbf{x}$	$\mathbf{x}\ d$	$c \in \mathbf{x}$	
		$s\ d$ $\mathbf{x}'$ b			$x_1 = s$	$c \notin \mathbf{x}$
		$\mathbf{x}\ d$ b			$x_1 \neq s$	$c \notin \mathbf{x}$

TABLE 4. Reduction when $n \geqslant 1$

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