UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS

JIA-JUN MA, BINYONG SUN, AND CHEN-BO ZHU

Abstract. Let $\mathbf{G}$ be a complex orthogonal or complex symplectic group, and let G be a real form of $\mathbf{G}$, namely G is a real orthogonal group, a real symplectic group, a quaternionic orthogonal group, or a quaternionic symplectic group. For a fixed parity $\mathbb{p} \in \mathbb{Z}/2\mathbb{Z}$, we define a set $\mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$ of nilpotent $\mathbf{G}$-orbits in $\mathfrak{g}$ (the Lie algebra of $\mathbf{G}$). When $\mathbb{p}$ is the parity of the dimension of the standard module of $\mathbf{G}$, this is the set of the stably trivial special nilpotent orbits, which includes all rigid special nilpotent orbits. For each $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$, we construct all unipotent representations of G (or its metaplectic cover when G is a real symplectic group and $\mathbb{p}$ is odd) attached to $\mathcal{O}$, in the sense of Barbasch and Vogan, via the method of theta lifting and show in particular that they are unitary.

Contents

1. Introduction and the main result

A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [35, 36]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [5, 35]. For more general Lie groups, work of Mackey and Duflo [20, 46] suggest that one should focus attention on reductive Lie groups. As expounded by Vogan in his writings (see for example [68, 70, 71]), the problem finally is to quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We mention two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [3, 4], Adams, Barbasch and Vogan established some

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important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [2]. The problem of finding (integral) special unipotent representations for complex semisimple groups (as well as their distribution characters) was solved earlier by Barbasch and Vogan [12] and the unitarity of these representations was established by Barbasch for complex classical groups [6]. In a similar vein, Barbasch outlined his proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [7]. The second major development is Vogan's theory of associated varieties [69] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a representation attached to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [31], the description of discrete spectrum by Adams [1] and the preservation of unitarity in stable range theta lifting by Li [42]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [10, 13, 26, 28, 29, 56, 59, 61, 67]. Particularly worth mentioning were the work of Przebinda [59] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [26] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [21, 44, 51, 53], which among other things demonstrate the tight link between the orbit method and Howe's theory.

In the present article we will demonstrate that the orbit method and Howe's theory in fact have perfect synergy when it comes to unipotent representations. (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [10, 26, 49, 67].) We will restrict our attention to a real classical group G of orthogonal or symplectic type and we will construct all unipotent representations of $\widetilde{G}$ attached to $\mathcal{O}$ (in the sense of Barbasch and Vogan) via the method of theta lifting. Here $\widetilde{G}$ is either G, or the metaplectic cover of G if G is a real symplectic group, and $\mathcal{O}$ is a member of our preferred set of complex nilpotent orbits, large enough to include all those which are rigid special. Here we wish to emphasize that there have been extensive investigations of unipotent representations for real reductive groups by Vogan and his collaborators (see e.g. [2, 68, 69]), only for unitary groups complete results are known (*cf.* [11, 67]), in which case all such representations may also be described in terms of cohomological induction.

We now proceed to our setup in order to state the main technical result of this article.

Let $\mathbf{G}$ be either a complex orthogonal group or a complex symplectic group. Put $\epsilon = 1$ or -1, when $\mathbf{G}$ is respectively orthogonal or symplectic. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$ and let $\mathcal{O}$ be a nilpotent $\mathbf{G}$-orbit in $\mathfrak{g}^*$. Unless otherwise specified, a superscript "$*$" indicates the dual space. By identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via the trace form, $\mathcal{O}$ is also viewed as a nilpotent $\mathbf{G}$-orbit in $\mathfrak{g}$. Denote by $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ the set of nilpotent $\mathbf{G}$-orbits in $\mathfrak{g}$.

In the usual way, a nilpotent $\mathbf{G}$-orbit $\mathcal{O}$ in $\mathfrak{g}$ is parameterized by its Young diagram. In this article, we will label a Young diagram $\mathbf{d}$ by its column partition $[c_0, c_1, \cdots, c_k]$, where

$$c_0 \geqslant c_1 \geqslant \cdots \geqslant c_k > 0$$

represent lengths of the columns of $\mathbf{d}$. If $\mathbf{G}$ is a trivial group, then the only nilpotent $\mathbf{G}$-orbit is labeled by the empty sequence $\varnothing$, and we set $k = -1$ in this case.

Fix a parity $\mathbb{p} \in \mathbb{Z}/2\mathbb{Z}$ throughout the paper. Let $\mathcal{P}_{\mathbf{G}}$ be the set of partitions/Young diagrams parameterizing the set $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$. Set $\epsilon_l = \epsilon(-1)^l$ for $l \geqslant 0$ and consider the following subset of $\mathcal{P}_{\mathbf{G}}$:

$$\mathcal{P}^{\mathbb{p}}_{\mathbf{G}} := \left\{ \mathbf{d} = [c_0, c_1, \cdots, c_k] \in \mathcal{P}_{\mathbf{G}} \;\middle|\; \begin{array}{l} \bullet \text{ all } c_i\text{'s have parity } \mathbb{p}; \\ \bullet\ c_l \geqslant c_{l+1} + 2, \text{ if } 0 \leqslant l \leqslant k-1 \\ \quad \text{and } \epsilon_l = 1. \end{array} \right\}. \tag{1.1}$$

Let $\mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$ be the subset of $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ corresponding to partitions in $\mathcal{P}^{\mathbb{p}}_{\mathbf{G}}$. Write $\mathbf{V}$ for the standard module of $\mathbf{G}$. Note that if the group $\mathbf{G}$ is orthogonal and nontrivial, then there is no partition in $\mathcal{P}_{\mathbf{G}}$ satisfying the first condition in (1.1) unless $\mathbb{p}$ equals the parity of $\dim \mathbf{V}$.

Remarks. 1. If $\mathbb{p}$ is the parity of $\dim \mathbf{V}$, then $\mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$ is precisely the set of stably trivial special nilpotent orbits in $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$. If $\mathbf{G}$ is a symplectic group, then all nilpotent orbits in $\mathrm{Nil}^{1}_{\mathbf{G}}(\mathfrak{g})$ are metaplectic-special, as defined in [34]. See also [48].
2. All rigid special nilpotent orbits in $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ are stably trivial. The reader is referred to [10, Section 2] for facts on stably trivial special nilpotent orbits.
3. Every orbit $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$ is irreducible as an algebraic variety [65, IV 2.27].

Let m be the rank of $\mathbf{G}$. We identify a Cartan subalgebra of $\mathfrak{g}$ with $\mathbb{C}^m$, using the standard coordinates. Let a superscript group indicate the space of invariant vectors under the group action. Then Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{\mathbf{G}} = (\mathrm{S}(\mathbb{C}^m))^{W_m},$$

where "U" indicates the universal enveloping algebra, "S" indicates the symmetric algebra, and W_m denotes the subgroup of $\mathrm{GL}_m(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. We may thus parameterize algebraic characters of $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ by $(\mathbb{C}^m)^*/W_m \cong \mathbb{C}^m/W_m$. Unless $\mathbf{G}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$.

For any positive integer r, denote

$$\rho^{\epsilon}_r = \begin{cases} (\frac{r}{2}-1, \frac{r}{2}-2, \cdots, \frac{r}{2} - \lfloor \frac{r}{2} \rfloor) & \in (\frac{1}{2}\mathbb{Z})^{\lfloor \frac{r}{2} \rfloor}, \quad \text{if } \epsilon = 1; \\ (\frac{r}{2}, \frac{r}{2}-1, \cdots, \frac{r}{2} - \lfloor \frac{r-1}{2} \rfloor) & \in (\frac{1}{2}\mathbb{Z})^{\lfloor \frac{r+1}{2} \rfloor}, \quad \text{if } \epsilon = -1. \end{cases}$$

Definition 1.1. *For a nilpotent orbit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ given by the partition $\mathbf{d} = [c_0, c_1, \cdots, c_k]$, define the following element $\lambda_{\mathcal{O}}$ of $\mathbb{C}^m$:*

$$\lambda_{\mathcal{O}} := (\rho^{\epsilon_0}_{c_0}, \rho^{\epsilon_1}_{c_1}, \cdots, \rho^{\epsilon_k}_{c_k}) \in (\tfrac{1}{2}\mathbb{Z})^m.$$

By abuse of notation, $\lambda_{\mathcal{O}}$ also represents its image in $\mathbb{C}^m/W_m$, as well as the corresponding algebraic character of $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$.

Remark. If $\mathbb{p}$ is the parity of $\dim \mathbf{V}$, then for every $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$, $\lambda_{\mathcal{O}}$ equals the algebraic character of $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ determined by $\frac{1}{2}{}^L h$ as in [12, Equation (1.5)] (*cf.* [67, Proposition 1.12] and [64, Section 7]).

We will work in the category of Casselman-Wallach representations [14, 73]. Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation if its Harish-Chandra module has finite length.

Let G be a real form of $\mathbf{G}$, namely, G is the fixed point group of an anti-holomorphic involutive automorphism of $\mathbf{G}$. Thus G is a real orthogonal group, a real symplectic group, a quaternionic orthogonal group, or a quaternionic symplectic group. When G is a real symplectic group, write

$$1 \longrightarrow \{1, \varepsilon_G\} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

for the metaplectic cover of G. It does not split unless G is trivial. We say that a Casselman-Wallach representation π of $\widetilde{G}$ is $\mathbb{p}$-genuine if ε_G acts on π through the scalar multiplication by $(-1)^{\mathbb{p}}$. In all other cases, we simply set $\widetilde{G} := G$, and all Casselman-Wallach representations of $\widetilde{G}$ are defined to be $\mathbb{p}$-genuine. When no confusion is possible, the notion of "$\mathbb{p}$-genuine" will be used in similar settings without further explanation.

For a Casselman-Wallach representation π of $\widetilde{G}$, denote $\mathrm{Ann}(\pi)$ its annihilator ideal in $\mathrm{U}(\mathfrak{g})$. Let $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$. Following Barbasch and Vogan [2, 12], we make the following definition.

Definition 1.2. *An irreducible Casselman-Wallach representation π of $\widetilde{G}$ is said to be $\mathcal{O}$-unipotent if*

- *the associated variety of* $\mathrm{Ann}(\pi)$ *is contained in the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$,*
- *the action of* $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ *on π is given by the character $\lambda_{\mathcal{O}}$, and*
- *π is $\mathbb{p}$-genuine.*

Remarks. 1. The first two conditions of Definition 1.2 may also be given in terms of a certain primitive ideal $I_{\mathcal{O}}$ of $\mathrm{U}(\mathfrak{g})$. We adopt a more direct approach.
2. We comment on the third condition of Definition 1.2 by the example of $G = \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. Take $\mathcal{O}$ the principal nilpotent orbit, then up to isomorphisms, there are six irreducible Casselman-Wallach representations of $\widetilde{G}$ satisfying the first two conditions. Four of these representations also satisfy the third condition and are thus $\mathcal{O}$-unipotent in our definition. They are the irreducible components of the two oscillator representations. The other two descend to the two irreducible principal series representations of G with infinitesimal character $\lambda_{\mathcal{O}}$. One of them is unitarizable and the other is not. They are not considered as unipotent representations.

Let $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ denote the set of isomorphism classes of $\mathcal{O}$-unipotent irreducible Casselman-Wallach representations of $\widetilde{G}$. Write $\mathcal{K}_{\mathcal{O}}(\widetilde{G})$ for the Grothedieck group of the category of Casselman-Wallach representations π of $\widetilde{G}$ such that the associated variety of $\mathrm{Ann}(\pi)$ is contained in $\overline{\mathcal{O}}$. We view $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ as a subset of $\mathcal{K}_{\mathcal{O}}(\widetilde{G})$.

Fix a maximal compact subgroup K of G. Write $\mathbf{K}$ for the Zariski closure of K in $\mathbf{G}$, which is a universal complexification of K. Write $\widetilde{K} \to K$ for the one or two fold covering map induced by the covering map $\widetilde{G} \to G$. Denote by $\widetilde{\mathbf{K}}$ the universal complexification of $\widetilde{K}$.

Write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of $\mathbf{K}$, and $\mathfrak{p}$ is its orthogonal complement in $\mathfrak{g}$ with respect to the trace form. Under the adjoint action of $\mathbf{K}$, the complex variety $\mathcal{O} \cap \mathfrak{p}$ is a union of finitely many orbits, each of dimension $\frac{\dim_{\mathbb{C}} \mathcal{O}}{2}$. For any $\mathbf{K}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}$, let $\mathcal{K}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ denote the Grothedieck group of the category of $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}}) := \bigoplus_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}_{\mathscr{O}}(\widetilde{\mathbf{K}}).$$

Following Vogan [69, Section 8], we make the following definition.

Definition 1.3. *Let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_X$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_X$;*
- *$\mathcal{E}$ is $\mathbb{p}$-genuine.*

Here $X \in \mathscr{O}$, $\mathcal{E}_X$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_X$ denotes the Lie algebra of the stabilizer of X in $\mathbf{K}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_X$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$.

Note that in the situation of classical groups we consider here, all admissible orbit data are line bundles. Denote by $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}})$. Put

$$\mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) := \bigsqcup_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) \subset \mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

According to Vogan [69, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{Ch}_{\mathcal{O}} : \mathcal{K}_{\mathcal{O}}(\widetilde{G}) \to \mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}}). \tag{1.2}$$

The main result of this article is the following theorem.

Theorem 1.4. *Let $\mathcal{O} \in \mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{g})$. The homomorphism* (1.2) *restricts to a bijective map*

$$\mathrm{Ch}_{\mathcal{O}} : \Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G}) \longrightarrow \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}). \tag{1.3}$$

Moreover, every representation in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ is unitarizable.

Remarks. 1. When G is a quaternionic orthogonal group or an quaternionic symplectic group, the first assertion of Theorem 1.4 is proved in Theorems 6 and 10 of [47]. See also [67, Theorem 3.1].
2. The bijectivity in Equation (1.3) implies in particular that the associated variety of a representation in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ is the closure of a single nilpotent $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$.

Key to the proof of Theorem 1.4 is the existence of unitarizable unipotent representations with the prescribed nilpotent admissible orbit data. More precisely we shall first prove the following result (see Theorem 5.1):

- for every $\mathcal{E} \in \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$, there exists a unitarizable $\mathcal{O}$-unipotent representation π of $\widetilde{G}$ such that $\mathrm{Ch}_{\mathcal{O}}(\pi) = \mathcal{E}$.

Our construction of the $\mathcal{O}$-unipotent representation π for a given $\mathcal{E} \in \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ will be guided by a geometric process which we call descent. We describe it briefly. Write $\mathbf{G} = \mathbf{G}_{\mathbf{V}}$, where $\mathbf{V}$ is the standard module of $\mathbf{G}$ as before, which is an ϵ-symmetric bilinear space. From the pair $(\mathbf{V}, \mathcal{O})$, the decent process yields another pair $(\mathbf{V}', \mathcal{O}')$, where $\mathbf{V}'$ is an ϵ'-symmetric bilinear space, and $\mathcal{O}'$ is a nilpotent $\mathbf{G}'$-orbit in $\mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}'}(\mathfrak{g}')$. Here $\epsilon\epsilon' = -1$, and $\mathbf{G}' = \mathbf{G}_{\mathbf{V}'}$. The real form G of $\mathbf{G}$ is specified by an additional structure on $\mathbf{V}$, which we call $(\epsilon, \dot{\epsilon})$-space, where $\dot{\epsilon} = \pm 1$. Assume that $\mathscr{O}$ is a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$. In a similar way, the decent process yields from the pair $(\mathbf{V}, \mathscr{O})$ another pair $(\mathbf{V}', \mathscr{O}')$, where $\mathbf{V}'$ is an $(\epsilon', \dot{\epsilon}')$-space $(\dot{\epsilon}\dot{\epsilon}' = -1)$, giving rise to a real form G' of $\mathbf{G}'$.

The geometric process inverse to the descent will be called lifting. Apart from lifting a nilpotent orbit, one can in fact lift an equivariant algebraic vector bundle on the nilpotent orbit which has the additional property of sending an admissible orbit datum over $\mathscr{O}'$ to an admissible orbit datum over $\mathscr{O}$. Moreover, by twisting with characters of the component group of $\mathbf{K}$ if necessary, this lifting map of admissible orbit data is surjective (see Lemma 2.21). A key observation here is that the geometric processes outlined are completely governed by a double fiberation of moment maps mentioned earlier. (Some of the basic facts about this double fiberation first appeared in a paper of Kraft and Procesi on the geometry of conjugate classes in classical groups [37].) This summarises the geometric preparations regarding our nilpotent orbits and concludes our discussion on the contents of Section 2.

To construct the $\mathcal{O}$-unipotent representation π, we will apply theta lifting for the dual pair (G', G), starting from an $\mathcal{O}'$-unipotent representation of $\widetilde{G'}$ with an associated character $\mathcal{E}' \in \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}'}(\widetilde{\mathbf{K}}')$. Our model of theta lifting is provided by matrix coefficient integrals against the oscillator representation in the so-called convergent range. The critical task before us is to show that the lifted representation will not vanish and furthermore the associated character $\mathcal{E}$ of the lifted representation is simply the lift of $\mathcal{E}'$. As a first step, we show an upper bound of $\mathcal{E}$ by the lift of $\mathcal{E}'$ by some largely geometric considerations. To achieve the equality, we adapt the idea of He mentioned earlier, namely by applying another theta lifting to an appropriate group (of the same type as G') and then relating the double theta lift, via a variant of the well-known doubling method (*cf.* [50, 60]), to a certain degenerate principal series representation of an appropriately large classical group (whose structure is explicitly described in [40, 41, 75]). The end result is that the lifted representation must have the anticipated associated character, by comparing associated cycles within the said degenerate principal series representation. To summarize the contents, we take care of the analytic preparations, namely growth of matrix coefficients, their integrals against oscillator representations as well as degenerate principal series in Section 3, and we combine geometry and analysis to determine the associated characters of theta lifts in the convergent range (in a general setting) in Section 4.

In Section 5, we complete the construction of unipotent representations by an iterative procedure. For the exhaustion part of the main theorem, we use in a crucial way the result of Barbasch [7, 9] counting unipotent representations for a real symplectic group and a split real odd orthogonal group. Putting together our construction of unipotent representations with the prescribed nilpotent admissible orbit data, the counting result of Barbasch for the specific cases just mentioned, as well as a general technique of embedding unitary representations (via stable range theta lifting), our main theorem follows for all the cases considered in this article.

Finally the unitarizability of all $\mathcal{O}$-unipotent representations is a consequence of the exhaustion and the construction by our concrete models. Specifically we achieve this by showing the nonnegativity of a certain matrix coefficient integral, using a general technique of Harris, Li and Sun [24]. Note that the unitarizability of the constructed representations also follows from the earlier work of He on theta lifting in the so-called strongly semistable range ([26, Chapter 5] or [27]).

Notation. We adopt the following convention throughout the article. Boldface letters denote objects defined over $\mathbb{C}$ such as a complex vector space or a complex Lie group. If an object with " $'$" (e.g. $\mathbf{V}'$) is defined, all related notions will be implicitly defined by adding a prime (e.g. $\mathbf{G}'$). Epsilons, e.g. $\epsilon, \dot{\epsilon}, \epsilon'$, always mean a sign taking values in $\{\pm 1\}$. A superscript " ^{t}" indicates the transpose and "tr" the trace, of a matrix. A superscript " $^{\vee}$" indicates the contragredient representation in the appropriate contexts. Unless otherwise specified, "Ind" and "ind" indicate the normalized smooth and Schwartz induction[1], respectively. "$\langle\, ,\, \rangle$" denotes natural parings or Hermitian inner products in the appropriate contexts. All measures appearing in integrals on real reductive groups are the Haar measures. The largest integer less than or equal to a real number a is denoted by $\lfloor a \rfloor$. Let "id" indicate the identity map in various contexts. Let $\mathbf{i}$ denote a fixed $\sqrt{-1}$.

[1]For Schwartz induction, see [15].

2. Classical groups and their nilpotent orbits

2.1. **Complex orthogonal and symplectic groups.** Let $\epsilon \in \{\pm 1\}$. Let $\mathbf{V}$ be a finite dimensional complex vector space equipped with an ϵ-symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{V}}$, to be called an ϵ-symmetric bilinear space. Denote its isometry group by

$$\mathbf{G} := \mathbf{G}_{\mathbf{V}} := \{ g \in \mathrm{End}_{\mathbb{C}}(\mathbf{V}) \mid \langle g \cdot v_1, g \cdot v_2 \rangle_{\mathbf{V}} = \langle v_1, v_2 \rangle_{\mathbf{V}}, \quad \text{for all } v_1, v_2 \in \mathbf{V} \}.$$

The Lie algebra of $\mathbf{G}$ is given by

$$\mathfrak{g} := \mathfrak{g}_{\mathbf{V}} := \{ X \in \mathrm{End}_{\mathbb{C}}(\mathbf{V}) \mid \langle X \cdot v_1, v_2 \rangle_{\mathbf{V}} + \langle v_1, X \cdot v_2 \rangle_{\mathbf{V}} = 0, \quad \text{for all } v_1, v_2 \in \mathbf{V} \}.$$

Note that $\mathbf{G}_{\mathbf{V}}$ is a trivial group if and only if $\mathbf{V} = \{0\}$.

2.2. **Real form, Cartan involution, and $(\epsilon, \dot\epsilon)$-space.** We recall some facts about real structures and Cartan involutions on the complex classical group $\mathbf{G}$. See [54, Section 1.2-1.3]. Let $\dot\epsilon \in \{\pm 1\}$.[2]

Definition 2.1. *An $\dot\epsilon$-real form of $\mathbf{V}$ is a conjugate linear automorphism J of $\mathbf{V}$ such that*

$$J^2 = \epsilon\dot\epsilon \quad \text{and} \quad \langle Jv_1, Jv_2 \rangle_{\mathbf{V}} = \overline{\langle v_1, v_2 \rangle_{\mathbf{V}}}, \quad \text{for all } v_1, v_2 \in \mathbf{V}.$$

For J as in Definition 2.1, write $\mathbf{G}^J$ for the centralizer of J in $\mathbf{G}$ (we will use similar notation without further explanation). It is a real form of $\mathbf{G}$ as in Table 1.

ϵ \ $\dot\epsilon$	1	-1
1	real orthogonal group	quaternionic orthogonal group
-1	quaternionic symplectic group	real symplectic group

Table 1. The real classical group $\mathbf{G}^J$

Remarks. 1. The real form $\mathbf{G}^J$ may also be described as the isometry group of a $\dot\epsilon$-Hermitian space over a division algebra D, where D is $\mathbb{R}$ if $\epsilon\dot\epsilon = 1$ and the quaternion algebra $\mathbb{H}$ if $\epsilon\dot\epsilon = -1$.
2. Every real form of $\mathbf{G}$ is of the form $\mathbf{G}^J$ for some J.
3. The conjugate linear map J is an equivalent formulation of Adams-Barbasch-Vogan's notion of strong real forms [2, Definition 2.13].

Definition 2.2. *An $\dot\epsilon$-Cartan form of $\mathbf{V}$ is a linear automorphism L of $\mathbf{V}$ such that*

$$L^2 = \dot\epsilon \quad \text{and} \quad \langle Lv_1, Lv_2 \rangle_{\mathbf{V}} = \langle v_1, v_2 \rangle_{\mathbf{V}}, \quad \text{for all } v_1, v_2 \in \mathbf{V}.$$

Given an $\dot\epsilon$-Cartan form L of $\mathbf{V}$, conjugation by L yields an involution of the algebraic group $\mathbf{G}$. Thus we have a symmetric subgroup $\mathbf{G}^L$ of $\mathbf{G}$.

We omit the proof of the following lemma (*cf.* [54, Section 1.3]).

Lemma 2.3. *For every $\dot\epsilon$-real form J of $\mathbf{V}$, up to conjugation by $\mathbf{G}^J$, there exists a unique $\dot\epsilon$-Cartan form L of $\mathbf{V}$ such that*

(a) $LJ = JL$;

(b) the Hermitian form $(u, v) \mapsto \langle Lu, Jv \rangle_{\mathbf{V}}$ on $\mathbf{V}$ is positive definite.

Similarly, for every $\dot\epsilon$-Cartan form L of $\mathbf{V}$, up to conjugation by $\mathbf{G}^L$, there exists a unique $\dot\epsilon$-real form J of $\mathbf{V}$ such that the above two conditions hold.

For L as in Lemma 2.3, conjugation by L induces a Cartan involution on $\mathbf{G}^J$.

[2]In [54], $\dot\epsilon$ is denoted by ω.

Definition 2.4. *An ϵ-symmetric bilinear space* $\mathbf{V}$ *with a pair* (J, L) *as in Lemma 2.3 is called an* $(\epsilon, \dot{\epsilon})$-space.

In view of Lemma 2.3 and when no confusion is possible, we also name the pairs $(\mathbf{V}, L)$ and $(\mathbf{V}, J)$ as $(\epsilon, \dot{\epsilon})$-spaces. We define the notion of a non-degenerate $(\epsilon, \dot{\epsilon})$-subspace and isomorphisms of $(\epsilon, \dot{\epsilon})$-spaces in an obvious way.

Definition 2.5. *The signature of an* $(\epsilon, \dot{\epsilon})$*-space* $(\mathbf{V}, J, L)$ *is defined by the following recipe:*

$$\mathrm{Sign}(\mathbf{V}) := \mathrm{Sign}(\mathbf{V}, L) := (n^+, n^-),$$

where n^+ *and* n^- *are respectively the dimensions of* $+1$ *and* -1 *eigenspaces of* L *if* $\dot{\epsilon} = 1$*, and the dimensions of* $+\mathbf{i}$ *and* $-\mathbf{i}$ *eigenspaces of* L *if* $\dot{\epsilon} = -1$*. For every* L*-stable subquotient* $\mathbf{E}$ *of* $\mathbf{V}$*, the signature* $\mathrm{Sign}(\mathbf{E})$ *is defined analogously.*

For a fixed pair $(\epsilon, \dot{\epsilon})$, the isomorphism class of an $(\epsilon, \dot{\epsilon})$-space $(\mathbf{V}, J, L)$ is determined by $\mathrm{Sign}(\mathbf{V})$. In view of Lemma 2.3, we define the signatures of the pairs $(\mathbf{V}, L)$ and $(\mathbf{V}, J)$ as that of $(\mathbf{V}, J, L)$. The value of $\mathrm{Sign}(\mathbf{V})$ is listed in Table 2 for the real classical group $\mathbf{G}^J$.

$\mathbf{G}^J$	$\mathrm{Sign}(\mathbf{V})$
$\mathrm{O}(p, q)$	(p, q)
$\mathrm{Sp}(2n, \mathbb{R})$	(n, n)
$\mathrm{O}^*(2n)$	(n, n)
$\mathrm{Sp}(p, q)$	$(2p, 2q)$

TABLE 2. Signature of $(\epsilon, \dot{\epsilon})$-spaces

2.3. **Metaplectic cover.** Let (J, L) be as in Lemma 2.3. Put $G := \mathbf{G}^J$ and $\mathbf{K} := \mathbf{G}^L$. Then $K := G \cap \mathbf{K}$ is a maximal compact subgroup of both G and $\mathbf{K}$. As a slight modification of G, we define

$$\widetilde{G} := \begin{cases} \text{the metaplectic double cover of } G, & \text{if } G \text{ is a real symplectic group;} \\ G, & \text{otherwise.} \end{cases}$$

Here "G is a real symplectic group" means that $(\epsilon, \dot{\epsilon}) = (-1, -1)$, and similar terminologies will be used later on. In the case of a real symplectic group, we use ε_G to denote the nontrivial element in the kernel of the covering homomorphism $\widetilde{G} \to G$. In general, we use "$\sim$" over a subgroup of G to indicate its inverse image under the covering map $\widetilde{G} \to G$. For example, $\widetilde{K}$ is the a maximal compact subgroup of $\widetilde{G}$ with a covering map $\widetilde{K} \to K$ of degree 1 or 2. Similar notation will be used for other covering groups similar to $\widetilde{G}$.

When there is a need to indicate the dependence of various objects introduced on the $(\epsilon, \dot{\epsilon})$-space $\mathbf{V}$, we will add the subscript $\mathbf{V}$ in various notations (e.g. $\widetilde{G}_{\mathbf{V}}$ and $\widetilde{K}_{\mathbf{V}}$).

2.4. **Young diagrams and complex nilpotent orbits.** Let $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ be the set of nilpotent $\mathbf{G}$-orbits in $\mathfrak{g}$. For $n \in \mathbb{N} := \{0, 1, 2, \cdots\}$, let $\mathcal{P}_\epsilon(n)$ be the set of Young diagrams of size n such that

i) rows of even length appear in even times if $\epsilon = 1$;
ii) rows of odd length appear in even times if $\epsilon = -1$.

In this paper, a Young diagram $\mathbf{d}$ will be labeled by a sequence $[c_0, c_1, \cdots, c_k]$ of integers enumerating its columns, where $k \geqslant -1$. By convention, $[c_0, c_1, \cdots, c_k]$ denotes the empty sequence $\varnothing$ which labels the empty Young diagram when $k = -1$. It is easy to see that $\mathcal{P}_\epsilon(n)$ consists of sequences of the form $[c_0, c_1, \cdots, c_k]$ such that

$$\begin{cases} c_0 \geqslant c_1 \geqslant \cdots \geqslant c_k > 0, \\ \sum_{l=0}^{k} c_l = n, \text{ and} \\ \sum_{l=i}^{k} c_l \text{ is even, when } i \equiv \frac{1+\epsilon}{2} \pmod 2 \text{ and } 0 \leqslant i \leqslant k, \end{cases}$$

and $\mathcal{P}_\epsilon(0) := \{ \varnothing \}$ by convention.

Definition 2.6. *For a nilpotent element X in $\mathfrak{g}$, set*

$$\operatorname{depth}(X) := \begin{cases} \max\{\, l \in \mathbb{N} \mid X^l \neq 0 \,\}, & \text{if } \mathbf{V} \neq \{0\}; \\ -1, & \text{if } \mathbf{V} = \{0\}. \end{cases} \tag{2.1}$$

Given $\mathcal{O} \in \operatorname{Nil}_{\mathbf{G}}(\mathfrak{g})$, pick any $X \in \mathcal{O}$. Set $\operatorname{depth}(\mathcal{O}) := \operatorname{depth}(X)$ and define the Young diagram

$$\mathbf{d}_{\mathcal{O}} := [c_0, c_1, \cdots, c_k], \qquad (k := \operatorname{depth}(\mathcal{O}) \geqslant -1)$$

where

$$c_l := \dim(\operatorname{Ker}(X^{l+1})/\operatorname{Ker}(X^l)), \quad \text{for all } 0 \leqslant l \leqslant k.$$

We have the one-one correspondence: ([16, Chapter 5])

$$\begin{aligned} \operatorname{Nil}_{\mathbf{G}}(\mathfrak{g}) &\longrightarrow \mathcal{P}_\epsilon(\dim \mathbf{V}), \\ \mathcal{O} &\longmapsto \mathbf{d}_{\mathcal{O}}. \end{aligned} \tag{2.2}$$

Let

$$\operatorname{Nil}_\epsilon := \left(\bigsqcup_{\mathbf{V}} \operatorname{Nil}_{\mathsf{G}_\mathbf{V}}(\mathfrak{g}_\mathbf{V}) \right) \Big/ \sim$$

where $\mathbf{V}$ runs over all ϵ-symmetric bilinear spaces, and $\sim$ denotes the equivalence relation induced by isomorphisms of ϵ-symmetric bilinear spaces. Let

$$\mathcal{P}_\epsilon := \bigsqcup_{n \geqslant 0} \mathcal{P}_\epsilon(n).$$

The correspondence (2.2) induces a bijection

$$\mathbf{D} \colon \operatorname{Nil}_\epsilon \longrightarrow \mathcal{P}_\epsilon.$$

Definition 2.7. *Define the descent of a Young diagram by*

$$\begin{aligned} \boldsymbol{\nabla} \colon \mathcal{P}_\epsilon &\longrightarrow \mathcal{P}_{-\epsilon} \\ [c_0, c_1, \cdots, c_k] &\longmapsto [c_1, \cdots, c_k], \end{aligned}$$

namely by removing the left most column of the diagram. By convention, $\boldsymbol{\nabla}(\varnothing) := \varnothing$.

Fix $\mathbb{p} \in \mathbb{Z}/2\mathbb{Z}$ as in the Introduction. As highlighted in the Introduction, we will consider the following set of partitions/Young diagrams/nilpotent orbits.

Definition 2.8. *Denote $\epsilon_l := \epsilon(-1)^l$, for $l \geqslant 0$. Let*

$$\mathcal{P}_\epsilon^{\mathbb{p}} := \left\{ \mathbf{d} = [c_0, \cdots, c_k] \in \mathcal{P}_\epsilon \;\middle|\; \begin{array}{l} \bullet \text{ all } c_i\text{'s have parity } \mathbb{p}; \\ \bullet\ c_l \geqslant c_{l+1} + 2, \text{ if } 0 \leqslant l \leqslant k-1 \text{ and} \\ \quad \epsilon_l = 1. \end{array} \right\}.$$

By convention, $\varnothing \in \mathcal{P}_\epsilon^{\mathbb{p}}$. Denote by $\operatorname{Nil}_{\mathbf{G}}^{\mathbb{p}}(\mathfrak{g})$ the subset of $\operatorname{Nil}_{\mathbf{G}}(\mathfrak{g})$ corresponding to partitions in $\mathcal{P}_{\mathbf{G}}^{\mathbb{p}} := \mathcal{P}_\epsilon^{\mathbb{p}} \cap \mathcal{P}_\epsilon(\dim \mathbf{V})$.

2.5. **Signed Young diagrams and rational nilpotent orbits.** Recall that $(\mathbf{V}, J, L)$ is an $(\epsilon, \dot\epsilon)$-space. Under conjugation by L, the complex Lie algebra $\mathfrak{g}$ decomposes into ±1-eigenspaces:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k}_{\mathbf{V}} \oplus \mathfrak{p}_{\mathbf{V}}. \tag{2.3}$$

Let $\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ be the set of nilpotent $\mathbf{K}$-orbits in $\mathfrak{p}$.

2.5.1. *Parametrization.* In this section, we explain the parameterization of $\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$.

Definition 2.9. *Let $\mathbb{Z}^2_{\geqslant 0}$ be the set of pairs of non-negative integers, whose elements are called* signatures. *For a signature $n = (n^+, n^-)$, its* dual signature *is defined to be $\check{n} := (n^-, n^+)$. Define a partial order on $\mathbb{Z}^2_{\geqslant 0}$ by*

$$(n_1^+, n_1^-) \succeq (n_2^+, n_2^-) \quad \textit{if and only if} \quad n_1^+ \geqslant n_2^+ \textit{ and } n_1^- \geqslant n_2^-.$$

Recall that a signed Young diagram is a Young diagram in which every box is labeled with a + or − sign in such a way that signs alternate across rows. For a signed Young diagram which contains n_i^+ number of "+" signs and n_i^- number of "−" signs in the i-th column, we will label it by the sequence of signatures $[d_0, d_1, \cdots, d_k]$, where $k \geqslant -1$ and $d_i = (n_i^+, n_i^-)$. By convention $[d_0, d_1, \cdots, d_k]$ labels the empty diagram when $k = -1$.

The set of all signed Young diagrams will then correspond to the set

$$\ddot{\mathcal{P}} := \bigsqcup_{k \geqslant -1} \{ [d_0, \cdots, d_k] \in (\mathbb{Z}^2_{\geqslant 0} \backslash \{(0,0)\})^{k+1} \mid d_l \succeq \check{d}_{l+1} \text{ for all } 0 \leqslant l \leqslant k-1 \}.$$

Similar to Definition 2.7, we define the descent map

$$\nabla \colon \ddot{\mathcal{P}} \longrightarrow \ddot{\mathcal{P}}, \quad [d_0, d_1, \cdots, d_k] \mapsto [d_1, \cdots, d_k]. \tag{2.4}$$

Definition 2.10. *Given $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, pick any $X \in \mathscr{O}$. Set* $\mathrm{depth}(\mathscr{O}) := \mathrm{depth}(X)$ *(see* (2.1)*) and define the signed Young diagram*

$$\ddot{\mathbf{d}}_{\mathscr{O}} := [d_0, \cdots, d_k], \qquad (k := \mathrm{depth}(\mathscr{O}) \geqslant -1)$$

where

$$d_l := \mathrm{Sign}(\mathrm{Ker}(X^{l+1})/\mathrm{Ker}(X^l)), \quad \textit{for all } 0 \leqslant l \leqslant k.$$

We parameterize $\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ via the injective map (*cf.* [18])

$$\ddot{\mathbf{D}} \colon \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p}) \longrightarrow \ddot{\mathcal{P}}, \quad \mathscr{O} \mapsto \ddot{\mathbf{d}}_{\mathscr{O}}.$$

2.5.2. *Stabilizers.* Let $\mathfrak{sl}_2(\mathbb{C})$ be the complex Lie algebra consisting of 2×2 complex matrices of trace zero. Write

$$\mathring{L} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathring{X} := \begin{pmatrix} 1/2 & \mathbf{i}/2 \\ \mathbf{i}/2 & -1/2 \end{pmatrix}.$$

Let $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ with $\ddot{\mathbf{D}}(\mathscr{O}) = [d_0, \cdots, d_k]$ and pick any element $X \in \mathscr{O}$. By [62] (also see [69, Section 6]), there is a Lie algebra homomorphism (unique up to $\mathbf{K}$-conjugation)

$$\phi_{\mathfrak{k}} \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$$

such that

- $\phi_{\mathfrak{k}}$ intertwines the conjugation of $\mathring{L}$ on $\mathfrak{sl}_2(\mathbb{C})$ and the conjugation of L on $\mathfrak{g}$; and
- $\phi_{\mathfrak{k}}(\mathring{X}) = X$;

We call $\phi_{\mathfrak{k}}$ an L-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple attached to X.

For each $l \geqslant 0$, let $\phi_l\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{l+1}(\mathbb{C})$ denote an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ realized on $\mathbb{C}^{l+1}$. Fix an $\mathrm{SL}_2(\mathbb{C})$-invariant $(-1)^l$-symmetric non-degenerate bilinear form $\langle\ ,\ \rangle_l$ on $\mathbb{C}^{l+1}$. As an $\mathfrak{sl}_2(\mathbb{C})$-module via $\phi_{\mathfrak{k}}$, we have

$$\mathbf{V} = \bigoplus_{l=0}^{k} {}^{l}\mathbf{V} \otimes_{\mathbb{C}} \mathbb{C}^{l+1}, \qquad (k := \mathrm{depth}(\mathscr{O}) \geqslant -1) \tag{2.5}$$

where $\mathbb{C}^{l+1}$ is viewed as an $\mathfrak{sl}_2(\mathbb{C})$-module via the differential of ϕ_l, and

$${}^{l}\mathbf{V} := \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V})$$

is the multiplicity space.

Let

$$L_l := (-1)^{\lfloor \frac{l}{2} \rfloor}\, \phi_l(\mathring{L}).$$

Then $(\mathbb{C}^{l+1}, L_l)$ is a $((-1)^l, (-1)^l)$-space and the L_l-stable subspace

$$(\mathbb{C}^{l+1})^{\mathring{X}} := \{v \in \mathbb{C}^{l+1} \mid \mathring{X} \cdot v = 0\}$$

has signature $(1, 0)$.

Define the $(-1)^l\epsilon$-symmetric bilinear form $\langle\ ,\ \rangle_{{}^{l}\mathbf{V}}$ on ${}^{l}\mathbf{V}$ by requiring that

$$\langle\ ,\ \rangle_{\mathbf{V}} = \bigoplus_{l} \langle\ ,\ \rangle_{{}^{l}\mathbf{V}} \otimes \langle\ ,\ \rangle_l\,.$$

Define

$${}^{l}L(T) := L \circ T \circ L_l^{-1}, \qquad \text{for all } T \in {}^{l}\mathbf{V} = \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V}).$$

It is routine to check that $({}^{l}\mathbf{V}, {}^{l}L)$ is a $((-1)^l\epsilon, (-1)^l\dot{\epsilon})$-space and it has signature $d_l - \check{d}_{l+1}$ if l is even and has signature $\check{d}_l - d_{l+1}$ if l is odd.

Let $\mathbf{K}_X := \mathrm{Stab}_{\mathbf{K}}(X)$ be the stabilizer of X in $\mathbf{K}$, and $\mathbf{R}_X$ the stabilizer of $\phi_{\mathfrak{k}}$ in $\mathbf{K}$. Then $\mathbf{K}_X = \mathbf{R}_X \ltimes \mathbf{U}_X$, where $\mathbf{U}_X$ denotes the unipotent radical of $\mathbf{K}_X$.

Set ${}^{l}\mathbf{K} := (\mathbf{G}_{{}^{l}\mathbf{V}})^{{}^{l}L}$. Using the decomposition (2.5), we get the following lemma from the discussion above.

Lemma 2.11. *There is a canonical isomorphism*

$$\mathbf{R}_X \cong \prod_{l=0}^{k} {}^{l}\mathbf{K}.$$

Let $A := G/G^{\circ}$ be the component group of G, where G° denotes the identity connected component of G. It is isomorphic to the component group of $\mathbf{K}$ and is trivial unless G is a real orthogonal group.

Suppose we are in the case when $G = \mathrm{O}(p, q)$ $(p, q \geqslant 0)$ is a real orthogonal group. Then $\mathbf{K} = \mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C})$ is a product of two complex orthogonal groups. For every pair $\eta := (\eta^+, \eta^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, let sgn^{η} denote the character $\det^{\eta^+} \boxtimes \det^{\eta^-}$ of $\mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C})$, where det denotes the sign character of an orthogonal group. It obviously induces characters on $\mathrm{O}(p) \times \mathrm{O}(q)$, $\mathrm{O}(p, q)$ and A, which are still denoted by sgn^{η}. Then

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2 \to \widehat{A}, \quad \eta \mapsto \mathrm{sgn}^{\eta}$$

is a surjective homomorphism. Here and henceforth, "$\widehat{\ \ }$" indicates the group of characters of a finite abelian group.

We record the following lemma.

Lemma 2.12. *Suppose $G = \mathrm{O}(p,q)$ $(p,q \geqslant 0)$. Write* $\mathrm{Sign}({}^l\mathbf{V}) = ({}^lp^+, {}^lp^-)$ *for* $0 \leqslant l \leqslant k$. *Then there is an obvious isomorphism*

$$
{}^l\mathbf{K} \cong \begin{cases} \mathrm{O}({}^lp^+, \mathbb{C}) \times \mathrm{O}({}^lp^-, \mathbb{C}), & \text{if } l \text{ is even;} \\ \mathrm{GL}({}^lp^+, \mathbb{C}), & \text{if } l \text{ is odd.} \end{cases} \tag{2.6}
$$

Moreover for $\eta := (\eta^+, \eta^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, *the character* sgn^η *of* $\mathbf{K}$ *has trivial restriction to* $\mathbf{U}_X$, *and with respect to the isomorphism* (2.6),

$$
\mathrm{sgn}^\eta|_{{}^l\mathbf{K}} = \begin{cases} \det^{\eta^+} \boxtimes \det^{\eta^-}, & \text{if } l \text{ is even;} \\ 1, & \text{if } l \text{ is odd.} \end{cases}
$$

Proof. The first assertion follows by using Table 2. Since $\mathbf{U}_X$ is unipotent, it has no nontrivial algebraic character. Thus sgn^η of $\mathbf{K}$ has trivial restriction to $\mathbf{U}_X$. The last assertion of the lemma is routine to check, which we omit. □

2.6. **Dual pairs, descent and lift of nilpotent orbits.**

Definition 2.13 (Dual pair). *(i) A* complex dual pair *is a pair consisting of an ϵ-symmetric bilinear space and an ϵ'-symmetric bilinear space, where* $\epsilon, \epsilon' \in \{\pm 1\}$ *with* $\epsilon\epsilon' = -1$.

(ii) A rational dual pair *is a pair consisting of an $(\epsilon, \dot\epsilon)$-space and an $(\epsilon', \dot\epsilon')$-space, where* $\epsilon, \epsilon', \dot\epsilon, \dot\epsilon' \in \{\pm 1\}$ *with* $\epsilon\epsilon' = \dot\epsilon\dot\epsilon' = -1$.

Let $(\mathbf{V}, \mathbf{V}')$ be a complex dual pair. Define a (-1)-symmetric bilinear space

$$
\mathbf{W} := \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}'),
$$

with the form

$$
\langle T_1, T_2 \rangle_{\mathbf{W}} := \mathrm{tr}(T_1^{\star} T_2), \qquad \text{for all } T_1, T_2 \in \mathbf{W}.
$$

Here $\star\colon \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}') \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}', \mathbf{V})$ is the adjoint map induced by the non-degenerate forms on $\mathbf{V}$ and $\mathbf{V}'$:

$$
\langle Tv, v' \rangle_{\mathbf{V}'} = \langle v, T^{\star} v' \rangle_{\mathbf{V}}, \qquad \text{for all } v \in \mathbf{V}, v' \in \mathbf{V}', T \in \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}').
$$

The pair of groups $(\mathbf{G}, \mathbf{G}') := (\mathbf{G}_{\mathbf{V}}, \mathbf{G}_{\mathbf{V}'})$ is a *complex reductive dual pair* in $\mathrm{Sp}(\mathbf{W})$ in the sense of Howe [30].

Further suppose that $(\mathbf{V}, J, L)$ and $(\mathbf{V}', J', L')$ form a rational dual pair. Then the complex symplectic space $\mathbf{W} = \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}')$ is naturally a $(-1,-1)$-space by defining

$$
J_{\mathbf{W}}(T) := J' \circ T \circ J^{-1} \quad \text{and} \quad L_{\mathbf{W}}(T) := \dot\epsilon\, L' \circ T \circ L^{-1}, \qquad \text{for all } T \in \mathbf{W}.
$$

The pair of groups $(G, G') = (\mathbf{G}_{\mathbf{V}}^J, \mathbf{G}_{\mathbf{V}'}^{J'})$ is then a (real) reductive dual pair in $\mathrm{Sp}(W)$, where $W := \mathbf{W}^{J_{\mathbf{W}}}$.

For the rest of this section, we work in the setting of rational dual pairs and introduce a notion of descent and lift of nilpotent orbits in this context.

2.6.1. *Moment maps.* We define the following *moment maps* $\mathbf{M}$ and $\mathbf{M}'$ with respect to a complex dual pair $(\mathbf{V}, \mathbf{V}')$:

$$
\begin{array}{ccccc}
\mathfrak{g} & \xleftarrow{\ \mathbf{M}\ } & \mathbf{W} & \xrightarrow{\ \mathbf{M}'\ } & \mathfrak{g}', \\
T^{\star}T & \longleftarrow\!\shortmid & T & \longmapsto & TT^{\star}.
\end{array}
$$

When we have a rational dual pair, we decompose

$$
\mathbf{W} = \mathcal{X} \oplus \mathcal{Y}
$$

where $\mathcal{X}$ and $\mathcal{Y}$ are $+\mathbf{i}$ and $-\mathbf{i}$ eigenspaces of $L_{\mathbf{W}}$, respectively. Restriction on $\mathcal{X}$ induces a pair of maps (see (2.3) for the definition of $\mathfrak{p}$ and $\mathfrak{p}'$):

$$\mathfrak{p} \xleftarrow{M:=\mathbf{M}|_{\mathcal{X}}} \mathcal{X} \xrightarrow{M':=\mathbf{M}'|_{\mathcal{X}}} \mathfrak{p}',$$

which are also called *moment maps* (with respect to the rational dual pair). By classical invariant theory (see [74] or [52, Lemma 2.1]), the image $M(\mathcal{X})$ is Zariski closed in $\mathfrak{p}$, and the moment map M induces an isomorphism

$$\mathbf{K}'\backslash\backslash\mathcal{X} \cong M(\mathcal{X}) \tag{2.7}$$

of affine algebraic varieties, where $\mathbf{K}'\backslash\backslash\mathcal{X}$ denotes the affine quotient.[3] Thus [57, Corollary 4.7] implies that M maps every $\mathbf{K}'$-stable Zariski closed subset of $\mathcal{X}$ onto a Zariski closed subset of $\mathfrak{p}$. Similar statement holds for M'. We will use these basic facts freely.

2.6.2. *Lifts and descents of nilpotent orbits.* Let

$$\mathbf{W}^\circ := \{\, T \in \mathbf{W} \mid T \text{ is a surjective map from } \mathbf{V} \text{ onto } \mathbf{V}' \,\}.$$

Clearly $\mathbf{W}^\circ \neq \varnothing$ only if $\dim \mathbf{V} \geqslant \dim \mathbf{V}'$.

Suppose $\mathbf{e} \in \mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathbf{e}' \in \mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. We call $\mathbf{e}'$ (resp. $\mathcal{O}'$) a descent of $\mathbf{e}$ (resp. $\mathcal{O}$), if there exits $T \in \mathbf{W}^\circ$ such that

$$\mathbf{M}(T) = \mathbf{e} \quad \text{and} \quad \mathbf{M}'(T) = \mathbf{e}'.$$

Put

$$\mathcal{X}^\circ := \mathbf{W}^\circ \cap \mathcal{X},$$

and write $\mathbf{K} := \mathbf{K}_{\mathbf{V}}$ and $\mathbf{K}' := \mathbf{K}_{\mathbf{V}'}$. Suppose $X \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ and $X' \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. We call X' (resp. $\mathscr{O}'$) a descent of X (resp. $\mathscr{O}$), if there exits $T \in \mathcal{X}^\circ$ such that

$$M(T) = X \quad \text{and} \quad M'(T) = X'.$$

In all cases, we will say that T realizes the descent, and $\mathcal{O}$ (resp. $\mathscr{O}$) is the lift of $\mathcal{O}'$ (resp. $\mathscr{O}'$). In the notation of Definition 2.7, we then have

$$\boldsymbol{\nabla}(\mathbf{d}_{\mathcal{O}}) = \mathbf{d}_{\mathcal{O}'} \quad \text{and} \quad \nabla(\ddot{\mathbf{d}}_{\mathscr{O}}) = \ddot{\mathbf{d}}_{\mathscr{O}'}.$$

Hence the notion of descent (for nilpotent orbits) defined here agrees with that of Definition 2.7 and (2.4) (for Young diagrams). We will thus write

$$\mathcal{O}' = \boldsymbol{\nabla}(\mathcal{O}) = \boldsymbol{\nabla}_{\mathbf{V},\mathbf{V}'}(\mathcal{O}) \quad \text{and} \quad \mathscr{O}' = \nabla(\mathscr{O}) = \nabla_{\mathbf{V},\mathbf{V}'}(\mathscr{O}).$$

We record a key property on descent and lift:

$$\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}} \quad \text{and} \quad M(M'^{-1}(\overline{\mathscr{O}'})) = \overline{\mathscr{O}}, \tag{2.8}$$

where "$\overline{}$" means taking Zariski closure. This is checked by using explicit formulas in [17, 37] (for complex dual pairs) and [54, Lemma 14] (for rational dual pairs).

In fact by [17, Theorem 1.1], the notion of lift can be extended to an arbitrary complex dual pair and an arbitrary complex nilpotent orbit: for any $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$, $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'}))$ equals to the closure of a unique nilpotent orbit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$. We call $\mathcal{O}$ the *theta lift* of $\mathcal{O}'$, written as

$$\mathcal{O} = \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}(\mathcal{O}'). \tag{2.9}$$

[3] See [57, Section 4.4] for the definition of affine quotient.

2.6.3. *Generalized descent of nilpotent orbits.* Let

$$\mathbf{W}^{\mathrm{gen}} := \{\, T \in \mathbf{W} \mid \text{the image of } T \text{ is a non-degenerate subspace of } \mathbf{V}' \,\}$$

and

$$\mathcal{X}^{\mathrm{gen}} := \mathbf{W}^{\mathrm{gen}} \cap \mathcal{X}.$$

Suppose $X \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ and $X' \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. We call X' (resp. $\mathscr{O}'$) a generalized descent of X (resp. $\mathscr{O}$), if there exits $T \in \mathcal{X}^{\mathrm{gen}}$ such that

$$M(T) = X \quad \text{and} \quad M'(T) = X'.$$

As before, we say that T realizes the generalized descent.

It is easy to see that for each nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, the following three assertions are equivalent (*cf.* [18, Table 4]).

- The orbit $\mathscr{O}$ has a generalized descent.
- The orbit $\mathscr{O}$ is contained in the image of the moment map M.
- Write $\ddot{\mathbf{d}}_{\mathscr{O}} = [d_0, d_1, \cdots, d_k] \in \ddot{\mathcal{P}}$, then

$$\mathrm{Sign}(\,\mathbf{V}') \geq \sum_{i=1}^{k} d_i.$$

When this is the case, $\mathscr{O}$ has a unique generalized descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$, and

$$\ddot{\mathbf{d}}_{\mathscr{O}'} = [d_1 + s, d_2, \cdots, d_k], \quad \text{where } s := \mathrm{Sign}(\,\mathbf{V}') - \sum_{i=1}^{k} d_i. \tag{2.10}$$

We write $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})$. On the other hand, different nilpotent orbits may map to a same nilpotent orbit under $\nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}$.

Analogously, suppose $\mathbf{e} \in \mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathbf{e}' \in \mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. We call $\mathbf{e}'$ (resp. $\mathcal{O}'$) a generalized descent of $\mathbf{e}$ (resp. $\mathcal{O}$), if there exits an element $T \in \mathbf{W}^{\mathrm{gen}}$ such that

$$\mathbf{M}(T) = \mathbf{e} \quad \text{and} \quad \mathbf{M}'(T) = \mathbf{e}'.$$

When this is the case, $\mathcal{O}'$ is determined by $\mathcal{O}$ and we write $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$.

Lemma 2.14. *Assume that $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ has a generalized descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Let $X \in \mathscr{O}$ and $T \in \mathcal{X}^{\mathrm{gen}}$ such that $M(T) = X$. Then $\mathbf{K}' \cdot T$ is the unique closed $\mathbf{K}'$-orbit in $M^{-1}(X)$. Moreover,*

$$\mathscr{O}' = M'(M^{-1}(\mathscr{O})) \cap \mathcal{O}' = M'(M^{-1}(\mathscr{O})) \cap \overline{\mathcal{O}'}, \tag{2.11}$$

where $\mathcal{O}' := \mathbf{G}' \cdot \mathscr{O}'$, which is the generalized descent of $\mathcal{O} := \mathbf{G} \cdot \mathscr{O}$.

Proof. It is elementary to check that $\mathbf{K}' \cdot T$ is Zariski closed in $\mathcal{X}$. Then the first assertion follows by using the isomorphism (2.7). Note that $\mathscr{O}'$ is the only $\mathbf{K}'$-orbit in $\overline{\mathcal{O}'} \cap \mathfrak{p}'$ whose Zariski closure contains $\mathscr{O}'$. Thus the first assertion implies the second one. □

In this article, we will need to consider the following special types of nilpotent orbits.

Definition 2.15. *A nilpotent oribit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ with $\mathrm{d}_{\mathcal{O}} = [c_0, c_1 \cdots, c_k] \in \mathcal{P}_\epsilon$ is said to be good for generalized descent if $k \geqslant 1$ and $c_0 = c_1$. A nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is said to be* good *for generalized descent if the nilpotent orbit $\mathbf{G} \cdot \mathscr{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is good for generalized descent.*

The following lemma exhibits a certain maximality property of nilpotent orbits which are good for generalized descent.

Lemma 2.16. *If $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is good for generalized descent, and $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$, then $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}$. Consequently, if $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is good for generalized descent and $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})$, then $\mathscr{O}$ is an open $\mathbf{K}$-orbit in $M(M'^{-1}(\overline{\mathscr{O}'}))$.*

Proof. This is easy to check using the explicit description of $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'}))$ in [17, Theorem 5.2 and 5.6]. ☐

2.6.4. *Map between isotropy groups.* Suppose $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ admits a descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. According to [37, Proposition 11.1] and [54, Lemmas 13 and 14], $M^{-1}(\mathscr{O})$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit contained in $\mathcal{X}^\circ$ and $M'(M^{-1}(\mathscr{O})) = \mathscr{O}'$. Moreover $\mathbf{K}'$ acts on $M^{-1}(\mathscr{O})$ freely.

Fix $T \in M^{-1}(\mathscr{O})$ which realizes the descent from $X := M(T) \in \mathscr{O}$ to $X' := M'(T) \in \mathscr{O}'$. Denote the respective isotropy subgroups by

$$\mathbf{S}_T := \mathrm{Stab}_{\mathbf{K}\times\mathbf{K}'}(T), \quad \mathbf{K}_X := \mathrm{Stab}_{\mathbf{K}}(X) \quad \text{and} \quad \mathbf{K}'_{X'} := \mathrm{Stab}_{\mathbf{K}'}(X').$$

Then there is a unique homomorphism

$$\alpha\colon \mathbf{K}_X \mapsto \mathbf{K}'_{X'} \tag{2.12}$$

such that $\mathbf{S}_T$ is the graph of α:

$$\mathbf{S}_T = \{\, (k, \alpha(k)) \in \mathbf{K}_X \times \mathbf{K}'_{X'} \mid k \in \mathbf{K}_X \,\}.$$

The homomorphism α is uniquely determined by the requirement that

$$\alpha(k)(Tv) = T(kv) \quad \text{for all } v \in \mathbf{V},\ k \in \mathbf{K}_X.$$

We recall the notation in Section 2.5.2 where $\phi_{\mathfrak{k}}$ is an L-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple attached to X. Let $\mathring{H} := \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$. Then there is a unique L'-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple $\phi_{\mathfrak{k}'}$ attached to X' such that (see [21, Section 5.2])

$$\phi_{\mathfrak{k}'}(\mathring{H}) \circ T - T \circ \phi_{\mathfrak{k}}(\mathring{H}) = T.$$

As an $\mathfrak{sl}_2(\mathbb{C})$-module via $\phi_{\mathfrak{k}'}$,

$$\mathbf{V}' = \bigoplus_{l \geqslant 0}^{k-1} {}^{l}\mathbf{V}' \otimes \mathbb{C}^{l+1}.$$

We adopt notations in Section 2.5.2 to $X' \in \mathscr{O}'$, via $\phi_{\mathfrak{k}'}$. In particular, we have $\mathbf{K}_X = \mathbf{R}_X \ltimes \mathbf{U}_X$ and $\mathbf{K}'_{X'} = \mathbf{R}_{X'} \ltimes \mathbf{U}_{X'}$. Here $\mathbf{U}_X$ and $\mathbf{U}'_{X'}$ are the unipotent radicals, and $\mathbf{R}_X = \prod_{l=0}^{k} {}^{l}\mathbf{K}$ and $\mathbf{R}_{X'} = \prod_{l=0}^{k-1} {}^{l}\mathbf{K}'$ are Levi factors of $\mathbf{K}_X$ and $\mathbf{K}'_{X'}$, respectively.

For each irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}^{l+1}$ fix a nonzero vector $v_l \in (\mathbb{C}^{l+1})^{\mathring{X}}$. For each $l \geqslant 0$, the map $\nu \mapsto \nu(v_l)$ identifies ${}^{l}\mathbf{V} = \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V})$ (resp. ${}^{l}\mathbf{V}'$) with a subspace ${}^{l}\mathbf{V}_0$ of $\mathbf{V}$ (resp. ${}^{l}\mathbf{V}'_0$ of $\mathbf{V}'$). For $1 \leqslant l \leqslant k$, $T^{\star}$ induces a vector space isomorphism[4]

$$\tau_l\colon {}^{l-1}\mathbf{V}' = {}^{l-1}\mathbf{V}'_0 \xrightarrow{v \mapsto T^{\star}(v)} {}^{l}\mathbf{V}_0 = {}^{l}\mathbf{V}.$$

This results in an isomorphism (which is independent of the choices of v_l and v_{l-1})

$$\alpha_l : {}^{l}\mathbf{K} \xrightarrow{\cong} {}^{l-1}\mathbf{K}', \qquad h_l \mapsto (\tau_l)^{-1} \circ h_l \circ \tau_l. \tag{2.13}$$

[4]In fact, it is a similitude between the two formed spaces and satisfis $L \circ \tau_l = \mathbf{i}\, \tau_l \circ L'$.

Lemma 2.17. *The homomorphism α maps $\mathbf{R}_X$ into $\mathbf{R}_{X'}$ and maps $\mathbf{U}_X$ into $\mathbf{U}_{X'}$. Moreover, the map $\alpha|_{\mathbf{R}_X}$ is given by*

$$\begin{aligned} \alpha|_{\mathbf{R}_X}:\ \prod_{l=0}^{k} {}^{l}\mathbf{K} &\longrightarrow \prod_{l=0}^{k-1} {}^{l}\mathbf{K}', \\ (h_0, h_1, \cdots h_k) &\longmapsto (\alpha_1(h_1), \cdots, \alpha_k(h_k)), \qquad h_l \in {}^{l}\mathbf{K}, \end{aligned}$$

where α_l $(1 \leqslant l \leqslant k)$ is given in (2.13).

Proof. Note that α is a surjection since $M^{-1}(X)$ is a $\mathbf{K}'$-orbit (see Lemma A.1 (ii) or [54, Lemma 13]). So $\alpha(\mathbf{U}_X)$ is a unipotent normal subgroup in $\mathbf{K}'_{X'}$ which must be contained in $\mathbf{U}_{X'}$. Note that (see [16, Lemma 3.4.4])

$$\mathbf{R}_X = \mathrm{Stab}_{\mathbf{K}}(\phi_{\mathfrak{k}}(\mathring{X})) \cap \mathrm{Stab}_{\mathbf{K}}(\phi_{\mathfrak{k}}(\mathring{H})).$$

By the defintion of α and $\phi_{\mathfrak{k}'}$, we see

$$\alpha(\mathbf{R}_X) \subset \mathrm{Stab}_{\mathbf{K}'}(\phi_{\mathfrak{k}'}(\mathring{X})) \cap \mathrm{Stab}_{\mathbf{K}'}(\phi_{\mathfrak{k}'}(\mathring{H})) = \mathbf{R}_{X'}.$$

The rest follows from the discussions before the lemma. □

Let A, A_X and $A'_{X'}$ be the component groups of G, $\mathbf{K}_X$ and $\mathbf{K}'_{X'}$ respectively. We will identify A with the component group of $\mathbf{K}$ and let $\chi|_{A_X}$ denote the pullback of $\chi \in \widehat{A}$ via the natural map $A_X \to A$. The homomorphism $\alpha\colon \mathbf{K}_X \mapsto \mathbf{K}'_{X'}$ induces a homomorphism $\alpha\colon A_X \mapsto A'_{X'}$, which further yields a homomorphism

$$\widehat{A'_{X'}} \to \widehat{A_X}, \quad \rho' \mapsto \rho' \circ \alpha.$$

Lemma 2.18. *The following map is surjective:*

$$\begin{aligned} \widehat{A} \times \widehat{A'_{X'}} &\twoheadrightarrow \widehat{A_X}, \\ (\chi, \rho') &\longmapsto \chi|_{A_X} \cdot (\rho' \circ \alpha). \end{aligned}$$

Proof. This follows easily from Lemma 2.17 and Lemma 2.12. □

2.7. **Lifting of equivariant vector bundles and admissible orbit data.** Recall from the Introduction the complexification $\widetilde{\mathbf{K}}$ of $\widetilde{K}$, which is $\mathbf{K}$ except when G is a real symplectic group. For a nilpotent $\mathbf{K}$-orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, let $\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ denote the Grothendieck group of $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaves on $\mathscr{O}$. This is a free abelian group with a free basis consisting of isomorphism classes of irreducible $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Taking the isotropy representation at a point $X \in \mathscr{O}$ yields an identification

$$\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) = \mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}_X), \tag{2.14}$$

where the right hand side denotes the Grothendieck group of the category of $\mathbb{p}$-genuine algebraic representations of the stabilizer group $\widetilde{\mathbf{K}}_X$.

For $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$, let $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ denote the Grothendieck group of $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaves on $\mathcal{O} \cap \mathfrak{p}$. Since $\mathcal{O} \cap \mathfrak{p}$ is the finite union of its $\mathbf{K}$-orbits, we have

$$\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) = \bigoplus_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{2.15}$$

There is a natural partial order $\succeq$ on $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ and $\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$: we say $c_1 \succeq c_2$ (or $c_2 \preceq c_1$) if $c_1 - c_2$ is represented by a $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaf. The same notation obviously applies to other Grothendieck groups.

2.7.1. *An algebraic character.* Attached to a rational dual pair $(\mathbf{V}, \mathbf{V}')$, there is a distinguished character $\varsigma_{\mathbf{V},\mathbf{V}'}$ of $\widetilde{\mathbf{K}} \times \widetilde{\mathbf{K}}'$ arising from the oscillator representation, which we shall describe.

When G is a real symplectic group, let $\mathcal{X}_{\mathbf{V}}$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}}$ is identified with

$$\{ (g, c) \in \mathrm{GL}(\mathcal{X}_{\mathbf{V}}) \times \mathbb{C}^\times \mid \det(g) = c^2 \}.$$

It has a character $(g, c) \mapsto c$, which is denoted by $\det^{\frac{1}{2}}_{\mathcal{X}_{\mathbf{V}}}$.

When G is a quaternionic orthogonal group, still let $\mathcal{X}_V$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}} = \mathrm{GL}(\mathcal{X}_{\mathbf{V}})$. Let $\det_{\mathcal{X}_{\mathbf{V}}}$ denote its determinant character.

Write $\mathrm{Sign}(\mathbf{V}') = (n'^+, n'^-)$. Then the character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}}$ is given by the following formula:

$$\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}} := \begin{cases} \left(\det^{\frac{1}{2}}_{\mathcal{X}_{\mathbf{V}}}\right)^{n'^+ - n'^-}, & \text{if } G \text{ is a real symplectic group;} \\ \det_{\mathcal{X}_{\mathbf{V}}}^{\frac{n'^+ - n'^-}{2}}, & \text{if } G \text{ is a quaternionic orthogonal group;} \\ \text{the trivial character,} & \text{otherwise.} \end{cases}$$

The character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'}$ is given by a similar formula with $n'^+ - n'^-$ replaced by $n^- - n^+$, where $(n^+, n^-) = \mathrm{Sign}(\mathbf{V})$.

2.7.2. *Lift of algebraic vector bundles.* In the rest of this section, we assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$. Then $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}}$ and $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'}$ are $\mathbb{p}$-genuine.

Suppose $T \in \mathcal{X}^\circ$ realizes the descent from $X = M(T) \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ to $X' = M'(T) \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Let $\alpha\colon \mathbf{K}_X \to \mathbf{K}'_{X'}$ be the homomorphism as in (2.12).

Let ρ' be a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}'_{X'}$. Then the representation $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'_{X'}} \otimes \rho'$ of $\widetilde{\mathbf{K}}'_{X'}$ descends to a representation of $\mathbf{K}'_{X'}$. Define

$$\vartheta_T(\rho') := \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}_X} \otimes (\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'_{X'}} \otimes \rho') \circ \alpha, \tag{2.16}$$

which is a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}_X$.

Clearly ϑ_T induces a homomorphism from $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}'_{X'})$ to $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}_X)$. In view of (2.14), we thus have a homomorphism

$$\vartheta_{\mathscr{O}',\mathscr{O}}\colon \mathcal{K}^{\mathbb{p}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{2.17}$$

This is independent of the choice of T.

Suppose $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}' = \nabla(\mathcal{O}) \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. Using decomposition (2.15), we define a homomorphism

$$\vartheta_{\mathcal{O}',\mathcal{O}} := \sum_{\substack{\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p} \\ \mathscr{O}' = \nabla(\mathscr{O}) \subset \mathfrak{p}'}} \vartheta_{\mathscr{O}',\mathscr{O}}\colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}'}(\widetilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) \tag{2.18}$$

where the summation is over all pairs $(\mathscr{O}, \mathscr{O}')$ such that $\mathscr{O}' \subset \mathfrak{p}'$ is the descent of $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}$.

Now suppose $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the generalized decent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. From the discussion in Section 2.6.3, there is an $(\epsilon', \dot{\epsilon}')$-space decomposition $\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$ and an element

$$T \in \mathcal{X}^\circ_1 := \{ w \in \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_1) \mid w \text{ is surjective} \} \cap \mathcal{X} \subseteq \mathcal{X}^{\mathrm{gen}}$$

such that $\mathscr{O}'_1 := \mathbf{K}'_1 \cdot X' \in \mathrm{Nil}_{\mathbf{K}'_1}(\mathfrak{p}'_1)$ is the descent of $\mathscr{O}$. Here $X' := M'(T) \in \mathscr{O}'$, $\mathbf{K}'_i := \mathbf{G}^{L'}_{\mathbf{V}'_i}$ is a subgroup of $\mathbf{K}'$ for $i = 1, 2$, and $\mathfrak{p}'_1 := \mathfrak{p}_{\mathbf{V}'_1}$.

Let $X := M(T) \in \mathscr{O}$ and

$$\alpha_1 \colon \mathbf{K}_X \to \mathbf{K}'_{1,X'} \tag{2.19}$$

be the (surjective) homomorphism defined in (2.12) with respect to the descent from $\mathscr{O}$ to $\mathscr{O}'_1$. Then the stabilizer $\mathbf{S}_T := \mathrm{Stab}_{\mathbf{K}\times\mathbf{K}'}(T)$ of T is given by

$$\mathbf{S}_T = \{\, (k, \alpha_1(k)k'_2) \in \mathbf{K} \times \mathbf{K}' \mid k \in \mathbf{K} \text{ and } k'_2 \in \mathbf{K}'_2 \,\}. \tag{2.20}$$

For a $\mathbb{p}$-genuine algebraic representation ρ' of $\widetilde{\mathbf{K}}'_{X'}$, define a representation

$$\vartheta^{\mathrm{gen}}_T(\rho') := \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}_X} \otimes \left((\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'_{X'}} \otimes \rho')^{\mathbf{K}'_2} \right) \circ \alpha_1. \tag{2.21}$$

Clearly, (2.21) is a generalization of (2.16), as $\mathbf{K}'_2$ is the trivial group in the latter case. As in the descent case, $\vartheta^{\mathrm{gen}}_T$ induces a homomorphism

$$\vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}'}(\mathscr{O}') \to \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}}(\mathscr{O}).$$

Furthermore, (2.18) is extended to the generalized descent case: for every pair $(\mathcal{O}, \mathcal{O}') \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g}) \times \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ with $\mathcal{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$, we define a homomorhism

$$\vartheta_{\mathcal{O}',\mathcal{O}} := \sum_{\substack{\mathscr{O} \subset \mathcal{O}\cap\mathfrak{p},\\ \mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \subset \mathfrak{p}'}} \vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}'}(\mathcal{O}') \to \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}}(\mathcal{O}).$$

2.7.3. *Lift of admissible orbit data.* Now let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$, where $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$. Let $X \in \mathscr{O}$. The component group $A_X := \mathbf{K}_X/\mathbf{K}^\circ_X$ is an elementary abelian 2-group by Section 2.5.2. Let $\mathfrak{k}_X$ be the Lie algebra of $\mathbf{K}_X$. Denote by $\widetilde{\mathbf{K}}_X \to \mathbf{K}_X$ the covering map induced by the covering $\widetilde{\mathbf{K}} \to \mathbf{K}$.

We make the following definition.

Definition 2.19 ([69, Definition 7.13])**.** *Let γ_X denote the one-dimensional $\mathbf{K}_X$-module $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$ and let $\mathrm{d}\gamma_X$ be its differential.* [5] *An irreducible representation ρ of $\widetilde{\mathbf{K}}_X$ is called* admissible *if*

(i) its differential $\mathrm{d}\rho$ is isomorphic to a multiple of $\frac{1}{2}\mathrm{d}\gamma_X$, equivalently,

$$\rho(\exp(x)) = \gamma_X(\exp(x/2)) \cdot \mathrm{id}, \quad \textit{for all } x \in \mathfrak{k}_X, \textit{ and}$$

(ii) it is $\mathbb{p}$-genuine.

Let Φ_X denote the set of all isomorphism classes of admissible irreducible representations of $\widetilde{\mathbf{K}}_X$.

Definition 2.19 is obviously consistent with Definition 1.3, since a representation $\rho \in \Phi_X$ determines an admissible orbit datum $\mathcal{E} \in \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$, where $\mathcal{E}$ is a $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundle on $\mathscr{O}$ whose isotropy representation $\mathcal{E}_X$ at X is isomorphic to ρ. We therefore have an identification

$$\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) = \Phi_X. \tag{2.22}$$

From the structure of $\mathbf{K}_X$ in Lemma 2.11 and Lemma 2.12, it is easy to see that Φ_X consists of one-dimensional representations.

Lemma 2.20. *The tensor product yields a simply transitive action of the character group $\widehat{A_X}$ on the set Φ_X.*

[5] $\mathrm{d}\gamma_X$ is the same as $\mathrm{d}\gamma_{\mathfrak{k}}$ in [69, Theorem 7.11] since $\mathfrak{k}$ is reductive.

Proof. It is clear that we only need to show that Φ_X is nonempty. Consider a rational dual pair $(\mathbf{V}, \mathbf{V}')$ such that $\mathscr{O}' = \nabla(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the descent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. Suppose that $T \in \mathcal{X}^\circ$ realizes the descent from X to $X' \in \mathscr{O}'$. The proof of [44, Proposition 6.1] shows that if $\rho' \in \Phi_{X'}$ is admissible, then $\vartheta_T(\rho')$ is admissible. Therefore, we may do reductions and eventually reduce the problem to the case when $\mathbf{V}$ is the zero space. It is clear that Φ_X is a singleton in this case. □

Remark. By Lemma 2.20, the set $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ of admissible orbit data over $\mathscr{O}$ has 2^r elements, where r is the number of orthogonal groups appearing in the decomposition of $\mathbf{R}_X$ in Lemma 2.11.

As in the proof of Lemma 2.20, the homomorphism (2.17) restricts to a map

$$\vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \to \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{2.23}$$

Lemma 2.21. *Let A be the component group of G which is identified with the component group of $\mathbf{K}$. Then the following map is surjective:*[6]

$$\begin{aligned} \widehat{A} \times \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') &\longrightarrow \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}), \\ (\chi, \mathcal{E}') &\longmapsto \chi \otimes \vartheta_{\mathscr{O}',\mathscr{O}}(\mathcal{E}'). \end{aligned}$$

In particular, $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ is a singleton if G is a quaternionic group.

Proof. This follows from Lemma 2.18 and Lemma 2.20. □

3. Matrix coefficient integrals and degenerate principal series

In this section, we define the notion of a ν-bounded representation and investigate the matrix coefficient integrals of such representations against the oscillator representation as well as certain degenerate principle series.

3.1. Matrix coefficients: growth and positivity.

In this subsection, let G be an arbitrary real reductive group.

Definition 3.1. *A (complex valued) function ℓ on G is said to be of* logarithmic growth *if there is a continuous homomorphism $\sigma \colon G \to \mathrm{GL}_n(\mathbb{R})$ for some $n \geqslant 1$, and an integer $d \geqslant 0$ such that*

$$|\ell(g)| \leqslant \big(\log(1 + \mathrm{tr}((\sigma(g))^{\mathrm{t}} \cdot \sigma(g)))\big)^d \quad \textit{for all } g \in G.$$

Assume that a maximal compact subgroup K of G is given. Let Ξ_G denote Harish-Chandra's Ξ-function on G associated to K.

Definition 3.2. *Let $\nu \in \mathbb{R}$. A function f on G is said to be ν-*bounded *if there is a positive function ℓ on G of logarithmic growth such that*

$$|f(g)| \leqslant \ell(g) \cdot (\Xi_G(g))^\nu \quad \textit{for all } g \in G.$$

We remark that the definition is independent of the choice of K.

Lemma 3.3. *Assume that the identity connected component of G has a compact center. Then for all $\nu > 2$, every ν-bounded continuous function on G is integrable (with respect to a Haar measure).*

Proof. This follows easily from the well-known estimate of Harish-Chandra's Ξ-function, see [72, Theorem 4.5.3]. □

[6] Under the identification (2.22), the tensor product of a character $\chi \in \widehat{A}$ on $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ is identified with the tensor product of $\chi|_{A_X}$ with the isotropy representation.

Definition 3.4. *Let $\nu \in \mathbb{R}$. A Casselman-Wallach representation π of G is said to be* ν-bounded *if there is a ν-bounded positive function f on G, and continuous seminorms $|\ |_\pi$ on π and $|\ |_{\pi^\vee}$ on $\pi^\vee$ such that*

$$|\langle g\cdot u, v\rangle| \leqslant f(g)\cdot |u|_\pi \cdot |v|_{\pi^\vee},$$

for all $g \in G$, $u \in \pi$, $v \in \pi^\vee$.

We also recall the following positivity result on diagonal matrix coefficients, which is a special case of [24, Theorem A. 5].

Proposition 3.5. *Let G be a real reductive group with a maximal compact subgroup K. Let π_1 and π_2 be two unitary representations of G such that π_2 is weakly contained in the regular representation. Let*

$$u := \sum_{i=1}^{s} u_i \otimes v_i \in \pi_1 \otimes \pi_2.$$

Assume that

(a) for all $i, j = 1, 2, \cdots, s$, the function $g \mapsto \langle g\cdot u_i, u_j\rangle \Xi_G(g)$ on G is absolutely integrable with respect to a Haar measure $\mathrm{d}g$ on G;

(b) $v_1, v_2, \cdots, v_s$ are all K-finite.

Then the integral

$$\int_G \langle g\cdot u, u\rangle\, \mathrm{d}g$$

absolutely converges to a nonnegative real number.

3.2. **Theta lifting via matrix coefficient integrals.** We return to the setting of Section 2.6 so that $\mathbf{V}$ is an $(\epsilon, \dot\epsilon)$-space, $\mathbf{V}'$ is a $(-\epsilon, -\dot\epsilon)$-space, and $(\mathbf{W}, J_{\mathbf{W}}, L_{\mathbf{W}})$ is a $(-1,-1)$-space with $\mathbf{W} := \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}')$. Recall that $(G, G') = (\mathbf{G}_{\mathbf{V}}^{J}, \mathbf{G}_{\mathbf{V}'}^{J'})$ is a dual pair in $\mathrm{Sp}(W)$, where $W = \mathbf{W}^{J_{\mathbf{W}}}$.

Let $\mathrm{H}(W) := W \times \mathbb{R}$ denote the Heisenberg group attached to W, with group multiplication

$$(u,t)\cdot(u',t') := (u+u', t+t'+\langle u, u'\rangle_{\mathbf{W}}), \qquad u, u' \in W,\ t, t' \in \mathbb{R}.$$

Note that $\mathbf{W}$ is naturally identified with a subspace of the complexified Lie algebra of $\mathrm{H}(W)$. There is a totally complex polarization $\mathbf{W} = \mathcal{X} \oplus \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are the $+\mathbf{i}$ and $-\mathbf{i}$ eigenspaces of $L_{\mathbf{W}}$ respectively. We have a Cartan decomposition:

$$G_{\mathbf{W}} = \mathrm{Sp}(W) = K_{\mathbf{W}} \times S_{\mathbf{W}}.$$

where $S_{\mathbf{W}} := \left\{ \exp(X) \;\middle|\; X \in \mathfrak{g}_{\mathbf{W}}^{J_{\mathbf{W}}},\ XL_{\mathbf{W}} + L_{\mathbf{W}}X = 0 \right\}$.

3.2.1. *The oscillator representation.* The group $\widetilde{G} \times \widetilde{G}'$ acts on $\mathrm{H}(W)$ as group automorphisms through the natural action of $G \times G'$ on W. Using this, we form the semidirect product $(\widetilde{G} \times \widetilde{G}') \ltimes \mathrm{H}(W)$. As in Section 2.7, we assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$.

Definition 3.6. *A smooth oscillator representation associated to $(\mathbf{V}, \mathbf{V}')$ is a smooth Fréchet representation $\omega_{\mathbf{V},\mathbf{V}'}$ of $(\widetilde{G} \times \widetilde{G}') \ltimes \mathrm{H}(W)$ of moderate growth such that*

- *as a representation of $\mathrm{H}(W)$, it is irreducible with central character $t \mapsto e^{\mathbf{i}t}$;*
- *$\widetilde{K} \times \widetilde{K}'$ acts on $\omega_{\mathbf{V},\mathbf{V}'}^{\mathcal{X}}$ (the space of vectors in $\omega_{\mathbf{V},\mathbf{V}'}$ which are annihilated by $\mathcal{X}$) through the character $\varsigma_{\mathbf{V},\mathbf{V}'}$.*

Note that the first condition in Definition 3.6 implies that the space $\omega^{\mathcal{X}}_{\mathbf{V},\mathbf{V}'}$ is one-dimensional. If G is a nontrivial real symplectic group, a quaternionic symplectic group, or a quaternionic orthogonal group which is not isomorphic to $\mathrm{O}^*(2)$, then the first condition also implies that $\widetilde{K}$ acts on $\omega^{\mathcal{X}}_{\mathbf{V},\mathbf{V}'}$ through the character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}}$. Similar result holds for $\widetilde{K}'$. The smooth oscillator representation always exists and is unique up to isomorphism. From now on we fix a smooth oscillator representation $\omega_{\mathbf{V},\mathbf{V}'}$ for each rational dual pair $(\mathbf{V},\mathbf{V}')$.

Let $\Psi_{\mathbf{W}}$ be the positive function on $G_{\mathbf{W}}$ which is bi-$K_{\mathbf{W}}$-invariant such that

$$\Psi_{\mathbf{W}}(g) = \prod_a (1+a)^{-\frac{1}{2}}, \quad \text{for every } g \in S_{\mathbf{W}},$$

where a runs over all eigenvalues of g, counted with multiplicities. By abuse of notation, we still use $\Psi_{\mathbf{W}}$ to denote the pullback function through the covering map $\widetilde{G}_{\mathbf{W}} \to G_{\mathbf{W}}$.

Lemma 3.7. *Extend the irreducible representation $\omega_{\mathbf{V},\mathbf{V}'}|_{\mathrm{H}(W)}$ to the group $\widetilde{G}_{\mathbf{W}} \ltimes \mathrm{H}(W)$. Then there exists continuous seminorms $|\cdot|_1$ on $\omega_{\mathbf{V},\mathbf{V}'}$ and $|\cdot|_2$ on $\omega^{\vee}_{\mathbf{V},\mathbf{V}'}$ such that*

$$|\langle g\cdot\phi, \phi'\rangle| \leqslant \Psi_{\mathbf{W}}(g)\cdot|\phi|_1\cdot|\phi'|_2, \quad \text{for all } g \in \widetilde{G}_{\mathbf{W}},\ \phi \in \omega_{\mathbf{V},\mathbf{V}'},\ \phi' \in \omega^{\vee}_{\mathbf{V},\mathbf{V}'}.$$

Proof. This is in the proof of [42, Theorem 3.2]. □

Define

$$\dim^{\circ}\mathbf{V} := \begin{cases} \dim\mathbf{V}, & \text{if } G \text{ is real symplectic;}\\ \dim\mathbf{V}-1, & \text{if } G \text{ is quaternionic symplectic;}\\ \dim\mathbf{V}-2, & \text{if } G \text{ is real orthogonal;}\\ \dim\mathbf{V}-3, & \text{if } G \text{ is quaternionic orthogonal.}\end{cases}$$

Let $\Psi_{\mathbf{W}}|_{\widetilde{G}\times\widetilde{G}'}$ denote the pull back of $\Psi_{\mathbf{W}}$ through the natural homomorphism $\widetilde{G}\times\widetilde{G}' \to \widetilde{G}_{\mathbf{W}}$.

Lemma 3.8. *Assume that both* $\dim^{\circ}\mathbf{V}$ *and* $\dim^{\circ}\mathbf{V}'$ *are positive. Then the pointwise inequality*

$$\Psi_{\mathbf{W}}|_{\widetilde{G}\times\widetilde{G}'} \leqslant (\Xi_{\widetilde{G}})^{\frac{\dim\mathbf{V}'}{\dim^{\circ}\mathbf{V}}}\cdot(\Xi_{\widetilde{G}'})^{\frac{\dim\mathbf{V}}{\dim^{\circ}\mathbf{V}'}}$$

holds.

Proof. This is easy to check, by using the well-known estimate of Harish-Chandra's Ξ-function (*cf.* [72, Theorem 4.5.3].)

□

3.2.2. *Matrix coefficient integrals against oscillator representations.* Let π be a Casselman-Wallach representation of $\widetilde{G}$.

Definition 3.9. *Assume that* $\dim^{\circ}\mathbf{V} > 0$. *The pair* $(\pi,\mathbf{V}')$ *is said to be in the* convergent range *if*

- *π is ν_π-bounded for some $\nu_\pi > 2 - \frac{\dim\mathbf{V}'}{\dim^{\circ}\mathbf{V}}$;*
- *if G is a real symplectic group, then ε_G acts on π through the scalar multiplication by $(-1)^{\dim\mathbf{V}'}$.*

In the rest of this section, assume that $\dim^{\circ}\mathbf{V} > 0$ and the pair $(\pi,\mathbf{V}')$ is in the convergent range. Consider the following integral:

$$\begin{aligned}(\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}) \times (\pi^{\vee}\widehat{\otimes}\omega^{\vee}_{\mathbf{V},\mathbf{V}'}) &\longrightarrow \mathbb{C},\\ (u,v) &\longmapsto \int_{\widetilde{G}}\langle g\cdot u, v\rangle\,\mathrm{d}g.\end{aligned} \tag{3.1}$$

Lemma 3.10. *The integrals in* (3.1) *are absolutely convergent and yield a continuous bilinear map.*

Proof. This is implied by Lemma 3.8 and Lemma 3.3. ☐

In view of Lemma 3.10, we define

$$\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi) := \frac{\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}}{\text{the left kernel of (3.1)}}. \tag{3.2}$$

This is a Casselman-Wallach representation of $\widetilde{G}'$, since it is a quotient of the full theta lift $(\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'})_G$ (the Hausdorff coinvariant space).

Lemma 3.11. *Assume that* $\dim^\circ \mathbf{V}' > 0$*. Then the representation* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is* $\frac{\dim \mathbf{V}}{\dim^\circ \mathbf{V}'}$*-bounded.*

Proof. This is also implied by Lemma 3.8 and Lemma 3.3. ☐

3.2.3. *Unitarity.*

Theorem 3.12. *Assume that* $\dim \mathbf{V}' \geqslant \dim^\circ \mathbf{V}$ *and* π *is* ν_π*-bounded for some*

$$\nu_\pi > \begin{cases} 2 - \frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}, & \textit{if } \dim^\circ \mathbf{V} \textit{ is even,} \\ 2 - \frac{\dim \mathbf{V}'-1}{\dim^\circ \mathbf{V}}, & \textit{if } \dim^\circ \mathbf{V} \textit{ is odd.} \end{cases}$$

Then $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is unitarizable if* π *is unitarizable. Moreover, when* π *is irreducible and unitarizable, and* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is nonzero, then* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is irreducible unitarizable.*

Proof. Assume that π is unitarizable. For the first claim, it suffices to show that

$$\int_{\widetilde{G}} \langle g\cdot u, u\rangle \, \mathrm{d}g \geqslant 0 \quad \text{for all } u \text{ in a dense subspace of } \pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}. \tag{3.3}$$

Here $\langle\, ,\, \rangle$ denotes a Hermitian inner product on $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}$ which is invariant under $\widetilde{G} \times ((\widetilde{G}\times\widetilde{G}')\ltimes \mathrm{H}(W))$.

Take an orthogonal decomposition

$$\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$$

which is stable under both J' and L' such that

$$\dim \mathbf{V}'_2 = \begin{cases} \dim^\circ \mathbf{V}, & \text{if } \dim^\circ \mathbf{V} \text{ is even.} \\ \dim^\circ \mathbf{V} + 1, & \text{if } \dim^\circ \mathbf{V} \text{ is odd.} \end{cases}$$

(When $\dim^\circ \mathbf{V}$ is odd, $\dim \mathbf{V}'$ is always even.) Write

$$\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'} = (\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'_1})\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'_2}.$$

Note that the Hilbert space completions of $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'_1}$ and $\omega_{\mathbf{V},\mathbf{V}'_2}$, viewed as unitary representations of $\widetilde{G}$, satisfy the hypothesis for π_1 and π_2 in Proposition 3.5. For the latter, see [42, Theorem 3.2]. Thus (3.3) follows by Proposition 3.5.

The second claim follows from the fact that $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is a quotient of $(\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\pi)_G$ which is of finite length and has a unique irreducible quotient [32]. ☐

3.3. **Degenerate principal series.** Let $(\mathbf{U}, J_{\mathbf{U}}, L_{\mathbf{U}})$ be a nonzero $(\epsilon, \dot{\epsilon})$-space which is split in the sense that there is a $J_{\mathbf{U}}$-stable totally isotropic subspace $\mathbf{E}$ of $\mathbf{U}$ whose dimension equals half of that of $\mathbf{U}$. Then we have a complete polarization

$$\mathbf{U} = \mathbf{E} \oplus \mathbf{E}', \quad \text{where } \mathbf{E}' := L_{\mathbf{U}}(\mathbf{E}) \text{ is also totally isotropic.} \tag{3.4}$$

Define notations as in Section 2.3, for example $G_{\mathbf{U}} := \mathbf{G}_{\mathbf{U}}^{J_{\mathbf{U}}}$. The parabolic subgroup $P_{\mathbf{E}}$ of $G_{\mathbf{U}}$ stabilizing $\mathbf{E}$, has a Levi decomposition

$$P_{\mathbf{E}} = \mathrm{GL}_{\mathbf{E}} \ltimes N_{\mathbf{E}},$$

where $N_{\mathbf{E}}$ is the unipotent radical and $\mathrm{GL}_{\mathbf{E}} \cong \mathrm{GL}(\mathbf{E})^{J_{\mathbf{U}}}$ is the stabilizer of the polarization (3.4) in $G_{\mathbf{U}}$. Since the covering $\widetilde{P_{\mathbf{E}}} \to P_{\mathbf{E}}$ uniquely splits over $N_{\mathbf{E}}$, we view $N_{\mathbf{E}}$ as a subgroup of $\widetilde{P_{\mathbf{E}}}$. Thus we have a decomposition

$$\widetilde{P_{\mathbf{E}}} = \widetilde{\mathrm{GL}_{\mathbf{E}}} \ltimes N_{\mathbf{E}}.$$

In this section, we assume that $\mathbb{p}$ is the parity of $\dim \mathbf{E} + 1$ if $G_{\mathbf{U}}$ is real orthogonal or real symplectic, and the parity of $\dim \mathbf{E}$ if $G_{\mathbf{U}}$ is quaternionic. Let $\chi_{\mathbf{E}}$ be a character of $\widetilde{\mathrm{GL}_{\mathbf{E}}}$ as in Table 3. In the real symplectic case, $\varepsilon_{\mathbf{U}}$ denotes the nontrivial element in the kernel of the covering $\widetilde{G}_{\mathbf{U}} \to G_{\mathbf{U}}$, and there are two choices of $\chi_{\mathbf{E}}$ if $\mathbf{U}$ is moreover nonzero. In the quaternionic case, $\det : \mathrm{GL}_{\mathbf{E}} \to \mathbb{R}_+^\times$ denotes the reduced norm.

$G_{\mathbf{U}}$	$\chi_{\mathbf{E}}$
real orthogonal	1
real symplectic	$\chi_{\mathbf{E}}^4 = 1,\ \ \chi_{\mathbf{E}}(\varepsilon_{\mathbf{U}}) = (-1)^{\mathbb{p}}$
quaternionic orthogonal	$\det^{\frac{1}{2}}$
quaternionic symplectic	$\det^{-\frac{1}{2}}$

TABLE 3. The character $\chi_{\mathbf{E}}$

For each character χ of $\widetilde{\mathrm{GL}_{\mathbf{E}}}$, define the degenerate principal series

$$\mathrm{I}(\chi) := \mathrm{Ind}_{\widetilde{P_{\mathbf{E}}}}^{\widetilde{G}_{\mathbf{U}}} \chi.$$

We are particularly interested in $\mathrm{I}(\chi_{\mathbf{E}})$, which plays a critical role in the study of theta correspondence.

Define a positive function on $\widetilde{G}_{\mathbf{U}}$ by setting

$$\Xi_{\mathbf{U}}(g) := \int_{\widetilde{K}_{\mathbf{U}}} f_0(xg)\,\mathrm{d}x, \qquad g \in \widetilde{G}_{\mathbf{U}},$$

where f_0 denotes the element of $\mathrm{Ind}_{\widetilde{P_{\mathbf{E}}}}^{\widetilde{G}_{\mathbf{U}}} |\chi_{\mathbf{E}}|$ whose restriction to $\widetilde{K}_{\mathbf{U}}$ is the constant function 1, and $\mathrm{d}x$ denotes the normalized Haar measure on $\widetilde{K}_{\mathbf{U}}$. This function is independent of $\mathbf{E}$.

Identify the representation $\mathrm{I}(\chi_{\mathbf{E}})^\vee$ with $\mathrm{I}(\chi_{\mathbf{E}}^{-1})$ via the paring

$$(f_1, f_2) \mapsto \langle f_1, f_2 \rangle := \int_{\widetilde{K}_{\mathbf{U}}} f_1(x) f_2(x)\mathrm{d}x, \quad f_1 \in \mathrm{I}(\chi_{\mathbf{E}}),\ f_2 \in \mathrm{I}(\chi_{\mathbf{E}}^{-1}).$$

The following lemma is clear from the definition of $\Xi_{\mathbf{U}}$.

Lemma 3.13. *For all $f_1 \in \mathrm{I}(\chi_{\mathbf{E}})$, $f_2 \in \mathrm{I}(\chi_{\mathbf{E}}^{-1})$ and $g \in \widetilde{G}_{\mathbf{U}}$,*

$$|\langle g \cdot f_1, f_2 \rangle| \leqslant \Xi_{\mathbf{U}}(g) \cdot \max_{x \in \widetilde{K}_{\mathbf{U}}} |f_1(x)| \cdot \max_{x \in \widetilde{K}_{\mathbf{U}}} |f_2(x)|. \quad \square$$

3.4. **Matrix coefficient integrals against degenerate principal series.** Now we assume that $(\mathbf{V}, J, L)$ is a non-degenerate $(\epsilon, \dot\epsilon)$-subspace of $\mathbf{U}$ so that

$$J = J_{\mathbf{U}}|_{\mathbf{V}} \quad \text{and} \quad L = L_{\mathbf{U}}|_{\mathbf{V}}.$$

Then $\widetilde{G} := \widetilde{G}_{\mathbf{V}}$ is naturally a closed subgroup of $\widetilde{G}_{\mathbf{U}}$. Further assume that

$$\dim \mathbf{V} \leqslant \dim \mathbf{E}.$$

We are interested in the growth of $\Xi_{\mathbf{U}}|_{\widetilde{G}}$.

Lemma 3.14. *Assume that* $\dim^\circ \mathbf{V} > 0$. *Then* $\Xi_{\mathbf{U}}|_{\widetilde{G}}$ *is* $\nu_{\mathbf{U},\mathbf{V}}$*-bounded, where*

$$\nu_{\mathbf{U},\mathbf{V}} := \begin{cases} \frac{\dim \mathbf{U}-2}{2\dim^\circ \mathbf{V}}, & \text{if } G \text{ is real orthogonal;} \\ \frac{\dim \mathbf{U}+2}{2\dim^\circ \mathbf{V}}, & \text{if } G \text{ is real symplectic;} \\ \frac{\dim \mathbf{U}}{2\dim^\circ \mathbf{V}}, & \text{if } G \text{ is quaternionic.} \end{cases} \tag{3.5}$$

Proof. In view of the estimate of Harish-Chandra's Ξ-function as in [72, Theorem 4.5.3], and the estimate of $\Xi_{\mathbf{U}}$ using [72, Corollary 3.6.8], the lemma is routine to check. □

Let π be a $\mathbb{p}$-genuine Casselman-Wallach representation of $\widetilde{G}$. Further assume that $\dim^\circ \mathbf{V} > 0$ and

$$\pi \text{ is } \nu_\pi\text{-bounded for some } \nu_\pi > 2 - \nu_{\mathbf{U},\mathbf{V}}, \tag{3.6}$$

where $\nu_{\mathbf{U},\mathbf{V}}$ is as in (3.5).

Let $\mathcal{J}$ be a subquotient of the degenerate principal series $\mathrm{I}(\chi_{\mathbf{E}})$.

Lemma 3.15. *The integrals in*

$$\begin{array}{rcl} (\pi\widehat{\otimes}\mathcal{J}) \times (\pi^\vee\widehat{\otimes}\mathcal{J}^\vee) & \to & \mathbb{C} \\ (u,v) & \mapsto & \int_{\widetilde{G}} \langle g\cdot u, v\rangle \,\mathrm{d}g, \end{array} \tag{3.7}$$

are absolutely convergent and define a continuous bilinear map.

Proof. By Lemma 3.13, there is a continuous seminorm $|\cdot|_{\mathcal{J}}$ on $\mathcal{J}$ and a continuous seminorm $|\cdot|_{\mathcal{J}^\vee}$ on $\mathcal{J}^\vee$ such that

$$|\langle g\cdot v_1, v_2\rangle| \leqslant \Xi_{\mathbf{U}}(g)\cdot |v_1|_{\mathcal{J}} \cdot |v_2|_{\mathcal{J}^\vee}, \quad \text{for all } v_1 \in \mathcal{J},\ v_2 \in \mathcal{J}^\vee,\ g \in \widetilde{G}_{\mathbf{U}}.$$

Take a positive function ℓ on $\widetilde{G}$ of logarithmic growth, and continuous seminorms $|\ |_\pi$ on π and $|\ |_{\pi^\vee}$ on $\pi^\vee$ such that

$$|\langle g\cdot u_1, u_2\rangle| \leqslant \ell(g)\cdot (\Xi_{\widetilde{G}}(g))^{\nu_\pi}\cdot |u_1|_\pi \cdot |u_2|_{\pi^\vee},$$

for all $g \in \widetilde{G}$, $u_1 \in \pi$, $u_2 \in \pi^\vee$.

Let $|\cdot|_{\pi\widehat{\otimes}\mathcal{J}}$ denote the continuous seminorm of $\pi\widehat{\otimes}\mathcal{J}$ which is the projective product of $|\cdot|_\pi$ and $|\cdot|_{\mathcal{J}}$. Likewise let $|\cdot|_{\pi^\vee\widehat{\otimes}\mathcal{J}^\vee}$ denote the continuous seminorm of $\pi^\vee\widehat{\otimes}\mathcal{J}^\vee$ which is the projective product of $|\cdot|_{\pi^\vee}$ and $|\cdot|_{\mathcal{J}^\vee}$. Then it is easy to see that

$$|\langle g\cdot w_1, w_2\rangle| \leqslant \ell(g)\cdot (\Xi_{\widetilde{G}}(g))^{\nu_\pi}\cdot \Xi_{\mathbf{U}}(g)\cdot |w_1|_{\pi\widehat{\otimes}\mathcal{J}} \cdot |w_2|_{\pi^\vee\widehat{\otimes}\mathcal{J}^\vee}, \tag{3.8}$$

for all $g \in \widetilde{G}$, $w_1 \in \pi\widehat{\otimes}\mathcal{J}$, $w_2 \in \pi^\vee\widehat{\otimes}\mathcal{J}^\vee$. Thus the lemma follows by Lemma 3.3 and Lemma 3.14. □

The orthogonal complement $\mathbf{V}^\perp$ of $\mathbf{V}$ in $\mathbf{U}$ is naturally an $(\epsilon, \dot\epsilon)$-space with $J_{\mathbf{V}^\perp} := J_{\mathbf{U}}|_{\mathbf{V}^\perp}$ and $L_{\mathbf{V}^\perp} := L_{\mathbf{U}}|_{\mathbf{V}^\perp}$. Define

$$\mathcal{R}_{\mathcal{J}}(\pi) := \frac{\pi\widehat{\otimes}\mathcal{J}}{\text{the left kernel of the bilinear map (3.7)}}. \tag{3.9}$$

It is a smooth Fréchet representation of $\widetilde{G}_{\mathbf{V}^\perp}$ of moderate growth.

Let $(\mathbf{V}^-, J_{\mathbf{V}^-}, L_{\mathbf{V}^-})$ denote the $(\epsilon,\dot\epsilon)$-space which equals $\mathbf{V}$ as a vector space, and is equipped with the form $-\langle\ ,\ \rangle_{\mathbf{V}}$, the conjugate linear map $J_{\mathbf{V}^-} = J$, and the linear map $L_{\mathbf{V}^-} = -L$. Then we have an obvious identification $G_{\mathbf{V}^-} = G_{\mathbf{V}}$. We identify $\mathbf{V}^-$ with an $(\epsilon,\dot\epsilon)$-subspace of $\mathbf{V}^\perp$ and fix a $J_{\mathbf{U}}$-stable subspace $\mathbf{E}_0$ in $\mathbf{E}$ such that

$$\mathbf{V}^\perp = \mathbf{V}^- \oplus (\mathbf{E}_0 \oplus \mathbf{E}_0'), \quad \mathbf{E} = \mathbf{V}^\triangle \oplus \mathbf{E}_0 \quad \text{and} \quad \mathbf{E}' = \mathbf{V}^\nabla \oplus \mathbf{E}_0', \tag{3.10}$$

where $\mathbf{E}_0' = L_{\mathbf{V}^\perp}(\mathbf{E}_0)$,

$$\mathbf{V}^\triangle := \{(v,v) \in \mathbf{V} \oplus \mathbf{V}^- \mid v \in \mathbf{V}\} \quad \text{and} \quad \mathbf{V}^\nabla := \{(v,-v) \in \mathbf{V} \oplus \mathbf{V}^- \mid v \in \mathbf{V}\}.$$

In particular, $G := G_{\mathbf{V}}$ is identified with a subgroup of $G_{\mathbf{V}^\perp}$. Let $\mathrm{GL}_{\mathbf{E}_0} := \mathrm{GL}(\mathbf{E}_0)^{J_{\mathbf{V}^\perp}|_{\mathbf{E}_0}}$ and let

$$P_{\mathbf{E}_0} = M_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$$

denote the parabolic subgroup of $G_{\mathbf{V}^\perp}$ stabilizing $\mathbf{E}_0$, where $N_{\mathbf{E}_0}$ denotes the unipotent radical, and $M_{\mathbf{E}_0} = G \times \mathrm{GL}_{\mathbf{E}_0}$ is the Levi subgroup stabilizing the first decomposition in (3.10).

Let $\chi_{\mathbf{E}_0}$ denotes the restriction of $\chi_{\mathbf{E}}$ to $\widetilde{\mathrm{GL}_{\mathbf{E}_0}}$. Then $\pi \otimes \chi_{\mathbf{E}_0}$, which is preliminarily a representation of $\widetilde{G} \times \widetilde{\mathrm{GL}_{\mathbf{E}_0}}$, descends to a representation of $\widetilde{M}_{\mathbf{E}_0}$.

The rest of this section is devoted to a proof of the following proposition.

Proposition 3.16. *There is an isomorphism*

$$\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \cong \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})$$

of representations of $\widetilde{G}_{\mathbf{V}^\perp}$.

Identify $\mathrm{I}(\chi_{\mathbf{E}})$ with the space of smooth sections of the line bundle

$$\mathcal{L}(\chi_{\mathbf{E}}) := \widetilde{P}_{\mathbf{E}} \backslash (\widetilde{G}_{\mathbf{U}} \times (\chi_{\mathbf{E}} \otimes \rho_{\mathbf{E}}))$$

over $\widetilde{P}_{\mathbf{E}} \backslash \widetilde{G}_{\mathbf{U}}$, where $\rho_{\mathbf{E}}$ denotes the positive character of $\widetilde{P}_{\mathbf{E}}$ whose square is the modular character.

Note that $\widetilde{G}^\circ_{\mathbf{U}} := \widetilde{P}_{\mathbf{E}} \widetilde{G} \widetilde{G}_{\mathbf{V}^\perp}$ is open in $\widetilde{G}_{\mathbf{U}}$. Put

$$\mathrm{I}^\circ(\chi_{\mathbf{E}}) := \left\{ f \in \mathrm{I}(\chi_{\mathbf{E}}) \;\middle|\; \begin{array}{l} (Df)|_{\widetilde{G}_{\mathbf{U}} \backslash \widetilde{G}^\circ_{\mathbf{U}}} = 0 \text{ for all left invariant} \\ \text{differential operators } D \text{ on } \widetilde{G}_{\mathbf{U}} \end{array} \right\}.$$

It is identical to the space of Schwartz sections of $\mathcal{L}(\chi_{\mathbf{E}})$ over $\widetilde{P}_{\mathbf{E}} \backslash \widetilde{G}^\circ_{\mathbf{U}}$. Similarly we define a subspace $\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$ of $\mathrm{I}(\chi_{\mathbf{E}}^{-1})$.

As in (3.7), we define a continuous bilinear map

$$(\pi \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}})) \times (\pi^\vee \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})) \to \mathbb{C}. \tag{3.11}$$

Using this map, we define a representation $\mathcal{R}_{\mathrm{I}^\circ(\chi_{\mathbf{E}})}(\pi)$ of $\widetilde{G}_{\mathbf{V}^\perp}$ as in (3.9).

Put

$$P'_{\mathbf{E}_0} := \mathrm{GL}_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}. \tag{3.12}$$

Lemma 3.17. *There is a canonical isomorphism*

$$\pi \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}) \cong \mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}) \tag{3.13}$$

of representations of $\widetilde{G} \times \widetilde{G}_{\mathbf{V}^\perp}$, *where* $\widetilde{G}$ *acts diagonally on the left hand side of* (3.13), *and it acts on the right hand side of* (3.13) *by*

$$(g \cdot f)(x) = g \cdot (f(g^{-1}x)), \quad f \in \mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}),\ g \in \widetilde{G},\ x \in \widetilde{G}_{\mathbf{V}^\perp}.$$

Proof. Note that

$$\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}^\circ_{\mathbf{U}} = \widetilde{P}'_{\mathbf{E}_0}\backslash\widetilde{G}_{\mathbf{V}^\perp}$$

and thus

$$\mathrm{I}^\circ(\chi_{\mathbf{E}}) \cong \mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} \chi_{\mathbf{E}_0} \tag{3.14}$$

as representations of $\widetilde{G}\times\widetilde{G}_{\mathbf{V}^\perp}$, where $\widetilde{G}$ acts on the right hand side of (3.14) by

$$(g\cdot f)(x) = f(g^{-1}x), \quad f\in \mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} \chi_{\mathbf{E}_0},\ g\in\widetilde{G},\ x\in\widetilde{G}_{\mathbf{V}^\perp}.$$

The lemma then easily follows by (3.14). □

Lemma 3.18. *The representation $\mathcal{R}_{\mathrm{I}^\circ(\chi_{\mathbf{E}})}(\pi)$ is isomorphic to $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})$.*

Proof. Similar to (3.13), we have a canonical isomorphism

$$\pi^\vee\widetilde{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1}) \cong \mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1}). \tag{3.15}$$

It is clear that the continuous linear map

$$\begin{array}{rcl}\xi:\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}) & \longrightarrow & \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}),\\ f & \mapsto & \left(x\mapsto \int_{\widetilde{P}_{\mathbf{E}_0}/\widetilde{P}'_{\mathbf{E}_0}}(\rho_{\mathbf{E}_0}(g))\cdot(g\cdot(f(g^{-1}x)))\,\mathrm{d}g\right),\end{array}$$

is $\widetilde{G}_{\mathbf{V}^\perp}$-intertwining and surjective, where $\rho_{\mathbf{E}_0}$ denotes the positive character of $\widetilde{P}_{\mathbf{E}_0}$ whose square is the modular character, and $\mathrm{d}g$ denotes a $\widetilde{P}_{\mathbf{E}_0}$-invariant positive Borel measure on $\widetilde{\mathrm{P}}(\mathbf{E}_0)/\widetilde{\mathrm{P}}'(\mathbf{E}_0)$. Similarly, we define a $\widetilde{G}_{\mathbf{V}^\perp}$-intertwining surjective continuous linear map

$$\xi':\ \mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1}) \to \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1}). \tag{3.16}$$

It is elementary to see that, when the invariant measures are suitably normalized,

$$\langle\xi(f),\xi'(f')\rangle = \int_{\widetilde{G}}\langle g\cdot f, f'\rangle\,\mathrm{d}g, \tag{3.17}$$

for all f and f' in the domains of ξ and ξ', respectively. Note that under isomorphisms (3.13) and (3.15), the paring between the domains of ξ and ξ' as in the right hand side of (3.17) agrees with the paring (3.11) between $\pi\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}})$ and $\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$. Thus the lemma follows as the natural pairing between $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})$ and $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1})$ is non-degenerate. □

Lemma 3.19. *Let $u\in\pi\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}})$. If*

$$\int_{\widetilde{G}}\langle g\cdot u, v\rangle\,\mathrm{d}g = 0 \tag{3.18}$$

for all $v\in\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$, then (3.18) *also holds for all $v\in\pi^\vee\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}}^{-1})$.*

Proof. Take a sequence $(\eta_1,\eta_2,\eta_3,\cdots)$ of real valued smooth functions on $\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}_{\mathbf{U}}$ such that

- for all $i\geqslant 1$, $\mathrm{supp}(\eta_i)\subset\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}^\circ_{\mathbf{U}}$ and $0\leqslant\eta_i(x)\leqslant\eta_{i+1}(x)\leqslant 1$ for all $x\in\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}_{\mathbf{U}}$;
- $\bigcup_{i=1}^\infty\eta_i^{-1}(1) = \widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}^\circ_{\mathbf{U}}$.

Let $v \in \pi^\vee \widehat{\otimes} \mathrm{I}(\chi_{\mathbf{E}}^{-1})$. Note that $\eta_i \mathrm{I}(\chi_{\mathbf{E}}^{-1}) \subset \mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$. Thus $\eta_i v \in \pi^\vee \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$.

In the proof of Lemma 3.15, take $\mathcal{J} = \mathrm{I}(\chi_{\mathbf{E}})$ and

$$|f_1|_{\mathcal{J}} = \max_{x \in \widetilde{K}_{\mathbf{U}}} \{|f_1(x)| \mid x \in \widetilde{K}_{\mathbf{U}}\} \quad \text{and} \quad |f_2|_{\mathcal{J}^\vee} = \max_{x \in \widetilde{K}_{\mathbf{U}}} \{|f_2(x)| \mid x \in \widetilde{K}_{\mathbf{U}}\},$$

for all $f_1 \in \mathrm{I}(\chi_{\mathbf{E}})$, $f_2 \in \mathrm{I}(\chi_{\mathbf{E}}^{-1})$. Then

$$|\eta_i v|_{\pi^\vee \widehat{\otimes} \mathcal{J}^\vee} \leqslant |v|_{\pi^\vee \widehat{\otimes} \mathcal{J}^\vee}.$$

Thus by (3.8) and Lebesgue's dominated convergence theorem,

$$\int_{\widetilde{G}} \langle g \cdot u, v \rangle \, \mathrm{d}g = \lim_{i \to +\infty} \int_{\widetilde{G}} \langle g \cdot u, \eta_i v \rangle \, \mathrm{d}g = 0.$$

□

Lemma 3.20. *Let $u \in \pi \widehat{\otimes} \mathrm{I}(\chi_{\mathbf{E}})$. Then there is a vector $u' \in \pi \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}})$ such that*

$$\int_{\widetilde{G}} \langle g \cdot u, v \rangle \, \mathrm{d}g = \int_{\widetilde{G}} \langle g \cdot u', v \rangle \, \mathrm{d}g$$

for all $v \in \pi^\vee \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$.

Proof. Similar to Lemma 3.19, we know that for every $v \in \pi^\vee \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$, if

$$\int_{\widetilde{G}} \langle g \cdot u', v \rangle \, \mathrm{d}g = 0 \quad \text{for all } u' \in \pi \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}), \tag{3.19}$$

then (3.19) also holds for all $u' \in \pi \widehat{\otimes} \mathrm{I}(\chi_{\mathbf{E}})$. The proof of Lemma 3.18 shows that v satisfies (3.19) if and only if it is in the kernel of ξ'. Here ξ' is as in (3.16), and $\pi^\vee \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$ is identified with $\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi^\vee \otimes \chi_{\mathbf{E}_0}^{-1})$ as in (3.15). Thus the continuous linear functional

$$\begin{aligned} \pi^\vee \widehat{\otimes} \mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1}) &\longrightarrow \mathbb{C}, \\ v &\longmapsto \textstyle\int_{\widetilde{G}} \langle g \cdot u, v \rangle \, \mathrm{d}g \end{aligned}$$

descents to a continuous linear functional

$$\varphi_u : \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi^\vee \otimes \chi_{\mathbf{E}_0}^{-1}) \longrightarrow \mathbb{C}.$$

Using the theorem of Dixmier-Malliavin [19, Theorem 3.3], we write

$$u = \sum_{i=1}^{s} \int_{\widetilde{G}_{\mathbf{V}^\perp}} \phi_i(g) g \cdot u_i \, \mathrm{d}g,$$

where ϕ_i's are compactly supported smooth functions on $\widetilde{G}_{\mathbf{V}^\perp}$, and u_i's are elements of $\pi \widehat{\otimes} \mathrm{I}(\chi_{\mathbf{E}})$. Here $\widetilde{G}_{\mathbf{V}^\perp}$ acts on $\pi \widehat{\otimes} \mathrm{I}(\chi_{\mathbf{E}})$ through the restriction of the action of $\widetilde{G}_{\mathbf{U}}$ on $\mathrm{I}(\chi_{\mathbf{E}})$. Then it is easily checked that for every $v' \in \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi^\vee \otimes \chi_{\mathbf{E}_0}^{-1})$,

$$\varphi_u(v') = \sum_{i=1}^{s} \varphi_{u_i} \left(\int_{\widetilde{G}_{\mathbf{V}^\perp}} \phi_i(g^{-1}) g \cdot v' \, \mathrm{d}g \right).$$

This equals the evaluation at v' of the linear functional

$$\sum_{i=1}^{s} \int_{\widetilde{G}_{\mathbf{V}^\perp}} \phi_i(g) g \cdot \varphi_{u_i} \, \mathrm{d}g.$$

By [66, Lemma 3.5], the above functional equals the pairing with an element of $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})$, and the lemma then follows by the proof of Lemma 3.18. □

Now Proposition 3.16 follows by Lemmas 3.18, 3.19 and 3.20.

3.5. **Degenerate principal series and Rallis quotients.** We are in the setting of Section 3.2 and Section 3.3. Assume that $\dim^\circ \mathbf{V}' > 0$. Recall $\mathbf{U} = \mathbf{E} \oplus \mathbf{E}'$ from (3.4) and assume that

$$\dim \mathbf{E} = \dim \mathbf{V}' + \delta, \quad \text{where } \delta := \begin{cases} 1, & \text{if } G_{\mathbf{U}} \text{ is real orthogonal;} \\ -1, & \text{if } G_{\mathbf{U}} \text{ is real symplectic;} \\ 0, & \text{if } G_{\mathbf{U}} \text{ is quaternionic.} \end{cases} \tag{3.20}$$

The Rallis quotient $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}$ is an irreducible unitarizable representation of $\widetilde{G}_{\mathbf{U}}$ (*cf.* [38, 60, 75]). Let $1_{\mathbf{V}'}$ denote the trivial representation of $\widetilde{G}'$. Then $(1_{\mathbf{V}'}, \mathbf{U})$ is in the convergent range.

Lemma 3.21. *One has that*

$$\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'}) \cong (\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}.$$

Proof. Since $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}$ is irreducible and $\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})$ is obviously a quotient of $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}$, it suffices to show that $\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})$ is nonzero.

Write V_0' for the standard real or quaternionic representation of G'. As usual, realize both $\omega_{\mathbf{V}',\mathbf{U}}$ and $\omega^\vee_{\mathbf{V}',\mathbf{U}}$ on the space of Schwartz functions on ${V_0'}^d$, where $d = \dim \mathbf{E}$ if G' is real orthogonal or real symplectic, and $d = \frac{1}{2}\dim \mathbf{E}$ if G' is quaternionic. Take a positive valued Schwartz function ϕ on ${V_0'}^d$. Then

$$\langle g \cdot \phi, \phi \rangle = \int_{{V_0'}^d} \phi(g^{-1} \cdot x) \cdot \phi(x)\, \mathrm{d}x > 0, \quad \text{for all } g \in \widetilde{G}'.$$

Thus

$$\int_{\widetilde{G}'} \langle g \cdot \phi, \phi \rangle \, \mathrm{d}g \neq 0,$$

and the lemma follows. □

Let $\chi_{\mathbf{E}}$ be as in Section 3.3. The relationship between the degenerate principle series representation $\mathrm{I}(\chi_{\mathbf{E}})$ and Rallis quotients is summarized in the following lemma (see [39, Theorem 2.4], [40, Introduction], [41, Theorem 6.1] and [75, Sections 9 and 10]).

Lemma 3.22. *(a) If $G_{\mathbf{U}}$ is real orthogonal, then*

$$\mathrm{I}(\chi_{\mathbf{E}}) \cong (\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'} \oplus ((\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'} \otimes \mathrm{sgn}_{\mathbf{U}}),$$

where $\mathrm{sgn}_{\mathbf{U}}$ *denotes the sign character of the orthogonal group $\widetilde{G}_{\mathbf{U}}$.*

(b) If $G_{\mathbf{U}}$ is real symplectic, then

$$\mathrm{I}(\chi_{\mathbf{E}}) \oplus \mathrm{I}(\chi'_{\mathbf{E}}) \cong \bigoplus_{\mathbf{V}''} (\omega_{\mathbf{V}'',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}''}},$$

where $\chi'_{\mathbf{E}} \neq \chi_{\mathbf{E}}$ is the character which also satisfies the condition in Table 3, and $\mathbf{V}''$ runs over the isomorphism classes of $(1,1)$-spaces whose dimension equals that of $\mathbf{V}'$.

(c) If $G_{\mathbf{U}}$ is quaternionic symplectic, then

$$\mathrm{I}(\chi_{\mathbf{E}}) \cong (\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}.$$

(d) If $G_{\mathbf{U}}$ is quaternionic orthogonal, then there is an exact sequence

$$0 \longrightarrow \bigoplus\nolimits_{\mathbf{V}''} (\omega_{\mathbf{V}'',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}''}} \longrightarrow \mathrm{I}(\chi_{\mathbf{E}}) \longrightarrow \bigoplus\nolimits_{\mathbf{V}'''} (\omega_{\mathbf{V}''',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}'''}} \longrightarrow 0,$$

where $\mathbf{V}''$ runs over the isomorphism classes of $(-1,1)$-spaces whose dimension equals that of $\mathbf{V}'$, and $\mathbf{V}'''$ runs over the isomorphism classes of $(-1,1)$-spaces whose dimension equals $\dim \mathbf{V}' - 2$. □

Thanks to Lemma 3.21, $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}'}}$ and $(\omega_{\mathbf{V}'',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}''}}$ in Lemma 3.22 can be replaced by $\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})$ and $\bar{\Theta}_{\mathbf{V}'',\mathbf{U}}(1_{\mathbf{V}''})$ respectively.

4. Associated characters of theta lifts

In this section we will prove Theorem 4.4 which gives a formula for the associated character of a certain theta lift in the convergent range.

Throughout this section, we fix a rational dual pair $(\mathbf{V}, \mathbf{V}')$ and we retain the notation in Section 2.6 and Section 3.2. As in Section 2.7, we also assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$.

4.1. An upper bound on the associated character of certain theta lift.

4.1.1. *Associated characters and associated cycles.* We recall fundamental results of Vogan [69]. Suppose $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is a nilpotent orbit. A finite length $(\mathfrak{g}, \widetilde{K})$-module π is said to be *$\mathcal{O}$-bounded* (or *bounded by $\mathcal{O}$*) if the associated variety of the annihilator ideal $\mathrm{Ann}(\pi)$ is contained in $\overline{\mathcal{O}}$. It follows from [69, Theorem 8.4] that π is $\mathcal{O}$-bounded if and only if its associated variety $\mathrm{AV}(\pi)$ is contained in $\overline{\mathcal{O}} \cap \mathfrak{p}$. Let $\mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K})$ denote the category of $\mathbb{p}$-genuine $\mathcal{O}$-bounded finite length $(\mathfrak{g}, \widetilde{K})$-modules, and write $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K})$ for its Grothendieck group. Recall the group $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ from Section 2.7. From [69, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{Ch}_{\mathcal{O}} \colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K}) \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

For a $\mathbb{p}$-genuine $\mathcal{O}$-bounded $(\mathfrak{g}, \widetilde{K})$-module π of finite length, we call $\mathrm{Ch}_{\mathcal{O}}(\pi)$ the associated character of π.

Let $\mu \colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) \to \mathbb{Z}[(\mathcal{O} \cap \mathfrak{p})/\mathbf{K}]$ be the map of taking dimensions of the isotropy representations. Post-composing μ with $\mathrm{Ch}_{\mathcal{O}}$ gives the associated cycle map:

$$\mathrm{AC}_{\mathcal{O}} \colon \mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K}) \longrightarrow \mathbb{Z}[(\mathcal{O} \cap \mathfrak{p})/\mathbf{K}].$$

For any π in $\mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K})$ and a nilpotent $\mathbf{K}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}$, let $\mathrm{c}_{\mathscr{O}}(\pi)$ denote the multiplicity of $\mathscr{O}$ in $\mathrm{AC}_{\mathcal{O}}(\pi)$. For a $\mathbb{p}$-genuine Casselman-Wallach representation of $\widetilde{G}$, the notion of $\mathcal{O}$-boundedness, and its associated character is defined by using its Harish-Chandra module.

4.1.2. *Algebraic theta lifting.* Write $\mathscr{Y}_{\mathbf{V},\mathbf{V}'}$ for the subspace of $\omega_{\mathbf{V},\mathbf{V}'}$ consisting of all the vectors which are annihilated by some powers of $\mathcal{X}$. This is a dense subspace of $\omega_{\mathbf{V},\mathbf{V}'}$ and is naturally a $((\mathfrak{g} \times \mathfrak{g}') \ltimes \mathfrak{h}(W), \widetilde{K} \times \widetilde{K}')$-module. Here $\mathfrak{h}(W)$ denotes the complexified Lie algebra of $\mathrm{H}(W)$.

Definition 4.1. *For each $\mathbb{p}$-genuine $(\mathfrak{g}', \widetilde{K}')$-module π' of finite length, define*

$$\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi') := (\mathscr{Y}_{\mathbf{V},\mathbf{V}'} \otimes \pi')_{\mathfrak{g}', \widetilde{K}'}, \qquad \text{(the coinvariant space).}$$

For each $\mathbb{p}$-genuine finite length $(\mathfrak{g}', \widetilde{K}')$-module π', the $(\mathfrak{g}, \widetilde{K})$-module $\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi')$, also abbreviated as $\check{\Theta}(\pi')$, is $\mathbb{p}$-genuine and of finite length [32]. Recall the notations in Section 2.6.2, in particular the map $\boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}$ in (2.9). We have the following estimate of the size of $\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi')$.

Lemma 4.2 ([44, Theorem B and Corollary E])**.** *For each $\mathbb{p}$-genuine $(\mathfrak{g}', \widetilde{K}')$-module π' of finite length,*

$$\mathrm{AV}(\check{\Theta}(\pi')) \subset M(M'^{-1}(\mathrm{AV}(\pi'))).$$

Consequently, if π *is* $\mathcal{O}'$*-bounded for a nilpotent orbit* $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$*, then* $\check{\Theta}(\pi')$ *is* $\boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}(\mathcal{O}')$*-bounded.*

4.1.3. *The bound in the descent and good generalized descent cases.* We will prove an upper bound of associated characters in the descent and good generalized descent cases.

Theorem 4.3. *Let* $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ *and* $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ *. Suppose that*

(a) $\mathcal{O}'$ *is a descent of* $\mathcal{O}$*, that is,* $\mathcal{O}' = \boldsymbol{\nabla}(\mathcal{O})$*, or*

(b) $\mathcal{O}$ *is good for generalized descent (see Definition 2.15) and* $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$*.*

Then for every $\mathbb{p}$*-genuine* $\mathcal{O}'$*-bounded* $(\mathfrak{g}', \widetilde{K}')$*-module* π' *of finite length,* $\check{\Theta}(\pi')$ *is* $\mathcal{O}$*-bounded and*

$$\mathrm{Ch}_{\mathcal{O}}\, \check{\Theta}(\pi') \preceq \vartheta_{\mathcal{O}',\mathcal{O}}(\mathrm{Ch}_{\mathcal{O}'}(\pi')).$$

Proof. With the geometric properties investigated in Appendix A.1, the proof to be given is similar to that of [44, Theorem C].

We recall some results in [44]. There are natural good filtrations on π' and $\check{\Theta}(\pi')$ generated by the minimal degree $\widetilde{K}'$-types and $\widetilde{K}$-types, respectively. Define

$$\mathscr{A} := \mathrm{Gr}\,\pi' \otimes \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}'} \quad \text{and} \quad \mathscr{B} := \mathrm{Gr}\,\check{\Theta}(\pi') \otimes \varsigma^{-1}_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}}.$$

We view $\mathscr{A}$ and $\mathscr{B}$ as a $\mathbf{K}'$-equivariant coherent sheaf on $\mathfrak{p}'^* \cong \mathfrak{p}'$ and a $\mathbf{K}$-equivariant coherent sheaf on $\mathfrak{p}^* \cong \mathfrak{p}$, respectively. Moreover, $\mathrm{AV}(\pi') = \mathrm{Supp}(\mathscr{A})$ and $\mathrm{AV}(\check{\Theta}(\pi')) = \mathrm{Supp}(\mathscr{B})$.

Recall the moment maps defined in Section 2.6.1. Define the following right exact functor

$$\mathcal{L}\colon \mathscr{F} \mapsto (M_*(M'^*(\mathscr{F})))^{\mathbf{K}'}$$

from the category of $\mathbf{K}'$-equivariant quasi-coherent sheaves on $\mathfrak{p}'$ to the category of $\mathbf{K}$-equivariant quasi-coherent sheaves on $\mathfrak{p}$, where $\mathscr{F}$ is a $\mathbf{K}'$-equivariant quasi-coherent sheave on $\mathfrak{p}'$, and M'^* and M_* denote the pull-back and push-forward functors respectively. There is a canonical surjective morphism of $\mathbf{K}$-equivariant sheaves on $\mathfrak{p}$ as follows: (*cf.* [44, Equation (16)])

$$\mathcal{Q}\colon \mathcal{L}(\mathscr{A}) \longrightarrow\!\!\!\!\rightarrow \mathscr{B}.$$

Note that the first equality of (2.8) and the first assertion of Lemma 2.16 imply that $\mathrm{Supp}(\mathcal{L}(\mathscr{A}))$ is contained in $M(M'^{-1}(\mathrm{Supp}(\mathscr{A}))) \subset \overline{\mathcal{O}} \cap \mathfrak{p}$. In particular, $\check{\Theta}(\pi)$ is $\mathcal{O}$-bounded.

According to [44, Proposition 4.3], we may fix a finite filtration

$$0 = \mathscr{A}_0 \subset \cdots \subset \mathscr{A}_l \subset \cdots \subset \mathscr{A}_f = \mathscr{A} \qquad (f \geqslant 0)$$

of $\mathscr{A}$ by $\mathbf{K}'$-equivariant coherent sheaves on $\mathfrak{p}'$ such that for any $1 \leqslant l \leqslant f$, the space of global sections of $\mathscr{A}^l := \mathscr{A}_l/\mathscr{A}_{l-1}$ is an irreducible $(\mathbb{C}[\overline{\mathscr{O}'_l}], \mathbf{K}')$-module for a nilpotent $\mathbf{K}'$-orbit $\mathscr{O}'_l$ in $\overline{\mathcal{O}'} \cap \mathfrak{p}'$.

Let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$. If $\mathscr{O}$ does not admit a generalized descent, then

$$\mathscr{O} \cap M(\mathcal{X}) = \varnothing,$$

and $\mathscr{O}$ does not appear in the support of $\mathrm{Ch}_{\mathcal{O}}(\check{\Theta}(\pi))$.

Now suppose that $\mathscr{O}$ admits a generalized descent $\mathscr{O}' := \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Retain the notation in Section 2.7.2 where an $(\epsilon', \dot{\epsilon}')$-space decomposition $\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$ and an element

$$T \in \mathcal{X}^\circ_1 := \{\, w \in \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_1) \mid w \text{ is surjective} \,\} \cap \mathcal{X} \subseteq \mathcal{X}^{\mathrm{gen}}$$

is fixed such that $X := M(T) \in \mathscr{O}$ and $X' := M'(T) \in \mathscr{O}'$.

For $1 \leqslant l \leqslant f$, if $\mathscr{O}'_l = \mathscr{O}'$, then Lemma A.2 and Lemma A.4 ensure that there exists a Zariski open set U in $\mathfrak{p}$ such that $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$ and the quasi-coherent sheaf

$\mathcal{L}(\mathscr{A}^l)|_U$ descends to a quasi-coherent sheaf on the closed subvariety $\mathscr{O}$ of U. (Thus $\mathcal{L}(\mathscr{A}^l)$ is generically reduced around X in the sense of [69, Proposition 2.9].)

Claim. *Assume that $\mathscr{O}'_l = \mathscr{O}'$ ($1 \leqslant l \leqslant f$). Then we have the following isomorphism of algebraic representations of $\mathbf{K}_X$:*

$$i_X^*(\mathcal{L}(\mathscr{A}^l)) \cong (i_{X'}^*(\mathscr{A}^l))^{\mathbf{K}'_2} \circ \alpha_1.$$

Here α_1 is defined in (2.19), *$i_X\colon \{X\} \hookrightarrow \mathfrak{p}$ and $i_{X'}\colon \{X'\} \hookrightarrow \mathfrak{p}'$ are the inclusion maps, and i_X^* and $i_{X'}^*$ are the associated pull-back functors of the quasi-coherent sheaves.*

Let $\mathrm{m}_{\mathfrak{p}}(X)$ be the maximal ideal of $\mathbb{C}[\mathfrak{p}]$ associated to X, and $\kappa(X) := \mathbb{C}[\mathfrak{p}]/\mathrm{m}_{\mathfrak{p}}(X)$ be the residual field at X. Let

$$Z_{X,\overline{\mathscr{O}'}} := \left\{ w \in \mathcal{X} \;\middle|\; M(w) = X, M'(w) \in \overline{\mathscr{O}'} \right\}.$$

Then $T \in Z_{X,\overline{\mathscr{O}'}}$ and

$$\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}] = \mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}]\otimes\mathbb{C}[\mathfrak{p}']} (\kappa(X) \otimes \mathbb{C}[\overline{\mathscr{O}'}])$$

by Lemma A.1 and Lemma A.3.

Let A^l be the module of global sections of $\mathscr{A}^l$. We have

$$\begin{aligned} i_X^*(\mathcal{L}(\mathscr{A}^l)) &= \kappa(X) \otimes_{\mathbb{C}[\mathfrak{p}]} \left(\mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}']} \mathbb{C}[\overline{\mathscr{O}'}] \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l\right)^{\mathbf{K}'} \\ &= \left(\mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}]\otimes\mathbb{C}[\mathfrak{p}']} (\kappa(X) \otimes \mathbb{C}[\overline{\mathscr{O}'}]) \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l\right)^{\mathbf{K}'} \\ &= (\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}] \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l)^{\mathbf{K}'}. \end{aligned}$$

Let $\rho' := i_{X'}^*(\mathscr{A}^l)$ be the isotropy representation of $\mathscr{A}^l$ at X'. Since $Z_{X,\overline{\mathscr{O}'}}$ is the $\mathbf{K}' \times \mathbf{K}_X$-orbit of T by Lemma A.1 (iii) and Lemma A.3 (ii), by considering the diagram (*cf.* [44, Section 4.2])

$$\begin{tikzcd} \{X'\} \arrow[d, hook] & \{T\} \arrow[l] \arrow[d, hook] \arrow[dr] & \\ \overline{\mathscr{O}'} & Z_{X,\overline{\mathscr{O}'}} \arrow[l, "M'"'] \arrow[r, "M"] & \{X\}, \end{tikzcd}$$

we know that $\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}] \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l$ is isomorphic to the algebraically induced representation $\mathrm{Ind}_{\mathbf{S}_T}^{\mathbf{K}'\times\mathbf{K}_X} \rho'$, where $\mathbf{S}_T$ is the stabilizer group as in (2.20), and ρ' is viewed as an $\mathbf{S}_T$-module via the natural projection $\mathbf{S}_T \to \mathbf{K}'_{X'}$. Now by Frobenius reciprocity, we have

$$i_X^*(\mathcal{L}(\mathscr{A}^l)) \cong (\mathrm{Ind}_{\mathbf{S}_T}^{\mathbf{K}'\times\mathbf{K}_X} \rho')^{\mathbf{K}'} = (\mathrm{Ind}_{\mathbf{K}'_2\times(\mathbf{K}'_{1,X'_1}\times_{\alpha_1}\mathbf{K}_X)}^{\mathbf{K}'\times\mathbf{K}_X} \rho')^{\mathbf{K}'} = (\rho')^{\mathbf{K}'_2} \circ \alpha_1.$$

This finishes the proof of the claim.

On the other hand, if $\mathscr{O}'_l \neq \mathscr{O}'$, then (2.11) implies that

$$\mathscr{O} \cap M(M'^{-1}(\overline{\mathscr{O}'_l})) = \varnothing,$$

and hence $\mathscr{O}$ is not contained in $\mathrm{Supp}(\mathcal{L}(\mathscr{A}^l))$. In conlcusion, the following inequality holds in the Grothedieck group of the category of algebraic representations of $\mathbf{K}_X$:

$$\chi(X, \mathscr{B}) \preceq \chi(X, \mathcal{L}(\mathscr{A})) \preceq \sum_{l=1}^{f} \chi(X, \mathcal{L}(\mathscr{A}^l)) = \sum_{l=1}^{f} (i_{X'}^*(\mathscr{A}^l))^{\mathbf{K}'_2} \circ \alpha_1 = \chi(X', \mathscr{A})^{\mathbf{K}'_2} \circ \alpha_1,$$

where $\chi(X, \cdot\,)$ indicates the virtual character of $\mathbf{K}_X$ attached to a $\mathbf{K}$-equivariant coherent sheaf on $\mathfrak{p}$ whose support is contained in $\overline{\mathcal{O}} \cap \mathfrak{p}$, and similarly for $\chi(X', \cdot\,)$ (cf. [69, Definition 2.12]). This finishes the proof of the theorem. $\square$

4.2. **An equality of associated characters in the convergent range.** In this subsection, we assume that $\dim^\circ \mathbf{V} > 0$. The purpose of this section is to prove the following theorem.

Theorem 4.4. *Let $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ such that $\mathcal{O}$ is the descent of $\mathcal{O}'$. Write $\mathbf{D}(\mathcal{O}') = [c_0, c_1, \cdots, c_k]$ ($k \geqslant 1$). Assume that $c_0 > c_1$ when G is a real symplectic group. Then for every $\mathcal{O}$-bounded Casselman-Wallach representation π of $\widetilde{G}$ such that $(\pi, \mathbf{V}')$ is in the convergent range (see Definition 3.9), $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is $\mathcal{O}'$-bounded and*

$$\mathrm{Ch}_{\mathcal{O}'}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) = \vartheta_{\mathcal{O},\mathcal{O}'}(\mathrm{Ch}_{\mathcal{O}}(\pi)). \tag{4.1}$$

Remark. When $(\mathbf{V}, \mathbf{V}')$ is in the stable range, Theorem 4.4 is proved for unitary representations π in [44].

We keep the setting of Theorem 4.4. First observe that the Harish-Chandra module of $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is isomorphic to a quotient of $\check{\Theta}(\pi^{\mathrm{al}})$, where π^{al} denotes the Harish-Chandra module of π. Thus Theorem 4.3 implies that $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is $\mathcal{O}'$-bounded and

$$\mathrm{Ch}_{\mathcal{O}'}\,\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi) \preceq \vartheta_{\mathcal{O},\mathcal{O}'}(\mathrm{Ch}_{\mathcal{O}}(\pi)). \tag{4.2}$$

We will devote the rest of this section to prove that the equality in (4.2) holds.

4.2.1. *Doubling.* We consider a two-step theta lifting. Let $\mathbf{U}$ be as in (3.20). We realize $\mathbf{V}$ as a non-genenerate $(\epsilon, \dot{\epsilon})$-subspace of $\mathbf{U}$ and write $\mathbf{V}^\perp$ for the orthogonal complement of $\mathbf{V}$ in $\mathbf{U}$, which is also an $(\epsilon, \dot{\epsilon})$-space.

Note that $\dim^\circ \mathbf{V}' > 0$ and Lemma 3.11 implies that $(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi), \mathbf{V}^\perp)$ is in the convergent range. Comparing (3.5) with (3.20), we have $\nu_{\mathbf{U},\mathbf{V}} = \frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}$. Thus (3.6) holds and $\mathcal{R}_{\bar{\Theta}_{\mathbf{U}}(1_{\mathbf{V}'})}(\pi)$ is defined by Lemma 3.15.

Lemma 4.5. *One has that*

$$\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \cong \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})}(\pi). \tag{4.3}$$

Proof. Note that the integral in

$$\begin{array}{rcl} (\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega_{\mathbf{V}',\mathbf{V}^\perp}) \times (\pi^\vee\widehat{\otimes}\omega^\vee_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega^\vee_{\mathbf{V}',\mathbf{V}^\perp}) & \longrightarrow & \mathbb{C} \\ (u,v) & \longmapsto & \int_{\widetilde{G}\times\widetilde{G}'}\langle g\cdot u, v\rangle\,\mathrm{d}g \end{array} \tag{4.4}$$

is absolutely convergent and defines a continuous bilinear map. In view of Fubini's theorem and Lemma 3.21, the lemma follows as both sides of (4.3) are isomorphic to the quotient of $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega_{\mathbf{V}',\mathbf{V}^\perp}$ by the left radical of the pairing (4.4). □

We will use (4.3) freely in the rest of this section.

4.2.2. *On certain induced orbits.* The main step in the proof of Theorem 4.4 consists of comparison of bound of the associated cycle of $\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))$ with the formula (due to Barbasch) of the wavefront cycle of a certain parabolically induced representation.

In this section, let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^J$ be the Lie algebra of G, which is identified with its dual space $\mathfrak{g}_{\mathbb{R}}^*$ under the trace from. Let $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}})$ denote the set of nilpotent G-orbits in $\mathbf{i}\mathfrak{g}_{\mathbb{R}}$. Similar notation will be used without further explanation. For example, for every Levi subgroup M of G, $\mathfrak{m}_{\mathbb{R}}$ denotes its Lie algebra, which is identified with the dual space $\mathfrak{m}_{\mathbb{R}}^*$ by using the trace form on $\mathfrak{g}$.

Let

$$\mathsf{KS}\colon \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}}) \to \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$$

be the natural bijection given by the Kostant-Sekiguchi correspondence (*cf.* [63, Equation (6.7)]). By abuse of notation, we also let $\ddot{\mathbf{D}}$ denote the map $\ddot{\mathbf{D}} \circ \mathsf{KS}\colon \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}}) \to \ddot{\mathcal{P}}$. See Section 2.5 for the parametrization map $\ddot{\mathbf{D}}\colon \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p}) \to \ddot{\mathcal{P}}$.

Theorem 4.6 (*cf.* Barbasch [8, Corollary 5.0.10]). *Let G_1 be an arbitrary real reductive group and let P_1 be a real parabolic subgroup of G_1, namely a closed subgroup of G_1 whose Lie algebra is a parabolic subalgebra of $\mathfrak{g}_{1,\mathbb{R}}$. Let N_1 be the unipotent radical of P_1 and $M_1 := P_1/N_1$. Write*

$$r_1 : \mathbf{i}(\mathfrak{g}_{1,\mathbb{R}}/\mathfrak{n}_{1,\mathbb{R}})^* \longrightarrow \mathbf{i}\,\mathfrak{m}_{1,\mathbb{R}}^*$$

for the natural map. Let π_1 be a Casselman-Wallach representation of M_1 with the wave-front cycle

$$\mathrm{WF}(\pi_1) = \sum_{\mathscr{O}_{\mathbb{R}} \in \mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}_{1,\mathbb{R}}^*)} \mathrm{c}_{\mathscr{O}_{\mathbb{R}}}[\mathscr{O}_{\mathbb{R}}].$$

Then

$$\mathrm{WF}(\mathrm{Ind}_{P_1}^{G_1} \pi) = \sum_{(\mathscr{O}_{\mathbb{R}}, \mathscr{O}'_{\mathbb{R}})} \mathrm{c}_{\mathscr{O}_{\mathbb{R}}} \frac{\# C_{G_1}(v')}{\# C_{P_1}(v')} [\mathscr{O}'_{\mathbb{R}}],$$

where the summation runs over all pairs $(\mathscr{O}_{\mathbb{R}}, \mathscr{O}'_{\mathbb{R}})$ such that $\mathscr{O}_{\mathbb{R}} \in \mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}_{1,\mathbb{R}}^*)$ and $\mathscr{O}'_{\mathbb{R}} \in \mathrm{Nil}_{G_1}(\mathbf{i}\mathfrak{g}_{1,\mathbb{R}}^*)$ is an induced orbit of $\mathscr{O}_{\mathbb{R}}$, v' is an element of $\mathscr{O}'_{\mathbb{R}} \cap r_1^{-1}(\mathscr{O}_{\mathbb{R}})$, and $C_{G_1}(v')$ and $C_{P_1}(v')$ are the component groups of the centralizers of v' in G_1 and P_1, respectively.

Remarks. 1. While Barbasch proved the theorem when G_1 is the real points of a connected reductive algebraic group, his proof still works in the slightly more general setting of Theorem 4.6.
2. The notion of induced nilpotent orbits was introduced by Lusztig and Spaltenstein for complex reductive groups [45]. For a real reductive group G_1, a nilpotent orbit $\mathscr{O}'_{\mathbb{R}} \in \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{1,\mathbb{R}}^*)$ is called an induced orbit of a nilpotent orbit $\mathscr{O}_{\mathbb{R}} \in \mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}_{1,\mathbb{R}}^*)$ if $\mathscr{O}'_{\mathbb{R}} \cap r_1^{-1}(\mathscr{O}_{\mathbb{R}})$ is open in $r_1^{-1}(\mathscr{O}_{\mathbb{R}})$ (see [8, Definition 5.0.7] or [55]). We write $\mathrm{Ind}_{P_1}^{G_1} \mathscr{O}_{\mathbb{R}}$ for the set of all induced orbits of $\mathscr{O}_{\mathbb{R}}$. Similar notation applies for a complex reductive group.
3. According to the fundamental result of Schimd-Vilonen [63], the wave front cycle and the associated cycle agree under the Kostant-Sekiguchi correspondence. We thank Professor Vilonen for confirming that their result extends to nonlinear groups. Therefore the associated cycle of a parabolically induced representation as in the above theorem of Barbasch is also determined.

We now consider induced orbits appearing in our cases. Retain the setting in Section 3.4 (see (3.10) and onwards), where

$$\mathbf{V}^{\perp} = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}'_0,$$

$M_{\mathbf{E}_0} = G \times \mathrm{GL}_{\mathbf{E}_0}$ and the parabolic subgroup of $G_{\mathbf{V}^\perp}$ stabilizing $\mathbf{E}_0$ is $P_{\mathbf{E}_0} = M_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$.

Recall that $\mathbf{D}(\mathcal{O}') = [c_0, \cdots, c_k]$ and $\mathbf{D}(\mathcal{O}) = [c_1, \cdots, c_k]$. Put $l := \dim \mathbf{E}_0$. In the notation of (3.20), we have

$$l = c_0 + \delta \geqslant c_1. \tag{4.5}$$

Note that if $\mathbf{G}$ is an orthogonal group, then $c_0 - c_1$ is even. Hence $l - c_1$ is odd if G is real orthogonal and $l - c_1$ is even if G is quaternionic orthogonal. View $\mathcal{O}$ as a nilpotent orbit in $\mathfrak{m}_{\mathbf{E}_0}$ via inclusion. The following lemma is clear (*cf.* [16, Section 7.3]).

Lemma 4.7. *(i) If G is a real orthogonal group, then*

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}) = [l+1, l-1, c_1, \cdots, c_k].$$

(ii) Otherwise,

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}) = [l, l, c_1, \cdots, c_k].$$

By [17, Theorem 5.2 and 5.6], the complex nilpotent orbit

$$\mathcal{O}^\perp := \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}^\perp}(\mathcal{O}') = \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'}(\mathcal{O}))$$

is given by

$$\mathbf{D}(\mathcal{O}^\perp) = \begin{cases} [c_0+2, c_0, c_1, \cdots, c_k], & \text{if } G \text{ is a real orthogonal group;} \\ [c_0-1, c_0-1, c_1, \cdots, c_k], & \text{if } G \text{ is a real symplectic group;} \\ [c_0, c_0, c_1, \cdots, c_k], & \text{otherwise.} \end{cases}$$

This implies that

$$\mathcal{O}^\perp = \operatorname{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}.$$

View each orbit in $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R})$ as an orbit in $\mathrm{Nil}_{M_{\mathbf{E}_0}}(\mathbf{i}\,\mathfrak{m}_{\mathbf{E}_0}^J)$ via inclusion. We state the result for the induction of real nilpotent orbits in the following lemma. The proof will be given in Appendix A.2, which is by elementary matrix manipulations.

Lemma 4.8. *Let $\mathscr{O}_\mathbb{R} \in \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R})$ be a real nilpotent orbit in $\mathcal{O}$ with $\ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}) = [d_1, \cdots, d_k]$.*

(i) Suppose G is a real orthogonal group. Then $\operatorname{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$ consists of a single orbit $\mathscr{O}_\mathbb{R}^\perp$ with

$$\ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}^\perp) = [d_1 + s + (1,1), \check{d}_1 + \check{s}, d_1, \cdots, d_k],$$

where $s := (\frac{l-c_1-1}{2}, \frac{l-c_1-1}{2}) \in \mathbb{Z}^2_{\geqslant 0}$. Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is injective and its image has index 2 in $C_{G_{\mathbf{V}^\perp}}(X)$ for $X \in \mathscr{O}_\mathbb{R}^\perp \cap (\mathscr{O}_\mathbb{R} + \mathbf{i}\,\mathfrak{n}_{\mathbf{E}_0}^J)$.

(ii) Otherwise, $\operatorname{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$ equals the set

$$\left\{ \mathscr{O}_\mathbb{R}^\perp \in \mathrm{Nil}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\mathfrak{g}_{\mathbf{V}^\perp}^J) \;\middle|\; \begin{array}{l} \ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}^\perp) = [d_1 + s, \check{d}_1 + \check{s}, d_1, \cdots, d_k] \textit{ for a signature} \\ s \textit{ of a } (-\epsilon, -\dot{\epsilon})\textit{-space of dimension } l - c_1 \end{array} \right\}.$$

Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is an isomorphism for $X \in \mathscr{O}_\mathbb{R}^\perp \cap (\mathscr{O}_\mathbb{R} + \mathbf{i}\,\mathfrak{n}_{\mathbf{E}_0}^J)$.

Here $C_{P_{\mathbf{E}_0}}(X)$ and $C_{G_{\mathbf{V}^\perp}}(X)$ are the component groups as in Theorem 4.6. Under the setting of Lemma 4.8, we see that $\operatorname{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$ consists of a single orbit when G is a real orthogonal group or a quaternionic symplectic group.

For each $\mathscr{O} \in \mathrm{Nil}_\mathbf{K}(\mathfrak{p})$, let

$$\operatorname{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O} := \left\{ \mathsf{KS}(\mathscr{O}_\mathbb{R}) \;\middle|\; \mathscr{O}_\mathbb{R} \in \operatorname{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}}(\mathsf{KS}^{-1}(\mathscr{O})) \right\}.$$

4.2.3. *Finishing the proof when G is a real orthogonal group.* Let $\mathrm{sgn}_\mathbf{V}$ and $\mathrm{sgn}_{\mathbf{V}^\perp}$ be the sign character of the orthogonal groups G and $G_{\mathbf{V}^\perp}$ respectively. By Proposition 3.16, Lemma 3.22 (a) and Lemma 3.21, we have

$$\begin{aligned} &\operatorname{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}) \\ &\cong \mathcal{R}_{\mathrm{I}(\chi_\mathbf{E})}(\pi) \cong \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})}(\pi) \oplus \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})\otimes \mathrm{sgn}_\mathbf{U}}(\pi) \\ &\cong \bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \oplus (\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi \otimes \mathrm{sgn}_\mathbf{V}))) \otimes \mathrm{sgn}_{\mathbf{V}^\perp}. \end{aligned} \tag{4.6}$$

By Theorem 4.6, Lemma 4.7, Lemma 4.8 (i) and [63, Theorem 1.4], the representation $\operatorname{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}))$ is $\mathcal{O}^\perp$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^\perp}(\operatorname{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})) = 2 \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{4.7}$$

where the summation runs over all $\mathbf{K}$-orbits $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^{\perp}$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^{\perp}}} \mathscr{O}$.

On the other hand, by the explicit formula for the descents of nilpotent orbits, we have

$$\nabla_{\mathbf{V}', \mathbf{V}}(\nabla_{\mathbf{V}^{\perp}, \mathbf{V}'}(\mathscr{O}^{\perp})) = \mathscr{O}.$$

Applying Theorem 4.3 twice, we have

$$\begin{aligned} &\mathrm{AC}_{\mathcal{O}^{\perp}}(\bar{\Theta}_{\mathbf{V}', \mathbf{V}^{\perp}}(\bar{\Theta}_{\mathbf{V}, \mathbf{V}'}(\pi))) \preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^{\perp}], \quad \text{and} \\ &\mathrm{AC}_{\mathcal{O}^{\perp}}(\bar{\Theta}_{\mathbf{V}', \mathbf{V}^{\perp}}(\bar{\Theta}_{\mathbf{V}, \mathbf{V}'}(\pi \otimes \mathrm{sgn}_{\mathbf{V}}))) \otimes \mathrm{sgn}_{\mathbf{V}^{\perp}}) \preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^{\perp}] \end{aligned} \tag{4.8}$$

where the summations run over the same set as the right hand side of (4.7).

In view of (4.6) to (4.8), we conclude that both inequalities in (4.8) are equalities. Thus (4.1) follows.

4.2.4. *Finishing the proof when G is a real symplectic group.* Let $\mathbf{V}'_1, \mathbf{V}'_2, \cdots, \mathbf{V}'_s$ be a list of representatives of the isomorphic classes of all $(1,1)$-spaces with dimension $\dim \mathbf{V}'$.

By Proposition 3.16, Lemma 3.22 (b) and Lemma 3.21, we have

$$\begin{aligned} &\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^{\perp}}} (\pi \otimes \chi_{\mathbf{E}_0}) \oplus \mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^{\perp}}} \left(\pi \otimes \chi'_{\mathbf{E}_0}\right) \\ &\cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \oplus \mathcal{R}_{\mathrm{I}(\chi'_{\mathbf{E}})}(\pi) \\ &\cong \bigoplus_{i=1}^{s} \mathcal{R}_{\bar{\Theta}_{\mathbf{V}'_i, \mathbf{U}}(1_{\mathbf{V}'_i})}(\pi) \cong \bigoplus_{i=1}^{s} \bar{\Theta}_{\mathbf{V}'_i, \mathbf{V}^{\perp}}(\bar{\Theta}_{\mathbf{V}, \mathbf{V}'_i}(\pi)). \end{aligned} \tag{4.9}$$

By Theorem 4.6, Lemma 4.7, Lemma 4.8 (ii) and [63, Theorem 1.4], $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^{\perp}}} (\pi \otimes \chi_{\mathbf{E}_0})$ and $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^{\perp}}} \left(\pi \otimes \chi'_{\mathbf{E}_0}\right)$ are $\mathcal{O}^{\perp}$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^{\perp}}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^{\perp}}} (\pi \otimes \chi_{\mathbf{E}_0})) = \mathrm{AC}_{\mathcal{O}^{\perp}}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^{\perp}}} \left(\pi \otimes \chi'_{\mathbf{E}_0}\right)) = \sum_{\mathscr{O}, \mathscr{O}^{\perp}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^{\perp}], \tag{4.10}$$

where the summation runs over all pairs $(\mathscr{O}, \mathscr{O}^{\perp})$, where $\mathscr{O}$ is a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$ and $\mathscr{O}^{\perp} \in \mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^{\perp}}} \mathscr{O}$.

Applying Theorem 4.3 twice, we see that $\bar{\Theta}_{\mathbf{V}'_i, \mathbf{V}^{\perp}}(\bar{\Theta}_{\mathbf{V}, \mathbf{V}'_i}(\pi))$ is $\mathcal{O}^{\perp}$-bounded $(1 \leqslant i \leqslant s)$, and

$$\sum_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^{\perp}}(\bar{\Theta}_{\mathbf{V}'_i, \mathbf{V}^{\perp}}(\bar{\Theta}_{\mathbf{V}, \mathbf{V}'_i}(\pi))) \preceq \sum_{i=1}^{s} \sum_{\mathscr{O}, \mathscr{O}^{\perp}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^{\perp}],$$

where the inner summation runs over all pairs of orbits $(\mathscr{O}, \mathscr{O}^{\perp})$ such that

$$\nabla_{\mathbf{V}'_i, \mathbf{V}}(\nabla^{\mathrm{gen}}_{\mathbf{V}^{\perp}, \mathbf{V}'_i}(\mathscr{O}^{\perp})) = \mathscr{O}. \tag{4.11}$$

By Lemma 4.8 (ii) and (2.10), such kind of pairs $(\mathscr{O}, \mathscr{O}^{\perp})$ are the same as those of the right hand side of (4.10). Suppose $\ddot{\mathbf{D}}(\mathscr{O}^{\perp}) = [d, \check{d}, d_1, \cdots, d_k]$, then $\ddot{\mathbf{D}}(\mathscr{O}) = [d_1, \cdots, d_k]$ and (4.11) holds for exactly two $\mathbf{V}'_i$, having signature $\check{d} + \mathrm{Sign}(\mathbf{V}) + (1,0)$ and $\check{d} + \mathrm{Sign}(\mathbf{V}) + (0,1)$ respectively. Hence we have

$$\sum_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^{\perp}}(\bar{\Theta}_{\mathbf{V}'_i, \mathbf{V}^{\perp}}(\bar{\Theta}_{\mathbf{V}, \mathbf{V}'_i}(\pi))) \preceq \sum_{\mathscr{O}, \mathscr{O}^{\perp}} 2\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^{\perp}], \tag{4.12}$$

where the summation is as in the right hand side of (4.10).

In view of (4.9), (4.10) and (4.12), the equality holds in (4.12). Thus (4.1) holds.

4.2.5. *Finishing the proof when G is a quaternionic symplectic group.* Using Proposition 3.16, Lemma 3.22 (c), Lemma 3.21, Theorem 4.6, Lemma 4.8 (ii), [63, Theorem 1.4] and Theorem 4.3, and a similar argument as in Section 4.2.3 shows that

$$\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi \otimes \chi_{\mathbf{E}_0}) \cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \cong \bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \tag{4.13}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi \otimes \chi_{\mathbf{E}_0})) = \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and} \tag{4.14}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))) \preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{4.15}$$

Here the summations run over all $\mathbf{K}_{\mathbf{V}}$-orbit $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^\perp$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}$. Note that

$$\nabla_{\mathbf{V}',\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'}(\mathscr{O}^\perp)) = \mathscr{O}.$$

Now (4.13), (4.14) and (4.15) imply that (4.15) is an equality and so (4.1) holds.

4.2.6. *Finishing the proof when G is a quaternionic orthogonal group.* Let $\mathbf{V}'_1, \mathbf{V}'_2, \cdots, \mathbf{V}'_s$ be a list of representatives of the isomorphic classes of all $(-1,1)$-spaces with dimension $\dim \mathbf{V}'$.

Lemma 4.9. *One has that*

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi)) = \sum_{i=1}^{s} \mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{\bar{\Theta}_{\mathbf{V}'_i,\mathbf{U}}(1_{\mathbf{V}'_i})}(\pi)).$$

Proof. For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in Lemma 3.22 (d). Note that $\mathcal{R}_{I_2}(\pi)$ is $\mathcal{O}^\perp$-bounded by Proposition 3.16, Theorem 4.6 and [63, Theorem 1.4]. By the definition in (3.9), we could view $\mathcal{R}_{I_1}(\pi)$ as a subrepresentation of $\mathcal{R}_{I_2}(\pi)$ which is also $\mathcal{O}^\perp$-bounded.

Since G acts on $\mathcal{R}_{I_2}(\pi)$ trivially, the natural homomorphism

$$\pi\widehat{\otimes} I_3 \cong \pi\widehat{\otimes} I_2/\pi\widehat{\otimes} I_1 \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)$$

descents to a surjective homomorphism

$$(\pi\widehat{\otimes} I_3)_G \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi).$$

For each $(-1,1)$-space $\mathbf{V}'''$ of dimension $\dim \mathbf{V}' - 2$, by applying Lemma 4.2 twice, we see that

$$(\pi\widehat{\otimes}(\omega_{\mathbf{V}''',\mathbf{U}})_{G_{\mathbf{V}'''}})_G \cong ((\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'''})_G\widehat{\otimes}\omega_{\mathbf{V}''',\mathbf{V}^\perp})_{G_{\mathbf{V}'''}}$$

is bounded by $\boldsymbol{\vartheta}_{\mathbf{V}''',\mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'''}(\mathcal{O}))$. Using formulas in [17, Theorem 5.2 and 5.6], one checks that the latter set is contained in the boundary of $\mathcal{O}^\perp$. Hence

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)) = 0$$

and the lemma follows. □

Using Theorem 4.6, Lemma 4.8 (ii), [63, Theorem 1.4], Theorem 4.3 and a similar argument as in Section 4.2.4, we have

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi \otimes \chi_{\mathbf{E}_0})) = \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and} \tag{4.16}$$

$$\bigoplus_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi))) \preceq \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp]. \tag{4.17}$$

Here the summations run over all pairs $(\mathscr{O}, \mathscr{O}^\perp)$ such that $\mathscr{O}^\perp$ is a $\mathbf{K}_{\mathbf{V}^\perp}$-orbit in $\mathcal{O}^\perp \cap \mathfrak{p}_{\mathbf{V}^\perp}$ and $\mathscr{O} = \nabla_{\mathbf{V}'_i, \mathbf{V}}(\nabla_{\mathbf{V}^\perp, \mathbf{V}'_i}(\mathscr{O}^\perp))$ for a unique $(-1,1)$-space $\mathbf{V}'_i$.

In view of Proposition 3.16, Lemma 4.9, (4.16) and (4.17), we see that the equality holds in (4.17). Thus (4.1) holds.

5. The unipotent representations: construction and exhaustion

In this section, we will prove Theorem 1.4. Recall that $\mathbf{V}$ is an $(\epsilon, \dot\epsilon)$-space and $\mathcal{O} \in \mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{g})$. Since Theorem 1.4 is obvious when $\mathcal{O}$ is the zero orbit, we assume without loss of generality that $\mathcal{O}$ is not the zero orbit.

5.1. **The construction.** Suppose there is a $\mathbf{K}$-orbit $\mathscr{O} \in \mathcal{O} \cap \mathfrak{p}$. Define a sequence $(\mathbf{V}_0, \mathscr{O}_0), (\mathbf{V}_1, \mathscr{O}_1), \cdots, (\mathbf{V}_k, \mathscr{O}_k)$ $(k \geqslant 1)$ such that

- $\mathbf{V}_j$ is a nonzero $((-1)^j\epsilon, (-1)^j\dot\epsilon)$-space for all $0 \leqslant j \leqslant k$;
- $(\mathbf{V}_0, \mathscr{O}_0) = (\mathbf{V}, \mathscr{O})$, and $\mathscr{O}_j = \nabla_{\mathbf{V}_{j-1}, \mathbf{V}_j}(\mathscr{O}_{j-1}) \in \mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_j}}(\mathfrak{p}_{\mathbf{V}_j})$ for all $1 \leqslant j \leqslant k$;
- $\mathscr{O}_k$ is the zero orbit.

Let $\mathcal{O}_j$ denote the nilpotent $G_{\mathbf{V}_j}$-orbit containing $\mathscr{O}_j$ $(0 \leqslant j \leqslant k)$. To ease the notation, we also let $\mathbf{V}_{k+1}$ be the zero $((-1)^{k+1}\epsilon, (-1)^{k+1}\dot\epsilon)$-space and let $\mathscr{O}_{k+1} \in \mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_{k+1}}}(\mathfrak{p}_{\mathbf{V}_{k+1}})$ and $\mathcal{O}_{k+1} \in \mathrm{Nil}_{\mathbf{G}_{\mathbf{V}_{k+1}}}(\mathfrak{g}_{\mathbf{V}_{k+1}})$ be $\{0\}$.

Let

$$\eta = \chi_0 \boxtimes \chi_1 \boxtimes \cdots \boxtimes \chi_k \tag{5.1}$$

be a character of $G_{\mathbf{V}_0} \times G_{\mathbf{V}_1} \times \cdots \times G_{\mathbf{V}_k}$. For $0 \leqslant j \leqslant k$, put

$$\eta_j := \chi_j \boxtimes \chi_{j+1} \boxtimes \cdots \boxtimes \chi_k.$$

For $0 \leqslant j < k$, write

$$\omega_{\mathscr{O}_j} := \omega_{\mathbf{V}_j, \mathbf{V}_{j+1}} \widehat{\otimes} \omega_{\mathbf{V}_{j+1}, \mathbf{V}_{j+2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{\mathbf{V}_{k-1}, \mathbf{V}_k},$$

and one checks that the integrals in

$$\begin{aligned} (\omega_{\mathscr{O}_j} \otimes \eta_j) \times (\omega^\vee_{\mathscr{O}_j} \otimes \eta_j^{-1}) &\longrightarrow \mathbb{C}, \\ (u, v) &\longmapsto \int_{\widetilde{G}_{\mathbf{V}_{j+1}} \times \widetilde{G}_{\mathbf{V}_{j+2}} \times \cdots \times \widetilde{G}_{\mathbf{V}_k}} \langle g \cdot u, v \rangle \, \mathrm{d}g, \end{aligned} \tag{5.2}$$

are absolutely convergent and define a continuous bilinear map using Lemma 3.7. Define

$$\pi_{\mathscr{O}_j, \eta_j} := \frac{\omega_{\mathscr{O}_j} \otimes \eta_j}{\text{the left kernel of } (5.2)},$$

which is a Casselman-Wallach representation of $\widetilde{G}_{\mathbf{V}_j}$, as in (3.2). Set $\pi_{\mathscr{O}_k, \eta_k} := \chi_k$ by convention.

For $0 \leqslant j \leqslant k$, let $\chi_i|_{\widetilde{\mathbf{K}}_{\mathbf{V}_i}}$ denote the algebraic character whose restriction to $\widetilde{K}_{\mathbf{V}_i}$ equals the pullback of χ_i through the natural homomorphism $\widetilde{K}_{\mathbf{V}_i} \to G_{\mathbf{V}_i}$. Let $\mathcal{E}_{\mathscr{O}_k, \chi_k}$ denote the $\widetilde{\mathbf{K}}_{\mathbf{V}_k}$-equivariant algebraic line bundle on the zero orbit $\mathscr{O}_k$ corresponding to $\chi_k|_{\widetilde{\mathbf{K}}_{\mathbf{V}_k}}$. Inductively define

$$\mathcal{E}_{\mathscr{O}_j, \eta_j} := \chi_j|_{\widetilde{\mathbf{K}}_{\mathbf{V}_j}} \otimes \vartheta_{\mathscr{O}_{j+1}, \mathscr{O}_j}(\mathcal{E}_{\mathscr{O}_{j+1}, \eta_{j+1}}), \quad 0 \leqslant j < k.$$

This is an admissible orbit datum over $\mathscr{O}_i$ by (2.23).

Theorem 5.1. *For each $0 \leqslant j \leqslant k$, $\pi_{\mathscr{O}_j,\eta_j}$ is an irreducible, unitarizable, $\mathcal{O}_j$-unipotent representation whose associated character*

$$\mathrm{Ch}_{\mathcal{O}_j}(\pi_{\mathscr{O}_j,\eta_j}) = \mathcal{E}_{\mathscr{O}_j,\eta_j}. \tag{5.3}$$

Moreover,

$$\bar{\Theta}_{\mathbf{V}_{j+1},\mathbf{V}_j}(\pi_{\mathscr{O}_{j+1},\eta_{j+1}}) \otimes \chi_j = \pi_{\mathscr{O}_j,\eta_j} \quad \textit{for all } 0 \leqslant j < k-1. \tag{5.4}$$

Proof. Since $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{p})$, one verifies that

- $\dim^\circ \mathbf{V}_j > 0$ for $0 \leqslant j < k$,
- $\dim \mathbf{V}_{j+1} + \dim \mathbf{V}_{j-1} > 2\dim^\circ \mathbf{V}_j$ for $1 \leqslant j \leqslant k$, and
- $\pi_{\mathscr{O}_j,\eta_j}$ is $\mathbb{p}$-genuine for $0 \leqslant j \leqslant k$.

The representation $\pi_{\mathcal{O}_k,\eta_k}$ clearly satisfies all claims in the theorem. For $j = k-1$, $\pi_{\mathscr{O}_{k-1},\eta_{k-1}}$ is the twist by χ_{k-1} of the theta lift of character χ_k in the stable range. It is known that the statement of Theorem 5.1 holds in this case (*cf.* [42, Section 2] and [44, Section 1.8]).

We now prove the theorem by induction. Assume the theorem holds for $j+1$ with $1 \leqslant j+1 \leqslant k-1$. Applying Lemma 3.10 and Lemma 3.11, we see that $(\pi_{\mathscr{O}_{j+1},\eta_{j+1}}, \mathbf{V}_j)$ is in the convergent range. Thus (5.4) holds by Fubini's theorem. By [58, Theorem 1.19], $\mathrm{U}(\mathfrak{g}_{\mathbf{V}_j})^{\mathbf{G}_{\mathbf{V}_j}}$ acts on $\pi_{\mathscr{O}_j,\eta_j}$ through the character $\lambda_{\mathcal{O}_j}$. By Theorem 4.4, $\pi_{\mathscr{O}_j,\eta_j}$ is $\mathcal{O}_j$-bounded and (5.3) holds. The unitarity and irreducibility of $\pi_{\mathscr{O}_j,\eta_j}$ follows from Theorem 3.12.

By Definition 1.2, $\pi_{\mathscr{O}_j,\eta_j}$ is thus $\mathcal{O}_j$-unipotent. This finishes the proof of the theorem. □

5.2. **The exhaustion.** In this section, we will complete the proof of Theorem 1.4.

Put

$$\mathfrak{U}_{\mathcal{O}} := \{\, \pi_{\mathscr{O},\eta} \mid \mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p},\ \eta \text{ is a character as in (5.1)} \,\}.$$

By Theorem 5.1 and Lemma 2.21, we have a containment and a surjection:

$$\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G}) \supset \mathfrak{U}_{\mathcal{O}} \twoheadrightarrow \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

Thus in order to complete the proof of Theorem 1.4, it suffices to show that

$$\#(\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})) \leqslant \#\mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}). \tag{5.5}$$

We will prove this case by case.

5.2.1. *The case of quaternionic groups.* Recall the following result for quaternionic groups.

Proposition 5.2 ([47, Theorems 6 and 10], see also [67, Theorem 3.1])**.** *Assume that G is a quaternionic orthogonal group or a quaternionic symplectic group. Then*

$$\#(\Pi^{\mathrm{unip}}_{\mathcal{O}}(G)) = \textit{the number of } \mathbf{K}\textit{-orbits in } \mathcal{O} \cap \mathfrak{p}.$$

Note that $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\mathbf{K})$ is a singleton for each $\mathbf{K}$-orbit $\mathscr{O} \in \mathcal{O} \cap \mathfrak{p}$. Hence, the equality in (5.5) holds.

5.2.2. *The case of real symplectic and real orthogonal groups.* Now we consider the real symplectic groups and real orthogonal groups. First recall the following result.

Proposition 5.3. *Assume that G is a real symplectic group and $\mathbb{p}$ is even, or G is a split real odd orthogonal group and $\mathbb{p}$ is odd. Then*

$$\#(\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})) = \#\mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

Proof. In view of Lemma 2.20, this is equivalent to the assertion in [7, Theorem 5.3], which is proved in [9, Proposition 9.5]. $\square$

By Proposition 5.3, Theorem 1.4 holds if G is a real symplectic group and $\mathbb{p}$ is even, or G is a split real odd orthogonal group and $\mathbb{p}$ is odd.

Lemma 5.4. *The inequality* (5.5) *holds when G is a real even orthogonal group.*

Proof. Let $\mathbf{V}'$ be a $(-1,-1)$-space of dimension $2\dim\mathbf{V}$. Then (G, G') is in the stable range. Let $\mathcal{O}' := \boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$ and

$$\Pi'_\circ := \left\{ \pi' \in \Pi^{\mathrm{unip}}_{\mathcal{O}'}(\widetilde{G}') \,\middle|\, \mathrm{Ch}_{\mathcal{O}'}(\pi') \in \bigoplus_{\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}'),\ \mathscr{O}' \text{ descends to a } \mathbf{K}\text{-orbit in } \mathcal{O}\cap\mathfrak{p}} \mathcal{K}^{\mathbb{p}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \right\}.$$

Then by Theorem 1.4 for $(G', \mathcal{O}')$ and the discussions in Section 2.7.3, we have that

$$\#(\Pi'_\circ) = \#\mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\mathbf{K}).$$

By Theorem 4.3, we see that the stable range theta lifting [42]

$$\pi \mapsto \text{the unique irreducible quotient of } (\pi \widehat{\otimes} \omega_{\mathbf{V},\mathbf{V}'})_G$$

yields an injective map

$$\Pi^{\mathrm{unip}}_{\mathcal{O}}(G) \hookrightarrow \Pi'_\circ.$$

Thus the lemma follows. $\square$

Lemma 5.5. *The inequality* (5.5) *holds when G is a real symplectic group and $\mathbb{p}$ is odd.*

Proof. Let $\mathbf{V}'$ be a $(1,1)$-space of dimension $2\dim\mathbf{V}+1$ such that G' is a split real orthogonal group. Then (G, G') is in the stable range.

Let $\mathcal{O}' = \boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$. Then each $\mathbf{K}'$-orbit $\mathscr{O}'$ in $\mathcal{O}' \cap \mathfrak{p}'$ has a descent $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}$. Fix an element $X' \in \mathscr{O}'$ and a compatible $\mathfrak{sl}_2(\mathbb{C})$-triple attached to X' as in Section 2.5.2. Under the notation of Section 2.5.2, we write $\mathbf{K}'_{X'} = \mathbf{R}_{X'} \ltimes \mathbf{U}_{X'}$ and $\mathbf{R}_{X'} = \prod_{l=0}^{k+1} {}^l\mathbf{K}'$. Let $\mathcal{K}^{\mathbb{p};\circ}_{\mathscr{O}'}(\mathbf{K}')$ denote the subgroup of $\mathcal{K}^{\mathbb{p}}_{\mathscr{O}'}(\mathbf{K}')$ generated by all $\mathbb{p}$-genuine $\mathbf{K}'$-equivariant algebraic vector bundles $\mathcal{E}$ on $\mathscr{O}'$ such that ${}^0\mathbf{K}'$ acts trivially on the fibre $\mathcal{E}_{X'}$.

Put

$$\Pi'_\circ := \left\{ \pi' \in \Pi^{\mathrm{unip}}_{\mathcal{O}'}(G') \,\middle|\, \mathrm{Ch}_{\mathcal{O}'}(\pi') \in \bigoplus_{\mathscr{O}' \text{ is a } \mathbf{K}'\text{-orbit } \mathscr{O}' \text{ in } \mathcal{O}'\cap\mathfrak{p}'} \mathcal{K}^{\mathbb{p},\circ}_{\mathscr{O}'}(\mathbf{K}') \right\}.$$

Then by Theorem 1.4 for G' and the discussions in Section 2.7.3, we have that

$$\#(\Pi'_\circ) = \#\mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

The rest of the proof is the same as that of Lemma 5.4. $\square$

Lemma 5.5 implies that Theorem 1.4 holds when G is a real symplectic group and $\mathbb{p}$ is odd. Then the same proof as that of Lemma 5.4 implies that (5.5) holds when G is a real odd orthogonal group.

The inequality (5.5) and therefore Theorem 1.4 are now proved in all cases. $\square$

5.3. **Concluding remarks.** We record the following result, which is a form of automatic continuity.

Proposition 5.6. *Retain the setting in Section 5.1. The representation $\pi_{\mathscr{O},\eta}$ is isomorphic to*

$$(\omega_{\mathbf{V}_0,\mathbf{V}_1}\widehat{\otimes}\omega_{\mathbf{V}_1,\mathbf{V}_2}\widehat{\otimes}\cdots\widehat{\otimes}\omega_{\mathbf{V}_{k-1},\mathbf{V}_k}\otimes\eta)_{\widetilde{G}_{\mathbf{V}_1}\times\widetilde{G}_{\mathbf{V}_2}\times\cdots\times\widetilde{G}_{\mathbf{V}_k}},\tag{5.6}$$

and its underlying Harish-Chandra module is isomorphic to

$$(\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1}\otimes\mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2}\otimes\cdots\otimes\mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k}\otimes\eta)_{(\mathfrak{g}_{\mathbf{V}_1}\times\mathfrak{g}_{\mathbf{V}_2}\times\cdots\times\mathfrak{g}_{\mathbf{V}_k},\widetilde{\mathbf{K}}_{\mathbf{V}_1}\times\widetilde{\mathbf{K}}_{\mathbf{V}_2}\times\cdots\times\widetilde{\mathbf{K}}_{\mathbf{V}_k})}.\tag{5.7}$$

Proof. Note that the representation $\pi_{\mathscr{O},\eta}$ is a quotient of the representation (5.6), and the underlying Harish-Chandra module of (5.6) is a quotient of (5.7). Thus it suffices to show that (5.7) is irreducible. We prove by induction on k. Assume that $k\geqslant 1$ and

$$\pi_1:=(\mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2}\otimes\cdots\otimes\mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k}\otimes\eta_1)_{(\mathfrak{g}_{\mathbf{V}_2}\times\cdots\times\mathfrak{g}_{\mathbf{V}_k},\widetilde{\mathbf{K}}_{\mathbf{V}_2}\times\cdots\times\widetilde{\mathbf{K}}_{\mathbf{V}_k})}$$

is irreducible. Then π_1 is isomorphic to the Harish-Chandra module of $\pi_{\mathscr{O}_1,\eta_1}$. The representation (5.7) is isomorphic to

$$\pi_0:=\chi_0\otimes(\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1}\otimes\pi_1)_{(\mathfrak{g}_{\mathbf{V}_1},\widetilde{\mathbf{K}}_{\mathbf{V}_1})}.$$

We know that $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ acts on $\pi_{\mathscr{O},\eta}$ through the character $\lambda_{\mathcal{O}}$ (*cf.* [58]), and by Lemma 4.2, π_0 is $\mathcal{O}$-bounded. Thus every irreducible subquotient of π_0 is $\mathcal{O}$-unipotent. Theorem 4.3 implies that $\mathrm{AC}_{\mathcal{O}}(\pi_0)$ is bounded by $1\cdot[\mathscr{O}]$. Now Theorem 1.4 implies that π_0 must be irreducible. □

Finally, we remark that the Whittaker cycles (attached to G-orbits in $\mathcal{O}\cap\mathbf{i}\mathfrak{g}_{\mathbb{R}}$) of all unipotent representations in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ can be calculated by using [21, Theorem 1.1]. These agree with the associated cycles under the Kostant-Sekiguchi correspondence.

Appendix A. A few geometric facts

A.1. **Geometry of moment maps.** In this section, let $(\mathbf{V},\mathbf{V}')$ be a rational dual pair. We use the notation of Section 2.6. Write $\breve{\mathfrak{p}}:=\mathfrak{p}\oplus\mathfrak{p}'$ and

$$\breve{M}:=M\times M'\colon\mathcal{X}\longrightarrow\breve{\mathfrak{p}}=\mathfrak{p}\times\mathfrak{p}'.$$

When there is a morphism $f\colon A\to B$ between smooth algebraic varieties, let $\mathrm{d}f_a\colon \mathrm{T}_aA\to\mathrm{T}_{f(a)}B$ denote the tangent map at a closed point $a\in A$, where T_aA and $\mathrm{T}_{f(a)}B$ are the tangent spaces. For a sub-scheme S of B, let $A\times_f S$ denote the scheme theoretical inverse image of S under f, which is a subscheme of A. Along the proof, we will use the Jacobian criterion for regularity in various places (see [43, Theorem 2.19]).

A.1.1. *Scheme theoretical results on descents.*

Lemma A.1 (*cf.* [54, Lemma 13 and Lemma 14])**.** *Suppose $\mathscr{O}'\in\mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the descent of $\mathscr{O}\in\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. Fix $X\in\mathscr{O}$ and let*

$$\mathcal{Z}_X:=\mathcal{X}\times_M\{X\}$$

be the scheme theoretical inverse image of X under the moment map $M:\mathcal{X}\to\mathfrak{p}$. Then

(i) $\mathcal{Z}_X$ is smooth (and hence reduced);
(ii) $\mathcal{Z}_X$ is a single free $\mathbf{K}'$-orbit in $\mathcal{X}^\circ$;
(iii) $M'(\mathcal{Z}_X)=\mathscr{O}'$ and $\mathcal{Z}_X=\mathcal{X}\times_{\breve{M}}(\{X\}\times\overline{\mathscr{O}'})$.

Proof. Fix a point $w \in \mathcal{X}^\circ$ such that $M(w) = X$. It is elementary to see that

$$M^{-1}(X) = \mathbf{K}' \cdot w$$

is a single $\mathbf{K}'$-orbit. This orbit is clearly Zariski closed and the stabilizer $\mathrm{Stab}_{\mathbf{K}'}(w)$ of w under the $\mathbf{K}'$-action is trivial. This proves part (ii) and the first assertion of part (iii).

We identify the tangent space $\mathrm{T}_w\mathcal{X}$ with $\mathcal{X}$. Let $N \subset \mathcal{X}$ be a complement of $\mathrm{T}_w(\mathbf{K}' \cdot w)$ in $\mathrm{T}_w\mathcal{X}$. By [57, Section 6.3], there is an affine open subvariety N° of N containing 0 such that the diagram

$$\begin{tikzcd}
\mathbf{K}' \times N^\circ \arrow[r, "(k',n)\mapsto k'\cdot(w+n)"] \arrow[d, "\text{projection}"'] & \mathcal{X} \arrow[d, "M"] \\
N^\circ \arrow[r, "n\mapsto M(w+n)"] & M(\mathcal{X})
\end{tikzcd}$$

is Cartesian and the horizontal morphisms are étale. Therefore the scheme $\mathcal{X} \times_M \{X\}$ is a smooth algebraic variety. This proves part (i), and the second assertion of (iii) then easily follows. $\square$

Remark. Retain the setting of Lemma A.1. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from X to $X' \in \mathfrak{p}'$. Let $\check{X} := (X, X') \in \mathfrak{p} \oplus \mathfrak{p}'$ and $\mathbf{K}'_{X'}$ be the stabilizer of X' under the $\mathbf{K}'$-action. By the $\mathbf{K}'$-equivariance, the map $\mathcal{Z}_X \to \mathscr{O}'$ is a smooth morphism between smooth schemes. Hence the scheme theoretical fibre

$$\mathcal{Z}_{\check{X}} := \mathcal{X} \times_{\check{M}} \{\check{X}\} = \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{X'\}$$

is smooth and equals the orbit $\mathbf{K}'_{X'} \cdot w$. Consequently, we have an exact sequence by the Jacobian criterion for regularity:

$$0 \longrightarrow \mathrm{T}_w\mathcal{Z}_{\check{X}} \longrightarrow \mathrm{T}_w\mathcal{X} \xrightarrow{\mathrm{d}\check{M}_w} \mathrm{T}_X\mathfrak{p} \oplus \mathrm{T}_{X'}\mathfrak{p}'. \tag{A.1}$$

Lemma A.2. *Retain the setup in Lemma A.1. Let*

$$U := \mathfrak{p} \backslash ((\overline{\mathcal{O}} \cap \mathfrak{p}) \backslash \mathscr{O})$$

and $\mathcal{Z}_{U,\overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$. *Then*

(i) U *is a Zariski open subset of* $\mathfrak{p}$ *such that* $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$;
(ii) $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ *is smooth and it is a single* $\mathbf{K} \times \mathbf{K}'$*-orbit.*

Proof. Part (i) is clear by (2.8). It is also clear that the underlying set of $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit by Lemma A.1 (iii).

Note that $M'|_{\mathcal{X}^\circ}$ is smooth because $\mathrm{d}M'_w \colon \mathrm{T}_w\mathcal{X}^\circ \longrightarrow \mathrm{T}_{M'(w)}\mathfrak{p}'$ is surjective for each $w \in \mathcal{X}^\circ$ (*cf.* [25, Proposition 10.4]). Since smooth morphisms are stable under base changes and compositions, the scheme $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$ is a smooth algebraic variety. Also note that $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is an open subscheme of $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$. Thus it is smooth. $\square$

Remark. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from $X \in \mathscr{O}$ to $X' \in \mathscr{O}' \subset \mathfrak{p}'$. By the proof of Lemma A.2,

$$\mathcal{Z}_{U,X'} := \mathcal{X} \times_{\check{M}} (U \times \{X'\}) = \mathbf{K}\mathbf{K}'_{X'} \cdot w$$

is smooth. Similar to the remark of Lemma A.1, this yields an exact sequence:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A,A')\mapsto A'w - wA]{} \mathcal{X} \xrightarrow[b\mapsto bw^{\star} + wb^{\star}]{\mathrm{d}M'_w} \mathfrak{p}'. \tag{A.2}$$

A.1.2. *Scheme theoretical results on generalized descents of good orbits.*

Lemma A.3. *Suppose $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is good for generalized descent (see Definition 2.15) and $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Fix $X \in \mathscr{O}$ and let $\mathcal{Z}_X := \mathcal{X} \times_{\check{M}} (\{ X \} \times \overline{\mathscr{O}'})$. Then*

(i) $\mathcal{Z}_X$ is smooth;
(ii) $\mathcal{Z}_X$ is a single $\mathbf{K}'$-orbit;
(iii) $\mathcal{Z}_X$ is contained in $\mathcal{X}^{\mathrm{gen}}$ and $M'(\mathcal{Z}_X) = \mathscr{O}'$.

Proof. Part (ii) follows from [18, Table 4]. Part (iii) follows from part (ii). By the generic smoothness, in order to prove part (i), it suffices to show that $\mathcal{Z}_X$ is reduced. Note that the $\mathbf{K}'$-equivariant morphism $\mathcal{Z}_X \xrightarrow{M'} \mathscr{O}'$ is flat by generic flatness [22, Théorème 6.9.1]. By [23, Proposition 11.3.13], to show that $\mathcal{Z}_X$ is reduced, it suffices to show that

$$\mathcal{Z}_{\check{X}} := \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{ X' \} = \mathcal{X} \times_{\check{M}} \{ \check{X} \}$$

is reduced, where $X' \in \mathscr{O}'$ and $\check{X} := (X, X')$. Let $w \in \mathcal{Z}_{\check{X}}$ be a closed point. Then $\mathbf{K}' \cdot w$ is the underlying set of $\mathcal{Z}_X$, and $Z_{\check{X}} := \mathbf{K}'_{X'} \cdot w$ is the underlying set of $\mathcal{Z}_{\check{X}}$. By the Jacobian criterion for regularity, to complete the proof of the lemma, it remains to prove the following claim.

Claim. *The following sequence is exact:*

$$0 \longrightarrow \mathrm{T}_w Z_{\check{X}} \longrightarrow \mathrm{T}_w \mathcal{X} \xrightarrow{\mathrm{d}\check{M}} \mathrm{T}_X \mathfrak{p} \oplus \mathrm{T}_{X'} \mathfrak{p}'.$$

We prove the above claim in what follows. Identify $\mathrm{T}_w\mathcal{X}$, $\mathrm{T}_X\mathfrak{p}$ and $\mathrm{T}_{X'}\mathfrak{p}'$ with $\mathcal{X}, \mathfrak{p}$ and $\mathfrak{p}'$ respectively and view $\mathrm{T}_w Z_{\check{X}}$ as a quotient of $\mathfrak{k}'_{X'}$. It suffices to show that the following sequence is exact:

$$\mathfrak{k}'_{X'} \xrightarrow[A' \mapsto A'w]{} \mathcal{X} \xrightarrow[b \mapsto (b^{\star} w + w^{\star} b, bw^{\star} + wb^{\star})]{\mathrm{d}\check{M}_w} \mathfrak{p} \oplus \mathfrak{p}' = \check{\mathfrak{p}}. \tag{A.3}$$

In fact, we will show that the following sequence

$$\mathfrak{g}'_{X'} \longrightarrow \mathbf{W} \longrightarrow \mathfrak{g} \oplus \mathfrak{g}' \tag{A.4}$$

is exact, where $\mathfrak{g}'_{X'}$ is the Lie algebra of the stabilizer group $\mathrm{Stab}_{\mathbf{G}'}(X')$ and the arrows are defined by the same formulas in (A.3). Then the exactness of (A.3) will follow since (A.4) is compatible with the natural (L, L')-actions.

We now prove the exactness of (A.4). Let $\mathbf{V}'_1 := w\mathbf{V}$ and $\mathbf{V}'_2$ be the orthogonal complement of $\mathbf{V}'_1$ so that

$$\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2.$$

Let $\mathbf{W}_i := \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_i)$ and $\mathfrak{g}'_i := \mathfrak{g}_{\mathbf{V}'_i}$ for $i = 1, 2$. Let $*\colon \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \to \mathrm{Hom}(\mathbf{V}'_2, \mathbf{V}'_1)$ denote the adjoint map. We have $\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2$ and $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxtimes}$, where

$$\mathfrak{g}^{\boxtimes} := \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \right\}.$$

Now w and X' are naturally identified with an element w_1 in $\mathbf{W}_1$ and an element X'_1 in $\mathfrak{g}'_1$, respectively. We have $\mathfrak{g}'_{X'} = \mathfrak{g}'_{1,X'_1} \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxtimes}_{X'_1}$, where

$$\mathfrak{g}^{\boxtimes}_{X'_1} = \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2),\ EX'_1 = 0 \right\}.$$

Now maps in (A.4) have the following forms respectively:

$$\mathfrak{g}'_{X'} \ni A' = (A'_1, A'_2, \left(\begin{smallmatrix} 0 & -E^* \\ E & 0 \end{smallmatrix}\right)) \mapsto A'w = (A'_1 w_1, E w_1) \in \mathbf{W} \qquad \text{and}$$

$$\mathbf{W} \ni (b_1, b_2) \mapsto \left(b_1^{\star} w_1 + w_1^{\star} b_1, \begin{pmatrix} b_1 w_1^{\star} + w_1 b_1^{\star} & w_1 b_2^{\star} \\ b_2 w_1^{\star} & 0 \end{pmatrix} \right) \in \mathfrak{g} \oplus \mathfrak{g}'.$$

Applying (A.1) to the complex dual pair $(\mathbf{V}, \mathbf{V}'_1)$, we see that the sequence

$$\mathfrak{g}'_{1,X'_1} \xrightarrow{A'_1 \mapsto A'_1 w_1} \mathbf{W}_1 \xrightarrow{b_1 \mapsto (b_1^{\star} w_1 + w_1^{\star} b_1, b_1 w_1^{\star} + w_1 b_1^{\star})} \mathfrak{g} \oplus \mathfrak{g}'_1$$

is exact. The task is then reduced to show that the following sequence is exact:

$$\text{(A.5)} \qquad \mathfrak{g}^{\boxslash}_{X'_1} \xrightarrow[\text{①}]{\left(\begin{smallmatrix} 0 & -E^* \\ E & 0 \end{smallmatrix}\right) \mapsto E w_1} \mathbf{W}_2 \xrightarrow[\text{②}]{b_2 \mapsto b_2 w_1^{\star}} \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2).$$

Note that w_1 is a surjection, hence ① is an injection. We have

$$\dim \mathfrak{g}^{\boxslash}_{X'_1} = \dim \mathbf{V}'_2 \cdot \dim \mathrm{Ker}(X'_1)$$

and

$$\dim \mathrm{Ker}(\text{②}) = \dim \mathbf{V}'_2 \cdot (\dim \mathbf{V} - \dim \mathrm{Im}(w_1)) = \dim \mathbf{V}'_2 \cdot (\dim \mathbf{V} - \dim \mathbf{V}'_1).$$

Note that $\dim \mathrm{Ker}(X'_1) = c_1$, and $\dim \mathbf{V} - \dim \mathbf{V}'_1 = c_0$. Since we are in the setting of good generalized descent (Definition 2.15), we have $c_0 = c_1$. Now the exactness of (A.5) follows by dimension counting. □

Remark. Suppose $\mathscr{O}$ is not good for generalized descent. Then the underlying set of $\mathcal{Z}_X$ may not be a single $\mathbf{K}'$-orbit. Even if it is a single orbit, the scheme $\mathcal{Z}_X$ may not be reduced.

Lemma A.4. *Retain the notation in Lemma A.3. Let*

$$U := \mathfrak{p} \backslash ((\overline{\mathcal{O}} \cap \mathfrak{p}) \backslash \mathscr{O})$$

and $\mathcal{Z}_{U,\overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$. *Then*

(i) U *is a Zariski open subset of* $\mathfrak{p}$ *such that* $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$;
(ii) $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ *is smooth and it is a single* $\mathbf{K} \times \mathbf{K}'$*-orbit.*

Proof. Part (i) is follows from Lemma 2.16. By Lemma A.3 (ii), the underlying set of $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit whose image under M' is contained in $\mathscr{O}'$. By generic flatness [22, Théorème 6.9.1], the morphism $\mathcal{Z}_{U,\overline{\mathscr{O}'}} \longrightarrow \mathscr{O}'$ is flat. To prove the scheme theoretical claim in part (ii), it suffices to show that

$$\mathcal{Z}_{U,X'} := \mathcal{X} \times_{\check{M}} (U \times \{ X' \})$$

is reduced, where $X' \in \mathscr{O}'$.

Let $w \in \mathcal{Z}_{U,X'}$ be a closed point. As in the proof of Lemma A.3, by the Jacobian criterion for regularity, it suffices to show that the following sequence is the exact:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A,A') \mapsto A'w - wA]{} \mathcal{X} \xrightarrow[b \mapsto b w^{\star} + w b^{\star}]{\mathrm{d}M'_w} \mathfrak{p}'.$$

The proof of the exactness follows the same line as the proof of Lemma A.3 utilizing (A.5) and the established exact sequence (A.2) in the descent case. We leave the details to the reader. □

A.2. **Proof of Lemma 4.8.** We will use notations of Section 3.3 and Section 4.2.2. Let Hom_J denote the space of homomorphisms commuting with the $\dot\epsilon$-real from J, and $\mathfrak{n}_\mathbb{R}$ denote the Lie algebra of $N_{\mathbf{E}_0}$.

We first explicitly construct some elements in the induced orbits in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$. Fix an element $\mathbf{i}\, X \in \mathscr{O}_\mathbb{R}$ and a J-invariant complement $\mathbf{V}_+$ of $\mathrm{Im}(X)$ in the vector space $\mathbf{V}^- = \mathbf{V}$ so that

$$\mathbf{V}^- = \mathbf{V}_+ \oplus \mathrm{Im}(X).$$

Fix any J-invariant decompositions

$$\mathbf{E}_0 = \mathbf{L}_1 \oplus \mathbf{L}_0 \quad \text{and} \quad \mathbf{E}_0' = \mathbf{L}_1' \oplus \mathbf{L}_0'$$

such that $\dim \mathbf{L}_1 = \dim \mathbf{V}_+$, and $(\mathbf{L}_1 \oplus \mathbf{L}_1') \perp (\mathbf{L}_0 \oplus \mathbf{L}_0')$. (This is possible due to the dimension inequality in (4.5).)

Fix a linear isomorphism

$$S \in \mathrm{Hom}_J(\mathbf{L}_1', \mathbf{V}_+),$$

and view it as an element of $\mathrm{Hom}_J(\mathbf{E}_0', \mathbf{V}^-)$ via the aforementioned decompositions. Put

$$\mathscr{T} := \{\, T \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0) \mid T^* + T = 0 \,\} \quad \text{and}$$
$$\mathscr{T}^\circ := \{\, T \in \mathscr{T} \mid T \text{ has maximal possible rank} \,\}$$

to be viewed as subsets of $\mathrm{Hom}_J(\mathbf{E}_0', \mathbf{E}_0)$ as before, where $T^* \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0)$ is specified by requiring that

$$\langle T \cdot u, v \rangle_{\mathbf{V}^\perp} = \langle u, T^* v \rangle_{\mathbf{V}^\perp}, \quad \text{for all } u, v \in \mathbf{L}_0', \tag{A.6}$$

and $\langle\, ,\, \rangle_{\mathbf{V}^\perp}$ denotes the ϵ-symmetric bilinear form on $\mathbf{V}^\perp$.

Each $T \in \mathscr{T}$ defines a $(-\epsilon)$-symmetric bilinear form $\langle\, ,\, \rangle_T$ on $\mathbf{L}_0'$ where

$$\langle v_1, v_2 \rangle_T := -\langle v_1, T v_2 \rangle, \quad \text{for all } v_1, v_2 \in \mathbf{L}_0'.$$

Put

$$X_{S,T} := \begin{pmatrix} 0 & S^* & T \\ & X & S \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_\mathbb{R} \tag{A.7}$$

according to the decomposition $\mathbf{V}^\perp = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}_0'$, where $S^* \in \mathrm{Hom}_J(\mathbf{V}^-, \mathbf{E}_0)$ is similarly defined as in (A.6).

Now suppose that $T \in \mathscr{T}^\circ$.

In case (ii) of Lemma 4.8, the form $\langle\, ,\, \rangle_T$ must be non-degenerate and $((\mathbf{L}_0', \langle\, ,\, \rangle_T), J|_{\mathbf{L}_0'})$ becomes a $(-\epsilon, -\dot\epsilon)$-space. Conversely, up to isomorphism, every $(-\epsilon, -\dot\epsilon)$-space of dimension $l - c_1$ is isomorphic to $((\mathbf{L}_0', \langle\, ,\, \rangle_T), J|_{\mathbf{L}_0'})$ for some $T \in \mathscr{T}$.

Let s be the signature of $((\mathbf{L}_0', \langle\, ,\, \rangle_T), J|_{\mathbf{L}_0'})$ and let $G_T := \mathbf{G}_{\mathbf{L}_0'}^J$ denote the corresponding real group. Then $\mathbf{i}\, X_{S,T}$ generates an induced orbit of $\mathscr{O}_\mathbb{R}$ with signed Young diagram $[d_1 + s, \check{d}_1 + \check{s}, d_1, \cdots, d_k]$.[7] One checks that the reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T})$ and $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\, X_{S,T})$ are both canonically isomorphic to $R \times G_T$, where R is the reductive quotient of $\mathrm{Stab}_G(X)$. Hence $C_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T}) \cong C_{G_{\mathbf{V}^\perp}}(\mathbf{i}\, X_{S,T})$.

In case (i) of Lemma 4.8, $\langle\, ,\, \rangle_T$ is skew symmetric, and T has rank $\dim \mathbf{L}_0 - 1 = l - c_1 - 1$ since $l - c_1$ is odd. One checks that $\mathbf{i}\, X_{S,T}$ generates an induced orbit of $\mathscr{O}_\mathbb{R}$ with the signed Young diagram prescribed in (i) of Lemma 4.8.

Fix decompositions

$$\mathbf{L}_0 = \mathbf{L}_2 \oplus \mathbf{L}_3 \quad \text{and} \quad \mathbf{L}_0' = \mathbf{L}_2' \oplus \mathbf{L}_3'$$

[7] The signed Young diagram may be computed using the explicit description of Kostant-Sekiguchi correspondence in [18, Propositions 6.2 and 6.4].

which are dual to each other such that $\mathrm{Ker}(T) = \mathbf{L}_3'$ and $\langle\ ,\ \rangle_T\,|_{\mathbf{L}_2'\times\mathbf{L}_2'}$ is a non-degenerate skew symmetric bilinear form on $\mathbf{L}_2'$. Let $G_T := \mathbf{G}^J_{\mathbf{L}_2'}$. Then G_T is a real symplectic group. The reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i}X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathrm{GL}(\mathbf{L}_3)^J$ and the reductive quotient of $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i}X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathbf{G}^J_{\mathbf{L}_3\oplus\mathbf{L}_3'}$, where R is the reductive quotient of $\mathrm{Stab}_G(\mathbf{i}X)$ and $\mathbf{G}^J_{\mathbf{L}_3\oplus\mathbf{L}_3'} \cong \mathrm{O}(1,1)$. The homomorphism

$$C_{P_{\mathbf{E}_0}}(\mathbf{i}\,X_{S,T}) \longrightarrow C_{G_{\mathbf{V}^\perp}}(\mathbf{i}X_{S,T})$$

is therefore an injection whose image has index 2.

To finish the proof of the lemma, it suffices to show that

$$\text{(A.8)} \qquad \mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}} = \{\, G_{\mathbf{V}^\perp}\cdot \mathbf{i}X_{S,T} \mid T \in \mathscr{T}^\circ \,\}.$$

Consider the set

$$\mathfrak{A} := \left\{ \begin{pmatrix} 0 & B^* & C \\ & X & B \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_{\mathbb{R}} \;\middle|\; X \oplus B \in \mathrm{Hom}_J(\,\mathbf{V}^- \oplus \mathbf{E}_0',\ \mathbf{V}^-) \text{ is surjective} \right\}.$$

Clearly $P'_{\mathbf{E}_0} := \mathrm{GL}_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$ acts on $\mathfrak{A}$ (*cf.* (3.12)). By suitable matrix manipulations, one sees that every element in $\mathfrak{A}$ is conjugated to an element in $\{\, X_{S,T} \mid T \in \mathscr{T} \,\}$ under the $P'_{\mathbf{E}_0}$-action. Hence $P'_{\mathbf{E}_0}\cdot\{\, X_{S,T} \mid T \in \mathscr{T}^\circ \,\}$ is open dense in $\mathfrak{A}$. Now (A.8) follows since $P_{\mathbf{E}_0}\cdot \mathbf{i}\mathfrak{A}$ is open dense in $\mathscr{O}_{\mathbb{R}} + \mathbf{i}\mathfrak{n}_{\mathbb{R}}$, $\square$

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School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China

Email address: hoxide@sjtu.edu.cn

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China

Email address: sun@math.ac.cn

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076

Email address: matzhucb@nus.edu.sg