SPECIAL UNIPOTENT REPRESENTATIONS : ORTHOGONAL AND SYMPLECTIC GROUPS

DAN M. BARBASCH, JIA-JUN MA, BINYONG SUN, AND CHEN-BO ZHU

Abstract. Let G be a real classical group of type B, C, D (including the real metaplectic group). We consider a nilpotent adjoint orbit $\check{\mathcal{O}}$ of $\check{G}$, the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group). We classify all special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. When $\check{\mathcal{O}}$ is of good parity, we construct all such representations of G via the method of theta lifting. As a consequence of the construction and the classification, we conclude that all special unipotent representations of G are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture.

Contents

1. Introduction and the main results

1.1. **Unitary representations and the orbit method.** A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [46, 47]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [7, 46]. For more general Lie groups, work of Mackey and Duflo [31, 59] suggest that one should focus attention on reductive Lie groups. As expounded by Vogan in his writings (see for example [85, 87, 88]), the problem finally is to

2000 *Mathematics Subject Classification.* 22E45, 22E46.

Key words and phrases. orbit method, unitary dual, special unipotent representation, classical group, theta lifting, moment map.

quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We highlight two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [5,6], Adams, Barbasch and Vogan established some important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [3]. The problem of classifying (integral) special unipotent representations for complex semisimple groups was solved earlier by Barbasch and Vogan [17] and the unitarity of these representations was established by Barbasch for complex classical groups [8, Section 10]. Shortly after, Barbasch outlined a proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [9]. The second major development is Vogan's theory of associated varieties [86] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a representation "attached" to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [42], the description of discrete spectrum by Adams [1] and Li [54], and the preservation of unitarity in stable range theta lifting by Li [53]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [13,18,37,39,40,71,74,77,83]. We would also like to mention the work of Przebinda [74] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [37] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [32,56,65,67], which among other things demonstrate the tight link between the orbit method and Howe's theory.

As mentioned earlier, there have been extensive investigations of unipotent representations for real reductive groups, by Vogan and his collaborators (see e.g. [3,85,86]). Nevertheless, the subject remains extraordinarily mysterious thus far.

In the present article we will demonstrate that Howe's (constructive) theory has immense implications for the orbit method, and in particular has near perfect synergy with special unipotent representations (in the case of classical groups). (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [13,37,62,83].) We will restrict our attention to a real classical group G of type B, C or D (which in our terminology includes a real metaplectic group), and we will classify all special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. Here $\check{\mathcal{O}}$ is a nilpotent adjoint orbit of $\check{G}$, the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group [14]). When $\check{\mathcal{O}}$ is of good parity [63], we will construct all special unipotent representations of G attached to $\check{\mathcal{O}}$ via the method of theta lifting. As a direct consequence of the construction and the classification, we conclude that all special unipotent representations of G are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([3, Introduction]).

1.2. **Special unipotent representations of classical groups of type B, C or D.** In this article, we aim to classify special unipotent representations of classical groups of type B, C or D. As the cases of complex orthogonal groups and complex symplectic groups are well-understood (see [17] and [13]), we will focus on the following groups:

$$\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p,q),\ \mathrm{O}^*(2n), \tag{1.1}$$

where $p,q,n \in \mathbb{N} := \{0,1,2,\cdots\}$. Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the real metaplectic group, namely the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n = 0$.

Let G be one of the groups in (1.1). As usual, we view G as a real form of $G_{\mathbb{C}}$ (or a double cover of a real form of $G_{\mathbb{C}}$ in the metaplecitic case), where

$$G_{\mathbb{C}} := \begin{cases} \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \text{ or } \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2p+2q}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ for the Lie algebras of G and $G_{\mathbb{C}}$, respectively, and view $\mathfrak{g}_{\mathbb{R}}$ as a real form of $\mathfrak{g}$.

Denote $r_{\mathfrak{g}}$ the rank of $\mathfrak{g}$. Let $W_{r_{\mathfrak{g}}}$ be the subgroup of $\mathrm{GL}_{r_{\mathfrak{g}}}(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. Then as usual, Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} = (\mathrm{S}(\mathbb{C}^{r_{\mathfrak{g}}}))^{W_{r_{\mathfrak{g}}}}. \tag{1.2}$$

Here and henceforth, “U” indicates the universal enveloping algebra of a Lie algebra, a superscript group indicate the space of invariant vectors under the group action, and “S” indicates the symmetric algebra. Unless $G_{\mathbb{C}}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. By (1.2), we have the following parameterization of characters of $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$:

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}, \mathbb{C}) = W_{r_{\mathfrak{g}}}\backslash(\mathbb{C}^{r_{\mathfrak{g}}})^* = W_{r_{\mathfrak{g}}}\backslash\mathbb{C}^{r_{\mathfrak{g}}}$$

Here “$\mathrm{Hom}_{\mathrm{alg}}$” indicates the set of $\mathbb{C}$-algebra homomorphisms, and a superscript “ $*$ ” over a vector space indicates the dual space.

We define the Langlands dual of G to be the complex group

$$\check{G} := \begin{cases} \mathrm{Sp}_{p+q-1}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is odd}; \\ \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is even}; \\ \mathrm{O}_{2n+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{O}_{2p+2q+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$.

Remark. The authors have defined the notion of metaplectic dual for a real metaplectic group [14]. In this article, we have chosen to use the uniform terminology of Langlands dual (rather than metaplectic dual in the case of a real metaplectic group).

Denote by $\mathrm{Nil}(\check{\mathfrak{g}})$ the set of $\check{G}$-orbits of nilpotent matrices in $\check{\mathfrak{g}}$. When no confusion is possible, we will not distinguish a nilpotent orbit for $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ with its corresponding Young diagram. In particular, the zero orbit is represented by the Young diagram consisting of one nonempty column at most.

Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. It determines a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$ as in what follows. For every $a \in \mathbb{N}$, write

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd;} \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even;} \end{cases}$$

By convention, $\rho(1)$ and $\rho(0)$ are the empty sequence. Write $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ ($s \geqslant 0$) for the row lengths of $\check{\mathcal{O}}$. Define

$$\chi(\check{\mathcal{O}}) := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0), \tag{1.3}$$

to be viewed as a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$. Here the number of 0's is

$$\left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor.$$

Recall the following well-known result of Dixmier ([19, Section 3]): for every algebraic character χ of $\mathrm{Z}(\mathfrak{g})$, there exists a unique maximal ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of χ. As an easy consequence, there is a unique maximal $G_{\mathbb{C}}$-stable ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$, for $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.

Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation ([22, 90]) if its Harish-Chandra module has finite length. When $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a metaplectic group, write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then a representation of G is said to be genuine if ε_G acts via the scalar multiplication by -1. The notion of "genuine" will be used in similar situations without further explanation. Following Barbasch and Vogan ([3, 17]), we make the following definition.

Definition 1.1. *Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. An irreducible Casselman-Wallach representation π of G is attached to $\check{\mathcal{O}}$ if*

- *$I_{\check{\mathcal{O}}}$ annihilates π; and*
- *π is genuine if G is a metaplectic group.*

Write $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ for the set of isomorphism classes of irreducible Casselman-Wallach representations of G that are attached to $\check{\mathcal{O}}$. We say that an irreducible Casselman-Wallach representation of G is special unipotent if it is attached to $\check{\mathcal{O}}$, for some $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. As mentioned earlier, we will construct all special unipotent representations of G, and will show that all of them are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([3, Introduction]).

1.3. **Combinatorial construct: painted bipartitions.** We introduce a symbol $\star$, taking values in $\{B, C, D, \widetilde{C}, C^*, D^*\}$, to specify the type of the groups that we are considering as in (1.1), namely odd real orthogonal groups, real symplectic groups, even real orthogonal groups, real metaplectic groups, quaternionic symplectic groups and quaternionic orthogonal groups, respectively.

For a Young diagram $\imath$, write

$$\mathbf{r}_1(\imath) \geqslant \mathbf{r}_2(\imath) \geqslant \mathbf{r}_3(\imath) \geqslant \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty} \mathbf{r}_i(\imath)$ the total size of $\imath$.

For any Young diagram $\imath$, we introduce the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as the following subset of $\mathbb{N}^+ \times \mathbb{N}^+$ ($\mathbb{N}^+$ denotes the set of positive integers):

$$\mathrm{Box}(\imath) := \left\{ (i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \mid j \leqslant \mathbf{r}_i(\imath) \right\}. \tag{1.4}$$

We also introduce five symbols $\bullet$, s, r, c and d, and make the following definition.

Definition 1.2. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *when $S = \{s\}$ or $\{r\}$, every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$;*
- *when $S = \{c\}$ or $\{d\}$, every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$.*

A painted Young diagram is then a pair $(\imath, \mathcal{P})$, consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$.

Example. Suppose that $\imath = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, then there are $25 + 12 + 6 + 2 = 45$ paintings on $\imath$ in total as listed below.

$$\begin{array}{|c|c|}\hline \bullet & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \alpha, \beta \in \{\bullet, s, r, c, d\} \qquad\qquad \begin{array}{|c|c|}\hline s & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \alpha \in \{r, c, d\}, \beta \in \{s, r, c, d\}$$

$$\begin{array}{|c|c|}\hline r & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \alpha \in \{c, d\}, \beta \in \{r, c, d\} \qquad\qquad \begin{array}{|c|c|}\hline c & \alpha \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \alpha \in \{c, d\}$$

When no confusion is possible, we write $\alpha \times \beta \times \gamma$ for a triple (α, β, γ). We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.3. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- *$\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$;*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \textit{if } \alpha = B^+ \textit{ or } B^-; \\ \{\bullet, r, c, d\}, & \textit{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \textit{if } \alpha = D; \\ \{\bullet, s, c\}, & \textit{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \textit{if } \alpha = C^*; \\ \{\bullet, s\}, & \textit{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \textit{if } \alpha = B^+ \textit{ or } B^-; \\ \{\bullet, s\}, & \textit{if } \alpha = C; \\ \{\bullet\}, & \textit{if } \alpha = D; \\ \{\bullet, r, d\}, & \textit{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \textit{if } \alpha = C^*; \\ \{\bullet, r\}, & \textit{if } \alpha = D^*. \end{cases}$$

For any painted bipartition τ as in Definition 1.3, we write

$$\imath_\tau := \imath, \ \mathcal{P}_\tau := \mathcal{P}, \ \jmath_\tau := \jmath, \ \mathcal{Q}_\tau := \mathcal{Q}, \ \alpha_\tau := \alpha,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

We further attach some objects to τ in what follows:

$$|\tau|\,,\ (p_\tau,\ q_\tau),\ G_\tau,\ \dim\tau,\ \varepsilon_\tau.$$

$|\tau|$: This is the natural number

$$|\tau| := |\imath| + |\jmath|\,.$$

(p_τ, q_τ) : If $\star_\tau \in \{B, D, C^*\}$, this is a pair of natural numbers given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\left\{\begin{array}{l} p_\tau := (\#\bullet) + 2(\#r) + (\#c) + (\#d) + (\#B^+); \\ q_\tau := (\#\bullet) + 2(\#s) + (\#c) + (\#d) + (\#B^-). \end{array}\right. \tag{1.5}$$

Here

$$\#\bullet := \#(\mathcal{P}^{-1}(\bullet)) + \#(\mathcal{Q}^{-1}(\bullet)) \qquad \text{(\# indicates the cardinality of a finite set)},$$

and the other terms are similarly defined. If $\star_\tau \in \{C, \widetilde{C}, D^*\}$, $p_\tau := q_\tau := |\tau|$.

G_τ: This is a classical group given by

$$G_\tau := \begin{cases} \mathrm{O}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

$\dim\tau$: This is the dimension of the standard representation of the complexification of G_τ, or equivalently,

$$\dim\tau := \begin{cases} 2\,|\tau| + 1, & \text{if } \star_\tau = B; \\ 2\,|\tau|\,, & \text{otherwise.} \end{cases}$$

ε_τ: This is the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\tau = 0 \Leftrightarrow \text{the symbol } d \text{ occurs in the first column of } (\imath, \mathcal{P}) \text{ or } (\jmath, \mathcal{Q}).$$

The triple $\mathsf{s}_\tau = (\star_\tau, p_\tau, q_\tau) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ will also be referred to as the classical signature attached to τ.

Example. Suppose that

$$\tau = \begin{array}{|c|c|} \hline \bullet & c \\ \hline \bullet \\ \cline{1-1} c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & s & r \\ \hline \bullet \\ \cline{1-1} r \\ \cline{1-1} r \\ \cline{1-1} \end{array} \times B^+.$$

Then

$$\begin{cases} |\tau| = 10; \\ p_\tau = 4 + 6 + 2 + 0 + 1 = 13; \\ q_\tau = 4 + 2 + 2 + 0 + 0 = 8; \\ G_\tau = \mathrm{O}(13, 8); \\ \dim\tau = 21; \\ \varepsilon_\tau = 1. \end{cases}$$

1.4. **Counting special unipotent representations by painted bipartitions.** We fix a classical group G which has type $\star$. Following [63, Definition 4.1], we say that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity if

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \check{G} \text{ is a complex symplectic group; and} \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \check{G} \text{ is a complex orthogonal group.} \end{cases}$$

In general, the study of the special unipotent representations attached to $\check{\mathcal{O}}$ will be reduced to the case when $\check{\mathcal{O}}$ has $\star$-good parity. We refer the reader to [15].

For the rest of this subsection, assume that $\check{\mathcal{O}}$ has $\star$-good parity. Equivalently we consider $\check{\mathcal{O}}$ as a Young diagram that has $\star$-good parity in the following sense:

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \star \in \{B, \widetilde{C}\}; \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \star \in \{C, D, C^*, D^*\}; \text{ and} \\ \text{the total size } |\check{\mathcal{O}}| \text{ is odd if and only if } \star \in \{C, C^*\}. \end{cases}$$

Definition 1.4. *A $\star$-pair is a pair $(i, i+1)$ of consecutive positive integers such that*

$$\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be
- *vacant in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*
- *tailed in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and even.*

Denote $\mathrm{PP}_\star(\check{\mathcal{O}})$ the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$.

We attach to $\check{\mathcal{O}}$ a pair of Young diagrams

$$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) := (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})),$$

as follows.

The case when $\star = B$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star = \widetilde{C}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star \in \{C, C^*\}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

The case when $\star \in \{D, D^*\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0, \end{cases}$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0\right), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}\right), & \text{otherwise.} \end{cases}$$

Define

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) := \left\{ \tau \text{ is a painted bipartition} \mid \star_\tau = \star \text{ and } (\imath_\tau, \jmath_\tau) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})) \right\}.$$

We also define the following extended parameter set:

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) := \begin{cases} \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})\}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\varnothing\}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

We use $\boldsymbol{\tau} = (\tau, \wp)$ to denote an element in $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$.

Put

$$\mathrm{Unip}_\star(\check{\mathcal{O}}) := \begin{cases} \bigsqcup_{p,q \in \mathbb{N}, p+q = |\check{\mathcal{O}}|+1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = B; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{|\check{\mathcal{O}}|-1}(\mathbb{R})), & \text{if } \star = C; \\ \bigsqcup_{p,q \in \mathbb{N}, p+q = |\check{\mathcal{O}}|} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = D; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{Sp}}_{|\check{\mathcal{O}}|}(\mathbb{R})), & \text{if } \star = \widetilde{C}; \\ \bigsqcup_{p,q \in \mathbb{N}, 2p+2q = |\check{\mathcal{O}}|-1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q)), & \text{if } \star = C^*; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}^*(|\check{\mathcal{O}}|)), & \text{if } \star = D^*. \end{cases}$$

The main result of [15], on counting of special unipotent representations, is as follows.

Theorem 1.5. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then*

$$\#(\mathrm{Unip}_\star(\check{\mathcal{O}})) = \begin{cases} 2\#(\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})), & \textit{if } \star \in \{B, D\} \textit{ and } (\star, \check{\mathcal{O}}) \neq (D, \varnothing); \\ \#(\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})), & \textit{otherwise}. \end{cases}$$

1.5. **Descending painted bipartitions and constructing representations by induction.** In the sequel, we will use $\varnothing$ to denote the empty set (in the usual way), as well as the empty Young diagram or the painted Young diagram whose underlying Young diagram is empty.

Let $\star$ and $\check{\mathcal{O}}$ be as before so that $\check{\mathcal{O}}$ has $\star$-good parity. Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or } \ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or } \ D^*.$$

We call $\star'$ the Howe dual of $\star$. Define the naive dual descent of $\check{\mathcal{O}}$ to be

$$\check{\nabla}_{\mathrm{naive}}(\check{\mathcal{O}}) := \text{the Young diagram obtained from } \check{\mathcal{O}} \text{ by removing the first row.}$$

By convention, $\check{\nabla}_{\mathrm{naive}}(\varnothing) = \varnothing$. The dual descent of $\check{\mathcal{O}}$ is defined to be

$$\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}}) := \check{\nabla}_\star(\check{\mathcal{O}}) := \begin{cases} \square, & \text{if } \star \in \{D, D^*\} \text{ and } \check{\mathcal{O}} = \varnothing; \\ \check{\nabla}_{\mathrm{naive}}(\check{\mathcal{O}}), & \text{otherwise,} \end{cases}$$

where $\square$ denotes the Young diagram that has total size 1.

Note that $\check{\mathcal{O}}'$ has $\star'$-good parity, and

$$\mathrm{PP}_{\star'}(\check{\mathcal{O}}') = \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \mathrm{PP}_\star(\check{\mathcal{O}})\}.$$

Define the dual descent of a subset $\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})$ to be the subset

$$\wp' := \check{\nabla}(\wp) := \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \wp\} \subset \mathrm{PP}_{\star'}(\check{\mathcal{O}}'). \tag{1.6}$$

In Section 2, we will define the descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

Let $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$. We define its descent be the element

$$\boldsymbol{\tau}' := (\tau', \wp') := \nabla(\boldsymbol{\tau}) := (\nabla(\tau), \check{\nabla}(\wp)) \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

Let $(W_{\tau,\tau'}, \langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\tau'})$ be a real symplectic space of dimension $\dim\tau \cdot \dim\tau'$. As usual, there are continuous homomorphisms $G_\tau \to \mathrm{Sp}(W_{\tau,\tau'})$ and $G_{\tau'} \to \mathrm{Sp}(W_{\tau,\tau'})$ whose images form a reductive dual pair in $\mathrm{Sp}(W_{\tau,\tau'})$. We form the semidirect product

$$J_{\tau,\tau'} := (G_\tau \times G_{\tau'}) \ltimes \mathrm{H}(W_{\tau,\tau'}),$$

where

$$\mathrm{H}(W_{\tau,\tau'}) := W_{\tau,\tau'} \times \mathbb{R}$$

is the Heisenberg group with group multiplication

$$(w, t)(w', t') := (w + w', t + t' + \langle w, w'\rangle_{\tau,\tau'}), \quad w, w' \in W_{\tau,\tau'},\ t, t' \in \mathbb{R}.$$

Let $\omega_{\tau,\tau'}$ be a suitably normalized smooth oscillator representation of $J_{\tau,\tau'}$ such that every $t \in \mathbb{R} \subset J_{\tau,\tau'}$ acts on it through the scalar multiplication by $e^{\sqrt{-1}\,t}$. See Section 4.1 for details. For any Casselman-Wallach representation π' of $G_{\tau'}$, write

$$\check{\Theta}^{\tau}_{\tau'}(\pi') := (\omega_{\tau,\tau'} \widehat{\otimes} \pi')_{G_{\tau'}} \qquad \text{(the Hausdorff coinvariant space)},$$

where $\widehat{\otimes}$ indicates the complete projective tensor product. This representation is clearly Fréchet, smooth, and of moderate growth, and so a Casselman-Wallach representation by the fundamental result of Howe [43].

In what follows we will construct a representation $\pi_{\boldsymbol{\tau}}$ of G_τ by the method of theta lifting. For the initial case when $\check{\mathcal{O}}$ is the empty Young diagram, define

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \text{the determinant character,} & \text{if } \alpha_\tau = B^- \text{ so that } G_\tau = \mathrm{O}(0,1); \\ \text{the one dimensional genuine representation,} & \text{if } \star = \widetilde{C} \text{ so that } G_\tau = \widetilde{\mathrm{Sp}}_0(\mathbb{R}); \\ \text{the one dimensional trivial representation,} & \text{otherwise.} \end{cases}$$

When $\check{\mathcal{O}} \neq \varnothing$, we define the representation $\pi_{\boldsymbol{\tau}}$ of G_τ by induction on the number of nonempty rows of $\check{\mathcal{O}}$:

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'}) \otimes (1^{+,-}_{p_\tau,q_\tau})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'} \otimes \det{}^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'}), & \text{if } \star = C^* \text{ or } D^*. \end{cases} \tag{1.7}$$

Here $1^{+,-}_{p_\tau,q_\tau}$ denotes the character of $\mathrm{O}(p_\tau, q_\tau)$ whose restriction to $\mathrm{O}(p_\tau) \times \mathrm{O}(q_\tau)$ equals $1 \otimes \det$ (1 stands for the trivial character), and $\varepsilon_\wp$ denote the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\wp = 1 \Leftrightarrow (1,2) \in \wp.$$

We are now ready to state our first main theorem.

Theorem 1.6. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For every $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, the representation $\pi_{\boldsymbol{\tau}}$ of G_τ in (1.7) is irreducible and attached to $\check{\mathcal{O}}$.
(b) If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ (\boldsymbol{\tau}, \epsilon) & \mapsto & \pi_{\boldsymbol{\tau}} \otimes \det{}^{\epsilon} \end{array}$$

is bijective.
(c) In all other cases, the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}) & \rightarrow & \mathrm{Unip}_{\star}(\check{\mathcal{O}}), \\ \tau & \mapsto & \pi_{\tau} \end{array}$$

is bijective.

By the above theorem, we have explicitly constructed all special unipotent representations in $\mathrm{Unip}_{\star}(\check{\mathcal{O}})$, when $\check{\mathcal{O}}$ has $\star$-good parity. The method of matrix coefficient integrals (Section 4) will imply the following unitarity result.

Corollary 1.7. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then all special unipotent representations in $\mathrm{Unip}_{\star}(\check{\mathcal{O}})$ are unitarizable.*

The reduction of [15] allows us to classify all special unipotent presentations attached to a general $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ from those which have $\star$-good parity, by using irreducible unitary parabolic inductions. We thus conclude the following generalization of Corollary 1.7.

Theorem 1.8. *All special unipotent representations of the classical groups in* (1.1) *are unitarizable.*

1.6. **Computing associated cycles of the constructed representations.** Let $G, G_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}, \check{\mathfrak{g}}$ and $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ be as in Section 1.2. As in Section 1.4, we assume that G has type $\star$, and $\check{\mathcal{O}}$ has $\star$-good parity. Fix a maximal compact subgroup K of G. We equipping $\mathfrak{g}$ with the trace form. Then we have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the complexified Lie algebra of K. By taking the dual spaces, we also have a decomposition

$$\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*.$$

Write $K_{\mathbb{C}}$ for the complexification of the compact group K. It is a complex algebraic group with an obvious algebraic action on $\mathfrak{p}^*$. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by using the trace form, and denote by $\mathrm{Nil}(\mathfrak{g}) = \mathrm{Nil}(\mathfrak{g}^*)$ the set of $G_{\mathbb{C}}$-orbits of nilpotent matrices in $\mathfrak{g} = \mathfrak{g}^*$.

For any $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$, write $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}^*)$ for the Barbarsch-Vogan dual of $\check{\mathcal{O}}$ so that its Zariski closure in $\mathfrak{g}^*$ equals the associated variety of the ideal $I_{\check{\mathcal{O}}} \subset \mathrm{U}(\mathfrak{g})$. See [14, 17] and Section 3.4. The algebraic variety $\mathcal{O} \cap \mathfrak{p}^*$ is a finite union of $K_{\mathbb{C}}$-orbits. Given any such orbit $\mathscr{O}$, write $\mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O})$ for the Grothendieck group of the category of $K_{\mathbb{C}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O}).$$

We say that a Casselman-Wallach representation of G is $\mathcal{O}$-bounded if the associated variety of its annihilator ideal is contained in the Zariski closure of $\mathcal{O}$. Note that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are $\mathcal{O}$-bounded. Write $\mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}}$ for the Grothendieck group of the category of all such representations. From [86, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{AC}_{\mathcal{O}} \colon \mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}} \longrightarrow \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}).$$

We call $\mathrm{AC}_{\mathcal{O}}(\pi)$ the associated cycle of π, where π is an $\mathcal{O}$-bounded Casselman-Wallach representation of G. This is a fundamental invariant attached to π.

Following Vogan [86, Section 8], we make the following definition.

Definition 1.9. *Let $\mathscr{O}$ be a $K_{\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}^*$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $K_{\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- $\mathcal{E}_{\mathbf{e}}$ *is isomorphic to a multiple of* $(\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}})^{\frac{1}{2}}$ *as a representation of* $\mathfrak{k}_{\mathbf{e}}$*;*
- $\mathcal{E}$ *is genuine if* $\star = \widetilde{C}$*.*

Here $\mathbf{e} \in \mathscr{O}$*,* $\mathcal{E}_{\mathbf{e}}$ *is the fibre of* $\mathcal{E}$ *at* $\mathbf{e}$*,* $\mathfrak{k}_{\mathbf{e}}$ *denotes the Lie algebra of the stabilizer of* $\mathbf{e}$ *in* $K_{\mathbb{C}}$*, and* $(\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}})^{\frac{1}{2}}$ *is a one-dimensional representation of* $\mathfrak{k}_{\mathbf{e}}$ *whose tensor square is the top degree wedge product* $\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}}$*.*

Note that in the situation of the classical groups we consider in this article, all admissible orbit data are line bundles. Denote by $\mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$. Put

$$\mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O}) \subset \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}). \tag{1.8}$$

Recall that a nilpotent orbit in $\check{\mathfrak{g}}$ is said to be distinguished if it is has empty intersection with every proper Levi subalgebra of $\check{\mathfrak{g}}$. Combinatorially, this is equivalent to saying that no pair of rows of the Young diagram have equal nonzero length. Note that all distinguished nilpotent orbits in $\check{\mathfrak{g}}$ has $\star$-good parity.

Definition 1.10. *(a) The orbit* $\check{\mathcal{O}}$ *(which has* $\star$*-good parity) is said to be quasi-distinguished if there is no* $\star$*-pair that is balanced in* $\check{\mathcal{O}}$*.*
(b) If $\star \in \{B, D, D^*\}$*, then* $\check{\mathcal{O}}$ *is said to be weakly-distinguished if there is no positive even integer* i *such that* $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = \mathbf{r}_{i+2}(\check{\mathcal{O}}) = \mathbf{r}_{i+3}(\check{\mathcal{O}}) > 0$*.*
(c) If $\star \in \{C, \widetilde{C}, C^*\}$ *so that its Howe dual* $\star' \in \{B, D, D^*\}$*, then* $\check{\mathcal{O}}$ *is said to be weakly-distinguished if its dual descent* $\check{\mathcal{O}}'$ *is weakly-distinguished.*

We will compute the associated cycle of π_τ for every painted bipartition τ associated to $\check{\mathcal{O}}$ which has $\star$-good parity. The computation will be a key ingredient in the proof of Theorem 1.6. Our second main theorem is the following result concerning associated cycles of the special unipotent representations.

Theorem 1.11. *Suppose that* $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ *has* $\star$*-good parity.*
(a) For every $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$*, the associated cycle* $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$ *is a nonzero sum of admissible orbit data over pairwise distinct* $K_{\mathbb{C}}$*-orbits in* $\mathcal{O} \cap \mathfrak{p}^*$*.*
(b) If $\check{\mathcal{O}}$ *is weakly-distinguished, then the map*

$$\mathrm{AC}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$$

is injective.
(c) If $\check{\mathcal{O}}$ *is quasi-distinguished, then the map* $\mathrm{AC}_{\mathcal{O}}$ *induces a bijection*

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}).$$

Remark. Suppose that $\star \in \{C^*, D^*\}$ so that G is quaternionic. Then there is precisely one admissible orbit datum over $\mathscr{O}$ for each $K_{\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}^*$. Thus

$$\mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}) = K_{\mathbb{C}} \backslash (\mathcal{O} \cap \mathfrak{p}^*).$$

If $\check{\mathcal{O}}$ is not quasi-distinghuished, then $\mathcal{O} \cap \mathfrak{p}^*$ is empty (see [26, Theorems 9.3.4 and 9.3.5]), and hence $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is also empty.

2. Descents of painted bipartitions: combinatorics

In this section, we define the descent of a painted bipartition, as alluded to in Section 1.5. As before, let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity. Recall the dual descent $\check{\mathcal{O}}'$ of $\mathcal{O}$ which has $\star'$-good parity, where $\star'$ is the Howe dual of $\star$.

For a Young diagram $\imath$, its naive descent, which is denoted by $\nabla_{\mathrm{naive}}(\imath)$, is defined to be the Young diagram obtained from $\imath$ by removing the first column. By convention, $\nabla_{\mathrm{naive}}(\varnothing) = \varnothing$.

2.1. **Naive descent of a painted bipartition.** In this subsection, let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ be a painted bipartition such that $\star_\tau = \star$. Put

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ does not occur in the first column of } (\imath, \mathcal{P}); \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ occurs in the first column of } (\imath, \mathcal{P}); \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{2.1}$$

Lemma 2.1. *If $\star \in \{ B, C, C^* \}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\imath, \nabla_{\mathrm{naive}}(\jmath))$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,
$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}(i,j), & \text{if } \mathcal{P}(i,j) \notin \{\bullet, s\}; \end{cases}$$
- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,
$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j+1), & \text{if } \mathcal{Q}(i,j+1) \notin \{\bullet, s\}. \end{cases}$$

Proof. First assume that the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$. Then the image of $\mathcal{P}$ is in fact contained in $\{\bullet\}$, and $(\imath, \jmath)$ is right interlaced in the sense that

$$\mathbf{c}_1(\jmath) \geqslant \mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\jmath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\jmath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots .$$

Hence $(\imath', \jmath') := (\imath, \nabla(\jmath))$ is left interlaced in the sense that

$$\mathbf{c}_1(\imath') \geqslant \mathbf{c}_1(\jmath') \geqslant \mathbf{c}_2(\imath') \geqslant \mathbf{c}_2(\jmath') \geqslant \mathbf{c}_3(\imath') \geqslant \mathbf{c}_3(\jmath') \geqslant \cdots .$$

Then it is clear that there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ such that images of $\mathcal{P}'$ and $\mathcal{Q}'$ are both contained in $\{\bullet, s\}$. This proves the lemma in the special case when the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$.

The proof of the lemma in the general case is easily reduced to this special case. □

Lemma 2.2. *If $\star \in \{ \widetilde{C}, D, D^* \}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\nabla_{\mathrm{naive}}(\imath), \jmath)$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,
$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{P}(i,j+1), & \text{if } \mathcal{P}(i,j+1) \notin \{\bullet, s\}; \end{cases}$$
- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,
$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j), & \text{if } \mathcal{Q}(i,j) \notin \{\bullet, s\}. \end{cases}$$

Proof. The proof is similar to that of Lemma 2.1. □

Definition 2.3. *In the notation of Lemma 2.1 and 2.2, we call τ' the naive descent of τ, to be denoted by $\nabla_{\mathrm{naive}}(\tau)$.*

Example. If

$$\tau = \begin{ytableau} \bullet & \bullet & \bullet & c \\ \bullet & s & c \\ s \\ c \end{ytableau} \times \begin{ytableau} \bullet & \bullet & \bullet \\ \bullet & r & d \\ d & d \end{ytableau} \times \widetilde{C},$$

then

$$\nabla_{\mathrm{naive}}(\tau) = \begin{ytableau} \bullet & \bullet & c \\ \bullet & c \end{ytableau} \times \begin{ytableau} \bullet & \bullet & s \\ \bullet & r & d \\ d & d \end{ytableau} \times B^-.$$

2.2. **Descents of painted bipartitions.** Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ and write

$$\tau'_{\mathrm{naive}} = (\imath', \mathcal{P}'_{\mathrm{naive}}) \times (\jmath', \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'$$

for the naive descent of τ. This is clearly an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$.

The following two lemmas are easily verified and we omit the proofs. We will give an example for each case.

Lemma 2.4. *Suppose that*

$$\begin{cases} \alpha = B^+; \\ \mathbf{r}_2(\check{\mathcal{O}}) > 0; \\ \mathcal{Q}(\mathbf{c}_1(\jmath), 1) \in \{ r, d \}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} s, & \text{if } (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{ytableau} \bullet & c \\ c \end{ytableau} \times \begin{ytableau} \bullet & r \\ r & d \end{ytableau} \times B^+,$$

then

$$\tau'_{\mathrm{naive}} = \begin{ytableau} s & c \\ c \end{ytableau} \times \begin{ytableau} r \\ d \end{ytableau} \times \widetilde{C} \qquad \text{and} \qquad \tau' = \begin{ytableau} s & c \\ s \end{ytableau} \times \begin{ytableau} r \\ d \end{ytableau} \times \widetilde{C}.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $4, 4, 4, 2$.

Lemma 2.5. *Suppose that*

$$\begin{cases} \alpha = D; \\ \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0; \\ \big(\mathcal{P}(\mathbf{c}_2(\imath), 1), \mathcal{P}(\mathbf{c}_2(\imath), 2)\big) = (r, c); \\ \mathcal{P}(\mathbf{c}_1(\imath), 1) \in \{ r, d \}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} r, & \text{if } (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & s \\ \hline \bullet & s \\ \hline r & c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet \\ \cline{1-1} \bullet \\ \cline{1-1} \end{array} \times D,$$

then

$$\tau'_{\mathrm{naive}} = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet \\ \cline{1-1} \bullet \\ \cline{1-1} \end{array} \times C, \qquad \text{and} \qquad \tau' = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline r \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet \\ \cline{1-1} \bullet \\ \cline{1-1} \end{array} \times C.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $7, 7, 7, 3$.

Definition 2.6. *We define the descent of τ to be*

$$\nabla(\tau) := \begin{cases} \tau', & \text{if either of the condition of Lemma 2.4 or 2.5 holds;} \\ \nabla_{\mathrm{naive}}(\tau), & \text{otherwise,} \end{cases}$$

which is an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$. Here τ' is as in Lemmas 2.4 and 2.5.

In conclusion, we have by now a well-defined descent map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

The following injectivity result will be proved in Section **??**.

Proposition 2.7. *If $\star \in \{ B, D, C^* \}$, then the map*

$$\begin{array}{rcl} \mathrm{PBP}_{\star}(\check{\mathcal{O}}) & \to & \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array} \tag{2.2}$$

is injective. If $\star \in \{ C, \widetilde{C}, D^* \}$, then the map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \quad \to \quad \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \tag{2.3}$$

is injective.

3. From painted bipartitions to associated cycles

3.1. **Classical spaces.** Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ as before. Put

$$(\epsilon, \dot{\epsilon}) := (\epsilon_\star, \dot{\epsilon}_\star) := \begin{cases} (1, 1), & \text{if } \star \in \{B, D\}; \\ (-1, -1), & \text{if } \star \in \{C, \widetilde{C}\}; \\ (-1, 1), & \text{if } \star = C^*; \\ (1, -1), & \text{if } \star = D^*. \end{cases} \tag{3.1}$$

A classical signature is defined to be a triple $\mathsf{s} = (\star, p, q) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ such that

$$\begin{cases} p + q \text{ is odd }, & \text{if } \star = B; \\ p + q \text{ is even }, & \text{if } \star = D; \\ p = q, & \text{if } \star \in \{C, \widetilde{C}, D^*\}; \\ \text{both } p \text{ and } q \text{ are even}, & \text{if } \star = C^*. \end{cases}$$

Suppose that $\mathsf{s} = (\star, p, q)$ is a classical signature in the rest of this section. Set

$$|\mathsf{s}| := p + q,$$

and we call $\star$ the type of s.

We omit the proof of the following lemma (*cf.* [68, Section 1.3]).

Lemma 3.1. *There is quadruple $(V, \langle\,,\,\rangle, J, L)$ satisfying the following conditions:*

- *V is a complex vector space of dimension $|\mathsf{s}|$;*
- *$\langle\,,\,\rangle$ is an ϵ-symmetric non-degenerate bilinear form on V;*
- *$J : V \to V$ is a conjugate linear automorphism of V such that $J^2 = \epsilon \cdot \dot{\epsilon}$;*
- *$L : V \to V$ is a linear automorphism of V such that $L^2 = \dot{\epsilon}$;*
- *$\langle Ju, Jv\rangle = \overline{\langle u, v\rangle}$, for all $u, v \in V$;*
- *$\langle Lu, Lv\rangle = \langle u, v\rangle$, for all $u, v \in V$;*
- *$LJ = JL$;*
- *the Hermitian form $(u, v) \mapsto \langle Lu, Jv\rangle$ on V is positive definite;*
- *if $\dot{\epsilon} = 1$, then $\dim\{v \in V \mid Lv = v\} = p$ and $\dim\{v \in V \mid Lv = -v\} = q$.*

Moreover, such a quadruple is unique in the following sense: if $(V', \langle\,,\,\rangle', J', L')$ is another quadruple satisfying the analogous conditions, then there is a linear isomorphism $\phi : V \to V'$ that respectively transforms $\langle\,,\,\rangle$, J and L to $\langle\,,\,\rangle'$, J' and L'.

In the notation of Lemma 3.1, we call $(V, \langle\,,\,\rangle, J, L)$ a classical space of signature s, and denote it by $(V_{\mathsf{s}}, \langle\,,\,\rangle_{\mathsf{s}}, J_{\mathsf{s}}, L_{\mathsf{s}})$.

Denote by $G_{\mathsf{s},\mathbb{C}}$ the isometry group of $(V_{\mathsf{s}}, \langle\,,\,\rangle_{\mathsf{s}})$, which is an complex orthogonal group if $\epsilon = 1$ and a complex symplectic group if $\epsilon = -1$. Respectively denote by $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ and $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ the centralizes of J_{s} and L_{s} in $G_{\mathsf{s},\mathbb{C}}$. Then $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ is a real classical group isomorphic with

$$\begin{cases} \mathrm{O}(p,q), & \text{if } \star = B \text{ or } D; \\ \mathrm{Sp}_{2p}(\mathbb{R}), & \text{if } \star \in \{C, \widetilde{C}\}; \\ \mathrm{Sp}(\frac{p}{2}, \frac{q}{2}), & \text{if } \star = C^*; \\ \mathrm{O}^*(2p), & \text{if } \star = D^*. \end{cases}$$

Put

$$G_{\mathsf{s}} := \begin{cases} \text{the metaplectic double cover of } G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{if } \star = \widetilde{C}; \\ G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{otherwise.} \end{cases}$$

Denote by K_{s} the inverse image of $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ under the natural homomorphism $G_{\mathsf{s}} \to G_{\mathsf{s},\mathbb{C}}$, which is a maximal compact subgroup of G_{s}. Write $K_{\mathsf{s},\mathbb{C}}$ for the complexification of K_{s}, which is a reductive complex linear algebraic group.

For every $\lambda \in \mathbb{C}$, write $V_{\mathsf{s},\lambda}$ for the eigenspace of L_{s} with eigenvalue λ. Write

$$\det_\lambda : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s},\lambda}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times. \tag{3.2}$$

If $\star = \widetilde{C}$, then the natural homomorphism $K_{\mathsf{s},\mathbb{C}} \to \mathrm{GL}(V_{\mathsf{s},\sqrt{-1}})$ is a double cover, and there is a unique genuine algebraic character $\det_{\sqrt{-1}}^{\frac{1}{2}} : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$ whose square equals $\det_{\sqrt{-1}}$.

Write

$$\det : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s}}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times. \tag{3.3}$$

This is the trivial character unless $\star = B$ or D.

Denote by $\mathfrak{g}_{\mathsf{s}}$ the Lie algebra of $G_{\mathsf{s},\mathbb{C}}$. Then we have a decomposition

$$\mathfrak{g}_{\mathsf{s}} = \mathfrak{k}_{\mathsf{s}} \oplus \mathfrak{p}_{\mathsf{s}},$$

where $\mathfrak{k}_{\mathsf{s}}$ is the Lie algebra of $K_{\mathsf{s},\mathbb{C}}$, and $\mathfrak{p}_{\mathsf{s}}$ is the orthogonal complement of $\mathfrak{k}_{\mathsf{s}}$ in $\mathfrak{g}_{\mathsf{s}}$ under the trace form. Using the trace form, we also identify $\mathfrak{g}_{\mathsf{s}}^*$ with $\mathfrak{g}_{\mathsf{s}}$. Denote by $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ the set of nilpotent $G_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathsf{s}}$, and by $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$ the set of nilpotent $K_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathsf{s}}$.

Given a $K_{\mathsf{s},\mathbb{C}}$-orbit $\mathscr{O}$ in $\mathfrak{p}_{\mathsf{s}}$, write $\mathcal{K}_{\mathsf{s}}(\mathscr{O})$ for the Grothendieck group of the categogy of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundles over $\mathscr{O}$. Denote by $\mathcal{K}^+_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}(\mathscr{O})$ the submonoid generated by all these equivariant algebraic vector bundles. The notion of admissible orbit datum over $\mathscr{O}$ is defined as in Definition 1.9. Write

$$\mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}^+_{\mathsf{s}}(\mathscr{O})$$

for the set of isomorphism classes of admissible orbit data over $\mathscr{O}$.

Let $\mathcal{O}$ be a $G_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s},\mathbb{C}}$. It is well-known that $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ has only finitely many $K_{\mathsf{s},\mathbb{C}}$-orbits. Put

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}_{\mathsf{s}}(\mathscr{O}),$$

and

$$\mathcal{K}^+_{\mathsf{s}}(\mathcal{O}) := \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}^+_{\mathsf{s}}(\mathscr{O}).$$

Put

$$\mathrm{AOD}_{\mathsf{s}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}^+_{\mathsf{s}}(\mathcal{O}).$$

Define a partial order $\leq$ on $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$ such that

$$\mathcal{E}_1 \leq \mathcal{E}_2 \Leftrightarrow \mathcal{E}_2 - \mathcal{E}_1 \in \mathcal{K}^+_{\mathsf{s}}(\mathcal{O}) \qquad (\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})).$$

For every algebraic character χ of $K_{\mathsf{s},\mathbb{C}}$, the twisting map

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \qquad \mathcal{E} \mapsto \mathcal{E} \otimes \chi$$

is obviously defined.

3.2. **The moment maps.** Recall that $\star'$ is the Howe dual of $\star$. Suppose that $\mathsf{s}' = (\star', p', q')$ is another classical signature. Put

$$W_{\mathsf{s},\mathsf{s}'} := \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_{\mathsf{s}'}).$$

Then we have the adjoint map

$$W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s}',\mathsf{s}}, \qquad \phi \mapsto \phi^*$$

that is specified by requiring

$$\langle \phi v, v' \rangle_{\mathsf{s}'} = \langle v, \phi^* v' \rangle_{\mathsf{s}}, \qquad \text{for all } v \in V_{\mathsf{s}},\ v' \in V_{\mathsf{s}'},\ \phi \in W_{\mathsf{s},\mathsf{s}'}. \tag{3.4}$$

Define three maps

$$\langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \times W_{\mathsf{s},\mathsf{s}'} \to \mathbb{C}, \quad (\phi_1, \phi_2) \mapsto \mathrm{tr}(\phi_1^* \phi_2),$$

$$J_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto J_{\mathsf{s}'} \circ \phi \circ J_{\mathsf{s}}^{-1};$$

and

$$L_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto \dot\epsilon L_{\mathsf{s}'} \circ \phi \circ L_{\mathsf{s}}^{-1}.$$

It is routine to check that

$$(W_{\mathsf{s},\mathsf{s}'}, \langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'}, J_{\mathsf{s},\mathsf{s}'}, L_{\mathsf{s},\mathsf{s}'})$$

is a classical space of signature $(C, \frac{|\mathsf{s}|\cdot|\mathsf{s}'|}{2}, \frac{|\mathsf{s}|\cdot|\mathsf{s}'|}{2})$.

Write

$$W_{\mathsf{s},\mathsf{s}'} = \mathcal{X}_{\mathsf{s},\mathsf{s}'} \oplus \mathcal{Y}_{\mathsf{s},\mathsf{s}'}, \tag{3.5}$$

where $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ and $\mathcal{Y}_{\mathsf{s},\mathsf{s}'}$ are the eigenspaces of $L_{\mathsf{s},\mathsf{s}'}$ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Then we have the following two well-defined algebraic maps:

$$
\begin{array}{ccccc}
\mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}:=M_{\mathsf{s}}^{\mathsf{s},\mathsf{s}'}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}:=M_{\mathsf{s}'}^{\mathsf{s},\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}, \\
\phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*.
\end{array} \tag{3.6}
$$

These two maps M_{s} and $M_{\mathsf{s}'}$ are called the moment maps. They are both $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-equivariant. Here $K_{\mathsf{s}',\mathbb{C}}$ acts trivially on $\mathfrak{p}_{\mathsf{s}}$, $K_{\mathsf{s},\mathbb{C}}$ acts trivially on $\mathfrak{p}'_{\mathsf{s}}$, and all the other actions are the obvious ones.

Put

$$W^{\circ}_{\mathsf{s},\mathsf{s}'} := \{\phi \in W_{\mathsf{s},\mathsf{s}'} \mid \text{the image of } \phi \text{ is non-degenerate with respect to } \langle\,,\,\rangle_{\mathsf{s}'}\}$$

and

$$\mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W^{\circ}_{\mathsf{s},\mathsf{s}'}.$$

Lemma 3.2 (*cf.* [68, Lemma 13], [94, Lemma 3.4])**.** *Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}. Then the set*

$$M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} \tag{3.7}$$

is a single $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-orbit. Moreover, for every element ϕ in $M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'}$, there is an exact sequence of algebraic groups:

$$1 \to (K_{\mathsf{s}',\mathbb{C}})_\phi \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi \xrightarrow{\textit{the projection to the first factor}} (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1,$$

where $\mathbf{e} := M_{\mathsf{s}}(\phi) \in \mathscr{O}$, $(K_{\mathsf{s}',\mathbb{C}})_\phi$ is the stabilizer of ϕ in $K_{\mathsf{s}',\mathbb{C}}$, $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi$ is the stabilizer of ϕ in $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$, and $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$ is the stabilizer of $\mathbf{e}$ in $K_{\mathsf{s},\mathbb{C}}$.

In the notation of Lemma 3.2, write

$$\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) := \text{the image of the set (3.7) under the moment map } M_{\mathsf{s}'},$$

which is a $K_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}'}$. This is called the descent of $\mathscr{O}$. It is an element of $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}'})$ if $\mathscr{O} \in \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$.

Similar to (3.6), we have the following two well-defined algebraic maps:

$$
\begin{array}{ccccc}
\mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}:=\tilde{M}_{\mathsf{s}}^{\mathsf{s},\mathsf{s}'}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}:=\tilde{M}_{\mathsf{s}'}^{\mathsf{s},\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\
\phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*.
\end{array} \tag{3.8}
$$

These two maps are also called the moment maps. Similar to the maps in (3.6), they are both $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-equivariant.

We record the following lemma for later use.

Lemma 3.3. *Suppose that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$. Then $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$ if and only if*

$$\delta := |\mathsf{s}'| - |\nabla_{\mathrm{naive}}(\mathcal{O})| \geqslant 0.$$

When this is the case, the Young diagram of $\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ is obtained from that of $\nabla_{\mathrm{naive}}(\mathcal{O})$ by adding δ boxes in the first column.

Proof. This is implied by [28, Theorem 3.6]. □

Now we suppose that the $G_{\mathsf{s},\mathbb{C}}$-orbit $\mathcal{O} \subset \mathfrak{g}_{\mathsf{s}}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Similar to the first assertion of Lemma 3.2, the set

$$\tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap W^{\circ}_{\mathsf{s},\mathsf{s}'} \tag{3.9}$$

is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit. Write

$$\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) := \text{the image of the set (3.9) under the moment map } \tilde{M}_{\mathsf{s}'},$$

which is a $G_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s}'}$. This is called the descent of $\mathcal{O}$. It is an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ if $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$.

3.3. **Geometric theta lift.** Let $\zeta_{\mathsf{s},\mathsf{s}'}$ denote the algebraic character on $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ such that

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s},\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{B, D, C^*\}; \\ (\det_{\sqrt{-1}})^{\frac{p'-q'}{2}}, & \text{if } \star \in \{C, D^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{p'-q'}, & \text{if } \star = \widetilde{C}, \end{cases}$$

and

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s}',\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{C, \widetilde{C}, D^*\}; \\ (\det_{\sqrt{-1}})^{\frac{q-p}{2}}, & \text{if } \star \in \{D, C^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{q-p}, & \text{if } \star = B. \end{cases}$$

Here and henceforth, when no confusion is possible, we use 1 to indicate the trivial representation of a group (we also use 1 to denote the identity element of a group). Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$ as before. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}, and write $\mathscr{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O})$. Let $\phi, \mathbf{e}$ be as in Lemma 3.2 and let $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. We have an exact sequence

$$1 \to (K_{\mathsf{s}',\mathbb{C}})_\phi \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi \to (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1$$

as in Lemma 3.2.

Let $\mathcal{E}'$ be a $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}'$. Its fibre $\mathcal{E}'_{\mathbf{e}}$ at $\mathbf{e}'$ is an algebraic representation of the stabilizer group $(K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}$. We also view it as a representation of the group $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi$ by the pull-back through the homomorphism

$$(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi \xrightarrow{\text{the projection to the second factor}} (K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}.$$

Then $\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'}$ is a representation of $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi$, and the coinvariant space

$$(\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{(K_{\mathsf{s}',\mathbb{C}})_\phi}$$

is an algebraic representation of $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$. Write $\mathcal{E} := \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$ for the $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$ whose fibre at $\mathbf{e}$ equals this coinvariant space representation. In this way, we get an exact functor $\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}$ from the category of $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}'$ to the category of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$. This exact functor induces a homomorphism of the Grothendieck groups:

$$\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'} : \mathcal{K}_{\mathsf{s}'}(\mathscr{O}') \to \mathcal{K}_{\mathsf{s}}(\mathscr{O}).$$

The above homomorphism is independent of the choice of ϕ.

Recall that $\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})$, and finally we define the geometric theta lift to be the homomorphism

$$\check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'} : \mathcal{K}_{\mathsf{s}'}(\mathcal{O}') \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

such that

$$\check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathcal{E}') = \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}},\ \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) = \mathscr{O}'} \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}'),$$

for all $K_{\mathsf{s}',\mathbb{C}}$-orbit $\mathscr{O}'$ in $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$, and all $\mathcal{E}' \in \mathcal{K}_{\mathsf{s}'}(\mathscr{O}')$.

3.4. **Associated cycles of painted bipartitions.** Let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.

Definition 3.4. *The Barbasch-Vogan dual* $\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})$ *of* $\check{\mathcal{O}}$ *is the Young diagram satisfying*

$$\begin{aligned}&\left(\mathbf{c}_i(\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})), \mathbf{c}_{i+1}(\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}}))\right)\\ =&\begin{cases}(\mathbf{r}_i(\check{\mathcal{O}}), \mathbf{r}_{i+1}(\check{\mathcal{O}})), & \textit{if } (i,i+1) \textit{ is a } \star\textit{-pair that is vacant or balanced in } \check{\mathcal{O}};\\ (\mathbf{r}_i(\check{\mathcal{O}})-1, 0), & \textit{if } (i,i+1) \textit{ is a } \star\textit{-pair that is tailed in } \check{\mathcal{O}};\\ (\mathbf{r}_i(\check{\mathcal{O}})-1, \mathbf{r}_{i+1}(\check{\mathcal{O}})+1), & \textit{if } (i,i+1) \textit{ is a } \star\textit{-pair that is primitive in } \check{\mathcal{O}},\end{cases}\end{aligned}$$

and

$$\mathbf{c}_1(\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})) = \begin{cases}0, & \textit{if } \star \in \{D, D^*\} \textit{ and } \check{\mathcal{O}} = \varnothing;\\ \mathbf{r}_1(\check{\mathcal{O}})+1, & \textit{if either } \star = B, \textit{ or } \star \in \{D, D^*\} \textit{ and } \check{\mathcal{O}} \neq \varnothing.\end{cases}$$

Assume that

$$\left|\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})\right| = |\mathsf{s}|\,.$$

Then $\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})$ is identified with an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$. Recall the ideal $I_{\check{\mathcal{O}}}$ of $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})$ from Section 1.2.

Lemma 3.5. *The associated variety of* $I_{\check{\mathcal{O}}}$ *equals* $\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})$*.*

Proof. This is implied by [17, Corollary A3] and [14, Theorem B]. □

As before, let $\check{\mathcal{O}}'$ be the dual descent of $\check{\mathcal{O}}$, which has $\star'$-good parity. Assume that

$$\left|\mathrm{d}_{\mathrm{BV}}^{\star'}(\check{\mathcal{O}}')\right| = |\mathsf{s}'|\,.$$

Then $\mathrm{d}_{\mathrm{BV}}^{\star'}(\check{\mathcal{O}}')$ is identified with an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$.

Lemma 3.6. *The orbit* $\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})$ *is contained in the image of the moment map* $\tilde{M}_{\mathsf{s}}$*, and*

$$\nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}})) = \mathrm{d}_{\mathrm{BV}}^{\star'}(\check{\mathcal{O}}').$$

Proof. This is elementary to check by using Lemmas 3.3. □

In view of Lemma 3.6, we suppose that

$$\mathcal{O} = \mathrm{d}_{\mathrm{BV}}^{\star}(\check{\mathcal{O}}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}}) \quad \text{and} \quad \mathcal{O}' = \mathrm{d}_{\mathrm{BV}}^{\star'}(\check{\mathcal{O}}') \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'}).$$

Put

$$\mathrm{PBP}_{\star}^{\mathrm{ext}}(\check{\mathcal{O}}, \mathsf{s}) := \{(\tau, \wp) \in \mathrm{PBP}_{\star}^{\mathrm{ext}}(\check{\mathcal{O}}) \mid (p_\tau, q_\tau) = (p,q)\}.$$

Let $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}_{\star}^{\mathrm{ext}}(\check{\mathcal{O}})$. In what follows we will define the associated cycle $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ of $\boldsymbol{\tau}$.

For the initial case when $\check{\mathcal{O}}$ is the empty Young diagram so that $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ is a singleton, we define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ to be the element that corresponds to the following representation of $K_{\mathsf{s},\mathbb{C}}$:

$$\begin{cases}\text{the determinant character}, & \text{if } \alpha_\tau = B^- \text{ so that } K_{\mathsf{s},\mathbb{C}} = \mathrm{O}(0,1);\\ \text{the one dimensional genuine representation}, & \text{if } \star = \widetilde{C} \text{ so that } K_{\mathsf{s},\mathbb{C}} = \{\pm 1\};\\ \text{the one dimensional trivial representation}, & \text{otherwise.}\end{cases}$$

Write $\boldsymbol{\tau}' \in \mathrm{PBP}_{\star'}^{\mathrm{ext}}(\check{\mathcal{O}}')$ for the descent of $\boldsymbol{\tau}$, and assume that $\boldsymbol{\tau}' \in \mathrm{PBP}_{\star'}^{\mathrm{ext}}(\check{\mathcal{O}}', \mathsf{s}')$. When $\check{\mathcal{O}} \neq \varnothing$, similar to the definition of $\pi_{\boldsymbol{\tau}}$ in the introductory section, we inductively define

$\mathrm{AC}(\uptau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ by

$$\mathrm{AC}(\uptau) := \begin{cases} \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\uptau')) \otimes (\det_{-1})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\uptau') \otimes \det^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\uptau')), & \text{if } \star = C^* \text{ or } D^*. \end{cases}$$

We have the following three analogous results of Theorem 1.11.

Proposition 3.7. *For every $\uptau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, the associated cycle $\mathrm{AC}(\uptau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ is a nonzero sum of admissible orbit data over pairwise distinct $K_{\mathsf{s},\mathbb{C}}$-orbits in $\mathcal{O} \cap \mathfrak{p}$.*

Proof. This follows from Lemma 6.23 and Lemma 10.3. □

Proposition 3.8. *Suppose that $\check{\mathcal{O}}$ is weakly-distinguished. If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \quad (\uptau, \epsilon) \mapsto \mathrm{AC}(\uptau) \otimes \det^{\epsilon}$$

is injective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

is injective.

Proof. When $\star \in \{ B, D \}$, this follows from Lemma 10.6 (b). When $\star = C^*$, this follows from Lemma 10.4. When $\star \in \{ C, \widetilde{C}, D^* \}$, this follows from Lemma 10.5 (b). □

Proposition 3.9. *Suppose that $\check{\mathcal{O}}$ is quasi-distinguished. If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{AOD}_{\mathsf{s}}(\mathcal{O}), \quad (\uptau, \epsilon) \mapsto \mathrm{AC}(\uptau) \otimes \det^{\epsilon}$$

is well-defined and bijective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \to \mathrm{AOD}_{\mathsf{s}}(\mathcal{O})$$

is well-defined and bijective.

Proof. When $\star \in \{ B, D \}$, this follows from Lemma 10.6 (c). When $\star = C^*$, this follows from Lemma 10.4. When $\star \in \{ C, \widetilde{C}, D^* \}$, this follows from Lemma 10.5 (c). □

The following two propositions will also be important for us.

Proposition 3.10. *Suppose that $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$. Let $\uptau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$ and $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$. If*

$$\mathrm{AC}(\uptau_1) \otimes \det^{\epsilon_1} = \mathrm{AC}(\uptau_2) \otimes \det^{\epsilon_2},$$

then

$$\epsilon_1 = \epsilon_2 \qquad \text{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Proof. This follows from Lemma 10.6 (a). □

Proposition 3.11. *Suppose that $\star \in \{C, \widetilde{C}, D^*\}$. Let $\uptau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$, and write $\uptau'_i = (\tau'_i, \wp'_i)$ for its descent $(i = 1, 2)$. If*

$$\mathrm{AC}(\uptau_1) = \mathrm{AC}(\uptau_2),$$

then

$$(p_{\tau'_1}, q_{\tau'_1}) = (p_{\tau'_2}, q_{\tau'_2}) \qquad \text{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

Proof. This follows from Lemma 10.5 (a). □

3.5. **Distinguishing the constructed representations.** Let $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$ as before. Recall that π_τ is the Casselman-Wallach representation of G_s, defined in the introductory section. We will prove the following theorem in Section 7.4.

Theorem 3.12. *The representation π_τ is irreducible, unitarizable and attached to $\check{\mathcal{O}}$. Moreover,*

$$\mathrm{AC}_\mathcal{O}(\pi_\tau) = \mathrm{AC}(\tau) \in \mathcal{K}_\mathsf{s}(\mathcal{O}).$$

Recall from the introductory section, the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G_\mathsf{s})$ of isomorphism classes of irreducible Casselman-Wallach representations of G_s that are attached to $\check{\mathcal{O}}$.

Theorem 3.13. *If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map*

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_\mathsf{s}), \quad (\tau, \epsilon) \mapsto \pi_\tau \otimes \det{}^\epsilon \tag{3.10}$$

is bijective. In all other cases, the map

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_\mathsf{s}), \quad \tau \mapsto \pi_\tau \tag{3.11}$$

is bijective.

Proof. In view of Theorem 1.5, we only need to show the injectivity of the two maps in the statement of the theorem. We prove by induction on the number of nonempty rows of $\check{\mathcal{O}}$. The theorem is trivially true in the case when $\check{\mathcal{O}} = \varnothing$. So we assume that $\check{\mathcal{O}} \neq \varnothing$ and the theorem has been proved for the dual descent $\check{\mathcal{O}}'$. Suppose that $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$, $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$. Write $\tau_i' = (\tau_i', \wp_i')$ for the descent of τ_i.

First assume that $\star \in \{B, D\}$, and

$$\pi_{\tau_1} \otimes \det{}^{\epsilon_1} \cong \pi_{\tau_2} \otimes \det{}^{\epsilon_2}.$$

Theorem 3.12 implies that

$$\mathrm{AC}(\tau_1) \otimes \det{}^{\epsilon_1} = \mathrm{AC}(\tau_2) \otimes \det{}^{\epsilon_2}.$$

By Proposition 3.10, we know that

$$\epsilon_1 = \epsilon_2 \qquad \text{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Then the definition of π_{τ_1} and π_{τ_2} implies that

$$\check{\Theta}^{\tau_1}_{\tau_1'}(\pi_{\tau_1'}) \cong \check{\Theta}^{\tau_2}_{\tau_2'}(\pi_{\tau_2'}).$$

Consequently,

$$\pi_{\tau_1'} \cong \pi_{\tau_2'}$$

by the injectivity property of the theta correspondence. Hence $\tau_1' = \tau_2'$ by the induction hypothesis, and Proposition 2.7 finally implies $\tau_1 = \tau_2$. This proves that the map (3.10) is injective.

A slightly simplified argument shows that the map (3.11) is injective when $\star = C^*$.

Now assume that $\star \in \{C, \widetilde{C}, D^*\}$. Suppose that

$$\pi_{\tau_1} \cong \pi_{\tau_2}.$$

Theorem 3.12 implies that

$$\mathrm{AC}(\tau_1) = \mathrm{AC}(\tau_2).$$

By Proposition 3.11, we know that

$$(p_{\tau_1'}, q_{\tau_1'}) = (p_{\tau_2'}, q_{\tau_2'}) \qquad \text{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

Then the definition of π_{τ_1} and π_{τ_2} implies that

$$\check{\Theta}^{\tau_1}_{\tau_1'}(\pi_{\tau_1'} \otimes \det{}^{\varepsilon_{\wp_1}}) \cong \check{\Theta}^{\tau_2}_{\tau_2'}(\pi_{\tau_2'} \otimes \det{}^{\varepsilon_{\wp_1}}).$$

The injectivity property of the theta correspondence then implies that

$$\pi_{\tau_1'} \otimes \det^{\varepsilon_{\wp_1}} \cong \pi_{\tau_2'} \otimes \det^{\varepsilon_{\wp_2}},$$

which further implies that $\pi_{\tau_1'} \cong \pi_{\tau_2'}$. Hence $\tau_1' = \tau_2'$ by the induction hypothesis, and Proposition 2.7 finally implies $\tau_1 = \tau_2$. This proves that the map (3.11) is injective.

A slightly simplified argument shows that the map (3.11) is injective when $\star = D^*$. This finishes the proof of the theorem.

□

Finally, our first main theorem (Theorem 1.6) follows from Theorems 3.12 and 3.13. Our second main theorem (Theorem 1.11) follows from Propositions 3.7 and 3.8, and Theorems 3.12 and 3.13.

4. Theta lifts via matrix coefficient integrals

In this section, we study theta lift via matrix coefficient integrals. Let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$.

4.1. The oscillator representation.

We use the notation of Section 3.2. Write $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \subset W_{\mathsf{s},\mathsf{s}'}$ for the fixed point set of $J_{\mathsf{s},\mathsf{s}'}$. It is a real symplectic space under the restriction of the form $\langle\, , \,\rangle_{\mathsf{s},\mathsf{s}'}$. Let $H_{\mathsf{s},\mathsf{s}'} := W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \times \mathbb{R}$ denote the Heisenberg group attached to $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$, with group multiplication

$$(\phi, t) \cdot (\phi', t') := (\phi + \phi', t + t' + \langle \phi, \phi' \rangle_{\mathsf{s},\mathsf{s}'}), \qquad \phi, \phi' \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}, \quad t, t' \in \mathbb{R}.$$

Denote by $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$ the complexified Lie algebra of $H_{\mathsf{s},\mathsf{s}'}$. Then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ (as in (3.5)) is an abelian Lie subalgebra of $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$.

The following is the smooth version of the Stone-von Neumann Theorem.

Lemma 4.1 (*cf.* [24, Theorem 5.1] and [2, Section 4]). *Up to isomorphism, there exists a unique irreducible smooth Fréchet representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ of moderate growth with central character*

$$\mathbb{R} \to \mathbb{C}^\times, \; t \mapsto e^{\sqrt{-1}t}.$$

Moreover, the space

$$\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}} := \{ v \in \omega_{\mathsf{s},\mathsf{s}'} \mid x \cdot v = 0 \quad \textit{for all } x \in \mathcal{X}_{\mathsf{s},\mathsf{s}'} \}$$

is one-dimensional.

The group $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ acts on $H_{\mathsf{s},\mathsf{s}'}$ as group automorphisms via the following natural action of $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ on $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$:

$$(g, g') \cdot \phi := g' \circ \phi \circ g^{-1}, \qquad (g, g') \in G_{\mathsf{s}} \times G_{\mathsf{s}'}, \; \phi \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}.$$

From this action, we form the semidirect product $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Recall the character $\zeta_{\mathsf{s},\mathsf{s}'}$ from Section 3.3.

Lemma 4.2 (*cf.* [2, Proposition 7.5]). *The representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ in Lemma 4.1 uniquely extends to a smooth representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ such that $K_{\mathsf{s}} \times K_{\mathsf{s}'}$ acts on $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$ through the scalar multiplication by $\zeta_{\mathsf{s},\mathsf{s}'}$.*

Let $\omega_{\mathsf{s},\mathsf{s}'}$ denote the representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ as in Lemma 4.2, henceforth called the smooth oscillator representation or simply the oscillator representation. As is well-known, this representation is unitarizable [91]. Fix an invariant continuous Hermitian inner product $\langle\, , \,\rangle$ on it, which is unique up to a positive scalar multiplication. Denote by $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ the completion of $\omega_{\mathsf{s},\mathsf{s}'}$ with respect to this inner product, which is a unitary representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Let $\omega_{\mathsf{s},\mathsf{s}'}^{\vee}$ denote the contragredient of the oscillator representation $\omega_{\mathsf{s},\mathsf{s}'}$. It is the smooth Fréchet representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ of moderate growth specified by the following conditions:

- there is given a $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$-invariant, non-degenerate, continuous bilinear form
$$\langle\, , \,\rangle : \omega_{\mathsf{s},\mathsf{s}'} \times \omega_{\mathsf{s},\mathsf{s}'}^{\vee} \to \mathbb{C};$$
- $\omega_{\mathsf{s},\mathsf{s}'}^{\vee}$ is irreducible as a representation of $H_{\mathsf{s},\mathsf{s}'}$.

Since $\omega_{\mathsf{s},\mathsf{s}'}$ is contained in the unitary representation $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$, $\omega_{\mathsf{s},\mathsf{s}'}^{\vee}$ is identified with the complex conjugation of $\omega_{\mathsf{s},\mathsf{s}'}$.

Put
$$\mathsf{s}'^{-} := (\star', q', p'),$$
Fix a linear isomorphism
$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^{-}}, \tag{4.1}$$
such that $(-\langle\, , \,\rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, -L_{\mathsf{s}'})$ corresponds to $(\langle\, , \,\rangle_{\mathsf{s}'^{-}}, J_{\mathsf{s}'^{-}}, L_{\mathsf{s}'^{-}})$ under this isomorphism. This induces an isomorphism
$$G_{\mathsf{s}'} \to G_{\mathsf{s}'^{-}}, \qquad g' \mapsto g'^{-}. \tag{4.2}$$
Then we have a group isomorphism
$$\begin{array}{rcl} (G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'} & \to & (G_{\mathsf{s}'^{-}} \times G_{\mathsf{s}}) \ltimes H_{\mathsf{s}'^{-},\mathsf{s}}, \\ (g, g', (\phi, t)) & \mapsto & (g'^{-}, g, (\phi^* \circ \iota_{\mathsf{s}'}^{-1}, t)). \end{array} \tag{4.3}$$
It is easy to see that the irreducible representation $\omega_{\mathsf{s},\mathsf{s}'}$ corresponds to the irreducible representation $\omega_{\mathsf{s}'^{-},\mathsf{s}}$ under this isomorphism.

For every Casselman-Wallach representation π' of $G_{\mathsf{s}'}$, put
$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi') := (\omega_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \pi')_{G_{\mathsf{s}'}} \qquad \text{(the Hausdorff coinvariant space)}.$$
This is a Casselman-Wallach representation of G_{s}.

4.2. **Growth of Casselman-Wallach representations.** Write $\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}}$ for the centralizer of J_{s} in $\mathfrak{p}_{\mathsf{s}}$, which is a real form of $\mathfrak{p}_{\mathsf{s}}$. The Cartan decomposition asserts that
$$G_{\mathsf{s}} = K_{\mathsf{s}} \cdot \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}}).$$
Denote by Ψ_{s} the function of G_{s} satisfying the following conditions:

- it is both left and right K_{s}-invariant;
- for all $g \in \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}})$,
$$\Psi_{\mathsf{s}}(g) = \prod_a \left(\frac{1+a}{2}\right)^{-\frac{1}{2}},$$
where a runs over all eigenvalues of $g : V_{\mathsf{s}} \to V_{\mathsf{s}}$, counted with multiplicities.

Note that all the eigenvalues of $g \in \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}})$ are positive real numbers, and $0 < \Psi_{\mathsf{s}}(g) \leqslant 1$ for all $g \in G_{\mathsf{s}}$.

Set
$$\nu_{\mathsf{s}} := \begin{cases} |\mathsf{s}|, & \text{if } \star \in \{C, \widetilde{C}\}; \\ |\mathsf{s}| - 1, & \text{if } \star = C^*; \\ |\mathsf{s}| - 2, & \text{if } \star \in \{B, D\}; \\ |\mathsf{s}| - 3, & \text{if } \star = D^*. \end{cases}$$
Denote by Ξ_{s} the bi-K_{s}-invariant Harish-Chandra's Ξ function on G_{s}.

Lemma 4.3. *There exists a real number $C_{\mathsf{s}} > 0$ such that*

$$\Psi_{\mathsf{s}}^{\nu_{\mathsf{s}}}(g) \leqslant C_{\mathsf{s}} \cdot \Xi_{\mathsf{s}}(g) \quad \text{for all } g \in G_{\mathsf{s}}.$$

Proof. This is implied by the well-known estimate of Harish-Chandra's Ξ function ([89, Theorem 4.5.3]). □

Lemma 4.4. *Let f be a bi-K_{s}-invariant positive function on G_{s} such that*

$$\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}} \to \mathbb{R}, \qquad x \mapsto f(\exp(x))$$

is a polynomial function. Then for every real number $\nu > 0$, the function $f \cdot \Psi_{\mathsf{s}}^{\nu} \cdot \Xi_{\mathsf{s}}^2$ is integrable with respect to a Haar measure on G_{s}.

Proof. This follows from the integral formula for G_{s} under the Cartan decomposition (see [89, Lemma 2.4.2]), as well as the estimate of Harish-Chandra's Ξ function ([89, Theorem 4.5.3]). □

For every Casselman-Wallach representation π of G_{s}, write $\pi^\vee$ for its contragredient representation, which is a Casselman-Wallach representation of G_{s} equipped with a G_{s}-invariant, non-degenerate, continuous bilinear form

$$\langle\,,\,\rangle : \pi \times \pi^\vee \to \mathbb{C}.$$

Definition 4.5. *Let $\nu \in \mathbb{R}$. A positive function Ψ on G_{s} is ν-bounded if there is a function f as in Lemma 4.4 such that*

$$\Psi(g) \leqslant f(g) \cdot \Psi_{\mathsf{s}}^{\nu}(g) \cdot \Xi_{\mathsf{s}}(g) \qquad \text{for all } g \in G_{\mathsf{s}}.$$

A Casselman-Wallach representation π of G_{s} is said to be ν-bounded if there exist a ν-bounded positive function Ψ on G_{s}, and continuous seminorms $|\cdot|_\pi$ and $|\cdot|_{\pi^\vee}$ on π and $\pi^\vee$ respectively such that

$$|\langle g \cdot u, v\rangle| \leqslant \Psi(g) \cdot |u|_\pi \cdot |v|_{\pi^\vee}$$

for all $u \in \pi$, $v \in \pi^\vee$, and $g \in G_{\mathsf{s}}$.

Let X_{s} be a maximal J_{s}-stable totally isotropic subspace of V_{s}, which is unique up to the action of K_{s}. Put $Y_{\mathsf{s}} := L_{\mathsf{s}}(X_{\mathsf{s}})$. Then $X_{\mathsf{s}} \cap Y_{\mathsf{s}} = \{0\}$. Write

$$P_{\mathsf{s}} = R_{\mathsf{s}} \ltimes N_{\mathsf{s}}$$

for the parabolic subgroup of G_{s} stabilizing X_{s}, where R_{s} is the Levi subgroup stabilizing both X_{s} and Y_{s}, and N_{s} is the unipotent radical. For every character $\chi : R_{\mathsf{s}} \to \mathbb{C}^\times$, view it as a character of P_{s} that is trivial on N_{s}. Write

$$I(\chi) := \mathrm{Ind}_{P_{\mathsf{s}}}^{G_{\mathsf{s}}} \chi \qquad \text{(normalized smooth induction)},$$

which is a Casselman-Wallach representation of G_{s} under the right translations. Note that the representations $I(\chi)$ and $I(\chi^{-1})$ are contragredients of each other with the G_{s}-invariant pairing

$$\langle\,,\,\rangle : I(\chi) \times I(\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\mathsf{s}}} f(g) \cdot f'(g)\, \mathrm{d}g.$$

Let ν_χ be a real number such that $|\chi|$ equals the composition of

$$R_{\mathsf{s}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(X_{\mathsf{s}}) \xrightarrow{|\det|^{\nu_\chi}} \mathbb{C}^\times. \tag{4.4}$$

Definition 4.6. *A classical signature s is split if $V_{\mathsf{s}} = X_{\mathsf{s}} \oplus Y_{\mathsf{s}}$.*

When s is split, P_{s} is called a Siegel parabolic subgroup of G_{s}.

Lemma 4.7. *Suppose that s is split and let $\chi : P_{\mathsf{s}} \to \mathbb{C}^\times$ be a character. Then there is a positive function Ψ on G_{s} with the following properties:*

- Ψ *is* $(1-2|\nu_\chi|-\frac{|\mathsf{s}|}{2})$*-bounded if* $\star\in\{B,C,D,\widetilde{C}\}$*, and* $(2-2|\nu_\chi|-\frac{|\mathsf{s}|}{2})$*-bounded if* $\star\in\{C^*,D^*\}$*;*
- *for all* $f\in I(\chi)$*,* $f'\in I(\chi^{-1})$ *and* $g\in G_{\mathsf{s}}$*,*

$$|\langle g.f,f'\rangle|\leqslant\Psi(g)\cdot|f|_{K_{\mathsf{s}}}|_\infty\cdot|f'|_{K_{\mathsf{s}}}|_\infty\qquad(|\ |_\infty \text{ stands for the suppernorm}).\tag{4.5}$$

Proof. Let f_0 denote the element in $I(|\chi|)$ such that $(f_0)|_{K_{\mathsf{s}}}=1$, and likewise let f_0' denote the element in $I(|\chi^{-1}|)$ such that $(f_0')|_{K_{\mathsf{s}}}=1$. Put

$$\Psi(g):=\langle g\cdot f_0,f_0'\rangle,\qquad g\in G_{\mathsf{s}}.$$

Then it is easy to see that (4.5) holds. Note that Ψ is an elementary spherical function, and the lemma then follows by using the well-known estimate of the elementary spherical functions ([89, Lemma 3.6.7]). ☐

4.3. **Matrix coefficient integrals against the oscillator representation.** We begin with the following lemma.

Lemma 4.8. *There exist continuous seminorms* $|\cdot|_{\mathsf{s},\mathsf{s}'}$ *and* $|\cdot|^\vee_{\mathsf{s},\mathsf{s}'}$ *on* $\omega_{\mathsf{s},\mathsf{s}'}$ *and* $\omega^\vee_{\mathsf{s},\mathsf{s}'}$ *respectively such that*

$$|\langle(g,g')\cdot u,v\rangle|\leqslant\Psi_{\mathsf{s}}^{|\mathsf{s}'|}(g)\cdot\Psi_{\mathsf{s}'}^{|\mathsf{s}|}(g')\cdot|u|_{\mathsf{s},\mathsf{s}'}\cdot|v|^\vee_{\mathsf{s},\mathsf{s}'}$$

for all $u\in\omega_{\mathsf{s},\mathsf{s}'}$*,* $v\in\omega^\vee_{\mathsf{s},\mathsf{s}'}$*, and* $(g,g')\in G_{\mathsf{s}}\times G_{\mathsf{s}'}$*.*

Proof. This follows from the proof of [53, Theorem 3.2]. ☐

Definition 4.9. *A Casselman-Wallach representation of* $G_{\mathsf{s}'}$ *is convergent for* $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ *if it is* ν*-bounded for some* $\nu>\nu_{\mathsf{s}'}-|\mathsf{s}|$*.*

Example. Suppose that $\star'\neq\widetilde{C}$. Then the trivial representation of $G_{\mathsf{s}'}$ is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ if $|\mathsf{s}|>2\nu_{\mathsf{s}'}$.

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$. Consider the integrals

$$\begin{array}{rcl}(\pi'\times\omega_{\mathsf{s},\mathsf{s}'})\times(\pi'^\vee\times\omega^\vee_{\mathsf{s},\mathsf{s}'}) & \to & \mathbb{C},\\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}}\langle g\cdot u,u'\rangle\cdot\langle g\cdot v,v'\rangle\,\mathrm{d}g.\end{array}\tag{4.6}$$

Unless otherwise specified, all the measures on Lie groups occurring in this article are Haar measures.

Lemma 4.10. *The integrals in* (4.6) *are absolutely convergent and the map* (4.6) *is continuous and multi-linear.*

Proof. This is a direct consequence of Lemmas 4.3, 4.4 and 4.8. ☐

By Lemma 4.10, the integrals in (4.6) yield a continuous bilinear form

$$(\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'})\times(\pi'^\vee\widehat{\otimes}\omega^\vee_{\mathsf{s},\mathsf{s}'})\to\mathbb{C}.\tag{4.7}$$

Put

$$\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi'):=\frac{\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}}{\text{the left kernel of (4.7)}}.\tag{4.8}$$

Proposition 4.11. *The representation* $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ *of* G_{s} *is a quotient of* $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$*, and is* $(|\mathsf{s}'|-\nu_{\mathsf{s}})$*-bounded.*

Proof. Note that the bilinear form (4.7) is $(G_{\mathsf{s}'} \times G_{\mathsf{s}'})$-invariant, as well as G_{s}-invariant. Thus $\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')$ is a quotient of $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')$, and is therefore a Casselman-Wallach representation. Its contragedient representation is identified with

$$(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi'))^{\vee} := \frac{\pi'^{\vee}\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}^{\vee}}{\text{the right kernel of (4.7)}}.$$

Lemmas 4.4 and 4.8 implies that there are continuous seminorms $|\cdot|_{\pi',\mathsf{s},\mathsf{s}'}$ and $|\cdot|_{\pi'^{\vee},\mathsf{s},\mathsf{s}'}$ on $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$ and $\pi'^{\vee}\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}^{\vee}$ respectively such that

$$|\langle g.u, v\rangle| \leqslant \Psi_{\mathsf{s}}^{|\mathsf{s}'|}(g) \cdot |u|_{\pi',\mathsf{s},\mathsf{s}'} \cdot |v|_{\pi'^{\vee},\mathsf{s},\mathsf{s}'}$$

for all $u \in \pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$, $v \in \pi'^{\vee}\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}^{\vee}$, and $g \in G_{\mathsf{s}}$. The proposition then easily follows in view of Lemma 4.3. □

4.4. **Unitarity.** For the notion of weakly containment of unitary representations, see [27] for example.

Lemma 4.12. *Suppose that $|\mathsf{s}| \geqslant \nu_{\mathsf{s}'}$. Then as a unitary representation of $G_{\mathsf{s}'}$, $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ is weakly contained in the regular representation.*

Proof. This has been known to experts (see [53, Theorem 3.2]). Lemmas 4.3, 4.4 and 4.8 implies that for a dense subspace of $\hat{\omega}_{\mathsf{s},\mathsf{s}'}|_{G_{\mathsf{s}'}}$, the diagonal matrix coefficients are almost square integrable. Thus the lemma follows form [27, Theorem 1]. □

However, if $\star' \in \{B, C^*, D^*\}$, $|\mathsf{s}| \neq \nu_{\mathsf{s}'}$ for the parity reason. For this reason, we introduce

$$\nu_{\mathsf{s}'}^{\circ} := \begin{cases} \nu_{\mathsf{s}'}, & \text{if } \star' \in \{C, D, \widetilde{C}\}; \\ \nu_{\mathsf{s}'} + 1, & \text{if } \star' \in \{B, C^*, D^*\}. \end{cases}$$

The following definition is a slight variation of definition 4.9.

Definition 4.13. *A Casselman-Wallach representation of $G_{\mathsf{s}'}$ is overconvergent for $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}$ if it is ν-bounded for some $\nu > \nu_{\mathsf{s}'}^{\circ} - |\mathsf{s}|$.*

We will prove the following unitarity result in the rest of this subsection.

Theorem 4.14. *Assume that $|\mathsf{s}| \geqslant \nu_{\mathsf{s}'}^{\circ}$. Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is overconvergent for $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}$. If π' is unitarizable, then so is $\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')$.*

Recall the following positivity result of matrix coefficient integrals, which is a special case of [35, Theorem A. 5].

Lemma 4.15. *Let G be a real reductive group with a maximal compact subgroup K. Let π_1 and π_2 be two unitary representations of G such that π_2 is weakly contained in the regular representation. Let $u_1, u_2, \cdots, u_r$ ($r \in \mathbb{N}$) be vectors in π_1 such that for all $i, j = 1, 2, \cdots, r$, the integral*

$$\int_G \langle g \cdot u_i, u_j\rangle\, \Xi_G(g)\, \mathrm{d}g$$

is absolutely convergent, where Ξ_G is the bi-K-invariant Harish-Chandra's Ξ function on G. Let $v_1, v_2, \cdots, v_r$ be K-finite vectors in π_2. Put

$$u := \sum_{i=1}^{r} u_i \otimes v_i \in \pi_1 \otimes \pi_2.$$

Then the integral

$$\int_G \langle g \cdot u, u\rangle\, \mathrm{d}g$$

absolutely converges to a nonnegative real number.

Now we come to the proof of Theorem 4.14.

Proof of Theorem 4.14. Fix an invariant continuous Hermitian inner product on π', and write $\hat{\pi}'$ for the completion of π' with respect to this Hermitian inner product. The space $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$ is equipped with the inner product $\langle\,,\,\rangle$ that is the tensor product of the ones on π' and $\omega_{\mathsf{s},\mathsf{s}'}$. It suffices to show that

$$\int_{G_{\mathsf{s}'}} \langle g\cdot u, u\rangle \,\mathrm{d}g \geqslant 0$$

for all u in a dense subspace of $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$.

If $\nu^\circ_{\mathsf{s}'} < 0$, then $\star \in \{D, D^*\}$ and $|\mathsf{s}'| = 0$. The theorem is trivial in this case. Thus we assume that $\nu^\circ_{\mathsf{s}'} \geqslant 0$. We also assume that $\star = B$. The proof in the other cases is similar and is omitted.

Note that $\nu^\circ_{\mathsf{s}'} = \nu_{\mathsf{s}'} = |\mathsf{s}'|$ is even. Let $\mathsf{s}_1 = (B, p_1, q_1)$ and $\mathsf{s}_2 = (D, p_2, q_2)$ be two classical signatures such that

$$(p_1, q_1) + (p_2, q_2) = (p, q) \quad \text{and} \quad |\mathsf{s}_2| = \nu^\circ_{\mathsf{s}'}.$$

View $\hat{\omega}_{\mathsf{s}_2,\mathsf{s}'}$ as a unitary representation of the symplectic group $G_{(C,p',q')}$. Define π_2 to be its pull-back through the covering homomorphism $G_{\mathsf{s}'} \to G_{(C,p',q')}$. By Lemma 4.12, the representation π_2 of $G_{\mathsf{s}'}$ is weakly contained in the regular representation.

Put

$$\pi_1 := \hat{\pi}'\widehat{\otimes}_{\mathrm{h}}(\hat{\omega}_{\mathsf{s}_1,\mathsf{s}'}|_{G_{\mathsf{s}'}}) \qquad (\widehat{\otimes}_{\mathrm{h}} \text{ indicates the Hilbert space tensor product}).$$

Lemmas 4.4 and 4.8 imply that the integral

$$\int_{G_{\mathsf{s}'}} \langle g\cdot u, v\rangle \cdot \Xi_{G_{\mathsf{s}'}}(g)\,\mathrm{d}g$$

is absolutely convergent for all $u, v \in \pi'\otimes\omega_{\mathsf{s}_1,\mathsf{s}'}$. The theorem then follows by Lemma 4.15. ☐

5. Double theta lifts and degenerate principal series

In this section, we relate double theta lifts with degenerate principal series representations.

Let

$$\dot{\mathsf{s}} = (\dot{\star}, \dot{p}, \dot{q}), \qquad \mathsf{s}' = (\star', p', q') \qquad \text{and} \qquad \mathsf{s}'' = (\star'', p'', q'')$$

be classical signatures such that

- $(q', p') + (p'', q'') = (\dot{p}, \dot{q})$;
- if $\dot{\star} \in \{B, D\}$, then $\star', \star'' \in \{B, D\}$;
- if $\dot{\star} \notin \{B, D\}$, then $\star' = \star'' = \dot{\star}$.

As in Section 4.1, put $\mathsf{s}'^- := (\star', q', p')$. We view $V_{\mathsf{s}'^-}$ and $V_{\mathsf{s}''}$ as subspaces of $V_{\dot{\mathsf{s}}}$ such that $(\langle\,,\,\rangle_{\dot{\mathsf{s}}}, J_{\dot{\mathsf{s}}}, L_{\dot{\mathsf{s}}})$ extends both $(\langle\,,\,\rangle_{\mathsf{s}'^-}, J_{\mathsf{s}'^-}, L_{\mathsf{s}'^-})$ and $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$, and $V_{\mathsf{s}'^-}$ and $V_{\mathsf{s}''}$ are perpendicular to each other under the form $\langle\,,\,\rangle_{\dot{\mathsf{s}}}$. Then we have an orthogonal decomposition

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''},$$

and both $G_{\mathsf{s}'^-}$ and $G_{\mathsf{s}''}$ are identified with subgroups of $G_{\dot{\mathsf{s}}}$.

Fix a linear isomorphism

$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^-} \tag{5.1}$$

as in (4.1), which induces an isomorphism

$$G_{\mathsf{s}'} \to G_{\mathsf{s}'^-}, \quad g \mapsto g^-. \tag{5.2}$$

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$. Write π'^{-} for the representation of $G_{\mathsf{s}'^{-}}$ that corresponds to π' under the isomorphism (5.2).

5.1. **Matrix coefficient integrals and double theta lifts.** We begin with the following lemma.

Lemma 5.1. *The function $(\Xi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}}$ on $G_{\mathsf{s}'}$ is $|\mathsf{s}''|$-bounded.*

Proof. This follows from the estimate of Harish-Chandra's Ξ function ([89, Theorem 4.5.3]). $\square$

Definition 5.2. *The Casselman-Wallach representation π'^{-} of $G_{\mathsf{s}'^{-}}$ is convergent for a Casselman-Wallach representation $\dot{\pi}$ of $G_{\dot{\mathsf{s}}}$ if there are real numbers ν' and $\dot{\nu}$ such that π' is ν'-bounded, $\dot{\pi}$ is $\dot{\nu}$-bounded, and*

$$\nu' + \dot{\nu} > -|\mathsf{s}''|.$$

Let $\dot{\pi}$ be a Casselman-Wallach representation of $G_{\dot{\mathsf{s}}}$ such that π'^{-} is convergent for $\dot{\pi}$. Consider the integrals

$$\begin{array}{rcl} (\pi'^{-}\times\dot{\pi})\times((\pi'^{-})^{\vee}\times\dot{\pi}^{\vee}) & \rightarrow & \mathbb{C}, \\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}}\langle g^{-}\cdot u,u'\rangle\cdot\langle g^{-}\cdot v,v'\rangle\,\mathrm{d}g. \end{array} \tag{5.3}$$

Lemma 5.3. *The integrals in* (5.3) *are absolutely convergent and the map* (5.3) *is continuous and multi-linear.*

Proof. Note that $(\Psi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}} = \Psi_{\mathsf{s}'}$. Thus the lemma follows from Lemmas 4.4 and 5.1. $\square$

By Lemma 5.3, the integrals in (5.3) yield a continuous bilinear form

$$\langle\,,\,\rangle : (\pi'^{-}\widehat{\otimes}\dot{\pi})\times((\pi'^{-})^{\vee}\widehat{\otimes}\dot{\pi}^{\vee})\rightarrow\mathbb{C}. \tag{5.4}$$

Put

$$\pi'^{-} * \dot{\pi} := \frac{\pi'^{-}\widehat{\otimes}\dot{\pi}}{\text{the left kernel of (5.4)}}. \tag{5.5}$$

This is a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

For every classical signature $\mathsf{s} = (\star, p, q)$, put

$$[\mathsf{s}] := \begin{cases} (C,p,q), & \text{if } \star = \tilde{C}; \\ \mathsf{s}, & \text{if } \star \neq \tilde{C}. \end{cases}$$

Proposition 5.4. *Suppose that $\mathsf{s} = (\star, p, q)$ is a classical signature such that both $\star'$ and $\star''$ equals the Howe dual of $\star$, and $2\nu_{\mathsf{s}} < |\dot{\mathsf{s}}|$. Assume that π' is ν'-bounded for some $\nu' > \nu_{\mathsf{s}'} - |\mathsf{s}|$. Then*

- *π' is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$;*
- *$\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ is convergent for $\check{\Theta}^{\mathsf{s}''}_{\mathsf{s}}$;*
- *the trivial representation 1 of $G_{[\mathsf{s}]}$ is convergent for $\check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}$;*
- *π'^{-} is convergent for $\bar{\Theta}^{\dot{\mathsf{s}}}_{[\mathsf{s}]}(1)$;*
- *as representations of $G_{\mathsf{s}''}$,*

$$\bar{\Theta}^{\mathsf{s}''}_{\mathsf{s}}(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')) \cong \pi'^{-} * \bar{\Theta}^{\dot{\mathsf{s}}}_{[\mathsf{s}]}(1). \tag{5.6}$$

Proof. The first four claims in the proposition are obvious. We only need to prove the last one. Note that the integrals in

$$\begin{array}{rcl} (\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}})\times((\pi')^{\vee}\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}^{\vee}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}}^{\vee}) & \rightarrow & \mathbb{C}, \\ (u,v) & \mapsto & \int_{G_{\mathsf{s}'}\times G_{\mathsf{s}}}\langle(g',g)\cdot u,v\rangle\,\mathrm{d}g'\,\mathrm{d}g \end{array} \tag{5.7}$$

are absolutely convergent and defines a continuous bilinear map. Also note that

$$\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}_2,\mathsf{s}} \cong \omega_{\dot{\mathsf{s}},\mathsf{s}}.$$

In view of Fubini's theorem, the lemma follows as both sides of (5.6) are isomorphic to the quotient of $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}}$ by the left kernel of the pairing (5.7).

□

5.2. **Matrix coefficient integrals against degenerate principal series.** We are particularly interested in the case when $\dot{\pi}$ is a degenerate principal series representation. Suppose that $\dot{\mathsf{s}}$ is split, and

$$p' \leqslant p'' \qquad \text{and} \qquad q' \leqslant q''.$$

Then $\star' = \star''$, and there is a split classical signature $\mathsf{s}_0 = (\star_0, p_0, q_0)$ such that

- $\star_0 = \dot{\star}$;
- $(p', q') + (p_0, q_0) = (p'', q'')$.

We view $V_{\mathsf{s}'}$ and V_{s_0} as subspaces of $V_{\mathsf{s}''}$ such that $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$ extends both $(\langle\,,\,\rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, L_{\mathsf{s}'})$ and $(\langle\,,\,\rangle_{\mathsf{s}_0}, J_{\mathsf{s}_0}, L_{\mathsf{s}_0})$, and $V_{\mathsf{s}'}$ and V_{s_0} are perpendicular to each other under the form $\langle\,,\,\rangle_{\mathsf{s}_2}$.

Put

$$V_{\mathsf{s}'}^{\triangle} := \{\iota_{\mathsf{s}'}(v) + v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\} \quad \text{and} \quad V_{\mathsf{s}'}^{\nabla} := \{\iota_{\mathsf{s}'}(v) - v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\}.$$

As before, we have that

$$V_{\mathsf{s}_0} = X_{\mathsf{s}_0} \oplus Y_{\mathsf{s}_0},$$

where X_{s_0} is a maximal J_{s_0}-stable totally isotropic subspace of V_{s_0}, and $Y_{\mathsf{s}_0} := L_{\mathsf{s}_0}(X_{\mathsf{s}_0})$. Suppose that

$$X_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0} \quad \text{and} \quad Y_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}.$$

In summary, we have decompositions

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}'} \oplus V_{\mathsf{s}_0} = (V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0}) \oplus (V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}). \tag{5.8}$$

As before, $X_{\dot{\mathsf{s}}}$ and X_{s_0} yield the Siegel parabolic subgroups

$$P_{\dot{\mathsf{s}}} = R_{\dot{\mathsf{s}}} \ltimes N_{\dot{\mathsf{s}}} \subset G_{\dot{\mathsf{s}}} \qquad \text{and} \qquad P_{\mathsf{s}_0} = R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}_0} \subset G_{\mathsf{s}_0}.$$

Write

$$P_{\mathsf{s}'',\mathsf{s}_0} = R_{\mathsf{s}'',\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0} \tag{5.9}$$

for the parabolic subgroup of $G_{\mathsf{s}''}$ stabilizing X_{s_0}, where $R_{\mathsf{s}'',\mathsf{s}_0}$ is the Levi subgroup stabilizing both X_{s_0} and Y_{s_0}, and $N_{\mathsf{s}'',\mathsf{s}_0}$ is the unipotent radical. We have an obvious homomorphism

$$G_{\mathsf{s}'} \times R_{\mathsf{s}_0} \to R_{\mathsf{s}'',\mathsf{s}_0}, \tag{5.10}$$

which is a two fold covering map when $\dot{\star} = \widetilde{C}$, and an isomorphism in the other cases. For every $g \in G_{\mathsf{s}'}$, write $g^{\triangle} := g^- g$, which is an element of $R_{\dot{\mathsf{s}}}$.

Let $\dot{\chi} : R_{\dot{\mathsf{s}}} \to \mathbb{C}^\times$ be a character. It yields a degenerate principal series representation

$$I(\dot{\chi}) := \mathrm{Ind}_{P_{\dot{\mathsf{s}}}}^{G_{\dot{\mathsf{s}}}} \dot{\chi}$$

of $G_{\dot{\mathsf{s}}}$. Write

$$\chi_0 := \dot{\chi}|_{R_{\mathsf{s}_0}}, \tag{5.11}$$

and define a character

$$\chi' : G_{\mathsf{s}'} \to \mathbb{C}^\times, \quad g \mapsto \dot{\chi}(g^{\triangle}) \qquad \text{(this is a quadratic character).} \tag{5.12}$$

Recall the representations π' of $G_{\mathsf{s}'}$ and π'^- of $G_{\mathsf{s}'^-}$. If $\dot{\mathsf{s}} = \widetilde{C}$, we assume that both π' and $\dot{\chi}$ are genuine. Then $(\pi' \otimes \chi') \otimes \chi_0$ descends to a Casselman-Wallach representation of $R_{\mathsf{s}'',\mathsf{s}_0}$. View it as a representation of $P_{\mathsf{s}'',\mathsf{s}_0}$ via the trivial action of $N_{\mathsf{s}'',\mathsf{s}_0}$, and form the representation $\mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi_1 \otimes \chi') \otimes \chi_0)$

Let $\nu_{\dot\chi} \in \mathbb{R}$ be as in (4.4). The rest of this subsection is devoted to a proof of the following theorem.

Theorem 5.5. *Assume that the Casselman-Wallach representation π' of $G_{\mathsf{s}'}$ is ν'-bounded for some*

$$\nu' > \begin{cases} 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 1, & \text{if } \dot\star \in \{B, C, D, \widetilde{C}\}; \\ 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 2, & \text{if } \dot\star \in \{C^*, D^*\}. \end{cases}$$

Then π'^- is convergent for $I_{\dot{\mathsf{s}}}(\dot\chi)$, and

$$\pi'^- * I(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Let the notation and assumptions be as in Theorem 5.5. Recall that the representations $I(\dot\chi)$ and $I(\dot\chi^{-1})$ are contragredients of each other with the $G_{\dot{\mathsf{s}}}$-invariant pairing

$$\langle\, , \rangle : I(\dot\chi) \times I(\dot\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\dot{\mathsf{s}}}} f(g) \cdot f'(g)\, \mathrm{d}g,$$

The first assertion of Theorem 5.5 is a direct consequence of Lemma 4.7. As in (5.4), we have a continuous bilinear form

$$\begin{array}{rcl} (\pi'^- \widehat{\otimes} I(\dot\chi)) \times ((\pi'^-)^\vee \widehat{\otimes} I(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^- \cdot u, v\rangle\, \mathrm{d}g \end{array} \tag{5.13}$$

so that

$$\pi'^- * I(\dot\chi) := \frac{\pi'^- \widehat{\otimes} I(\dot\chi)}{\text{the left kernel of (5.13)}}. \tag{5.14}$$

Note that

$$G_{\dot{\mathsf{s}}}^\circ := P_{\dot{\mathsf{s}}} \cdot G_{\mathsf{s}''}$$

is open and dense in $G_{\dot{\mathsf{s}}}$, and its complement has measure zero in $G_{\dot{\mathsf{s}}}$. Moreover,

$$P_{\dot{\mathsf{s}}} \backslash G_{\dot{\mathsf{s}}}^\circ = (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}) \backslash G_{\mathsf{s}''}. \tag{5.15}$$

Form the normalized Schwartz induction

$$I^\circ(\dot\chi) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0.$$

The reader is referred to [23, Section 6.2] for the general notion of Schwartz inductions (in a slightly different unnormalized setting). Similarly, put

$$I^\circ(\dot\chi^{-1}) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0^{-1}.$$

Note that

(5.16) the modulus character of $P_{\dot{\mathsf{s}}}$ restricts to the modulus character of $R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$.

In view of (5.15) and (5.16), by extension by zero, $I^\circ(\dot\chi)$ is viewed as a closed subspace of $I(\dot\chi)$, and $I^\circ(\dot\chi^{-1})$ is viewed as a closed subspace of $I(\dot\chi^{-1})$. These two closed subspaces are $(G_{\mathsf{s}'} \times G_{\mathsf{s}''})$-stable.

Similar to (5.13), we have a continuous bilinear form

$$\begin{array}{rcl} (\pi'^- \widehat{\otimes} I^\circ(\dot\chi)) \times ((\pi'^-)^\vee \widehat{\otimes} I^\circ(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^- \cdot u, v\rangle\, \mathrm{d}g. \end{array} \tag{5.17}$$

Put

$$\pi'^- * I^\circ(\dot\chi) := \frac{\pi'^- \widehat{\otimes} I^\circ(\dot\chi)}{\text{the left kernel of (5.17)}}, \tag{5.18}$$

which is still a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

Lemma 5.6. *As representations of $G_{\mathsf{s}''}$,*

$$\pi'^{-} * I^\circ(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Proof. As Fréchet spaces, π'^{-} is obviously identified with π'. Note that the natural map $G_{\mathsf{s}'} \to (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0})\backslash G_{\mathsf{s}''}$ is proper and hence the following integrals are absolutely convergent and yield a $G_{\mathsf{s}''}$-equivariant continuous linear map:

$$\begin{array}{rcl} \xi : \pi'^{-}\widehat{\otimes} I^\circ(\dot\chi) & \to & \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0), \\ v \otimes f & \mapsto & \left(h \mapsto \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f(g^{-1}h)(g \cdot v)\,\mathrm{d}g\right). \end{array}$$

Moreover, this map is surjective (*cf.* [23, Section 6.2]). It is thus open by the open mapping theorem. Similarly, we have a open surjective $G_{\mathsf{s}''}$-equivariant continuous linear map

$$\begin{array}{rcl} \xi' : (\pi'^{-})^\vee \widehat{\otimes} I^\circ(\dot\chi^{-1}) & \to & \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi'^\vee \otimes \chi') \otimes \chi_0^{-1}), \\ v' \otimes f' & \mapsto & \left(h \mapsto \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f'(g^{-1}h)(g \cdot v')\,\mathrm{d}g\right). \end{array}$$

For all $v \otimes f \in \pi'^{-}\widehat{\otimes} I^\circ(\dot\chi)$ and $v' \otimes f' \in (\pi'^{-})^\vee \widehat{\otimes} I^\circ(\dot\chi^{-1})$, we have that

$$\begin{aligned} & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot (v \otimes f), v' \otimes f' \rangle\,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot v, v' \rangle \cdot \langle g^{-} \cdot f, f' \rangle\,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot v, v' \rangle \cdot \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} (g^{-} \cdot f)(hx) \cdot f'(hx)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot v, v' \rangle \cdot \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f(g^{-1}hx) \cdot f'(hx)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}g \\ = & \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} \int_{G_{\mathsf{s}'}} \langle (h^{-}(g^{-})^{-1}) \cdot v, v' \rangle \cdot \chi'(hg^{-1}) \cdot f(gx) \cdot f'(hx)\,\mathrm{d}g\,\mathrm{d}h\,\mathrm{d}x \\ = & \int_{K_{\mathsf{s}''}} \langle \xi(v \otimes f)(x), \xi'(v' \otimes f')(x) \rangle\,\mathrm{d}x \\ = & \langle \xi(v \otimes f), \xi'(v' \otimes f') \rangle. \end{aligned}$$

This implies the lemma. □

Lemma 5.7. *Let $u \in \pi'^{-}\widehat{\otimes} I(\dot\chi)$. Assume that*

$$\int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, v \rangle\,\mathrm{d}g = 0 \tag{5.19}$$

for all $v \in (\pi'^{-})^\vee \widehat{\otimes} I^\circ(\dot\chi^{-1})$. Then (5.19) *also holds for all $v \in (\pi'^{-})^\vee \widehat{\otimes} I(\dot\chi^{-1})$.*

Proof. Take a sequence $(\eta_1, \eta_2, \eta_3, \cdots)$ of real valued smooth functions on $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$ such that

- for all $i \geqslant 1$, the support of η_i is contained in $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^\circ$;
- for all $i \geqslant 1$ and $x \in P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$, $0 \leqslant \eta_i(x) \leqslant \eta_{i+1}(x) \leqslant 1$;
- $\bigcup_{i=1}^\infty \eta_i^{-1}(1) = P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^\circ$.

Let $v \in (\pi'^{-})^\vee \widehat{\otimes} I^\circ(\dot\chi^{-1})$. Note that $\eta_i I(\dot\chi^{-1}) \subset I^\circ(\dot\chi^{-1})$. Thus $\eta_i v \in (\pi'^{-})^\vee \widehat{\otimes} I^\circ(\dot\chi^{-1})$.

Lemma 4.7 and Lebesgue's dominated convergence theorem imply that

$$\int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, v \rangle\,\mathrm{d}g = \lim_{i \to +\infty} \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, \eta_i v \rangle\,\mathrm{d}g = 0.$$

This proves the lemma. □

Lemma 5.7 implies that we have a natural continuous linear map

$$\mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi'\otimes\chi')\otimes\chi_0)=\pi'^{-}*I^{\circ}(\dot\chi)\to\pi'^{-}*I(\dot\chi).$$

This map is clearly injective and $G_{\mathsf{s}''}$-equivariant. Similarly to Lemma 5.7, we know that the natural pairing

$$(\pi'^{-}*I(\dot\chi))\times((\pi'^{-})^{\vee}*I^{\circ}(\dot\chi^{-1}))\to\mathbb{C}$$

is well-defined and non-degenerate. Thus Theorem 5.5 follows by the following lemma.

Lemma 5.8. *Let G be a real reductive group. Let π be a Casselman-Wallach representation of G, and let $\tilde\pi$ be a smooth Fréchet representation of G with a G-equivariant injective continuous linear map*

$$\phi:\pi\to\tilde\pi.$$

Assume that there is a non-degenerate G-invariant continuous bilinear map

$$\langle\,,\,\rangle:\tilde\pi\times\pi^{\vee}\to\mathbb{C},$$

such that the composition of

$$\pi\times\pi^{\vee}\xrightarrow{(u,v)\mapsto(\phi(u),v)}\tilde\pi\times\pi^{\vee}\xrightarrow{\langle\,,\,\rangle}\mathbb{C}$$

is the natural pairing. Then ϕ is a topological isomorphism.

Proof. Let $u\in\tilde\pi$. Using the theorem of Dixmier-Malliavin [30, Theorem 3.3], we write

$$u=\sum_{i=1}^{s}\int_G\varphi_i(g)(g\cdot u_i)\,\mathrm{d}g\qquad(s\in\mathbb{N},\ u_i\in\tilde\pi),$$

where φ_i's are compactly supported smooth functions on G. As a continuous linear functional on $\pi^{\vee}$, we have that

$$\langle u,\,\cdot\rangle=\sum_{i=1}^{s}\int_G\varphi_i(g)\cdot\langle g\cdot u_i,\,\cdot\rangle\,\mathrm{d}g.$$

By [82, Lemma 3.5], the right-hand side functional equals $\langle u_0,\,\cdot\rangle$ for a unique $u_0\in\pi$. Thus $u=u_0$, and the lemma follows by the open mapping theorem. □

5.3. **Double theta lifts and parabolic induction.** Let $\star$ be the Howe dual of $\star'=\star''$. Let $k\in\mathbb{N}$ and we assume that the set

$$S_{\star,k}:=\{\mathsf{s}_1\text{ is a classical signature}\mid\mathsf{s}_1\text{ has type }\star,\text{ and }|\mathsf{s}_1|=k\}$$

is non-empty. Thus k is odd if $\star=B$, and k is even in all other cases. In this subsection, we further assume that $k\geqslant 2$ and

$$\dot p=\dot q=\begin{cases}k-1, & \text{if }\star\in\{B,D\};\\ k+1, & \text{if }\star\in\{C,\widetilde C\};\\ k & \text{if }\star\in\{C^*,D^*\}.\end{cases}\tag{5.20}$$

We also assume that the character $\dot\chi:R_{\dot{\mathsf{s}}}\to\mathbb{C}^{\times}$ satisfies the following conditions:

$$\begin{cases}\dot\chi\text{ is genuine and }\dot\chi^4=1, & \text{if }\star=B;\\ \dot\chi^2=1, & \text{if }\star=D;\\ \dot\chi=1, & \text{if }\star\in\{C,\widetilde C\};\\ \dot\chi^2\text{ equals the composition of }R_{\dot{\mathsf{s}}}\xrightarrow{\text{natural map}}\mathrm{GL}(X_{\dot{\mathsf{s}}})\xrightarrow{\det}\mathbb{C}^{\times}, & \text{if }\star=C^*;\\ \dot\chi^2\text{ equals the composition of }R_{\dot{\mathsf{s}}}\xrightarrow{\text{natural map}}\mathrm{GL}(Y_{\dot{\mathsf{s}}})\xrightarrow{\det}\mathbb{C}^{\times}, & \text{if }\star=D^*.\end{cases}\tag{5.21}$$

If $\star \notin \{B, D\}$, the character $\dot\chi$ is uniquely determined by (5.21). If $\dot\star \in \{B, D\}$, there are two characters satisfying (5.21), and we let $\dot\chi'$ denote the one other than $\dot\chi$.

The relationship between the degenerate principle series representation $I(\dot\chi)$ and Rallis quotients is summarized in the following lemma.

Lemma 5.9. *(a) If $\star \in \{B, D\}$, then*

$$I(\dot\chi) \oplus I(\dot\chi') \cong \bigoplus_{\mathsf{s}_1 \in S_{\star,k}} \check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1).$$

(b) If $\star \in \{C, \widetilde{C}\}$, then

$$I(\dot\chi) \cong \check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1) \oplus (\check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1) \otimes \det), \qquad \textit{where}\;\; \mathsf{s}_1 = (C, \frac{k}{2}, \frac{k}{2}).$$

(c) If $\star = D^*$, then

$$I(\dot\chi) \cong \check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1), \qquad \textit{where}\;\; \mathsf{s}_1 = (D^*, \frac{k}{2}, \frac{k}{2}).$$

(d) If $\star = C^*$, then there is an exact sequence of representations of $G_{\dot{\mathsf{s}}}$:

$$0 \to \bigoplus_{\mathsf{s}_1 \in S_{\star,k}} \check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1) \to I(\dot\chi) \to \bigoplus_{\mathsf{s}_2 \in S_{\star,k-2}} \check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_2}(1) \to 0.$$

(e) All the representations $\check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1)$and $\check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_2}(1)$ appearing in (a), (b), (c) and (d) are irreducible and unitarizable.

Proof. See [50, Theorem 2.4], [51, Introduction], [52, Theorem 6.1] and [93, Sections 9 and 10]. ☐

Lemma 5.10. *For all s_1 appearing in Lemma 5.9, the trivial representation 1 of G_{s_1} is overconvergent for $\check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}$, and*

$$\bar\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1) = \check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1).$$

Proof. Note that the trivial representation 1 of G_{s_1} is $(-\nu_{\mathsf{s}_1})$-bounded, and $-\nu_{\mathsf{s}_1} > \nu^\circ_{\mathsf{s}_1} - |\dot{\mathsf{s}}|$. This implies the first assertion. Since $\check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1)$ is irreducible and $\bar\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1)$ is a quotient of $\check\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1)$, for the proof of the second assertion, it suffices to show that $\bar\Theta^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1)$ is nonzero.

Put

$$V_{\mathsf{s}_1}(\mathbb{R}) := \begin{cases} \text{the fixed point set of } J_{\mathsf{s}_1} \text{ in } V_{\mathsf{s}_1}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ V_{\mathsf{s}_1}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

As usual, realize $\hat\omega_{\dot{\mathsf{s}},\mathsf{s}_1}$ on the space of square integrable functions on $(V_{\mathsf{s}_1}(\mathbb{R}))^{\dot p}$ so that $\omega_{\dot{\mathsf{s}},\mathsf{s}_1}$ is identified with the space of the Schwartz functions, and G_{s_1} acts on it through the obvious transformation.

Take a positive valued Schwartz function ϕ on $(V_{\mathsf{s}_1}(\mathbb{R}))^{\dot p}$. Then

$$\langle g\cdot\phi, \phi\rangle = \int_{(V_{\mathsf{s}}(\mathbb{R}))^{\dot p}} \phi(g^{-1}\cdot x)\cdot\phi(x)\,\mathrm{d}x > 0, \quad \text{for all } g \in G_{\mathsf{s}_1}.$$

Thus

$$\int_{G_{\mathsf{s}_1}} \langle g\cdot\phi, \phi\rangle \,\mathrm{d}g \neq 0,$$

and the lemma follows. ☐

Lemma 5.11. *Assume that π' is ν'-bounded for some*

$$\nu' > \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \textit{if } \star = C^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \textit{otherwise}. \end{cases}$$

Then for all s_1 *appearing in Lemma 5.9 and all unitary character* γ' *of* $G_{\mathsf{s}'}$*, the representation* $\pi' \otimes \gamma'$ *is convergent for* $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}$*, and* $\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi' \otimes \gamma')$ *is overconvergent for* $\check{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}$*.*

Proof. The first assertion follows by noting that

$$\nu_{\mathsf{s}'} - |\mathsf{s}_1| = \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \text{if } \star = C^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \text{otherwise.} \end{cases} \tag{5.22}$$

The proof of second assertion is similar to the proof of the first assertion of Lemma 5.10. □

Recall the character $\chi_0 := \dot{\chi}|_{R_{\mathsf{s}_0}}$ from (5.11). Similarly we define $\chi'_0 := \dot{\chi}'|_{R_{\mathsf{s}_0}}$. Recall that π' is assumed to be genuine when $\dot{\star} = \widetilde{C}$.

Theorem 5.12. *Assume that* π' *is* ν'*-bounded for some* $\nu' > -\frac{|\mathsf{s}_0|}{2} - 1$.
(a) If $\star \in \{B, D\}$*, then*

$$\bigoplus_{\mathsf{s}_1 \in S_{\star,k}} \bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \oplus \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi'_0).$$

(b) If $\star \in \{C, \widetilde{C}\}$*, then*

$$\bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')) \oplus \Big((\bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi' \otimes \det))) \otimes \det\Big) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s}_1 = (\star, \frac{k}{2}, \frac{k}{2})$.
(c) If $\star = D^*$*, then*

$$\bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s}_1 = (D^*, \frac{k}{2}, \frac{k}{2})$.
(d) If $\star = C^*$*, then there is an exact sequence*

$$0 \to \bigoplus_{\mathsf{s}_1 \in S_{\star,k}} \bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')) \to \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \to \mathcal{J} \to 0$$

of representations of $G_{\mathsf{s}''}$ *such that* $\mathcal{J}$ *is a quotient of*

$$\bigoplus_{\mathsf{s}_2 \in S_{\star,k-2}} \check{\Theta}_{\mathsf{s}_2}^{\mathsf{s}''}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_2}(\pi')).$$

Proof. Using Lemma 4.7, we know that π'^{-} is convergent for $I(\dot{\chi})$ (and convergent for $I(\dot{\chi}')$ when $\star \in \{B, D\}$). Also note that the character χ' (see (5.12)) is trivial.

If we are in the situation (a), (b) or (c), by using Theorem 5.5, Lemma 5.10 and Proposition 5.4, the theorem follows by applying the operation $\pi'^{-} * (\,\cdot\,)$ to the isomorphism in Lemma 5.9.

Now assume that we are in the situation (d). For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in part (e) of Lemma 5.9. Since π'^{-} is nuclear as a Fréchet space, the sequence

$$0 \to \pi'^{-}\widehat{\otimes} I_1 \to \pi'^{-}\widehat{\otimes} I_2 \to \pi'^{-}\widehat{\otimes} I_3 \to 0$$

is also topologically exact. Note that the natural map

$$\pi'^{-} * I_1 \to \pi'^{-} * I_2$$

is injective, and the natural map

$$\pi'^{-}\widehat{\otimes} I_3 \cong \frac{\pi'^{-}\widehat{\otimes} I_2}{\pi'^{-}\widehat{\otimes} I_1} \longrightarrow \frac{\pi'^{-} * I_2}{\pi'^{-} * I_1}$$

descends to a surjective map

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'^{-}}} \to \frac{\pi'^{-} * I_2}{\pi'^{-} * I_1}.$$

Note that

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'^{-}}} \cong \bigoplus_{\mathsf{s}_2 \in S_{\star,k-1}} \check{\Theta}_{\mathsf{s}_2}^{\mathsf{s}''}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_2}(\pi')).$$

Theorem 5.5 implies that

$$\pi'^{-} * I_2 \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0).$$

Lemma 5.10 and Proposition 5.4 imply that

$$\pi'^{-} * I_1 \cong \bigoplus_{\mathsf{s}_1 \in S_{\star,k}} \bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')).$$

Therefore the theorem follows. □

6. Regular descents and bounding the associated cycles

In this section, let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$. Let $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$, whose Zariski closure in $\mathfrak{g}_{\mathsf{s}}$ is denoted by $\overline{\mathcal{O}}$.

6.1. **Associated cycles.** In this subsection, we will recall the definition of associated cycles from [86].

Definition 6.1. *Let $H_{\mathbb{C}}$ be a complex linear algebraic group. Let $\mathcal{A}$ be a commutative $\mathbb{C}$-algebra carrying a locally algebraic linear action of $H_{\mathbb{C}}$ by algebra automorphisms. An $(\mathcal{A}, H_{\mathbb{C}})$-module is an $\mathcal{A}$-module σ equipped with a locally algebraic linear action of $H_{\mathbb{C}}$ such that the module structure map $\mathcal{A} \otimes \sigma \to \sigma$ is $H_{\mathbb{C}}$-equivariant. An $(\mathcal{A}, H_{\mathbb{C}})$-module is said to be finitely generated if it is so as an $\mathcal{A}$-module.*

For every affine complex algebraic variety Z, write $\mathbb{C}[Z]$ for the algebra of regular functions on Z. By convention, all elements of complex algebraic varieties are closed. Given a finitely generated $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module σ°, for each $K_{\mathsf{s},\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$, the set

$$\bigsqcup_{\mathbf{e} \in \mathscr{O}} \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]} \sigma^\circ$$

is naturally a $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$, where $\mathbb{C}_{\mathbf{e}}$ denotes the complex number field $\mathbb{C}$ viewing as a $\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]$-algebra via the evaluation map at $\mathbf{e}$. We define $\mathrm{AC}_{\mathscr{O}}(\sigma^\circ) \in \mathcal{K}_{\mathsf{s}}(\mathscr{O})$ to be the Grothendieck group element associated to this bundle. Define the associated cycle of σ° to be

$$\mathrm{AC}_{\mathcal{O}}(\sigma^\circ) := \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathrm{AC}_{\mathscr{O}}(\sigma^\circ) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

Write $I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}}$ for the radical ideal of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$ corresponding to the closed subvariety $\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}$. Every $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module is clearly an $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module. On the other hand, for each $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module σ,

$$\frac{(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i \cdot \sigma}{(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^{i+1} \cdot \sigma} \qquad (i \in \mathbb{N})$$

is naturally a $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module. We say that σ is $\mathcal{O}$-bounded if

$$(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i \cdot \sigma = 0 \qquad \text{for some } i \in \mathbb{N}.$$

When σ is finitely generated, this is equivalent to saying that the support of σ is contained in $\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}$.

When σ is finitely generated and $\mathcal{O}$-bounded, we define its associated cycle to be

$$\mathrm{AC}_{\mathcal{O}}(\sigma) := \sum_{i\in\mathbb{N}} \mathrm{AC}_{\mathcal{O}}\left(\frac{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i\cdot\sigma}{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^{i+1}\cdot\sigma}\right) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

The assignment $\mathrm{AC}_{\mathcal{O}}$ is additive in the following sense: the equality

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\sigma') + \mathrm{AC}_{\mathcal{O}}(\sigma'')$$

holds for every exact sequence $0 \to \sigma' \to \sigma \to \sigma'' \to 0$ of finitely generated $\mathcal{O}$-bounded $(\mathrm{S}(\mathfrak{p}_{\mathsf{s}}), K_{\mathsf{s},\mathbb{C}})$-modules.

Given a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module ρ of finite length, pick a filtration

$$\mathcal{F}: \quad \cdots \subset \rho_{-1} \subset \rho_0 \subset \rho_1 \subset \rho_2 \subset \cdots \tag{6.1}$$

of ρ that is good in the following sense:

- for each $i \in \mathbb{Z}$, ρ_i is a finite-dimensional K_{s}-stable subspace of ρ;
- $\mathfrak{g}_{\mathsf{s}}\cdot\rho_i \subset \rho_{i+1}$ for all $i\in\mathbb{Z}$, and $\mathfrak{g}_{\mathsf{s}}\cdot\rho_i = \rho_{i+1}$ when $i\in\mathbb{Z}$ is sufficiently large;
- $\bigcup_{i\in\mathbb{Z}}\rho_i = \rho$, and $\rho_i = 0$ for some $i\in\mathbb{Z}$.

Then the grading

$$\mathrm{Gr}(\rho) := \mathrm{Gr}(\rho,\mathcal{F}) := \bigoplus_{i\in\mathbb{Z}} \rho_i/\rho_{i+1}$$

is naturally a finitely generated $(\mathrm{S}(\mathfrak{p}_{\mathsf{s}}), K_{\mathsf{s},\mathbb{C}})$-module, where $\mathrm{S}(\mathfrak{p}_{\mathsf{s}})$ denotes the symmetric algebra of $\mathfrak{p}_{\mathsf{s}}$. Recall that we have identified $\mathfrak{p}_{\mathsf{s}}$ with its dual space $\mathfrak{p}_{\mathsf{s}}^*$ by using the trace from. Hence $\mathrm{S}(\mathfrak{p}_{\mathsf{s}})$ is identified with $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$, and $\mathrm{Gr}(\rho)$ is a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module.

Recall that ρ is said to be $\mathcal{O}$-bounded if the associated variety of its annihilator ideal in $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})$ is contained in $\overline{\mathcal{O}}$.

Lemma 6.2. *Let ρ be a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module of finite length, and let $\mathcal{F}$ be a good filtration of ρ. Then ρ is $\mathcal{O}$-bounded if and only if $\mathrm{Gr}(\rho,\mathcal{F})$ is $\mathcal{O}$-bounded as a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module.*

Proof. This is a direct consequence of [86, Theorem 8.4]. □

When ρ is $\mathcal{O}$-bounded, it associated cycle is defined to be

$$\mathrm{AC}_{\mathcal{O}}(\rho) := \mathrm{AC}_{\mathcal{O}}(\mathrm{Gr}(\rho)) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

This is independent of the good filtration (6.1). Moreover, taking the associated cycles is additive in the following sense: the equality

$$\mathrm{AC}_{\mathcal{O}}(\rho) = \mathrm{AC}_{\mathcal{O}}(\rho') + \mathrm{AC}_{\mathcal{O}}(\rho'')$$

holds for every exact sequence $0\to\rho'\to\rho\to\rho''\to 0$ of $\mathcal{O}$-bounded $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-modules of finite length.

For every Cassleman-Wallach representation π of G_{s}, write π^{alg} for the underlying $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module of π. When π is $\mathcal{O}$-bounded, we define the associated cycle $\mathrm{AC}_{\mathcal{O}}(\pi) := \mathrm{AC}_{\mathcal{O}}(\pi^{\mathrm{alg}})$.

6.2. **Algebraic theta lifts and commutative theta lifts.** We retain the notation of Section 3 and Section 4. Denote by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}}$ the $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$-submodule of $\omega_{\mathsf{s},\mathsf{s}'}$ generated by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$. This is an $(\mathfrak{g}_{\mathsf{s}}\times\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}}\times K_{\mathsf{s}'})$-module. For every $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module ρ', define its full theta lift (the algebraic theta lift) to be the $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho') := (\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}}\otimes\rho')_{\mathfrak{g}_{\mathsf{s}'},K_{\mathsf{s}'}} \qquad \text{(the coinvariant space)}.$$

It has finite length whenever ρ' has finite length ([43, Section 4]).

Recall from (3.6) the moment maps

$$\begin{array}{ccccc} \mathfrak{p}_{\mathsf{s}}^* = \mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}^* = \mathfrak{p}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array}$$

We view $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$ as an $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}] \otimes \mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]$-algebra by using the moment maps. The natural action of $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ on $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ yields a locally algebraic linear action of $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ on $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$. Thus $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$ is naturally a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}] \otimes \mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})$-module.

For every $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module σ', define its commutative theta lift to be

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma') := (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(the coinvariant space)},$$

which is naturally a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module.

Lemma 6.3. *Suppose that σ' is a finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then the $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$ is also finitely generated.*

Proof. This follows from the structural analysis of the space of joint harmonics, as in [43, Section 4]. $\square$

Lemma 6.4. *Suppose that ρ' is a $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then there exist a good filtration $\mathcal{F}'$ on ρ', a good filtration $\mathcal{F}$ on $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')$, and a surjective $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module homomorphism*

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\mathrm{Gr}(\rho', \mathcal{F}')) \to \mathrm{Gr}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho'), \mathcal{F}).$$

Proof. This follows from the discussions in [56, Section 3.2]. $\square$

Recall from (3.8) the moment maps

$$\begin{array}{ccccc} \mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array}$$

Lemma 6.5. *Suppose that $\mathcal{O}' \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ and $M_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'}) \subset M_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}})$, where $\overline{\mathcal{O}'}$ denotes the Zariski closure of $\mathcal{O}'$ in $\mathfrak{g}_{\mathsf{s}'}$. Then*

- *$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$ is $\mathcal{O}$-bounded for every $\mathcal{O}'$-bounded $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module σ';*
- *$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')$ is $\mathcal{O}$-bounded for every $\mathcal{O}'$-bounded $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module ρ' of finite length;*
- *$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')$ is $\mathcal{O}$-bounded for every $\mathcal{O}'$-bounded Casselman-Wallach representation π' of $G_{\mathsf{s}'}$.*

Proof. The assumption of the lemma implies that

$$(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i \cdot \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \subset I_{\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}} \cdot \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \qquad \text{for some } i \in \mathbb{N},$$

where $I_{\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}}$ denotes the radical ideal of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$ corresponding to the closed subvariety $\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}$. This implies the first assertion of the lemma. In view of Lemma 6.2, the second assertion is implied by the first one and Lemma 6.4. The third assertion is implied by the second one since there is a surjective $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module homomorphisms

$$\check{\Theta}_{\mathsf{s}}^{\mathsf{s}'}(\pi'^{\mathrm{alg}}) \to \left(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')\right)^{\mathrm{alg}}.$$

$\square$

6.3. **Regular descent of a nilpotent orbit.**

Definition 6.6. *The orbit* $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ *is regular for* $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ *if either*

$$|\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|,$$

or

$$|\mathsf{s}'| > |\nabla_{\mathrm{naive}}(\mathcal{O})| \qquad \textit{and} \qquad \mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O}).$$

In the rest of this section we assume that $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Then Lemma 3.3 implies that $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Put $\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$, and let $\overline{\mathcal{O}'}$ denote the Zariski closure of $\mathcal{O}'$ in $\mathfrak{g}_{\mathsf{s}'}$.

Lemma 6.7. *As subsets of* $\mathfrak{g}_{\mathsf{s}}$,

$$\tilde{M}_{\mathsf{s}}(\tilde{M}_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}.$$

Proof. This is implied by [28, Theorems 5.2 and 5.6]. □

Lemma 6.8. *Suppose that* σ' *is an* $\mathcal{O}'$*-bounded* $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$*-module. Then the* $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$*-module* $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')$ *is* $\mathcal{O}$*-bounded.*

Proof. This follows from Lemmas 6.7 and 6.5. □

Put

$$\partial := \overline{\mathcal{O}} \backslash \mathcal{O} \qquad \text{and} \qquad \bar{\partial} := \mathfrak{g}_{\mathsf{s}} \backslash \partial.$$

Then $\bar{\partial}$ is an open subvariety of $\mathfrak{g}_{\mathsf{s}}$ and $\mathcal{O}$ is a closed subvariety of $\bar{\partial}$. Put

$$W^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} := \tilde{M}_{\mathsf{s}}^{-1}(\bar{\partial}) \qquad \text{and} \qquad W^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} := \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap \tilde{M}_{\mathsf{s}'}^{-1}(\mathcal{O}').$$

Then $W^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}$ is an open subvariety of $W_{\mathsf{s},\mathsf{s}'}$, and the following lemma implies that $W^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ is a closed subvariety of $W^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}$.

Lemma 6.9. *The set* $W^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ *is a single* $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$*-orbit. Moreover, it is contained in* $W^{\circ}_{\mathsf{s},\mathsf{s}'}$ *and equals*

$$W^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \cap \tilde{M}_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'}).$$

Proof. The first assertion is implied by [28, Theorem 3.6]. Recall from (3.9) that $W^{\circ}_{\mathsf{s},\mathsf{s}'} \cap \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O})$ is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit whose image under the moment map $\tilde{M}_{\mathsf{s}'}$ equals $\mathcal{O}'$. Thus

$$W^{\circ}_{\mathsf{s},\mathsf{s}'} \cap W^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} = W^{\circ}_{\mathsf{s},\mathsf{s}'} \cap \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}),$$

which is also a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit. Hence $W^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} \subset W^{\circ}_{\mathsf{s},\mathsf{s}'}$.

In fact, [28, Theorem 3.6] also implies that

$$W^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} = \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap \tilde{M}_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'}).$$

Thus the last assertion is a direct consequence of Lemma 6.7. □

Write

$$\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \qquad \text{and} \qquad \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}.$$

Then $\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}$ is an open subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ and $\mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ is a closed subvariety of $\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}$. We have a decomposition

$$\mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} = \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}} \text{ that is contained in the image of } M_{\mathsf{s}}} \mathcal{X}^{\mathscr{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'},$$

where

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'} := M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'},$$

which is a Zariski open and closed subset of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$. Lemmas 3.2 and 6.9 imply that $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'}$ is a single $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-orbit.

We defer the proof of the following proposition to Sections 6.6 and 6.7.

Proposition 6.10. *The scheme theoretic fibre product*

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'})$$

is reduced, where $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$ is viewed as a $\mathfrak{p}_{\mathsf{s}'}$-scheme via the moment map $M_{\mathsf{s}'}$.

By Lemma 6.9 and Proposition 6.10, we know that

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} = \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}) = \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}), \tag{6.2}$$

which is a smooth closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$.

For every element $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$, write $K_{\mathbf{e}}$ for the stabilizer of $\mathbf{e}$ in $K_{\mathsf{s},\mathbb{C}}$. Let $\mathbb{C}_{\mathbf{e}}$ be the field $\mathbb{C}$ viewing as a $\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]$-algebra via the evaluation map at $\mathbf{e}$. Put

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'} := M_{\mathsf{s}}^{-1}(\mathbf{e}) \cap M_{\mathsf{s}'}^{-1}(\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}) = M_{\mathsf{s}}^{-1}(\mathbf{e}) \cap M_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}),$$

which is a closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$. If $\mathbf{e}$ is not contained in the image of M_{s}, then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'} = \varnothing$. Otherwise, it is a single $(K_{\mathbf{e}} \times K_{\mathsf{s}',\mathbb{C}})$-orbit, and Lemma 3.2 implies that it is also a single $K_{\mathsf{s}',\mathbb{C}}$-orbit.

Lemma 6.11. *For every element $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$,*

$$\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}] = \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}].$$

Proof. As schemes, we have that

$$\begin{aligned} & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}) \\ = \ & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}) \\ = \ & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \\ = \ & \{\mathbf{e}\} \times_{\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} ((\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}) \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}) \\ = \ & \{\mathbf{e}\} \times_{\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \\ = \ & \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}. \end{aligned}$$

The last equality holds because the morphism $M_{\mathsf{s}} : \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \to \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ is $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-equivariant and hence smooth in the sense of algebraic geometry. This proves the lemma. $\square$

6.4. **Commutative theta lifts and geometric theta lifts.** The purpose of this subsection is to prove the following proposition.

Proposition 6.12. *Suppose that σ' is a finitely generated $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')) = \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma'))$$

as elements of $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$.

By a quasi-coherent module over a scheme Z, we mean a quasi-coherent module over the structure sheaf of Z. We say that a quasi-coherent module $\mathcal{M}$ over a scheme Z descends to a closed subscheme Z_1 of Z if $\mathcal{M}$ is isomorphic to the push-forward of a quasi-coherent module over Z_1 via the closed embedding $Z_1 \to Z$. This is equivalent to saying that the ideal sheaf defining Z_1 annihilates $\mathcal{M}$.

When Z is an affine complex algebraic variety and σ is a $\mathbb{C}[Z]$-module, we write $\mathcal{M}_\sigma$ for the quasi-coherent module over Z corresponding to σ.

Lemma 6.13. *Let σ be a finitely generated $\mathcal{O}$-bounded $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s}})$-module. Assume that $(\mathcal{M}_\sigma)|_{\partial\cap\mathfrak{p}_{\mathsf{s}}}$ descends to $\mathcal{O}\cap\mathfrak{p}_{\mathsf{s}}$. Then*

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\mathbb{C}[\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}]\otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]}\sigma).$$

Proof. Let $\mathbf{e}\in\mathcal{O}\cap\mathfrak{p}_{\mathsf{s}}$. Note that for every ideal I of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$,

$$(I\cdot\sigma)_{\mathbf{e}} = (I_{\mathbf{e}})\cdot\sigma_{\mathbf{e}}\subset\sigma_{\mathbf{e}},$$

where a subscript $\mathbf{e}$ indicates the localization at $\mathbf{e}$. The assumption of the lemma implies that

$$((I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i)_{\mathbf{e}}\cdot\sigma_{\mathbf{e}} = 0 \qquad \text{for all } i\in\mathbb{N}^+.$$

Thus

$$((I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i\cdot\sigma)_{\mathbf{e}} = 0,$$

which implies the lemma. □

Lemma 6.14. *Suppose that Z is an affine complex algebraic variety with a transitive algebraic action of $K_{\mathsf{s},\mathbb{C}}$. Let σ be a finitely generated $(\mathbb{C}[Z], K_{\mathsf{s},\mathbb{C}})$-module. Let $z\in Z$ and write K_z for the stabilizer of z in $K_{\mathsf{s},\mathbb{C}}$. Then the natural map*

$$\sigma^{K_{\mathsf{s},\mathbb{C}}}\to(\mathbb{C}_z\otimes_{\mathbb{C}[Z]}\sigma)^{K_z}$$

is a linear isomorphism. Here $\mathbb{C}_z$ is the field $\mathbb{C}$ viewing as a $\mathbb{C}[Z]$-algebra via the evaluation map at z, and a superscript group indicates the space of invariant vectors under the group action.

Proof. Note that the set

$$\bigsqcup_{x\in Z}\mathbb{C}_x\otimes_{\mathbb{C}[Z]}\sigma$$

is naturally a $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over Z, and σ is identified with the space of algebraic sections of this bundle. Thus

$$\sigma\cong{}^{\mathrm{alg}}\,\mathrm{Ind}_{K_z}^{K_{\mathsf{s},\mathbb{C}}}(\mathbb{C}_x\otimes_{\mathbb{C}[Z]}\sigma) \qquad \text{(algebraically induced representation).}$$

The lemma then follows by the algebraic version of the Frobenius reciprocity. □

Lemma 6.15. *Suppose that σ' is a finitely generated $(\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module, and write $\sigma := \check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$. Then the quasi-coherent module $(\mathcal{M}_\sigma)|_{\partial\cap\mathfrak{p}_{\mathsf{s}}}$ descends to $\mathcal{O}\cap\mathfrak{p}_{\mathsf{s}}$.*

Proof. Write

$$\tilde{\sigma} := \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]\otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]}\sigma'\otimes\zeta_{\mathsf{s},\mathsf{s}'} = (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]\otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]}\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}])\otimes_{\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}]}(\sigma'\otimes\zeta_{\mathsf{s},\mathsf{s}'}),$$

to be viewed as a $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$-module. Write $\tilde{\sigma}_0 := \sigma$, viewing as a $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$-module via the moment map M_{s}.

Proposition 6.10 and the equalities in (6.2) imply that $(\mathcal{M}_{\tilde{\sigma}})|_{\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\partial}}$ descends to $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$. This implies that $(\mathcal{M}_{\tilde{\sigma}_0})|_{\partial\cap\mathfrak{p}_{\mathsf{s}}}$ descends to $\mathcal{O}\cap\mathfrak{p}_{\mathsf{s}}$. The lemma then follows since σ is a direct summand of $\tilde{\sigma}_0$. □

We are now ready to prove Proposition 6.12.

Proof of Proposition 6.12. Let σ' be a finitely generated $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module, and write $\sigma := \check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$. Lemmas 6.15 and 6.13 imply that

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma).$$

Suppose that $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$. Then

$$\begin{aligned} & \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma) \\ = \; & \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma \\ = \; & \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}',\mathbb{C}}} \\ = \; & \left((\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]) \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{K_{\mathsf{s}',\mathbb{C}}} \\ = \; & \left(\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(by Lemma 6.11).} \end{aligned}$$

If $\mathbf{e}$ is not in the image of M_{s}, then the above module is zero. Now we assume that $\mathbf{e}$ is in the image of M_{s} so that $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}$ is a single $K_{\mathsf{s}',\mathbb{C}}$-orbit. Pick an arbitrary element $\phi \in \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}$, and write $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. Let

$$1 \to (K_{\mathsf{s}',\mathbb{C}})_{\phi} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \xrightarrow{\text{the projection to the first factor}} (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1$$

be the exact sequence as in Lemma 3.2.

Let $\mathbb{C}_{\phi}$ denote the field $\mathbb{C}$ viewing as a $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}]$-algebra via the evaluation map at ϕ. Then we have that

$$\begin{aligned} & \left(\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{K_{\mathsf{s}',\mathbb{C}}} \\ = \; & \left(\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)^{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(because } K_{\mathsf{s}',\mathbb{C}} \text{ is reductive)} \\ = \; & \left(\mathbb{C}_{\phi} \otimes_{\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}]} \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)^{(K_{\mathsf{s}',\mathbb{C}})_{\phi}} \qquad \text{(by Lemma 6.14)} \\ = \; & \left(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)^{(K_{\mathsf{s}',\mathbb{C}})_{\phi}} \\ = \; & \left(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{(K_{\mathsf{s}',\mathbb{C}})_{\phi}} \qquad \text{(because } (K_{\mathsf{s}',\mathbb{C}})_{\phi} \text{ is reductive).} \end{aligned}$$

Putting all things together, we have an identification

$$\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma) = \left(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{(K_{\mathsf{s}',\mathbb{C}})_{\phi}}.$$

It is routine to check that this identification respects the $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$-actions, and $(K_{\mathsf{s}',\mathbb{C}})_{\phi}$ acts trivially on the both sides. Thus the identification respects the $K_{\mathbf{e}}$-actions. In view of Lemmas 6.13 and 6.15, this proves the proposition. □

6.5. **Algebraic theta lifts and geometric theta lifts.** Proposition 6.12 has the following consequence.

Proposition 6.16. *Suppose that σ' is a finitely generated $\mathcal{O}'$-bounded $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')) \preceq \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma'))$$

in $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$.

Proof. Suppose that $0 \to \sigma'_1 \to \sigma'_2 \to \sigma'_3 \to 0$ be an exact sequence of $\mathcal{O}'$-bounded finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-modules. Then

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma'_1) \to \check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma'_2) \to \check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma'_3) \to 0$$

is an exact sequence of $\mathcal{O}$-bounded finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-modules. Thus

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma'_2)) \preceq \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma'_1)) + \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma'_3)).$$

On the other hand, it is clear that

$$\vartheta_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma'_2)) = \vartheta_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma'_1)) + \vartheta_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma'_3)).$$

The proposition then easily follows by Proposition 6.12. □

Finally, we are ready to prove the main result of this section. Recall that $\mathcal{O}$ is assumed to be regular for $\nabla_{\mathsf{s}'}^{\mathsf{s}}$.

Theorem 6.17. *Let ρ' be an $\mathcal{O}'$-bounded $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')$ is $\mathcal{O}$-bounded, and*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')) \preceq \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\rho')).$$

Proof. Let $\mathcal{F}'$ and $\mathcal{F}$ be good filtrations on ρ' and $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')$ respectively as in Lemma 6.4 so that there exists a surjective $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module homomorphism

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\mathrm{Gr}(\rho', \mathcal{F}')) \to \mathrm{Gr}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho'), \mathcal{F}). \tag{6.3}$$

The first assertion then follows from Lemmas 6.2 and 6.8.

Put $\sigma' := \mathrm{Gr}(\rho', \mathcal{F}')$. Then

$$\begin{array}{rll} & \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')) & \\ = & \mathrm{AC}_{\mathcal{O}}(\mathrm{Gr}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho'), \mathcal{F})) & \\ \preceq & \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')) & \text{(by (6.3))} \\ \preceq & \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma')) & \text{(by Proposition 6.16)} \\ = & \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\rho')). & \end{array}$$

This proves the theorem. □

6.6. **Geometries of regular descent.** The following lemma is a form of the Jacobian criterion for regularity (see [55, Theorem 2.19]).

Lemma 6.18. *Let Z and Z' be smooth complex algebraic varieties, and let $z' \in Z$. Let $f : Z \to Z'$ be a morphism of algebraic varieties such that $f^{-1}(z')$ is a smooth subvariety of Z. Then the scheme theoretic fibre product $Z \times_{Z'} \{z'\}$ is reduced if and only if the sequence*

$$\mathrm{T}_z(f^{-1}(z')) \xrightarrow{\text{inclusion}} \mathrm{T}_z(Z) \xrightarrow{\text{the differential of } f} \mathrm{T}_{z'}(Z') \tag{6.4}$$

is exact for all $z \in f^{-1}(z')$.

Here and as usual T indicates the tangent space.

Recall that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is regular for $\nabla_{\mathsf{s}'}^{\mathsf{s}}$. Let $\mathbf{e}' \in \mathcal{O}' = \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O})$. Denote by $G_{\mathbf{e}'}$ the stabilizer of $\mathbf{e}'$ in $G_{\mathsf{s}',\mathbb{C}}$, and by $\mathfrak{g}_{\mathbf{e}'}$ the Lie algebra of $G_{\mathbf{e}'}$. Consider the map

$$\tilde{M}' := (\tilde{M}_{\mathsf{s}'})|_{W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}} : W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \to \mathfrak{g}_{\mathsf{s}'}.$$

By Lemma 6.9, its fibre at $\mathbf{e}'$ equals the set

$$W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathbf{e}'} := \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap \tilde{M}_{\mathsf{s}'}^{-1}(\mathbf{e}'),$$

and this set is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathbf{e}'}$-orbit. Thus the set $W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathbf{e}'}$ is a smooth closed subvariety of $W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$.

Let $\phi \in W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathbf{e}'}$. Then we have a sequence

$$\mathrm{T}_\phi(W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathbf{e}'}) \xrightarrow{\text{inclusion}} \mathrm{T}_\phi(W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}) \xrightarrow{\text{the differential of } \tilde{M}'} \mathrm{T}_{\mathbf{e}'}(\mathfrak{g}_{\mathsf{s}'}) \tag{6.5}$$

as in (6.4). The tangent spaces $\mathrm{T}_\phi(W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}})$ and $\mathrm{T}_{\mathbf{e}'}(\mathfrak{g}_{\mathsf{s}'})$ are respectively identified with $W_{\mathsf{s},\mathsf{s}'}$ and $\mathfrak{g}_{\mathsf{s}'}$ as usual. Then the second map in (6.5) equals the map

$$W_{\mathsf{s},\mathsf{s}'} \to \mathfrak{g}_{\mathsf{s}'}, \qquad b \mapsto b\phi^* + \phi b^*. \tag{6.6}$$

Note that the image of the first map in (6.5) is identified with the image of the following map

$$\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathbf{e}'} \to W_{\mathsf{s},\mathsf{s}'}, \qquad (a, a') \mapsto a'\phi - \phi a. \tag{6.7}$$

Thus the sequence (6.5) is exact if and only if the following sequence is exact:

$$\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathbf{e}'} \xrightarrow{(6.7)} W_{\mathsf{s},\mathsf{s}'} \xrightarrow{(6.6)} \mathfrak{g}_{\mathsf{s}'}. \tag{6.8}$$

Lemma 6.19. *The sequence* (6.5) *is exact when* $|\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|$.

Proof. The map (6.6) equals the composition of

$$W_{\mathsf{s},\mathsf{s}'} \xrightarrow{b \mapsto b\phi^*} \mathfrak{gl}(V_{\mathsf{s}'}) \xrightarrow{x \mapsto x - x^*} \mathfrak{g}_{\mathsf{s}'}. \tag{6.9}$$

Here $\mathfrak{gl}(V_{\mathsf{s}'})$ denotes the algebra of linear endomorphisms of $V_{\mathsf{s}'}$, and $x^* \in \mathfrak{gl}(V_{\mathsf{s}'})$ is specified by requiring that

$$\langle xu, v \rangle_{\mathsf{s}'} = \langle u, x^* v \rangle_{\mathsf{s}'}, \qquad \text{for all } u, v \in V_{\mathsf{s}'}. \tag{6.10}$$

The second map in (6.9) is always surjective.

Note that $|\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|$ if and only if $\phi : V_{\mathsf{s}} \to V_{\mathsf{s}'}$ is surjective. Assume this is the case. Then ϕ^* is injective, and consequently the first map in (6.9) is surjective. Thus the map (6.6) is also surjective. Since $\phi \in W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathbf{e}'}$ is arbitrary, this implies that the scheme theoretic fibre product $W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{g}_{\mathsf{s}'}} \{\mathbf{e}'\}$ is reduced (see [36, Proposition 10.4]). Lemma 6.18 then implies that the sequence (6.5) is exact.

□

Lemma 6.20. *The sequence* (6.5) *is exact when* $\mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O})$.

Proof. Write

$$V_{\mathsf{s}'} = V_1' \oplus V_2', \tag{6.11}$$

where $V_1' := \phi(V_{\mathsf{s}})$, which is a non-degenerate subspace of $V_{\mathsf{s}'}$, and V_2' is the orthogonal complement of V_1' in $V_{\mathsf{s}'}$. Then

$$W_{\mathsf{s},\mathsf{s}'} = \left\{ \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \mid b_1 \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_1'),\ b_2 \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_2') \right\} \qquad \text{and} \qquad \phi = \begin{bmatrix} \phi_1 \\ 0 \end{bmatrix},$$

where $\phi_1 \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_1')$ is a surjective linear map.

Using the decomposition (6.11), we also have that

$$\mathfrak{g}_{\mathsf{s}'} := \left\{ \begin{bmatrix} a_1 & -\psi^* \\ \psi & a_2 \end{bmatrix} \mid a_1 \in \mathfrak{g}_1',\ a_2 \in \mathfrak{g}_2',\ \psi \in \mathrm{Hom}_{\mathbb{C}}(V_1', V_2') \right\},$$

where $\mathfrak{g}_1'$ and $\mathfrak{g}_2'$ are respectively the Lie algebras of the isometry groups of V_1' and V_2'. Here and henceforth, the adjoint operation $\psi \mapsto \psi^*$ is defined as in (6.10) and (3.4).

Note that

$$\mathbf{e}' = \begin{bmatrix} \mathbf{e}_1' & 0 \\ 0 & 0 \end{bmatrix}, \qquad \text{where } \mathbf{e}_1' := \phi_1\phi_1^* \in \mathfrak{g}_1',$$

and

$$\mathfrak{g}_{\mathbf{e}'} := \left\{ \begin{bmatrix} a_1 & -\psi^* \\ \psi & a_2 \end{bmatrix} \mid a_1 \in \mathfrak{g}_{\mathbf{e}_1'},\ a_2 \in \mathfrak{g}_2',\ \psi \in \mathrm{Hom}_{\mathbb{C}}(V_1', V_2')^{\mathbf{e}_1'} \right\},$$

where $\mathfrak{g}_{\mathbf{e}_1'}$ denotes the centralizer of $\mathbf{e}_1'$ in $\mathfrak{g}_1'$, and

$$\mathrm{Hom}_{\mathbb{C}}(V_1', V_2')^{\mathbf{e}_1'} := \{\psi \in \mathrm{Hom}_{\mathbb{C}}(V_1', V_2') \mid \psi\mathbf{e}_1' = 0\}.$$

The two maps in (6.8) are respectively given by

$$\left(a, \begin{bmatrix} a_1 & -\psi^* \\ \psi & a_2 \end{bmatrix}\right) \mapsto \begin{bmatrix} a_1\phi_1 - \phi_1 a \\ \psi\phi_1 \end{bmatrix}$$

and

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \mapsto \begin{bmatrix} b_1\phi_1^* + \phi_1 b_1^* & \phi_1 b_2^* \\ b_2\phi_1^* & 0 \end{bmatrix}.$$

In order to prove the lemma, it suffices to show that the sequences

$$\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathbf{e}_1'} \xrightarrow{(a,a_1)\mapsto a_1\phi_1 - \phi_1 a} \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_1') \xrightarrow{b_1 \mapsto b_1\phi_1^* + \phi_1 b_1^*} \mathfrak{g}_1'$$

and

$$\mathrm{Hom}_{\mathbb{C}}(V_1', V_2')^{\mathbf{e}_1'} \xrightarrow{\psi\mapsto\psi\phi_1} \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_2') \xrightarrow{b_2\mapsto b_2\phi_1^*} \mathrm{Hom}_{\mathbb{C}}(V_1', V_2') \tag{6.12}$$

are both exact. Lemma 6.19 implies that the first sequence is exact.

Note that the Young diagram of the nilpotent orbit in $\mathfrak{g}_1'$ containing $\mathbf{e}_1'$ equals $\nabla_{\mathrm{naive}}(\mathcal{O})$. Thus

$$\dim(V_1'/\mathbf{e}_1'(V_1')) = \mathbf{c}_1(\nabla_{\mathrm{naive}}(\mathcal{O})) = \mathbf{c}_2(\mathcal{O}).$$

On the other hand,

$$\begin{aligned} \dim(V_{\mathsf{s}}/\phi_1^*(V_1')) &= \dim(V_{\mathsf{s}}) - \dim(\phi(V_{\mathsf{s}})) \\ &= \dim(\mathrm{Ker}(\phi)) \\ &= \dim(\mathrm{Ker}(\phi^*\phi)) \\ &= \mathbf{c}_1(\mathcal{O}). \end{aligned}$$

The first map of (6.12) is injective since ϕ_1 is surjective. Then we have that

$$\begin{aligned} & \dim(\mathrm{Hom}_{\mathbb{C}}(V_1', V_2')^{\mathbf{e}_1'}) \\ = {} & \dim V_2' \cdot \dim(V_1'/\mathbf{e}_1'(V_1')) \\ = {} & \dim V_2' \cdot \mathbf{c}_2(\mathcal{O}) \\ = {} & \dim V_2' \cdot \mathbf{c}_1(\mathcal{O}) \\ = {} & \dim V_2' \cdot \dim(V_{\mathsf{s}}/\phi_1^*(V_1')) \\ = {} & \text{dimension of the kernel of the second map of (6.12).} \end{aligned}$$

Therefore the sequence (6.12) is exact since the composition of (6.12) is the zero map. This finishes the proof the lemma.

□

6.7. **A proof of Proposition 6.10.** Now we suppose that $\mathbf{e}' \in \mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$. Denote by $K_{\mathbf{e}'}$ the stabilizer of $\mathbf{e}'$ in $K_{\mathsf{s}',\mathbb{C}}$. Consider the map

$$M' := M_{\mathsf{s}'}|_{\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}} : \mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \to \mathfrak{p}_{\mathsf{s}'}.$$

As before, its fibre at $\mathbf{e}'$ equals the variety

$$\mathcal{X}^{\mathcal{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'} := M_{\mathsf{s}}^{-1}(\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}) \cap M_{\mathsf{s}'}^{-1}(\mathbf{e}').$$

This variety is a finite union of open and closed $K_{\mathsf{s},\mathbb{C}} \times K_{\mathbf{e}'}$-orbits:

$$\mathcal{X}^{\mathcal{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'} = \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}} \text{ such that } \mathscr{O} \subset M_{\mathsf{s}}(\mathcal{X}_{\mathsf{s},\mathsf{s}'}) \text{ and } \mathbf{e}' \in \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O})} \mathcal{X}^{\mathscr{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'},$$

where

$$\mathcal{X}^{\mathscr{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'} := M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\mathcal{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'}.$$

Thus $\mathcal{X}^{\mathcal{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'}$ is a smooth closed subvariety of $\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}$.

Lemma 6.21. *Suppose that $\phi \in \mathcal{X}^{\mathcal{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'}$. Then the sequence*

$$\mathrm{T}_\phi(\mathcal{X}^{\mathcal{O},\mathbf{e}'}_{\mathsf{s},\mathsf{s}'}) \xrightarrow{\text{inclusion}} \mathrm{T}_\phi(\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}) \xrightarrow{\text{the differential of } M'} \mathrm{T}_{\mathbf{e}'}(\mathfrak{p}_{\mathsf{s}'}) \tag{6.13}$$

is exact.

Proof. Recall $\dot{\epsilon} \in \{\pm 1\}$ form (3.1). We have commutative diagrams

$$\begin{array}{ccccc}
\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathbf{e}'} & \xrightarrow{(6.7)} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{(6.6)} & \mathfrak{g}_{\mathsf{s}'} \\
\Big\downarrow{\scriptstyle (a,a') \mapsto (L_{\mathsf{s}} \circ a \circ L_{\mathsf{s}}^{-1},\, L_{\mathsf{s}'} \circ a' \circ L_{\mathsf{s}'}^{-1})} & & \Big\downarrow{\scriptstyle b \mapsto -\dot{\epsilon}\sqrt{-1} L_{\mathsf{s}'} \circ b \circ L_{\mathsf{s}}^{-1}} & & \Big\downarrow{\scriptstyle a' \mapsto -L_{\mathsf{s}'} \circ a' \circ L_{\mathsf{s}'}^{-1}} \\
\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathbf{e}'} & \xrightarrow{(6.7)} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{(6.6)} & \mathfrak{g}_{\mathsf{s}'}.
\end{array}$$

Lemmas 6.19 and 6.20 imply that the horizontal sequences are exact. The lemma then follows by taking the fixed points of the vertical arrows.

□

Lemmas 6.18 and 6.21 implies that the scheme theoretic fibre product $\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \{\mathbf{e}'\}$ is reduced. In other words,

$$\left(\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'})\right) \times_{\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}} \{\mathbf{e}'\}$$

is reduced. Since $\mathbf{e}' \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}'}$ is arbitrary, this further implies that the scheme theoretic fibre product

$$\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'})$$

is reduced (see [34, Proposition 11.3.13] and [33, Théorème 6.9.1]). Proposition 6.10 is now proved.

6.8. **Geometric theta lifts of admissible orbit data.** Recall that $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Suppose that $\phi \in \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$, $\mathbf{e} = M_{\mathsf{s}}(\phi)$ and $\mathbf{e}' = M_{\mathsf{s}'}(\phi)$. As before, $K_{\mathbf{e}}$ and $K_{\mathbf{e}'}$ are respectively the stabilizers of $\mathbf{e}$ and $\mathbf{e}'$ in $K_{\mathsf{s},\mathbb{C}}$ and $K_{\mathsf{s}',\mathbb{C}}$. Write $\mathfrak{k}_{\mathbf{e}}$ and $\mathfrak{k}_{\mathbf{e}'}$ for the Lie algebras of $K_{\mathbf{e}}$ and $K_{\mathbf{e}'}$, respectively. As before, $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi$ denotes the stabilizer of ϕ in $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$. Write $\mathfrak{s}_\phi$ for the Lie algebra of $(K_{\mathsf{s}} \times K_{\mathsf{s}'})_\phi$.

Lemma 6.22. *As representations of $\mathfrak{s}_\phi$,*

$$\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}} \otimes \bigwedge^{\mathrm{top}} \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cong \bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}'}. \tag{6.14}$$

Proof. We retain the notation in the proof of Lemma 6.20. Since $\phi \in \mathcal{X}_{\mathsf{s},\mathsf{s}'}$, both V_1' and V_2' are $L_{\mathsf{s}'}$-stable. Write

$$\mathfrak{g}_i' = \mathfrak{k}_i' \oplus \mathfrak{p}_i' \qquad (i = 1, 2),$$

where $\mathfrak{k}_i'$ is the centralizer of $L_{\mathsf{s}'}$ in $\mathfrak{g}_i'$, and $\mathfrak{p}_i'$ is its orthogonal complement in $\mathfrak{g}_i'$ under the trace form.

The sequence (6.13) leads to a short exact sequence

$$0 \longrightarrow (\mathfrak{k}_{\mathsf{s}} \oplus \mathfrak{k}_{\mathbf{e}'})/\mathfrak{s}_\phi \longrightarrow \mathcal{X}_{\mathsf{s},\mathsf{s}'} \longrightarrow \mathfrak{p}_{\mathsf{s}'}/\mathfrak{p}_2' \longrightarrow 0$$

of representations of $(K_{\mathsf{s}} \times K_{\mathsf{s}'})_\phi$. By Lemma 3.2, we have an exact sequence

$$0 \to \mathfrak{k}_2' \to \mathfrak{s}_\phi \to \mathfrak{k}_{\mathbf{e}} \to 1$$

of representations of $(K_{\mathsf{s}} \times K_{\mathsf{s}'})_\phi$. Therefore, as representations of $(K_{\mathsf{s}} \times K_{\mathsf{s}'})_\phi$, we have that

$$\begin{aligned}
& \bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}} \otimes \bigwedge^{\mathrm{top}} \mathcal{X}_{\mathsf{s},\mathsf{s}'} \\
\cong\ & \bigwedge^{\mathrm{top}} (\mathfrak{p}_{\mathsf{s}'}/\mathfrak{p}_2') \otimes \bigwedge^{\mathrm{top}} (\mathfrak{k}_{\mathsf{s}} \oplus \mathfrak{k}_{\mathbf{e}'}) \otimes \left(\bigwedge^{\mathrm{top}} \mathfrak{k}_2'\right)^{-1} \\
\cong\ & \bigwedge^{\mathrm{top}} (\mathfrak{p}_{\mathsf{s}'}/\mathfrak{p}_2') \otimes \bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathsf{s}} \otimes \left(\bigwedge^{\mathrm{top}} \mathfrak{k}_2'\right)^{-1} \otimes \bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}'}.
\end{aligned}$$

The lemma then follows since $\mathfrak{s}_\phi$ acts trivially on $\bigwedge^{\mathrm{top}} (\mathfrak{p}_{\mathsf{s}'}/\mathfrak{p}_2') \otimes \bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathsf{s}} \otimes \left(\bigwedge^{\mathrm{top}} \mathfrak{k}_2'\right)^{-1}$. $\square$

Lemma 6.23. *Suppose that $\mathscr{O}$ is a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ that is contained in the image of M_{s}, and $\mathscr{O}' = \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) \subset \mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$. Then for every $\mathcal{E}' \in \mathrm{AOD}_{\mathsf{s}'}(\mathscr{O}')$, the geometric theta lift $\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$ is either zero or an element of $\mathrm{AOD}_{\mathsf{s}}(\mathscr{O})$.*

Proof. Note that $(\zeta_{\mathsf{s},\mathsf{s}'})^2 \cong (\bigwedge^{\mathrm{top}} \mathcal{X}_{\mathsf{s},\mathsf{s}'})^{-1}$ as representations of $\mathfrak{k}_{\mathsf{s}} \times \mathfrak{k}_{\mathsf{s}'}$, and $\mathcal{E}'$ is represented by a line bundle. Thus the lemma is a direct consequence of Lemma 6.22. $\square$

7. Equating the associated cycles and a proof of Theorem 3.12

As before, $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ are classical signatures such that $\star'$ is the Howe dual of $\star$.

7.1. Good descents and doubling method.

Definition 7.1. *An nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is good for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ if it is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ and satisfies the following condition:*

$$\begin{cases} \mathbf{c}_1(\mathcal{O}) > \mathbf{c}_2(\mathcal{O}), & \text{if } \star \in \{B, D\}; \\ |\mathsf{s}'| - |\nabla_{\mathrm{naive}}(\mathcal{O})| \in \{0, 1\}, & \text{if } \star \in \{C, \widetilde{C}\}; \\ |\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

In the rest of this section we suppose that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is good for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Put $\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})$ as before. Write

$$p_0 := q_0 := \begin{cases} |\mathsf{s}| - |\mathsf{s}'| - 1, & \text{if } \star \in \{B, D\}; \\ |\mathsf{s}| - |\mathsf{s}'| + 1, & \text{if } \star \in \{C, \widetilde{C}\}; \\ |\mathsf{s}| - |\mathsf{s}'|, & \text{if } \star \in \{C^*, D^*\}, \end{cases}$$

which is a non-negative integer. Similar to (5.22), we have that

$$-p_0 - 1 = \begin{cases} \nu_{\mathsf{s}'} - |\mathsf{s}| + 2, & \text{if } \star = C^*; \\ \nu_{\mathsf{s}'} - |\mathsf{s}|, & \text{otherwise}. \end{cases} \tag{7.1}$$

Thus π' is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ whenever π' is a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is ν'-bounded for some $\nu' > -p_0 - 1$.

We will prove the following theorem in this section.

Theorem 7.2. *Let π' be an $\mathcal{O}'$-bounded Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is ν'-bounded for some $\nu' > -p_0 - 1$. Then $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$, $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ and $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi'^{\mathrm{alg}})$ are all $\mathcal{O}$-bounded, and*

$$\mathrm{AC}_{\mathcal{O}}(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')) = \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')) = \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi'^{\mathrm{alg}})) = \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi')) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

Define three classical signatures

$$\dot{\mathsf{s}} := (\dot{\star}, \dot{p}, \dot{q}) := \begin{cases} (\star', |\mathsf{s}'| + p_0, |\mathsf{s}'| + q_0), & \text{if } \star' \neq B; \\ (D, |\mathsf{s}'| + p_0, |\mathsf{s}'| + q_0), & \text{if } \star' = B, \end{cases}$$

$$\mathsf{s}'' := (\star', p' + p_0, q' + q_0) \qquad \text{and} \qquad \mathsf{s}_0 := (\dot{\star}, p_0, q_0).$$

Then we have decompositions

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}'} \oplus V_{\mathsf{s}_0} = (V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0}) \oplus (V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}).$$

as in (5.8).

Theorem 7.2 is obviously true when $|\mathsf{s}| \leqslant 1$. Thus we assume in the rest of this subsection that $|\mathsf{s}| \geqslant 2$. As in (5.20), we have that

$$\dot{p} = \dot{q} = \begin{cases} |\mathsf{s}| - 1, & \text{if } \star \in \{B, D\}; \\ |\mathsf{s}| + 1, & \text{if } \star \in \{C, \widetilde{C}\}; \\ |\mathsf{s}| & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

Thus we are in the situation of Section 5.3 (with $|\mathsf{s}|$ equals the integer k of Section 5.3).

Let

$$P_{\mathsf{s}'',\mathsf{s}_0} = R_{\mathsf{s}'',\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0} = (G_{\mathsf{s}'} \cdot R_{\mathsf{s}_0}) \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$$

be the parabolic subgroup of $G_{\mathsf{s}''}$ as in (5.9) and (5.10). Write $\mathfrak{r}_{\mathsf{s}_0}$ and $\mathfrak{n}_{\mathsf{s}'',\mathsf{s}_0}$ for the complexified Lie algebras of R_{s_0} and $N_{\mathsf{s}'',\mathsf{s}_0}$ respectively so that the complexified Lie algebra of $P_{\mathsf{s}'',\mathsf{s}_0}$ equals $(\mathfrak{g}_{\mathsf{s}'} \times \mathfrak{r}_{\mathsf{s}_0}) \ltimes \mathfrak{n}_{\mathsf{s}'',\mathsf{s}_0}$. Define

$$\mathcal{O}'' := \mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}} \mathcal{O}'$$

to be the unique nilpotent orbit in $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}''})$ that contains a non-empty Zariski open subset of $\mathcal{O}' + \mathfrak{n}_{\mathsf{s}'',\mathsf{s}_0}$ (see [26, Theorem 7.1.1]).

Lemma 7.3. *The nilpotent orbit $\mathcal{O}''$ is good for $\nabla^{\mathsf{s}''}_{\mathsf{s}}$ and $\nabla^{\mathsf{s}''}_{\mathsf{s}}(\mathcal{O}'') = \mathcal{O}$.*

Proof. This follows from the description of induced orbits in [26, Section 7.3]. □

Lemma 7.4. *Suppose that $\mathcal{E} \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ and $0 \preceq \mathcal{E} \preceq \check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathcal{E}')$ for some $\mathcal{E}' \in \mathcal{K}^+_{\mathsf{s}'}(\mathcal{O}')$. If $\mathcal{E} \neq 0$, then $\check{\vartheta}^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s},\mathcal{O}}(\mathcal{E}) \neq 0$.*

Proof. Without loss of generality we may assume that $\mathcal{E}'$ is irreducible and supported on a nilpotent orbit $\mathscr{O}' \in \mathrm{Nil}_{\mathsf{s}'}(\mathcal{O}')$. We also can assume that $\mathcal{E}$ is supported on a nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathsf{s}}(\mathcal{O})$. By the definition of geometric lift, $\mathscr{O}' = \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O})$. Using Proposition 8.1, one can check that there is a nilpotent orbit $\mathscr{O}'' \in \mathrm{Nil}_{\mathsf{s}''}(\mathcal{O}'')$ such that $\mathscr{O} = \nabla^{\mathsf{s}''}_{\mathsf{s}}(\mathscr{O}'')$. When $\mathcal{O} = \nabla_{\mathrm{naive}}(\mathcal{O}'')$, the non-vanishing of $\check{\vartheta}^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s},\mathcal{O}}(\mathcal{E})$ follows from the inejctivity of $\check{\vartheta}^{\mathscr{O}''}_{\mathscr{O}}$. When $|\mathcal{O}| > |\nabla_{\mathrm{naive}}(\mathcal{O}'')|$, we have $\star \in \{B, D\}$, $\mathscr{O}' = \nabla_{\mathrm{naive}}(\mathscr{O})$ and so $\mathcal{E} = \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$. Now is straightforward to check that $\vartheta^{\mathsf{s},\mathscr{O}''}_{\mathsf{s},\mathscr{O}}(\mathcal{E}) \neq 0$. □

7.2. **Weak associated cycles.** For every classical signature s_1 and every $\mathcal{O}_1 \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}_1})$, write $\mathcal{K}^{\mathrm{weak}}_{\mathsf{s}_1}(\mathcal{O}_1)$ for the free abelian group generated by the set of K_{s_1}-orbits in $\mathcal{O}_1 \cap \mathfrak{p}_{\mathsf{s}_1}$. Define a homomorphism

$$\mathcal{K}_{\mathsf{s}_1}(\mathcal{O}_1) \to \mathcal{K}^{\mathrm{weak}}_{\mathsf{s}_1}(\mathcal{O}_1), \qquad \mathcal{E} \mapsto \mathcal{E}^{\mathrm{weak}}$$

such that if $\mathcal{E}$ is represented by a $K_{\mathsf{s}_1,\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}_0$ over a $K_{\mathsf{s}_1,\mathbb{C}}$-orbit $\mathscr{O}_1 \subset \mathcal{O}_1 \cap \mathfrak{p}_{\mathsf{s}_1}$, then

$$\mathcal{E}^{\mathrm{weak}} = (\text{rank of } \mathcal{E}_0) \cdot \mathscr{O}_1.$$

For all $\mathsf{s}_1 \in S_{\star,|\mathsf{s}|}$, $\mathfrak{g}_{\mathsf{s}_1}$ is identified with $\mathfrak{g}_{\mathsf{s}}$ and we consider $\mathcal{O}$ also as an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}_1})$.

Note that for every Cassleman-Wallach representation π_1 of $R_{\mathsf{s}'',\mathsf{s}_0} = G_{\mathsf{s}'} \cdot R_{\mathsf{s}_0}$ such that $(\pi_1)|_{G_{\mathsf{s}'}}$ is an $\mathcal{O}'$-bounded Casselman-Wallach representation, the parabolic induction $\mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}} \pi_1$ is an $\mathcal{O}''$-bounded Casselman-Wallach representation of $G_{\mathsf{s}''}$ (see [11, Corollary 5.0.10]).

Proposition 7.5. *Let π' be an $\mathcal{O}'$-bounded Casselman-Wallach representation of $G_{\mathsf{s}'}$, and let $\chi_1 : R_{\mathsf{s}_0} \to \mathbb{C}^\times$ be a character. Assume that π' is genuine when $\star' = \widetilde{C}$.*
(a) If $\star \in \{B, D\}$, then

$$\bigoplus_{\mathsf{s}_1 \in S_{\star,|\mathsf{s}|}} \left(\check{\vartheta}^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s}_1,\mathcal{O}}(\check{\vartheta}^{\mathsf{s}_1,\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi')))\right)^{\mathrm{weak}} = 2 \cdot \left(\mathrm{AC}_{\mathcal{O}'}(\mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}}(\pi' \otimes \chi_1))\right)^{\mathrm{weak}}.$$

(b) If $\star \in \{C, \widetilde{C}\}$, then

$$\begin{aligned}&\left(\check{\vartheta}^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s},\mathcal{O}}(\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi')))\right)^{\mathrm{weak}} + \left(\check{\vartheta}^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s},\mathcal{O}}(\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi' \otimes \det)))\right)^{\mathrm{weak}}\\ = &\left(\mathrm{AC}_{\mathcal{O}''}(\mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}}(\pi' \otimes \chi_1))\right)^{\mathrm{weak}}.\end{aligned}$$

(c) If $\star = D^*$, then

$$\left(\check{\vartheta}^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s},\mathcal{O}}(\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi')))\right)^{\mathrm{weak}} = \left(\mathrm{AC}_{\mathcal{O}''}(\mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}}(\pi' \otimes \chi_1))\right)^{\mathrm{weak}}.$$

(d) If $\star = C^*$, then

$$\bigoplus_{\mathsf{s}_1 \in S_{\star,|\mathsf{s}|}} \left(\check{\vartheta}^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s}_1,\mathcal{O}}(\check{\vartheta}^{\mathsf{s}_1,\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi')))\right)^{\mathrm{weak}} = \left(\mathrm{AC}_{\mathcal{O}'}(\mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}}(\pi' \otimes \chi_1))\right)^{\mathrm{weak}}.$$

Proof. The proposition follows by comaparing Lemma 8.6 with the direct computations of the left hand side of the equations. □

Lemma 7.6. *Let π' be an $\mathcal{O}'$-bounded Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is ν'-bounded for some $\nu' > -p_0 - 1$. Then*

$$\begin{aligned}&\left(\bigoplus_{\mathsf{s}_1 \in S_{\star,|\mathsf{s}|}} \mathrm{AC}_{\mathcal{O}''}(\bar{\Theta}^{\mathsf{s}''}_{\mathsf{s}_1}(\bar{\Theta}^{\mathsf{s}_1}_{\mathsf{s}'}(\pi')))\right)^{\mathrm{weak}}\\ = &\left(\bigoplus_{\mathsf{s}_1 \in S_{\star,|\mathsf{s}|}} \vartheta^{\mathsf{s}'',\mathcal{O}''}_{\mathsf{s}_1,\mathcal{O}}(\vartheta^{\mathsf{s}_1,\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\pi')))\right)^{\mathrm{weak}}.\end{aligned}$$

Proof. There is no loss of generality to assume that π' is genuine when $\star' = \widetilde{C}$. When $\star \neq C^*$, the lemma is a direct consequence of Theorem 5.12 and Proposition 7.5.

Now assume that $\star = C^*$. Let $\mathcal{J}$ be as in Theorem 5.12. In view of Lemma 6.5, and using formulas in [28, Theorems 5.2 and 5.6], one checks that $\mathcal{J}$ is $\mathcal{O}_2$-bounded for a certain oribit $\mathcal{O}_2 \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}''})$ that is contained in the boundary $\overline{\mathcal{O}''} \backslash \mathcal{O}''$ of $\mathcal{O}''$. Thus $\mathcal{J}$ is $\mathcal{O}''$-bounded, and $\mathrm{AC}_{\mathcal{O}''}(\mathcal{J}) = 0$. Hence the lemma in the case when $\star = C^*$ is also a consequence of Theorem 5.12 and Proposition 7.5.

□

7.3. **A proof of Theorem 7.2.** We are now ready to prove Theorem 7.2. Let π' be as in Theorem 7.2. For every $\mathsf{s}_1 \in S_{\star,|s|}$ we have surjective $(\mathfrak{g}_{\mathsf{s}_1}, K_{\mathsf{s}_1})$-module homomorphisms

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi'^{\mathrm{alg}}) \to \left(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')\right)^{\mathrm{alg}} \to \left(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')\right)^{\mathrm{alg}}.$$

Thus Theorem 6.17 implies that $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\rho')$, $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')$ and $\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')$ are all $\mathcal{O}$-bounded, and

$$\mathrm{AC}_{\mathcal{O}}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')) \preceq \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')) \preceq \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\rho')) \preceq \check{\vartheta}_{\mathsf{s}',\mathcal{O}'}^{\mathsf{s}_1,\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi')). \tag{7.2}$$

Consequently,

$$\mathrm{AC}_{\mathcal{O}}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')) \preceq \check{\vartheta}_{\mathsf{s}',\mathcal{O}'}^{\mathsf{s}_1,\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi')). \tag{7.3}$$

By Lemma 5.11, $\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')$ is over-convergent for $\check{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}$ so that $\bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi'))$ is defined. Similar to (7.3), we have that

$$\mathrm{AC}_{\mathcal{O}''}(\bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi'))) \preceq \check{\vartheta}_{\mathsf{s}_1,\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\mathrm{AC}_{\mathcal{O}}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi'))). \tag{7.4}$$

Combining (7.3) and (7.4), we obtain that

$$\mathrm{AC}_{\mathcal{O}''}(\bar{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi'))) \preceq \check{\vartheta}_{\mathsf{s}_1,\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\check{\vartheta}_{\mathsf{s}',\mathcal{O}'}^{\mathsf{s}_1,\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi'))). \tag{7.5}$$

Then Lemma 7.6 implies that the equality in (7.5) holds. In particular,

$$\mathrm{AC}_{\mathcal{O}''}(\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi'))) = \check{\vartheta}_{\mathsf{s},\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\check{\vartheta}_{\mathsf{s}',\mathcal{O}'}^{\mathsf{s},\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi'))). \tag{7.6}$$

In view of (7.3), write

$$\check{\vartheta}_{\mathsf{s}',\mathcal{O}'}^{\mathsf{s},\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi')) = \mathrm{AC}_{\mathcal{O}}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) + \mathcal{E}, \qquad \text{where } \ \mathcal{E} \in \mathcal{K}_{\mathsf{s}}^{+}(\mathcal{O}).$$

Then

$$\begin{aligned} \check{\vartheta}_{\mathsf{s},\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\check{\vartheta}_{\mathsf{s}',\mathcal{O}'}^{\mathsf{s},\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi'))) &= \mathrm{AC}_{\mathcal{O}''}(\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi'))) \\ &\preceq \check{\vartheta}_{\mathsf{s},\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\mathrm{AC}_{\mathcal{O}}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi'))) \\ &\preceq \check{\vartheta}_{\mathsf{s},\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\mathrm{AC}_{\mathcal{O}}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) + \mathcal{E}) \\ &= \check{\vartheta}_{\mathsf{s},\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\check{\vartheta}_{\mathsf{s}',\mathcal{O}'}^{\mathsf{s},\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi'))). \end{aligned} \tag{7.7}$$

In particular, the equality holds in (7.7). Therefore $\check{\vartheta}_{\mathsf{s},\mathcal{O}}^{\mathsf{s}'',\mathcal{O}''}(\mathcal{E}) = 0$, and Lemma 7.4 imply that $\mathcal{E} = 0$. This proves Theorem 7.2.

7.4. **A proof of Theorem 3.12.** As in Section 3.5, $\check{\mathcal{O}}$ is a Young diagram that has $\star$-good parity, and $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}_{\star}^{\mathrm{ext}}(\check{\mathcal{O}}, \mathsf{s})$. We have the dual descent

$$\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}}),$$

and the descent

$$\boldsymbol{\tau}' := (\tau', \wp') := \nabla(\boldsymbol{\tau}) := (\nabla(\tau), \check{\nabla}(\wp)) \in \mathrm{PBP}^{\mathrm{ext}}{}_{\star'}(\check{\mathcal{O}}').$$

Set

$$\mathrm{d}'_{\tau} := \dim \tau'.$$

Suppose that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ equals the Barbasch-Vogan dual of $\check{\mathcal{O}}$. Recall that π_τ is the Casselman-Wallach representation of G_{s}, defined in the introductory section. Also recall the element $\mathrm{AC}(\tau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ from Section 3.4.

Recall the character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g}_{\mathsf{s}})^{G_{\mathsf{s},\mathbb{C}}} \to \mathbb{C}$ defined in the Introduction.

Lemma 7.7. *The algebra* $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})^{G_{\mathsf{s},\mathbb{C}}}$ *acts on* π_τ *through the character* $\chi(\check{\mathcal{O}})$.

Proof. By induction, this is implied by [73, Theorem 1.19]. □

Recall the ideal $I_{\check{\mathcal{O}}} \subset \mathrm{U}(\mathfrak{g}_{\mathsf{s}})$ from the Introduction.

Lemma 7.8. *Let* π *be an* $\mathcal{O}$*-bounded irreducible Casselman-Wallach representation of* G_{s} *such that* $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})^{G_{\mathsf{s},\mathbb{C}}}$ *acts on it through the character* $\chi(\check{\mathcal{O}})$. *Then* $\mathrm{AC}_{\mathcal{O}}(\pi) \neq 0$, *and* $I_{\check{\mathcal{O}}}$ *annihilated* π.

Proof. This is implied by [20, Korollar 3.6] and [86, Theorem 8.4].

□

We will prove in the rest of this subsection the following theorem, which, together with Lemmas 7.7 and 7.8, implies Theorem 3.12.

Theorem 7.9. *The representation* π_τ *is irreducible, unitarizable,* $\mathcal{O}$*-bounded and* $(\mathrm{d}'_\tau - \nu_{\mathsf{s}})$*-bounded. Moreover,*

$$\mathrm{AC}_{\mathcal{O}}(\pi_\tau) = \mathrm{AC}(\tau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

We prove Theorem 7.9 by induction on $\mathbf{c}_1(\check{\mathcal{O}})$ (which equals the number of nonempty rows of $\check{\mathcal{O}}$). Theorem 7.9 is obvious when $\mathbf{c}_1(\check{\mathcal{O}}) = 0$. Now we assume $\mathbf{c}_1(\check{\mathcal{O}}) > 0$ and Theorem 7.9 holds for smaller $\mathbf{c}_1(\check{\mathcal{O}})$.

Lemma 7.10. *The orbit* $\mathcal{O}$ *is good for* $\nabla^{\mathsf{s}}_{\mathsf{s}_{\tau'}}$, *and* $\nabla^{\mathsf{s}}_{\mathsf{s}_{\tau'}}(\mathcal{O})$ *equals the Barbasch-Vogan dual of* $\check{\mathcal{O}}'$.

Proof. The first assertion is routine to check, and the second one is Lemma 3.6. □

In view of Lemma 7.10, we assume that $\mathsf{s}' = \mathsf{s}_{\tau'}$. Then

$$\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) = \text{the Barbasch-Vogan dual of } \check{\mathcal{O}}'.$$

By the induction hypothesis, the representation $\pi_{\tau'}$ is irreducible, unitarizable, $\mathcal{O}'$-bounded and $(\mathrm{d}'_{\tau'} - \nu_{\mathsf{s}'})$-bounded, and

$$\mathrm{AC}_{\mathcal{O}}(\pi_{\tau'}) = \mathrm{AC}(\tau') \in \mathcal{K}_{\mathsf{s}'}(\mathcal{O}').$$

It is routine to check case by case that

$$\mathrm{d}'_{\tau'} - \nu_{\mathsf{s}'} > \max\{\nu^\circ_{\mathsf{s}'} - |\mathsf{s}|, -p_0 - 1\} = \begin{cases} \nu^\circ_{\mathsf{s}'} - |\mathsf{s}| + 1, & \text{if } \star = C^*; \\ \nu^\circ_{\mathsf{s}'} - |\mathsf{s}|, & \text{otherwise.} \end{cases}$$

Thus $\pi_{\tau'} \otimes \chi$ is over-convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ for every unitary character χ of G_{s}.

Recall from (1.7) that

$$\pi_\tau := \begin{cases} \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi_{\tau'}) \otimes (1^{+,-}_{p_\tau,q_\tau})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi_{\tau'} \otimes \det^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi_{\tau'}), & \text{if } \star = C^* \text{ or } D^*. \end{cases}$$

Similarly define

$$\bar{\pi}_\tau := \begin{cases} \bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi_{\tau'}) \otimes (1^{+,-}_{p_\tau,q_\tau})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi_{\tau'} \otimes \det^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\bar{\Theta}}^{\mathsf{s}}_{\mathsf{s}'}(\pi_{\tau'}), & \text{if } \star = C^* \text{ or } D^*. \end{cases}$$

Theorem 4.14 implies that $\bar{\pi}_{\tau}$ is unitarizable. Theorem 7.2 and Proposition 3.7 imply that $\bar{\pi}_{\tau}$ and π_{τ} are $\mathcal{O}$-bounded, and

$$\mathrm{AC}_{\mathcal{O}}(\bar{\pi}_{\tau}) = \mathrm{AC}_{\mathcal{O}}(\pi_{\tau}) = \mathrm{AC}(\tau) \neq 0.$$

Since $\bar{\pi}_{\tau}$ is a completely reducible quotient of π_{τ}, and π_{τ} has at most one irreducible quotient by the fundamental result of Howe ([43, Theorem 1A]), we conclude that $\bar{\pi}_{\tau}$ is irreducible.

Write $\mathcal{J}_{\tau}$ for the kernel of the natural surjective homomorphism $\pi_{\tau} \to \bar{\pi}_{\tau}$. By Lemma 7.7, $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})^{G_{\mathsf{s},\mathbb{C}}}$ acts on $\mathcal{J}_{\tau}$ through the character $\chi(\check{\mathcal{O}})$. Moreover, $\mathcal{J}_{\tau}$ is $\mathcal{O}$-bounded and $\mathrm{AC}(\mathcal{J}_{\tau}) = 0$. This implies that $\mathcal{J}_{\tau} = 0$ by Lemma 7.8. Thus $\pi_{\tau} = \bar{\pi}_{\tau}$. Finally, Proposition 4.11 implies that $\bar{\pi}_{\tau}$ is $(\mathrm{d}'_{\tau} - \nu_{\mathsf{s}})$-bounded. This finishes the proof of Theorem 7.9.

8. Nilpotent orbits: odds and ends

In this section, we review the combinatorial parametrization of nilpotent orbits, and describe several constructions related to nilpotent orbits in combinatorial terms.

8.1. **Signed Young diagrams and the descent map.**

Let $(V_{\mathsf{s}}, \langle\, ,\, \rangle\, J_{\mathsf{s}}, L_{\mathsf{s}})$ be a classical space with signature s, as in Section 3.

The set $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ of nilpotent $G_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathsf{s}}$ is parameterized by Young diagrams: the nilpotent obit $G_{\mathsf{s},\mathbb{C}} \cdot \mathbf{e}$ generated by a nilpotent element $\mathbf{e}$ in $\mathfrak{g}_{\mathsf{s}}$ is attached to the Young diagram $\mathcal{O}$ such that

$$\mathbf{c}_i(\mathcal{O}) = \dim(\mathrm{Ker}(\mathbf{e}^i)/\,\mathrm{Ker}(\mathbf{e}^{i-1})) \quad \text{for all } \ i \in \mathbb{N}^+.$$

See [26, Section 5.1]. By abuse of notation, we identify the Young diagram $\mathcal{O}$ with the orbit $G_{\mathsf{s},\mathbb{C}} \cdot \mathbf{e}$. Let $\mathsf{YD}_{\star}$ denote the set of all Young diagrams of type $\star$.

Let $\mathfrak{sl}_2(\mathbb{C})$ be the complex Lie algebra consisting of 2×2 complex matrices of trace zero. Put

$$\mathring{\mathbf{e}} := \begin{pmatrix} 1/2 & \sqrt{-1}/2 \\ \sqrt{-1}/2 & -1/2 \end{pmatrix}, \quad \text{and} \quad \mathring{X} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

By the Jacobson-Morozov theorem, there is a Lie algebra homomorphism

$$\varphi \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}_{\mathsf{s}} \subset \mathfrak{gl}(V_{\mathsf{s}}) \quad \text{such that } \varphi(\mathring{\mathbf{e}}) = \mathbf{e}. \tag{8.1}$$

Let $\mathrm{St}_i = \mathrm{Sym}^{i-1}(\mathbb{C}^2)$ denote the realization of the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module φ_i on the $(i-1)^{\mathrm{th}}$-symmetric power of the standard representation $\mathbb{C}^2$.

As an $\mathfrak{sl}_2(\mathbb{C})$-module via φ, we have

$$V_{\mathsf{s}} = \bigoplus_{i \in \mathbb{N}^+} {}^{i}V \otimes_{\mathbb{C}} \mathrm{St}_i, \tag{8.2}$$

where

$${}^{i}V := \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathrm{St}_i, V_{\mathsf{s}})$$

is the multiplicity space and $\dim {}^{i}V = \mathbf{c}_i(\mathcal{O}) - \mathbf{c}_{i-1}(\mathcal{O})$. We view $\mathcal{O}$ as the function

$$\mathcal{O} \colon \mathbb{N}^+ \to \mathbb{N} \text{ such that } \mathcal{O}(i) = \dim {}^{i}V.$$

We equip St_i with a "standard" classical space structure $(\mathrm{St}_i, \langle\, ,\, \rangle_{\mathrm{St}_i}, J_{\mathrm{St}_i}, L_{\mathrm{St}_i})$. Specifically,

- the classical signature of St_i is $(B, (i+1)/2, (i-1)/2)$ if i is odd and $(C, i/2, i/2)$ if i is even,
- J_{St_i} is the complex conjugation on $\mathrm{St}_i = \mathrm{Sym}^{i-1}(\mathbb{C}^2)$,
- $L_{\mathrm{St}_i} = (-1)^{\lfloor \frac{i-1}{2} \rfloor}\, \mathrm{Sym}^{i-1}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$,
- $\langle\, ,\, \rangle_{\mathrm{St}_i}$ is uniquely determined by J_{St_i} and L_{St_i} up to a positive scalar. (We fix one for each i.)

Suppose we have a rational nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathsf{s}}(\mathcal{O})$, and $\mathbf{e} \in \mathscr{O}$. Then the Lie-algebra homomorphism (8.1) can be made compatible with the classical space structure on V_{s} so that $\mathfrak{sl}_2(\mathbb{C})$-isotypic components ${}^iV \otimes \mathrm{St}_i$ are invariant under the J_{s} and L_{s} actions. See [78] (and also [86, Section 6]). When this is done, the multiplicity space ${}^iV$ will have a unique classical space structure $({}^iV, \langle\, ,\, \rangle_{{}^iV}, J_i, L_i)$ with classical signature $\mathsf{s}_i = (\star_i, p_i, q_i)$ such that $\star_i \neq \widetilde{C}$ and when restricted on ${}^iV \otimes \mathrm{St}_i$, the following conditions hold:

- $\langle\, ,\, \rangle$ is given by $\langle\, ,\, \rangle_{{}^iV} \otimes \langle\, ,\, \rangle_{\mathrm{St}_i}$,
- J_{s} is given by $J_i \otimes J_{\mathrm{St}_i}$,
- L_{s} is given by $L_i \otimes L_{\mathrm{St}_i}$,
- the symbol $\star_i$ satisfies the following "type" condition

$$\begin{cases} \star_i \sim \star_{\mathsf{s}} & \text{if } i \text{ is odd} \\ \star_i \sim \star'_{\mathsf{s}} & \text{if } i \text{ is even} \end{cases} \tag{8.3}$$

Here $\star'_{\mathsf{s}}$ is the Howe dual of $\star_{\mathsf{s}}$, and $\sim$ is the equivalent relation on $\{ B, C, \widetilde{C}, C^*, D, D^* \}$ such that $B \sim D$ and $C \sim \widetilde{C}$. Moreover the classical signature s_i (for each $i \in \mathbb{N}^+$) is independent of the choice of φ and together they uniquely determine the orbit $\mathscr{O}$.

Let

$$\mathsf{CS} := \{ (\star, p, q) \mid \star \neq \widetilde{C} \text{ and } (\star, p, q) \text{ is a classical signature} \}.$$

We identify $\mathscr{O}$ with the function

$$\mathscr{O} \colon \mathbb{N}^+ \to \mathsf{CS}, \qquad i \mapsto \mathsf{s}_i.$$

Let $\mathsf{m} \colon \mathsf{CS} \to \mathbb{N}$ be the map given by $(\star, p, q) \mapsto p + q$. The complexification $G_{\mathsf{s},\mathbb{C}} \cdot \mathscr{O}$ of a rational nilpotent orbit $\mathscr{O}$ is then given by $\mathcal{O} := \mathsf{m} \circ \mathscr{O}$.

We now recall the signed Young diagram classification of rational nilpotent orbits [26, Chapter 9]. Define

$$\begin{aligned} \mathsf{SYD}_\star(\mathcal{O}) &:= \left\{ \mathscr{O} \colon \mathbb{N}^+ \to \mathsf{CS} \;\middle|\; \begin{array}{l} \star_{\mathscr{O}(i)} \text{ satisfies (8.3),} \\ \mathsf{m} \circ \mathscr{O} = \mathcal{O} \end{array} \right\} \\ \mathsf{SYD}_\star &:= \bigsqcup_{\mathcal{O} \in \mathrm{Nil}_\star} \mathsf{SYD}_\star(\mathcal{O}). \end{aligned}$$

Then $\mathsf{SYD}_\star$ is naturally identified with the set $\bigsqcup_{\mathsf{s}} \mathrm{Nil}_{K_{\mathsf{s}}}(\mathfrak{p}_{\mathsf{s}})$ where s runs over all classical signatures of type $\star$.

An important property of a compatible map φ is that the Kostant-Sekiguchi correspondence is given by

$$G_{\mathsf{s}} \cdot \varphi(\mathring{X}) \longleftrightarrow K_{\mathsf{s}} \cdot \varphi(\mathring{\mathbf{e}}).$$

Via this correspondence, we may therefore make the following identification:

$$\mathrm{Nil}_{G_{\mathsf{s}}}(\mathfrak{g}_{\mathsf{s},\mathbb{R}}) = \mathsf{SYD}_{\mathsf{s}} = \mathrm{Nil}_{K_{\mathsf{s}}}(\mathfrak{p}_{\mathsf{s}}). \tag{8.4}$$

For any $\mathscr{O} \in \mathsf{SYD}_\star$, we define $\mathrm{Sign}(\mathscr{O}) = (p, q)$, again called the signature, if $\mathscr{O}$ corresponds to a nilpotent orbit in $\mathrm{Nil}_{\mathsf{s}}(\mathfrak{p}_{\mathsf{s}})$ with the classical signature $\mathsf{s} = (\star, p, q)$. Also we write

$$\mathsf{SYD}_{\mathsf{s}}(\mathcal{O}) := \{ \mathscr{O} \in \mathsf{SYD}_\star(\mathcal{O}) \mid \mathrm{Sign}(\mathscr{O}) = (p, q) \}.$$

which is identified with $\mathrm{Nil}_{\mathsf{s}}(\mathcal{O})$.

The signature $\mathrm{Sign}(\mathscr{O})$ can be computed from the signatures of $\mathscr{O}(i)$ ($i \in \mathbb{N}^+$) by the following formula:

$$\begin{aligned}\mathrm{Sign}(\mathscr{O}) = &\sum_{i\in\mathbb{N}^+} \left(i \cdot p_{2i} + i \cdot q_{2i}, i \cdot p_{2i} + i \cdot q_{2i}\right) \\ &+ \sum_{i\in\mathbb{N}^+} \left(i \cdot p_{2i-1} + (i-1) \cdot q_{2i-1}, (i-1) \cdot p_{2i-1} + i \cdot q_{2i-1}\right),\end{aligned}$$

where $\mathscr{O}(i) = (\star_i, p_i, q_i)$. The signature ${}^l\mathrm{Sign}(\mathscr{O})$ of the "leftmost column" of the signed Young diagram $\mathscr{O}$ (see Example 8.3) is also useful, which is computed by

$${}^l\mathrm{Sign}(\mathscr{O}) = \sum_{i\in\mathbb{N}^+} \left(q_{2i}, p_{2i}\right) + \sum_{i\in\mathbb{N}^+} \left(p_{2i-1}, q_{2i-1}\right).$$

For each $\mathscr{O} \in \mathsf{SYD}_\star$ define the naive descent $\nabla_{\mathrm{naive}}(\mathscr{O}) \in \mathsf{SYD}_{\star'}$ by

$$\left(\nabla_{\mathrm{naive}}(\mathscr{O})\right)(i) = \mathscr{O}(i+1) \quad \text{for } i \in \mathbb{N}^+.$$

We can now explicitly describe the decent map $\nabla^{\mathsf{s}}_{\mathsf{s}'}$.

Proposition 8.1. *Suppose $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star, p', q')$. A nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{K_\mathsf{s}}(\mathfrak{p}_\mathsf{s})$ is in the image of moment map if and only if*

$$(p_0, q_0) := (p', q') - \mathrm{Sign}(\nabla_{\mathrm{naive}}(\mathscr{O})) \geq (0, 0). \tag{8.5}$$

When (8.5) is satisfied, the descent $\mathscr{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O})$ is given by

- *$\mathscr{O}'(1) = (\star'_1, p_2 + p_0, q_2 + q_0)$ where $\star'_1 \sim \star'$ and $(\star_2, p_2, q_2) = \mathscr{O}(2)$;*
- *$\mathscr{O}'(i) = \mathscr{O}(i+1)$, for $i \geqslant 2$.* □

8.2. **Marked Young diagram classification of equivariant line bundles.** Suppose $\mathscr{O} \in \mathrm{Nil}_{K_\mathsf{s}}(\mathfrak{p}_\mathsf{s})$ and φ is compatible with $\mathscr{O}$. Let $\mathbf{e} := \varphi(\mathring{\mathbf{e}}) \in \mathscr{O}$,

$$K_{[\mathsf{s}],\mathbb{C}} := G_{\mathsf{s},\mathbb{C}}^{L_\mathsf{s}}, \quad \text{and} \quad (K_{[\mathsf{s}],\mathbb{C}})_\mathbf{e} := \mathrm{Stab}_{K_{[\mathsf{s}],\mathbb{C}}}(\mathbf{e}),$$

the stabilizer of $\mathbf{e}$ in $K_{[\mathsf{s}],\mathbb{C}}$. Using the signed Young diagram classification of $\mathscr{O}$, we have the following well-known fact on the isotropy subgroup $(K_{[\mathsf{s}],\mathbb{C}})_\mathbf{e}$.

Lemma 8.2. *The isotropy subgroup have decomposition $(K_{[\mathsf{s}],\mathbb{C}})_\mathbf{e} = R_\mathbf{e} \ltimes U_\mathbf{e}$, where $U_\mathbf{e}$ is the unipotent radical of $(K_{[\mathsf{s}],\mathbb{C}})_\mathbf{e}$ and $R_\mathbf{e}$ is a reductive group canonically identified with*

$$\prod_{i\in\mathbb{N}^+} K_{[\mathsf{s}_{\mathscr{O}(i)}],\mathbb{C}}.$$

Consequently $R_\mathbf{e}$ is a product of complex general linear groups with complex symplectic groups or orthogonal groups. □

Recall that K_s is a double cover of $K_{[\mathsf{s}],\mathbb{C}}$ if $\star = \widetilde{C}$ and $K_\mathsf{s} = K_{[\mathsf{s}],\mathbb{C}}$ otherwise. Fix a one-dimensional character χ of $(K_\mathsf{s})_\mathbf{e}$. We assume χ is a genuine character if $\star = \widetilde{C}$.

Let $(K_\mathsf{s})^\circ_\mathbf{e}$ be the connected component of $(K_\mathsf{s})_\mathbf{e}$ and

$$\mathsf{LB}_\chi(\mathscr{O}) := \left\{\, K_\mathsf{s}\text{-equivariant line bundle } \mathcal{E} \text{ on } \mathscr{O} \text{ such that } \mathcal{E}_\mathbf{e} |_{(K_\mathsf{s})^\circ_\mathbf{e}} = \chi |_{(K_\mathsf{s})^\circ_\mathbf{e}} \,\right\}.$$

By the structure of $R_\mathbf{e}$, a line bundle $\mathcal{E} \in \mathsf{LB}_\chi(\mathscr{O})$ is completely determined by the restriction of the $(K_\mathsf{s})_\mathbf{e}$-module $\mathcal{E}_\mathbf{e}$ on the orthogonal group factors of $R_\mathbf{e}$. We introduce marked Young diagrams to parameterize the restrictions.

Let

$$\widetilde{\mathsf{CS}} = \mathsf{CS} \cup \{\, (\star, r, s) \mid \star \in \{\, B, D \,\}, r, s \in \mathbb{Z} \,\}.$$

Suppose $\mathscr{O}(i) = (\star_i, p_i, q_i)$. We identify $\mathcal{E} \in \mathsf{LB}_\chi(\mathscr{O})$ with the map

$$\mathcal{E} \colon \mathbb{N}^+ \to \widetilde{\mathsf{CS}}$$

such that

- $\mathcal{E}(i) := \mathscr{O}(i)$ if $\star_i \notin \{B, D\}$;
- $\mathcal{E}(i) := (\star_i, (-1)^{\epsilon_i^+} p_i, (-1)^{\epsilon_i^-} q_i)$ if $\star_i \in \{B, D\}$ and the factor $K_{[\mathcal{E}(i)],\mathbb{C}} = \mathrm{O}_{p_i}(\mathbb{C}) \times \mathrm{O}_{q_i}(\mathbb{C})$ acts by $\det^{\epsilon_i^+} \otimes \det^{\epsilon_i^-}$ on
$$\begin{cases} \mathcal{E}, & \text{when } \star = D, \\ \mathcal{E} \otimes \det_{\sqrt{-1}}^{-1}, & \text{when } \star = B. \end{cases}$$

Let

$$\mathscr{F}\colon \widetilde{\mathsf{CS}} \to \mathsf{CS} \qquad (\star, r, s) \mapsto (\star, |r|, |s|), \tag{8.6}$$

and

$$\mathsf{MYD}(\mathscr{O}) := \left\{ \mathcal{E}\colon \mathbb{N}^+ \to \widetilde{\mathsf{CS}} \,\middle|\, \mathscr{F} \circ \mathcal{E} = \mathscr{O} \right\}.$$

From the structure of $R_{\mathbf{e}}$, the above recipe will then yield an identification of $\mathsf{MYD}(\mathscr{O})$ with the set $\mathsf{LB}_\chi(\mathscr{O})$ whenever the later set is non-empty.

Let

$$\mathsf{MYD}_{\mathsf{s}}(\mathcal{O}) := \bigcup_{\mathscr{O} \in \mathsf{SYD}_{\mathsf{s}}(\mathcal{O})} \mathsf{MYD}(\mathscr{O}).$$

Recall the set

$$\mathrm{AOD}_{\mathsf{s}}(\mathcal{O}) = \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathrm{AOD}_{K_{\mathsf{s}}}(\mathscr{O}),$$

where $\mathrm{AOD}_{\mathsf{s}}(\mathscr{O})$ is the set of isomorphism classes of admissible orbit data over $\mathscr{O}$ (Definition 1.9). We may thus identify $\mathrm{AOD}_{\mathsf{s}}(\mathcal{O})$ with a subset of $\mathsf{MYD}_{\mathsf{s}}(\mathcal{O})$. We also set

$$\mathsf{MYD}_\star := \bigcup_{\mathsf{s}, \mathcal{O} \in \mathsf{YD}_\star} \mathsf{MYD}_{\mathsf{s}}(\mathcal{O})$$

where s runs over all possible classical signatures of type $\star$.

The map $\mathscr{F}$ natrually extends to a homomorphism

$$\mathscr{F}\colon \mathbb{Z}[\mathsf{MYD}_\star] \to \mathbb{Z}[\mathsf{SYD}_\star] \tag{8.7}$$

sending $\mathcal{E} \in \mathsf{MYD}_\star$ to $\mathscr{F} \circ \mathcal{E} \in \mathsf{SYD}_\star$.

Suppose $\mathcal{E} \in \mathsf{MYD}_\star$ such that $\mathcal{E}(i) = (\mathsf{s}_i, p_i, q_j)$. To ease the notation, we will use

$$(i_1^{(p_{i_1}, q_{i_1})} i_2^{(p_{i_2}, q_{i_2})} \cdots i_k^{(p_{i_k}, q_{i_k})})_\star$$

to represent $\mathcal{E}$ where $i_1, \cdots, i_k$ are all the indices i such that $(p_i, q_i) \neq (0, 0)$. A Young diagram in $\mathsf{YD}_\star$ is represented similarly.

We adopt the usual convention to depict a signed Young diagram where the signature (p_i, q_i) equals the total signature for the rightmost columns of length i rows. We use "$\times$" and "$\diagup$" instead of "$+$" and "$-$" to mark the rows when p_i or q_i is negative.

Example 8.3. *The expression $(1^{(r,s)})_\star$ represents the marked Young diagram in $\mathsf{MYD}_\star$ such that*

$$(1^{(r,s)})_\star(i) := \begin{cases} (\star, r, s) & \text{if } i = 1 \\ (\star_i, 0, 0) & \text{if } i > 0 \end{cases}$$

Here the symbols $\star_i$ are uniquely determined by (8.3).

The marked Young diagram $\mathcal{E} := (1^{(1,-1)} 2^{(2,0)} 3^{(0,-1)})_B$ is depicted as follows:

$$\begin{array}{|c|c|c|} \hline \diagup & \times & \diagup \\ \hline - & + & \multicolumn{1}{c}{} \\ \cline{1-2} - & + & \multicolumn{1}{c}{} \\ \cline{1-2} + & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \diagup & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The singed Young diagram $\mathscr{O} := \mathscr{F}\circ\mathcal{E}$ is $(1^{(1,1)}2^{(2,0)}3^{(0,1)})_B$ and the complexification of $\mathscr{O}$ is $(1^2\,2^2\,3^1)_B$.

8.3. **Theta lifts of admissible orbit data: combinatorial descriptions.** In this section, we will first define some operations on marked Young diagrams. We then use these operations to describe theta lifts of admissible orbit data.

Suppose $\star \in \{B, D\}$. We define a $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ action on $\mathsf{MYD}_\star$, as follows. Given $\mathcal{E}\in\mathsf{MYD}_\star$ such that $(\star_i, p_i, q_i) := \mathcal{E}(i)$, and $(\epsilon^+,\epsilon^-)\in\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, we define $\mathcal{E}\otimes(\epsilon^+,\epsilon^-)\in\mathsf{MYD}_\star$ by

$$(\mathcal{E}\otimes(\epsilon^+,\epsilon^-))(i) := \begin{cases} (\star_i, (-1)^{\frac{\epsilon^+(i+1)+\epsilon^-(i-1)}{2}}p_i, (-1)^{\frac{\epsilon^+(i-1)+\epsilon^-(i+1)}{2}}q_i) & \text{if } i \text{ is odd,}\\ (\star_i, p_i, q_i) & \text{otherwise.}\end{cases}$$

For each $\mathcal{E}\in\mathrm{AOD}_{K_{\mathsf{s}}}(\mathcal{O})$, its twist $\mathcal{E}\otimes(1^{-,+}_{p,q})^{\epsilon^+}\otimes(1^{+,-}_{p,q})^{\epsilon^-}$ is also in $\mathrm{AOD}_{K_{\mathsf{s}}}(\mathcal{O})$. As elements in $\mathsf{MYD}_\star$, we have

$$\mathcal{E}\otimes(1^{-,+}_{p,q})^{\epsilon^+}\otimes(1^{+,-}_{p,q})^{\epsilon^-} = \mathcal{E}\otimes(\epsilon^+,\epsilon^-).$$

Suppose $\star\in\{C,\widetilde{C},D^*\}$. We define an involution $\maltese$ on $\mathsf{MYD}_\star$: For $\mathcal{E}\in\mathsf{MYD}_\star$ such that $(\star_i,p_i,q_i) := \mathcal{E}(i)$, we define $\maltese\mathcal{E}\in\mathsf{MYD}_\star$ such that

$$(\maltese\mathcal{E})(i) := \begin{cases}(\star_i, -p_i, -q_i) & \text{if } i\equiv 2 \pmod 4 \text{ and } \star_i\in\{B,D\},\\ (\star_i,p_i,q_i) & \text{otherwise.}\end{cases}$$

For each $\mathcal{E}\in\mathsf{LB}_\chi(\mathscr{O})$, its twist $\mathcal{E}\otimes\det^{-1}_{\sqrt{-1}}$ is a line bundle in $\mathsf{LB}_{\chi'}(\mathscr{O})$ where χ' is a certain character. As elements in $\mathsf{MYD}_\star$, we have

$$\mathcal{E}\otimes\det{}^{-1}_{\sqrt{-1}} = \maltese(\mathcal{E}).$$

We now define the argumentation operation on marked Young diagrams. Suppose that $(p_0,q_0)\in\mathbb{Z}\times\mathbb{Z}$ and there is a $\star'_0\sim\star'$ such that $\mathsf{s}_0 := (\star'_0, p_0, q_0)\in\mathsf{CS}$. For each $\mathcal{E}\in\mathsf{MYD}_\star$, let $\mathcal{E}\cdot(p_0,q_0)$ be the marked Young diagram in $\mathsf{MYD}_{\star'}$ given by

$$\big(\mathcal{E}\cdot(p_0,q_0)\big)(i) := \begin{cases}\mathsf{s}_0 & \text{if } i=1,\\ \mathcal{E}(i-1) & \text{if } i>1.\end{cases}$$

We will use the same notation to denote the natural extension of above operation to $\mathbb{Z}[\mathsf{MYD}_\star]$.

We introduce a partial order $\succeq$ on $\mathbb{Z}\times\mathbb{Z}$ by

$$(a,b)\succeq(c,d)\Leftrightarrow\begin{cases}a\geqslant c\geqslant 0\\ b\geqslant d\geqslant 0\end{cases} \quad\text{or}\quad \begin{cases}a\leqslant c\leqslant 0\\ b\leqslant d\leqslant 0\end{cases}.$$

We also define an involution on $\mathbb{Z}\times\mathbb{Z}$ by $(a,b)\mapsto(a,b)^\sharp := (b,a)$. For a $\mathcal{E}\in\mathsf{MYD}_\star$ and $(p_0,q_0)\in\mathbb{Z}\times\mathbb{Z}$, we write

$$\mathcal{E}\sqsupseteq(p_0,q_0)\Leftrightarrow(p_1,q_1)\succeq(p_0,q_0) \text{ where } \mathcal{E}(1)=(\star_1,p_1,q_1). \tag{8.8}$$

Suppose $\mathcal{E}\sqsupseteq(p_0,q_0)$, we define the truncation $\Lambda_{(p_0,q_0)}\mathcal{E}$ by

$$(\Lambda_{(p_0,q_0)}(\mathcal{E}))(i) = \begin{cases}(\star_1, p_1-p_0, q_1-q_0) & \text{if } i=1,\\ \mathcal{E}(i) & \text{if } i>1.\end{cases}$$

We extend the truncation operation $\Lambda_{(p_0,q_0)}$ to $\mathbb{Z}[\mathsf{MYD}_\star]$ by setting $\Lambda_{p_0,q_0}(\mathcal{E}) := 0$ when $(p_0,q_0)\not\sqsubseteq\mathcal{E}\in\mathsf{MYD}_\star$. Note that $\Lambda_{(0,0)}$ is the identity map.

The augmentation and truncation of a signed Young diagram are defined by the same formula.

We are now ready to describe theta lifting of admissible orbit data in terms of marked Young diagrams.

Let s and s' be two fixed classical signatures. Suppose $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ and $\mathcal{O}' \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ such that $\mathcal{O}' = \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})$. The theta lifting of marked Young diagrams is a homomorphism

$$\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'} \colon \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}'}(\mathcal{O}')] \to \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}}(\mathcal{O})]$$

defined as follows. Set

$$t := \mathbf{c}_1(\mathcal{O}') - \mathbf{c}_2(\mathcal{O}).$$

Suppose $\mathsf{s}' = (\star', n, n)$ and $\mathsf{s} = (\star, p, q)$ such that $\star' \in \{ C, \widetilde{C}, D^* \}$ and $\star \in \{ D, B, C^* \}$. For $\mathcal{E}' \in \mathsf{MYD}_{\mathsf{s}}(\mathcal{O}')$, we define

$$\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathcal{E}') := \begin{cases} \left(\maltese^{\frac{p-q}{2}}(\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}'))\right) \cdot (p_0, q_0) & \text{if } (p_0,q_0) \geq (0,0) \text{ and } \star = D, \\ \left(\maltese^{\frac{p-q+1}{2}}(\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}'))\right) \cdot (p_0, q_0) & \text{if } (p_0,q_0) \geq (0,0) \text{ and } \star = B, \\ \left(\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}')\right) \cdot (p_0, q_0) & \text{if } (p_0,q_0) \geq (0,0) \text{ and } \star = C^*. \\ 0 & \text{otherwise} \end{cases} \tag{8.9}$$

Here $(p_0, q_0) := (p, q) - (n, n) - {}^l\mathrm{Sign}(\mathcal{E}')^\sharp + (\frac{t}{2}, \frac{t}{2})$. Note that t is a nonnegative even integer for the mentioned cases.

Suppose $\mathsf{s}' = (\star', p, q)$ and $\mathsf{s} = (\star, n, n)$ such that $\star' \in \{ D, B, C^* \}$ and $\star \in \{ C, \widetilde{C}, D^* \}$. For $\mathcal{E}' \in \mathsf{MYD}_{\mathsf{s}}(\mathcal{O}')$, we define

$$\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}(\mathcal{E}') := \begin{cases} \maltese^{\frac{p-q}{2}}\left(\sum_{j=0}^{t} \Lambda_{(j,t-j)}(\mathcal{E}') \cdot (n_0, n_0)\right) & \text{if } \star = C, \\ \maltese^{\frac{p-q-1}{2}}\left(\sum_{j=0}^{t} \Lambda_{(j,t-j)}(\mathcal{E}') \cdot (n_0, n_0)\right) & \text{if } \star = \widetilde{C}, \\ \sum_{j=0}^{\frac{t}{2}} \Lambda_{(2j,t-2j)}(\mathcal{E}') \cdot (n_0, n_0) & \text{if } \star = D^*. \end{cases} \tag{8.10}$$

Here $2n_0 = \mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O})$.

Suppose $\mathcal{O}$ is regular with respect to $\nabla^{\mathsf{s}}_{\mathsf{s}'}$, then admissible orbit data are lifted to admissible orbit data, in view of Lemma 6.23. The following lemma is now routine to check, which we skip.

Lemma 8.4. *Suppose $\mathcal{O}$ is regular with respect to $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Then the following diagram is commute*

$$\begin{array}{ccc} \mathbb{Z}[\mathrm{AOD}_{\mathsf{s}'}(\mathcal{O}')] & \xrightarrow{\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}} & \mathbb{Z}[\mathrm{AOD}_{\mathsf{s}}(\mathcal{O})] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}'}(\mathcal{O}')] & \xrightarrow{\check{\vartheta}^{\mathsf{s},\mathcal{O}}_{\mathsf{s}',\mathcal{O}'}} & \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}'}(\mathcal{O}')]. \end{array}$$

Here the vertical arrows are given by the parameterization of admissible orbit data of this section, and the top horizontal map is defined in Section 3.3. □

8.4. **Induced orbits and double geometric lifts.** In this section, we describe the weak associated cycles of the induced representations appearing in Theorem 5.12.

We first recall the notion of induction of real nilpotent orbits. Let G_1 be an arbitrary real reductive group and let P_1 be a real parabolic subgroup of G_1, namely a closed subgroup of G_1 whose Lie algebra is a parabolic subalgebra of $\mathfrak{g}_{1,\mathbb{R}}$. Let N_1 be the unipotent radical of P_1 and $R_1 := P_1/N_1$. Let $\mathfrak{p}_{1,\mathbb{R}} := \mathrm{Lie}(P_1)$, $\mathfrak{r}_{1,R} := \mathrm{Lie}(R_1)$

$$r_1 : \mathfrak{p}_{1,\mathbb{R}} \longrightarrow \mathfrak{r}_{1,\mathbb{R}}$$

be the natural quotient map.

Define a morphism

$$\mathrm{Ind}_{R_1}^{G_1} \colon \mathbb{Z}[\mathrm{Nil}_{R_1}(\mathfrak{r}_{1,\mathbb{R}})] \longrightarrow \mathbb{Z}[\mathrm{Nil}_{G_1}(\mathfrak{g}_{1,\mathbb{R}})]$$

by

$$\mathrm{Ind}_{R_1}^{G_1}(\mathscr{O}'_{\mathbb{R}}) = \sum_{\mathscr{O}_{\mathbb{R}}} \frac{\#C_{G_1}(v)}{\#C_{P_1}(v)} \mathscr{O}_{\mathbb{R}}$$

where $\mathscr{O}_{\mathbb{R}}$ runs over all open orbits in $G_1 \cdot r_1^{-1}(\mathscr{O}'_{\mathbb{R}})$, $v \in \mathscr{O}_{\mathbb{R}}$, $C_{G_2}(v)$ and $C_{P_1}(v)$ are the component groups of the centralizers of v in G_1 and P_1 respectively. It is well-known that this definition only depends on the Levi subgroup R_1 but independent of the choice of the parobolic subgroup P_1.

Via the identification in (8.4), we translate $\mathrm{Ind}_{R_1}^{G_1}$ to a morphism

$$\mathrm{Ind}_{R_1}^{G_1} \colon \mathbb{Z}[\mathrm{Nil}_{K_{R_1,\mathbb{C}}}(\mathfrak{p}_{\mathfrak{r}_1})] \to \mathbb{Z}[\mathrm{Nil}_{K_{G_1,\mathbb{C}}}(\mathfrak{p}_{\mathfrak{g}_1})].$$

Here $K_{R_1,\mathbb{C}}$ and $K_{G_1,\mathbb{C}}$ are maximal compact subgroups of R_1 and G_1 respectively; $\mathfrak{p}_{\mathfrak{r}_1}$ and $\mathfrak{p}_{\mathfrak{g}_1}$ are the non-compact parts of $\mathfrak{r}_1$ and $\mathfrak{g}_1$ respectively.

We remark that the notion of induced nilpotent orbits was first introduced by Lusztig and Spaltenstein for complex reductive groups [57]. Let $r_{1,\mathbb{C}} \colon \mathfrak{p}_1 \to \mathfrak{r}_1$ be the natural quotient map where $\mathfrak{p}_1$ and $\mathfrak{r}_1$ are the complexification of $\mathfrak{p}_{1,\mathbb{R}}$ and $\mathfrak{r}_{1,\mathbb{R}}$ respectively. For a complex nilpotent orbit $\mathcal{O}' \in \mathrm{Nil}_{R_{1,\mathbb{C}}}(\mathbf{r}_1)$, there is a unique open orbit in $G_{1,\mathbb{C}} \cdot r_{1,\mathbb{C}}^{-1}(\mathcal{O}')$ which is, by definition, the induced orbit $\mathrm{Ind}_{R_{1,\mathbb{C}}}^{G_{1,\mathbb{C}}}(\mathcal{O}')$.

Lemma 8.5 (*cf.* Barbasch [11, Corollary 5.0.10])**.** *Let $\mathcal{O}' \in \mathrm{Nil}_{R_{1,\mathbb{C}}}(\mathbf{r}_1)$ and π' be an $\mathcal{O}'$-bounded Casselman-Wallach representation of R_1. Then $\mathrm{Ind}_{P_1}^{G_1} \pi'$ is $\mathcal{O} := \mathrm{Ind}_{R_{1,\mathbb{C}}}^{G_{1,\mathbb{C}}}(\mathcal{O}')$-bounded. Moreover*

$$\mathrm{WF}^{\mathrm{weak}}(\mathrm{Ind}_{P_1}^{G_1} \pi') = \mathrm{Ind}_{R_1}^{G_1}(\mathrm{WF}^{\mathrm{weak}}(\pi'))$$

and

$$\mathrm{AC}_{\mathcal{O}}^{\mathrm{weak}}(\mathrm{Ind}_{P_1}^{G_1} \pi') = \mathrm{Ind}_{R_1}^{G_1}(\mathrm{AC}_{\mathcal{O}'}^{\mathrm{weak}}(\pi')).$$

Proof. Barbasch proved the required formula for weak wavefront cycles when G_1 is the group of real points of a connected reductive algebraic group. The proof carries over in the slightly more general setting here.

According to the fundamental result of Schimd-Vilonen [79], the weak wave front cycle and the weak associated cycle agree under the Kostant-Sekiguchi correspondence. Therefore, we obtain the required formula for weak associated cycles. $\square$

Remark. We thank Professor Vilonen for confirming that the results of [79] carry over to nonlinear groups.

We retain the notation in Section 5.2 and proceed to describe inductions of certain nilpotent orbits in terms of signed Young diagram parameterization. These results are used in the proof of Proposition 7.5.

Let $\mathcal{O}' \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$. Suppose that $\mathsf{s}_0 = (\star_0, p_0, q_0)$ such that $p_0 = q_0 \geqslant \mathbf{c}_2(\mathcal{O}')$. Recall that $\mathsf{s}' = (\star', p', q')$, $\mathsf{s}'' = (\star', p' + p_0, q' + q_0)$ and the Levi subgroup $R_{\mathsf{s}'',\mathsf{s}_0} = R_{\mathsf{s}_0} \times G_{\mathsf{s}'}$. We identify $\mathscr{O}' \in \mathrm{Nil}_{G_{\mathsf{s}'}}(\mathfrak{g}_{\mathsf{s}'})$ with the nilpotent orbit $\{ 0 \} \times \mathscr{O}'$ of $R_{\mathsf{s}'',\mathsf{s}_0}$.

Lemma 8.6. *Let* $\mathcal{O}'' := \mathrm{Ind}_{R_{\mathsf{s}'',\mathsf{s}_0,\mathbb{C}}}^{G_{\mathsf{s}'',\mathbb{C}}}(\mathcal{O}')$, $\mathscr{O}' \in \mathrm{Nil}_{\mathsf{s}'}(\mathcal{O}')$ *and* $t = p_0 - \mathbf{c}_1(\mathcal{O}')$. *Then* $\mathcal{O}''$ *satisfies* $\mathbf{c}_i(\mathcal{O}'') = \mathbf{c}_{i-2}(\mathcal{O}')$ *for* $i \geqslant 4$.

(a) Suppose $\star' \in \{B, D\}$. *Then*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (p_0, p_0, \mathbf{c}_1(\mathcal{O}')) & \textit{if } t \geqslant 0 \textit{ is even,} \\ (p_0+1, p_0-1, \mathbf{c}_1(\mathcal{O}')) & \textit{if } t \geqslant 0 \textit{ is odd,} \\ (\mathbf{c}_1(\mathcal{O}'), p_0, p_0) & \textit{if } t < 0. \end{cases}$$

We have

$$\mathrm{Ind}_{R_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \mathscr{O}' \cdot (\frac{t}{2}, \frac{t}{2}) \cdot (0,0) & \textit{if } t \geqslant 0 \textit{ is even,} \\ 2\left(\mathscr{O}' \cdot (\frac{t-1}{2}, \frac{t-1}{2}) \cdot (1,1)\right) & \textit{if } t \geqslant 0 \textit{ is odd,} \\ \sum_{a+b=-t} (\Lambda_{(a,b)}\mathscr{O}') \cdot (0,0) \cdot (a,b) & \textit{if } t < 0. \end{cases}$$

(b) Suppose $\star' \in \{C, \widetilde{C}\}$. *Then*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (p_0, p_0, \mathbf{c}_1(\mathcal{O}')) & \textit{if } t \geqslant 0, \\ (\mathbf{c}_1(\mathcal{O}'), p_0, p_0) & \textit{if } t < 0 \textit{ is even,} \\ (\mathbf{c}_1(\mathcal{O}'), p_0+1, p_0-1) & \textit{if } t < 0 \textit{ is odd.} \end{cases}$$

We have

$$\mathrm{Ind}_{R_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \sum_{t=a+b} \mathscr{O}' \cdot (a,b) \cdot (0,0) & \textit{if } t \geqslant 0, \\ (\Lambda_{(-\frac{t}{2},-\frac{t}{2})}\mathscr{O}') \cdot (0,0) \cdot (-\frac{t}{2}, -\frac{t}{2}) & \textit{if } t < 0 \textit{ is even,} \\ \sum_{a+b=2} (\Lambda_{(-\frac{t-1}{2},-\frac{t-1}{2})}\mathscr{O}') \cdot (a,b) \cdot (\frac{1-t}{2}, \frac{1-t}{2}) & \textit{if } t < 0 \textit{ is odd.} \end{cases}$$

(c) Suppose $\star' \in \{C^*, D^*\}$. *Note that* p_0 *and* t *are even integers in this case. Then*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (p_0, p_0, \mathbf{c}_1(\mathcal{O}')) & \textit{if } t \geqslant 0, \\ (\mathbf{c}_1(\mathcal{O}'), p_0, p_0) & \textit{if } t < 0. \end{cases}$$

When $\star' = D^*$, *we have*

$$\mathrm{Ind}_{R_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \sum_{a+b=\frac{t}{2}} \mathscr{O}' \cdot (2a, 2b) \cdot (0,0) & \textit{if } t \geqslant 0, \\ \left(\Lambda_{(-\frac{t}{2},-\frac{t}{2})}\mathscr{O}'\right) \cdot (0,0) \cdot (-\frac{t}{2}, -\frac{t}{2}) & \textit{if } t < 0. \end{cases}$$

When $\star' = C^*$, *we have*

$$\mathrm{Ind}_{R_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \mathscr{O}' \cdot (\frac{t}{2}, \frac{t}{2}) \cdot (0,0) & \textit{if } t \geqslant 0, \\ \sum_{a+b=-\frac{t}{2}} \left(\Lambda_{(2a,2b)}\mathscr{O}'\right) \cdot (0,0) \cdot (2a, 2b) & \textit{if } t < 0. \end{cases}$$

□

9. Properties of painted bipartitions

In this section, we study properties of the descent maps. The results of this section are crucial for the calculation of associated cycles in Section 10.

9.1. **Vanishing proposition.**

Because of the following vanishing proposition, we will only need to consider quasi-distinguished nilpotent orbits $\check{\mathcal{O}}$ when $\star \in \{C^*, D^*\}$.

Proposition 9.1. *Suppose that* $\star \in \{C^*, D^*\}$. *If the set* $\mathrm{PBP}_\star(\check{\mathcal{O}})$ *is nonempty, then* $\check{\mathcal{O}}$ *is quasi-distinguished.*

Proof. Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. If $\star = C^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_i(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots .$$

This forces that $\check{\mathcal{O}}$ is quasi-distinguished.

If $\star = D^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_{i+1}(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots .$$

This also forces that $\check{\mathcal{O}}$ is quasi-distinguished. □

9.2. **Tails of painted bipartitions.** In this subsection, we assume that $\star \in \{B, D, C^*\}$ and define the notion of "tail" of a painted bipartition. Let $(\imath, \jmath) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}}))$. Put

$$\star_{\mathbf{t}} := \begin{cases} D, & \text{if } \star \in \{B, D\}; \\ C^*, & \text{if } \star = C^*. \end{cases} \quad \text{and} \quad k := \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} + 1 & \text{if } \star \in \{B, D\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} - 1 & \text{if } \star = C^*. \end{cases}$$

We have $k = \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, $\mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath)$, and $\mathbf{c}_1(\imath) - \mathbf{c}_1(\jmath)$ when $\star = B, D, C^*$, respectively.

From the pair $(\star, \check{\mathcal{O}})$, we define a Young diagram $\check{\mathcal{O}}_{\mathbf{t}}$ as follows:

- If $\star \in \{ B, D \}$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of two rows with lengths $2k - 1$ and 1.
- If $\star = C^*$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of one row with length $2k + 1$.

Note that in all three cases $\check{\mathcal{O}}_{\mathbf{t}}$ has $\star_{\mathbf{t}}$-good parity and every element in $\mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ has the form

$$\begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \varnothing \times D, \qquad \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \varnothing \times D \qquad \text{or} \qquad \varnothing \times \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times C^*, \tag{9.1}$$

according to $\star = B, D$ or C^*, respectively.

Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. The tail $\tau_{\mathbf{t}}$ of τ will be a painted bipartition in $\mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$, which is defined below case by case.

The case when $\star = B$: In this case, we define the tail $\tau_{\mathbf{t}}$ to be the first painted bipartition in (9.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{Q}(j, 1) \mid \mathbf{c}_1(\imath) + 1 \leqslant j \leqslant \mathbf{c}_1(\jmath) \}$$

with the set

$$\begin{cases} \{ c \}, & \text{if } \alpha = B^+, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ s \}, & \text{if } \alpha = B^-, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ \mathcal{Q}(\mathbf{c}_1(\imath), 1) \}, & \text{otherwise.} \end{cases}$$

The case when $\star = D$: In this case, we define the tail $\tau_{\mathbf{t}}$ to be the second painted bipartition in (9.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{P}(j, 1) \mid \mathbf{c}_1(\jmath) + 2 \leqslant j \leqslant \mathbf{c}_1(\imath) \}$$

with the set

$$\begin{cases} \{ c \}, & \text{if } \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}), \\ & \quad (\mathcal{P}(\mathbf{c}_1(\jmath) + 1, 1), \mathcal{P}(\mathbf{c}_1(\jmath) + 1, 2)) = (r, c) \\ & \quad \text{and } \mathcal{P}(\mathbf{c}_1(\imath), 1) \in \{ r, d \}; \\ \{ \mathcal{P}(\mathbf{c}_1(\jmath) + 1, 1) \}, & \text{otherwise.} \end{cases}$$

The case $\star = C^*$*:* If $k = 0$, we define the tail $\tau_{\mathbf{t}}$ to be $\varnothing \times \varnothing \times C^*$. If $k > 0$, we define the tail $\tau_{\mathbf{t}}$ to be the third painted bipartition in (9.1) such that

$$(x_1, x_2, \cdots, x_k) = (\mathcal{Q}(\mathbf{c}_1(\imath) + 1, 1), \mathcal{Q}(\mathbf{c}_1(\imath) + 2, 1), \cdots, \mathcal{Q}(\mathbf{c}_1(\jmath), 1)).$$

When $\star \in \{ B, D \}$, the symbol in the last box of the tail $\tau_{\mathbf{t}} \in \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ will be important for us. We write x_τ for it, namely

$$x_\tau := \mathcal{P}_{\tau_{\mathbf{t}}}(k, 1).$$

The following lemma is easy to check.

Lemma 9.2. *If* $\star = B$*, then*

$$x_\tau = s \Longleftrightarrow \begin{cases} \alpha = B^-; \\ \mathbf{c}_1(\jmath) = 0 \ \textit{or} \ \mathcal{Q}(\mathbf{c}_1(\jmath), 1) \in \{ \bullet, s \}, \end{cases}$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{Q}(\mathbf{c}_1(\jmath), 1) = d.$$

If $\star = D$*, then*

$$x_\tau = s \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath), 1) = s,$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath), 1) = d.$$

□

9.3. **Key properties of the descent map.** In this subsection, we establish some relevant properties of the descent of painted bipartitions. These properties will be used extensively in Section 10. Recall that $\check{\mathcal{O}}$ is assumed to be quasi-distinguished when $\star \in \{ C^*, D^* \}$.

The key properties of the descent map when $\star \in \{ C, \widetilde{C}, D^* \}$ are summarized in the following proposition.

Proposition 9.3. *Suppose that* $\star \in \{ C, \widetilde{C}, D^* \}$ *and consider the descent map*

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}'). \tag{9.2}$$

(a) If $\star = D^*$ *or* $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$*, then the map* (9.2) *is bijective.*

(b) If $\star \in \{C, \widetilde{C}\}$ *and* $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$*, then the map* (9.2) *is injective and its image equals*

$$\left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, x_{\tau'} \neq s \right\}.$$

Proof. We give the detailed proof of part (b) when $\star = \widetilde{C}$. The proofs in the other cases are similar, which we skip.

By the definition of descent map (see (2.1)), we have a well-defined map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\}. \tag{9.3}$$

Suppose that τ' is an element in the codomain of the map (9.3). Similar to the proof of Lemma 2.1, there is a unique element in $\tau := \nabla^{-1}(\tau') \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ such that for all $i = 1, 2, \cdots, \mathbf{c}_1(\imath)$,

$$\mathcal{P}_\tau(i, 1) \in \{\bullet, s\},$$

and for all $(i, j) \in \mathrm{Box}(\tau')$,

$$\mathcal{P}_\tau(i, j+1) = \begin{cases} \bullet \text{ or } s, & \text{if } \ \mathcal{P}_{\tau'}(i, j) \in \{\bullet, s\}; \\ \mathcal{P}_{\tau'}(i, j), & \text{if } \ \mathcal{P}_{\tau'}(i, j) \notin \{\bullet, s\}, \end{cases}$$

and

$$\mathcal{Q}_\tau(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}_{\tau'}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}_{\tau'}(i,j), & \text{if } \mathcal{Q}_{\tau'}(i,j) \notin \{\bullet, s\}. \end{cases}$$

Note that τ is in the domain of (9.3). It is then routine to check that the map

$$\nabla^{-1} : \{\, \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \mid \alpha_{\tau'} = B^+ \,\} \to \{\, \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \mid \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq c \,\}$$

and the map (9.3) are inverse to each other. Hence the map (9.3) is bijective.

Similarly, the map

$$\nabla : \{\, \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \mid \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) = c \,\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \begin{array}{l} \alpha_{\tau'} = B^- \text{ and} \\ \mathcal{Q}_{\tau'}(\mathbf{c}_1(\imath), 1) \in \{r, d\} \end{array} \right\} \tag{9.4}$$

is well-defined, and shown to be bijective by constructing its inverse. In view of Lemma 9.2, this proves the proposition for the case concerned. ☐

Now we discuss the case when $\star \in \{\, B, D, C^* \,\}$, which is more involved.

The following lemma deals with the case when the Barbasch-Vogan dual $\mathcal{O}$ of $\check{\mathcal{O}}$ has only one column. This is the most basic case, which can be verified easily.

Lemma 9.4. *Suppose $k \in \mathbb{N}^+$, $\star \in \{\, B, D, C^* \,\}$. Let $\check{\mathcal{O}}$ be the $\star$-good parity orbit such that $\mathbf{r}_3(\check{\mathcal{O}}) = 0$ and*

$$(\mathbf{r}_1(\check{\mathcal{O}}), \mathbf{r}_2(\check{\mathcal{O}})) = \begin{cases} (2k-2, 0) & \text{if } \star = B, \\ (2k-1, 1) & \text{if } \star = D, \\ (2k-1, 0) & \text{if } \star = C^*. \end{cases}$$

Then $\check{\mathcal{O}}$ is the regular nilpotent orbit and $\mathcal{O} = (1^{\mathbf{r}_1(\check{\mathcal{O}})+1})_\star$.

When $\star \in \{\, B, D \,\}$, the following map is bijective:

$$\begin{array}{ccl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \longrightarrow & \left\{ (p, q, \varepsilon) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z} \,\middle|\, \begin{array}{l} p + q = |\mathcal{O}| \text{ and} \\ \varepsilon = 1 \text{ if } pq = 0 \end{array} \right\}, \\ \tau & \mapsto & (p_\tau, q_\tau, \varepsilon_\tau). \end{array}$$

When $\star = C^*$, the following map is bijective:

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \{\, (p, q) \mid p, q \in 2\mathbb{N} \text{ and } p + q = |\mathcal{O}| \,\}, \quad \tau \mapsto (p_\tau, q_\tau).$$

For every painted bipartition $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, write

$$\mathrm{Sign}(\tau) := (p_\tau, q_\tau).$$

The key properties of the descent map when $\star \in \{\, D, B, C^* \,\}$ are summarized in the following lemma.

Lemma 9.5. *Suppose that $\star \in \{\, D, B, C^* \,\}$ and $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Write $\check{\mathcal{O}}'' := \check{\nabla}(\check{\mathcal{O}}')$ and consider the map*

$$\delta \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}), \qquad \tau \mapsto (\nabla^2(\tau), \tau_{\mathbf{t}}). \tag{9.5}$$

(a) Suppose that $\star = C^*$ or $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then the map (9.5) is bijective, and for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}), \mathbf{c}_2(\mathcal{O})) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{9.6}$$

(b) *Suppose that* $\star \in \{B, D\}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$*. Then the map* (9.5) *is an injection and its image equals*

$$\left\{ (\tau'', \tau_0) \in \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}}) \;\middle|\; \begin{array}{l} \textit{either } x_{\tau''} = d, \textit{ or} \\ x_{\tau''} \in \{r, c\} \textit{ and } \mathcal{P}_{\tau_0}^{-1}(\{s, c\}) \neq \varnothing \end{array} \right\}. \tag{9.7}$$

Moreover, for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$*,*

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{9.8}$$

Proof. We give the detailed proof of part (b) when $\star = B$. The proofs in the other cases are similar, which we skip. Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ and $\tau'' = \nabla^2(\tau)$.

According to the descent algorithm (see Section 2), we have

$$\begin{aligned} \mathcal{P}(i, j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{P}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 1; \end{cases} \\ \mathcal{Q}(i, j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{Q}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 2; \end{cases} \end{aligned} \tag{9.9}$$

We label the boxes in $\mathcal{P}$ and $\mathcal{Q}$ which are not considered in (9.9) as the following:

$$\tau : \quad \boxed{x_0} \quad \times \quad \begin{array}{|c|c|} \hline x_1 & x'' \\ \hline x_2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} x_k & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad \times \alpha \quad \mapsto \quad \tau'' : \quad \varnothing \quad \times \quad \boxed{y''} \quad \times \alpha'', \tag{9.10}$$

where $k := \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, x_0, x_1 and y'' have coordinate $(\mathbf{c}_1(\imath), 1)$ in the corresponding painted Young diagram. It is easy to check that $x'' = y''$ always holds. Also note that $\mathbf{c}_2(\mathcal{O}) = 2\mathbf{c}_1(\imath)$. To establish (9.8), we examine case by case according to the symbol x_1.

When $x_1 = s$, we have $(x_0, \alpha'') = (c, B^-)$. So

$$\begin{aligned} &\mathrm{Sign}(\tau) - \mathrm{Sign}(\tau'') \\ &= (2\mathbf{c}_1(\imath) - 2, 2\mathbf{c}_1(\imath) - 2) + (1, 1) + (0, 2) - (0, 1) + \mathrm{Sign}(\alpha) + \mathrm{Sign}(x_2 \cdots x_k) \\ &= (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\tau_{\mathbf{t}}). \end{aligned} \tag{9.11}$$

Here Sign counts the signature of various symbols in the same way as (1.5). In the second line of the above formula, the terms count the signatures of extra bullets, x_0, x_1, α'', α and $x_1 \cdots x_k$ respectively.

When $x_1 = \bullet$, we have $(x_0, \alpha'') = (\bullet, B^+)$. When $x_1 \in \{r, d\}$, we have $x'' = y'' = d$, $(x_0, \alpha'') = (c, \alpha)$. In these two cases, the signature formula can be checked similarly.

It is clear that the image of δ is in (9.7) and is straightforward to construct the inverse of δ from (9.7) to $\mathrm{PBP}_\star(\check{\mathcal{O}})$. To illustrate the idea, we give the detail construction of $\tau := (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha := \delta^{-1}(\tau'', \tau_0)$ when $\mathcal{P}_{\tau_0}(k, 1) = c$. We set $\alpha := B^-$. Most of the entries in $\mathcal{P}$ and $\mathcal{Q}$ are already determined by (9.9). We refer to (9.10) for the labeling of the rest of the entries in $\mathcal{P}$ and $\mathcal{Q}$. We set $x'' := \mathcal{Q}_{\tau''}(\mathbf{c}_1(\imath), 1)$,

$$(x_0, x_1) := \begin{cases} (\bullet, \bullet) & \text{if } \alpha'' = B^+, \\ (c, s) & \text{if } \alpha'' = B^-, \end{cases} \quad \text{and} \quad x_i := \mathcal{P}_{\tau_0}(i - 1, 1) \quad \text{for } i = 2, 3, \cdots, k.$$

This gives $\tau := \delta^{-1}(\tau'', \tau_0)$ and finishes the proof for the case concerned. □

Combining the injectivity results in Lemma 9.5 and Proposition 9.3, we get the following lemma.

Lemma 9.6. *Suppose that* $\star \in \{ D, B, C^* \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. *Then the map*

$$\delta' \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathrm{PBP}_{\star_\mathbf{t}}(\check{\mathcal{O}}_\mathbf{t}) \qquad \tau \mapsto (\nabla(\tau), \tau_\mathbf{t}) \tag{9.12}$$

is injective. Moreover, (9.12) *is bijective unless* $\star \in \{ D, B \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$.

Combining Lemmas 9.4, 9.5 and 9.6, we obtain the following proposition.

Proposition 9.7. *If* $\star \in \{ B, D, C^* \}$, *then the map*

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \rightarrow & \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array} \tag{9.13}$$

is injective. ☐

10. Properties of associated cycles: proofs

In this section, we study properties of the associated cycles $\mathrm{AC}(\uptau)$ defined in Section 3.4. Let $\uptau = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, $\uptau' := (\tau', \wp') := \nabla(\uptau)$ and $\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}})$.

10.1. Computing $\mathrm{AC}(\uptau)$ via $\mathcal{L}_\uptau$.

Lemma 10.1. *The cycle* $\mathrm{AC}(\uptau)$ *is an element in* $\mathbb{N}[\mathrm{AOD}_{\mathsf{s}_\tau}(\mathcal{O})]$.

Proof. We prove by induction on the number of rows in $\check{\mathcal{O}}$. The lemma clearly holds for the initial case by the definition in Section 3.4. For the general case, we have $\mathrm{AC}(\uptau') \in \mathbb{N}[\mathrm{AOD}_{\mathsf{s}_{\tau'}}(\mathcal{O}')]$ by the induction hypothesis and $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}_\tau}_{\mathsf{s}_{\tau'}}$ by Lemma 7.10. Therefore $\mathrm{AC}(\uptau) \in \mathbb{N}[\mathrm{AOD}_{\mathsf{s}_\tau}(\mathcal{O})]$ by Lemma 6.23 ☐

Let $\mathcal{L}_\uptau \in \mathbb{N}[\mathsf{MYD}_\star]$ be the element corresponding to the associated cycle $\mathrm{AC}(\uptau)$, as in Section 8.2. The study of $\mathrm{AC}(\uptau)$ is now translated to the corresponding combinatorial problem for $\mathcal{L}_\uptau$. Via Lemma 8.4, $\mathcal{L}_\uptau$ may also be defined combinatorially as follows:

- When $|\check{\mathcal{O}}| = 0$,
$$\mathcal{L}_\uptau := \begin{cases} (1^{\mathrm{Sign}(\tau)})_\star \otimes (0,1), & \text{if } \star = B; \\ (1^{(0,0)})_\star, & \text{otherwise.} \end{cases}$$
- When $|\check{\mathcal{O}}| > 0$,
$$\mathcal{L}_\uptau := \begin{cases} \check{\vartheta}^{\mathsf{s}_\tau, \mathcal{O}}_{\mathsf{s}_{\tau'}, \mathcal{O}'}(\mathcal{L}_{\uptau'} \otimes (\varepsilon_\wp, \varepsilon_\wp)) & \text{if } \star \in \{ C, \widetilde{C} \}; \\ \check{\vartheta}^{\mathsf{s}_\tau, \mathcal{O}}_{\mathsf{s}_{\tau'}, \mathcal{O}'}(\mathcal{L}_{\uptau'}) \otimes (0, \varepsilon_\tau) & \text{if } \star \in \{ B, D \}; \\ \check{\vartheta}^{\mathsf{s}_\tau, \mathcal{O}}_{\mathsf{s}_{\tau'}, \mathcal{O}'}(\mathcal{L}_{\uptau'}) & \text{if } \star \in \{ C^*, D^* \}. \end{cases}$$

  Here $\check{\vartheta}^{\mathsf{s}_\tau, \mathcal{O}}_{\mathsf{s}_{\tau'}, \mathcal{O}'}$ is defined by (8.9) and (8.10).

For $\star \in \{ B, D \}$ and $\mathcal{E} \in \mathbb{Z}[\mathsf{MYD}_\star]$, we write

$$\mathcal{E}^+ := \Lambda_{(1,0)}(\mathcal{E}) \quad \text{and} \quad \mathcal{E}^- := \Lambda_{(0,1)}(\mathcal{E}).$$

We adopt the convention that $\mathcal{E} \cdot \varnothing = 0$ for $\mathcal{E} \in \mathbb{Z}[\mathsf{MYD}]$. We have the following more explicit formulas for $\mathcal{L}_\uptau$.

Lemma 10.2. *Suppose that* $\check{\mathcal{O}}$ *such that* $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. *Let* $\uptau = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ *and* $\uptau' = \nabla(\uptau)$.

(a) Suppose that $\star \in \{ C, \widetilde{C} \}$. *Then*

$$\mathcal{L}_\uptau = \begin{cases} \maltese^y\big((\mathcal{L}_{\uptau'} \otimes (\varepsilon_\wp, \varepsilon_\wp)) \cdot (n_0, n_0)\big), & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}); \\ \maltese^y(\mathcal{L}^+_{\uptau'} \cdot (0,0)) + \maltese^y(\mathcal{L}^-_{\uptau'} \cdot (0,0)), & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}); \end{cases} \tag{10.1}$$

Here $n_0 = \frac{\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O})}{2}$, $y = \frac{p_{\tau'} - q_{\tau'}}{2}$ *if* $\star = C$ *and* $y = \frac{p_{\tau'} - q_{\tau'} - 1}{2}$ *if* $\star = \widetilde{C}$.

(b) Suppose that $\star \in \{B, D\}$ *and* $\mathbf{r}_2(\mathcal{O}) > \mathbf{r}_3(\mathcal{O})$. *Then*

$$\mathcal{L}_\tau = \big((\maltese^y \mathcal{L}_{\tau'}) \cdot (p_{\tau_\mathbf{t}}, q_{\tau_\mathbf{t}})\big) \otimes (0, \varepsilon_\tau) \tag{10.2}$$

Here $y = \frac{p_\tau - q_\tau + 1}{2}$ *if* $\star = B$ *and* $y = \frac{p_\tau - q_\tau}{2}$ *if* $\star = D$. *In particular,*

$$\mathcal{E}(1) = \mathcal{L}_{(\tau_\mathbf{t}, \varnothing)}(1) \tag{10.3}$$

for each $\mathcal{E} \in \mathsf{MYD}_\star$ *has nonzero multiplicity in* $\mathcal{L}_\tau$.

(c) Suppose that $\star \in \{B, D\}$ *and* $\mathbf{r}_2(\mathcal{O}) = \mathbf{r}_3(\mathcal{O})$. *Then*

$$\mathcal{L}_\tau = \big((\maltese^t(\mathcal{L}^+_{\tau''} \cdot (0,0)) \cdot \mathcal{T}^+_\tau + (\maltese^t(\mathcal{L}^-_{\tau''} \cdot (0,0))) \cdot \mathcal{T}^-_\tau\big) \otimes (0, \varepsilon_\tau). \tag{10.4}$$

Here $\tau'' = \nabla^2(\tau)$ *and*

$$t = \frac{|\mathrm{Sign}(\tau_\mathbf{t})|}{2},$$

$$\mathcal{T}^+_\tau = \begin{cases} (p_{\tau_\mathbf{t}}, q_{\tau_\mathbf{t}} - 1) & \textit{when } q_{\tau_\mathbf{t}} - 1 \geqslant 0, \\ \varnothing & \textit{otherwise}, \end{cases}$$

$$\mathcal{T}^-_\tau = \begin{cases} (p_{\tau_\mathbf{t}} - 1, q_{\tau_\mathbf{t}}) & \textit{when } p_{\tau_\mathbf{t}} - 1 \geqslant 0, \\ \varnothing & \textit{otherwise}. \end{cases}$$

(d) Suppose that $\star = D^*$. *Then*

$$\mathcal{L}_\tau = \mathcal{L}_{\tau'} \cdot (n_0, n_0),$$

where $n_0 = \frac{\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O})}{2}$.

(e) Suppose that $\star = C^*$. *Then*

$$\mathcal{L}_\tau = \mathcal{L}_{\tau'} \cdot (p_{\tau_\mathbf{t}}, q_{\tau_\mathbf{t}}).$$

Proof. The lemma follows from (8.9), (8.10), and the signature formulas (9.6), (9.8) in Lemma 9.5. □

10.2. **Properties of $\mathcal{L}_\tau$.** In this subsection, we summarize the key properties of $\mathcal{L}_\tau$. The proofs will be given in the next three sections, and will go by induction on the number of columns in $\mathcal{O}$.

Recall that $\check{\mathcal{O}}$ is quasi-distinguished when $\star = C^*, D^*$. See Proposition 9.1. All the claims of this subsection for $\star \in \{C^*, D^*\}$ are relatively straightforward to verify, and as such we shall only give the detailed proofs for the non-quaternionic cases in the next three sections.

We say an element $\mathcal{A}$ in $\mathbb{Z}[\mathsf{MYD}_\star]$ is multiplicity free if $\mathscr{F}(\mathcal{A})$ is a sum of distinct elements in $\mathsf{SYD}_\star$, where $\mathscr{F}$ is defined in (8.7).

The following lemma will imply the non-vanishing and multiplicity freeness of $\mathrm{AC}(\tau)$, as in Proposition 3.7.

Lemma 10.3. *For each* $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, $\mathcal{L}_\tau \neq 0$ *and* $\mathcal{L}_\tau$ *is multiplicity free.*

The following three lemmas will imply Proposition 3.8, Proposition 3.9, Proposition 3.10, and Proposition 3.11.

Define the map

$$\mathcal{L}\colon \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \longrightarrow \mathbb{Z}[\mathsf{MYD}_\star(\mathcal{O})], \quad \tau \mapsto \mathcal{L}_\tau.$$

Lemma 10.4. *Suppose that* $\star = C^*$. *Then the map* $\mathcal{L}$ *is injective with image* $\mathsf{MYD}_\star(\mathcal{O})$.

Lemma 10.5. *Suppose that* $\star \in \{C, \widetilde{C}, D^*\}$ *and* $\check{\mathcal{O}} \neq \varnothing$.

(a) *Let* $\tau_1 = (\tau_1, \wp_1)$ *and* $\tau_2 = (\tau_2, \wp_2)$ *be two elements in* $\mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$ *such that* $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. *Then*

$$\mathrm{Sign}(\tau_1') = \mathrm{Sign}(\tau_2') \quad \textit{and} \quad \varepsilon_{\wp_1} = \varepsilon_{\wp_2},$$

where $\tau_i' := \nabla(\tau_i)$ *for* $i = 1, 2$.

(b) *When* $\check{\mathcal{O}}$ *is weakly-distinguished, then the map* $\mathcal{L}$ *is injective.*

(c) *When* $\check{\mathcal{O}}$ *is quasi-distinguished, then the image of the map* $\mathcal{L}$ *is* $\mathsf{MYD}_\star(\mathcal{O})$.

Lemma 10.6. *Suppose* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$. *Define the map*

$$\tilde{\mathcal{L}}\colon \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}[\mathsf{MYD}_\star(\mathcal{O})], \quad (\tau, \epsilon) \mapsto \mathcal{L}_\tau \otimes (\epsilon, \epsilon).$$

(a) *Let* $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ *and* $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$ *such that*

$$\mathcal{L}_{\tau_1} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{\tau_2} \otimes (\epsilon_2, \epsilon_2).$$

Then

$$\epsilon_1 = \epsilon_2, \quad \varepsilon_{\tau_1} = \varepsilon_{\tau_2} \quad \textit{and} \quad \mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}.$$

(b) *When* $\check{\mathcal{O}}$ *is weakly-distinguished, then the map* $\tilde{\mathcal{L}}$ *is injective.*

(c) *When* $\check{\mathcal{O}}$ *is quasi-distinguished, then the image of the map* $\tilde{\mathcal{L}}$ *is* $\mathsf{MYD}_\star(\mathcal{O})$.

We will also need to use the following technical lemmas.

Lemma 10.7. *Suppose that* $\star \in \{B, D\}$ *and* $\check{\mathcal{O}} \neq \varnothing$.

(a) *If* $x_\tau = s$, *then* $\mathcal{L}^+_\tau = 0$ *and* $\mathcal{L}^-_\tau = 0$.

(b) *If* $x_\tau \in \{r, c\}$, *then* $\mathcal{L}^+_\tau \neq 0$ *and* $\mathcal{L}^-_\tau = 0$.

(c) *If* $x_\tau = d$, *then* $\mathcal{L}^+_\tau \neq 0$ *and* $\mathcal{L}^-_\tau \neq 0$.

Lemma 10.8. *Suppose that* $\star \in \{B, D\}$, $\check{\mathcal{O}} \neq \varnothing$ *and* $\check{\mathcal{O}}$ *is weakly-distinguished.*

(a) *The map*

$$\begin{array}{rcl} \Upsilon_{\check{\mathcal{O}}}\colon \{\mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \textit{ s.t. } \mathcal{L}^+_\tau \neq 0\} & \to & \mathbb{Z}[\mathsf{MYD}] \times \mathbb{Z}[\mathsf{MYD}] \\ \mathcal{L}_\tau & \mapsto & (\mathcal{L}^+_\tau, \mathcal{L}^-_\tau) \end{array} \tag{10.5}$$

is injective.

(b) *When* $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$, *the maps*

$$\begin{array}{rcll} \Upsilon^+_{\check{\mathcal{O}}}\colon \{\mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \textit{ s.t. } \mathcal{L}^+_\tau \neq 0\} & \to & \{\mathcal{L}^+_\tau\} & \\ \mathcal{L}_\tau & \mapsto & \mathcal{L}^+_\tau, & \textit{and} \\ \Upsilon^-_{\check{\mathcal{O}}}\colon \{\mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \textit{ s.t. } \mathcal{L}^+_\tau \neq 0\} & \to & \{\mathcal{L}^-_\tau\} & \\ \mathcal{L}_\tau & \mapsto & \mathcal{L}^-_\tau & \end{array}$$

are injective.

(c) *Suppose that* $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$ *and* $x_\tau = d$. *Then there is* $\mathcal{E} \in \mathsf{MYD}$ *with a non-zero coefficient in* $\mathcal{L}^-_\tau$ *such that* $\mathcal{E} \sqsupseteq (1, 0)$. *In particular, there does not exist* $\tau_2 \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ *such that*

$$\mathcal{L}^+_{\tau_2} \cdot (0, 0) = \maltese(\mathcal{L}^-_\tau \cdot (0, 0)). \tag{10.6}$$

10.3. **Establishing properties of $\mathcal{L}_\tau$: the initial cases.** Suppose $\mathbf{c}_1(\mathcal{O}) = 0$. Then the group $G_{\mathsf{s}_\tau, \mathbb{C}}$ is the trivial group and everything are clear.

We now assume that $\mathbf{c}_1(\mathcal{O}) > 0$ and $\mathbf{c}_2(\mathcal{O}) = 0$. Suppose that $\star \in \{C, \widetilde{C}\}$. There there is $k \in \mathbb{N}^+$ such that $\mathcal{O} = (1^{2k})_\star$ and $\check{\mathcal{O}}$ is the regular nilpotent orbit in $\check{\mathfrak{g}}$. Now $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ has only one element, say τ_0, for which we have $\mathcal{L}_{\tau_0} = (1^{(k,k)})_\star$.

Suppose that $\star \in \{B, D\}$. There is $k \in \mathbb{N}^+$ such that $\mathcal{O}$ and $\check{\mathcal{O}}$ are given as in Lemma 9.4. We also have

$$\mathcal{L}_\tau = (1^{(p_\tau, (-1)^{\varepsilon_\tau} q_\tau)})_\star, \quad \text{for all } \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}).$$

Thanks to Lemma 9.4, it is easy to see that

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \rightarrow & \mathsf{MYD}_\star(\mathcal{O}) \\ (\tau, \epsilon) & \mapsto & \mathcal{L}_\tau \otimes (\epsilon, \epsilon) \end{array}$$

is a bijection.

In all initial cases being considered, the orbit $\check{\mathcal{O}}$ is quasi-distinguished and the rest of claims in all lemmas of Section 10.2 are straightforward to verify.

10.4. **Establishing properties of $\mathcal{L}_\tau$: the general case for $\star \in \{ C, \widetilde{C} \}$.** Retain the notation of Lemma 10.5. We now assume $\mathbf{c}_2(\mathcal{O}) > 0$ and all the lemmas in Section 10.2 hold for $\check{\mathcal{O}}'$.

Suppose $\sum_{j=1}^b m_j \mathcal{E}_j \in \mathbb{Z}[\mathsf{MYD}_\star]$ such that $m_j \neq 0$ and $\mathcal{E}_j \in \mathsf{MYD}_\star$. We define the "descent signature" of $\sum_{j=1}^b m_j \mathcal{E}_j$ as a subset of $\mathbb{N} \times \mathbb{N}$:

$$^d\mathrm{Sign}(\sum_{j=1}^b m_j \mathcal{E}_j) := \{ \mathrm{Sign}(\mathscr{O}'_j) \mid j = 1, 2, \cdots, b \}.$$

Here $\mathscr{O}'_j \in \mathsf{SYD}_{\star'}$ is given by $\mathscr{O}'_j(i) := \mathscr{F}(\mathcal{E}_j)(i+1)$ for all $i \in \mathbb{N}^+$.

The case when $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$.

Proof of Lemma 10.3. Since $\mathcal{L}_{\tau'} \neq 0$ and is multiplicity free, $\mathcal{L}_\tau \neq 0$ and is multiplicity free by (10.1). This proves Lemma 10.3.

Proof of Lemma 10.5 (a). Suppose $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (10.1), $\{ \mathrm{Sign}(\tau'_1) \} = {}^d\mathrm{Sign}(\mathcal{L}_{\tau_1}) = {}^d\mathrm{Sign}(\mathcal{L}_{\tau_2}) = \{ \mathrm{Sign}(\tau'_2) \}$. In particular, $\mathrm{Sign}(\tau'_1) = \mathrm{Sign}(\tau'_2)$ and the integer "y" in (10.1) is the same for τ_1 and τ_2. By (10.1), we get $\mathcal{L}_{\tau'_1} \otimes (\varepsilon_{\wp_1}, \varepsilon_{\wp_1}) = \mathcal{L}_{\tau'_2} \otimes (\varepsilon_{\wp_1}, \varepsilon_{\wp_2})$. So $\varepsilon_{\wp_1} = \varepsilon_{\wp_2}$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$ by Lemma 10.6 (a) for $\mathcal{O}'$.

Proof of Lemma 10.5 (b). When $\check{\mathcal{O}}$ is weakly-distinguished, $\check{\mathcal{O}}'$ is also weakly-distinguished and we get $\tau'_1 = \tau'_2$ by Lemma 10.6 (b) for $\mathcal{O}'$. Hence $\tau_1 = \tau_2$ by Proposition 9.3.

Proof of Lemma 10.5 (c). By (10.1), $\check{\vartheta}^{\mathsf{s}_\tau, \mathcal{O}}_{\mathsf{s}_{\tau'}, \mathcal{O}'}$ induces a bijection from $\mathsf{MYD}_{\star'}(\mathcal{O}')$ to $\mathsf{MYD}_\star(\mathcal{O})$. When $\check{\mathcal{O}}$ is quasi-distinguished, $\check{\mathcal{O}}'$ is quasi-distinguished and the claim follows from Lemma 10.6 (c).

The case when $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}) > 0$.

Proof of Lemma 10.3. According to Proposition 9.3, $x_{\tau'} \neq s$. Thanks to Lemma 10.7, $\mathcal{L}^+_{\tau'} \neq 0$ and so $\mathcal{L}_\tau \neq 0$ by (10.1).

We now prove the multiplicity freeness of $\mathcal{L}_\tau$. To ease the notation, we write

$$\mathcal{L}^+_\tau := \maltese^y(\mathcal{L}^+_{\tau'} \cdot (0,0)) \quad \text{and} \quad \mathcal{L}^-_\tau := \maltese^y(\mathcal{L}^-_{\tau'} \cdot (0,0)).$$

Then

$$\begin{aligned} {}^d\mathrm{Sign}(\mathcal{L}_\tau) &= {}^d\mathrm{Sign}(\mathcal{L}^+_\tau) \cup {}^d\mathrm{Sign}(\mathcal{L}^-_\tau), \\ {}^d\mathrm{Sign}(\mathcal{L}^+_\tau) &= \{ \mathrm{Sign}(\tau') - (1,0) \}, \quad \text{and} \\ {}^d\mathrm{Sign}(\mathcal{L}^-_\tau) &= \begin{cases} \{ \mathrm{Sign}(\tau') - (0,1) \} & \text{if } x_{\tau'} = d, \\ \varnothing & \text{otherwise.} \end{cases} \end{aligned} \tag{10.7}$$

Since ${}^d\mathrm{Sign}(\mathcal{L}^+_\tau) \cap {}^d\mathrm{Sign}(\mathcal{L}^-_\tau) = \varnothing$, $\mathscr{F}(\mathcal{L}^+_\tau)$ and $\mathscr{F}(\mathcal{L}^-_\tau)$ have no common terms. Now the multiplicity freeness of $\mathcal{L}_\tau$ follows from the multiplicity freeness of $\mathcal{L}_{\tau'}$ and the fact that the operations $\Lambda_{(1,0)}$, $\Lambda_{(0,1)}$ and "$\cdot(0,0)$" preserve multiplicity freeness.

Proof of Lemma 10.5 (a). Suppose $\mathcal{L}_{\boldsymbol{\tau}_1} = \mathcal{L}_{\boldsymbol{\tau}_2}$. By (10.7), we conclude that $\mathrm{Sign}(\boldsymbol{\tau}'_1) = \mathrm{Sign}(\boldsymbol{\tau}'_2)$. Since we have $\varepsilon_{\wp_1} = \varepsilon_{\wp_2} = 0$, Lemma 10.5 (a) is proven.

Proof of Lemma 10.5 (b). By considering the ${}^d\mathrm{Sign}$, we get $\mathcal{L}^+_{\boldsymbol{\tau}_1} = \mathcal{L}^+_{\boldsymbol{\tau}_2}$ and $\mathcal{L}^-_{\boldsymbol{\tau}_1} = \mathcal{L}^-_{\boldsymbol{\tau}_2}$. So $\mathcal{L}^+_{\boldsymbol{\tau}'_1} = \mathcal{L}^+_{\boldsymbol{\tau}'_2}$ and $\mathcal{L}^-_{\boldsymbol{\tau}'_1} = \mathcal{L}^-_{\boldsymbol{\tau}'_2}$ by (10.1). Suppose $\check{\mathcal{O}}$ is weakly-distinguished, then $\check{\mathcal{O}}'$ is weakly-distinguished. Due to the injectivity of $\Upsilon_{\check{\mathcal{O}}'}$ in Lemma 10.8 (a), we get $\mathcal{L}_{\boldsymbol{\tau}'_1} = \mathcal{L}_{\boldsymbol{\tau}'_2}$ and so $\boldsymbol{\tau}'_1 = \boldsymbol{\tau}'_2$ by Lemma 10.6 (b). Now Proposition 9.3 implies $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$, which proves Lemma 10.5 (b).

Note that $\check{\mathcal{O}}$ is not quasi-distinguished, so Lemma 10.5 (c) is vacant.

10.5. **Establishing properties of $\mathcal{L}_{\boldsymbol{\tau}}$: the general case for $\star \in \{B, D\}$.** To ease the discussion, we adopt the following notation (c.f. (8.8)). Suppose $\mathcal{A} \in \mathbb{Z}[\mathsf{MYD}]$. We write $\mathcal{A} \sqsupseteq (p_0, q_0)$ if there is an marked Young diagram $\mathcal{E}$ with a non-zero coefficient in $\mathcal{A}$ such that $\mathcal{E} \sqsupseteq (p_0, q_0)$.

Retain the notation in lemma 10.6 to 10.8. We now assume $\mathbf{c}_2(\mathcal{O}) > 0$ and all the lemmas in Section 10.2 hold for $\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}})$ and $\check{\mathcal{O}}'' := \check{\nabla}(\check{\mathcal{O}}')$.

The case when $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$.

Proof of Lemma 10.3. Since $\mathcal{L}_{\boldsymbol{\tau}'} \neq 0$ and is multiplicity free, we conclude that $\mathcal{L}_{\boldsymbol{\tau}} \neq 0$ and is multiplicity free by (10.2).

Proof of Lemma 10.6 (a). Suppose $\mathcal{L}_{\boldsymbol{\tau}_1} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{\boldsymbol{\tau}_2} \otimes (\epsilon_2, \epsilon_2)$. According to (10.3), we have $\mathcal{L}_{((\tau_1)_\mathbf{t}, \varnothing)} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{((\tau_2)_\mathbf{t}, \varnothing)} \otimes (\epsilon_2, \epsilon_2)$. By the initial cases (see Section 10.3), we get $\epsilon_1 = \epsilon_2$ and $\varepsilon_{\tau_1} = \varepsilon_{(\tau_1)_\mathbf{t}} = \varepsilon_{(\tau_2)_\mathbf{t}} = \varepsilon_{\tau_2}$. By $\varepsilon_{\tau_1} = \varepsilon_{\tau_2}$ and $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we deduce that $\mathcal{L}_{\boldsymbol{\tau}'_1} = \mathcal{L}_{\boldsymbol{\tau}'_2}$, and so $\mathcal{L}_{\boldsymbol{\tau}_1} = \mathcal{L}_{\boldsymbol{\tau}_2}$ by from (10.2).

Proof of Lemma 10.6 (b). Since $\check{\mathcal{O}}$ is weakly-distinguished, $\check{\mathcal{O}}'$ is also weakly-distinguished and so $\boldsymbol{\tau}'_1 = \boldsymbol{\tau}'_2$ by Lemma 10.5 (b). Applying Proposition 9.7, we conclude that $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$.

Proof of Lemma 10.6 (c). Suppose $\check{\mathcal{O}}$ is quasi-distinguished. Then $\check{\mathcal{O}}'$ is quasi-distinguished and $\mathcal{L}_{\boldsymbol{\tau}'} \in \mathsf{MYD}_{\star'}(\mathcal{O}')$. By Equation (10.2), $\mathcal{L}_{\boldsymbol{\tau}} \in \mathsf{MYD}_\star(\mathcal{O})$.

For $\mathcal{E} \in \mathsf{MYD}_\star$ with $(p, q) = \mathrm{Sign}(\mathcal{E})$, we shall construct $(\boldsymbol{\tau}, \epsilon) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z}$ such that $\mathcal{L}_{\boldsymbol{\tau}} \otimes (\epsilon, \epsilon) = \mathcal{E}$, which implies $\tilde{\mathcal{L}}(\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})) = \mathsf{MYD}_\star(\mathcal{O})$. By the initial cases (see Section 10.3), there is a unique pair (τ_0, ϵ) such that

$$\mathcal{E}(1) = \mathcal{L}_{(\tau_0, \varnothing)} \otimes (\epsilon, \epsilon).$$

By Lemma 10.5 (c), there is a unique $\boldsymbol{\tau}' = (\tau', \wp') \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$ such that

$$\big(((\maltese^{y'} \mathcal{L}_{\boldsymbol{\tau}'}) \cdot (p_{\tau_0}, q_{\tau_0})) \otimes (0, \varepsilon_{\tau_0})\big)(i) = \big(\mathcal{E} \otimes (\epsilon, \epsilon)\big)(i) \qquad \text{for all} \quad i \geqslant 2.$$

Here $y' = \frac{p-q+1}{2}$ if $\star = B$ and $y' = \frac{p-q+1}{2}$ if $\star = D$. Using Lemma 9.6, we obtain a unique τ such that $(\nabla(\tau), \tau_\mathbf{t}) = (\tau', \tau_0)$. Let $\wp \in \mathrm{PP}(\check{\mathcal{O}})$ be the unique element such that $\nabla(\wp) = \wp'$. Then $\boldsymbol{\tau} = (\tau, \wp)$ is the desired element.

Proof of Lemma 10.7 and 10.8. Using (10.3), all the claims follow from that of $\check{\mathcal{O}}_\mathbf{t}$. See Sections 9.2 and 10.3.

The case when $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$. We write $\tau'' := (\tau'', \wp'') := \nabla^2(\tau)$. To ease the notation, we write

$$\begin{aligned} \mathcal{L}_\tau &= \mathcal{L}_{\tau,+} + \mathcal{L}_{\tau,-} \qquad \text{with} \\ \mathcal{L}_{\tau,+} &:= \left(\maltese^t\left(\mathcal{L}^+_{\tau''} \cdot (0,0)\right) \cdot \mathcal{T}^+_\tau\right) \otimes (0, \varepsilon_\tau) \quad \text{and} \\ \mathcal{L}_{\tau,-} &:= \left(\maltese^t\left(\mathcal{L}^-_{\tau''} \cdot (0,0)\right) \cdot \mathcal{T}^-_\tau\right) \otimes (0, \varepsilon_\tau). \end{aligned} \tag{10.8}$$

Recall also the number $k = \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} + 1$ (Section 9.2).

Proof of Lemma 10.3. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \succeq (0,1)$, the term $\mathcal{L}_{\tau,+}$ in (10.8) does not vanish since $\mathcal{L}^+_{\tau''} \neq 0$, by Lemma 10.7 for $\check{\mathcal{O}}''$ and $\mathcal{T}^+_\tau \neq \emptyset$. Note that $x_{\tau''} \neq s$ by Lemma 9.5.

When $\mathrm{Sign}(\tau_{\mathbf{t}}) \not\succeq (0,1)$, we must have $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k, 0)$ and so $\mathcal{P}^{-1}_{\tau_{\mathbf{t}}}(\{ s, c \}) = \emptyset$. Hence $x_{\tau''} = d$ by Lemma 9.5. Now $\mathcal{L}^-_{\tau''} \neq 0$ by Lemma 10.7 and $\mathcal{L}_{\tau,-}$ in (10.8) does not vanish. This proves the non-vanishing of $\mathcal{L}_\tau$.

Note that $\mathscr{F}(\mathcal{L}_{\tau,+})$ and $\mathscr{F}(\mathcal{L}_{\tau,-})$ have no common terms since $\mathcal{T}^+_\tau \neq \mathcal{T}^-_\tau$. Therefore the multiplicity freeness of $\mathcal{L}_\tau$ follows from the multiplicity freeness of $\mathcal{L}_{\tau'}$. This proves Lemma 10.3

Proof of Lemma 10.7. It is clear that $\mathcal{L}^-_\tau = 0$ when $x_\tau \in \{ s, r, c \}$ because of the twist "$\otimes(0,1)$" in (10.4).

(a) Suppose $x_\tau = s$. We have $\mathrm{Sign}(\tau_{\mathbf{t}}) = (0, 2k)$. Therefore, $\mathcal{L}_{\tau,+} \neq 0$, $\mathcal{L}_{\tau,-} = 0$ and $\mathcal{E}(1) = (B, 0, -(2k-1))$ for every $\mathcal{E} \in \mathsf{MYD}_\star$ with a non-zero coefficient in $\mathcal{L}_\tau$. Hence $\mathcal{L}^+_\tau = 0$.

(b) Suppose $x_\tau \in \{ r, c \}$. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \succeq (1,1)$, the term $\mathcal{L}_{\tau,+}$ in (10.8) is non-zero. So

$$\mathcal{L}^+_\tau \geq \Lambda_{(1,0)}(\mathcal{L}_{\tau,+}) \neq 0.$$

When $\mathrm{Sign}(\tau_{\mathbf{t}}) \not\succeq (1,1)$, we have $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k, 0)$ and $\mathcal{P}^{-1}_{\tau_{\mathbf{t}}}(\{ s, c \}) = \emptyset$. So $x_{\tau''} = d$ by Lemma 9.5. Now $\mathcal{L}^-_{\tau''} \neq 0$ and $\mathcal{T}^-_\tau \succeq (1,0)$. Hence $\mathcal{L}_{\tau,-} \neq 0$ and

$$\mathcal{L}^+_\tau \geq \Lambda_{(1,0)}(\mathcal{L}_{\tau,-}) \neq 0.$$

In both cases, we conclude that $\mathcal{L}^+_\tau \neq 0$.

(c) Suppose $x_\tau = d$. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \succeq (1,2)$, the term $\mathcal{L}_{\tau,+} \neq 0$ and so $\mathcal{L}_\tau \sqsupseteq (1,1)$. Therefore $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau \neq 0$. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \not\succeq (1,2)$, we must have $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k-1, 1) \succeq (1,1)$. In particular, $\mathcal{P}^{-1}_{\tau_{\mathbf{t}}}(\{ s, c \}) = \emptyset$ and so $x_{\tau''} = d$ by Lemma 9.5. Now both $\mathcal{L}_{\tau,+}$ and $\mathcal{L}_{\tau,-}$ in (10.8) are non-zero, which leads to the desired conclusions.

Proof of Lemma 10.6 (a). The following proof only depends on Lemma 10.7. Suppose $\mathcal{L}' = \mathcal{L}_\tau \otimes (\epsilon, \epsilon)$ for $(\tau, \epsilon) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z}$. Let

$$\begin{aligned} \mathcal{L}'^+ &:= \Lambda_{(1,0)}(\mathcal{L}'), & \mathcal{L}'^- &:= \Lambda_{(0,1)}(\mathcal{L}'), \\ \widetilde{\mathcal{L}'}^+ &:= \Lambda_{(1,0)}(\mathcal{L}' \otimes (1,1)), \text{ and} & \widetilde{\mathcal{L}'}^- &:= \Lambda_{(0,1)}(\mathcal{L}' \otimes (1,1)). \end{aligned}$$

Note that $\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O}) > 0$, and so at least one of $\mathcal{L}'^+$, $\mathcal{L}'^-$, $\widetilde{\mathcal{L}'}^+$ and $\widetilde{\mathcal{L}'}^-$ is non-vanishing.

Using Lemma 10.7, we have the following non-vanishing criteria in terms of $(\varepsilon_\tau, \epsilon)$:

$$(\varepsilon_\tau,\epsilon)=(0,0)\Leftrightarrow\begin{cases}\mathcal{L}'^{+}\neq 0\\ \mathcal{L}'^{-}\neq 0\end{cases};$$

$$(\varepsilon_\tau,\epsilon)=(0,1)\Leftrightarrow\begin{cases}\widetilde{\mathcal{L}'}^{+}\neq 0\\ \widetilde{\mathcal{L}'}^{-}\neq 0\end{cases};$$

$$(\varepsilon_\tau,\epsilon)=(1,0)\Leftrightarrow\begin{cases}\mathcal{L}'^{+}\neq 0\\ \mathcal{L}'^{-}= 0\end{cases}\quad\text{or}\quad\begin{cases}\widetilde{\mathcal{L}'}^{+}= 0\\ \widetilde{\mathcal{L}'}^{-}\neq 0\end{cases};$$

$$(\varepsilon_\tau,\epsilon)=(1,1)\Leftrightarrow\begin{cases}\widetilde{\mathcal{L}'}^{+}\neq 0\\ \widetilde{\mathcal{L}'}^{-}= 0\end{cases}\quad\text{or}\quad\begin{cases}\mathcal{L}'^{+}= 0\\ \mathcal{L}'^{-}\neq 0\end{cases}.$$

This implies Lemma 10.6 (a).

Proof of Lemma 10.6 (b). The weakly-distinguishness of $\check{\mathcal{O}}$ implies that

$$\mathbf{r}_1(\check{\mathcal{O}}'')=\mathbf{r}_3(\check{\mathcal{O}})=\mathbf{r}_2(\check{\mathcal{O}})>\mathbf{r}_5(\check{\mathcal{O}})=\mathbf{r}_3(\check{\mathcal{O}}''). \tag{10.9}$$

Suppose that $\mathcal{L}_{\tau_i}\sqsupseteq(0,1)$ or $\mathcal{L}_{\tau_i}\sqsupseteq(0,-1)$. We must have $\mathrm{Sign}((\tau_1)_{\mathbf{t}})\geq(0,1)$, $\mathrm{Sign}((\tau_2)_{\mathbf{t}})\geq(0,1)$ and so $\mathcal{L}_{\tau_1,+}=\mathcal{L}_{\tau_2,+}\neq 0$. This implies $\mathcal{L}^{+}_{\tau_1''}=\mathcal{L}^{+}_{\tau_2''}\neq 0$. From (10.9), we see that $\Upsilon^{+}_{\check{\mathcal{O}}''}$ is injective by Lemma 10.8 (b), and so $\mathcal{L}_{\tau_1''}=\mathcal{L}_{\tau_2''}$.

Now consider the case when $\mathcal{L}_{\tau_i}\not\sqsupseteq(0,1)$ and $\mathcal{L}_{\tau_i}\not\sqsupseteq(0,-1)$. By Lemma 10.7, $x_\tau\neq d$. From (10.4), we conclude that $\{\,\mathrm{Sign}((\tau_i)_{\mathbf{t}})\mid i=1,2\,\}\subset\{\,(2k,0),(2k-1,1)\,\}$.

We now discuss case by case according to the set $\{\,\mathrm{Sign}((\tau_i)_{\mathbf{t}})\mid i=1,2\,\}$:

(i) Suppose $\{\,\mathrm{Sign}((\tau_i)_{\mathbf{t}})\mid i=1,2\,\}=\{\,(2k-1,1)\,\}$. We have $\mathcal{L}_{\tau_1,+}=\mathcal{L}_{\tau_2,+}\neq 0$ and so $\mathcal{L}^{+}_{\tau_1''}=\mathcal{L}^{+}_{\tau_2''}\neq 0$. By (10.9) and Lemma 10.8 (b), we deduce that $\mathcal{L}_{\tau_1''}=\mathcal{L}_{\tau_2''}$.

(ii) Suppose $\{\,\mathrm{Sign}((\tau_i)_{\mathbf{t}})\mid i=1,2\,\}=\{\,(2k,0)\,\}$. We have $\mathcal{L}_{\tau_1,+}=\mathcal{L}_{\tau_2,-}\neq 0$ and so $\mathcal{L}^{-}_{\tau_1''}=\mathcal{L}^{-}_{\tau_2''}\neq 0$. By (10.9) and Lemma 10.8 (b), we deduce that $\mathcal{L}_{\tau_1''}=\mathcal{L}_{\tau_2''}$.

(iii) Without loss of generality, we may assume that $\mathrm{Sign}((\tau_1)_{\mathbf{t}})=(2k-1,1)$ and $\mathrm{Sign}((\tau_2)_{\mathbf{t}})=(2k,0)$. By (10.4), we get $\mathcal{L}_{\tau_1,+}=\mathcal{L}_{\tau_2,-}$ and so

$$\mathcal{L}^{+}_{\tau_1''}\cdot(0,0)=\maltese(\mathcal{L}^{-}_{\tau_2''}\cdot(0,0)),$$

which contradicts Lemma 10.8 (c) for $\check{\mathcal{O}}''$.

In all the cases, we have $\mathcal{L}_{\tau_1''}=\mathcal{L}_{\tau_2''}$. Since $\mathcal{O}''$ is weakly-distinguished, $\tau_1''=\tau_2''$ by Lemma 10.6 (b). Note that $\varepsilon_{\tau_1}=\varepsilon_{\tau_2}$ by Lemma 10.6 (a) and $\mathrm{Sign}(\tau_1)=\mathrm{Sign}(\tau_2)$. We conclude that $\tau_1=\tau_2$ by combining Proposition 9.7 and Proposition 9.3. This finishes the proof of Lemma 10.6 (b).

Lemma 10.6 (c) is vacant in the current case.

Proof of Lemma 10.8 (b). Note that the condition $\mathbf{r}_1(\check{\mathcal{O}})>\mathbf{r}_3(\check{\mathcal{O}})=\mathbf{r}_2(\check{\mathcal{O}})$ implies $k\geqslant 2$.

We first prove the injectivity of $\Upsilon^{+}_{\check{\mathcal{O}}}$.

Note that $\mathcal{L}_\tau\sqsupseteq(2,0)$ is equivalent to $\mathcal{L}^{+}_\tau\sqsupseteq(1,0)$. By (10.4), $\mathcal{L}_\tau\sqsupseteq(2,0)$ implies that $\mathcal{L}_\tau\rhd(1,0)$, which means that $\mathcal{E}\sqsupseteq(1,0)$, for any marked Young diagram $\mathcal{E}$ with a non-zero coefficient in $\mathcal{L}_\tau$. So we obtain the following bijection

$$\{\,\mathcal{L}_\tau\mid\tau\in\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})\text{ and }\mathcal{L}_\tau\sqsupseteq(2,0)\,\}\to\{\,\mathcal{L}^{+}_\tau\mid\tau\in\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})\text{ and }\mathcal{L}^{+}_\tau\sqsupseteq(1,0)\,\}.$$

Now we consider the set

$$\{\,\mathcal{L}_\tau\mid\tau\in\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}),\mathcal{L}^{+}_\tau\neq 0\text{ and }\mathcal{L}_\tau\not\sqsupseteq(2,0)\,\}.$$

Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in this set such that $\mathcal{L}^{+}_{\tau_1}=\mathcal{L}^{+}_{\tau_2}$.

By (10.4), we have $\mathrm{Sign}((\tau_i)_{\mathbf{t}}) = (1, 2k-1)$ and

$$\mathcal{L}_{\tau_1,+} = \mathcal{L}_{\tau_2,+} \neq 0. \tag{10.10}$$

Note that $\mathcal{E}(1) = (B, 1, (-1)^{\varepsilon_{\tau_i}}(2k-2))$ for each marked Young diagram $\mathcal{E}$ having a non-zero coefficient in $\mathcal{L}_{\tau_1,+}$. So $\varepsilon_{\tau_1} = \varepsilon_{\tau_2}$. Note that $\check{\mathcal{O}}''$ is weakly-distinguished and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$, and so we deduce that $\tau_1'' = \tau_2''$, $\tau_1 = \tau_2$ and $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ as in the proof of Lemma 10.6 (b).

The injectivity of $\Upsilon^-_{\check{\mathcal{O}}}$ is proved similarly, which we give below.

Note that $\mathcal{L}_\tau \sqsupseteq (0,2)$ is equivalent to $\mathcal{L}^-_\tau \sqsupseteq (0,1)$. We get the bijection

$$\{\, \mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}_\tau \sqsupseteq (0,2) \,\} \to \{\, \mathcal{L}^-_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^-_\tau \sqsupseteq (0,1) \,\}.$$

Now we consider the set

$$\{\, \mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^-_\tau \neq 0 \text{ and } \mathcal{L}_\tau \not\sqsupseteq (0,2) \,\}.$$

Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in the set such that $\mathcal{L}^-_{\tau_1} = \mathcal{L}^-_{\tau_2} \neq 0$. By $k \geqslant 2$ and (10.4), we get

$$\{\, \mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i = 1,2 \,\} \subseteq \{\, (2k-1,1), (2k-2,2) \,\}. \tag{10.11}$$

Note that (10.11) cannot be equality since it will contradict Lemma 10.8 (c). We consider case by case according to the set $\{\, \mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i = 1,2 \,\}$:

(i) When $\{\, \mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i = 1,2 \,\} = \{\, (2k-1,1) \,\}$, we have $\mathcal{L}^-_{\tau_1''} = \mathcal{L}^-_{\tau_2''}$.
(ii) When $\{\, \mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i = 1,2 \,\} = \{\, (2k-2,2) \,\}$, we have $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$.

By the same argument as in the proof of Lemma 10.6 (b), we conclude that $\tau_1 = \tau_2$, $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ and therefore the injectivity of $\Upsilon^-_{\check{\mathcal{O}}}$.

Proof of Lemma 10.8 (c). Note that we have $k \geqslant 2$ and $x_\tau = d$ under the stated condition. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \succeq (1,2)$, $\mathcal{L}_{\tau,+} \sqsupseteq (1,1)$. Otherwise, $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k-1,1) \succeq (3,1)$. Since $\mathcal{L}^-_\tau \neq 0$, we have $\mathcal{L}_{\tau,-} \sqsupseteq (2,1)$. In all the cases, $\mathcal{L}_\tau \sqsupseteq (1,1)$ and so $\mathcal{L}^-_\tau \sqsupseteq (1,0)$.

Now there is a marked Young diagram $\mathcal{A}$ having a non-zero coefficient in ✠$(\mathcal{L}^-_\tau \cdot (0,0))$ such that $\mathcal{A}(2) = (\star_2, p_2, q_2)$ and $p_2 < 0$. On the other hand, every marked Young diagram $\mathcal{B}$ having a non-zero coefficient in $\mathcal{L}^+_{\tau_2} \cdot (0,0)$ satisfies $p_2' \geqslant 2$ with $\mathcal{B}(2) = (\star_2, p_2', q_2')$. Therefore (10.6) never holds.

Proof of Lemma 10.8 (a). When $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$, the injectivity follows from the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$.

We are left to consider the case when $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$. In this case, we have $k = 1$ and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$. Suppose $\tau_1, \tau_2 \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ such that $\Upsilon_{\check{\mathcal{O}}}(\mathcal{L}_{\tau_1}) = \Upsilon_{\check{\mathcal{O}}}(\mathcal{L}_{\tau_2})$.

When $\mathcal{L}^-_{\tau_1} = \mathcal{L}^-_{\tau_2} \neq 0$, we have $x_{\tau_1} = x_{\tau_2} = d$ and $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$ by (10.4).

When $\mathcal{L}^-_{\tau_1} = \mathcal{L}^-_{\tau_2} = 0$, by a similar argument as in the proof of Lemma 10.6 (b), we conclude that there are only two possibilities:

- $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$ and $x_{\tau_1} = x_{\tau_2} = c$;
- $\mathcal{L}^-_{\tau_1''} = \mathcal{L}^-_{\tau_2''}$ and $x_{\tau_1} = x_{\tau_2} = r$.

In all cases, we conclude that $\tau_1 = \tau_2$ and $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$, as in the proof of Lemma 10.6 (b).

References

[1] J. Adams, *Discrete spectrum of the reductive dual pair* $(O(p,q), Sp(2m))$, Invent. Math. **74** (1983), no. 3, 449–475.

[2] ———, *The theta correspondence over R*, Harmonic analysis, group representations, automorphic forms and invariant theory, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 12 (2007), 1–39.

[3] J. Adams, B. Barbasch, and D. A. Vogan, *The Langlands classification and irreducible characters for real reductive groups*, Progress in Math., vol. 104, Birkhauser, 1991.

[4] Jeffrey Adams and Fokko du Cloux, *Algorithms for representation theory of real reductive groups*, Journal of the Institute of Mathematics of Jussieu **8** (2009), no. 2, 209-259, DOI 10.1017/S1474748008000352.

[5] J. Arthur, *On some problems suggested by the trace formula*, Lie group representations, II (College Park, Md.), Lecture Notes in Math. 1041 (1984), 1–49.

[6] ———, *Unipotent automorphic representations: conjectures*, Orbites unipotentes et représentations, II, Astérisque **171-172** (1989), 13–71.

[7] L. Auslander and B. Kostant, *Polarizations and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

[8] D. Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. **96** (1989), no. 1, 103–176.

[9] ———, *Unipotent representations for real reductive groups*, Proceedings of ICM (1990), Kyoto (2000), 769–777.

[10] D. Barbasch and David Vogan, *Weyl Group Representations and Nilpotent Orbits*, Representation Theory of Reductive Groups: Proceedings of the University of Utah Conference 1982, 1983, pp. 21–33.

[11] D. Barbasch, *Orbital integrals of nilpotent orbits*, The mathematical legacy of Harish-Chandra, Proc. Sympos. Pure Math. **68** (2000), 97–110.

[12] ———, *The unitary spherical spectrum for split classical groups*, J. Inst. Math. Jussieu **9** (2010), 265–356.

[13] ———, *Unipotent representations and the dual pair correspondence*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 47–85.

[14] D. Barbasch, J.-J. Ma, B.-Y. Sun, and C.-B. Zhu, *On the notion of metaplectic Barbasch-Vogan duality*, arXiv:2010.16089.

[15] ———, *Counting special unipotent representations: orthogonal and symplectic groups*, in preparation.

[16] D. Barbasch and D. A. Vogan, *Weyl group representations and nilpotent orbits*, in Representation theory of reductive groups (Park City, Utah, 1982), Progress in Math. **40** (1983), 21–33.

[17] ———, *Unipotent representations of complex semisimple groups*, Annals of Math. **121** (1985), no. 1, 41–110.

[18] R. Brylinski, *Dixmier algebras for classical complex nilpotent orbits via Kraft-Procesi models. I*, The orbit method in geometry and physics (Marseille, 2000). Progress in Math. **213** (2003), 49–67.

[19] W. Borho, *Recent advances in enveloping algebras of semisimple Lie-algebras*, Séminaire Bourbaki, Exp. No. 489 (1976/77), 1–18.

[20] Walter Borho and Hanspeter Kraft, *Über die Gelfand-Kirillov-Dimension*, Math. Ann. **220** (1976), no. 1, 1–24, DOI 10.1007/BF01354525. MR412240

[21] R. W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993.

[22] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. **41** (1989), 385–438.

[23] Y. Chen and B. Sun, *Schwartz homologies of representations of almost linear Nash groups*, J. Funct. Anal. **280** (2021), 108817.

[24] F. du Cloux, *Représentations tempérées des groupes de Lie nilpotents*, J. Funct. Anal. **85** (1989), no. 2, 420–457.

[25] ———, *Sur les représentations différentiables des groupes de Lie algébriques*, Ann. Sci. École Norm. Sup. **24** (1991), no. 3, 257–318.

[26] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, Van Nostrand Reinhold Co., 1993.

[27] M. Cowling, U. Haagerup, and R. Howe, *Almost L2 matrix coefficients*, J. Reine Angew. Math. **387** (1988), 97–110.

[28] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda, *Nilpotent orbits and complex dual pairs*, J. Algebra **190** (1997), no. 2, 518 - 539.

[29] ______, *Dual pairs and Kostant-Sekiguchi correspondence. II. Classification of nilpotent elements*, Central European J. Math. **3** (2005), 430–474.

[30] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) **102** (1978), 307–330.

[31] M. Duflo, *Théorie de Mackey pour les groupes de Lie algébriques*, Acta Math. **149** (1982), no. 3-4, 153–213.

[32] R. Gomez and C.-B. Zhu, *Local theta lifting of generalized Whittaker models associated to nilpotent orbits*, Geom. Funct. Anal. **24** (2014), no. 3, 796–853.

[33] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algb́rique IV: Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[34] ______, *Éléments de géométrie algb́rique IV: Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966).

[35] M. Harris, J.-S. Li, and B. Sun, *Theta correspondences for close unitary groups*, Arithmetic Geometry and Automorphic Forms, Adv. Lect. Math. (ALM) **19** (2011), 265–307.

[36] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag, 1983.

[37] H. He, *Unipotent representations and quantum induction*, arXiv:math/0210372 (2002).

[38] ______, *Unitary representations and theta correspondence for type I classical groups*, J. Funct. Anal. **199** (2003), no. 1, 92–121.

[39] J.-S. Huang and J.-S. Li, *Unipotent representations attached to spherical nilpotent orbits*, Amer. J. Math. **121** (1999), no. 3, 497–517.

[40] J.-S. Huang and C.-B. Zhu, *On certain small representations of indefinite orthogonal groups*, Represent. Theory **1** (1997), 190–206.

[41] R. Howe, *θ-series and invariant theory*, Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math, vol. 33, 1979, pp. 275-285.

[42] ______, *On a notion of rank for unitary representations of the classical groups*, Harmonic analysis and group representations, Liguori, Naples (1982), 223-331.

[43] ______, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

[44] ______, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Piatetski-Shapiro, I. et al. (eds.), The Schur lectures (1992). Ramat-Gan: Bar-Ilan University, Isr. Math. Conf. Proc. 8, (1995), 1-182.

[45] D. Jiang, B. Liu, and G. Savin, *Raising nilpotent orbits in wave-front sets*, Represent. Theory **20** (2016), 419–450.

[46] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.

[47] B. Kostant, *Quantization and unitary representations*, Lectures in Modern Analysis and Applications III, Lecture Notes in Math. **170** (1970), 87–208.

[48] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.

[49] S. S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.

[50] S. S. Kudla, *Some extensions of the Siegel-Weil formula*, In: Gan W., Kudla S., Tschinkel Y. (eds) Eisenstein Series and Applications. Progress in Mathematics, vol 258. Birkhäuser Boston (2008), 205–237.

[51] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence II*, Israel J. Math. **100** (1997), 29–59.

[52] ______, *Degenerate principal series of metaplectic groups and Howe correspondence*, D. Prasad at al. (eds.), Automorphic Representations and L-Functions, Tata Institute of Fundamental Research, India, (2013), 379–408.

[53] J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), no. 2, 237–255.

[54] ______, *Theta lifting for unitary representations with nonzero cohomology*, Duke Math. J. **61** (1990), no. 3, 913–937.

[55] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2006.

[56] H. Y. Loke and J. Ma, *Invariants and K-spectrums of local theta lifts*, Compositio Math. **151** (2015), 179–206.

[57] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, j. London Math. Soc. **19** (1979), 41–52.

[58] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272, DOI 10.1007/BF01388564. MR732546

[59] G. W. Mackey, *Unitary representations of group extentions*, Acta Math. **99** (1958), 265–311.

[60] W. M McGovern, *Cells of Harish-Chandra modules for real classical groups*, Amer. J. of Math. **120** (1998), 211–228.

[61] C. Mœglin, *Front d'onde des représentations des groupes classiques p-adiques*, Amer. J. Math. **118** (1996), 1313–1346.

[62] ———, *Paquets d'Arthur Spéciaux Unipotents aux Places Archimédiennes et Correspondance de Howe*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 469–502.

[63] C. Mœglin and D. Renard, *Sur les paquets d'Arthur des groupes classiques réels*, J. Eur. Math. Soc. **22** (2020), 1827–1892.

[64] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer, 1987.

[65] K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*, Astérisque **273** (2001), 1–163.

[66] K. Nishiyama, H. Ochiai, and C.-B. Zhu, *Theta lifting of nilpotent orbits for symmetric pairs*, Trans. Amer. Math. Soc. **358** (2006), 2713–2734.

[67] K. Nishiyama and C.-B. Zhu, *Theta lifting of unitary lowest weight modules and their associated cycles*, Duke Math. J. **125** (2004), 415–465.

[68] T. Ohta, *The closures of nilpotent orbits in the classical symmetric pairs and their singularities*, Tohoku Math. J. **43** (1991), no. 2, 161–211.

[69] ———, *Induction of nilpotent orbits for real reductive groups and associated varieties of standard representations*, Hiroshima Math. J. **29** (1999), no. 2, 347–360.

[70] ———, *Nilpotent orbits of $\mathbb{Z}_4$-graded Lie algebra and geometry of moment maps associated to the dual pair* $(\mathrm{U}(p,q),\mathrm{U}(r,s))$, Publ. RIMS **41** (2005), no. 3, 723–756.

[71] A. Paul and P. Trapa, *Some small unipotent representations of indefinite orthogonal groups and the theta correspondence*, University of Aarhus Publ. Series **48** (2007), 103–125.

[72] V. L. Popov and E. B. Vinberg, *Invariant Theory*, Algebraic Geometry IV: Linear Algebraic Groups, Invariant Theory, Encyclopedia of Mathematical Sciences, vol. 55, Springer, 1994.

[73] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloq. Math. **70** (1996), 93–102.

[74] ———, *Characters, dual pairs, and unitary representations*, Duke Math. J. **69** (1993), no. 3, 547–592.

[75] S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.

[76] D. Renard and P. Trapa, *Irreducible genuine characters of the metaplectic group: Kazhdan-Lusztig algorithm and Vogan duality*, Represent. Theory **4** (2000), 245–295, DOI 10.1090/S1088-4165-00-00105-9.

[77] S. Sahi, *Explicit Hilbert spaces for certain unipotent representations*, Invent. Math. **110** (1992), no. 2, 409–418.

[78] J. Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138.

[79] W. Schmid and K. Vilonen, *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Annals of Math. **151** (2000), no. 3, 1071–1118.

[80] E. Sommers, *Lusztig's canonical quotient and generalized duality*, J. Algebra **243** (2001), no. 2, 790–812.

[81] T. A. Springer and R. Steinberg, *Seminar on algebraic groups and related finite groups; Conjugate classes*, Lecture Notes in Math., vol. 131, Springer, 1970.

[82] B. Sun and C.-B. Zhu, *A general form of Gelfand-Kazhdan criterion*, Manuscripta Math. **136** (2011), 185–197.

[83] P. Trapa, *Special unipotent representations and the Howe correspondence*, University of Aarhus Publication Series **47** (2004), 210–230.

[84] D. A. Vogan, *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. **49** (1982), no. 4, 943–1073. MR683010

[85] ———, *Unitary representations of reductive Lie groups*, Ann. of Math. Stud., vol. 118, Princeton University Press, 1987.

[86] ———, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups, Proc. Conf., Brunswick/ME (USA) 1989, Prog. Math. **101** (1991), 315–388.

[87] ———, *The method of coadjoint orbits for real reductive groups*, Representation theory of Lie groups (Park City, UT, 1998). IAS/Park City Math. Ser. **8** (2000), 179–238.

[88] ———, *Unitary representations of reductive Lie groups*, Mathematics towards the Third Millennium (Rome, 1999). Accademia Nazionale dei Lincei, (2000), 147–167.
[89] N. R. Wallach, *Real reductive groups I*, Academic Press Inc., 1988.
[90] ———, *Real reductive groups II*, Academic Press Inc., 1992.
[91] A. Weil, *Sur certain group d'operateurs unitaires*, Acta Math. **111** (1964), 143–211.
[92] H. Weyl, *The classical groups: their invariants and representations*, Princeton University Press, 1947.
[93] S. Yamana, *Degenerate principal series representations for quaternionic unitary groups*, Israel J. Math. **185** (2011), 77–124.
[94] C.-B. Zhu, *Local theta correspondence and nilpotent invariants*, Proceedings of Symposia in Pure Mathematics **101** (2019), 427–450.

the Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, New York 14853
Email address: dmb14@cornell.edu

School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China
Email address: hoxide@sjtu.edu.cn

Institute for Advanced Study in Mathematics, Zhejiang University, Hangzhou, 310058, China
Email address: sunbinyong@zju.edu.cn

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
Email address: matzhucb@nus.edu.sg