# Special unipotent representations of real classical groups and theta correspondence

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# Classical groups and special unipotent representations

| | $G$ | $\mathbf{G}$ | $\mathbf{G}^\vee$ | |
|---|---|---|---|---|
| $D_n$ | $\mathrm{O}(p, 2n-p)$ | $\mathrm{O}(2n, \mathbb{C})$ | $\mathrm{O}(2n, \mathbb{C})$ | $D_n$ |
| $C_n$ | $\mathrm{Sp}(2n, \mathbb{R})$ | $\mathrm{Sp}(2n, \mathbb{C})$ | $\mathrm{SO}(2n+1, \mathbb{C})$ | $B_n$ |
| $B_n$ | $\mathrm{O}(p, 2n+1-p)$ | $\mathrm{O}(2n+1, \mathbb{C})$ | $\mathrm{Sp}(2n, \mathbb{C})$ | $C_n$ |
| $\tilde{C}_n$ | $\mathrm{Mp}(2n, \mathbb{R})$ | $\mathrm{Sp}(2n, \mathbb{C})$ | $\mathrm{Sp}(2n, \mathbb{C})$ | $C_n$ |
| $D_n$ | $\mathrm{O}^*(n)$ | $\mathrm{SO}(2n, \mathbb{C})$ | $\mathrm{SO}(2n, \mathbb{C})$ | $D_n$ |
| $C_n$ | $\mathrm{Sp}(p, n-p)$ | $\mathrm{Sp}(2n, \mathbb{C})$ | $\mathrm{SO}(2n+1, \mathbb{C})$ | $B_n$ |
| $A_n$ | $\mathrm{U}(p, n-p)$ | $\mathrm{GL}(n, \mathbb{C})$ | $\mathrm{GL}(n, \mathbb{C})$ | $A_n$ |
| $A_m$ | $\mathrm{U}(r, m-r)$ | $\mathrm{GL}(m, \mathbb{C})$ | $\mathrm{GL}(m, \mathbb{C})$ | $A_m$ |

**Theorem (Barbasch-M.-Sun-Zhu)**

Arthur-Barbasch-Vogan's conj. on special unipotent repn. holds for $G$: All *specail unipotent representations* of $G$ are *unitarizable*.

# Barbasch-Vogan's definition of unipotent representation

$G$: a real reductive group.

Nilpotent orbit $\check{\mathcal{O}}$ in $\mathbf{G}^\vee$.

$\rightsquigarrow \varphi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathbf{G}^\vee$ (Jacobson-Morozov)

$\rightsquigarrow$ an infinitesimal character $\mathrm{d}\varphi(\frac{1}{2}\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)) \leftrightarrow \lambda_{\mathcal{O}^\vee}$

$\rightsquigarrow$ the maximal primitive ideal $\mathcal{I}_{\check{\mathcal{O}}}$ with inf. char. $\lambda_{\check{\mathcal{O}}}$

- *Definition* (Barbasch-Vogan):

  An irr. admissible $G$-repn. is called *special unipotent* if

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) = \mathcal{I}_{\check{\mathcal{O}}}.$$

  $\Longleftrightarrow \pi$ has inf. char. $\lambda_{\check{\mathcal{O}}}$ and $\mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}}$

- $\mathcal{O}$: the Lusztig-Spaltenstein-Barbasch-Vogan dual of $\check{\mathcal{O}}$, which is a *(metaplectic) special nilpotent orbit.*
- $\mathrm{Unip}_{\check{\mathcal{O}}}(G) := \{$ special unipotent repn. attached to $\check{\mathcal{O}}\}$.

# Conjecture/Open problems

- *Major open problem:* Classify the unitary dual of a reductive group:
$$\widehat{G}_{\text{unitary}} = \{ \text{ irr. unitary repn. of } G\}.$$
- *Philosophy:* $\mathrm{Unip}(G)$ = the building blocks of the unitary dual.
- *Conjecture*: $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ consists of unitary representations.
- Question: How many elements are there in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$?
- Question: How to construct elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$?
- *Barbasch-Vogan* 1985: Complete classification of unipotent repn. of complex reductive groups.
- Vogan 1986: Classify the unitary dual of $GL(n)$.
- Barbasch 1989: Classify the unitary dual of complex classical groups.
- *Altas of Lie group:* $\rightsquigarrow$ complete answer for exceptional groups.

# Counting $(\mathfrak{g}, K)$-module with a paticular asso. variety

- Fix regular inf. char. $\lambda \in \mathfrak{h}^*/W$
- integral Weyl group
$$W(\lambda) := \{\, w \in W \mid \langle \lambda - w\lambda, \check{\alpha} \rangle \in \mathbb{Z},\ \forall \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \,\}$$
Double cell $\mathcal{D}$ in $\widehat{W(\lambda)} \longleftrightarrow$ the specail repn. $\tau_0 \in \mathcal{D}$
$\longrightarrow$ truncated induction $J_{W(\lambda)}^{W} \tau_0$
$\xrightarrow{\text{Springer corr.}} \mathcal{O}$.
- Let $\mu \in \lambda + X^*$ ($X^*$ is the weight lattice),
$$W_\mu = \{\, w \in W \mid w \cdot \mu = \mu \,\}.$$
- $\mathscr{G}_\lambda(\mathfrak{g}, K)$: the Groth. gp. of $(\mathfrak{g}, K)$-modules with inf. char. $\lambda$.
- Lemma:
$$\begin{aligned} &\#\{\, \pi \in \mathrm{Irr}_\mu(\mathfrak{g}, K)(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \,\} \\ &= \sum_{\substack{\tau \in \mathcal{D} \\ \mathcal{D} \rightsquigarrow \mathcal{O}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathscr{G}_\lambda(\mathfrak{g}, K)] \end{aligned}$$

# Counting unipotent representations I

- Example: $G = \mathrm{Sp}(2n, \mathbb{R})$
- $\lambda_{\check{\mathcal{O}}} \in \rho(G) + X^*$

  $\rightsquigarrow$ special representation $\tau \leftrightarrow \mathcal{O}$
- $\mathscr{G}_\rho(G)$: the Groth. gp. of $(\mathfrak{g}, K)$-modules with inf. char. $\rho$.

$$\#\mathrm{Unip}_{\mathcal{O}^\vee}(G) = 2^l \cdot [\tau : \mathscr{G}_\rho(G)]$$

$$W(\mathrm{Sp}(2n)) = S_n \ltimes \{\pm 1\}^n,$$

$$\mathscr{G}_\rho(\mathrm{Sp}(2n, \mathbb{R})) = \sum_{\substack{p,q,t,s, \\ \sigma \in \widehat{S}_s}} \mathrm{Ind}_{S_t \times W_{2s} \times W_p \times W_q}^{W_n} \mathrm{sgn} \otimes (\sigma \times \sigma) \otimes \mathbf{1} \otimes \mathbf{1}.$$

- $[\tau : 1_{W_{\lambda_{\check{\mathcal{O}}}}}] = 1.$
- $[\tau : \mathscr{G}_\rho(G)]$ is counted by painted bi-partitions $\mathrm{PBP}(\check{\mathcal{O}})$.

# Example of PBP

$$\check{\mathcal{O}} = [7,5,5,5,3,1,1] = \begin{ytableau} \ & \ & \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ & \ & \ \\ \ & \ & \ \\ \ \\ \ \end{ytableau}$$

$\mathrm{PBP}_{\check{\mathcal{O}}}(\mathrm{Sp}(2n,\mathbb{R}))$ — Associated cycle

$$\begin{ytableau} \bullet & \bullet & \bullet \\ \bullet & c \\ r & d \\ d \end{ytableau} \times \begin{ytableau} \bullet & \bullet & \bullet \\ \bullet & s \end{ytableau} \quad \mapsto \quad \begin{ytableau} * & = & * & = & * & = \\ * & = & * & = & * & = \\ = & * & = & * \\ * & = & * & = \\ * & = & * & = \\ - & + \end{ytableau} \cup \begin{ytableau} * & = & * & = & * & = \\ * & = & * & = & * & = \\ = & * & = & * \\ = & * & = & * \\ * & = & * & = \\ + & - \end{ytableau}$$

$$\vdots \qquad\qquad\qquad \vdots$$

# Nilpotent orbits with "good/bad parity"

- Bad parity (must occurs with even multiplicity in $\check{\mathcal{O}}$):

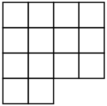
$$\begin{cases} \text{even number}, & \text{when } \mathbf{G}^\vee \text{ is type } B \text{ or } D \\ \text{odd number}, & \text{when } \mathbf{G}^\vee \text{ is type } C \end{cases}$$

- $\check{\mathcal{O}}$ has *"good parity"* if $\check{\mathcal{O}}$ only contains

$$\begin{cases} \text{odd rows}, & \text{when } \mathbf{G}^\vee \text{ is type } B \text{ or } D \\ \text{even rows}, & \text{when } \mathbf{G}^\vee \text{ is type } C \end{cases}$$

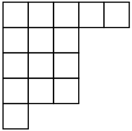
- $\lambda_{\check{\mathcal{O}}}$ is integral.
- Example of good parity:

$\mathcal{O}$



$\mathrm{Sp}(14, \mathbb{C})$

$\mathcal{O}^\vee$



$\mathrm{SO}(15, \mathbb{C})$

# Reduction to the "good parity"

- Consider $G = \mathrm{Sp}(2n, \mathbb{R})$.
- $\check{\mathcal{O}}$ decompose into two parts $\check{\mathcal{O}}_g$ (good parity) and $\check{\mathcal{O}}_b$ (bad parity).
- Assume $\check{\mathcal{O}}_b = \{r_1, r_1, \cdots, r_k, r_k\}$.

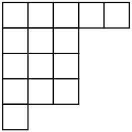
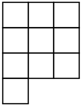
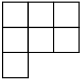
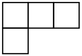
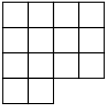
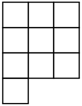
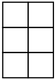
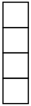
  Theorem (Let $\check{\mathcal{O}}'_b = \{\, r_1, \cdots, r_k \,\} \in \mathrm{Nil}_{\mathrm{GL}}$.)

$$
\begin{array}{ccc}
\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}_{\mathbb{R}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{Sp}_{\mathbb{R}}) & \xrightarrow{1-1} & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{\mathbb{R}}) \\
(\pi', \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}(|\check{\mathcal{O}}'_b|, \mathbb{R}) \times \mathrm{Sp}(2n_0, \mathbb{R}) \ltimes U}^{\mathrm{Sp}(2n, \mathbb{R})} \pi' \otimes \pi_0
\end{array}
$$

$$
\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}) = \left\{ \mathrm{Ind} \bigotimes_{j=1}^{k} \mathrm{sgn}_{\mathrm{GL}(r_j, \mathbb{R})}^{\epsilon_j} \;\middle|\; \epsilon_j \in \mathbb{Z}/2\mathbb{Z} \right\}
$$

- Use *theta correspondence* to construct $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
- We assume $\check{\mathcal{O}}$ has good parity from now on.

# Example of descent sequences

| | | | | |
|---|---|---|---|---|
| $\mathbf{G}_i^\vee$ | $\mathrm{SO}(15,\mathbb{C})$ | $\mathrm{O}(10,\mathbb{C})$ | $\mathrm{SO}(7,\mathbb{C})$ | $\mathrm{O}(4,\mathbb{C})$ |
| $\mathcal{O}_i^\vee$ |  |  |  |  |
| $\mathcal{O}_i$ |  |  |  |  |
| $\mathbf{G}_i$ | $\mathrm{Sp}(14,\mathbb{C})$ | $\mathrm{O}(10,\mathbb{C})$ | $\mathrm{Sp}(6,\mathbb{C})$ | $\mathrm{O}(4,\mathbb{C})$ |

Kraft-Procesi's resolution of singularities of the closure of complex nilpotent orbits.

# Descent of nilpotent orbits: $G = \mathrm{Sp}(2n, \mathbb{R})$

- Take $\check{\mathcal{O}} \in \mathrm{Nil}^{gp}(\mathfrak{g}^\vee)$ (nilpotent orbits with good parity).
- Descent sequence on the dual side:

$$\mathcal{O}^\vee = \mathcal{O}^\vee_{2a} \qquad \mathcal{O}^\vee_{2a-1} \qquad \cdots \qquad \mathcal{O}^\vee_0$$

  $\mathcal{O}^\vee_i$ = removing the first rows of $\mathcal{O}^\vee_{i+1}$.
- Descent sequence of real classical groups:

$$G = G_{2a} \qquad G_{2a-1} \qquad \cdots \qquad G_0$$

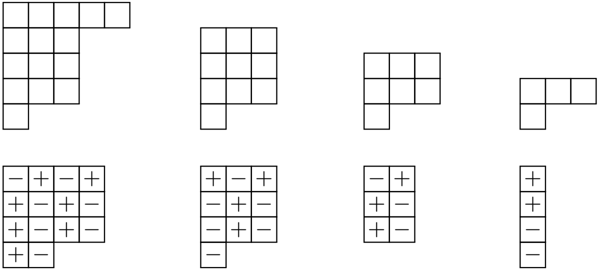
  - $G_{2k}$ is a symplectic group
    allow $G_0 = \mathrm{Sp}(0, \mathbb{R})$ = the trivial group.
  - $G_{2k-1} = \mathrm{O}(p_k, q_k)$
  - $\mathcal{O}^\vee_i$ is nilpotent orbit of $\mathbf{G}^\vee_i$
- $(G_i, G_{i-1})$ forms a reductive dual pair.
- $\mathcal{O}_i$ = delete the first col. of $\mathcal{O}_{i+1}$ and may add one box back.

# Example of descent sequences

| | | | | |
|---|---|---|---|---|
| $\mathbf{G}_i^\vee$ | $\mathrm{SO}(15,\mathbb{C})$ | $\mathrm{O}(10,\mathbb{C})$ | $\mathrm{SO}(7,\mathbb{C})$ | $\mathrm{O}(4,\mathbb{C})$ |

$\mathcal{O}_i^\vee$

$\mathcal{O}_i$




| | | | | |
|---|---|---|---|---|
| $G_i$ | $\mathrm{Sp}(14,\mathbb{R})$ | $\mathrm{O}(4,6)$ | $\mathrm{Sp}(6,\mathbb{R})$ | $\mathrm{O}(2,2)$ |

Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

## Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

- $\chi = \bigotimes_{j=0}^{2a} \chi_j$, a 1-dim repn. of $\prod_{j=0}^{2a} G_j$.
- $\chi_j \in \{\mathbf{1}, \mathrm{sgn}^{+,-}, \mathrm{sgn}^{-,+}, \det\}$
- Define a smooth repn. of $G = G_{2a}$ (the symplectic group).

$$\pi_\chi := (\omega_{G_{2a}, G_{2a-1}} \widehat{\otimes} \omega_{G_{2a-1}, G_{2a-2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1, G_0} \otimes \chi)_{G_{2a-1} \times G_{2a-2} \times \cdots \times G_0}$$

**Theorem (Barbasch-M.-Sun-Zhu)**

*Let $\check{\mathcal{O}}^\vee$ be an orbit with good parity. Then*

- *either $\pi_\chi = 0$ or*
- *$\pi_\chi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$ and unitarizable.*
- *Moreover,*

$$\mathrm{Unip}_{\mathcal{O}^\vee}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}.$$

# Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

$$
\begin{array}{cccc}
 & & \mathrm{Sp}(6, \mathbb{R}) & \\
\mathrm{O}(4,0) & & \theta(\mathrm{sgn}^{+,-}) & \\
\mathrm{O}(3,1) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(2,2) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(1,3) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(0,4) & & & \theta(\mathrm{sgn}^{-,+})
\end{array}
$$

# Some comments

- Many people have studied the problem

  Adams, Barbasch, He, Huang, Li, Loke, Mœglin, Paul, Przebinda, Trapa, ....
- Unitarity:
  - Estimate of matrix coefficients using the explicit realization of the Weil representations.

    Work of **Li**, **He**, and an idea of **Harris-Li-Sun** showing the nonnegativity of a matrix coefficient integral.
- non-vanishing and compute associated cycle:
  - **Geometry**: moment maps provide the upper bound.
  - **Analysis**: degenerate principal series force the lower bound.
  - **Geometry meets Analysis**: the equality.
- Exhaustion: Combinatorics (recent breakthrough!)
- Corollary: (using [Gomez-Zhu]) For $\pi_\chi$,

$$\text{Whittaker cycle} = \text{Wavefront cycle}.$$

# Associated cycle formula I

- Example $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$

$$
\begin{tikzcd}
 & M_{p,n} \oplus M_{q,n} \arrow[dl, "\varphi'"'] \arrow[dr, "\varphi"] & \\
M_{p,q} \cong \mathfrak{p}' & (A, B) \arrow[dl, mapsto] \arrow[dr, mapsto] & \mathfrak{p} \cong \mathrm{Sym}_n \oplus \mathrm{Sym}_n \\
X' = AB^T & & X = (A^TA, B^TB)
\end{tikzcd}
$$

- $\overline{\mathcal{O}} \cap \mathfrak{p} \supset \varphi(\varphi'^{-1}(\mathfrak{p}' \cap \mathcal{O}'))$ where $\mathcal{O}$ is a cplx. nil. $\mathbf{G}$-orbit.
- Upper bound of associated cycle: we can define

$$\vartheta^{\mathrm{geo}} \colon \mathcal{K}_{\mathcal{O}'}(G') \longrightarrow \mathcal{K}_{\mathcal{O}}(G)$$

such that

$$\mathrm{AC}(\Theta(\pi')) \preceq \vartheta^{\mathrm{geo}}(\mathrm{AC}(\pi')),$$

for any $\pi'$ with $\mathrm{AV}(\pi') \subset \overline{\mathcal{O}'}$

# Associated cycle formula II

- Recall $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$
- For $\mathscr{L}' \in \mathcal{K}_{\mathcal{O}'}(G')$, $\mathscr{L} = \vartheta(\mathscr{L}') \in \mathcal{K}_{\mathcal{O}}(G)$,

$$\mathscr{L}_X = \vartheta_T(\mathscr{L}_{X'}) := \det{}^{(p-q)/2}|_{K_X} \otimes (\mathscr{L}'_{X'})^{K'_{2,X'}} \circ \alpha,$$

  $\alpha\colon K_X \longrightarrow K'_{1,X'}$: a homomorphism between isotropic subgroups.
- The twisting is crucial.

  $\Rightarrow$ admissible orbit data $\rightsquigarrow$ admissible orbit data.
- Support of $\vartheta(\mathscr{L}')$ could be reducible.
- Stable range lifting trick: Suppose $n > p + q$.

$$\bigcup_{p,q} \mathrm{Unip}_{\mathcal{O}'^\vee}(\mathrm{O}(p,q)) \hookrightarrow \mathrm{Unip}_{\mathcal{O}^\vee}(\mathrm{Sp}(2n, \mathbb{R}))$$

# Matching unipotent representations with PBP

- $\mathrm{PBP}(\check{\mathcal{O}})$ is complicate.
- $\mathrm{LS}(\check{\mathcal{O}}) = \{\, \mathrm{AC}(\pi_\chi) \,\}$ is also complicate.
- Proof of Exhaustion

  Define descent of painted bi-partitions,

  compatible with the theta lifting!

$$
\begin{tikzcd}
\mathrm{LS}(\check{\mathcal{O}}) & \mathrm{PBP}(\check{\mathcal{O}}) \arrow[l, "\mathrm{AC}"'] \arrow[r, leftrightarrow] \arrow[d, "\nabla"'] & \mathrm{Unip}_{\check{\mathcal{O}}}(G) \\
\mathrm{LS}(\check{\mathcal{O}}') \arrow[u, "\vartheta^{\mathrm{geo}}"] & \mathrm{PBP}(\check{\mathcal{O}}') \arrow[l, "\mathrm{AC}"'] \arrow[r, leftrightarrow] & \mathrm{Unip}_{\check{\mathcal{O}}'}(G') \arrow[u, "\theta"']
\end{tikzcd}
$$

$$\pi_\tau := \Theta(\pi_{\nabla(\tau)} \otimes \chi'_\tau) \otimes \chi_\tau$$

- The injectivity of theta lifting is crucial!

# Unipotent Arthur packet

- Arthur parameter: $\psi\colon W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{G}^\vee \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

  Here $W_{\mathbb{R}} = \mathbb{C} \rtimes \langle j \rangle$.
- Arthur's Arthur packet $\Pi^A_\psi(G)$:

  $\{$local components of automorphic cusp. repn. $\}$

  They are unitary by definition!
- Unipotent Arthur parameter: $\psi|_{\mathbb{C}^\times}$ is trivial.

  Mœglin: $\pi_{\psi,\eta}$ is zero or multiplicity free ($\eta \in \mathrm{Irr}(\pi_1(Z_{\mathbf{G}^\vee}(\psi)))$).

  Warning: $\Pi^A_\psi(G) \cap \Pi^A_{\psi'}(G) \neq \emptyset$ in general.
- "Corollary":

$$\Pi^A_\psi(G) = \Pi^{ABV}_\psi(G)$$

- Question: How to describe $\pi_{\psi,\eta}$ explicitly?

Dan Barabasch, M. , Binyong Sun and Chen-Bo Zhu
Special unipotent representations: orthogonal and symplectic groups
ArXiv e-prints: https://arxiv.org/abs/1712.05552v2

**Thank you for your attention!**