1. Introduction and the main results

Let G be a real reductive group in the Harish-Chandra class, which may be linear or non-linear. Write $\mathfrak{g}$ for the complexified Lie algebra of G and let $\mathfrak{h}$ denote its universal Cartan subalgebra. Let $\lambda \in \mathfrak{h}^*$ (a superscript $*$ indicates the dual space). By Harish-Chandra isomorphism, it determines an algebraic character $\chi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$. Here $\mathcal{Z}(\mathfrak{g})$ denotes the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of irreducible Casselman-Wallach representations of G, and by $\mathrm{Irr}_\lambda(G)$ its subset consisting of the representations with infinitesimal character χ_λ. The latter set has finite cardinality.

Let $\mathrm{Nil}(\mathfrak{g}^*)$ denote the set of nilpotent elements in $\mathfrak{g}^*$. It has only finitely many orbits under the coadjoint action of the inner automorphism group $\mathrm{Inn}(\mathfrak{g})$ of $\mathfrak{g}$. Let S be an $\mathrm{Inn}(\mathfrak{g})$-stable Zariski closed subset of $\mathrm{Nil}(\mathfrak{g}^*)$. Put

$$\mathrm{Irr}_{\lambda,\mathsf{S}}(G) := \{\, \pi \in \mathrm{Irr}_\lambda(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) \subset \mathsf{S} \,\}.$$

Here $\mathrm{AV}_{\mathbb{C}}(\pi)$ denotes the complex associated variety of π, namely the associated variety of the annihilate ideal of π. It is an $\mathrm{Inn}(\mathfrak{g})$-stable Zariski closed subset of $\mathrm{Nil}(\mathfrak{g}^*)$. An interesting problem of representation theory is to understand the cardinality of the finite set $\mathrm{Irr}_{\lambda,\mathsf{S}}(G)$. The coherent continuation representation is a powerful tool for this problem.

1.1. **The coherent continuation representation.** Consider the category of Cassleman-Wallach representations of G that have generalized infinitesimal character λ and whose complex associated variety is contained in S. Write $\mathcal{K}_{\lambda,\mathsf{S}}(G)$ for the Grothendieck group with $\mathbb{C}$-coefficients of this category. Then

$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = \dim \mathcal{K}_{\lambda,\mathsf{S}}(G) \qquad (\sharp \text{ indicates the cardinality of a set}).$$

We also have that

$$\mathcal{K}_{\mathsf{S}}(G) = \bigoplus_{\mu \in W\backslash \mathfrak{h}^*} \mathcal{K}_{\mu,\mathsf{S}}(G) \qquad (W \text{ denotes the Weyl group}),$$

where $\mathcal{K}_{\mathsf{S}}(G)$ is the Grothendieck group, with $\mathbb{C}$-coefficients, of the category of Cassleman-Wallach representations of G whose complex associated variety is contained in S.

Let $\mathcal{K}(\mathfrak{g})$ be the Grothendieck group with $\mathbb{C}$-coefficients of the category of finite-dimensional algebraic representations of $\mathfrak{g}$. Write $\mathcal{R}(\mathfrak{g})$ be the subgroup of $\mathcal{K}(\mathfrak{g})$ generated by finite dimensional $\mathfrak{g}$-modules occur in the symmetric algebra $\mathrm{S}(\mathfrak{g})$. Write $\Delta \subset Q \subset \mathfrak{h}^*$ for the root system and the root lattice of $\mathfrak{g}$, respectively. It is a commutative $\mathbb{C}$-algebra under the tensor products and identified with $\mathbb{C}[Q/W]$.

By pulling back through the adjoint representation $G \to \mathrm{Inn}(\mathfrak{g})$, $\mathcal{R}(\mathfrak{g})$ is naturally identified with a subgroup of $\mathcal{K}(G)$. Then by using the tensor products, $\mathcal{K}_{\mathsf{S}}(G)$ is naturally a $\mathcal{R}(\mathfrak{g})$-module.

Let $[\lambda] \subset \mathfrak{h}^*$ be a Q-coset, and write $W_{[\lambda]}$ for its stabilizer group in W. Then $W_{[\lambda]}$ equals the Weyl group of the root system

$$\{\alpha \in \Delta \mid \langle \mu, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \mu \in [\lambda]\} \qquad (\alpha^\vee \text{ denotes the corresponding coroot}).$$

Definition 1.1. *Let $\mathcal{K}$ be a $\mathcal{R}(\mathfrak{g})$-module equipped with a family $\{\mathcal{K}_\mu\}_{\mu\in\mathfrak{h}^*}$ of subspaces such that $\mathcal{K}_{w\cdot\mu} = \mathcal{K}_\mu$ for all $w \in W_{[\lambda]}$ and $\mu \in \mathfrak{h}^*$. A $\mathcal{K}$-valued coherent family on $[\lambda]$ is a map*

$$\Theta : [\lambda] \to \mathcal{K}$$

satisfying the following two conditions:

- *for all $\mu \in [\lambda]$, $\Theta(\mu) \in \mathcal{K}_\mu$;*

- *for all finite-dimensional algebraic representations F of* $\mathrm{Inn}(\mathfrak{g})$*, and all* $\mu \in [\lambda]$*,*
$$F \cdot (\Theta(\mu)) = \sum_{\nu} \Theta(\mu + \nu),$$
where ν runs over all weights of F, counted with multiplicities, and F is obviously identified with an element of $\mathcal{R}(\mathfrak{g})$.

In the notation of Definition **??**, let $\mathrm{Coh}_{[\lambda]}(\mathcal{K})$ denote the vector space of all $\mathcal{K}$-valued coherent families on $[\lambda]$. It is a representation of $W_{[\lambda]}$ under the action
$$(w \cdot \Theta)(\mu) = \Theta(w^{-1} \cdot \mu), \qquad \text{for all } \ w \in W_{[\lambda]}, \ \mu \in [\lambda].$$
For simplicity in notation, put
$$\mathrm{Coh}_{[\lambda],\mathsf{S}}(G) := \mathrm{Coh}_{[\lambda]}(\mathcal{K}_{\mathsf{S}}(G)).$$

1.2. **Counting the irreducible representations with bounded complex associated varieties.** Suppose that $\lambda \in [\lambda]$ and denote by W_λ the stabilizer of λ in W. Then $W_\lambda \subset W_{[\lambda]}$. Write 1_{W_λ} for the trivial representation of W_λ.

The following Theorem is due to Vogan. We will provide a proof for the lack of reference.

Theorem 1.2. *The equality*
$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = [1_{W_\lambda} : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)]$$
holds.

Here and henceforth, $[\ :\ ]$ indicates the multiplicity of the first (irreducible) representation in the second one. Theorem **??** implies that
$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = \sum_{\sigma \in \mathrm{Irr}(W_{[\lambda]})} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)]. \tag{1.1}$$
Thus it suffices to understand the multiplicity $[\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)]$ for every $\sigma \in \mathrm{Irr}(W_{[\lambda]})$. Let $\sigma \in \mathrm{Irr}(W_{[\lambda]})$. Define the nilpotent orbit
$$\mathcal{O}_\sigma := \mathrm{Springer}(j_{W_{[\lambda]}}^W \sigma_0) \subset \mathrm{Nil}(\mathfrak{g}^*) \quad (\text{“Springer” indicates the Springer correspondence}),$$
where σ_0 denote the special irreducible representation of $W_{[\lambda]}$ that lies in the same double cell as σ, and $j_{W_{[\lambda]}}^W \sigma_0$ denotes the j-induction which is an irreducible representation of W.

Proposition 1.3. *Suppose that* $\sigma \in \mathrm{Irr}(W_{[\lambda]})$ *and* $\mathcal{O}_\sigma \not\subset \mathsf{S}$*. Then*
$$[\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)] = 0.$$

Combining Theorems **??** and **??**, we conclude that
$$\sharp(\mathrm{Irr}_{\lambda,\mathsf{S}}(G)) = \sum_{\sigma \in \mathrm{Irr}(W_{[\lambda]}), \mathcal{O}_\sigma \subset \mathsf{S}} [1_{W_\lambda} : \sigma] \cdot [\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)]. \tag{1.2}$$

Write
$$\mathrm{Coh}_{[\lambda]}(G) := \mathrm{Coh}_{[\lambda],\mathrm{Nil}(\mathfrak{g}^*)}(G),$$
which contains $\mathrm{Coh}_{[\lambda],\mathsf{S}}(G)$ as a subrepresentation. It is obvious that
$$[\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)] \leq [\sigma : \mathrm{Coh}_{[\lambda]}(G)]. \tag{1.3}$$
We expect that
$$\text{if } \ \mathcal{O}_\sigma \subset \mathsf{S} \ , \text{ then } \ [\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(G)] = [\sigma : \mathrm{Coh}_{[\lambda]}(G)]. \tag{1.4}$$

1.3. **Counting the irreducible representations annihilated by a maximal primitive ideal.** Write I_λ for the maximal ideal of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character λ. Its associated variety equals the Zariski closure $\overline{\mathcal{O}_\lambda}$ of an $\mathrm{Inn}(\mathfrak{g})$-orbit $\mathcal{O}_\lambda \subset \mathrm{Nil}(\mathfrak{g}^*)$. Note that an irreducible Casselman-Wallach representation of G lies in $\mathrm{Irr}_{\lambda,\overline{\mathcal{O}_\lambda}}(G)$ if and only if it is annihilated by I_λ.

Let ${}^L\mathscr{C}_\lambda \subset \mathrm{Irr}(W_{[\lambda]})$ be the subset consisting all the irreducible representations that occur in

$$(J_{W_\lambda}^{W_{[\lambda]}} \mathrm{sgn}) \otimes \mathrm{sgn},$$

where $J_{W_\lambda}^{W_{[\lambda]}}$ indicates the J-induction, and sgn denotes the sign character.

Proposition 1.4 ([?BVUni, (5.26), Proposition 5.28])**.** *The set*

$${}^L\mathscr{C}_\lambda = \left\{ \sigma \in \mathrm{Irr}(W_{[\lambda]}) \mid \mathcal{O}_\sigma \subset \overline{\mathcal{O}_\lambda},\ [1_{W_\lambda} : \sigma] \neq 0 \right\}.$$

Moreover, the multiplicity $[1_{W_\lambda} : \sigma]$ *is one when* $\sigma \in {}^L\mathscr{C}_\lambda$.

Combining (**??**), Proposition **??** and Proposition **??**, we obtain the following inequality (and the equality is expected).

Corollary 1.5. *The inequality*

$$\sharp(\mathrm{Irr}_{\lambda,\overline{\mathcal{O}_\lambda}}(G)) \leq \sum_{\sigma \in {}^L\mathscr{C}_\lambda} [\sigma : \mathrm{Coh}_{[\lambda]}(G)] \tag{1.5}$$

holds.

1.4. **Special unipotent representations of classical groups.** We are particularly interested in counting special unipotent representations of real classical groups.

Let $\star$ be one of the 14 symbols

$$A,\ A^{\mathbb{C}},\ A^{\mathbb{H}},\ A^*,\ B,\ D,\ B^{\mathbb{C}},\ D^{\mathbb{C}},\ C,\ C^{\mathbb{C}},\ D^*,\ C^*,\ \widetilde{C},\ \widetilde{C}^{\mathbb{C}}.$$

Suppose that G is a classical Lie group of type $\star$, namely G respectively equals one of the following Lie groups:

$$\begin{gathered} \mathrm{GL}_n(\mathbb{R}),\ \mathrm{GL}_n(\mathbb{C}),\ \mathrm{GL}_n(\mathbb{H}),\ \mathrm{U}(p,q), \\ \mathrm{SO}(p,q)\ (p+q\ \text{ is odd}),\ \mathrm{SO}(p,q)\ (p+q\ \text{ is even}), \\ \mathrm{SO}_n(\mathbb{C})\ (n\ \text{ is odd}),\ \mathrm{SO}_n(\mathbb{C})\ (n\ \text{ is even}), \\ \mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}^*(2n),\ \mathrm{Sp}(p,q),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}_{2n}(\mathbb{C}) \qquad (n,p,q \geq 0). \end{gathered}$$

Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the metaplectic double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n = 0$. As usual, the universal Cartan subalgebra $\mathfrak{h}$ of the complexified Lie algebra $\mathfrak{g}$ of G is respectively identified with

$$\begin{gathered} \mathbb{C}^n,\ \mathbb{C}^n \times \mathbb{C}^n,\ \mathbb{C}^{2n},\ \mathbb{C}^{p+q}, \\ \mathbb{C}^{\frac{p+q-1}{2}},\ \mathbb{C}^{\frac{p+q}{2}}, \\ \mathbb{C}^{\frac{p+q-1}{2}} \times \mathbb{C}^{\frac{p+q-1}{2}},\ \mathbb{C}^{\frac{p+q}{2}} \times \mathbb{C}^{\frac{p+q}{2}}, \\ \mathbb{C}^n,\ \mathbb{C}^n \times \mathbb{C}^n,\ \mathbb{C}^n,\ \mathbb{C}^{p+q},\ \mathbb{C}^n,\ \mathbb{C}^n \times \mathbb{C}^n. \end{gathered}$$

We define the Langlands dual $\check{G}$ of G to be the respective complex group

$$\begin{gathered} \mathrm{GL}_n(\mathbb{C}),\ \mathrm{GL}_n(\mathbb{C}),\ \mathrm{GL}_{2n}(\mathbb{C}),\ \mathrm{GL}_{p+q}(\mathbb{C}), \\ \mathrm{Sp}_{p+q-1}(\mathbb{C})\ (p+q\ \text{ is odd}),\ \mathrm{SO}_{p+q}(\mathbb{C})\ (p+q\ \text{ is even}), \\ \mathrm{Sp}_{n-1}(\mathbb{C})\ (n\ \text{ is odd}),\ \mathrm{SO}_n(\mathbb{C})\ (n\ \text{ is even}), \\ \mathrm{SO}_{2n+1}(\mathbb{C}),\ \mathrm{SO}_{2n+1}(\mathbb{C}),\ \mathrm{SO}_{2n}(\mathbb{C}),\ \mathrm{SO}_{2p+2q+1}(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})\ \text{or}\ \ \mathrm{Sp}_{2n}(\mathbb{C}). \end{gathered}$$

Write $n_{\check{G}}$ for the dimension of the standard representation of $\check{G}$, which respectively equals

$$n, n, 2n, p+q, p+q-1, p+q, n-1, n, 2n+1, 2n+1, 2n, 2p+2q+1, 2n, \text{ or } \ 2n.$$

For a Young diagram $\imath$, write

$$\mathbf{r}_1(\imath) \geq \mathbf{r}_2(\imath) \geq \mathbf{r}_3(\imath) \geq \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath) \geq \mathbf{c}_2(\imath) \geq \mathbf{c}_3(\imath) \geq \cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty} \mathbf{r}_i(\imath)$ the total size of $\imath$.

We recall the parameterization of the set $\mathrm{Nil}(\check{G})$ of nilpotent $\check{G}$ orbits in $\check{\mathfrak{g}} := \mathrm{Lie}(\check{G})$ by Young diagrams. Each nilpotent orbit $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G})$ is identified with a Young diagram, still called $\check{\mathcal{O}}$ by abuse of notation, together with a possible symbol I/II with the following property:

- $|\check{\mathcal{O}}| = n_{\check{G}}$;
- if $\star \in \{ D, D^{\mathbb{C}}, D^*, C, C^{\mathbb{C}}, C^* \}$, then all even rows occur in $\check{\mathcal{O}}$ with even multiplicity;
- if $\star \in \{ B, \widetilde{C}, B^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}} \}$, then all odd rows occur in $\check{\mathcal{O}}$ with even multiplicity.
- if $\star \in \{ D, D^{\mathbb{C}}, D^* \}$ and all rows of $\check{\mathcal{O}}$ are even (called a very even orbit), then one attach the symbol "I" or "II" to $\check{\mathcal{O}}$. We adopt the convention that the orbits with symbol I are those can be induced from the standard parabolic subgroups. [Here, $\check{G}$ is the dual group of G. Since we fixed an abstract Cartan $\mathfrak{h}$ and therefore a Borel subalgebra of G, the corresponding Cartan and Borel are also fixed for $\check{G}$!]

Since, the situation of orbits marked with I and II are symmetry (see **??**), in the following we will not consider the orbits with "II" and if $\star \in \{ D, D^{\mathbb{C}}, D^* \}$,

a partition is implicitly attached to the symbol "I" when necessary.

From now on, let $\check{\mathcal{O}}$ be a Young diagram which is naturally identified with an nilpotent orbit of $\check{\mathcal{G}}$.

We define an element $\lambda_{\check{\mathcal{O}}} := \lambda_{\star,\check{\mathcal{O}}} \in \mathfrak{h}^*$ as in what follows. For every $a \in \mathbb{N}^+$ (the set of positive integers), write

$$\rho(a) := \begin{cases} (\frac{a-1}{2}, \frac{a-3}{2}, \cdots, 2, 1), & \text{if } a \text{ is odd;} \\ (\frac{a-1}{2}, \frac{a-3}{2}, \cdots, \frac{3}{2}, \frac{1}{2}), & \text{if } a \text{ is even,} \end{cases}$$

and

$$\tilde{\rho}(a) := (\frac{a-1}{2}, \frac{a-3}{2}, \cdots, \frac{3-a}{2}, \frac{1-a}{2}).$$

By convention, $\rho(1)$ is the empty sequence. Write $a_1 \geq a_2 \geq \cdots \geq a_s > 0$ $(s \geq 0)$ for the nonzero row lengths of $\check{\mathcal{O}}$. Then we define

$$\lambda_{\check{\mathcal{O}}} := \begin{cases} (\tilde{\rho}(a_1), \tilde{\rho}(a_2), \cdots, \tilde{\rho}(a_s)), & \text{if } \star \in \{ A, A^{\mathbb{H}}, A^* \}; \\ (\tilde{\rho}(a_1), \tilde{\rho}(a_2), \cdots, \tilde{\rho}(a_s); \tilde{\rho}(a_1), \tilde{\rho}(a_2), \cdots, \tilde{\rho}(a_s)), & \text{if } \star = A^{\mathbb{C}}; \\ (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0^{l_1}), & \text{if } \star \in \{ B, C, D, D^*, C^*, \widetilde{C} \}; \\ (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0^{l_1}; \rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0^{l_1}), & \text{if } \star \in \{ B^{\mathbb{C}}, C^{\mathbb{C}}, D^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}} \}. \end{cases}$$

Here

$$l_1 := \left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor,$$

and 0^{l_1} denotes the sequence of 0's of length l_1.

Note that $\lambda_{\check{\mathcal{O}}}$ is conjugate to the neutral element in any $\mathfrak{sl}_2$ triple attached to $\check{\mathcal{O}}$. (put more explanation?)

By using Harish-Chandra isomorphism, we view $\lambda_{\check{\mathcal{O}}}$ as a character $\lambda_{\check{\mathcal{O}}} : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$. Write

$$I_{\check{\mathcal{O}}} := I_{\star,\check{\mathcal{O}}}$$

for the maximal ideal of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character $\lambda_{\check{\mathcal{O}}}$.

Finally, we define the set of the special unipotent representations of G attached to $\check{\mathcal{O}}$ by

$$\begin{aligned}\mathrm{Unip}_{\check{\mathcal{O}}}(G) :=& \mathrm{Unip}_{\star,\check{\mathcal{O}}}(G)\\ :=& \begin{cases}\{\pi \in \mathrm{Irr}(G) \mid \pi \text{ is genuine and annihilated by } I_{\check{\mathcal{O}}}\}, & \text{if } \star = \widetilde{C};\\ \{\pi \in \mathrm{Irr}(G) \mid \pi \text{ is annihilated by } I_{\check{\mathcal{O}}}\}, & \textit{otherwise}.\end{cases}\end{aligned}$$

Here "genuine" means that the representation π of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ does not descend to $\mathrm{Sp}_{2n}(\mathbb{R})$. Our ultimate goal is to parametrize the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ and to construct all the representations in this set ([?BMSZ2]). (Define the unipotent packet using the neutral element first?)

Remark 1.6. Suppose $\star = \{ D, D^{\mathbb{C}}, D^* \}$. Let $\mathcal{A}$ be an outer automorphism of G, which induces an automorphism on $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$. Let

$$I'_{\check{\mathcal{O}}} := \mathcal{A}(I_{\check{\mathcal{O}}}).$$

If $\check{\mathcal{O}}$ has only even rows (i.e. $\check{\mathcal{O}}$ is very even), then $I'_{\check{\mathcal{O}}}$ is the maximal primitive ideal attached to the nilpotent orbit $\check{\mathcal{O}}_{II}$. Otherwise $I_{\check{\mathcal{O}}} = I'_{\check{\mathcal{O}}}$.

1.5. **The cases of general linear groups and unitary groups.** For any Young diagram $\imath$, we introduce the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as the following subset of $\mathbb{N}^+ \times \mathbb{N}^+$:

$$\mathrm{Box}(\imath) := \left\{ (i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \;\middle|\; j \leq \mathbf{r}_i(\imath) \right\}. \tag{1.6}$$

We also introduce five symbols $\bullet$, s, r, c and d, and make the following definitions.

Definition 1.7. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *when $S = \{s\}$ or $\{r\}$, every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$;*
- *when $S = \{c\}$ or $\{d\}$, every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$.*

Definition 1.8. *Suppose that $\star \in \{A, A^{\mathbb{H}}, A^*\}$. A painting $\mathcal{P}$ on a Young diagram $\imath$ has type $\star$ if*

- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases}\{\bullet, c, d\}, & \text{if } \star = A;\\ \{\bullet\}, & \text{if } \star = A^{\mathbb{H}};\\ \{\bullet, s, r\}, & \text{if } \star = A^*,\end{cases}$$

- *if $\star = A$ or $A^{\mathbb{H}}$, then $\mathcal{P}^{-1}(\bullet)$ has even number of boxes in every column of $\imath$,*
- *if $\star = A^*$, then $\mathcal{P}^{-1}(\bullet)$ has even number of boxes in every row of $\imath$.*

To reconcile with the Barbasch-Vogan duality, we write $\mathrm{PP}_\star(\imath^t)$ for the set of paintings on $\imath$ that has type $\star$ where $\imath^t$ is the transpose of $\imath$,.

The special unipotent representations of the general linear groups are well-understood. In particular, we have the following result on the counting.

Theorem 1.9. *The equality*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \begin{cases} \sharp(\mathrm{PP}_\star(\check{\mathcal{O}})), & \textit{if } \star \in \{A, A^{\mathbb{H}}\}; \\ 1, & \textit{if } \star = A^{\mathbb{C}} \end{cases}$$

holds.

Remark 1.10. If $\star = A$, then

$$\sharp(\mathrm{PP}_\star(\check{\mathcal{O}})) = \prod_{i \in \mathbb{N}^+} (1 + \text{the number of rows of length } i \text{ in } \check{\mathcal{O}})$$

If $\star = A^{\mathbb{H}}$, then

$$\sharp(\mathrm{PP}_\star(\check{\mathcal{O}})) = \begin{cases} 1, & \text{if all row lengths of } \check{\mathcal{O}} \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $\imath$ is a Young diagram and $\mathcal{P}$ is a painting on $\imath$ that has type A^*. Let

$$\mathrm{AC}_{\mathcal{P}} : \mathrm{Box}(\imath) \to \{+, -\}$$

to be the map such that

- the symbols $+$ and $-$ occur alternatively in each row of $\imath$;
- for all $1 \leq i \leq \mathbf{c}_1(\imath)$,

$$\mathrm{AC}_{\mathcal{P}}(i, \mathbf{r}_i(\imath)) := \begin{cases} +, & \text{if } \mathcal{P}(i, \mathbf{r}_i(\imath)) = r; \\ -, & \text{if } \mathcal{P}(i, \mathbf{r}_i(\imath)) \in \{\bullet, s\}. \end{cases}$$

Define the signature of $\mathcal{P}$ to be the pair

$$\begin{aligned}(p_{\mathcal{P}}, q_{\mathcal{P}}) := &= (\sharp(\mathcal{P}^{-1}(r)) + \tfrac{1}{2}\sharp(\mathcal{P}^{-1}(\bullet)), \sharp(\mathcal{P}^{-1}(s)) + \tfrac{1}{2}\sharp(\mathcal{P}^{-1}(\bullet))) \\ &= (\sharp(\mathrm{AC}_{\mathcal{P}}^{-1}(+)), \sharp(\mathrm{AC}_{\mathcal{P}}^{-1}(-))).\end{aligned}$$

[The first equation is the true definition of signature. The second one is an easy consequence of the definition of $\mathrm{AC}_{\mathcal{P}}$.]

Example 1.11. *Suppose that*

$$\check{\mathcal{O}} = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \\ \cline{1-1} \end{array} \quad \textit{and} \quad \mathcal{P} = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet & r \\ \hline \bullet & \bullet \\ \cline{1-2} s & r \\ \cline{1-2} s \\ \cline{1-1} r \\ \cline{1-1} \end{array} \in \mathrm{PP}_{A^*}(\check{\mathcal{O}}).$$

Then

$$\mathrm{AC}_{\mathcal{P}} = \begin{array}{|c|c|c|c|c|} \hline + & - & + & - & + \\ \hline + & - \\ \cline{1-2} - & + \\ \cline{1-2} - \\ \cline{1-1} + \\ \cline{1-1} \end{array} \quad \textit{and} \quad (p_{\mathcal{P}}, q_{\mathcal{P}}) = (6, 5).$$

Given two Young diagrams $\imath$ and $\jmath$, write $\imath \overset{r}{\sqcup} \jmath$ for the Young diagram whose multiset of nonzero row lengths equals the union of those of $\imath$ and $\jmath$.

For unitary groups, we have the following result.

Theorem 1.12. *Assume that* $\star = A^*$ *so that* $G = \mathrm{U}(p,q)$. *If there is a decomposition*

$$\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}} \overset{r}{\sqcup} \check{\mathcal{O}}'_{\mathrm{b}} \overset{r}{\sqcup} \check{\mathcal{O}}'_{\mathrm{b}}$$

such that all nonzero row lengths of $\check{\mathcal{O}}_{\mathrm{g}}$ *have the same parity as* $p+q$, *and all nonzero row lengths of* $\check{\mathcal{O}}'_{\mathrm{b}}$ *have different parity as* $p+q$, *then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp\left\{ \mathcal{P} \in \mathrm{PP}_{\star}(\check{\mathcal{O}}_{\mathrm{g}}) \mid (p_{\mathcal{P}} + \left|\check{\mathcal{O}}'_{\mathrm{b}}\right|, q_{\mathcal{P}} + \left|\check{\mathcal{O}}'_{\mathrm{b}}\right|) = (p,q) \right\}$$

If there is no such decomposition, then $\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = 0$.

1.6. **Orthogonal and symplectic groups: reduction to good parity.** Now we assume that

$$\star \in \left\{ B, D, B^{\mathbb{C}}, D^{\mathbb{C}}, C, C^{\mathbb{C}}, D^*, C^*, \widetilde{C}, \widetilde{C}^{\mathbb{C}} \right\}.$$

Then there is a unique decomposition

$$\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}} \overset{r}{\sqcup} \check{\mathcal{O}}'_{\mathrm{b}} \overset{r}{\sqcup} \check{\mathcal{O}}'_{\mathrm{b}}$$

such that $\check{\mathcal{O}}_{\mathrm{g}}$ has $\star$-good parity in the sense that all its nonzero row lengths are

$$\begin{cases} \text{even}, & \text{if } \star \in \{ B, B^{\mathbb{C}}, \widetilde{C}, \widetilde{C}^{\mathbb{C}} \}; \\ \text{odd}, & \text{if } \star \in \{ C, D, C^{\mathbb{C}}, D^{\mathbb{C}}, D^*, C^* \}, \end{cases}$$

and $\check{\mathcal{O}}_{\mathrm{b}}$ has $\star$-bad parity in the sense that all its nonzero row lengths are

$$\begin{cases} \text{odd}, & \text{if } \star \in \{ B, B^{\mathbb{C}}, \widetilde{C}, \widetilde{C}^{\mathbb{C}} \}; \\ \text{even}, & \text{if } \star \in \{ C, D, C^{\mathbb{C}}, D^{\mathbb{C}}, D^*, C^* \}. \end{cases}$$

For simplicity, put

$$l := \left|\check{\mathcal{O}}_{\mathrm{b}}\right|,$$

and

$$G'_{\mathrm{b}} := \begin{cases} \mathrm{GL}_{\left|\check{\mathcal{O}}'_{\mathrm{b}}\right|}(\mathbb{R}) & \text{when } \star \in \{ B, C, \widetilde{C}, D \}, \\ \mathrm{GL}_{\left|\check{\mathcal{O}}'_{\mathrm{b}}\right|}(\mathbb{C}) & \text{when } \star \in \{ B^{\mathbb{C}}, C^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}}, D^{\mathbb{C}} \}. \\ \mathrm{GL}_{\frac{1}{2}\left|\check{\mathcal{O}}'_{\mathrm{b}}\right|}(\mathbb{H}) & \text{when } \star \in \{ C^*, D^* \}. \end{cases}$$

Note that G has a subgroup isomorphic to G'_{b} if and only if

$$\begin{cases} p, q \geq l & \text{when } G = \mathrm{SO}(p,q), \\ p, q \geq 2l & \text{when } G = \mathrm{Sp}(p,q), \\ \text{no conditions} & \text{otherwise.} \end{cases}$$

In such cases, G has a Levi subgroup isomorphic to $G'_{\mathrm{b}} \times G_{\mathrm{g}}$ where

$$G_{\mathrm{g}} := \begin{cases} \mathrm{SO}(p-l, q-l) & \text{when } \star \in \{ B, D \}, \\ \mathrm{SO}_{n-2l}(\mathbb{C}) & \text{when } \star \in \{ B^{\mathbb{C}}, D^{\mathbb{C}} \}, \\ \mathrm{O}^*(2n-2l) & \text{when } \star = D^*, \\ \mathrm{Sp}_{2n-2l}(\mathbb{R}) & \text{when } \star = C, \\ \widetilde{\mathrm{Sp}}_{2n-2l}(\mathbb{R}) & \text{when } \star = \widetilde{C}, \\ \mathrm{Sp}_{2n-2l}(\mathbb{C}) & \text{when } \star \in \{ C^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}} \}, \\ \mathrm{Sp}(p - \frac{1}{2}l, q - \frac{1}{2}l) & \text{when } \star = C^*. \end{cases}$$

Theorem 1.13. *Retain the notation above. When G has a subgroup isomorphic to G'_{b}, there is a bijection*

$$\begin{array}{rcl} \mathfrak{I}\colon\ \mathrm{Unip}_{G'}(\check{\mathcal{O}}'_b)\times \mathrm{Unip}_{G_g}(\check{\mathcal{O}}_g) & \longrightarrow & \mathrm{Unip}_G(\check{\mathcal{O}}) \\ (\pi',\pi_0) & \mapsto & \pi'\rtimes\pi_0. \end{array} \tag{1.7}$$

Otherwise,

$$\mathrm{Unip}_G(\check{\mathcal{O}})=\emptyset.$$

Combine the result for general linear groups, we list the consequence of the above theorem as the follows:

(a) Assume that $\star\in\{B,D\}$ so that $G=\mathrm{SO}(p,q)$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G))=\begin{cases}\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_l))\times\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(\mathrm{GL}_l(\mathbb{R}))), & \text{if } p,q\geq l;\\ 0, & \text{otherwise.}\end{cases}$$

(b) Assume that $\star=C^*$ so that $G=\mathrm{Sp}(p,q)$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G))=\begin{cases}\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_l)), & \text{if } p,q\geq \frac{l}{2};\\ 0, & \text{otherwise.}\end{cases}$$

(c) Assume that $\star\in\{C,\widetilde{C}\}$ so that $G=\mathrm{Sp}_{2n}(\mathbb{R})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G))=\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_l))\times\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(\mathrm{GL}_l(\mathbb{R}))).$$

(d) Assume that $\star\in\{B^{\mathbb{C}},D^{\mathbb{C}},C^{\mathbb{C}},\widetilde{C}^{\mathbb{C}},D^*\}$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G))=\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_l)).$$

1.7. **Orthogonal and symplectic groups: the case of good parity.** Now we further assume that $\check{\mathcal{O}}$ has $\star$-good parity, namely $\check{\mathcal{O}}=\check{\mathcal{O}}_{\mathrm{g}}$. By Theorem **??**, the counting problem in general is reduced to this case.

Definition 1.14. *A $\star$-pair is a pair $(i,i+1)$ of consecutive positive integers such that*

$$\begin{cases} i \text{ is odd}, & \text{if } \star\in\{C,\widetilde{C},C^*,C^{\mathbb{C}},\widetilde{C}^{\mathbb{C}}\};\\ i \text{ is even}, & \text{if } \star\in\{B,D,D^*,B^{\mathbb{C}},D^{\mathbb{C}}\}.\end{cases}$$

A $\star$-pair $(i,i+1)$ is said to be

- *vacant in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}})=\mathbf{r}_{i+1}(\check{\mathcal{O}})=0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}})=\mathbf{r}_{i+1}(\check{\mathcal{O}})>0$;*
- *tailed in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}})-\mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}})-\mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and even.*

Denote $\mathfrak{P}_\star(\check{\mathcal{O}})$ the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$.

We remark that, the set $\mathfrak{P}_\star(\check{\mathcal{O}})$ is the same as that in [?So, Section 5] (used to describe Lusztig's canonical quotient) when $\star\neq\widetilde{C}^{\mathbb{C}}$.

Theorem 1.15 (Barbasch-Vogan, Barbasch)**.** *Assume that $\star\in\{B^{\mathbb{C}},D^{\mathbb{C}},C^{\mathbb{C}},\widetilde{C}^{\mathbb{C}}\}$. Then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G))=2^{\sharp(\mathfrak{P}_\star(\check{\mathcal{O}}))}.$$

[Here is a tricky point: When $\star=\widetilde{C}^{\mathbb{C}}$, the Lusztig canonical quotient of $\check{\mathcal{O}}$ is a quotient by $\wp\sim\wp^c$ of $\mathbb{F}_2[\mathfrak{P}_\star(\check{\mathcal{O}})]$. However, the counting should still be correct!]

We attach to $\check{\mathcal{O}}$ a pair of Young diagrams

$$(\imath_{\check{\mathcal{O}}},\jmath_{\check{\mathcal{O}}}):=(\imath_\star(\check{\mathcal{O}}),\jmath_\star(\check{\mathcal{O}})),$$

as follows.

The case when $\star = \{B, B^{\mathbb{C}}\}$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star \in \{\widetilde{C}, \widetilde{C}^{\mathbb{C}}\}$. In this case, for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star \in \{C, C^*, C^{\mathbb{C}}\}$. In this case, for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

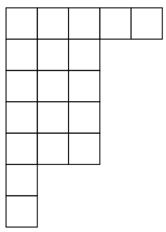
The case when $\star \in \{D, D^*, D^{\mathbb{C}}\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0, \end{cases}$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0\right), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}\right), & \text{otherwise.} \end{cases}$$

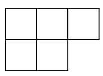
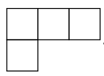
Example 1.16. *Suppose that $\star = C$, and $\check{\mathcal{O}}$ is the following Young diagram which has $\star$-good parity.*



Then

$$\mathfrak{P}_\star(\check{\mathcal{O}}) = \{(1,2), (5,6)\}$$

and

$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) =$  $\times$ .

Here and henceforth, when no confusion is possible, we write $\alpha \times \beta$ for a pair (α, β). We will also write $\alpha \times \beta \times \gamma$ for a triple (α, β, γ).

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.17. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- $\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$*;*

- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, r, c, d\}, & \text{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \text{if } \alpha = D; \\ \{\bullet, s, c\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \text{if } \alpha = C^*; \\ \{\bullet, s\}, & \text{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, s\}, & \text{if } \alpha = C; \\ \{\bullet\}, & \text{if } \alpha = D; \\ \{\bullet, r, d\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \text{if } \alpha = C^*; \\ \{\bullet, r\}, & \text{if } \alpha = D^*. \end{cases}$$

For any painted bipartition τ as in Definition **??**, we write

$$\imath_\tau := \imath,\ \mathcal{P}_\tau := \mathcal{P},\ \jmath_\tau := \jmath,\ \mathcal{Q}_\tau := \mathcal{Q},\ \alpha_\tau := \alpha,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

We further define a pair (p_τ, q_τ) of natural numbers given by the following recipe.

- If $\star_\tau \in \{B, D, C^*\}$, (p_τ, q_τ) is given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := (\#\bullet) + 2(\#r) + (\#c) + (\#d) + (\#B^+); \\ q_\tau := (\#\bullet) + 2(\#s) + (\#c) + (\#d) + (\#B^-). \end{cases} \tag{1.8}$$

  Here

$$\#\bullet := \#(\mathcal{P}^{-1}(\bullet)) + \#(\mathcal{Q}^{-1}(\bullet)) \qquad (\#\text{ indicates the cardinality of a finite set}),$$

  and the other terms are similarly defined.
- If $\star_\tau \in \{C, \widetilde{C}, D^*\}$, $p_\tau := q_\tau := |\tau|$.

We also define a classical group

$$G_\tau := \begin{cases} \mathrm{SO}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases} \tag{1.9}$$

Define

$$\mathsf{PBP}_\star(\check{\mathcal{O}}) := \{\,\tau \text{ is a painted bipartition} \mid \star_\tau = \star,\ \text{and } (\imath_\tau, \jmath_\tau) = (\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}})\,\}, \quad \text{and}$$
$$\mathsf{PBP}_G(\check{\mathcal{O}}) := \{\,\tau \in \mathsf{PBP}_\star(\check{\mathcal{O}}) \mid G_\tau = G\,\}.$$

Example 1.18. *Suppose that* $\star = B$ *and*

$$\check{\mathcal{O}} = \ydiagram{6,6,2,2,2}$$

Then

$$\tau := \begin{ytableau} \bullet & \bullet \\ \bullet \\ c \end{ytableau} \times \begin{ytableau} \bullet & \bullet & d \\ \bullet \\ d \end{ytableau} \times B^+ \in \mathrm{PBP}_\star(\check{\mathcal{O}}),$$

and

$$G_\tau = \mathrm{SO}(10, 9).$$

Theorem 1.19. *Assume that* $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$. *Then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) \leq 2^{\sharp(\mathfrak{P}_\star(\check{\mathcal{O}}))} \cdot \sharp\mathsf{PBP}_G(\check{\mathcal{O}}).$$

In [?BMSZ2], we will construct sufficiently many representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$, and show that the equality holds in **??**.