Special unipotent representations of real classical groups and theta correspondence

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Classical groups and special unipotent representations

	G	$\mathbf{G}$	$\mathbf{G}^\vee$	
D_n	$\mathrm{O}(p, 2n-p)$	$\mathrm{O}(2n, \mathbb{C})$	$\mathrm{O}(2n, \mathbb{C})$	D_n
C_n	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$	B_n
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D_n	$\mathrm{O}^*(n)$	$\mathrm{SO}(2n, \mathbb{C})$	$\mathrm{SO}(2n, \mathbb{C})$	D_n
C_n	$\mathrm{Sp}(p, n-p)$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$	B_n
A_n	$\mathrm{U}(p, n-p)$	$\mathrm{GL}(n, \mathbb{C})$	$\mathrm{GL}(n, \mathbb{C})$	A_n
A_m	$\mathrm{U}(r, m-r)$	$\mathrm{GL}(m, \mathbb{C})$	$\mathrm{GL}(m, \mathbb{C})$	A_m

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Arthur-Barbasch-Vogan's conj. on special unipotent repn. holds for G:
All *specail unipotent representations* of G are *unitarizable*.

Barbasch-Vogan's definition of unipotent representation

G: a real reductive group.

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- *Altas of Lie group:* $\rightsquigarrow$ complete answer for exceptional groups.

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- Lemma:

$$\begin{aligned} &\#\{\, \pi \in \mathrm{Irr}_\mu(\mathfrak{g}, K)(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \,\} \\ &= \sum_{\substack{\tau \in \mathcal{D} \\ \mathcal{D} \rightsquigarrow \mathcal{O}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathcal{K}_\lambda(\mathfrak{g}, K)] \end{aligned}$$

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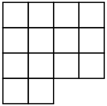
- Example of good parity:

$\mathcal{O}$



$\mathrm{Sp}(14, \mathbb{C})$

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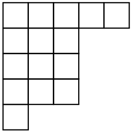

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- $\check{\mathcal{O}}$ decompose into two parts $\check{\mathcal{O}}_g$ (good parity) and $\check{\mathcal{O}}_b$ (bad parity).
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Theorem (Let $\check{\mathcal{O}}'_b = \{\, r_1, \cdots, r_k \,\} \in \mathrm{Nil}_{\mathrm{GL}}$.)

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\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}_{\mathbb{R}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{Sp}_{\mathbb{R}}) & \xrightarrow{1-1} & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{\mathbb{R}}) \\
(\pi_1, \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}(|\check{\mathcal{O}}'_b|, \mathbb{R}) \times \mathrm{Sp}(2n_0, \mathbb{R})}^{\mathrm{Sp}(2n, \mathbb{R})} \pi_1 \otimes \pi_0
\end{array}
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- Use *theta correspondence* to construct $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
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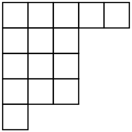
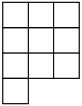
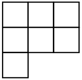
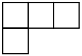
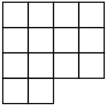
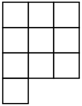
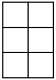
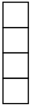
Example of descent sequences

$\mathbf{G}_i^\vee$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$
$\mathcal{O}_i^\vee$				
$\mathcal{O}_i$				
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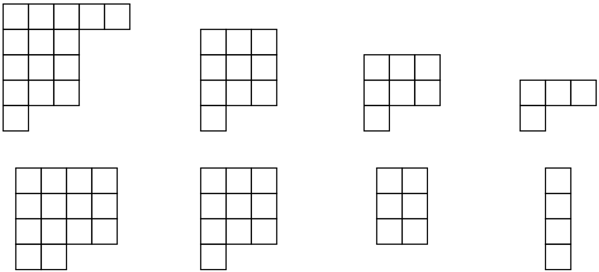

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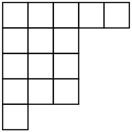
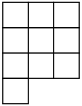
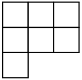
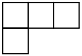
$\mathcal{O}_i^\vee$				
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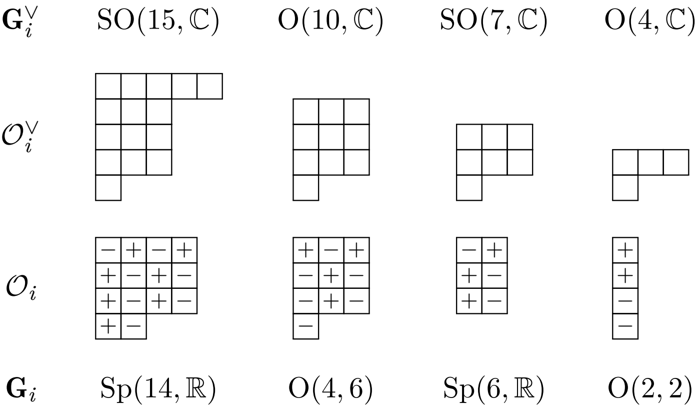

Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

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- *Moreover,*

$$\mathrm{Unip}_{\mathcal{O}^\vee}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}.$$

Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

		$\mathrm{Sp}(6, \mathbb{R})$	
$\mathrm{O}(4,0)$		$\theta(\mathrm{sgn}^{+,-})$	
$\mathrm{O}(3,1)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(2,2)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(1,3)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(0,4)$			$\theta(\mathrm{sgn}^{-,+})$

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- Corollary: (using [Gomez-Zhu]) For π_χ,

$$\text{Whittaker cycle} = \text{Wavefront cycle}.$$

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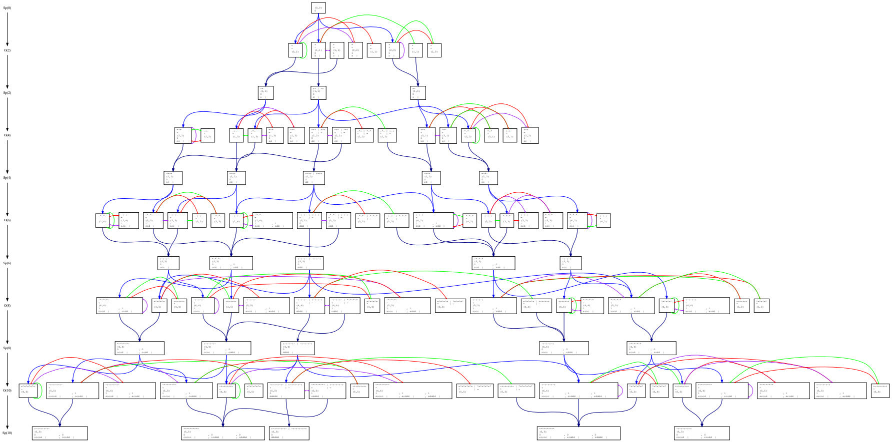
- The injectivity of theta lifting is crucial!

Example: $\mathcal{O}$ = regular nilpotent orbit



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- Question: How to describe $\pi_{\psi,\eta}$ explicitly?

Dan Barabasch, M. , Binyong Sun and Chen-Bo Zhu
Special unipotent representations: orthogonal and symplectic groups
ArXiv e-prints: https://arxiv.org/abs/1712.05552v2

Thank you for your attention!