# SPECIAL UNIPOTENT REPRESENTATIONS: COUNTING

DAN M. BARBASCH, JIA-JUN MA, BINYONG SUN, AND CHEN-BO ZHU

Contents



## 1.

## 2. Special non-special switch

2.0.1. *The case of* $\star = \widetilde{C}$. Suppose that $\star = \widetilde{C}$. We define an involution $\natural$ on the set $\{\bullet, s, r, c, d\}$ by sending $\bullet, s, r, c, d$ to $\bullet, r, s, d, c$ respectively.

Let $\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})$ such that $\wp \neq \emptyset$. Pick a pair $(2k-1, 2k) \in \wp$, and let $\wp' = \wp - \{(2k-1, 2k)\}$.

**Lemma 2.1.** *For each* $\tau' \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp')$, *there is a unique* $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp)$ *such that*

$$\begin{aligned}\mathcal{P}_\tau(i,j) &:= \begin{cases} \natural(\mathcal{Q}_{\tau'}(i,j)) & \text{if } i = k \\ \mathcal{P}_{\tau'}(i,j) & \text{otherwise}\end{cases} \\ \mathcal{Q}_\tau(i,j) &:= \begin{cases} \natural(\mathcal{P}_{\tau'}(i,j)) & \text{if } i = k \\ \mathcal{Q}_{\tau'}(i,j) & \text{otherwise}\end{cases}\end{aligned} \tag{2.1}$$

*Moreover, the map* $\tau' \mapsto \tau$ *yields a bijection*

$$\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp') \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp). \tag{2.2}$$

*Proof.* One first variety that $(\imath(\check{\mathcal{O}}, \wp), \mathcal{P}_\tau) \times (\jmath(\check{\mathcal{O}}, \wp), \mathcal{Q}_\tau) \times \widetilde{C}$ is a valid painted bi-partition. We can define the inverse of (2.2) by (2.1) (switching the role of $\tau$ and $\tau'$). This proves the bijectivity of (2.2). □

[ We now verify $\tau$ is a valid painted bipartition.

The relevant portion of $\mathcal{P}$ and $\mathcal{Q}$ are boxes with indexes $(i,j)$ such that $k-1 \leqslant i \leqslant k$ and $\frac{\mathbf{r}_{2k}(\check{\mathcal{O}})}{2} \leqslant j \leqslant \frac{\mathbf{r}_{2k-1}(\check{\mathcal{O}})}{2}$

$$\tau' : \begin{array}{c} w_1\ x_2 \\ * \\ * \\ * \\ * \\ w_2 \end{array} \times \begin{array}{cc} \multicolumn{2}{c}{w_3\ x_0} \\ * & r \\ * & \vdots \\ * & \vdots \\ * & r \\ \multicolumn{2}{c}{w_4\ x_1} \end{array} \qquad \bar{\tau} : \begin{array}{cc} \multicolumn{2}{c}{w_5\ y_0} \\ * & s \\ * & \vdots \\ * & \vdots \\ * & s \\ \multicolumn{2}{c}{w_6\ y_1} \end{array} \times \begin{array}{c} w_7\ y_2 \\ * \\ * \\ * \\ * \\ w_8 \end{array} , \tag{2.3}$$

---

2000 *Mathematics Subject Classification.* 22E45, 22E46.

*Key words and phrases.* orbit method, unitary dual, unipotent representation, classical group, theta lifting, moment map.

Either all $w_i$ are $\emptyset$ ($\star$ are all $\emptyset$) or all none-empty ($\star$ are all $\bullet$). When $w_i$ are all $\emptyset$ or $\bullet$, the validity of $\tau$ is easy.

For the rest cases, $w_1, W_3, w_5, w_7$ must be painted by $\bullet$. Therefore, it suffice to consider the switch

$$w_2 \qquad \times\ w_4\ x_1 \longleftrightarrow w_6\ y_1\ \times\ w_8 \qquad .$$

| | $\tau$ | $\tau^s$ | $\tau^{ns}$ | |
|---|---|---|---|---|
| | $\begin{array}{cc} x_0' & * \\ x_1' & x_2' \end{array} \times \begin{array}{c} * \end{array}$ | $\begin{array}{cc} x_0 & * \\ x_1 & x_2 \end{array} \times \begin{array}{cc} * & * \\ x_3 & \\ s & \\ \vdots & \\ s & \end{array}$ | $\begin{array}{cc} y_0 & * \\ r & y_2 \\ \vdots & \\ r & \\ y_1 & \\ y_3 & \end{array} \times \begin{array}{cc} * & * \end{array}$ | |
| $x_1' = s$ then $z_0 = \emptyset/s$, $x_0 = \emptyset/\bullet$, $x_1 = \bullet$, $x_2 = x_2'$ | $\begin{array}{cc} x_0' & * \\ s & x_2 \end{array} \times \begin{array}{c} * \end{array}$ | $\begin{array}{cc} x_0 & * \\ \bullet & x_2 \end{array} \times \begin{array}{cc} * & * \\ \bullet & \\ s & \\ \vdots & \\ s & \end{array}$ | $\begin{array}{cc} x_0 & * \\ r & x_2 \\ \vdots & \\ r & \\ c & \\ d & \end{array} \times \begin{array}{cc} * & * \end{array}$ | $x_2 \neq r$ |
| | | $\begin{array}{cc} x_0 & * \\ \bullet & r \end{array} \times \begin{array}{cc} * & * \\ \bullet & \\ s & \\ \vdots & \\ s & \end{array}$ | $\begin{array}{cc} x_0 & * \\ r & c \\ \vdots & \\ r & \\ r & \\ d & \end{array} \times \begin{array}{cc} * & * \end{array}$ | $x_2 = r$ |
| $x_1' \neq s$ then $x_1 = x_1'$, $x_2 = x_2'$ | $\begin{array}{cc} x_0' & * \\ x_1' & x_2 \end{array} \times \begin{array}{c} * \end{array}$ | $\begin{array}{cc} x_0 & * \\ x_1 & x_2 \end{array} \times \begin{array}{cc} * & * \\ s & \\ s & \\ \vdots & \\ s & \end{array}$ | $\begin{array}{cc} x_0 & * \\ r & x_2 \\ \vdots & \\ r & \\ r & \\ x_1 & \end{array} \times \begin{array}{cc} * & * \end{array}$ | $x_0' \neq c$ then $x_0' = \emptyset/s/r$, $x_0 = \emptyset/\bullet/r$ |
| | | $\begin{array}{cc} c & * \\ d & x_2 \end{array} \times \begin{array}{cc} * & * \\ s & \\ s & \\ \vdots & \\ s & \end{array}$ | $\begin{array}{cc} r & * \\ r & x_2 \\ \vdots & \\ r & \\ c & \\ d & \end{array} \times \begin{array}{cc} * & * \end{array}$ | $x_0' = c$ then $x_0 = x_0' = c$, $x_1 = x_1' = d$, $x_2 = x_2' = \emptyset/d$ |

TABLE 1. "special-non-special" switch

We can list all the cases:

| $\tau'$ | | | | $\tau$ | | | |
|---|---|---|---|---|---|---|---|
| $s$ | $\times$ | $r$ | $d$ | $s$ | $c$ | $\times$ | $r$ |
| $s$ | $\times$ | $d$ | $d$ | $s$ | $c$ | $\times$ | $d$ |
| $c$ | $\times$ | $r$ | $d$ | $c$ | $c$ | $\times$ | $r$ |
| $c$ | $\times$ | $d$ | $d$ | $c$ | $c$ | $\times$ | $d$ |

]

**Lemma 2.2.** *Suppose $\star \in \{B, D\}$ and $\wp \in \mathrm{PP}_\star(\check{\mathcal{O}})$. Let $\check{\mathcal{O}}_1$ be the $\star'$-good orbit having the rows*

$$\mathbf{r}_1(\check{\mathcal{O}}_1) := \mathbf{r}_1(\check{\mathcal{O}}) + 2, \quad \textit{and} \quad \mathbf{r}_{i+1}(\check{\mathcal{O}}_1) := \mathbf{r}_i(\check{\mathcal{O}}) \forall i = 1, 2, \cdots$$

*Let*

$$\wp_1 = \{ (i+1, i+2) \mid (i, i+1) \in \wp \}.$$

*Then the naive descent map yields a bijection*

$$\mathrm{PBP}_{\star'}(\check{\mathcal{O}}_1, \wp_1) \xrightarrow{\nabla_{\mathrm{naive}}} \mathrm{PBP}_{\star'}(\check{\mathcal{O}}, \wp)$$

*Proof.* Clear by the definition of $\nabla_{\mathrm{naive}}$. $\square$

**Proposition 2.3.** *Suppose $\star \in \{B, C, \widetilde{C}, D\}$. For $\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})$, $\left|\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp)\right| = \left|\mathrm{PBP}_\star(\check{\mathcal{O}}, \varnothing)\right|$*

*Proof.* Suppose $\star \in \{C, \widetilde{C}\}$. The proposition following by applying Lemma 2.1 finitely many times to reduce $\wp$ to $\varnothing$.

Suppose $\star \in \{B, D\}$. By Lemma 2.2, the problem translate to $\star' \in \{C, \widetilde{C}\}$ which we already proved. $\square$

Then $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}) \geqslant 0$.

For every $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \star \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, its leg is defined to be the pair

$$\mathrm{Leg}(\tau) := \Lambda_{\max(l'-1,0),1}(\imath, \mathcal{P}) \times \Lambda_{\max(l'-1,0),1}(\jmath, \mathcal{Q}),$$

and its body is defined to be the pair

$$\mathrm{Body}(\tau) := \bar{\Lambda}_{\max(l'-1,0),1}(\imath, \mathcal{P}) \times \bar{\Lambda}_{\max(l'-1,0),1}(\jmath, \mathcal{Q}).$$

Note that if $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp)$, then $\mathrm{Leg}(\tau)$ is represented by the first pair in (2.4) where

$$x_1 \neq \varnothing, \qquad x_0 = \varnothing \Leftrightarrow x_2 = \varnothing \Leftrightarrow \mathbf{r}_2(\check{\mathcal{O}}) = 0,$$

and the grey part consisting of $l - l' - 1$ boxes with label $r$.

$$\mathrm{Leg}(\tau): \quad \begin{array}{|c|}\hline x_2 \\ \hline \end{array} \times \begin{array}{|c|}\hline x_0 \\ \hline r \\ \hline \vdots \\ \vdots \\ \hline r \\ \hline x_1 \\ \hline \end{array} \qquad\qquad \mathrm{Leg}(\bar{\tau}): \quad \begin{array}{|c|}\hline y_0 \\ \hline s \\ \hline \vdots \\ \vdots \\ \hline s \\ \hline y_1 \\ \hline \end{array} \times \begin{array}{|c|}\hline y_2 \\ \hline \end{array}\,, \tag{2.4}$$

Likewise, for every $\bar{\tau} \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \bar{\wp})$, $\mathrm{Leg}(\bar{\tau})$ is represented by the second pair of (2.4) where

$$y_1 \neq \varnothing, \qquad y_0 = \varnothing \Leftrightarrow y_2 = \varnothing \Leftrightarrow \mathbf{r}_2(\check{\mathcal{O}}) = 0,$$

and the grey part consisting of $l - l' - 1$ boxes with label $s$.

The following proposition is much easier to check than Proposition **??**. We omit the details.

**Proposition 2.4.** *Suppose that* $\star = \widetilde{C}$ *and* $(1,2) \in \wp$. *For every* $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp)$ *such that* $\mathrm{Leg}(\tau)$ *is represented by the first pair in* (2.4), *there is a unique element* $\bar{\tau} \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \bar{\wp})$ *such that*

$$\mathrm{Body}(\bar{\tau}) = \mathrm{Body}(\tau)$$

*and* $\mathrm{Leg}(\bar{\tau})$ *is represented by the second pair in* (2.4) *with*

$$y_i = \bullet, r, s, d, c, \text{ or } \varnothing \qquad (i = 0, 1, 2),$$

*respectively if*

$$x_i = \bullet, s, r, c, d, \text{ or } \varnothing.$$

*Moreover, the map*

$$\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) \to \mathrm{PBP}_\star(\check{\mathcal{O}}, \bar{\wp}), \qquad \tau \mapsto \bar{\tau}.$$

*is bijective.*

## 3. Counting the multiplicity of non-special representations

In this section, we assume $\star \in \{ B, C, \widetilde{C}, D \}$. Let $\check{\mathcal{O}}$ be a $\star$-good parity orbit.

For $\star \in \{ B, D \}$, we write

$$\begin{aligned}
\mathrm{PBP}^d_\star(\check{\mathcal{O}}, \wp) &:= \{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) \mid x_\tau = d \} . \\
\mathrm{PBP}^{rc}_\star(\check{\mathcal{O}}, \wp) &:= \{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) \mid x_\tau \in \{ r, c \} \} . \\
\mathrm{PBP}^s_\star(\check{\mathcal{O}}, \wp) &:= \{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) \mid x_\tau = s \} . \\
\mathrm{PBP}^{-s}_\star(\check{\mathcal{O}}, \wp) &:= \{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) \mid x_\tau \neq s \} .
\end{aligned}$$

Note that $\mathrm{PBP}^{-s}_\star(\check{\mathcal{O}}, \wp) = \mathrm{PBP}^d_\star(\check{\mathcal{O}}, \wp) \sqcup \mathrm{PBP}^{rc}_\star(\check{\mathcal{O}}, \wp)$. We write

$$\mathrm{PBP}_{D,sc}(\check{\mathcal{O}}_0) := \{ \tau_0 \in \mathrm{PBP}_D(\check{\mathcal{O}}_0) \mid \mathcal{P}^{-1}_{\tau_0}(\{ s, c \}) \neq \varnothing \} .$$

We define $\mathrm{PBP}^\sharp_{D,sc}$ similarly.

We will prove the following counting lemma:

**Proposition 3.1.** *Let* $\wp \in \mathrm{PP}(\check{\mathcal{O}})$, *we have* $\left|\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp)\right| = \left|\mathrm{PBP}_\star(\check{\mathcal{O}}, \varnothing)\right|$. *When* $\star \in \{ B, D \}$, *for* $\sharp \in \{ -s, s, d, rc \}$, *we have*

$$\left|\mathrm{PBP}^\sharp_\star(\check{\mathcal{O}}, \wp)\right| = \left|\mathrm{PBP}^\sharp_\star(\check{\mathcal{O}}, \varnothing)\right| \tag{3.1}$$

*Proof.* We prove the proposition by induction. Assume for each good parity orbit $\check{\mathcal{O}}'$ such that $\mathbf{r}_k(\check{\mathcal{O}}') = 0$ the propitiation holds.

Suppose $\star \in \{ C, \widetilde{C} \}$.

When $(1,2) \in \mathrm{PP}_\star(\check{\mathcal{O}})$, and $(1,2) \in \wp$. Let $\bar{\wp} = \wp - \{ (1,2) \}$. By the switching algorithm, we have

$$|\mathrm{PBP}_\star(\mathcal{O}, \wp)| = \left|\mathrm{PBP}_\star(\check{\mathcal{O}}, \bar{\wp})\right| .$$

Therefore, we can assume $(1,2) \notin \wp$ without of loss of generality.

Suppose $(1,2) \in \mathrm{PP}_\star(\check{\mathcal{O}})$, we have a bijection

$$\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) \xrightarrow{\quad \nabla \quad} \mathrm{PBP}_{\star'}(\check{\mathcal{O}}', \wp')$$

for each $(1,2) \notin \wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})$. By the induction hypothesis,

$$\left|\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp)\right| = \left|\mathrm{PBP}_{\star'}(\check{\mathcal{O}}', \wp')\right| = \left|\mathrm{PBP}_{\star'}(\check{\mathcal{O}}', \varnothing)\right| = \left|\mathrm{PBP}_\star(\check{\mathcal{O}}, \varnothing)\right| .$$

Suppose $(1,2) \notin \mathrm{PP}_\star(\check{\mathcal{O}})$, we have a bijection

$$\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) \xrightarrow{\quad \nabla \quad} \mathrm{PBP}^{-s}_{\star'}(\check{\mathcal{O}}', \wp')$$

for each $\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})$. Therefore, by induction hypothesis, we have

$$\left|\mathrm{PBP}_\star(\check{\mathcal{O}},\wp)\right| = \left|\mathrm{PBP}^{-s}_{\star'}(\check{\mathcal{O}}',\wp')\right| = \left|\mathrm{PBP}^{-s}_{\star'}(\check{\mathcal{O}}',\emptyset)\right| = \left|\mathrm{PBP}_\star(\check{\mathcal{O}},\emptyset)\right|.$$

Note that under the

$$\mathrm{PBP}_{\star'}(\check{\mathcal{O}}',\wp') \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}'',\wp'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_0')$$

Suppose $\star \in \{B, D\}$. Assume $(2,3) \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. Then

$$\delta\colon \mathrm{PBP}_\star(\check{\mathcal{O}},\wp) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}',\wp') \times \mathrm{PBP}_D(\check{\mathcal{O}}_0)$$

is a bijection for each $\wp \in \mathrm{PP}_\star(\check{\mathcal{O}})$. Therefore

$$\left|\mathrm{PBP}_\star(\check{\mathcal{O}},\wp)\right| = \left|\mathrm{PBP}_{\star'}(\check{\mathcal{O}}',\wp')\right|\left|\mathrm{PBP}_D(\check{\mathcal{O}}_0)\right| = \left|\mathrm{PBP}_{\star'}(\check{\mathcal{O}}',\emptyset)\right|\left|\mathrm{PBP}_D(\check{\mathcal{O}}_0)\right| = \left|\mathrm{PBP}_\star(\check{\mathcal{O}},\emptyset)\right|.$$

Since $x_\tau = d \Leftrightarrow x_{\tau_0} = d$ and $x_\tau = s \Leftrightarrow x_{\tau_0} = s$, $\delta$ induces bijections

$$\mathrm{PBP}^\sharp_\star(\check{\mathcal{O}},\wp) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}',\wp') \times \mathrm{PBP}^\sharp_D(\check{\mathcal{O}}_0)$$

for $\sharp \in \{s, -s, d, rc\}$. Now (3.1) follows.

Assume $(2,3) \notin \mathrm{PBP}_\star(\check{\mathcal{O}})$. Note that the image of $\delta$ as the following

$$\begin{aligned}\mathrm{Im}(\delta) &= \left\{ (\tau'',\tau_0) \in \mathrm{PBP}_\star(\check{\mathcal{O}}'',\wp'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}},\emptyset) \,\middle|\, x_{\tau''} = d, \text{ or } \mathcal{P}^{-1}_{\tau_0}(\{s,c\}) \neq \emptyset \right\}\\ &= \mathrm{PBP}^d_\star(\check{\mathcal{O}}'',\wp'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_0) \sqcup \mathrm{PBP}^{rc}_\star(\check{\mathcal{O}}'',\wp'') \times \mathrm{PBP}_{D,sc}(\check{\mathcal{O}}_0).\end{aligned}$$

Now

$$\begin{aligned}&\left|\mathrm{PBP}_\star(\check{\mathcal{O}},\wp)\right|\\ &= \left|\mathrm{PBP}^d_\star(\check{\mathcal{O}},\wp'')\right|\left|\mathrm{PBP}_d(\check{\mathcal{O}}_0)\right| + \left|\mathrm{PBP}^{rc}_\star(\check{\mathcal{O}},\wp'')\right|\left|\mathrm{PBP}^{sc}_\star(\check{\mathcal{O}}_0)\right|\\ &= \left|\mathrm{PBP}_\star(\check{\mathcal{O}},\wp)\right|\end{aligned}$$

The equation (3.1) follows from the bijection:

$$\mathrm{PBP}^\sharp_\star(\check{\mathcal{O}},\wp) \xrightarrow{\ \delta\ } \mathrm{PBP}^d_\star(\check{\mathcal{O}}'',\wp'') \times \mathrm{PBP}^\sharp_D(\check{\mathcal{O}}_0) \sqcup \mathrm{PBP}^{rc}_\star(\check{\mathcal{O}}'',\wp'') \times \mathrm{PBP}^\sharp_{D,sc}(\check{\mathcal{O}}_0).$$

This finished the proof of the proposition. □

In the rest of this section we assume that $\check{\mathcal{O}} \neq \emptyset$. Let $\check{\mathcal{O}}'$ be the dual descent of $\check{\mathcal{O}}$ as defined in the Introduction. Then $\check{\mathcal{O}}'$ has $\star'$-good parity, where $\star'$ is the Howe dual of $\star$. Put

$$l' := l_{\star',\check{\mathcal{O}}'}.$$

The following equation of signatures will be crucial in our computation of the local system in the next section.

**Proposition 3.2.** *Suppose* $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. *Let* $\check{\mathcal{O}}'' := \check{\nabla}^2(\check{\mathcal{O}})$ *and* $\wp'' = \check{\nabla}^2(\wp)$. *Consider the map*

$$\delta\colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}'',\wp'') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}},\emptyset), \qquad \tau \mapsto (\nabla^2(\tau),\tau_{\mathbf{t}}) \tag{3.2}$$

*Then* $\delta$ *is an injection.*

- *When* $\star = C^*$ *or* $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$, *the map* $\delta$ *is a bijection. Moreover,*

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}),\mathbf{c}_2(\mathcal{O})) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}).$$

- *When* $\star \in \{B, D\}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$, *the map* $\delta$ *is an injection, whose image equals*

$$\left\{ (\tau'',\tau_0) \in \mathrm{PBP}_\star(\check{\mathcal{O}},\wp'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}},\emptyset) \,\middle|\, x_{\tau''} = d, \text{ or } \mathcal{P}^{-1}_{\tau_0}(\{s,c\}) \neq \emptyset \right\}.$$

*Moreover,*

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}) - 1,\mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}).$$

## References


[1] J. Adams, *Discrete spectrum of the reductive dual pair $(O(p,q), Sp(2m))$*, Invent. Math. **74** (1983), no. 3, 449–475.

[2] J. Adams, B. Barbasch, and D. A. Vogan, *The Langlands classification and irreducible characters for real reductive groups*, Progress in Math., vol. 104, Birkhauser, 1991.

[3] Jeffrey Adams and Fokko du Cloux, *Algorithms for representation theory of real reductive groups*, Journal of the Institute of Mathematics of Jussieu **8** (2009), no. 2, 209-259, DOI 10.1017/S1474748008000352.

[4] J. Arthur, *On some problems suggested by the trace formula*, Lie group representations, II (College Park, Md.), Lecture Notes in Math. 1041 (1984), 1–49.

[5] ———, *Unipotent automorphic representations: conjectures*, Orbites unipotentes et représentations, II, Astérisque **171-172** (1989), 13–71.

[6] L. Auslander and B. Kostant, *Polarizations and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

[7] D. Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. **96** (1989), no. 1, 103–176.

[8] ———, *Unipotent representations for real reductive groups*, Proceedings of ICM (1990), Kyoto (2000), 769–777.

[9] Dan Barbasch and David Vogan, *Weyl Group Representations and Nilpotent Orbits*, Representation Theory of Reductive Groups: Proceedings of the University of Utah Conference 1982, 1983, pp. 21–33.

[10] D. Barbasch, *Orbital integrals of nilpotent orbits*, The mathematical legacy of Harish-Chandra, Proc. Sympos. Pure Math. **68** (2000), 97–110.

[11] ———, *The unitary spherical spectrum for split classical groups*, J. Inst. Math. Jussieu **9** (2010), 265–356.

[12] ———, *Unipotent representations and the dual pair correspondence*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 47–85.

[13] D. Barbasch, J.-J. Ma, B.-Y. Sun, and C.-B. Zhu, *The counting the special unipotent representations*, in preparation.

[14] D. Barbasch and D. A. Vogan, *Weyl group representations and nilpotent orbits*, in Representation theory of reductive groups (Park City, Utah, 1982), Progress in Math. **40** (1983), 21–33.

[15] ———, *Unipotent representations of complex semisimple groups*, Annals of Math. **121** (1985), no. 1, 41–110.

[16] R. Brylinski, *Dixmier algebras for classical complex nilpotent orbits via Kraft-Procesi models. I*, The orbit method in geometry and physics (Marseille, 2000). Progress in Math. **213** (2003), 49–67.

[17] W. Borho, *Recent advances in enveloping algebras of semisimple Lie-algebras*, Séminaire Bourbaki, Exp. No. 489 (1976/77), 1–18.

[18] Roger W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993.

[19] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of $G$*, Canad. J. Math. **41** (1989), 385–438.

[20] F. Du Cloux, *Sur les représentations différentiables des groupes de Lie algébriques*, Ann. Sci. École Norm. Sup. **24** (1991), no. 3, 257–318.

[21] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, Van Nostrand Reinhold Co., 1993.

[22] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda, *Nilpotent orbits and complex dual pairs*, J. Algebra **190** (1997), no. 2, 518 - 539.

[23] ———, *Dual pairs and Kostant-Sekiguchi correspondence. II. Classification of nilpotent elements*, Central European J. Math. **3** (2005), 430–474.

[24] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) **102** (1978), 307–330.

[25] M. Duflo, *Théorie de Mackey pour les groupes de Lie algébriques*, Acta Math. **149** (1982), no. 3-4, 153–213.

[26] R. Gomez and C.-B. Zhu, *Local theta lifting of generalized Whittaker models associated to nilpotent orbits*, Geom. Funct. Anal. **24** (2014), no. 3, 796–853.

[27] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algb́rique IV: Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[28] ———, *Éléments de géométrie algbrique IV: Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966).

[29] M. Harris, J.-S. Li, and B. Sun, *Theta correspondences for close unitary groups*, Arithmetic Geometry and Automorphic Forms, Adv. Lect. Math. (ALM) **19** (2011), 265–307.

[30] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag, 1983.

[31] H. He, *Unipotent representations and quantum induction*, arXiv:math/0210372 (2002).

[32] ———, *Unitary representations and theta correspondence for type I classical groups*, J. Funct. Anal. **199** (2003), no. 1, 92–121.

[33] J.-S. Huang and J.-S. Li, *Unipotent representations attached to spherical nilpotent orbits*, Amer. J. Math. **121** (1999), no. 3, 497–517.

[34] J.-S. Huang and C.-B. Zhu, *On certain small representations of indefinite orthogonal groups*, Represent. Theory **1** (1997), 190–206.

[35] R. Howe, *$\theta$-series and invariant theory*, Automorphic Forms, Representations and $L$-functions, Proc. Sympos. Pure Math, vol. 33, 1979, pp. 275-285.

[36] ———, *On a notion of rank for unitary representations of the classical groups*, Harmonic analysis and group representations, Liguori, Naples (1982), 223-331.

[37] ———, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

[38] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Piatetski-Shapiro, I. et al. (eds.), The Schur lectures (1992). Ramat-Gan: Bar-Ilan University, Isr. Math. Conf. Proc. 8, (1995), 1-182.

[39] D. Jiang, B. Liu, and G. Savin, *Raising nilpotent orbits in wave-front sets*, Represent. Theory **20** (2016), 419–450.

[40] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.

[41] B. Kostant, *Quantization and unitary representations*, Lectures in Modern Analysis and Applications III, Lecture Notes in Math. **170** (1970), 87–208.

[42] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.

[43] S. S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.

[44] S. S. Kudla, *Some extensions of the Siegel-Weil formula*, In: Gan W., Kudla S., Tschinkel Y. (eds) Eisenstein Series and Applications. Progress in Mathematics, vol 258. Birkhäuser Boston (2008), 205–237.

[45] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence II*, Israel J. Math. **100** (1997), 29–59.

[46] ———, *Degenerate principal series of metaplectic groups and Howe correspondence*, D. Prasad at al. (eds.), Automorphic Representations and L-Functions, Tata Institute of Fundamental Research, India, (2013), 379–408.

[47] J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), no. 2, 237–255.

[48] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2006.

[49] H. Y. Loke and J. Ma, *Invariants and $K$-spectrums of local theta lifts*, Compositio Math. **151** (2015), 179–206.

[50] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, j. London Math. Soc. **19** (1979), 41–52.

[51] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272, DOI 10.1007/BF01388564. MR732546

[52] G. W. Mackey, *Unitary representations of group extentions*, Acta Math. **99** (1958), 265–311.

[53] W. M McGovern, *Cells of Harish-Chandra modules for real classical groups*, Amer. J. of Math. **120** (1998), 211–228.

[54] C. Mœglin, *Front d'onde des représentations des groupes classiques p-adiques*, Amer. J. Math. **118** (1996), 1313–1346.

[55] ———, *Paquets d'Arthur Spéciaux Unipotents aux Places Archimédiennes et Correspondance de Howe*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 469–502.

[56] C. Mœglin and D. Renard, *Sur les paquets d'Arthur des groupes classiques réels*, J. Eur. Math. Soc. **22** (2020), 1827–1892.

[57] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer, 1987.

[58] K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*, Astérisque **273** (2001), 1–163.
[59] K. Nishiyama, H. Ochiai, and C.-B. Zhu, *Theta lifting of nilpotent orbits for symmetric pairs*, Trans. Amer. Math. Soc. **358** (2006), 2713–2734.
[60] K. Nishiyama and C.-B. Zhu, *Theta lifting of unitary lowest weight modules and their associated cycles*, Duke Math. J. **125** (2004), 415–465.
[61] T. Ohta, *The closures of nilpotent orbits in the classical symmetric pairs and their singularities*, Tohoku Math. J. **43** (1991), no. 2, 161–211.
[62] ———, *Induction of nilpotent orbits for real reductive groups and associated varieties of standard representations*, Hiroshima Math. J. **29** (1999), no. 2, 347–360.
[63] ———, *Nilpotent orbits of $\mathbb{Z}_4$-graded Lie algebra and geometry of moment maps associated to the dual pair* $(\mathrm{U}(p,q),\mathrm{U}(r,s))$, Publ. RIMS **41** (2005), no. 3, 723–756.
[64] A. Paul and P. Trapa, *Some small unipotent representations of indefinite orthogonal groups and the theta correspondence*, University of Aarhus Publ. Series **48** (2007), 103–125.
[65] V. L. Popov and E. B. Vinberg, *Invariant Theory*, Algebraic Geometry IV: Linear Algebraic Groups, Invariant Theory, Encyclopedia of Mathematical Sciences, vol. 55, Springer, 1994.
[66] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloq. Math. **70** (1996), 93–102.
[67] ———, *Characters, dual pairs, and unitary representations*, Duke Math. J. **69** (1993), no. 3, 547–592.
[68] S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.
[69] David A. Renard and Peter E. Trapa, *Irreducible genuine characters of the metaplectic group: Kazhdan-Lusztig algorithm and Vogan duality*, Represent. Theory **4** (2000), 245–295, DOI 10.1090/S1088-4165-00-00105-9.
[70] S. Sahi, *Explicit Hilbert spaces for certain unipotent representations*, Invent. Math. **110** (1992), no. 2, 409–418.
[71] J. Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138.
[72] W. Schmid and K. Vilonen, *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Annals of Math. **151** (2000), no. 3, 1071–1118.
[73] E. Sommers, *Lusztig's canonical quotient and generalized duality*, J. Algebra **243** (2001), no. 2, 790–812.
[74] T. A. Springer and R. Steinberg, *Seminar on algebraic groups and related finite groups; Conjugate classes*, Lecture Notes in Math., vol. 131, Springer, 1970.
[75] B. Sun and C.-B. Zhu, *A general form of Gelfand-Kazhdan criterion*, Manuscripta Math. **136** (2011), 185–197.
[76] P. Trapa, *Special unipotent representations and the Howe correspondence*, University of Aarhus Publication Series **47** (2004), 210–230.
[77] D. A. Vogan, *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. **49** (1982), no. 4, 943–1073. MR683010
[78] ———, *Unitary representations of reductive Lie groups*, Ann. of Math. Stud., vol. 118, Princeton University Press, 1987.
[79] ———, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups, Proc. Conf., Brunswick/ME (USA) 1989, Prog. Math. **101** (1991), 315–388.
[80] ———, *The method of coadjoint orbits for real reductive groups*, Representation theory of Lie groups (Park City, UT, 1998). IAS/Park City Math. Ser. **8** (2000), 179–238.
[81] ———, *Unitary representations of reductive Lie groups*, Mathematics towards the Third Millennium (Rome, 1999). Accademia Nazionale dei Lincei, (2000), 147–167.
[82] N. R. Wallach, *Real reductive groups I*, Academic Press Inc., 1988.
[83] ———, *Real reductive groups II*, Academic Press Inc., 1992.
[84] H. Weyl, *The classical groups: their invariants and representations*, Princeton University Press, 1947.
[85] S. Yamana, *Degenerate principal series representations for quaternionic unitary groups*, Israel J. Math. **185** (2011), 77–124.

the Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, New York 14853
*Email address*: dmb14@cornell.edu

School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China
*Email address*: hoxide@sjtu.edu.cn

Institute for Advanced Study in Mathematics, Zhejiang University, Hangzhou, 310058, China
*Email address*: sunbinyong@zju.edu.cn

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
*Email address*: matzhucb@nus.edu.sg