1. Proof of theorem on counting

We do induction on the number of columns of the nilpotent orbits $\mathcal{O}$.

In this section, we let $\star \in \{B, D, C^*\}$. Now $\star' = \widetilde{C}, C, D^*$ respectively.

We assume all the claim had been proven when $\mathcal{O}$ has at most two columns, that is

- When $\star = D$, $\mathbf{r}_3(\check{\mathcal{O}}) = 0$;
- When $\star = B$, $\mathbf{r}_2(\check{\mathcal{O}}) = 0$;
- When $\star = C^*$, $\mathbf{r}_2(\check{\mathcal{O}}) = 0$.

Proposition 1.1. *We have:*

i) For each $\tau' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$, $\mathcal{L}_{\tau'} \neq 0$.

ii) For each $\tau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$, $\mathcal{L}_{\tau} \neq 0$.

iii) Let τ'_1, τ'_2 be two distinct elements in $\mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$. Then either

- *$\mathcal{L}_{\tau'_1} \neq \mathcal{L}_{\tau'_2}$; or*
- *$(\tau''_1, \varepsilon'_1) \neq (\tau''_2, \varepsilon'_2)$ and $\mathrm{Sign}(\tau''_1) = \mathrm{Sign}(\tau''_2)$ where $(\tau''_i, \varepsilon'_i) := \nabla(\tau'_i)$ for $i = 1, 2$.*

iv) Let τ_1, τ_2 be two distinct elements in $\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$. Then either

- *$\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$; or*
- *$\tau'_1 \neq \tau'_2$ and $\varepsilon_1 = \varepsilon_2$ where $(\tau'_i, \varepsilon_i) := \nabla(\tau_i)$ for $i = 1, 2$.*

v) When $\star \in \{B, D\}$ The local system $\mathcal{L}_\tau$ is disjoint with their determinant twist:

$$\{\mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})\} \cap \{\mathcal{L}_\tau \otimes \det \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})\} = \emptyset. \tag{1.1}$$

vi) We have

(i) If $x_\tau = s$, then $\mathcal{L}^+_\tau = \mathcal{L}^-_\tau = 0$.

(ii) If $x_\tau \in \{r, c\}$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau = 0$.

(iii) If $x_\tau = d$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau \neq 0$.

vii) When $\check{\mathcal{O}}'$ is noticed, the map $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}') \longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O}')$ is a bijection.

viii) When $\mathcal{O}$ is noticed, the map $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}) \longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O})$ is a bijection.

ix) When $\mathcal{O}$ is +noticed, the maps

$$\begin{aligned} \Upsilon^+_{\mathcal{O}}\colon \{\mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0\} &\longrightarrow \{\mathcal{L}^+_\tau\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}^+_\tau \qquad\qquad \textit{and} \\ \Upsilon^-_{\mathcal{O}}\colon \{\mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0\} &\longrightarrow \{\mathcal{L}^-_\tau\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}^-_\tau \end{aligned}$$

are injective.

x) When $\mathcal{O}$ is noticed, the map

$$\begin{aligned} \Upsilon_{\mathcal{O}}\colon \{\mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0\} &\longrightarrow \{(\mathcal{L}^+_\tau, \mathcal{L}^-_\tau) \mid \mathcal{L}^+_\tau \neq 0\} \\ \mathcal{L}_\tau &\longmapsto (\mathcal{L}^+_\tau, \mathcal{L}^-_\tau) \end{aligned} \tag{1.2}$$

is injective.[1]

1.1. The initial case:

When $k = 0$, $\mathcal{O} = (C_1 = 2c_1, C_0 = 2c_0)$ has at most two columns. Now $\pi_{\tau'}$ is the trivial representation of $\mathrm{Sp}(2c_0, \mathbb{R})$. The lift $\pi_{\tau'} \mapsto \bar{\Theta}(\pi_{\tau'})$ is in fact a stable range theta lift.For $\tau \in \mathrm{DRC}(\mathcal{O})$, let $(p_1, q_1) = \mathrm{Sign}(\mathbf{x}_\tau)$ and x_τ be the foot of τ. It is easy to see that (using associated character formula)

$$\mathcal{L}_{\pi_\tau} = \mathcal{T}_\tau \cdot \mathcal{P}_\tau, \text{ where } \mathcal{T}_\tau := \dagger\dagger_{c_0, c_0}, \text{ and } \mathcal{P}_\tau := \begin{cases} \ddagger_{p_1, q_1} & x_\tau \neq d, \\ \dagger_{p_1, q_1} & x_\tau = d. \end{cases}$$

[1] In fact, we only need to know whether $\mathcal{L}^-_\tau$ is non-zero or not.

Note that $\mathcal{L}_{\pi_\tau}$ is irreducible.

Let $\mathcal{O}_1 = (2(c_1 - c_0)) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. Using the above formula, one can check that the following maps are bijections

$$\begin{array}{ccccc} \mathrm{DRC}(\mathcal{O}_1) & \longleftarrow & \mathrm{DRC}(\mathcal{O}) & \longrightarrow & {}^{\ell}\mathrm{LS}(\mathcal{O}) \\ \mathbf{x}_\tau & \longleftarrow\!\shortmid & \tau & \longmapsto & \mathcal{L}_\tau. \end{array}$$

To see that $\mathrm{DRC}(\mathcal{O}) \ni \tau \mapsto \mathcal{L}_\tau$ is an injection, one check the following lemma holds:

Lemma 1.2. *The map* $\mathrm{DRC}(\mathcal{O}_1) \longrightarrow \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective (see* **??** *for the definition of* ϵ_τ*). Moreover,* $\epsilon_\tau = 0$ *only if* $\mathrm{Sign}(\tau) \geqslant (0,1)$.

Moreover, $\mathcal{L}_\tau \otimes \det \notin {}^{\ell}\mathrm{LS}(\mathcal{O})$ for any $\tau \in \mathrm{DRC}(\mathcal{O})$. In fact, if $\mathrm{Sign}(\mathbf{x}_\tau) \succeq (1,0)$, $\mathcal{L}_\tau$ on the 1-rows with $+$-signs is trivial and $\mathcal{L}_\tau \otimes \det$ has non-trivial restriction. When $\mathrm{Sign}(\mathbf{x}_\tau) \not\succeq (1,0)$, $\mathbf{x}_\tau = s \cdots s$, all 1-rows of $\mathcal{L}_\tau$ are marked by "=" and all 1-rows of $\mathcal{L}_\tau \otimes \det$ are marked by "−".

Therefore our main **??** and main proposition **??** holds for $\mathcal{O}$.

Let $C_i = \mathbf{c}_i(\check{\mathcal{O}})$ for $i = 1, 2, \cdots$.

1.2. **The descent case.** Now we assume $(2,3) \in \mathrm{PP}_\star(\check{\mathcal{O}})$.

1.2.1. We first calculate the local system attached to $\check{\mathcal{O}}'$.

Let $\tau' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$.

For $\mathcal{L}'' \in \mathrm{AOD}(\mathcal{O}'')$, let $(p_1, q_1) = {}^{l}\mathrm{Sign}(\mathcal{L}'')$ and $(p_2, q_2) = (C_2 - q_1, C_2 - p_1)$. Then

$$\vartheta(\mathcal{L}'') = (\maltese^{y} \dagger \mathcal{L}'') \cdot \dagger_{(n_0,n_0)}$$

where y is an integer determined by $\mathrm{Sign}(\tau'')$ and $n_0 = (C_2 - C_3)/2$.

Therefore, $\mathrm{LS}(\mathcal{O}'') \longrightarrow \mathrm{LS}(\mathcal{O}')$ given by $\mathcal{L} \mapsto \vartheta(\mathcal{L})$ is an injection between abelian groups.

By (1.1), we have a injection

$$\begin{array}{ccc} {}^{\ell}\mathrm{LS}(\mathcal{O}'') \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & {}^{\ell}\mathrm{LS}(\mathcal{O}') \\ (\mathcal{L}'', \epsilon') & \longmapsto & \vartheta(\mathcal{L}'' \otimes \det^{\epsilon'}). \end{array} \tag{1.3}$$

In particular, we have $\mathcal{L}_{\tau'} \neq 0$ for every $\tau' \in \mathrm{DRC}(\mathcal{O}')$.

Now suppose $\tau'_1 \neq \tau'_2$. Suppose $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (1.3),

$$\varepsilon_{\tau'_1} = \varepsilon_{\tau'_2} \tag{1.4}$$

and $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$. In particular, we have $\mathrm{Sign}(\tau''_1) = \mathrm{Sign}(\mathcal{L}_{\tau''_1}) = \mathrm{Sign}(\mathcal{L}_{\tau''_2}) = \mathrm{Sign}(\tau''_1)$. We now claim that $\tau''_1 \neq \tau''_2$. It suffice to consider the case where $\wp_{\tau''_1} = \wp_{\tau''_2}$. Then $\wp_{\tau'_1} = \wp_{\tau'_2}$ by (1.4) and so $\tau'_1 \neq \tau'_2$. By **??**, $\tau''_1 \neq \tau''_2$ and this proves the claim.

[Now suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (1.3), $\epsilon'_1 = \epsilon'_2$ and $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$. In particular, τ''_1 and τ''_2 have the same signature. By **??** and the induction hypothesis, $\tau''_1 \neq \tau''_2$ and so $\pi_{\tau''_1} \otimes \det^{\epsilon'_1} \neq \pi_{\tau''_2} \otimes \det^{\epsilon'_2}$. Now the injectivity of theta lifting yields

$$\pi_{\tau'_2} = \bar{\Theta}(\pi_{\tau''_1} \otimes \det^{\epsilon'_1}) \neq \bar{\Theta}(\pi_{\tau''_2} \otimes \det^{\epsilon'_2}) = \pi_{\tau'_2}$$

Suppose $\mathcal{O}'$ is noticed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is noticed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is an injection into $\mathrm{LS}(\mathcal{O}'')$. Now (1.3) implies $\tau' \mapsto \mathcal{L}_{\tau'}$ is also an injection.

Suppose $\mathcal{O}'$ is quasi-distingushed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is quasi-distingushed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is a bijection with $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}'')$. Since the component group of each K-nilpotent orbit in $\mathcal{O}'$ is naturally isomorphic to the component group of its descent. So we deduce that $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection onto $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}')$.

This proves the main proposition for $\mathcal{O}'$.]

1.2.2. Now we consider the local system attached to $\check{\mathcal{O}}$. Let $\tau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$. By the signature formula (??)

$$\begin{aligned}\mathrm{AC}(\tau) &= \left(\left(\maltese^{y'}(\mathrm{AC}(\tau'') \otimes (\det)^{\varepsilon_{\tau'}}) \cdot 1_{(n_0,n_0)}\right) \cdot 1_{\mathrm{Sign}(\tau_{\mathbf{t}})}\right) \otimes (1^{+,-})^{\varepsilon_\tau} \\ &= \left(\maltese^{y''}\mathcal{L}_{\tau'} \otimes (1^{+,-})^{\varepsilon_{\tau'}}\right) \cdot \mathcal{L}_{(\tau_{\mathbf{t}},\varnothing)}\end{aligned} \tag{1.5}$$

where $y' = (p_\tau - q_\tau - p_{\tau''} + q_{\tau''})/2$ and y'' is determined by $\mathrm{Sign}(\tau)$.

Now suppose $\tau_1 \neq \tau_2 \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$ and $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (1.5),

By (1.5), we have $\mathcal{L}_{\tau_{1\mathbf{t}}} = \mathcal{L}_{\tau_{2\mathbf{t}}}$ and . Therefore $\tau_{1\mathbf{t}} = \tau_{2\mathbf{t}}$, $\varepsilon_{\tau_1} = \varepsilon_{\tau_2}$ and $\mathcal{L}_{\tau_1'} = \mathcal{L}_{\tau_2'}$. Thanks to ??, $\tau_1' \neq \tau_2'$.

The claims in ?? vi) follows from (1.5) and the initial case for $\tau_{\mathbf{t}}$.

1.2.3. Note that $\mathcal{L}_\tau$ is obtained from $\mathcal{L}_{\tau'}$ by attaching the peduncle determined by $\mathbf{x}_\tau$. The claims about noticed and quasi-distinguished orbit are easy to verify. We leave them to the reader.

1.3. **The general descent case.** We assume $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$. Note that in this case, $\star \in \{B, D\}$.

1.3.1. We now describe the local systems $\mathcal{L}_{\tau'}$.

$$\mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''}) = \maltese^{\frac{|\mathrm{Sign}(\tau'')|}{2}}(\dagger\mathcal{L}^+_{\tau''} + \dagger\mathcal{L}^-_{\tau''}) \neq 0 \tag{1.6}$$

where ${}^l\mathrm{Sign}(\dagger\mathcal{L}^+_{\tau''}) = (q_1'', p_1'' - 1)$ and ${}^l\mathrm{Sign}(\dagger\mathcal{L}^-_{\tau''}) = (q_1'' - 1, p_1'')$ (if $\dagger\mathcal{L}^-_{\tau''} \neq 0$).

1.3.2. *Local systems.* We now describe the local systems $\mathcal{L}_{\tau'}$ and $\mathcal{L}_\tau$ more precisely using our formula of the associated character and the induction hypothesis. Note that these local systems are always non-zero which implies that $\pi_{\tau'}$ and π_τ are unipotent representations attached to $\mathcal{O}'$ and $\mathcal{O}$ respectively.

First note that in τ' and τ'', $x_{\tau''} \neq s$ by the definition of dot-r-c diagram. [2] By the induction hypothesis, $\mathcal{L}^+_{\tau''} \neq 0$.

Let $(p_1'', q_1'') := {}^l\mathrm{Sign}(\mathcal{L}_{\tau''})$. Then

$${}^l\mathrm{Sign}(\mathcal{L}^+_{\tau''}) = (p_0 - 1, q_0), \text{ and } {}^l\mathrm{Sign}(\mathcal{L}^-_{\tau''}) = (p_0, q_0 - 1) \text{ if } \mathcal{L}^-_{\tau''} \neq \varnothing. \tag{1.7}$$

$$\mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''}) = \maltese^{\frac{|\mathrm{Sign}(\tau'')|}{2}}(\dagger\mathcal{L}^+_{\tau''} + \dagger\mathcal{L}^-_{\tau''}) \neq 0 \tag{1.8}$$

where ${}^l\mathrm{Sign}(\dagger\mathcal{L}^+_{\tau''}) = (q_1'', p_1'' - 1)$ and ${}^l\mathrm{Sign}(\dagger\mathcal{L}^-_{\tau''}) = (q_1'' - 1, p_1'')$ (if $\dagger\mathcal{L}^-_{\tau''} \neq 0$).

Let $(p_1, q_1) := {}^l\mathrm{Sign}(\mathcal{L}_\tau)$ and $(e, f) := (p_1 - p_1'' + 1, q_1 - q_1'' + 1)$. Using ?? one can show that

$$(e, f) = \mathrm{Sign}(\mathbf{u}_\tau).$$

[It suffice to consider the most left three columns of the peduncle part: this part has signature ${}^l\mathrm{Sign}(\mathcal{P}_\tau) + (1,1) = \mathrm{Sign}(\mathbf{u}_\tau x_{\tau''}) = \mathrm{Sign}(\mathbf{u}_\tau) + {}^l\mathrm{Sign}(\mathcal{D}_{\tau''})$. Therfore,

$$\mathrm{Sign}(\mathbf{u}_\tau) = {}^l\mathrm{Sign}(\mathcal{P}_\tau) - {}^l\mathrm{Sign}(\mathcal{D}_{\tau''}) + (1,1) = {}^l\mathrm{Sign}(\mathcal{L}_\tau) - {}^l\mathrm{Sign}(\mathcal{L}_{\tau''}) + (1,1).$$

]

Now

$$\mathcal{L}_\tau = \begin{cases}(\mathbf{1}^{+,-} \otimes \dagger\maltese^t\dagger\mathcal{L}^+_{\tau''}) \cdot \mathcal{P}^+_\tau + (\mathbf{1}^{+,-} \otimes \dagger\maltese^t\dagger\mathcal{L}^-_{\tau''}) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau \neq d \\ (\dagger\maltese^t\dagger\mathcal{L}^+_{\tau''}) \cdot \mathcal{P}^+_\tau + (\dagger\maltese^t\dagger\mathcal{L}^-_{\tau''}) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau = d\end{cases} \tag{1.9}$$

[2]Suppose $\tau'' \in \mathrm{DRC}(\mathcal{O}'')$ has the shape in (??) and $x_{\tau''} = s$. By the induction hypothesis, $\mathcal{L}^+_{\tau''} = \mathcal{L}^-_{\tau''} = 0$ and so the theta lift of $\pi_{\tau''}$ vanishes.

where

$$t=\frac{|\mathrm{Sign}(\tau)-\mathrm{Sign}(\tau'')|}{2}=\frac{|\mathrm{Sign}(\mathbf{u}_\tau)|}{2}$$

$$\mathcal{P}_\tau^+=\begin{cases}\ddagger_{e,f-1} & \text{when } f\geqslant 1 \text{ and } x_\tau\neq d\\ \dagger_{e,f-1} & \text{when } f\geqslant 1 \text{ and } x_\tau=d\\ 0 & \text{otherwise}\end{cases}$$

$$\mathcal{P}_\tau^-=\begin{cases}\ddagger_{e-1,f} & \text{when } e\geqslant 1 \text{ and } x_\tau\neq d\\ \dagger_{e-1,f} & \text{when } e\geqslant 1 \text{ and } x_\tau=d\ .\\ 0 & \text{otherwise}\end{cases}$$

Lemma 1.3. *The local system $\mathcal{L}_\tau\neq 0$.*

Proof. Since $x_{\tau''}\neq s$, $\mathcal{L}^+_{\tau''}\neq 0$. When $\mathrm{Sign}(\mathbf{u}_\tau)\geq(0,1)$, the first term in (1.9) dose not vanish. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau)\not\succ(0,1)$. We have $x_{\tau''}=d$ by **??** and $\mathrm{Sign}(\mathbf{u}_\tau)\succ(1,0)$. Therefore, $\mathcal{L}^-_{\tau''}\neq 0$ by the induction hypothesis and the second term in (1.9) dose not vanish. Hence $\mathcal{L}_\tau\neq 0$ in all cases. □

Lemma 1.4. *The local system $\mathcal{L}_\tau$ satisfies the following properties:*

(i) When $x_\tau=s$, then $\mathcal{L}^+_\tau=\mathcal{L}^-_\tau=0$.
(ii) When $x_\tau=r/c$, then $\mathcal{L}^+_\tau\neq 0$ and $\mathcal{L}^-_\tau=0$.
(iii) When $x_\tau=d$, then $\mathcal{L}^+_\tau\neq 0$ and $\mathcal{L}^-_\tau\neq 0$.

Proof. If $x_\tau=s/r/c$, "$-$" mark dose not appear as a 1-row in $\mathcal{L}_\tau$ by (1.9). So $\mathcal{L}^-_\tau=0$ in these cases.

(i) Suppose $x_\tau=s$. We have $\mathbf{u}_\tau=s\cdots s$. Therefore, only the first term in (1.9) is non-zero and $\mathcal{P}^+_\tau$ consists of $=$ marks. Hence $\mathcal{L}^+_\tau=0$.

(ii) Suppose $x_\tau=r$. If $\mathrm{Sign}(\mathbf{u}_\tau)\succ(1,1)$, the first term in (1.9) is non-zero and $\mathcal{L}^+_\tau\succ\dagger_{(1,0)}$. So $\mathcal{L}^+_\tau\neq 0$. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau)\not\succ(1,1)$. Then $\mathbf{u}_\tau=r\cdots r$ and $x_{\tau''}=d$ by **??**. Now the second term of (1.9) is non-zero and $\mathcal{L}^-_\tau\succ\dagger_{(1,0)}$. So $\mathcal{L}^+_\tau\neq 0$.

(iii) Suppose $x_\tau=d$. Suppose $\mathrm{Sign}(\mathbf{u}_\tau)\succ(1,2)$. Then the first term in (1.9) is non-zero and $\mathcal{P}^+_\tau\succ\dagger_{(1,1)}$. So $\mathcal{L}^+_\tau\neq 0$ and $\mathcal{L}^-_\tau\neq 0$. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau)\not\succ(1,2)$. Then $\mathbf{u}_\tau=r\cdots rd$ [3] and $x_{\tau''}=d$ by **??**. So the both terms in (1.9) are non-zero. Since $\mathcal{P}^+_\tau\succ\dagger_{(1,0)}$ and $\mathcal{P}^-_\tau\succ\dagger_{(0,1)}$, we get the conclusion.

□

1.3.3. *Unipotent representations attached to $\mathcal{O}'$.* In this section, we let $\tau'\in\mathrm{DRC}(\mathcal{O}')$ and $\tau''=\overline{\nabla}_1(\tau')\in\mathrm{DRC}(\mathcal{O}'')$. First note that $x_{\tau''}\neq s$ and so $\mathcal{L}^+_{\tau''}$ always non-zero.

Recall (1.8). We claim that we can recover $\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''}$ from $\mathcal{L}_{\tau'}$:

Lemma 1.5. *The map $\Omega_{\mathcal{O}'}\colon\mathcal{L}_{\tau'}\mapsto(\mathcal{L}^+_{\tau''},\mathcal{L}^-_{\tau''})$ is a well defined map.*

Proof. When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has two elements, $x_{\tau''}=d$ and ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})=\{(q''_1,p''_1-1),(q''_1-1,p''_1)\}$. In (1.8), $\dagger\mathcal{L}^+_{\tau''}$ (resp. $\dagger\mathcal{L}^-_{\tau''}$) consists of components whose first column has siginature (q''_1,p''_1-1) (resp. (q''_1-1,p''_1)). When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has only one elements, $x_\tau=r/c$, ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})=\{(q''_1,p''_1-1)\}$, $\mathcal{L}_{\tau'}=\dagger\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''}=0$

In any case, we get

$$\mathrm{Sign}(\tau'')=(n_0,n_0)+(p''_1,q''_1)\quad\text{where } 2n_0=|\nabla(\mathcal{O}'')|\,. \tag{1.10}$$

Using the signature $\mathrm{Sign}(\tau'')$, we could recover the twisting characters in the theta lifting of local system. Therefore $\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''}$ can be recovered from $\mathcal{L}_{\tau'}$. □

[3] $\mathbf{u}_\tau=d$ if it has length 1.

Lemma 1.6. *Suppose* $\tau_1' \neq \tau_2' \in \mathrm{DRC}(\mathcal{O}')$. *Then* $\pi_{\tau_1'} \neq \pi_{\tau_2'}$.

Proof. It suffice to consider the case when $\mathcal{L}_{\tau_1'} = \mathcal{L}_{\tau_2'}$. By (1.10) in the proof of Lemma 1.5, $\mathrm{Sign}(\tau_1'') = \mathrm{Sign}(\tau_2'')$. On the other hand, $\epsilon_1 = \epsilon_2 = 0$ and $\tau_1'' \neq \tau_2''$ by **??**. So

$$\pi_{\tau_1'} = \bar{\Theta}(\pi_{\tau_1''}) \neq \bar{\Theta}(\pi_{\tau_2''}) = \pi_{\tau_2'}$$

by the injectivity of theta lift. $\square$

1.3.4. *Unipotent representations attached to* $\mathcal{O}$.

Lemma 1.7. *Suppose* $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$. *Then* $\pi_{\tau_1} \neq \pi_{\tau_2}$.

Proof. It suffice to consider the case that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. On the other hand, the twisting $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ since it is determined by the local system: $\epsilon_{\tau_i} = 0$ if and only if $\mathcal{L}_{\tau_i} \supset \boxed{-}$.

We conclude that $\tau_1'' \neq \tau_2''$ and $\tau_1' \neq \tau_2'$ by **??** and **??**. By Lemma 1.6 and the injectivity of theta lift,

$$\pi_{\tau_1} = \bar{\Theta}(\pi_{\tau_1'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_1}} \neq \bar{\Theta}(\pi_{\tau_2'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_2}} = \pi_{\tau_2}$$

$\square$

1.3.5. *Noticed orbits.* In this section, we prove the claims about noticed and +noticed orbits.

Lemma 1.8. *Suppose* $\mathcal{O}'$ *is noticed. Then*

(i) The map $\{ \mathcal{L}_{\tau''} \mid x_{\tau''} \neq s \} \longrightarrow \{ \mathcal{L}_{\tau'} \} = {}^{\ell}\mathrm{LS}(\mathcal{O}')$ *given by* $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''})$ *is a bijection.*

(ii) The map $\mathrm{DRC}(\mathcal{O}') \to {}^{\ell}\mathrm{LS}(\mathcal{O}')$ *given by* $\tau' \mapsto \mathcal{L}_{\tau'}$ *is a bijection.*

Proof. (i) Note that $\mathcal{O}''$ is noticed by definition. Hence $\Upsilon_{\mathcal{O}''}$ is invertible. Recall Lemma 1.5, we see that

$$(\Upsilon_{\mathcal{O}''})^{-1} \circ \Omega_{\mathcal{O}'} : \mathcal{L}_{\tau'} \mapsto (\mathcal{L}^{+}_{\tau''}, \mathcal{L}^{-}_{\tau''}) \mapsto \mathcal{L}_{\tau''}$$

gives the inverse of the map in the claim.

(ii) Suppose $\tau_1' \neq \tau_2' \in \mathrm{DRC}(\mathcal{O}')$. By **??**, $\tau_1'' \neq \tau_2''$. Hence $\mathcal{L}_{\tau_1''} \neq \mathcal{L}_{\tau_2''}$ by the induction hypothesis. Therefore, $\mathcal{L}_{\tau_1'} \neq \mathcal{L}_{\tau_2'}$ by (i) of the claim.

$\square$

Lemma 1.9. *Suppose* $\mathcal{O}$ *is noticed,* $\mathcal{L} \colon \mathrm{DRC}(\mathcal{O}) \mapsto {}^{\ell}\mathrm{LS}(\mathcal{O})$ *given by* $\tau \mapsto \mathcal{L}_\tau$ *is bijective.*

Proof. We prove the claim by contradiction. We assume $\tau_1 \neq \tau_2$ such that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$.

Recall (1.9): there is an pair of non-negative integers $(e_i, f_i) = \mathrm{Sign}(\mathbf{u}_{\tau_i})$ such that

$$\mathcal{L}_{\tau_i} = \dagger\dagger\mathcal{L}^{+}_{\tau_i''} \cdot \mathcal{P}^{+}_{\tau_i} + \dagger\dagger\mathcal{L}^{-}_{\tau''} \cdot \mathcal{P}^{-}_{\tau_i}.$$

(i) Suppose that $\mathcal{L}_{\tau_i}$ contains a 1-row marked by $-$ or $=$. Then we have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$, which can be read from the mark $-/=$. Moreover, the term $\dagger\dagger\mathcal{L}^{+}_{\tau_i''} \cdot \mathcal{P}^{+}_{\tau_i} \neq 0$ and we can recover it from $\mathcal{L}_{\tau_i}$. So $\mathcal{L}^{+}_{\tau_1''} = \mathcal{L}^{+}_{\tau_2''}$. Since $\mathcal{O}''$ is +noticed, $\tau_1'' = \tau_2''$ by (1.11) and the induction hypothesis. Note that $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by **??** which is contradict to our assumption.

(ii) Now we assume that $\mathcal{L}_{\tau_i}$ does not contains a 1-row marked by $-/=$. By (1.9), we conclude that $x_{\tau''} \neq d$ and $\mathbf{u}_\tau = r \cdots r$ or $r \cdots rc$. Therefore $\mathbf{p}_{\tau_i}$ must be one of the following form by **??**:

$$\mathbf{p}_1 = \begin{array}{|c|c|} \hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} r & \\ \cline{1-1} \end{array}, \quad \mathbf{p}_2 = \begin{array}{|c|c|} \hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} c & \\ \cline{1-1} \end{array}, \quad \text{and} \quad \mathbf{p}_3 = \begin{array}{|c|c|} \hline r & d \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} r & \\ \cline{1-1} \end{array}$$

We consider case by case according to the set $\{ \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \}$:

(a) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1\}$ or $\{\mathbf{p}_2\}$. In these cases, $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau_1'' \neq \tau_2''$ by **??**. On the other hand, $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$ by (1.9). Since $\mathcal{O}''$ is +noticed, we get $\tau_1'' = \tau_2''$ a contradiction.

(b) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_3\}$. We still have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau_1'' \neq \tau_2''$ by **??**. On the other hand, $\mathcal{L}^-_{\tau_1''} = \mathcal{L}^-_{\tau_2''}$ by (1.9). This again contradict to the injectivity of $\mathcal{L}_{\tau''} \mapsto \mathcal{L}^-_{\tau''}$.

(c) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1, \mathbf{p}_2\}$. By the argument above, we have $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$ Now $\tau_1'' \neq \tau_2''$ since $\{x_{\tau_1''}, x_{\tau_2''}\} = \{r, c\}$. This contradict to that $\mathcal{O}''$ is +noticed again.

(d) Suppose $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ or $\mathbf{p}_2, \mathbf{p}_{\tau_2} = \mathbf{p}_3$. Then

$$\mathcal{L}_{\tau_1} = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_1}|}{2}}\dagger\mathcal{L}^+_{\tau_1''} \cdot \mathcal{P}^+_{\tau_1''} \quad \text{and} \quad \mathcal{L}_{\tau_2} = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_2}|}{2}}\dagger\mathcal{L}^-_{\tau_2''} \cdot \mathcal{P}^-_{\tau_2''}.$$

This implies $|\mathrm{Sign}(\tau_1'')| \equiv |\mathrm{Sign}(\tau_2'')| + 2 \pmod 4$ and $\maltese\dagger\mathcal{L}^+_{\tau_1''} = \dagger\mathcal{L}^-_{\tau_2''}$.

Claim 1.10. *We have* $\mathcal{L}^-_{\tau_2''} \supset \boxed{+}$.

Proof. If $\mathcal{O}''$ is obtained by the usual descent, we have $\mathcal{L}_\tau \geq \begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array}$ and we are done. Now assume $\mathcal{O}''$ is obtained by generalized descent and $\mathcal{O}''$ is +noticed, i.e. $\mathbf{u}_{\tau''}$ has at least length 2. Suppose $\mathrm{Sign}(\mathbf{u}_{\tau''}) \succ (1,2)$. Applying (1.9) to τ'', we see that the first term is non-zero and $\mathcal{L}_{\tau''} \supseteq \dagger_{(1,1)}$.

Otherwise, $\mathrm{Sign}(\mathbf{u}_{\tau''}) = (2n_0 - 1, 1) \geq (3,1)$ where n_0 is the length of $\mathbf{u}_{\tau''}$. By (1.9), the second term is non-zero and $\mathcal{L}_{\tau''} \succ \dagger_{(2n_0-1,1)} \succ +$. □

The claim leads to a contradiction: Thanks to the character twist in the theta lifting formula of the local system, we see that the the associated character restricted on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$ of $\maltese\dagger\mathcal{L}^+_{\tau_1''}$ and $\dagger\mathcal{L}^-_{\tau_2''}$ must be are different, a contradiction.

We finished the proof of the lemma. □

1.3.6. *The maps* $\Upsilon^+_{\mathcal{O}}$, $\Upsilon^-_{\mathcal{O}}$ *and* $\Upsilon_{\mathcal{O}}$.

Lemma 1.11. *Suppose* $\mathcal{O}$ *is +noticed. The map* $\Upsilon^+_{\mathcal{O}}\colon \{\mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0\} \to \{\mathcal{L}^+_\tau \neq 0\}$ *is injective.*

Proof. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$. This is equivalent to $\mathcal{L}^+_\tau$ has an irreducible component $\succ \boxed{+}$. Now all irreducible components of $\mathcal{L}_\tau \succ \boxed{+}$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^+_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^+_\tau$.

(ii) Suppose that $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ do not contain a component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$.

By Lemma 1.9 $\tau_1 \neq \tau_2$.[4] We now show that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2} \neq 0$ leads to a contradiction. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. By (1.9) and checking the properties in **??**, we have

$$\mathbf{p}_{\tau_i} = \begin{array}{|c|c|}\hline s & x_{\tau''} \\ \hline \vdots & \\ \cline{1-1} s & \\ \cline{1-1} x_{\tau_i} & \\ \cline{1-1}\end{array} \qquad \text{where } x_{\tau_i''} = c/d, x_{\tau_i} = c/d.$$

On the other hand, when $\mathbf{p}_{\tau_i}$ have the above form, the first term of (1.9) is non-zero. So we conclude that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$ implies

$$\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}. \tag{1.11}$$

[4] $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ clearly implies $\tau_1 \neq \tau_2$.

Moreover, we see that every components of $\mathcal{L}_{\tau_i}$ has a 1-row of mark "$-/=$" and we can determine ϵ_{τ_i} by the mark $-/=$ of the 1-row appeared in $\mathcal{L}^+_{\tau_i}$. In particular, we have $\epsilon_1 = \epsilon_2$. Now **??** implies $\tau''_1 \neq \tau''_2$. This is contradict to (1.11) and our induction hypothesis since $\mathcal{O}''$ is also $+$noticed.

$\square$

Lemma 1.12. *Suppose $\mathcal{O}$ is noticed. Then the map $\Upsilon_{\mathcal{O}}\colon \mathcal{L}_\tau \mapsto (\mathcal{L}^+_\tau, \mathcal{L}^-_\tau)$ is injective.*

Proof. It suffice to consider the case where $\mathcal{O}$ is noticed but not $+$noticed, i.e. $C_{2k+1} = C_{2k} + 1$. In this case, $n = 1$. The peduncle $\mathbf{p}_\tau$ have four possible cases:

$$\begin{array}{|c|c|}\hline r & c \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline c & c \\ \hline \end{array}, \quad \begin{array}{|c|c|}\hline c & d \\ \hline \end{array}, \quad \text{or} \quad \begin{array}{|c|c|}\hline d & d \\ \hline \end{array}.$$

By (1.9), $\mathcal{L}^+_\tau = \dagger\dagger\mathcal{L}^+_{\tau''}$.

Now suppose $\tau_1 \neq \tau_2$ such that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$. Since $\mathcal{O}''$ is $+$noticed, we have $\tau''_1 = \tau''_2$ by the induction hypothesis (see Lemma 1.11). By **??**, this only happens when $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\begin{array}{|c|c|}\hline c & d \\ \hline \end{array}, \begin{array}{|c|c|}\hline d & d \\ \hline \end{array}\}$. Since one of $\mathcal{L}^-_{\tau_i}$ is zero and the other is non-zero, we conclude that $\Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_1}) \neq \Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_2})$. $\square$

Lemma 1.13. *Suppose $\mathcal{O}$ is $+$noticed. The map $\Upsilon^-_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0 \,\} \to \{\, \mathcal{L}^-_\tau \neq 0 \,\}$ is injective.*

Proof. The proof is similar to that of Lemma 1.11. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $\succ \begin{array}{|c|}\hline - \\ \hline - \\ \hline \end{array}$. This is equivalent to $\mathcal{L}^-_\tau$ has an irreducible component $\succ \begin{array}{|c|}\hline - \\ \hline \end{array}$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^-_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^-_\tau$.

(ii) Now we make a weaker assumption that $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$ such that $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ do not contain a component $\succ \begin{array}{|c|}\hline - \\ \hline - \\ \hline \end{array}$.

By (1.9) and the properties in **??**, we see that $\mathbf{p}_{\tau_i}$ must be one of the following

$$\mathbf{p}_1 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} d & \\ \cline{1-1} \end{array} \quad \text{or} \quad \mathbf{p}_2 = \begin{array}{|c|c|}\hline r & d \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} d & \\ \cline{1-1} \end{array}.$$

Without of loss of generality, we have the following possibilities:

(a) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_1$. We have $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ and obtain the contradiction.

(b) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_2$. We have $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$ and obtain the contradiction.

(c) $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ and $\mathbf{p}_{\tau_2} = \mathbf{p}_2$. Now we apply the same argument in the case (d) of the proof of Lemma 1.9. We get $\maltese\dagger\mathcal{L}^+_{\tau''_1} = \dagger\mathcal{L}^-_{\tau''_2}$, $\mathcal{L}^-_{\tau''_2} \supset +$ and a contradiction by looking at the associated character on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline \end{array}$.

This finished the proof. $\square$

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