SPECIAL UNIPOTENT REPRESENTATIONS : ORTHOGONAL AND SYMPLECTIC GROUPS

DAN M. BARBASCH, JIA-JUN MA, BINYONG SUN, AND CHEN-BO ZHU

Abstract. Let G be a real classical group of type B, C, D (including the real metaplectic group). We consider a nilpotent adjoint orbit $\check{\mathcal{O}}$ of $\check{G}$, the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group). We classify all special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. When $\check{\mathcal{O}}$ is of good parity, we construct all such representations of G via the method of theta lifting. As a direct consequence of the construction and the classification, we conclude that all special unipotent representations of G are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture.

Contents

1. Introduction and the main results 1
2. Descents of painted bipartitions: combinatorics 11
3. From painted bipartitions to associated cycles 14
4. Theta lifts via matrix coefficient integrals 20
5. Double theta lifts and degenerate principal series 26
6. Bounding the associated cycles 34
7. Induced orbits and double geometric lifts 43
8. The unipotent representations construction and iterated theta lifting 49
9. Concluding remarks 51
Appendix A. Combinatorial parameterization of special unipotent representations 51
Appendix B. A few geometric facts 53
References 58

1. Introduction and the main results

1.1. **Unitary representations and the orbit method.** A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [43, 44]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [6, 43]. For more general Lie groups, work of Mackey and Duflo [28, 56] suggest that one should focus attention on reductive Lie groups. As

2000 *Mathematics Subject Classification.* 22E45, 22E46.

Key words and phrases. orbit method, unitary dual, special unipotent representation, classical group, theta lifting, moment map.

expounded by Vogan in his writings (see for example [82, 84, 85]), the problem finally is to quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We highlight two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [4, 5], Adams, Barbasch and Vogan established some important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [2]. The problem of classifying (integral) special unipotent representations for complex semisimple groups was solved earlier by Barbasch and Vogan [16] and the unitarity of these representations was established by Barbasch for complex classical groups [7, Section 10]. Shortly after, Barbasch outlined a proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [8]. The second major development is Vogan's theory of associated varieties [83] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a representation "attached" to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [39], the description of discrete spectrum by Adams [1] and Li [51], and the preservation of unitarity in stable range theta lifting by Li [50]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [12, 17, 34, 36, 37, 68, 71, 74, 80]. We would also like to mention the work of Przebinda [71] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [34] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [29, 53, 62, 64], which among other things demonstrate the tight link between the orbit method and Howe's theory.

As mentioned earlier, there have been extensive investigations of unipotent representations for real reductive groups, by Vogan and his collaborators (see e.g. [2, 82, 83]). Nevertheless, the subject remains extraordinarily mysterious thus far.

In the present article we will demonstrate that Howe's (constructive) theory has immense implications for the orbit method, and in particular has near perfect synergy with special unipotent representations (in the case of classical groups). (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [12, 34, 59, 80].) We will restrict our attention to a real classical group G of type B, C or D (which in our terminology includes a real metaplectic group), and we will classify all special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. Here $\check{\mathcal{O}}$ is a nilpotent adjoint orbit of $\check{G}$, the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group [13]). When $\check{\mathcal{O}}$ is of good parity [60], we will construct all special unipotent representations of G attached to $\check{\mathcal{O}}$ via the method of theta lifting. As a direct consequence of the construction and the classification, we conclude that all special unipotent representations of G are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([2, Introduction]).

1.2. **Special unipotent representations of classical groups of type B, C or D.** In this article, we aim to classify special unipotent representations of classical groups of type B, C or D. As the cases of complex orthogonal groups and complex symplectic groups are well-understood (see [16] and [12]), we will focus on the following groups:

$$\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p,q),\ \mathrm{O}^*(2n), \tag{1.1}$$

where $p,q,n \in \mathbb{N} := \{0,1,2,\cdots\}$. Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the real metaplectic group, namely the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n=0$.

Let G be one of the groups in (1.1). As usual, we view G as a real form of $G_{\mathbb{C}}$ (or a double cover of a real form of $G_{\mathbb{C}}$ in the metaplecitic case), where

$$G_{\mathbb{C}} := \begin{cases} \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \text{ or } \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2p+2q}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ for the Lie algebras of G and $G_{\mathbb{C}}$, respectively, and view $\mathfrak{g}_{\mathbb{R}}$ as a real form of $\mathfrak{g}$.

Denote $r_{\mathfrak{g}}$ the rank of $\mathfrak{g}$. Let $W_{r_{\mathfrak{g}}}$ be the subgroup of $\mathrm{GL}_{r_{\mathfrak{g}}}(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. Then as usual, Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} = \left(\mathrm{S}(\mathbb{C}^{r_{\mathfrak{g}}})\right)^{W_{r_{\mathfrak{g}}}}. \tag{1.2}$$

Here and henceforth, "U" indicates the universal enveloping algebra of a Lie algebra, a superscript group indicate the space of invariant vectors under the group action, and "S" indicates the symmetric algebra. Unless $G_{\mathbb{C}}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. By (1.2), we have the following parameterization of characters of $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$:

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}, \mathbb{C}) = W_{r_{\mathfrak{g}}} \backslash (\mathbb{C}^{r_{\mathfrak{g}}})^* = W_{r_{\mathfrak{g}}} \backslash \mathbb{C}^{r_{\mathfrak{g}}}$$

Here "$\mathrm{Hom}_{\mathrm{alg}}$" indicates the set of $\mathbb{C}$-algebra homomorphisms, and a superscript " $*$ " over a vector space indicates the dual space.

We define the Langlands dual of G to be the complex group

$$\check{G} := \begin{cases} \mathrm{Sp}_{p+q-1}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is odd}; \\ \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is even}; \\ \mathrm{O}_{2n+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{O}_{2p+2q+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$.

Remark. The authors have defined the notion of metaplectic dual for a real metaplectic group [13]. In this article, we have chosen to use the uniform terminology of Langlands dual (rather than metaplectic dual in the case of a real metaplectic group).

Denote by $\mathrm{Nil}(\check{\mathfrak{g}})$ the set of nilpotent $\check{G}$-orbits in $\check{\mathfrak{g}}$. When no confusion is possible, we will not distinguish a nilpotent orbit in $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ with its corresponding Young diagram. In particular, the zero orbit is represented by the Young diagram consisting of one nonempty column.

Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. It determines a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$ as in what follows. For every $a \in \mathbb{N}$, write

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd;} \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even;} \end{cases}$$

By convention, $\rho(1)$ and $\rho(0)$ are the empty sequence. Write $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ ($s \geqslant 0$) for the row lengths of $\check{\mathcal{O}}$. Define

$$\chi(\check{\mathcal{O}}) := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0), \tag{1.3}$$

to be viewed as a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$. Here the number of 0's is

$$\left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor.$$

Recall the following well-known result of Dixmier ([18, Section 3]): for every algebraic character χ of $\mathrm{Z}(\mathfrak{g})$, there exists a unique maximal ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of χ. As an easy consequence, there will be a unique maximal $G_{\mathbb{C}}$-stable ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$, for $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.

Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation ([20,87]) if its Harish-Chandra module has finite length. When $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a metaplectic group, write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then a representation of G is said to be genuine if ε_G acts via the scalar multiplication by -1. The notion of "genuine" will be used in similar situations without further explanation. Following Barbasch and Vogan ([2,16]), we make the following definition.

Definition 1.1. *Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. An irreducible Casselman-Wallach representation π of G is attached to $\check{\mathcal{O}}$ if*

- *$I_{\check{\mathcal{O}}}$ annihilates π; and*
- *π is genuine if G is a metaplectic group.*

Write $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ for the set of isomorphism classes of irreducible Casselman-Wallach representations of G that are attached to $\check{\mathcal{O}}$. We say that an irreducible Casselman-Wallach representation of G is special unipotent if it is attached to $\check{\mathcal{O}}$, for some $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. As mentioned earlier, we will construct all special unipotent representations of G, and will show that all of them are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([2, Introduction]).

1.3. **Combinatorial construct: painted bipartitions.** We introduce a symbol $\star$, taking values in $\{B, C, D, \widetilde{C}, C^*, D^*\}$, to specify the type of the groups that we are considering as in (1.1), namely odd real orthogonal groups, real symplectic groups, even real orthogonal groups, real metaplectic groups, quaternionic symplectic groups and quaternionic orthogonal groups, respectively.

For a Young diagram $\imath$, write

$$\mathbf{r}_1(\imath) \geqslant \mathbf{r}_2(\imath) \geqslant \mathbf{r}_3(\imath) \geqslant \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty} \mathbf{r}_i(\imath)$ the total size of $\imath$.

For any Young diagram $\imath$, we introduce the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as the following subset of $\mathbb{N}^+ \times \mathbb{N}^+$ ($\mathbb{N}^+$ denotes the set of positive integers):

$$\mathrm{Box}(\imath) := \left\{ (i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \;\middle|\; j \leqslant \mathbf{r}_i(\imath) \right\}. \tag{1.4}$$

We also introduce five symbols $\bullet$, s, r, c and d, and make the following definition.

Definition 1.2. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *when $S = \{s\}$ or $\{r\}$, every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$;*
- *when $S = \{c\}$ or $\{d\}$, every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$.*

A painted Young diagram is then a pair $(\imath, \mathcal{P})$, consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$.

Example. Suppose that $\imath = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, then there are $25 + 12 + 6 + 2 = 45$ paintings on $\imath$ in total as listed below.

$\begin{array}{|c|c|}\hline \bullet & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\alpha, \beta \in \{\bullet, s, r, c, d\}$ $\qquad$ $\begin{array}{|c|c|}\hline s & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\alpha \in \{r, c, d\}, \beta \in \{s, r, c, d\}$

$\begin{array}{|c|c|}\hline r & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\alpha \in \{c, d\}, \beta \in \{r, c, d\}$ $\qquad$ $\begin{array}{|c|c|}\hline c & \alpha \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ $\alpha \in \{c, d\}$

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.3. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- *$\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$;*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, r, c, d\}, & \text{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \text{if } \alpha = D; \\ \{\bullet, s, c\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \text{if } \alpha = C^*; \\ \{\bullet, s\}, & \text{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, s\}, & \text{if } \alpha = C; \\ \{\bullet\}, & \text{if } \alpha = D; \\ \{\bullet, r, d\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \text{if } \alpha = C^*; \\ \{\bullet, r\}, & \text{if } \alpha = D^*. \end{cases}$$

For any painted bipartition τ as in Definition 1.3, we write

$$\imath_\tau := \imath,\ \mathcal{P}_\tau := \mathcal{P},\ \jmath_\tau := \jmath,\ \mathcal{Q}_\tau := \mathcal{Q},\ \alpha_\tau := \alpha,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

We further attach some objects to τ in what follows:

$$|\tau|,\ (p_\tau,\ q_\tau),\ G_\tau,\ \dim \tau,\ \varepsilon_\tau.$$

$|\tau|$**:** This is the natural number

$$|\tau| := |\imath| + |\jmath| \,.$$

(p_τ, q_τ)**:** If $\star_\tau \in \{B, D, C^*\}$, this is a pair of natural numbers given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := \# \bullet + 2\#r + \#c + \#d + \#B^+; \\ q_\tau := \# \bullet + 2\#s + \#c + \#d + \#B^-. \end{cases}$$

If $\star_\tau \in \{C, \widetilde{C}, D^*\}$, we let $p_\tau := q_\tau := |\tau|$.

G_τ**:** This is a classical group given by

$$G_\tau := \begin{cases} \mathrm{O}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

$\dim \tau$**:** This is the dimension of the standard representation of the complexification of G_τ, or equivalently,

$$\dim \tau := \begin{cases} 2\,|\tau| + 1, & \text{if } \star_\tau = B; \\ 2\,|\tau|\,, & \text{otherwise.} \end{cases}$$

ε_τ**:** This is the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\tau = 0 \Leftrightarrow \text{the symbol } d \text{ occurs in the first column of } (\imath, \mathcal{P}) \text{ or } (\jmath, \mathcal{Q}).$$

The triple $\mathsf{s}_\tau = (\star_\tau, p_\tau, q_\tau) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ will also be referred to as the classical signature attached to τ.

Example. Suppose that

$$\tau = \begin{array}{|c|c|}\hline \bullet & c \\ \hline \bullet & \\ \cline{1-1} c & \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|c|}\hline \bullet & s & r \\ \hline \bullet & & \\ \cline{1-1} r & & \\ \cline{1-1} r & & \\ \cline{1-1}\end{array} \times B^+.$$

Then

$$\begin{cases} |\tau| = 10; \\ p_\tau = 4 + 6 + 2 + 0 + 1 = 13; \\ q_\tau = 4 + 2 + 2 + 0 + 0 = 8; \\ G_\tau = \mathrm{O}(13, 8); \\ \dim \tau = 21; \\ \varepsilon_\tau = 1. \end{cases}$$

1.4. **Counting special unipotent representations by painted bipartitions.** We fix a classical group G which has type $\star$. Following [60, Definition 4.1], we say that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity if

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \check{G} \text{ is a complex symplectic group; and} \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \check{G} \text{ is a complex orthogonal group.} \end{cases}$$

In general, the study of the special unipotent representations attached to $\check{\mathcal{O}}$ will be reduced to the case when $\check{\mathcal{O}}$ has $\star$-good parity. We refer the reader to [14].

For the rest of this subsection, assume that $\check{\mathcal{O}}$ has $\star$-good parity. Equivalently we consider $\check{\mathcal{O}}$ as a Young diagram that has $\star$-good parity in the following sense:

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \star \in \{B, \widetilde{C}\}; \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \star \in \{C, D, C^*, D^*\}; \text{ and} \\ \text{the total size } \left|\check{\mathcal{O}}\right| \text{ is odd if and only if } \star \in \{C, C^*\}. \end{cases}$$

Definition 1.4. *A $\star$-pair is a pair $(i, i+1)$ of consecutive positive integers such that*

$$\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be

- *vacant in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*
- *tailed in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and even.*

Denote $\mathrm{PP}_\star(\check{\mathcal{O}})$ the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$.

For any $\check{\mathcal{O}}$, we attach a pair of Young diagrams

$$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) := (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})),$$

as follows.

The case when $\star = B$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}).$$

The case when $\star = \widetilde{C}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}).$$

The case when $\star \in \{C, C^*\}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0, 0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

The case when $\star \in \{D, D^*\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0, \end{cases}$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0, 0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

Define

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) := \left\{ \tau \text{ is a painted bipartition} \mid \star_\tau = \star \text{ and } (\imath_\tau, \jmath_\tau) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})) \right\}.$$

We also define the following extended parameter set:

$$\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}) := \begin{cases} \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \times \{\wp \subset \mathrm{PP}_{\star}(\check{\mathcal{O}})\}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \times \{\varnothing\}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

We use $\boldsymbol{\tau} = (\tau, \wp)$ to denote an element in $\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$.

Put

$$\mathrm{Unip}_{\star}(\check{\mathcal{O}}) := \begin{cases} \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|+1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = B; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{|\check{\mathcal{O}}|-1}(\mathbb{R})), & \text{if } \star = C; \\ \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = D; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{Sp}}_{|\check{\mathcal{O}}|}(\mathbb{R})), & \text{if } \star = \widetilde{C}; \\ \bigsqcup_{p,q\in\mathbb{N}, 2p+2q=|\check{\mathcal{O}}|-1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q)), & \text{if } \star = C^*; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}^*(|\check{\mathcal{O}}|)), & \text{if } \star = D^*. \end{cases}$$

The main result of [14], on counting of special unipotent representations, is as follows.

Theorem 1.5. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then*

$$\#(\mathrm{Unip}_{\star}\check{\mathcal{O}}) = \begin{cases} 2\#(\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})), & \textit{if } \star \in \{B, D\} \textit{ and } (\star, \check{\mathcal{O}}) \neq (D, \varnothing); \\ \#(\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})), & \textit{otherwise}\ . \end{cases}$$

1.5. **Descending painted bipartitions and constructing representations by induction.** Let $\star$ and $\check{\mathcal{O}}$ be as before. For every $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$, in what follows we will construct a representation $\pi_{\boldsymbol{\tau}}$ of G_{τ} by the method of theta lifting. For the initial case when $\check{\mathcal{O}}$ is the empty Young diagram, define

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \text{the one dimensional genuine representation}, & \text{if } \star = \widetilde{C} \text{ so that } G_{\tau} = \widetilde{\mathrm{Sp}}_0(\mathbb{R}); \\ \text{the one dimensional trivial representation}, & \text{if } \star \neq \widetilde{C}. \end{cases}$$

Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or } \ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or } \ D^*.$$

We call $\star'$ the Howe dual of $\star$. Assume now that $\check{\mathcal{O}}$ is nonempty, and define its dual descent to be

$$\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}}) := \text{the Young diagram obtained from } \check{\mathcal{O}} \text{ by removing the first row.}$$

Note that $\check{\mathcal{O}}'$ has $\star'$-good parity, and

$$\mathrm{PP}_{\star'}(\check{\mathcal{O}}') = \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \mathrm{PP}_{\star}(\check{\mathcal{O}})\}.$$

Define the dual descent of $\wp$ to be

$$\wp' := \check{\nabla}(\wp) := \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \wp\} \subset \mathrm{PP}_{\star'}(\check{\mathcal{O}}'). \tag{1.5}$$

In Section 2, we will define the descent map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

We then define the descent of $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$ to be the element

$$\boldsymbol{\tau}' := (\tau', \wp') := \nabla(\boldsymbol{\tau}) := (\nabla(\tau), \check{\nabla}(\wp)) \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

Let $(W_{\tau,\tau'}, \langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\tau'})$ be a real symplectic space of dimension $\dim\tau\cdot\dim\tau'$. As usual, there are continuous homomorphisms $G_\tau \to \mathrm{Sp}(W_{\tau,\tau'})$ and $G_{\tau'} \to \mathrm{Sp}(W_{\tau,\tau'})$ whose images form a reductive dual pair in $\mathrm{Sp}(W_{\tau,\tau'})$. We form the semidirect product

$$J_{\tau,\tau'} := (G_\tau \times G_{\tau'}) \ltimes \mathrm{H}(W_{\tau,\tau'}),$$

where

$$\mathrm{H}(W_{\tau,\tau'}) := W_{\tau,\tau'} \times \mathbb{R}$$

is the Heisenberg group with group multiplication

$$(w,t)(w',t') := (w+w', t+t'+\langle w,w'\rangle_{\tau,\tau'}), \quad w,w' \in W_{\tau,\tau'},\ t,t' \in \mathbb{R}.$$

Let $\omega_{\tau,\tau'}$ be a suitably normalized smooth oscillator representation of $J_{\tau,\tau'}$ such that every $t \in \mathbb{R} \subset J_{\tau,\tau'}$ acts on it through the scalar multiplication by $e^{2\pi\sqrt{-1}\,t}$ (the letter π often denotes a representation, but here it stands for the circumference ratio). See Section 4.1 for details.

For any Casselman-Wallach representation π' of $G_{\tau'}$, write

$$\check{\Theta}^{\tau}_{\tau'}(\pi') := (\omega_{\tau,\tau'}\widehat{\otimes}\pi')_{G_{\tau'}} \qquad \text{(the Hausdorff coinvariant space)},$$

where $\widehat{\otimes}$ indicates the complete projective tensor product. This representation is clearly Fréchet, smooth, and of moderate growth, and so a Casselman-Wallach representation by the fundamental result of Howe [40].

Now we define the representation $\pi_{\boldsymbol{\tau}}$ of G_τ by induction on the number of nonempty rows of $\check{\mathcal{O}}$:

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'}) \otimes (1^{+,-}_{p_\tau,q_\tau})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'} \otimes \det{}^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'}), & \text{if } \star = C^* \text{ or } D^*. \end{cases} \tag{1.6}$$

Here $1^{+,-}_{p_\tau,q_\tau}$ denotes the character of $\mathrm{O}(p_\tau,q_\tau)$ whose restriction to $\mathrm{O}(p_\tau)\times\mathrm{O}(q_\tau)$ equals $1\otimes\det$ (1 stands for the trivial character), and $\varepsilon_\wp$ denote the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\wp := 1 \Leftrightarrow (1,2) \in \wp.$$

In the sequel, we will use $\varnothing$ to denote the empty set (in the usual way), as well as the empty Young diagram or the painted Young diagram whose underlying Young diagram is empty. As always, we let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$.

We are now ready to state our first main theorem.

Theorem 1.6. *Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For every $\boldsymbol{\tau} = (\tau,\wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, the representation $\pi_{\boldsymbol{\tau}}$ of G_τ in (1.6) is irreducible and attached to $\check{\mathcal{O}}$.
(b) If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ (\boldsymbol{\tau},\epsilon) & \mapsto & \pi_{\boldsymbol{\tau}} \otimes \det{}^\epsilon \end{array}$$

is bijective.
(c) In all other cases, the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ \boldsymbol{\tau} & \mapsto & \pi_{\boldsymbol{\tau}} \end{array}$$

is bijective.

By the above theorem, we have explicitly constructed all special unipotent representations in $\mathrm{Unip}_\star(\check{\mathcal{O}})$, when $\check{\mathcal{O}}$ has $\star$-good parity. The method of matrix coefficient integrals (Section 4) will imply the following

Corollary 1.7. *Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then all special unipotent representations in $\mathrm{Unip}_\star(\check{\mathcal{O}})$ are unitarizable.*

The reduction of [14] allows us to classify all special unipotent presentations attached to a general $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ from those which have $\star$-good parity. We thus conclude

Theorem 1.8. *All special unipotent representations of the classical groups in* (1.1) *are unitarizable.*

1.6. **Computing associated cycles of the constructed representations.** Let $G, G_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}, \check{\mathfrak{g}}$ and $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ be as in Section 1.2. Fix a maximal compact subgroup K of G whose Lie algebra is denoted by $\mathfrak{k}_{\mathbb{R}}$. We equipping $\mathfrak{g}$ with the trace form. Then we have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the complexification of $\mathfrak{k}_{\mathbb{R}}$. By taking the dual spaces, we also have a decomposition

$$\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*,$$

Write $K_{\mathbb{C}}$ for the complexification of the compact group K. It is a complex algebraic group with an obvious algebraic action on $\mathfrak{p}^*$.

For any $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$, write $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}^*)$ for the Barbarsch-Vogan dual of $\check{\mathcal{O}}$ so that its Zariski closure in $\mathfrak{g}^*$ equals the associated variety of the ideal $I_{\check{\mathcal{O}}} \subset \mathrm{U}(\mathfrak{g})$. See [13, 16]. The algebraic variety $\mathcal{O} \cap \mathfrak{p}^*$ is a finite union of $K_{\mathbb{C}}$-orbits. Given any such orbit $\mathscr{O}$, write $\mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O})$ for the Grothendieck group of the category of $K_{\mathbb{C}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O}).$$

We say that a Casselman-Wallach representation of G is $\mathcal{O}$-bounded if the associated variety of its annihilator ideal is contained in the Zariski closure of $\mathcal{O}$. Note that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are $\mathcal{O}$-bounded. Write $\mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}}$ for the Grothendieck group of the category of all such representations. From [83, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{AC}_{\mathcal{O}} \colon \mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}} \longrightarrow \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}).$$

We call $\mathrm{AC}_{\mathcal{O}}(\pi)$ the associated cycle of π, where π is an $\mathcal{O}$-bounded Casselman-Wallach representation of G. This is a fundamental invariant attached to π.

Following Vogan [83, Section 8], we make the following definition.

Definition 1.9. *Let $\mathscr{O}$ be a $K_{\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}^*$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $K_{\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_X$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_X$;*
- *if $\star = \widetilde{C}$, then ε_G acts on $\mathcal{E}$ by the scalar multiplication by -1.*

Here $X \in \mathscr{O}$, $\mathcal{E}_X$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_X$ denotes the Lie algebra of the stabilizer of X in $K_{\mathbb{C}}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_X$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$.

Note that in the situation of the classical groups we consider in this article, all admissible orbit data are line bundles. Denote by $\mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$. Put

$$\mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O}) \subset \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}). \tag{1.7}$$

For the rest of this subsection, we suppose that G has type $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and $\check{\mathcal{O}}$ has $\star$-good parity. Recall that a nilpotent orbit in $\check{\mathfrak{g}}$ is said to be distinguished if it is has no nontrivial intersection with any proper Levi subalgebra of $\check{\mathfrak{g}}$. Combinatorially, this is equivalent to saying that no pair of rows of the Young diagram have equal nonzero length. Note that all distinguished nilpotent orbits in $\check{\mathfrak{g}}$ has $\star$-good parity.

Definition 1.10. *(a) The orbit $\check{\mathcal{O}}$ (which has $\star$-good parity) is said to be quasi-distinguished if there is no $\star$-pair that is balanced in $\check{\mathcal{O}}$.*
(b) If $\star \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be weakly-distinguished if there is no positive even integer i such that $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = \mathbf{r}_{i+2}(\check{\mathcal{O}}) = \mathbf{r}_{i+3}(\check{\mathcal{O}}) > 0$.
(c) If $\star \in \{C, \widetilde{C}, C^*\}$ so that its Howe dual $\star' \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be weakly-distinguished if either it is the empty Young diagram or it is nonempty and its dual descent $\check{\mathcal{O}}'$ (which is a Young diagram that has $\star'$-good parity) is weakly-distinguished.

We will compute the associated cycle of π_τ for every painted bipartition τ associated to $\check{\mathcal{O}}$ which has $\star$-good parity. The computation will be a key ingredient in the proof of Theorem 1.6. Our second main theorem is the following result concerning associated cycles of the special unipotent representations.

Theorem 1.11. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For any $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the associated cycle $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$ is a nonzero sum of pairwise distinct elements of $\mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O})$.
(b) If $\check{\mathcal{O}}$ is weakly-distinguished, then the map

$$\mathrm{AC}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$$

is injective.
(c) If $\check{\mathcal{O}}$ is quasi-distinguished, then the map $\mathrm{AC}_{\mathcal{O}}$ induces a bijection

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}).$$

Remark. Suppose that $\star \in \{C^*, D^*\}$ so that G is quaternionic. Then there is precisely one admissible orbit datum over $\mathscr{O}$ for each $K_{\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}^*$. Thus

$$\mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O}) = K_{\mathbb{C}} \backslash (\mathcal{O} \cap \mathfrak{p}^*).$$

If $\check{\mathcal{O}}$ is not quasi-distinghuished, then $\mathcal{O} \cap \mathfrak{p}^*$ is empty (see [23, Theorems 9.3.4 and 9.3.5]), and hence $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is also empty.

2. Descents of painted bipartitions: combinatorics

In this section, we define the descent of a painted bipartition, as alluded to in Section 1.5. As before, let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.

For a diagram $\imath$, its naive descent, which is denoted by $\nabla_{\mathrm{naive}}(\imath)$, is defined to be the Young diagram obtained from $\imath$ by removing the first column. By convention, $\nabla_{\mathrm{naive}}(\varnothing) = \varnothing$.

In the rest of this section, we assume that $\check{\mathcal{O}} \neq \varnothing$. Recall that we have its dual descent $\check{\mathcal{O}}'$, and $\check{\mathcal{O}}'$ has $\star'$-good parity, where $\star'$ is the Howe dual of $\star$.

2.1. Naive descents of painted bipartitions.

In this subsection, let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ be a painted bipartition such that $\star_\tau = \star$. Put

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ does not occur in the leading column of } \tau; \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ occurs in the leading column of } \tau; \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{2.1}$$

Lemma 2.1. *If $\star \in \{B, C, C^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\imath, \nabla_{\text{naive}}(\jmath))$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}(i,j), & \text{if } \mathcal{P}(i,j) \notin \{\bullet, s\}; \end{cases}$$

- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,

$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j+1), & \text{if } \mathcal{Q}(i,j+1) \notin \{\bullet, s\}. \end{cases}$$

Proof. First assume that the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$. Then the image of $\mathcal{P}$ is in fact contained in $\{\bullet\}$, and $(\imath, \jmath)$ is right interlaced in the sense that

$$\mathbf{c}_1(\jmath) \geqslant \mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\jmath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\jmath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots.$$

Hence $(\imath', \jmath') := (\imath, \nabla(\jmath))$ is left interlaced in the sense that

$$\mathbf{c}_1(\imath') \geqslant \mathbf{c}_1(\jmath') \geqslant \mathbf{c}_2(\imath') \geqslant \mathbf{c}_2(\jmath') \geqslant \mathbf{c}_3(\imath') \geqslant \mathbf{c}_3(\jmath') \geqslant \cdots.$$

Then it is clear that there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ such that images of $\mathcal{P}'$ and $\mathcal{Q}'$ are both contained in $\{\bullet, s\}$. This proves the lemma in the special case when the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$.

The proof of the lemma in the general case is easily reduced to this special case. □

Lemma 2.2. *If $\star \in \{\widetilde{C}, D, D^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\nabla_{\text{naive}}(\imath), \jmath)$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{P}(i,j+1), & \text{if } \mathcal{P}(i,j+1) \notin \{\bullet, s\}; \end{cases}$$

- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,

$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j), & \text{if } \mathcal{Q}(i,j) \notin \{\bullet, s\}. \end{cases}$$

Proof. The proof is similar to that of Lemma 2.1. □

Definition 2.3. *In the notation of Lemma 2.1 and 2.2, we call τ' the naive descent of τ, to be denoted by $\nabla_{\text{naive}}(\tau)$.*

Example. If

$$\tau = \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & \bullet & c \\ \hline \bullet & s & c \\ \cline{1-3} s \\ \cline{1-1} c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & r & d \\ \hline d & d \\ \cline{1-2} \end{array} \times \widetilde{C},$$

then

$$\nabla_{\text{naive}}(\tau) = \begin{array}{|c|c|c|}\hline \bullet & \bullet & c \\ \hline \bullet & c \\ \cline{1-2} \end{array} \times \begin{array}{|c|c|c|}\hline \bullet & \bullet & s \\ \hline \bullet & r & d \\ \hline d & d \\ \cline{1-2} \end{array} \times B^-.$$

2.2. **Descents of painted bipartitions.** Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ and write

$$\tau'_{\text{naive}} = (\imath', \mathcal{P}'_{\text{naive}}) \times (\jmath', \mathcal{Q}'_{\text{naive}}) \times \alpha'$$

for the naive descent of τ. This is clearly an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$.

The following two lemmas are easily verified and we omit the proofs. We will give an example for each case.

Lemma 2.4. *Suppose that*

$$\begin{cases} \alpha = B^+; \\ \mathbf{r}_2(\check{\mathcal{O}}) > 0; \\ \mathcal{Q}(\mathbf{c}_1(\jmath), 1) \in \{\, r, d \,\}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\text{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} s, & \text{if } (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\text{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|}\hline \bullet & r \\ \hline r & d \\ \hline \end{array} \times B^+,$$

then

$$\tau'_{\text{naive}} = \begin{array}{|c|c|}\hline s & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C} \qquad \text{and} \qquad \tau' = \begin{array}{|c|c|}\hline s & c \\ \hline s & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C}.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $4, 4, 4, 2$.

Lemma 2.5. *Suppose that*

$$\begin{cases} \alpha = D; \\ \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0; \\ \big(\mathcal{P}(\mathbf{c}_2(\imath), 1), \mathcal{P}(\mathbf{c}_2(\imath), 2)\big) = (r, c); \\ \mathcal{P}(\mathbf{c}_1(\imath), 1) \in \{\, r, d \,\}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\text{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} r, & \text{if } (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\text{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & s \\ \hline \bullet & s \\ \hline r & c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times D,$$

then

$$\tau'_{\text{naive}} = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \\ \cline{1-1} \bullet & \\ \cline{1-1} \end{array} \times C, \qquad \text{and} \qquad \tau' = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline r \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \\ \cline{1-1} \bullet & \\ \cline{1-1} \end{array} \times C.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $7, 7, 7, 3$.

Definition 2.6. *We define the descent of τ to be*

$$\nabla(\tau) := \begin{cases} \tau', & \text{if either of the condition of Lemma 2.4 or 2.5 holds;} \\ \nabla_{\text{naive}}(\tau), & \text{otherwise,} \end{cases}$$

which is an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$. *Here τ' is as in Lemmas 2.4 and 2.5.*

In conclusion, we have by now a well-defined descent map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

The following injectivity result will be important for us.

Proposition 2.7. *If $\star \in \{ B, D, C^* \}$, then the map*

$$\begin{array}{rcl} \mathrm{PBP}_{\star}(\check{\mathcal{O}}) & \to & \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array} \tag{2.2}$$

is injective. If $\star \in \{ C, \widetilde{C}, D^* \}$, then the map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \tag{2.3}$$

is injective.

3. From painted bipartitions to associated cycles

3.1. **Classical spaces.** Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ as before. Put

$$(\epsilon, \dot{\epsilon}) := (\epsilon_\star, \dot{\epsilon}_\star) := \begin{cases} (1,1), & \text{if } \star \in \{B, D\}; \\ (-1,-1), & \text{if } \star \in \{C, \widetilde{C}\}; \\ (-1,1), & \text{if } \star = C^*; \\ (1,-1), & \text{if } \star = D^*. \end{cases}$$

A classical signature is defined to be a triple $\mathsf{s} = (\star, p, q) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ such that

$$\begin{cases} p+q \text{ is odd }, & \text{if } \star = B; \\ p+q \text{ is even }, & \text{if } \star = D; \\ p = q, & \text{if } \star \in \{C, \widetilde{C}, D^*\}; \\ \text{both } p \text{ and } q \text{ are even}, & \text{if } \star = C^*. \end{cases}$$

Suppose that $\mathsf{s} = (\star, p, q)$ is a classical signature in the rest of this section.

We omit the proof of the following lemma (*cf.* [65, Section 1.3]).

Lemma 3.1. *Then there is quadruple $(V, \langle\, , \rangle, J, L)$ satisfying the following conditions:*

- *V is a complex vector space of dimension $p+q$;*
- *$\langle\, , \rangle$ is an ϵ-symmetric non-degenerate bilinear form on V;*
- *$J : V \to V$ is a conjugate linear automorphism of V such that $J^2 = \epsilon \cdot \dot{\epsilon}$;*
- *$L : V \to V$ is a linear automorphism of V such that $L^2 = \dot{\epsilon}$;*
- *$\langle Ju, Jv \rangle = \overline{\langle u, v \rangle}$, for all $u, v \in V$;*

- $\langle Lu, Lv\rangle = \langle u, v\rangle$, *for all* $u, v \in V$;
- $LJ = JL$;
- *the Hermitian form* $(u, v) \mapsto \langle Lu, Jv\rangle$ *on* V *is positive definite;*
- *if* $\dot{\epsilon} = 1$, *then* $\dim\{v \in V \mid Lv = v\} = p$ *and* $\dim\{v \in V \mid Lv = -v\} = q$.

Moreover, such a quadruple is unique in the following sense: if $(V', \langle\ ,\ \rangle', J', L')$ *is another quadruple satisfying the analogous conditions, then there is a linear isomorphism* $\phi : V \to V'$ *that respectively transforms* $\langle\ ,\ \rangle$, J *and* L *to* $\langle\ ,\ \rangle'$, J' *and* L'.

In the notation of Lemma 3.1, we call $(V, \langle\ ,\ \rangle, J, L)$ a classical space of signature s, and denote it by $(V_{\mathsf{s}}, \langle\ ,\ \rangle_{\mathsf{s}}, J_{\mathsf{s}}, L_{\mathsf{s}})$.

Denote by $G_{\mathsf{s},\mathbb{C}}$ the isometry group of $(V_{\mathsf{s}}, \langle\ ,\ \rangle_{\mathsf{s}})$, which is an complex orthogonal group if $\epsilon = 1$ and a complex symplectic group if $\epsilon = -1$. Respectively denote by $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ and $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ the centralizes of J_{s} and L_{s} in $G_{\mathsf{s},\mathbb{C}}$. Then $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ is a real classical group isomorphic with

$$\begin{cases} \mathrm{O}(p,q), & \text{if } \star = B \text{ or } D; \\ \mathrm{Sp}_{2p}(\mathbb{R}), & \text{if } \star \in \{C, \widetilde{C}\}; \\ \mathrm{Sp}(\frac{p}{2}, \frac{q}{2}), & \text{if } \star = C^*; \\ \mathrm{O}^*(2p), & \text{if } \star_\tau = D^*. \end{cases}$$

Put

$$G_{\mathsf{s}} := \begin{cases} \text{the metaplectic double cover of } G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{if } \star = \widetilde{C}; \\ G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{otherwise;}. \end{cases}$$

Denote by K_{s} the inverse image of $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ under the natural homomorphism $G_{\mathsf{s}} \to G_{\mathsf{s},\mathbb{C}}$, which is a maximal compact subgroup of G_{s}. Write $K_{\mathsf{s},\mathbb{C}}$ for the complexification of K_{s}, which is a reductive complex linear algebraic group.

For every $\lambda \in \mathbb{C}$, write $V_{\mathsf{s},\lambda}$ for the eigenspace of L_{s} with eigenvalue λ. Write

$$\det_\lambda : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s},\lambda}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times. \tag{3.1}$$

If $\star = \widetilde{C}$, then the natural homomorphism $K_{\mathsf{s},\mathbb{C}} \to \mathrm{GL}(V_{\mathsf{s},\sqrt{-1}})$ is a double cover, and there is a unique genuine algebraic character $\det_{\sqrt{-1}}^{\frac{1}{2}} : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$ whose square equals $\det_{\sqrt{-1}}$.

Write

$$\det : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s}}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times. \tag{3.2}$$

This is the trivial character unless $\star = B$ or D.

Denote by $\mathfrak{g}_{\mathsf{s}}$ the Lie algebra of $G_{\mathsf{s},\mathbb{C}}$. Then we have a decomposition

$$\mathfrak{g}_{\mathsf{s}} = \mathfrak{k}_{\mathsf{s}} \oplus \mathfrak{p}_{\mathsf{s}},$$

where $\mathfrak{k}_{\mathsf{s}}$ is the Lie algebra of $K_{\mathsf{s},\mathbb{C}}$, and $\mathfrak{p}_{\mathsf{s}}$ is the orthogonal complement of $\mathfrak{k}_{\mathsf{s}}$ in $\mathfrak{g}_{\mathsf{s}}$ under the trace form. Using the trace form, we also identify $\mathfrak{g}_{\mathsf{s}}^*$ with $\mathfrak{g}_{\mathsf{s}}$. Denote by $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ the set of nilpotent $G_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathsf{s}}$, and by $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$ the set of nilpotent $K_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathsf{s}}$.

Given a $K_{\mathsf{s},\mathbb{C}}$-orbit $\mathscr{O}$ in $\mathfrak{p}_{\mathsf{s}}$, write $\mathcal{K}_{\mathsf{s}}(\mathscr{O})$ for the Grothendieck group of the categogy of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundles over $\mathscr{O}$. Denote by $\mathcal{K}_{\mathsf{s}}^+(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}(\mathscr{O})$ the submonoid generated by all these equivariant algebraic vector bundles. The notion of admissible orbit datum over $\mathscr{O}$ is defined as in Definition 1.9. Write

$$\mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}^+(\mathscr{O})$$

for the set of isomorphism classes of admissible orbit data over $\mathscr{O}$.

Let $\mathcal{O}$ be a $G_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s},\mathbb{C}}$. It is well-known that $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ has only finitely many $K_{\mathsf{s},\mathbb{C}}$-orbits. Put

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}_{\mathsf{s}}(\mathscr{O}),$$

and

$$\mathcal{K}_{\mathsf{s}}^{+}(\mathcal{O}) := \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}_{\mathsf{s}}^{+}(\mathscr{O}).$$

Put

$$\mathrm{AOD}_{\mathsf{s}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}^{+}(\mathcal{O}).$$

Define a partial order $\preceq$ on $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$ such that

$$\mathcal{E}_1 \preceq \mathcal{E}_2 \Leftrightarrow \mathcal{E}_2 - \mathcal{E}_1 \in \mathcal{K}_{\mathsf{s}}^{+}(\mathcal{O}) \qquad (\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})).$$

For every algebraic character χ of $K_{\mathsf{s},\mathbb{C}}$, the twisting map

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \qquad \mathcal{E} \mapsto \mathcal{E} \otimes \chi$$

is obviously defined.

3.2. **The moment maps.** Recall that $\star'$ is the Howe dual of $\star$. Suppose that $\mathsf{s}' = (\star', p', q')$ is another classical signature. Put

$$W_{\mathsf{s},\mathsf{s}'} := \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_{\mathsf{s}'}).$$

Then we have the adjoint map

$$W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s}',\mathsf{s}}, \qquad \phi \mapsto \phi^*$$

that is specified by requiring

$$\langle \phi v, v' \rangle_{\mathsf{s}'} = \langle v, \phi^* v' \rangle_{\mathsf{s}}, \qquad \text{for all } v \in V_{\mathsf{s}},\ v' \in V_{\mathsf{s}'},\ \phi \in W_{\mathsf{s},\mathsf{s}'}.$$

Define three maps

$$\langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \times W_{\mathsf{s},\mathsf{s}'} \to \mathbb{C}, \quad (\phi_1, \phi_2) \mapsto \mathrm{tr}(\phi_1^* \phi_2),$$
$$J_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto J_{\mathsf{s}'} \circ \phi \circ J_{\mathsf{s}}^{-1};$$

and

$$L_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto \dot{\epsilon} L_{\mathsf{s}'} \circ \phi \circ L_{\mathsf{s}}^{-1}.$$

It is routine to check that

$$(W_{\mathsf{s},\mathsf{s}'}, \langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'}, J_{\mathsf{s},\mathsf{s}'}, L_{\mathsf{s},\mathsf{s}'})$$

is a classical space of signature $(C, \frac{(p+q)(p'+q')}{2}, \frac{(p+q)(p'+q')}{2})$.

Write

$$W_{\mathsf{s},\mathsf{s}'} = \mathcal{X}_{\mathsf{s},\mathsf{s}'} \oplus \mathcal{Y}_{\mathsf{s},\mathsf{s}'}, \tag{3.3}$$

where $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ and $\mathcal{Y}_{\mathsf{s},\mathsf{s}'}$ are the eigenspaces of $L_{\mathsf{s},\mathsf{s}'}$ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Then we have the following two well-defined algebraic maps:

$$\begin{array}{ccccc} \mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array} \tag{3.4}$$

These two maps M_{s} and $M_{\mathsf{s}'}$ are called the moment maps. They are both $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-equivariant. Here $K_{\mathsf{s}',\mathbb{C}}$ acts trivially on $\mathfrak{p}_{\mathsf{s}}$, $K_{\mathsf{s},\mathbb{C}}$ acts trivially on $\mathfrak{p}'_{\mathsf{s}}$, and all the other actions are the obvious ones.

Put

$$W^{\circ}_{\mathsf{s},\mathsf{s}'} := \{\phi \in W_{\mathsf{s},\mathsf{s}'} \mid \text{the image of } \phi \text{ is non-degenerate with respect to } \langle\, ,\, \rangle_{\mathsf{s}'}\}$$

and

$$\mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W^{\circ}_{\mathsf{s},\mathsf{s}'}.$$

Lemma 3.2 (*cf.* [65, Lemma 13], [91, Lemma 3.4])**.** *Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}. Then the set*

$$M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} \tag{3.5}$$

is a single $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-orbit. Moreover, for any element ϕ in $M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'}$, there is an exact sequence of algebraic groups:

$$1 \to K_{\mathsf{s}_1,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \xrightarrow{\textit{the projection to the first factor}} (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1,$$

where $\mathbf{e} := M_{\mathsf{s}}(\phi) \in \mathscr{O}$, s_1 is a certain classical signature of form $(\star', p_1, q_1)$ $(p_1, q_1 \in \mathbb{N})$, $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$ is the stabilizer of ϕ in $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$, and $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$ is the stabilizer of $\mathbf{e}$ in $K_{\mathsf{s},\mathbb{C}}$.

In the notation of Lemma 3.2, write

$$\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) := \text{the image of the set (3.5) under the moment map } M_{\mathsf{s}'},$$

which is a $K_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}'}$. This is called the descent of $\mathscr{O}$. It is an element of $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}'})$ if $\mathscr{O} \in \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$.

3.3. **Geometric theta lift.** Let $\zeta_{\mathsf{s},\mathsf{s}'}$ denote the algebraic character on $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ such that

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s},\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{B, D, C^*\}; \\ (\det_{\sqrt{-1}})^{\frac{q'-p'}{2}}, & \text{if } \star \in \{C, D^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{q'-p'}, & \text{if } \star = \widetilde{C}, \end{cases}$$

and

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s}',\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{C, \widetilde{C}, D^*\}; \\ (\det_{\sqrt{-1}})^{\frac{p-q}{2}}, & \text{if } \star \in \{D, C^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{p-q}, & \text{if } \star = B. \end{cases}$$

Here and henceforth, when no confusion is possible, we use 1 to indicate the trivial representation of a group (we also use 1 to denote the identity element of a group). Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$ as before. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}, and write $\mathscr{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O})$. Let $\phi, \mathbf{e}$ be as in Lemma 3.2 and let $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. We have an exact sequence

$$1 \to K_{\mathsf{s}_1,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \to (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1$$

as in Lemma 3.2.

Let $\mathcal{E}'$ be a $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}'$. Its fibre $\mathcal{E}'_{\mathbf{e}}$ at $\mathbf{e}'$ is an algebraic representation of the stabilizer group $(K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}$. We also view it as a representation of the group $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$ by the pull-back through the homomorphism

$$(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \xrightarrow{\text{the projection to the second factor}} (K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}.$$

Then $\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'}$ is a representation of $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$, and the coinvariant space

$$(\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}_1,\mathbb{C}}}$$

is an algebraic representation of $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$. Write $\mathcal{E} := \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$ for the $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$ whose fibre at $\mathbf{e}$ equals this coinvariant space representation. In this way, we get an exact functor $\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}$ from the category of $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic

vector bundle over $\mathscr{O}'$ to the category of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$. This exact functor induces a homomorphism of the Grothendieck groups:

$$\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'} : \mathcal{K}_{\mathsf{s}'}(\mathscr{O}') \to \mathcal{K}_{\mathsf{s}}(\mathscr{O}).$$

The above homomorphism is independent of the choice of ϕ.

Similar to (3.4), we have the following two well-defined algebraic maps:

$$\begin{array}{ccccc} \mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array} \tag{3.6}$$

These two maps are also called the moment maps. Similar to the maps in (3.4), they are both $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-equivariant.

Now we suppose that $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Similar to the first assertion of Lemma 3.2, the set

$$\tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap W^{\circ}_{\mathsf{s},\mathsf{s}'} \tag{3.7}$$

is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit. Write

$$\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) := \text{the image of the set (3.7) under the moment map } \tilde{M}_{\mathsf{s}'},$$

which is a $G_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s}'}$. This is called the descent of $\mathcal{O}$. It is an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ if $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$.

Finally, we define the geometric theta lift to be the homomorphism

$$\check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'} : \mathcal{K}_{\mathsf{s}'}(\mathcal{O}') \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

such that

$$\check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathcal{E}') = \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}},\ \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) = \mathscr{O}'} \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}'),$$

for all $K_{\mathsf{s}',\mathbb{C}}$-orbit $\mathscr{O}'$ in $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$, and all $\mathcal{E}' \in \mathcal{K}_{\mathsf{s}'}(\mathscr{O}')$.

3.4. **Associated cycles of painted bipartitions.** As before, let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity. Suppose that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is its Barbasch-Vogan dual, where $\mathsf{s} = (\star, p, q)$ is a classical signature. Put

$$\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) := \{(\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}) \mid (p_\tau, q_\tau) = (p, q)\}.$$

Let $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$. In what follows we will define the associated cycle $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ of $\boldsymbol{\tau}$.

First assume that $\check{\mathcal{O}} = \varnothing$. Then $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ is a singleton. If $\star = \widetilde{C}$, we define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ to be the element that corresponds to the one dimensional genuine representation of $K_{\mathsf{s},\mathbb{C}} = \{\pm 1\}$. In all other cases, we define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ to be the element that corresponds to the one dimensional trivial representation of $K_{\mathsf{s},\mathbb{C}}$.

Now we assume that $\check{\mathcal{O}} \neq \varnothing$. As before, let $\check{\mathcal{O}}'$ be its dual descent. Write $\boldsymbol{\tau}' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$ for the descent of $\boldsymbol{\tau}$, and assume that $\boldsymbol{\tau}' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}', \mathsf{s}')$.

Lemma 3.3. *The orbit $\mathcal{O}$ is in the image of the moment map $\tilde{M}_{\mathsf{s}}$, and*

$$\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) = \textit{the Barbasch-Vogan dual of } \check{\mathcal{O}}'.$$

By Lemma 3.3, $\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ equals the Barbasch-Vogan dual of $\check{\mathcal{O}}'$.

Similar to the definition of π_τ in the introductory section, we inductively define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ by

$$\mathrm{AC}(\boldsymbol{\tau}) := \begin{cases} \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\boldsymbol{\tau}')) \otimes (\det_{-1})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\boldsymbol{\tau}') \otimes \det^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\boldsymbol{\tau}')), & \text{if } \star = C^* \text{ or } D^*. \end{cases}$$

We have the following three analogous results of Theorem 1.11.

Proposition 3.4. *For every* $\tau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$*, the associated cycle* $\mathrm{AC}(\tau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ *is a nonzero sum of pairwise distinct elements of* $\mathrm{AOD}_{\mathcal{O}}(K_{\mathsf{s},\mathbb{C}})$*.*

Proposition 3.5. *Suppose that* $\check{\mathcal{O}}$ *is weakly-distinguished. If* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$*, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \quad (\tau, \epsilon) \mapsto \mathrm{AC}(\tau) \otimes \det{}^{\epsilon}$$

is injective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

is injective.

Proposition 3.6. *Suppose that* $\check{\mathcal{O}}$ *is quasi-distinguished. If* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$*, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}), \quad (\tau, \epsilon) \mapsto \mathrm{AC}(\tau) \otimes \det{}^{\epsilon}$$

is well-defined and bijective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

is well-defined and bijective.

The following two propositions will also be important for us.

Proposition 3.7. *Suppose that* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$*. Let* $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$ *and* $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ *(*$i = 1, 2$*). If*

$$\mathrm{AC}(\tau_1) \otimes \det{}^{\epsilon_1} = \mathrm{AC}(\tau_2) \otimes \det{}^{\epsilon_2},$$

then

$$\epsilon_1 = \epsilon_2 \qquad \text{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Proposition 3.8. *Suppose that* $\star \in \{C, \widetilde{C}, D^*\}$ *and* $\check{\mathcal{O}} \neq \varnothing$*. Let* $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$*, and write* $\tau'_i = (\tau'_i, \wp'_i)$ *for its descent (*$i = 1, 2$*). If*

$$\mathrm{AC}(\tau_1) = \mathrm{AC}(\tau_2),$$

then

$$(p_{\tau'_1}, q_{\tau'_1}) = (p_{\tau'_2}, q_{\tau'_2}) \qquad \text{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

3.5. **Distinguishing the constructed representations.** Let $\tau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$, where $\mathsf{s} = (\star, p, q)$ is a classical signature. Recall that π_τ is the Casselman-Wallach representation of G_{s}, defined in the introductory section. We will prove the following theorem in Section **??**.

Theorem 3.9. *The representation* π_τ *is irreducible, unitarizable and attached to* $\check{\mathcal{O}}$*. Moreover,*

$$\mathrm{AC}_{\mathcal{O}}(\pi_\tau) = \mathrm{AC}(\tau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

Recall from the introductory section, the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}})$ of isomorphism classes of irreducible Casselman-Wallach representations of G_{s} that are attached to $\check{\mathcal{O}}$.

Theorem 3.10. *If* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$*, then the map*

$$\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}}), \quad (\tau, \epsilon) \mapsto \pi_\tau \otimes \det{}^{\epsilon} \tag{3.8}$$

is bijective. In all other cases, the map

$$\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}}), \quad \tau \mapsto \pi_\tau \tag{3.9}$$

is bijective.

Proof. In view of Theorem 1.5, we only need to show the injectivity of the two maps in the statement of the theorem. We prove by induction on the number of nonempty rows of $\check{\mathcal{O}}$. The theorem is trivially true in the case when $\check{\mathcal{O}} = \varnothing$. So we assume that $\check{\mathcal{O}} \neq \varnothing$ and the theorem has been proved for the dual descent $\check{\mathcal{O}}'$. Suppose that $\boldsymbol{\tau}_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$, $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$. Write $\boldsymbol{\tau}'_i = (\tau'_i, \wp'_i)$ for the descent of $\boldsymbol{\tau}_i$.

First assume that $\star \in \{B, D\}$, and

$$\pi_{\boldsymbol{\tau}_1} \otimes \det{}^{\epsilon_1} \cong \pi_{\boldsymbol{\tau}_2} \otimes \det{}^{\epsilon_2}.$$

Theorem 3.9 implies that

$$\mathrm{AC}(\boldsymbol{\tau}_1) \otimes \det{}^{\epsilon_1} = \mathrm{AC}(\boldsymbol{\tau}_2) \otimes \det{}^{\epsilon_2}.$$

By Proposition 3.7, we know that

$$\epsilon_1 = \epsilon_2 \qquad \text{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Then the definition of $\pi_{\boldsymbol{\tau}_1}$ and $\pi_{\boldsymbol{\tau}_2}$ implies that

$$\check{\Theta}^{\tau_1}_{\tau'_1}(\pi_{\boldsymbol{\tau}'_1}) \cong \check{\Theta}^{\tau_2}_{\tau'_2}(\pi_{\boldsymbol{\tau}'_2}).$$

Consequently,

$$\pi_{\boldsymbol{\tau}'_1} \cong \pi_{\boldsymbol{\tau}'_2}$$

by the injectivity property of the theta correspondence. Hence $\boldsymbol{\tau}'_1 = \boldsymbol{\tau}'_2$ by the induction hypothesis, and Proposition 2.7 finally implies $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. This proves that the map (3.8) is injective.

A slightly simplified argument shows that the map (3.9) is injective when $\star = C^*$.

Now assume that $\star \in \{C, \widetilde{C}, D^*\}$. Suppose that

$$\pi_{\boldsymbol{\tau}_1} \cong \pi_{\boldsymbol{\tau}_2}.$$

Theorem 3.9 implies that

$$\mathrm{AC}(\boldsymbol{\tau}_1) = \mathrm{AC}(\boldsymbol{\tau}_2).$$

By Proposition 3.8, we know that

$$(p_{\tau'_1}, q_{\tau'_1}) = (p_{\tau'_2}, q_{\tau'_2}) \qquad \text{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

Then the definition of $\pi_{\boldsymbol{\tau}_1}$ and $\pi_{\boldsymbol{\tau}_2}$ implies that

$$\check{\Theta}^{\tau_1}_{\tau'_1}(\pi_{\boldsymbol{\tau}'_1} \otimes \det{}^{\varepsilon_{\wp_1}}) \cong \check{\Theta}^{\tau_2}_{\tau'_2}(\pi_{\boldsymbol{\tau}'_2} \otimes \det{}^{\varepsilon_{\wp_1}}).$$

The injectivity property of the theta correspondence then implies that

$$\pi_{\boldsymbol{\tau}'_1} \otimes \det{}^{\varepsilon_{\wp_1}} \cong \pi_{\boldsymbol{\tau}'_2} \otimes \det{}^{\varepsilon_{\wp_2}},$$

which further implies that $\pi_{\boldsymbol{\tau}'_1} \cong \pi_{\boldsymbol{\tau}'_2}$. Hence $\boldsymbol{\tau}'_1 = \boldsymbol{\tau}'_2$ by the induction hypothesis, and Proposition 2.7 finally implies $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. This proves that the map (3.9) is injective.

A slightly simplified argument shows that the map (3.9) is injective when $\star = D^*$. This finishes the proof of the theorem. □

Finally, our first main theorem (Theorem 1.6) follows from Theorems 3.9 and 3.10. Our second main theorem (Theorem 1.11) follows from Propositions 3.4 and 3.5, and Theorems 3.9 and 3.10.

4. Theta lifts via matrix coefficient integrals

In this section, we study theta lift via matrix coefficient integrals. Let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$.

4.1. **The oscillator representation.** We use the notation of Section 3.2. Write $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \subset W_{\mathsf{s},\mathsf{s}'}$ for the fixed point set of $J_{\mathsf{s},\mathsf{s}'}$. It is a real symplectic space under the restriction of the form $\langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'}$. Let $H_{\mathsf{s},\mathsf{s}'} := W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \times \mathbb{R}$ denote the Heisenberg group attached to $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$, with group multiplication

$$(\phi, t) \cdot (\phi', t') := (\phi + \phi', t + t' + \langle \phi, \phi' \rangle_{\mathsf{s},\mathsf{s}'}), \qquad \phi, \phi' \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}, \quad t, t' \in \mathbb{R}.$$

Denote by $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$ the complexified Lie algebra of $H_{\mathsf{s},\mathsf{s}'}$. Then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ (as in (3.3)) is an abelian Lie subalgebra of $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$.

The following is the smooth version of the Stone-von Neumann Theorem.

Lemma 4.1. *Up to isomorphism, there exists a unique irreducible smooth Fréchet representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ of moderate growth with central character*

$$\mathbb{R} \to \mathbb{C}^\times, \ t \mapsto e^{\sqrt{-1}\, t}.$$

Moreover, the space

$$\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}} := \{v \in \omega_{\mathsf{s},\mathsf{s}'} \mid x \cdot v = 0 \quad \textit{for all } x \in \mathcal{X}_{\mathsf{s},\mathsf{s}'}\}$$

is one-dimensional.

The group $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ acts on $H_{\mathsf{s},\mathsf{s}'}$ as group automorphisms via the following natural action of $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ on $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$:

$$(g, g') \cdot \phi := g' \circ \phi \circ g^{-1}, \qquad (g, g') \in G_{\mathsf{s}} \times G_{\mathsf{s}'}, \ \phi \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}.$$

From this action, we form the semidirect product $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Lemma 4.2. *The representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ in Lemma 4.1 uniquely extends to a smooth representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ such that $K_{\mathsf{s}} \times K_{\mathsf{s}'}$ acts on $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$ through the scalar multiplication by $\zeta_{\mathsf{s},\mathsf{s}'}$.*

Let $\omega_{\mathsf{s},\mathsf{s}'}$ denote the representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ as in Lemma 4.2, henceforth called the smooth oscillator representation or simply the oscillator representation. As is well-known, this representation is unitarizable [88]. Fix an invariant continuous Hermitian inner product $\langle\, ,\, \rangle$ on it, which is unique up to a positive scalar multiplication. Denote by $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ the completion of $\omega_{\mathsf{s},\mathsf{s}'}$ with respect to this inner product, which is a unitary representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Let $\omega_{\mathsf{s},\mathsf{s}'}^\vee$ denote the contragredient of the oscillator representation $\omega_{\mathsf{s},\mathsf{s}'}$. It is the smooth Fréchet representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ of moderate growth specified by the following conditions:

- there is given a $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$-invariant, non-degenerate, continuous bilinear form
$$\langle\, ,\, \rangle : \omega_{\mathsf{s},\mathsf{s}'} \times \omega_{\mathsf{s},\mathsf{s}'}^\vee \to \mathbb{C};$$
- $\omega_{\mathsf{s},\mathsf{s}'}^\vee$ is irreducible as a representation of $H_{\mathsf{s},\mathsf{s}'}$.

Since $\omega_{\mathsf{s},\mathsf{s}'}$ is contained in the unitary representation $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$, $\omega_{\mathsf{s},\mathsf{s}'}^\vee$ is identified with the complex conjugation of $\omega_{\mathsf{s},\mathsf{s}'}$.

Put

$$\mathsf{s}'^{-} := (\star', q', p'),$$

Fix a linear isomorphism

$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^-}, \tag{4.1}$$

such that $(-\langle\,,\,\rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, -L_{\mathsf{s}'})$ corresponds to $(\langle\,,\,\rangle_{\mathsf{s}'-}, J_{\mathsf{s}'-}, L_{\mathsf{s}'-})$ under this isomorphism. This induces an isomorphism

$$G_{\mathsf{s}'} \to G_{\mathsf{s}'-}, \qquad g' \mapsto g'^{-}. \tag{4.2}$$

Then we have a group isomorphism

$$\begin{array}{rcl} (G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'} & \to & (G_{\mathsf{s}'-} \times G_{\mathsf{s}}) \ltimes H_{\mathsf{s}'-,\mathsf{s}}, \\ (g, g', (\phi, t)) & \mapsto & (g'^{-}, g, (\phi^* \circ \iota_{\mathsf{s}'}^{-1}, t)). \end{array} \tag{4.3}$$

It is easy to see that the irreducible representation $\omega_{\mathsf{s},\mathsf{s}'}$ corresponds to the irreducible representation $\omega_{\mathsf{s}'-,\mathsf{s}}$ under this isomorphism.

For every Casselman-Wallach representation π' of $G_{\mathsf{s}'}$, put

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi') := (\omega_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \pi')_{G_{\mathsf{s}'}} \qquad \text{(the Hausdorff coinvariant space).}$$

This is a Casselman-Wallach representation of G_{s}.

4.2. **Growth of Casselman-Wallach representations.** Write $\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}}$ for the centralizer of J_{s} in $\mathfrak{p}_{\mathsf{s}}$, which is a real form of $\mathfrak{p}_{\mathsf{s}}$. The Cartan decomposition asserts that

$$G_{\mathsf{s}} = K_{\mathsf{s}} \cdot \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}}).$$

Denote by Ψ_{s} the function of G_{s} satisfying the following conditions:

- it is both left and right K_{s}-invariant;
- for all $g \in \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}})$,
$$\Psi_{\mathsf{s}} = \prod_a \left(\frac{1+a}{2}\right)^{-\frac{1}{2}},$$
where a runs over all eigenvalues of $g : V_{\mathsf{s}} \to V_{\mathsf{s}}$, counted with multiplicities.

Note that all the eigenvalues of $g \in \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}})$ are positive real numbers, and $0 < \Psi_{\mathsf{s}}(g) \leqslant 1$ for all $g \in G_{\mathsf{s}}$.

Set

$$|\mathsf{s}| := p + q,$$

and

$$\nu_{\mathsf{s}} := \begin{cases} |\mathsf{s}|\,, & \text{if } \star \in \{C, \widetilde{C}\}; \\ |\mathsf{s}| - 1, & \text{if } \star = C^*; \\ |\mathsf{s}| - 2, & \text{if } \star \in \{B, D\}; \\ |\mathsf{s}| - 3, & \text{if } \star = D^*. \end{cases}$$

Denote by Ξ_{s} the bi-K_{s}-invariant Harish-Chandra's Ξ function on G_{s}.

Lemma 4.3. *There exists a real number $C_{\mathsf{s}} > 0$ such that*

$$\Psi_{\mathsf{s}}^{\nu_{\mathsf{s}}}(g) \leqslant C_{\mathsf{s}} \cdot \Xi_{\mathsf{s}}(g) \quad \textit{for all } g \in G_{\mathsf{s}}.$$

Proof. This is implied by the well-known estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

Lemma 4.4. *Let f be a bi-K_{s}-invariant positive function on G_{s} such that*

$$\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}} \to \mathbb{R}, \qquad x \mapsto f(\exp(x))$$

is a polynomial function. Then for every real number $\nu > 0$, the function $f \cdot \Psi_{\mathsf{s}}^{\nu} \cdot \Xi_{\mathsf{s}}^2$ is integrable with respect to a Haar measure on G_{s}.

Proof. This follows from the integral formula for G_{s} under the Cartan decomposition (see [86, Lemma 2.4.2]), as well as the estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

For every Casselman-Wallach representation π of G_{s}, write $\pi^\vee$ for its contragredient representation, which is a Casselman-Wallach representation of G_{s} equipped with a G_{s}-invariant, non-degenerate, continuous bilinear form

$$\langle\, ,\, \rangle : \pi \times \pi^\vee \to \mathbb{C}.$$

Definition 4.5. *Let $\nu \in \mathbb{R}$. A positive function Ψ on G_{s} is ν-bounded if there is a function f as in Lemma 4.4 such that*

$$\Psi(g) \leqslant f(g) \cdot \Psi_{\mathsf{s}}^{\nu}(g) \cdot \Xi_{\mathsf{s}}(g) \qquad \text{for all } g \in G_{\mathsf{s}}.$$

A Casselman-Wallach representation π of G_{s} is said to be ν-bounded if there exist a ν-bounded positive function Ψ on G_{s}, and continuous seminorms $|\cdot|_\pi$ and $|\cdot|_{\pi^\vee}$ on π and $\pi^\vee$ respectively such that

$$|\langle g \cdot u, v\rangle| \leqslant \Psi(g) \cdot |u|_\pi \cdot |v|_{\pi^\vee}$$

for all $u \in \pi$, $v \in \pi^\vee$, and $g \in G_{\mathsf{s}}$.

Let X_{s} be a maximal J_{s}-stable totally isotropic subspace of V_{s}, which is unique up to the action of K_{s}. Put $Y_{\mathsf{s}} := L_{\mathsf{s}}(X_{\mathsf{s}})$. Then $X_{\mathsf{s}} \cap Y_{\mathsf{s}} = \{0\}$. Write

$$P_{\mathsf{s}} = R_{\mathsf{s}} \ltimes N_{\mathsf{s}}$$

for the parabolic subgroup of G_{s} stabilizing X_{s}, where R_{s} is the Levi subgroup stabilizing both X_{s} and Y_{s}, and N_{s} is the unipotent radical. For every character $\chi : R_{\mathsf{s}} \to \mathbb{C}^\times$, view it as a character of P_{s} that is trivial on N_{s}. Write

$$I(\chi) := \mathrm{Ind}_{P_{\mathsf{s}}}^{G_{\mathsf{s}}} \chi \qquad \text{(normalized smooth induction)},$$

which is a Casselman-Wallach representation of G_{s} under the right translations. Note that the representations $I(\chi)$ and $I(\chi^{-1})$ are contragredients of each other with the G_{s}-invariant pairing

$$\langle\, ,\, \rangle : I(\chi) \times I(\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\mathsf{s}}} f(g) \cdot f'(g)\, \mathrm{d}g.$$

Let ν_χ be a real number such that $|\chi|$ equals the composition of

$$R_{\mathsf{s}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(X_{\mathsf{s}}) \xrightarrow{|\det|^{\nu_\chi}} \mathbb{C}^\times. \tag{4.4}$$

Definition 4.6. *A classical signature s is split if $V_{\mathsf{s}} = X_{\mathsf{s}} \oplus Y_{\mathsf{s}}$.*

When s is split, P_{s} is called a Siegel parabolic subgroup of G_{s}.

Lemma 4.7. *Suppose that s is split and let $\chi : P_{\mathsf{s}} \to \mathbb{C}^\times$ be a character. Then there is a positive function Ψ on G_{s} with the following properties:*

- *Ψ is $(1 - 2\,|\nu_\chi| - \frac{|\mathsf{s}|}{2})$-bounded if $\star \in \{B, C, D, \widetilde{C}\}$, and $(2 - 2\,|\nu_\chi| - \frac{|\mathsf{s}|}{2})$-bounded if $\star \in \{C^*, D^*\}$;*
- *for all $f \in I(\chi)$, $f' \in I(\chi^{-1})$ and $g \in G_{\mathsf{s}}$,*

$$|\langle g.f, f'\rangle| \leqslant \Psi(g) \cdot |f|_{K_{\mathsf{s}}}|_\infty \cdot |f'|_{K_{\mathsf{s}}}|_\infty \qquad (|\ |_\infty \text{ stands for the suppernorm}). \tag{4.5}$$

Proof. Let f_0 denote the element in $I(|\chi|)$ such that $(f_0)|_{K_{\mathsf{s}}} = 1$, and likewise let f_0' denote the element in $I(|\chi^{-1}|)$ such that $(f_0')|_{K_{\mathsf{s}}} = 1$. Put

$$\Psi(g) := \langle g \cdot f_0, f_0'\rangle, \qquad g \in G_{\mathsf{s}}.$$

Then it is easy to see that (4.5) holds. Note that Ψ is an elementary spherical function, and the lemma then follows by using the well-known estimate of the elementary spherical functions ([86, Lemma 3.6.7]). $\square$

4.3. **Matrix coefficient integrals against the oscillator representation.** We begin with the following lemma.

Lemma 4.8. *There exist continuous seminorms* $|\cdot|_{\mathsf{s},\mathsf{s}'}$ *and* $|\cdot|^\vee_{\mathsf{s},\mathsf{s}'}$ *on* $\omega_{\mathsf{s},\mathsf{s}'}$ *and* $\omega^\vee_{\mathsf{s},\mathsf{s}'}$ *respectively such that*

$$|\langle (g,g')\cdot u, v\rangle| \leqslant \Psi_{\mathsf{s}}^{|\mathsf{s}'|}(g)\cdot \Psi_{\mathsf{s}'}^{|\mathsf{s}|}(g')\cdot |u|_{\mathsf{s},\mathsf{s}'}\cdot |v|^\vee_{\mathsf{s},\mathsf{s}'}$$

for all $u\in\omega_{\mathsf{s},\mathsf{s}'}$, $v\in\omega^\vee_{\mathsf{s},\mathsf{s}'}$, *and* $(g,g')\in G_{\mathsf{s}}\times G_{\mathsf{s}'}$.

Proof. This follows from the proof of [50, Theorem 3.2]. □

Definition 4.9. *A Casselman-Wallach representation of* $G_{\mathsf{s}'}$ *is convergent for* $\check\Theta^{\mathsf{s}}_{\mathsf{s}'}$ *if it is* ν*-bounded for some* $\nu>\nu_{\mathsf{s}'}-|\mathsf{s}|$.

Example. Suppose that $\star'\neq\widetilde{C}$. Then the trivial representation of $G_{\mathsf{s}'}$ is convergent for $\check\Theta^{\mathsf{s}}_{\mathsf{s}'}$ if $|\mathsf{s}|>2\nu_{\mathsf{s}'}$.

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is convergent for $\check\Theta^{\mathsf{s}}_{\mathsf{s}'}$. Consider the integrals

$$\begin{array}{rcl}(\pi'\times\omega_{\mathsf{s},\mathsf{s}'})\times(\pi'^\vee\times\omega^\vee_{\mathsf{s},\mathsf{s}'}) & \to & \mathbb{C},\\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}}\langle g\cdot u,u'\rangle\cdot\langle g\cdot v,v'\rangle\,\mathrm{d}g.\end{array}\tag{4.6}$$

Unless otherwise specified, all the measures on Lie groups occurring in this article are Haar measures.

Lemma 4.10. *The integrals in* (4.6) *are absolutely convergent and the map* (4.6) *is continuous and multi-linear.*

Proof. This is a direct consequence of Lemmas 4.3, 4.4 and 4.8. □

By Lemma 4.10, the integrals in (4.6) yield a continuous bilinear form

$$(\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'})\times(\pi'^\vee\widehat{\otimes}\omega^\vee_{\mathsf{s},\mathsf{s}'})\to\mathbb{C}.\tag{4.7}$$

Put

$$\bar\Theta^{\mathsf{s}}_{\mathsf{s}'}(\pi'):=\frac{\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}}{\text{the left kernel of (4.7)}}.\tag{4.8}$$

Proposition 4.11. *The representation* $\bar\Theta^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ *of* G_{s} *is a quotient of* $\check\Theta^{\mathsf{s}}_{\mathsf{s}'}(\pi')$, *and is* $(|\mathsf{s}'|-\nu_{\mathsf{s}})$*-bounded.*

Proof. Note that the bilinear form (4.7) is $(G_{\mathsf{s}'}\times G_{\mathsf{s}'})$-invariant, as well as G_{s}-invariant. Thus $\bar\Theta^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ is a quotient of $\check\Theta^{\mathsf{s}}_{\mathsf{s}'}(\pi')$, and is therefore a Casselman-Wallach representation. Its contragedient representation is identified with

$$(\bar\Theta^{\mathsf{s}}_{\mathsf{s}'}(\pi'))^\vee:=\frac{\pi'^\vee\widehat{\otimes}\omega^\vee_{\mathsf{s},\mathsf{s}'}}{\text{the right kernel of (4.7)}}.$$

Lemmas 4.4 and 4.8 implies that there are continuous seminorms $|\cdot|_{\pi',\mathsf{s},\mathsf{s}'}$ and $|\cdot|_{\pi'^\vee,\mathsf{s},\mathsf{s}'}$ on $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$ and $\pi'^\vee\widehat{\otimes}\omega^\vee_{\mathsf{s},\mathsf{s}'}$ respectively such that

$$|\langle g.u,v\rangle|\leqslant\Psi_{\mathsf{s}}^{|\mathsf{s}'|}(g)\cdot|u|_{\pi',\mathsf{s},\mathsf{s}'}\cdot|v|_{\pi'^\vee,\mathsf{s},\mathsf{s}'}$$

for all $u\in\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$, $v\in\pi'^\vee\widehat{\otimes}\omega^\vee_{\mathsf{s},\mathsf{s}'}$, and $g\in G_{\mathsf{s}}$. The proposition then easily follows in view of Lemma 4.3. □

4.4. **Unitarity.** For the notion of weakly containment of unitary representations, see [24] for example.

Lemma 4.12. *Suppose that $|\mathsf{s}| \geqslant \nu_{\mathsf{s}'}$. Then as a unitary representation of $G_{\mathsf{s}'}$, $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ is weakly contained in the regular representation.*

Proof. This has been known to experts (see [50, Theorem 3.2]). Lemmas 4.3, 4.4 and 4.8 implies that for a dense subspace of $\hat{\omega}_{\mathsf{s},\mathsf{s}'}|_{G_{\mathsf{s}'}}$, the diagonal matrix coefficients are almost square integrable. Thus the lemma follows form [24, Theorem 1]. □

However, if $\star' \in \{B, C^*, D^*\}$, $|\mathsf{s}| \neq \nu_{\mathsf{s}'}$ for the parity reason. For this reason, we introduce

$$\nu^{\circ}_{\mathsf{s}'} := \begin{cases} \nu_{\mathsf{s}'}, & \text{if } \star' \in \{C, D, \widetilde{C}\}; \\ \nu_{\mathsf{s}'} + 1, & \text{if } \star' \in \{B, C^*, D^*\}. \end{cases}$$

The following definition is a slight variation of definition 4.9.

Definition 4.13. *A Casselman-Wallach representation of $G_{\mathsf{s}'}$ is overconvergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ if it is ν-bounded for some $\nu > \nu^{\circ}_{\mathsf{s}'} - |\mathsf{s}|$.*

We will prove the following unitarity result in the rest of this subsection.

Theorem 4.14. *Assume that $|\mathsf{s}| \geqslant \nu^{\circ}_{\mathsf{s}'}$. Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is overconvergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$. If π' is unitarizable, then so is $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$.*

Recall the following positivity result of matrix coefficient integrals, which is a special case of [32, Theorem A. 5].

Lemma 4.15. *Let G be a real reductive group with a maximal compact subgroup K. Let π_1 and π_2 be two unitary representations of G such that π_2 is weakly contained in the regular representation. Let $u_1, u_2, \cdots, u_r$ ($r \in \mathbb{N}$) be vectors in π_1 such that for all $i, j = 1, 2, \cdots, r$, the integral*

$$\int_G \langle g \cdot u_i, u_j \rangle \Xi_G(g) \, \mathrm{d}g$$

is absolutely convergent, where Ξ_G is the bi-K-invariant Harish-Chandra's Ξ function on G. Let $v_1, v_2, \cdots, v_r$ be K-finite vectors in π_2. Put

$$u := \sum_{i=1}^{r} u_i \otimes v_i \in \pi_1 \otimes \pi_2.$$

Then the integral

$$\int_G \langle g \cdot u, u \rangle \, \mathrm{d}g$$

absolutely converges to a nonnegative real number.

Now we come to the proof of Theorem 4.14.

Proof of Theorem 4.14. Fix an invariant continuous Hermitian inner product on π', and write $\hat{\pi}'$ for the completion of π' with respect to this Hermitian inner product. The space $\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'}$ is equipped with the inner product $\langle \, , \, \rangle$ that is the tensor product of the ones on π' and $\omega_{\mathsf{s},\mathsf{s}'}$. It suffices to show that

$$\int_{G_{\mathsf{s}'}} \langle g \cdot u, u \rangle \, \mathrm{d}g \geqslant 0$$

for all u in a dense subspace of $\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'}$.

If $\nu^\circ_{\mathsf{s}'} < 0$, then $\star \in \{D, D^*\}$ and $|\mathsf{s}'| = 0$. The theorem is trivial in this case. Thus we assume that $\nu^\circ_{\mathsf{s}'} \geqslant 0$. We also assume that $\star = B$. The proof in the other cases is similar and is omitted.

Note that $\nu^\circ_{\mathsf{s}'} = \nu_{\mathsf{s}'} = |\mathsf{s}'|$ is even. Let $\mathsf{s}_1 = (B, p_1, q_1)$ and $\mathsf{s}_2 = (D, p_2, q_2)$ be two classical signatures such that

$$(p_1, q_1) + (p_2, q_2) = (p, q) \quad \text{and} \quad |\mathsf{s}_2| = \nu^\circ_{\mathsf{s}'}.$$

View $\hat{\omega}_{\mathsf{s}_2,\mathsf{s}'}$ as a unitary representation of the symplectic group $G_{(C,p',q')}$. Define π_2 to be its pull-back through the covering homomorphism $G_{\mathsf{s}'} \to G_{(C,p',q')}$. By Lemma 4.12, the representation π_2 of $G_{\mathsf{s}'}$ is weakly contained in the regular representation.

Put

$$\pi_1 := \hat{\pi}' \widehat{\otimes}_{\mathrm{h}} (\hat{\omega}_{\mathsf{s}_1,\mathsf{s}'}|_{G_{\mathsf{s}'}}) \qquad (\widehat{\otimes}_{\mathrm{h}} \text{ indicates the Hilbert space tensor product}).$$

Lemmas 4.4 and 4.8 imply that the integral

$$\int_{G_{\mathsf{s}'}} \langle g.u, v\rangle \cdot \Xi_{G_{\mathsf{s}'}}(g)\,\mathrm{d}g$$

is absolutely convergent for all $u, v \in \pi' \otimes \omega_{\mathsf{s}_1,\mathsf{s}'}$. The theorem then follows by Lemma 4.15. □

5. Double theta lifts and degenerate principal series

In this section, we relate double theta lifts with degenerate principal series representations.

Let

$$\dot{\mathsf{s}} = (\dot{\star}, \dot{p}, \dot{q}), \qquad \mathsf{s}' = (\star', p', q') \qquad \text{and} \qquad \mathsf{s}'' = (\star'', p'', q'')$$

be classical signatures such that

- $(q', p') + (p'', q'') = (\dot{p}, \dot{q})$;
- if $\dot{\star} \in \{B, D\}$, then $\star', \star'' \in \{B, D\}$;
- if $\dot{\star} \notin \{B, D\}$, then $\star' = \star'' = \dot{\star}$.

As in Section 4.1, put $\mathsf{s}'^- := (\star', q', p')$. We view $V_{\mathsf{s}'^-}$ and $V_{\mathsf{s}''}$ as subspaces of $V_{\dot{\mathsf{s}}}$ such that $(\langle\,,\,\rangle_{\dot{\mathsf{s}}}, J_{\dot{\mathsf{s}}}, L_{\dot{\mathsf{s}}})$ extends both $(\langle\,,\,\rangle_{\mathsf{s}'^-}, J_{\mathsf{s}'^-}, L_{\mathsf{s}'^-})$ and $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$, and $V_{\mathsf{s}'^-}$ and $V_{\mathsf{s}''}$ are perpendicular to each other under the form $\langle\,,\,\rangle_{\dot{\mathsf{s}}}$. Then we have an orthogonal decomposition

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''},$$

and both $G_{\mathsf{s}'^-}$ and $G_{\mathsf{s}''}$ are identified with subgroups of $G_{\dot{\mathsf{s}}}$.

Fix a linear isomorphism

$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^-} \tag{5.1}$$

as in (4.1), which induces an isomorphism

$$G_{\mathsf{s}'} \to G_{\mathsf{s}'^-}, \quad g \mapsto g^-. \tag{5.2}$$

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$. Write π'^- for the representation of $G_{\mathsf{s}'^-}$ that corresponds to π' under the isomorphism (5.2).

5.1. **Matrix coefficient integrals and double theta lifts.** We begin with the following lemma.

Lemma 5.1. *The function* $(\Xi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}}$ *on* $G_{\mathsf{s}'}$ *is* $|\mathsf{s}''|$*-bounded.*

Proof. This follows from the estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

Definition 5.2. *The Casselman-Wallach representation* π'^{-} *of* $G_{\mathsf{s}'-}$ *is convergent for a Casselman-Wallach representation* $\dot{\pi}$ *of* $G_{\dot{\mathsf{s}}}$ *if there are real number* ν' *and* $\dot{\nu}$ *such that* π' *is* ν'*-bounded,* $\dot{\pi}$ *is* $\dot{\nu}$*-bounded, and*

$$\nu' + \dot{\nu} > -|\mathsf{s}''|.$$

Let $\dot{\pi}$ be a Casselman-Wallach representation of $G_{\dot{\mathsf{s}}}$ such that π'^{-} is convergent for $\dot{\pi}$. Consider the integrals

$$\begin{array}{rcl} (\pi'^{-} \times \dot{\pi}) \times ((\pi'^{-})^{\vee} \times \dot{\pi}^{\vee}) & \to & \mathbb{C}, \\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, u' \rangle \cdot \langle g^{-} \cdot v, v' \rangle \mathrm{d}g. \end{array} \tag{5.3}$$

Lemma 5.3. *The integrals in* (5.3) *are absolutely convergent and the map* (5.3) *is continuous and multi-linear.*

Proof. Note that $(\Psi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}} = \Psi_{\mathsf{s}'}$. Thus the lemma follows from Lemmas 4.4 and 5.1. □

By Lemma 5.3, the integrals in (5.3) yield a continuous bilinear form

$$\langle\, , \,\rangle : (\pi'^{-} \widehat{\otimes} \dot{\pi}) \times ((\pi'^{-})^{\vee} \widehat{\otimes} \dot{\pi}^{\vee}) \to \mathbb{C}. \tag{5.4}$$

Put

$$\pi'^{-} * \dot{\pi} := \frac{\pi'^{-} \widehat{\otimes} \dot{\pi}}{\text{the left kernel of (5.4)}}. \tag{5.5}$$

This is a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

For every classical signature $\mathsf{s} = (\star, p, q)$, put

$$[\mathsf{s}] := \begin{cases} (C, p, q), & \text{if } \star = \tilde{C}; \\ \mathsf{s}, & \text{if } \star \neq \tilde{C}. \end{cases}$$

Proposition 5.4. *Suppose that* $\mathsf{s} = (\star, p, q)$ *is a classical signature such that both* $\star'$ *and* $\star''$ *equals the Howe dual of* $\star$*, and* $2\nu_{\mathsf{s}} < |\dot{\mathsf{s}}|$*. Assume that* π' *is* ν'*-bounded for some* $\nu' > \nu_{\mathsf{s}'} - |\mathsf{s}|$*. Then*

- π' *is convergent for* $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$*;*
- $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ *is convergent for* $\check{\Theta}^{\mathsf{s}''}_{\mathsf{s}}$*;*
- *the trivial representation* 1 *of* $G_{\bar{\mathsf{s}}}$ *is convergent for* $\check{\Theta}^{\dot{\mathsf{s}}}_{\bar{\mathsf{s}}}$*;*
- π'^{-} *is convergent for* $\bar{\Theta}^{\dot{\mathsf{s}}}_{[\mathsf{s}]}(1)$*;*
- *as representations of* $G_{\mathsf{s}''}$*, we have*

$$\bar{\Theta}^{\mathsf{s}''}_{\mathsf{s}}(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')) \cong \pi'^{-} * \bar{\Theta}^{\dot{\mathsf{s}}}_{[\mathsf{s}]}(1). \tag{5.6}$$

Proof. The first four claims in the proposition are obvious. We only need to prove the last one. Note that the integrals in

$$\begin{array}{rcl} (\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \omega_{\mathsf{s}'',\mathsf{s}}) \times ((\pi')^{\vee} \widehat{\otimes} \omega^{\vee}_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \omega^{\vee}_{\mathsf{s}'',\mathsf{s}}) & \to & \mathbb{C}, \\ (u,v) & \mapsto & \int_{G_{\mathsf{s}'} \times G_{\mathsf{s}}} \langle (g',g) \cdot u, v \rangle \mathrm{d}g' \, \mathrm{d}g \end{array} \tag{5.7}$$

are absolutely convergent and defines a continuous bilinear map. Also note that

$$\omega_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \omega_{\mathsf{s}_2,\mathsf{s}} \cong \omega_{\dot{\mathsf{s}},\mathsf{s}}.$$

In view of Fubini's theorem, the lemma follows as both sides of (5.6) are isomorphic to the quotient of $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}}$ by the left kernel of the pairing (5.7).

□

5.2. **Matrix coefficient integrals against degenerate principal series.** We are particularly interested in the case when $\dot\pi$ is a degenerate principal series representation. Suppose that $\dot{\mathsf{s}}$ is split, and

$$p' \leqslant p'' \qquad \text{and} \qquad q' \leqslant q''.$$

Then there is a split classical signature $\mathsf{s}_0 = (\star_0, p_0, q_0)$ such that

- $\star_0 = \dot\star$;
- $(p', q') + (p_0, q_0) = (p'', q'')$.

We view $V_{\mathsf{s}'}$ and V_{s_0} as subspaces of $V_{\mathsf{s}''}$ such that $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$ extends both $(\langle\,,\,\rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, L_{\mathsf{s}'})$ and $(\langle\,,\,\rangle_{\mathsf{s}_0}, J_{\mathsf{s}_0}, L_{\mathsf{s}_0})$, and $V_{\mathsf{s}'}$ and V_{s_0} are perpendicular to each other under the form $\langle\,,\,\rangle_{\mathsf{s}_2}$.

Put

$$V_{\mathsf{s}'}^{\triangle} := \{\iota_{\mathsf{s}'}(v) + v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\} \quad \text{and} \quad V_{\mathsf{s}'}^{\nabla} := \{\iota_{\mathsf{s}'}(v) - v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\}.$$

As before, we have that

$$V_{\mathsf{s}_0} = X_{\mathsf{s}_0} \oplus Y_{\mathsf{s}_0},$$

where X_{s_0} is a maximal J_{s_0}-stable totally isotropic subspace of V_{s_0}, and $Y_{\mathsf{s}_0} := L_{\mathsf{s}_0}(X_{\mathsf{s}_0})$. Suppose that

$$X_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0} \quad \text{and} \quad Y_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}.$$

In summary, we have decompositions

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}'} \oplus V_{\mathsf{s}_0} = (V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0}) \oplus (V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}).$$

As before, $X_{\dot{\mathsf{s}}}$ and X_{s_0} yield the Siegel parabolic subgroups

$$P_{\dot{\mathsf{s}}} = R_{\dot{\mathsf{s}}} \ltimes N_{\dot{\mathsf{s}}} \subset G_{\dot{\mathsf{s}}} \qquad \text{and} \qquad P_{\mathsf{s}_0} = R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}_0} \subset G_{\mathsf{s}_0}.$$

Write

$$P_{\mathsf{s}'',\mathsf{s}_0} = R_{\mathsf{s}'',\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$$

for the parabolic subgroup of $G_{\mathsf{s}''}$ stabilizing X_{s_0}, where $R_{\mathsf{s}'',\mathsf{s}_0}$ is the Levi subgroup stabilizing both X_{s_0} and Y_{s_0}, and $N_{\mathsf{s}'',\mathsf{s}_0}$ is the unipotent radical. We have an obvious homomorphism

$$G_{\mathsf{s}'} \times R_{\mathsf{s}_0} \to R_{\mathsf{s}'',\mathsf{s}_0},$$

which is a two fold covering map when $\dot\star = \widetilde{C}$, and an isomorphism in the other cases. For every $g \in G_{\mathsf{s}'}$, write $g^{\triangle} := g^- g$, which is an element of $R_{\dot{\mathsf{s}}}$.

Let $\dot\chi : R_{\dot{\mathsf{s}}} \to \mathbb{C}^\times$ be a character. It yields a degenerate principal series representation

$$I(\dot\chi) := \mathrm{Ind}_{P_{\dot{\mathsf{s}}}}^{G_{\dot{\mathsf{s}}}} \dot\chi$$

of $G_{\dot{\mathsf{s}}}$. Write

$$\chi_0 := \dot\chi|_{R_{\mathsf{s}_0}}, \tag{5.8}$$

and define a character

$$\chi' : G_{\mathsf{s}'} \to \mathbb{C}^\times, \quad g \mapsto \dot\chi(g^{\triangle}) \qquad \text{(this is a quadratic character)}. \tag{5.9}$$

Recall the representations π' of $G_{\mathsf{s}'}$ and π'^- of $G_{\mathsf{s}'^-}$. If $\dot{\mathsf{s}} = \widetilde{C}$, we assume that both π' and $\dot\chi$ are genuine. Then $(\pi' \otimes \chi') \otimes \chi_0$ descends to a Casselman-Wallach representation of $R_{\mathsf{s}'',\mathsf{s}_0}$. View it as a representation of $P_{\mathsf{s}'',\mathsf{s}_0}$ via the trivial action of $N_{\mathsf{s}'',\mathsf{s}_0}$, and form the representation $\mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi_1 \otimes \chi') \otimes \chi_0)$

Let $\nu_{\dot\chi} \in \mathbb{R}$ be as in (4.4). The rest of this subsection is devoted to a proof of the following theorem.

Theorem 5.5. *Assume that π' is ν' bounded for some*

$$\nu' > \begin{cases} 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 1, & \text{if } \dot\star \in \{B, C, D, \widetilde{C}\}; \\ 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 2, & \text{if } \dot\star \in \{C^*, D^*\}. \end{cases}$$

Then π'^{-} is convergent for $I_{\dot{\mathsf{s}}}(\dot\chi)$, and

$$\pi'^{-} * I(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}''},\mathsf{s}_0}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Let the notation and assumptions be as in Theorem 5.5. Recall that the representations $I(\dot\chi)$ and $I(\dot\chi^{-1})$ are contragredients of each other with the $G_{\dot{\mathsf{s}}}$-invariant pairing

$$\langle\, ,\, \rangle : I(\dot\chi) \times I(\dot\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\dot{\mathsf{s}}}} f(g) \cdot f'(g)\, \mathrm{d}g,$$

The first assertion of Theorem 5.5 is a direct consequence of Lemma 4.7. As in (5.4), we have a continuous bilinear form

$$\begin{array}{rcl} (\pi'^{-}\widehat{\otimes} I(\dot\chi)) \times ((\pi'^{-})^{\vee}\widehat{\otimes} I(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, v\rangle\, \mathrm{d}g \end{array} \tag{5.10}$$

so that

$$\pi'^{-} * I(\dot\chi) := \frac{\pi'^{-}\widehat{\otimes} I(\dot\chi)}{\text{the left kernel of (5.10)}}. \tag{5.11}$$

Note that

$$G_{\dot{\mathsf{s}}}^{\circ} := P_{\dot{\mathsf{s}}} \cdot G_{\mathsf{s}''}$$

is open and dense in $G_{\dot{\mathsf{s}}}$, and its complement has measure zero in $G_{\dot{\mathsf{s}}}$. Moreover,

$$P_{\dot{\mathsf{s}}} \backslash G_{\dot{\mathsf{s}}}^{\circ} = (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}) \backslash G_{\mathsf{s}''}. \tag{5.12}$$

Form the normalized Schwartz induction

$$I^{\circ}(\dot\chi) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0.$$

The reader is referred to [21, Section 6.2] for the general notion of Schwartz inductions (in a slightly different unnormalized setting). Similarly, put

$$I^{\circ}(\dot\chi^{-1}) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0^{-1}.$$

Note that

(5.13) the modulus character of $P_{\dot{\mathsf{s}}}$ restricts to the modulus character of $R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$.

In view of (5.12) and (5.13), by extension by zero, $I^{\circ}(\dot\chi)$ is viewed as a closed subspace of $I(\dot\chi)$, and $I^{\circ}(\dot\chi^{-1})$ is viewed as a closed subspace of $I(\dot\chi^{-1})$. These two closed subspaces are $(G_{\mathsf{s}'} \times G_{\mathsf{s}''})$-stable.

Similar to (5.10), we have a continuous bilinear form

$$\begin{array}{rcl} (\pi'^{-}\widehat{\otimes} I^{\circ}(\dot\chi)) \times ((\pi'^{-})^{\vee}\widehat{\otimes} I^{\circ}(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, v\rangle\, \mathrm{d}g. \end{array} \tag{5.14}$$

Put

$$\pi'^{-} * I^{\circ}(\dot\chi) := \frac{\pi'^{-}\widehat{\otimes} I^{\circ}(\dot\chi)}{\text{the left kernel of (5.14)}}, \tag{5.15}$$

which is still a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

Lemma 5.6. *As representations of $G_{\mathsf{s}''}$,*

$$\pi'^{-} * I^{\circ}(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}''},\mathsf{s}_0}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Proof. As Fréchet spaces, π'^{-} is obviously identified with π'. Note that the natural map $G_{\mathsf{s}'} \to (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0})\backslash G_{\mathsf{s}''}$ is proper and hence the following integrals are absolutely convergent and yield a $G_{\mathsf{s}''}$-equivariant continuous linear map:

$$\begin{aligned} \xi : \pi'^{-}\widehat{\otimes} I^{\circ}(\dot{\chi}) &\to \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0), \\ v \otimes f &\mapsto \left(h \mapsto \textstyle\int_{G_{\mathsf{s}'}} \chi'(g) \cdot f(g^{-1}h)(g \cdot v)\,\mathrm{d}g\right). \end{aligned}$$

Moreover, this map is surjective (*cf.* [21, Section 6.2]). It is thus open by the open mapping theorem. Similarly, we have a open surjective $G_{\mathsf{s}''}$-equivariant continuous linear map

$$\begin{aligned} \xi' : (\pi'^{-})^{\vee}\widehat{\otimes} I^{\circ}(\dot{\chi}^{-1}) &\to \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi'^{\vee} \otimes \chi') \otimes \chi_0^{-1}), \\ v' \otimes f' &\mapsto \left(h \mapsto \textstyle\int_{G_{\mathsf{s}'}} \chi'(g) \cdot f'(g^{-1}h)(g \cdot v')\,\mathrm{d}g\right). \end{aligned}$$

For all $v \otimes f \in \pi'^{-}\widehat{\otimes} I^{\circ}(\dot{\chi})$ and $v' \otimes f' \in (\pi'^{-})^{\vee}\widehat{\otimes} I^{\circ}(\dot{\chi}^{-1})$, we have that

$$\begin{aligned} & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot (v \otimes f), v' \otimes f' \rangle \,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot v, v' \rangle \cdot \langle g^{-} \cdot f, f' \rangle \,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot v, v' \rangle \cdot \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} (g^{-} \cdot f)(hx) \cdot f'(hx) \,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot v, v' \rangle \cdot \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f(g^{-1}hx) \cdot f'(hx) \,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}g \\ = & \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} \int_{G_{\mathsf{s}'}} \langle (h^{-}(g^{-})^{-1}) \cdot v, v' \rangle \cdot \chi'(hg^{-1}) \cdot f(gx) \cdot f'(hx) \,\mathrm{d}g\,\mathrm{d}h\,\mathrm{d}x \\ = & \int_{K_{\mathsf{s}''}} \langle \xi(v \otimes f)(x), \xi'(v' \otimes f')(x) \rangle \,\mathrm{d}x \\ = & \langle \xi(v \otimes f), \xi'(v' \otimes f') \rangle. \end{aligned}$$

This implies the lemma. □

Lemma 5.7. *Let $u \in \pi'^{-}\widehat{\otimes} I(\dot{\chi})$. Assume that*

$$\int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, v \rangle \,\mathrm{d}g = 0 \tag{5.16}$$

for all $v \in (\pi'^{-})^{\vee}\widehat{\otimes} I^{\circ}(\dot{\chi}^{-1})$. Then (5.16) also holds for all $v \in (\pi'^{-})^{\vee}\widehat{\otimes} I(\dot{\chi}^{-1})$.

Proof. Take a sequence $(\eta_1, \eta_2, \eta_3, \cdots)$ of real valued smooth functions on $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$ such that

- for all $i \geqslant 1$, the support of η_i is contained in $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^{\circ}$;
- for all $i \geqslant 1$ and $x \in P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$, $0 \leqslant \eta_i(x) \leqslant \eta_{i+1}(x) \leqslant 1$;
- $\bigcup_{i=1}^{\infty} \eta_i^{-1}(1) = P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^{\circ}$.

Let $v \in (\pi'^{-})^{\vee}\widehat{\otimes} I^{\circ}(\dot{\chi}^{-1})$. Note that $\eta_i I(\dot{\chi}^{-1}) \subset I^{\circ}(\dot{\chi}^{-1})$. Thus $\eta_i v \in (\pi'^{-})^{\vee}\widehat{\otimes} I^{\circ}(\dot{\chi}^{-1})$.

Lemma 4.7 and Lebesgue's dominated convergence theorem imply that

$$\int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, v \rangle \,\mathrm{d}g = \lim_{i \to +\infty} \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, \eta_i v \rangle \,\mathrm{d}g = 0.$$

This proves the lemma.

□

Lemma 5.7 implies that we have a natural continuous linear map

$$\mathrm{Ind}_{P_{\mathsf{s}''},\mathsf{s}_0}^{G_{\mathsf{s}''}}((\pi'\otimes\chi')\otimes\chi_0)=\pi'^{-}*I^{\circ}(\dot\chi)\to\pi'^{-}*I(\dot\chi).$$

This map is clearly injective and $G_{\mathsf{s}''}$-equivariant. Similarly to Lemma 5.7, we know that the natural pairing

$$(\pi'^{-}*I(\dot\chi))\times((\pi'^{-})^{\vee}*I^{\circ}(\dot\chi^{-1}))\to\mathbb{C}$$

is well-defined and non-degenerate. Thus Theorem 5.5 follows by the following lemma.

Lemma 5.8. *Let G be a real reductive group. Let π be a Casselman-Wallach representation of G, and let $\tilde\pi$ be a smooth Fréchet representation of G with a G-equivariant injective continuous linear map*

$$\phi:\pi\to\tilde\pi.$$

Assume that there is a non-degenerate G-invariant continuous bilinear map

$$\langle\,,\,\rangle:\tilde\pi\times\pi^{\vee}\to\mathbb{C},$$

such that the composition of

$$\pi\times\pi^{\vee}\xrightarrow{(u,v)\mapsto(\phi(u),v)}\tilde\pi\times\pi^{\vee}\xrightarrow{\langle,\rangle}\mathbb{C}$$

is the natural pairing. Then ϕ is a topological isomorphism.

Proof. Let $u\in\tilde\pi$. Using the theorem of Dixmier-Malliavin [27, Theorem 3.3], we write

$$u=\sum_{i=1}^{s}\int_G\varphi_i(g)(g\cdot u_i)\,\mathrm{d}g\qquad(s\in\mathbb{N},\ u_i\in\tilde\pi),$$

where φ_i's are compactly supported smooth functions on G. As a continuous linear functional on $\pi^{\vee}$, we have that

$$\langle u,\,\cdot\,\rangle=\sum_{i=1}^{s}\int_G\varphi_i(g)\cdot\langle g\cdot u_i,\,\cdot\,\rangle\mathrm{d}g.$$

By [79, Lemma 3.5], the right-hand side functional equals $\langle u_0,\,\cdot\,\rangle$ for a unique $u_0\in\pi$. Thus $u=u_0$, and the lemma follows by the open mapping theorem.

□

5.3. **Double theta lifts and parabolic induction.** In this subsection, we further assume that

$$\dot{\mathsf{s}}=(C,2k-1,2k-1),\ (D,2k-1,2k-1),\ (\widetilde{C},2k,2k),\ (C^*,2k,2k),\text{or}\ \ (D^*,2k,2k),$$

where $k\in\mathbb{N}^+$. We also assume that the character $\dot\chi:R_{\dot{\mathsf{s}}}\to\mathbb{C}^\times$ satisfies the following conditions:

$$\text{(5.17)}\qquad\begin{cases}\dot\chi=1, & \text{if }\dot\star\in\{B,D\};\\ \dot\chi^2=1, & \text{if }\dot\star=C;\\ \dot\chi\text{ is genuine and }\dot\chi^4=1, & \text{if }\dot\star=\widetilde{C};\\ \dot\chi^2\text{ equals the composition of }R_{\dot{\mathsf{s}}}\xrightarrow{\text{natural map}}\mathrm{GL}(Y_{\dot{\mathsf{s}}})\xrightarrow{\det}\mathbb{C}^\times, & \text{if }\dot\star=C^*;\\ \dot\chi^2\text{ equals the composition of }R_{\dot{\mathsf{s}}}\xrightarrow{\text{natural map}}\mathrm{GL}(X_{\dot{\mathsf{s}}})\xrightarrow{\det}\mathbb{C}^\times, & \text{if }\dot\star=D^*.\end{cases}$$

If $\dot\star\notin\{C,\widetilde{C}\}$, the character $\dot\chi$ is uniquely determined by (5.17). If $\dot\star\in\{C,\widetilde{C}\}$, there are two characters satisfying (5.17), and we let $\dot\chi'$ denote the one other than $\dot\chi$.

The relationship between the degenerate principle series representation $I(\dot\chi)$ and Rallis quotients is summarized in the following lemma.

Lemma 5.9. *(a) If* $\dot{\mathsf{s}} = (C, 2k-1, 2k-1)$*, then*

$$I(\dot{\chi}) \oplus I(\dot{\chi}') \cong \bigoplus_{\mathsf{s}=(D,p,q),\, p+q=2k} \check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1).$$

(b) If $\dot{\mathsf{s}} = (\widetilde{C}, 2k, 2k)$*, then*

$$I(\dot{\chi}) \oplus I(\dot{\chi}') \cong \bigoplus_{\mathsf{s}=(B,p,q),\, p+q=2k+1} \check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1).$$

(c) If $\dot{\mathsf{s}} = (D, 2k-1, 2k-1)$*, then*

$$I(\dot{\chi}) \cong \check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1) \oplus (\check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1) \otimes \det), \qquad \textit{where } \mathsf{s} = (C, k-1, k-1).$$

(d) If $\dot{\mathsf{s}} = (C^*, 2k, 2k)$*, then*

$$I(\dot{\chi}) \cong \check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1), \qquad \textit{where } \mathsf{s} = (D^*, k, k).$$

(e) If $\dot{\mathsf{s}} = (D^*, 2k, 2k)$*, then there is an exact sequence of representations of* $G_{\dot{\mathsf{s}}}$*:*

$$0 \to \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1) \to I(\dot{\chi}) \to \bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}_1}(1) \to 0.$$

(f) All the representations $\check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1)$ *and* $\check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}_0}(1)$ *appearing in (a), (b), (c), (d), (e) are irreducible and unitarizable.*

Proof. See [47, Theorem 2.4], [48, Introduction], [49, Theorem 6.1] and [90, Sections 9 and 10]. □

Lemma 5.10. *For all* s *appearing in Lemma 5.9, the trivial representation* 1 *of* G_{s} *is overconvergent for* $\check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}$*, and*

$$\bar{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1) = \check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1).$$

Proof. Note that the trivial representation 1 of G_{s} is 0-bounded, and $0 > \nu_{\mathsf{s}} + \nu^{\circ}_{\mathsf{s}} - |\dot{\mathsf{s}}|$. This implies the first assertion. Since $\check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1)$ is irreducible and $\bar{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1)$ is a quotient of $\check{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1)$, for the proof of the second assertion, it suffices to show that $\bar{\Theta}^{\dot{\mathsf{s}}}_{\mathsf{s}}(1)$ is nonzero.

Put

$$V_{\mathsf{s}}(\mathbb{R}) := \begin{cases} \text{the fixed point set of } J_{\mathsf{s}} \text{ in } V_{\mathsf{s}}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ V_{\mathsf{s}}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

As usual, realize $\hat{\omega}_{\dot{\mathsf{s}},\mathsf{s}}$ on the space of square integrable functions on $(V_{\mathsf{s}}(\mathbb{R}))^{\dot{p}}$ so that $\omega_{\dot{\mathsf{s}},\mathsf{s}}$ is identified with the space of the Schwartz functions, and G_{s} acts on it through the obvious transformation.

Take a positive valued Schwartz function ϕ on $(V_{\mathsf{s}}(\mathbb{R}))^{\dot{p}}$. Then

$$\langle g \cdot \phi, \phi \rangle = \int_{(V_{\mathsf{s}}(\mathbb{R}))^{\dot{p}}} \phi(g^{-1} \cdot x) \cdot \phi(x)\, \mathrm{d}x > 0, \quad \text{for all } g \in G_{\mathsf{s}}.$$

Thus

$$\int_{G_{\mathsf{s}}} \langle g \cdot \phi, \phi \rangle \, \mathrm{d}g \neq 0,$$

and the lemma follows. □

Lemma 5.11. *Assume that* π' *is* ν'*-bounded for some*

$$\nu' > \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \textit{if } \dot{\mathsf{s}} = D^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \textit{otherwise.} \end{cases}$$

Then for all s *appearing in Lemma 5.9 and all unitary character* α' *of* $G_{\mathsf{s}'}$*, the representation* $\pi' \otimes \alpha'$ *is convergent for* $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$*, and* $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi' \otimes \alpha')$ *is overconvergent for* $\check{\Theta}^{\mathsf{s}''}_{\mathsf{s}}$*.*

Proof. The first assertion follows by noting that

$$\nu_{\mathsf{s}'} - |\mathsf{s}| = \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \text{if } \dot{\mathsf{s}} = D^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \text{otherwise.} \end{cases}$$

The proof of second assertion is similar to the proof of the first assertion of Lemma 5.10. □

Recall the character $\chi_0 := \dot{\chi}|_{R_{\mathsf{s}_0}}$ from (5.8). Similarly we define $\chi_0' := \dot{\chi}'|_{R_{\mathsf{s}_0}}$. Recall that π' is assumed to be genuine when $\dot{\star} = \widetilde{C}$.

Theorem 5.12. *Assume that π' is ν'-bounded for some $\nu' > -\frac{|\mathsf{s}_0|}{2} - 1$.*
(a) If $\dot{\mathsf{s}} = (C, 2k-1, 2k-1)$, then

$$\bigoplus_{\mathsf{s}=(D,p,q),\, p+q=2k} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \oplus \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0').$$

(b) If $\dot{\mathsf{s}} = (\widetilde{C}, 2k, 2k)$, then

$$\bigoplus_{\mathsf{s}=(B,p,q),\, p+q=2k+1} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \oplus \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0').$$

(c) If $\dot{\mathsf{s}} = (D, 2k-1, 2k-1)$, then

$$\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \oplus \Big((\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi' \otimes \det))) \otimes \det \Big) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s} = (C, k-1, k-1)$ if $|\mathsf{s}'|$ is even, and $\mathsf{s} = (\widetilde{C}, k-1, k-1)$ if $|\mathsf{s}'|$ is odd.
(d) If $\dot{\mathsf{s}} = (C^*, 2k, 2k)$, then

$$\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s} = (D^*, k, k)$.
(e) If $\dot{\mathsf{s}} = (D^*, 2k, 2k)$, then there is an exact sequence

$$0 \to \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \to \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \to \mathcal{J} \to 0$$

of representations of $G_{\mathsf{s}''}$ such that $\mathcal{J}$ is a quotient of

$$\bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')).$$

Proof. Using Lemma 4.7, we know that π'^{-} is convergent for $I(\dot{\chi})$ (and convergent for $I(\dot{\chi}')$ when $\dot{\star} \in \{C, \widetilde{C}\}$). Also note that the character χ' (see (5.9)) is trivial.

If we are in the situation (a), (b), (c) or (d), by using Theorem 5.5, Lemma 5.10 and Proposition 5.4, the theorem follows by applying the operation $\pi'^{-}*(\,\cdot\,)$ to the isomorphism in Lemma 5.9.

Now assume that we are in the situation (e). For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in part (e) of Lemma 5.9. Since π'^{-} is nuclear as a Fréchet space, the sequence

$$0 \to \pi'^{-}\widehat{\otimes} I_1 \to \pi'^{-}\widehat{\otimes} I_2 \to \pi'^{-}\widehat{\otimes} I_3 \to 0$$

is also topologically exact. Note that the natural map

$$\pi'^{-} * I_1 \to \pi'^{-} * I_2$$

is injective, and the natural map

$$\pi'^{-}\widehat{\otimes} I_3 \cong \frac{\pi'^{-}\widehat{\otimes} I_2}{\pi'^{-}\widehat{\otimes} I_1} \longrightarrow \frac{\pi'^{-} * I_2}{\pi'^{-} * I_1}$$

descends to a surjective map

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'-}} \to \frac{\pi'^{-} * I_2}{\pi'^{-} * I_1}.$$

Note that

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'-}} \cong \bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}^{\mathsf{s}''}_{\mathsf{s}_1}(\check{\Theta}^{\mathsf{s}_1}_{\mathsf{s}'}(\pi')).$$

Theorem 5.5 implies that

$$\pi'^{-} * I_2 \cong \mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}}(\pi' \otimes \chi_0).$$

Lemma 5.10 and Proposition 5.4 imply that

$$\pi'^{-} * I_1 \cong \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \bar{\Theta}^{\mathsf{s}''}_{\mathsf{s}}(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')).$$

Therefore the theorem follows. □

6. Bounding the associated cycles

In this section, let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$. Let $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$, whose Zariski closure in $\mathfrak{g}_{\mathsf{s}}$ is denoted by $\overline{\mathcal{O}}$.

6.1. **Associated cycles.** In this subsection, we will recall the definition of associated cycles from [83].

Definition 6.1. *Let $H_{\mathbb{C}}$ be a complex linear algebraic group. Let $\mathcal{A}$ be a commutative $\mathbb{C}$-algebra carrying a locally algebraic linear action of $H_{\mathbb{C}}$ on it by algebra automorphisms. An $(\mathcal{A}, H_{\mathbb{C}})$-module is an $\mathcal{A}$-module σ equipped with a locally algebraic linear action of $H_{\mathbb{C}}$ on it such that the module structure map $\mathcal{A} \otimes \sigma \to \sigma$ is $H_{\mathbb{C}}$-equivariant. An $(\mathcal{A}, H_{\mathbb{C}})$-module is said to be finitely generated if it is so as an $\mathcal{A}$-module.*

For every affine complex algebraic variety Z, write $\mathbb{C}[Z]$ the algebra of regular functions on Z. Given a finitely generated $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module σ°, for each $K_{\mathsf{s},\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$, the set

$$\bigsqcup_{\mathbf{e}\in\mathscr{O}} \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}]} \sigma^\circ$$

is naturally a $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$, where $\mathbb{C}_{\mathbf{e}}$ denotes the complex number field $\mathbb{C}$ viewing as a $\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]$-algebra via the evaluation map at $\mathbf{e}$. We define $\mathrm{AC}_{\mathscr{O}}(\sigma^\circ) \in \mathcal{K}_{\mathsf{s}}(\mathscr{O})$ to be the Grothendieck group element associated to this bundle. Define the associated cycle of σ° to be

$$\mathrm{AC}_{\mathcal{O}}(\sigma^\circ) := \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O}\cap\mathfrak{p}_{\mathsf{s}}} \mathrm{AC}_{\mathscr{O}}(\sigma^\circ) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

Write $I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}}$ for the radical ideal of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$ corresponding to the closed subvariety $\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}$. Every $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}], K_{\mathsf{s},\mathbb{C}})$-module is certainly an $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module. On the other hands, for each $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module σ,

$$\frac{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i.\sigma}{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^{i+1}.\sigma} \qquad (i \in \mathbb{N})$$

is naturally a $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module. We say that σ is $\mathcal{O}$-bounded if

$$(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i.\sigma = 0 \qquad \text{for some } i \in \mathbb{N}.$$

When σ is finitely generated, this is equivalent to saying that the support of σ is contained in $\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}$.

When σ is finitely generated and $\mathcal{O}$-bounded, we define its associated cycle to be

$$\mathrm{AC}_{\mathcal{O}}(\sigma) := \sum_{i\in\mathbb{N}} \mathrm{AC}_{\mathcal{O}}\left(\frac{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i.\sigma}{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^{i+1}.\sigma}\right) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

The assignment $\mathrm{AC}_{\mathcal{O}}$ is additive in the following sense: the equality

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\sigma') + \mathrm{AC}_{\mathcal{O}}(\sigma'')$$

holds for every exact sequence $0 \to \sigma' \to \sigma \to \sigma'' \to 0$ of finitely generated $\mathcal{O}$-bounded $(\mathrm{S}(\mathfrak{p}_{\mathsf{s}}), K_{\mathsf{s},\mathbb{C}})$-modules.

Given a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module ρ of finite length, pick a filtration

$$\mathcal{F}: \quad \cdots \subset \rho_{-1} \subset \rho_0 \subset \rho_1 \subset \rho_2 \subset \cdots \tag{6.1}$$

of ρ that is good in the following sense:

- for each $i \in \mathbb{Z}$, ρ_i is a finite-dimensional K_{s}-stable subspace of ρ;
- $\mathfrak{g}_{\mathsf{s}}.\rho_i \subset \rho_{i+1}$ for all $i \in \mathbb{Z}$, and $\mathfrak{g}_{\mathsf{s}}.\rho_i = \rho_{i+1}$ when $i \in \mathbb{Z}$ is sufficiently large;
- $\bigcup_{i\in\mathbb{Z}} \rho_i = \rho$, and $\rho_i = 0$ for some $i \in \mathbb{Z}$.

Then the grading

$$\mathrm{Gr}(\rho) := \mathrm{Gr}(\rho, \mathcal{F}) := \bigoplus_{i\in\mathbb{Z}} \rho_i/\rho_{i+1}$$

is naturally a finitely generated $(\mathrm{S}(\mathfrak{p}_{\mathsf{s}}), K_{\mathsf{s},\mathbb{C}})$-module, where $\mathrm{S}(\mathfrak{p}_{\mathsf{s}})$ denotes the symmetric algebra. Recall that we have identify $\mathfrak{p}_{\mathsf{s}}$ with its dual space $\mathfrak{p}_{\mathsf{s}}^*$ by using the trace from. Hence $\mathrm{S}(\mathfrak{p}_{\mathsf{s}})$ is identified with $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$, and $\mathrm{Gr}(\rho)$ is a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module.

Recall that ρ is said to be $\mathcal{O}$-bounded if the associated variety of its annihilator ideal in $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})$ is contained in $\overline{\mathcal{O}}$.

Lemma 6.2. *Let ρ be a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module of finite length, and let $\mathcal{F}$ be a good filtration of ρ. Then ρ is $\mathcal{O}$-bounded if and only if $\mathrm{Gr}(\rho, \mathcal{F})$ is $\mathcal{O}$-bounded.*

Proof. This is a direct consequence of [83, Theorem 8.4]. □

When ρ is $\mathcal{O}$-bounded, it associated cycle is defined to be

$$\mathrm{AC}_{\mathcal{O}}(\rho) := \mathrm{AC}_{\mathcal{O}}(\mathrm{Gr}(\rho)) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

This is independent of the good filtration (6.1). Moreover, taking the associated cycles is additive in the following sense: the equality

$$\mathrm{AC}_{\mathcal{O}}(\rho) = \mathrm{AC}_{\mathcal{O}}(\rho') + \mathrm{AC}_{\mathcal{O}}(\rho'')$$

holds for every exact sequence $0 \to \rho' \to \rho \to \rho'' \to 0$ of $\mathcal{O}$-bounded $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-modules of finite length.

6.2. **Algebraic theta lifts and commutative theta lifts.** We retain the notation of Section 3 and Section 4. Denote by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}}$ the $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$-submodule of $\omega_{\mathsf{s},\mathsf{s}'}$ generated by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$. This is an $(\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}} \times K_{\mathsf{s}'})$-module. For every $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module ρ', define its full theta lift (the algebraic theta lift) to be the $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho') := (\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}} \otimes \rho')_{\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'}} \qquad \text{(the coinvariant space)}.$$

It has finite length whenever ρ' has finite length (Howe?).

Recall form (3.4) the moment maps

$$\begin{array}{ccccc} \mathfrak{p}_{\mathsf{s}}^* = \mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}^* = \mathfrak{p}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array}$$

We view $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$ as an $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]\otimes\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]$-algebra by using the moment maps. The natural action of $K_{\mathsf{s},\mathbb{C}}\times K_{\mathsf{s}',\mathbb{C}}$ on $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ yields a locally algebraic linear action of $K_{\mathsf{s},\mathbb{C}}\times K_{\mathsf{s}',\mathbb{C}}$ on $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$. Thus $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$ is naturally a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]\otimes\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s},\mathbb{C}}\times K_{\mathsf{s}',\mathbb{C}})$-module.

For every $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module σ', define its commutative theta lift to be

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma') := (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]\otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]}\sigma'\otimes\zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(the coinvariant space)},$$

which is naturally a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module.

Lemma 6.3. *Suppose that σ' is a finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then the $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')$ is also finitely generated.*

Proof. Howe. □

Lemma 6.4. *Suppose that ρ' is a $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then there exist a good filtration $\mathcal{F}'$ on ρ', a good filtration $\mathcal{F}$ on $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho')$, and a surjective $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module homomorphism*

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\mathrm{Gr}(\rho',\mathcal{F}')) \to \mathrm{Gr}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho'),\mathcal{F}).$$

Proof. Loke-Ma. □

6.3. **Regular descents of nilpotent orbits.** Recall from (3.6) the moment maps

$$\begin{array}{ccccc} \mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array}$$

Lemma 6.5. *The orbit $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$ if and only if*

$$\delta := |\mathsf{s}'| - |\nabla_{\mathrm{naive}}(\mathcal{O})| \geqslant 0.$$

When this is the case, the Young diagram of $\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})\in\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ is obtained from that of $\nabla_{\mathrm{naive}}(\mathcal{O})$ by adding δ boxes in the first column.

Proof. This is implied by [25, Theorem 3.6]. □

Definition 6.6. *The orbit $\mathcal{O}\in\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ if either*

$$|\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|,$$

or

$$|\mathsf{s}'| > |\nabla_{\mathrm{naive}}(\mathcal{O})| \qquad \textit{and} \qquad \mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O}).$$

In the rest of this section we assume that $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Then $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Put $\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})\in\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$, and let $\overline{\mathcal{O}'}$ denote the Zariski closure of $\mathcal{O}'$ in $\mathfrak{g}_{\mathsf{s}'}$.

Lemma 6.7. *As subsets of $\mathfrak{g}_{\mathsf{s}}$,*

$$\tilde{M}_{\mathsf{s}}(\tilde{M}^{-1}_{\mathsf{s}'}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}.$$

Proof. This is implied by [25, Theorems 5.2 and 5.6]. □

Lemma 6.8. *Suppose that σ' is an $\mathcal{O}'$-bounded $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then the $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')$ is $\mathcal{O}$-bounded.*

Proof. Lemma 6.9 implies that

$$M^{-1}_{\mathsf{s}}(\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}) \supset M^{-1}_{\mathsf{s}'}(\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}).$$

Thus

$$(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i\cdot\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \subset I_{\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}}\cdot\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \qquad \text{for some } i\in\mathbb{N},$$

and the lemma follows. Here $I_{\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}}$ denotes the radical ideal of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$ corresponding to the closed subvariety $\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}$. □

Put

$$\partial := \overline{\mathcal{O}}\backslash\mathcal{O} \qquad \text{and} \qquad \bar{\partial} := \mathfrak{g}_{\mathsf{s}}\backslash\partial.$$

Then $\bar{\partial}$ is an open subvariety of $\mathfrak{g}_{\mathsf{s}}$ and $\mathcal{O}$ is a closed subvariety of $\bar{\partial}$. Put

$$W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} := \tilde{M}_{\mathsf{s}}^{-1}(\bar{\partial}) \qquad \text{and} \qquad W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} := \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O})\cap\tilde{M}_{\mathsf{s}'}^{-1}(\mathcal{O}').$$

Then $W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$ is an open subvariety of $W_{\mathsf{s},\mathsf{s}'}$, and the following lemma implies that $W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$ is a closed subvariety of $W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$.

Lemma 6.9. *The set $W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$ is a single $G_{\mathsf{s},\mathbb{C}}\times G_{\mathsf{s}',\mathbb{C}}$-orbit. Moreover, it is contained in $W_{\mathsf{s},\mathsf{s}'}^{\circ}$ and equals*

$$W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}\cap\tilde{M}_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'}).$$

Proof. The first assertion is implied by [25, Theorem 3.6]. Recall from (3.7) that $W_{\mathsf{s},\mathsf{s}'}^{\circ}\cap\tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O})$ is a single $G_{\mathsf{s},\mathbb{C}}\times G_{\mathsf{s}',\mathbb{C}}$-orbit whose image under the moment map $\tilde{M}_{\mathsf{s}'}^{-1}$ equals $\mathcal{O}'$. Thus

$$W_{\mathsf{s},\mathsf{s}'}^{\circ}\cap W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} = W_{\mathsf{s},\mathsf{s}'}^{\circ}\cap\tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}),$$

which is also a single $G_{\mathsf{s},\mathbb{C}}\times G_{\mathsf{s}',\mathbb{C}}$-orbit. Hence $W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}\subset W_{\mathsf{s},\mathsf{s}'}^{\circ}$. The last assertion is a direct consequence of Lemma 6.9.

□

Write

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} := \mathcal{X}_{\mathsf{s},\mathsf{s}'}\cap W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \qquad \text{and} \qquad \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'}\cap W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}.$$

Then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$ is an open subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ and $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$ is a closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$. We have a decomposition

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} = \bigsqcup_{\mathscr{O}\text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O}\cap\mathfrak{p}_{\mathsf{s}} \text{ that is contained in the image of } M_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'},$$

where

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'} := M_{\mathsf{s}}^{-1}(\mathscr{O})\cap\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'},$$

which is a Zariski open and closed subset of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$. Lemmas 3.2 and 6.9 imply that $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'}$ is a single $K_{\mathsf{s},\mathbb{C}}\times K_{\mathsf{s}',\mathbb{C}}$-orbit.

We defer the proof of the following proposition to Section 6.6.

Proposition 6.10. *The scheme theoretic fibre product*

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}\times_{\mathfrak{p}_{\mathsf{s}'}}(\mathcal{O}'\cap\mathfrak{p}_{\mathsf{s}'})$$

is reduced, where $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$ is viewed as a $\mathfrak{p}_{\mathsf{s}'}$-scheme via the moment map $M_{\mathsf{s}'}$.

By Lemma 6.9 and Proposition 6.10, we know that

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} = \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}\times_{\mathfrak{p}_{\mathsf{s}'}}(\mathcal{O}'\cap\mathfrak{p}_{\mathsf{s}'}) = \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}\times_{\mathfrak{p}_{\mathsf{s}'}}(\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}), \tag{6.2}$$

which is a smooth closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$.

For every element $\mathbf{e}\in\mathcal{O}\cap\mathfrak{p}_{\mathsf{s}}$, write $K_{\mathbf{e}}$ for the stabilizer of $\mathbf{e}$ in $K_{\mathsf{s},\mathbb{C}}$. Let $\mathbb{C}_{\mathbf{e}}$ be the field $\mathbb{C}$ viewing a $\mathbb{C}[\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}]$-algebra via the evaluation map at $\mathbf{e}$. Put

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'} := M_{\mathsf{s}}^{-1}(\mathbf{e})\cap M_{\mathsf{s}'}^{-1}(\mathcal{O}'\cap\mathfrak{p}_{\mathsf{s}'}) = M_{\mathsf{s}}^{-1}(\mathbf{e})\cap M_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}),$$

which is a closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$. If $\mathbf{e}$ is not contained in the image of M_{s}, then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'} = \varnothing$. Otherwise, it is a single $(K_{\mathbf{e}} \times K_{\mathsf{s}',\mathbb{C}})$-orbit, and Lemma 3.2 implies that it is also a single $K_{\mathsf{s}',\mathbb{C}}$-orbit.

Lemma 6.11. *For every element* $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$,

$$\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}] = \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}].$$

Proof. As schemes, we have that

$$\begin{aligned}
& \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}) \\
= \; & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}) \\
= \; & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \\
= \; & \{\mathbf{e}\} \times_{\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} ((\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}) \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}) \\
= \; & \{\mathbf{e}\} \times_{\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \\
= \; & \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}.
\end{aligned}$$

The last equality holds because the morphism $M_{\mathsf{s}} : \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \to \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ is $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-equivariant and hence smooth in the sense of algebraic geometry. This proves the lemma. □

6.4. **Commutative theta lifts and geometric theta lifts.** This subsection is to prove the following proposition.

Proposition 6.12. *Suppose that* σ' *is a finitely generated* $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$*-module. Then*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')) = \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma'))$$

as elements of $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$.

By a quasi-coherent module over a scheme Z, we mean a quasi-coherent module over the structure sheaf of Z. We say that a quasi-coherent module $\mathcal{M}$ over a scheme Z descends to a closed subscheme Z_1 of Z if $\mathcal{M}$ is isomorphic to the push-forward of a quasi-coherent module over Z_1 via the closed embedding $Z_1 \to Z$. This is equivalent to saying that the ideal sheaf defining Z_1 annihilates $\mathcal{M}$.

When Z is an affine complex algebraic variety and σ is a $\mathbb{C}[Z]$-module, we write $\mathcal{M}_\sigma$ for the quasi-coherent module over Z corresponding to σ.

Lemma 6.13. *Let* σ *be a finitely generated* $\mathcal{O}$*-bounded* $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s}})$*-module. Assume that* $(\mathcal{M}_\sigma)|_{\bar{\partial} \cap \mathfrak{p}_{\mathsf{s}}}$ *descends to* $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$*. Then*

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma).$$

Proof. Let $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$. Note that for every ideal I of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$,

$$(I.\sigma)_{\mathbf{e}} = (I_{\mathbf{e}}).\sigma_{\mathbf{e}} \subset \sigma_{\mathbf{e}},$$

where a subscript $\mathbf{e}$ indicates the localization at $\mathbf{e}$. The assumption of the lemma implies that

$$((I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i)_{\mathbf{e}}.\sigma_{\mathbf{e}} = 0 \qquad \text{for all } i \in \mathbb{N}^+.$$

Thus

$$((I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i.\sigma)_{\mathbf{e}} = 0,$$

which implies the lemma. □

Lemma 6.14. *Suppose that Z is an affine complex algebraic variety with a transitive algebraic action of $K_{\mathsf{s},\mathbb{C}}$-action on it. Let σ be a finitely generated $(\mathbb{C}[Z], K_{\mathsf{s},\mathbb{C}})$-module. Let $z \in Z$ and write K_z for the stabilizer of z in $K_{\mathsf{s},\mathbb{C}}$. Then the natural map*

$$\sigma^{K_{\mathsf{s},\mathbb{C}}} \to (\mathbb{C}_z \otimes_{\mathbb{C}[Z]} \sigma)^{K_z}$$

is a linear isomorphism. Here $\mathbb{C}_z$ is the field $\mathbb{C}$ viewing as a $\mathbb{C}[Z]$-algebra via the evaluation map at z, and a superscript group indicates the space of invariant vectors under the group action.

Proof. Note that the set

$$\bigsqcup_{x \in Z} \mathbb{C}_x \otimes_{\mathbb{C}[Z]} \sigma$$

is naturally a $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over Z, and σ is identified with the space of algebraic sections of this bundle. Thus

$$\sigma \cong {}^{\mathrm{alg}}\mathrm{Ind}_{K_z}^{K_{\mathsf{s},\mathbb{C}}}(\mathbb{C}_x \otimes_{\mathbb{C}[Z]} \sigma) \qquad \text{(algebraically induced representation).}$$

The lemma then follows by the algebraic version of the Frobenius reciprocity. □

Lemma 6.15. *Suppose that σ' is a finitely generated $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module, and write $\sigma := \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')$. Then the quasi-coherent module $(\mathcal{M}_\sigma)|_{\bar{\partial} \cap \mathfrak{p}_{\mathsf{s}}}$ descends to $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$.*

Proof. Write

$$\tilde{\sigma} := \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'} = (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]) \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} (\sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}),$$

to be viewed as a $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$-module. Write $\tilde{\sigma}_0 := \sigma$, viewing as a $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$-module via the moment map M_{s}.

Proposition 6.10 and the equalities in (6.2) imply that $(\mathcal{M}_{\tilde{\sigma}})|_{\mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'}}$ descends to $\mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$. This implies that $(\mathcal{M}_{\tilde{\sigma}_0})|_{\bar{\partial} \cap \mathfrak{p}_{\mathsf{s}}}$ descends to $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$. The lemma then follows since σ is a direct summand of $\tilde{\sigma}_0$.

□

We are now ready to prove Proposition 6.12.

Proof of Proposition 6.12. Let σ' be a finitely generated $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module, and write $\sigma := \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')$. Lemmas 6.15 and 6.13 imply that

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma).$$

Suppose that $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$. Then

$$\begin{aligned}
& \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma) \\
= \; & \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma \\
= \; & \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}',\mathbb{C}}} \\
= \; & \Big((\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]) \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'} \Big)_{K_{\mathsf{s}',\mathbb{C}}} \\
= \; & \Big(\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'} \Big)_{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(by Lemma 6.11).}
\end{aligned}$$

If $\mathbf{e}$ is not in the image of M_{s}, then the above module is zero. Now we assume that $\mathbf{e}$ is in the image of M_{s} so that $\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ is a single $K_{\mathsf{s}',\mathbb{C}}$-orbit. Pick an arbitrary element $\phi \in \mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$, and write $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. Let

$$1 \to K_{\mathsf{s}_1,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi \xrightarrow{\text{the projection to the first factor}} (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1$$

be the exact sequence as in Lemma 3.2. Note that $K_{\mathsf{s}_1,\mathbb{C}}$ equals the stabilizer of ϕ in $K_{\mathsf{s}',\mathbb{C}}$.

Let $\mathbb{C}_\phi$ denote the field $\mathbb{C}$ viewing as a $\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}]$-algebra via the evaluation map at ϕ. Then we have that

$$\begin{aligned}
&\left(\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{K_{\mathsf{s}',\mathbb{C}}} \\
= &\left(\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)^{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(because } K_{\mathsf{s}',\mathbb{C}} \text{ is reductive)} \\
= &\left(\mathbb{C}_\phi \otimes_{\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}]} \mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)^{K_{\mathsf{s}_1,\mathbb{C}}} \qquad \text{(by Lemma 6.14)} \\
= &\left(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)^{K_{\mathsf{s}_1,\mathbb{C}}} \\
= &\left(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{K_{\mathsf{s}_1,\mathbb{C}}} \qquad \text{(because } K_{\mathsf{s}',\mathbb{C}} \text{ is reductive).}
\end{aligned}$$

In conclusion, we have an identification

$$\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma) = \left(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'}\cap\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\right)_{K_{\mathsf{s}_1,\mathbb{C}}}.$$

It is routine to check that this identification respects the $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi$-actions, and thus respects the $K_{\mathbf{e}}$-actions. This proves the proposition. □

6.5. **Algebraic theta lifts and geometric theta lifts.** Proposition 6.12 has the following consequence.

Proposition 6.16. *Suppose that σ' is a finitely generated $\mathcal{O}'$-bounded $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')) \preceq \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\sigma'))$$

in $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$.

Proof. Suppose that $0 \to \sigma'_1 \to \sigma'_2 \to \sigma'_3 \to 0$ be an exact sequence of $\mathcal{O}'$-bounded finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-modules. Then

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_1) \to \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_2) \to \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_3) \to 0$$

is an exact sequence of $\mathcal{O}$-bounded finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-modules. Thus

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_2)) \preceq \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_1)) + \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_3)).$$

On the other hand, it is clear that

$$\vartheta^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\sigma'_2)) = \vartheta^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\sigma'_1)) + \vartheta^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\sigma'_3)).$$

The proposition then easily follows by Proposition 6.12. □

Finally, we are prepared to prove the following main result of this section. Recall that $\mathcal{O}$ is assumed to be regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$.

Theorem 6.17. *Let ρ' be an $\mathcal{O}'$-bounded $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho')$ is $\mathcal{O}$-bounded, and*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho')) \preceq \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\rho')).$$

Proof. Let $\mathcal{F}'$ and $\mathcal{F}$ be good filtrations on ρ' and $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho')$ respectively as in Lemma 6.4 so that there exists a surjective $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module homomorphism

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\mathrm{Gr}(\rho', \mathcal{F}')) \to \mathrm{Gr}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho'), \mathcal{F}). \tag{6.3}$$

The first assertion then follows from Lemmas 6.2 and 6.8.

Put $\sigma' := \mathrm{Gr}(\rho', \mathcal{F}')$. Then

$$\begin{aligned} & \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho')) \\ = \ & \mathrm{AC}_{\mathcal{O}}(\mathrm{Gr}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho'), \mathcal{F})) \\ \leq \ & \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')) \qquad \text{(by (6.3))} \\ \leq \ & \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\sigma')) \qquad \text{(by Proposition 6.16)} \\ = \ & \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\rho')). \end{aligned}$$

This proves the theorem. □

6.6. **Geometries of regular descent.** In this section, let (V, V') be a complex dual pair. We investigate the geometric properties of regular descent.

Let $\phi \in \tilde{M}_{\mathsf{s}'}^{-1}(\mathcal{O}) \cap W^{\circ}_{\mathsf{s},\mathsf{s}'}$, and $\mathbf{e}' = \tilde{M}_{\mathsf{s}'}(\phi)$. By the Jacobian criterion for regularity (see [52, Theorem 2.19]), the following sequence

$$0 \longrightarrow T_{\phi}\tilde{M}^{-1}_{\mathsf{s},\mathsf{s}'}(\mathbf{e}') \longrightarrow T_{\phi}W_{\mathsf{s},\mathsf{s}'} \xrightarrow{\mathrm{d}\tilde{M}_{\mathsf{s}'}} T_{\mathbf{e}'}\mathfrak{g}_{\mathsf{s}} \tag{6.4}$$

is exact if and only if ϕ is regular at $W_{\mathsf{s},\mathsf{s}'\mathbf{e}'}$.

We identify $T_{\phi}W_{\mathsf{s},\mathsf{s}'}$ and $T_{\mathbf{e}'}\mathfrak{g}_{\mathsf{s}'}$ with $W_{\mathsf{s},\mathsf{s}'}$ and $\mathfrak{g}_{\mathsf{s}'}$ respectively. Since $(G_{\mathsf{s}} \times (G_{\mathsf{s}'})_{\mathbf{e}'}) \cdot \phi$ is open in $\tilde{M}_{\mathsf{s}}^{-1}(\mathbf{e}')$, the map

$$\mathfrak{g}_{\mathsf{s}} \times (\mathfrak{g}_{\mathsf{s}'})_{\mathbf{e}'} \xrightarrow{(A,A') \mapsto A'\phi - \phi A} T_{\phi}\tilde{M}_{\mathsf{s}'}^{-1}(\mathbf{e}')$$

is surjective, where we identify $T_{\phi}\tilde{M}_{\mathsf{s}'}^{-1}(\mathbf{e}')$ as a subspace of $W_{\mathsf{s},\mathsf{s}'}$. Then the sequence (6.4) is exact if and only if the sequence (6.5) is exact.

$$\mathfrak{g} \oplus (\mathfrak{g}_{\mathsf{s}})_{\mathbf{e}'} \xrightarrow{(A,A') \mapsto A'\phi - \phi A} W \xrightarrow{b \mapsto \phi b^* + b\phi^*} \mathfrak{g}_{\mathsf{s}}. \tag{6.5}$$

Lemma 6.18. *The sequence* (6.5) *is exact and* $W_{\mathbf{e}'}$ *is regular at* ϕ.

Proof. Let $V_{\mathsf{s}'_0} := \phi V_{\mathsf{s}}$ and $V_{\mathsf{s}'_1}$ be the orthogonal complement of $V_{\mathsf{s}'_0}$ so that $V_{\mathsf{s}'} = V_{\mathsf{s}'_0} \oplus V_{\mathsf{s}'_1}$.

Suppose $|\mathsf{s}'| = \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})$. Then every element $\phi \in W_{U,\mathbf{e}'} \subset \mathrm{Hom}(V, V')$ is a surjective morphism and $V_{\mathsf{s}'_1} = 0$. Therefore $\mathrm{d}M_{\mathsf{s}'} \colon T_{\phi}W_{\mathsf{s},\mathsf{s}'} \longrightarrow T_{\mathbf{e}'}\mathfrak{g}_{\mathsf{s}'}$ is surjective, which implies $M_{\mathsf{s}'}$ is smooth at ϕ (*cf.* [33, Proposition 10.4]). Pulling back $M_{\mathsf{s}'}$ via the inclusion $\mathbf{e}' \hookrightarrow \mathfrak{g}_{\mathsf{s}}$, we conclude that $W_{\bar{\partial},\mathbf{e}'}$ is smooth and so regular at ϕ. This also establishs the exactness of (6.5).

Now suppose $|\mathsf{s}'| > \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O})$ and $\mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O})$. We now establish the exactness of (6.5), which concludes the lemma.

Let $W_{\mathsf{s},\mathsf{s}'_i} := \mathrm{Hom}(V_{\mathsf{s}}, V_{\mathsf{s}'_i})$ and $\mathfrak{g}_{\mathsf{s}'_i} := \mathfrak{g}_{\mathsf{s}_i}$ for $i = 1, 2$. Let $*\colon \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}) \to \mathrm{Hom}(V_{\mathsf{s}'_1}, V_{\mathsf{s}'_0})$ denote the adjoint map. We have $W_{\mathsf{s},\mathsf{s}'} = W_{\mathsf{s},\mathsf{s}'_0} \oplus W_{\mathsf{s},\mathsf{s}'_1}$ and $\mathfrak{g}_{\mathsf{s}'} = \mathfrak{g}_{\mathsf{s}'_0} \oplus \mathfrak{g}_{\mathsf{s}'_1} \oplus \mathfrak{g}^{\boxslash}_{\mathsf{s}'}$, where

$$\mathfrak{g}^{\boxslash} := \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}) \right\}.$$

Now ϕ and $\mathbf{e}'$ are naturally identified with an element ϕ_0 in $W_{\mathsf{s},\mathsf{s}'_0}$ and an element $\mathbf{e}'_0$ in $\mathfrak{g}_{\mathsf{s}'_0}$ respectively. We have $(\mathfrak{g}_{\mathsf{s}'})_{\mathbf{e}'} = (\mathfrak{g}_{\mathsf{s}'_0})_{\mathbf{e}'_0} \oplus \mathfrak{g}_{\mathsf{s}'_1} \oplus \mathfrak{g}^{\boxslash}_{\mathbf{e}'_0}$, where

$$\mathfrak{g}^{\boxslash}_{\mathsf{s}'\mathbf{e}'_0} = \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}) \text{ and } E\mathbf{e}'_0 = 0 \right\}.$$

Now maps in (6.5) have the following forms respectively:

$$\mathfrak{g}_{\mathsf{s}} \oplus (\mathfrak{g}_{\mathsf{s}'})_{\mathbf{e}'} \ni (A, A') = \left(A, (A'_0, A'_1, \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix})\right) \mapsto A'\phi - \phi A = (A'_0\phi_0 - \phi_0 A, E\phi_0) \in W_{\mathsf{s},\mathsf{s}'}$$

and

$$W_{\mathsf{s},\mathsf{s}'} = \mathrm{Hom}(V_{\mathsf{s}}, V_{\mathsf{s}'_0}) \oplus \mathrm{Hom}(Vs, V_{\mathsf{s}'_1}) \ni (b_0, b_1) \mapsto \left(b_0^*\phi_0 + \phi_0^* b_0, \begin{pmatrix} b_0\phi_0^* + \phi_0 b_0^* & \phi_0 b_1^* \\ b_1\phi_0^* & 0 \end{pmatrix} \right) \in \mathfrak{g}_{\mathsf{s}} \oplus \mathfrak{g}_{\mathsf{s}'}.$$

Note that $G_{\mathsf{s}'_0} \cdot \mathbf{e}'_0 = \mathcal{O}'_0 := \nabla^{\mathsf{s}}_{\mathsf{s}'_0}(\mathcal{O})$, we have the exactness of the sequence

$$\mathfrak{g}_{\mathsf{s}} \oplus (\mathfrak{g}_{\mathsf{s}'_0})_{\mathbf{e}'_0} \longrightarrow W_{\mathsf{s},\mathsf{s}'_0} \longrightarrow \mathfrak{g}_{\mathsf{s}} \oplus \mathfrak{g}_{\mathsf{s}'_0}.$$

It left to show that the following sequence is exact:

$$\mathfrak{g}^{\boxslash}_{\mathsf{s}'\mathbf{e}'_0} \xrightarrow{\eta_1} W_{\mathsf{s},\mathsf{s}'_1} \xrightarrow{\eta_2} \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}). \tag{6.6}$$

Here η_1 is given by $\begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \mapsto E\phi_0$ and η_2 is given by $b_2 \mapsto b_2\phi_0^*$. We have

$$\begin{aligned} \dim \mathrm{Im}(\eta_1) &= \dim \mathfrak{g}^{\boxslash}_{\mathsf{s}'\phi_0} \qquad (\phi_0 \text{ is surjective.}) \\ &= \dim V_{\mathsf{s}'_1} \cdot \dim(V_{\mathsf{s}'_0}/\mathrm{Im}(\mathbf{e}'_0)) \\ &= \dim V_{\mathsf{s}'_1} \cdot \dim \mathrm{Ker}(\mathbf{e}'_0) \\ &= \dim V_{\mathsf{s}'_1} \cdot \dim \mathbf{c}_2(\mathcal{O}) \end{aligned}$$

and

$$\begin{aligned} \dim \mathrm{Ker}(\eta_2) &= \dim V_{\mathsf{s}'_1} \cdot \dim(V_{\mathsf{s}}/\mathrm{Im}(\phi_0^*)) \\ &= \dim V_{\mathsf{s}'_1} \cdot (\dim V_{\mathsf{s}} - \dim V_{\mathsf{s}'_0}) \\ &= \dim V_{\mathsf{s}'_1} \cdot \mathbf{c}_1(\mathcal{O}) \\ &= \dim V_{\mathsf{s}'_1} \cdot \mathbf{c}_2(\mathcal{O}). \end{aligned}$$

Now the exactness of (6.6) and (6.5) follow by dimension counting. □

Now take $\phi \in \mathcal{X}_{\mathsf{s},\mathsf{s}'}$. Then the sequence (6.5) is compatible with the $(L_{\mathsf{s}}, L_{\mathsf{s}'})$-actions.

Lemma 6.19. *Suppose $\phi \in \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ and $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. Then following sequence is exact.*

$$\mathfrak{k}_{\mathsf{s}} \oplus (\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \longrightarrow \mathcal{X}_{\mathsf{s},\mathsf{s}'} \longrightarrow \mathfrak{p}_{\mathsf{s}'}. \tag{6.7}$$

In particular, $\mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ is regular at ϕ.

Proof. The exactness of the sequence is clear by taking eign-spaces of (6.6) under the $(L_{\mathsf{s}}, L_{\mathsf{s}'})$-action. Since $K_{\mathsf{s}}K_{\mathsf{s}'} \cdot \phi$ is open in $M_{\mathsf{s}'}^{-1}(\mathbf{e}')$, $\mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ is regular at ϕ by Jacobian criterion. □

We now recall the following lemma in EGA.

Lemma 6.20 ([31, Proposition 11.3.13])**.** *Suppose $f\colon X \to Y$ is a morphism between schemes, x is a point in X such that f is flat at x. Let $y = f(x)$.*

(a) If X is reduced at the point x, then Y is reduced at y.

(b) We assume f is of finite presentation at the porint x. If Y is reduced at x and x is reduced at the scheme theoretical fiber X_y, then X is reduced at x. □

Proof of Proposition 6.10. Consider the $K_{\mathsf{s}} \times K_{\mathsf{s}'}$-equivariant morphism

$$\mathcal{X}^{\bar{\partial},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}) \longrightarrow \mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$$

Since $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$ is a finite union of $K_{\mathsf{s}'}$-orbits, the above morphism is flat by generic flatness [30, Théorème 6.9.1]. By Lemma 6.19, $\mathcal{X}^{\bar{\partial},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ is regular at every point ϕ since it is a open subscheme of $\mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ and $\phi \in \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$. Clearly $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$ is reduced. We conclude that $\mathcal{X}^{\bar{\partial},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ is reduced by Lemma 6.20. □

Using the same argument we also have the following lemma which we will not use in the paper.

Lemma 6.21. *Suppose $\mathcal{O}$ regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Then*

$$W_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{g}_{\mathsf{s}}\times\mathfrak{g}_{\mathsf{s}'}} \big((\mathfrak{g}_{\mathsf{s}} - \partial\mathcal{O}) \times \mathcal{O}'\big)$$

is a reduced scheme. □

Lemma 6.22. *Suppose $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$, $\mathscr{O}$ is a K_{s}-orbit in $\mathcal{O}\cap\mathfrak{p}_{\mathsf{s}}$ and $\mathscr{O}' = \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) \subset \mathcal{O}'\cap\mathfrak{p}_{\mathsf{s}'}$. Suppose $\mathcal{E}' \in \mathrm{AOD}_{\mathsf{s}'}(\mathscr{O}')$. Then $\vartheta^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$ is either zero or in $\mathrm{AOD}_{\mathsf{s}}(\mathscr{O})$.*

Proof. Note that $(\zeta_{\mathsf{s},\mathsf{s}'})^2 = (\bigwedge^{\mathrm{top}}\mathcal{X}_{\mathsf{s},\mathsf{s}'})^{-1}$. The lemma follows from Lemma 6.23 below. □

Lemma 6.23. *Suppose $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ and $\phi \in \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$. Let $\mathbf{e} = M_{\mathsf{s}}(\phi)$ and $\mathbf{e}' = M_{\mathsf{s}'}(\phi)$. Then*

$$\bigwedge^{\mathrm{top}}(\mathfrak{k}_{\mathsf{s}})_{\mathbf{e}} = \bigwedge^{\mathrm{top}}(\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \otimes \Big(\bigwedge^{\mathrm{top}}\mathcal{X}_{\mathsf{s},\mathsf{s}'}\Big)^{-1} \tag{6.8}$$

as $(K_{\mathsf{s}}\times K_{\mathsf{s}'})_{\phi}$-module.

Proof. We retain the notation in the proof of Lemma 6.18 abd Lemma 6.19. Let $S_{\phi} := (K_{\mathsf{s}}\times K_{\mathsf{s}'})_{\phi}$, $\mathfrak{s}_{\phi} = \mathrm{Lie}(S_{\phi})$, $S_{\phi_0} := (K_{\mathsf{s}}\times K_{\mathsf{s}'_0})_{\phi_0}$ and $\mathfrak{s}_{\phi_0} := \mathrm{Lie}(S_{\phi_0})$. Then $\mathfrak{s}_{\phi} = \mathfrak{s}_{\phi_0}\oplus\mathfrak{k}_{\mathsf{s}'_1}$ and $\mathfrak{s}_{\phi_0} \xrightarrow{\cong} (\mathfrak{k}_{\mathsf{s}})_{\mathbf{e}}$ as S_{ϕ}-module.

The sequence (6.7) leads to the following short exact sequence

$$0 \longrightarrow (\mathfrak{k}_{\mathsf{s}}\oplus(\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'})/\mathfrak{s}_{\phi} \longrightarrow \mathcal{X}_{\mathsf{s},\mathsf{s}'} \longrightarrow \mathfrak{p}_{\mathsf{s}'_0}\oplus\mathfrak{p}^{\boxslash}_{\mathsf{s}'} \longrightarrow 0,$$

where $\mathfrak{p}^{\boxslash}_{\mathsf{s}'} = \mathfrak{g}^{\boxslash}_{\mathsf{s}'}\cap\mathfrak{p}$.

Note that the S_{ϕ}-modules $\bigwedge^{\mathrm{top}}\mathfrak{k}_{\mathsf{s}}$, $\bigwedge^{\mathrm{top}}\mathfrak{k}_{\mathsf{s}'_1}$, $\bigwedge^{\mathrm{top}}\mathfrak{p}_{\mathsf{s}'_1}$ and $\bigwedge^{\mathrm{top}}\mathfrak{g}_{\mathsf{s}'}$ are trivial. Now

$$\begin{aligned}
&\bigwedge^{\mathrm{top}}(\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \otimes \Big(\bigwedge^{\mathrm{top}}\mathcal{X}_{\mathsf{s},\mathsf{s}'}\Big)^{-1}\\
&= \bigwedge^{\mathrm{top}}(\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \otimes \Big(\bigwedge^{\mathrm{top}}\big((\mathfrak{k}_{\mathsf{s}}\oplus(\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'})/\mathfrak{s}_{\phi}\big)\otimes\bigwedge^{\mathrm{top}}\mathfrak{p}_{\mathsf{s}'_0}\otimes\bigwedge^{\mathrm{top}}\mathfrak{p}^{\boxslash}_{\mathsf{s}'}\Big)^{-1}\\
&= \bigwedge^{\mathrm{top}}(\mathfrak{s}_{\phi_0}\oplus\mathfrak{k}_{\mathsf{s}'_1})\otimes\Big(\bigwedge^{\mathrm{top}}(\mathfrak{k}_{\mathsf{s}}\oplus\mathfrak{k}_{\mathsf{s}'}\oplus\mathfrak{p}_{\mathsf{s}'_0}\oplus\mathfrak{p}^{\boxslash}_{\mathsf{s}'})\Big)^{-1}\\
&= \bigwedge^{\mathrm{top}}\mathfrak{s}_{\phi_0}\otimes\Big(\bigwedge^{\mathrm{top}}\mathfrak{g}_{\mathsf{s}'}\Big)^{-1}\\
&\qquad (\text{by } \mathfrak{k}_{\mathsf{s}'}\oplus\mathfrak{p}_{\mathsf{s}'_0}\oplus\mathfrak{p}^{\boxslash}_{\mathsf{s}'}\oplus\mathfrak{p}_{\mathsf{s}'_1} = \mathfrak{g}_{\mathsf{s}'}\ )\\
&= \bigwedge^{\mathrm{top}}(\mathfrak{k}_{\mathsf{s}})_{\mathbf{e}}.
\end{aligned}$$

This proves the lemma. □

7. Induced orbits and double geometric lifts

In this subsection, we discuss various formulas for the induced orbits and double geometric theta lifts.

Suppose $\mathsf{s}' = (\star', p', q')$. In this sectoin, we let $\mathcal{O}' \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$. Fix $l \in \mathbb{N}$ such that $l \geqslant \mathbf{c}_2(\mathcal{O}')$.

It is elementary to check the following lemma on the orbit induction.

Lemma 7.1. *Let $\mathsf{s}'' = (\star', p'+l, q'+l)$ and $t = l - \mathbf{c}_1(\mathcal{O}')$. Let $\mathcal{O}'' := \mathrm{Ind}^{G_{\mathsf{s}'',\mathbb{C}}}_{G_{\mathsf{s}',\mathbb{C}}}(\mathcal{O}')$ and . $\mathscr{O}' \in \mathrm{Nil}_{\mathsf{s}'}(\mathcal{O}')$.*

(a) *Suppose* $\star' \in \{B, D\}$. *Then* $\mathcal{O}''$ *satisfies* $\mathbf{c}_i(\mathcal{O}'') = \mathbf{c}_{i-2}(\mathcal{O}')$ *for* $i \geqslant 4$ *and*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (l, l, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0 \text{ is even}, \\ (l+1, l-1, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0 \text{ is odd}, \\ (\mathbf{c}_1(\mathcal{O}'), l, l) & \text{if } t < 0. \end{cases}$$

We have

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \mathscr{O}' \cdot (\frac{t}{2}, \frac{t}{2}) \cdot (0,0) & \text{if } t \geqslant 0 \text{ is even}, \\ 2\left(\mathscr{O}' \cdot (\frac{t-1}{2}, \frac{t-1}{2}) \cdot (1,1)\right) & \text{if } t \geqslant 0 \text{ is odd}, \\ \sum_{p_0+q_0=-t} (\Lambda_{(p_0,q_0)}\mathscr{O}') \cdot (0,0) \cdot (p_0, q_0) & \text{if } t < 0. \end{cases}$$

(b) *Suppose* $\star' \in \{C, \widetilde{C}\}$. *Then* $\mathcal{O}''$ *satisfies* $\mathbf{c}_i(\mathcal{O}'') = \mathbf{c}_{i-2}(\mathcal{O}')$ *for* $i \geqslant 4$ *and*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (l, l, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0, \\ (\mathbf{c}_1(\mathcal{O}'), l, l) & \text{if } t < 0 \text{ is even}, \\ (\mathbf{c}_1(\mathcal{O}'), l+1, l-1) & \text{if } t < 0 \text{ is odd}. \end{cases}$$

We have

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \sum_{t=p_0+q_0} \mathscr{O}' \cdot (p_0, q_0) \cdot (0,0) & \text{if } t \geqslant 0, \\ (\Lambda_{(-\frac{t}{2}, -\frac{t}{2})}\mathscr{O}') \cdot (0,0) \cdot (-\frac{t}{2}, -\frac{t}{2}) & \text{if } t < 0 \text{ is even}, \\ \sum_{p_0+q_0=2} (\Lambda_{(-\frac{t-1}{2}, -\frac{t-1}{2})}\mathscr{O}') \cdot (p_0, q_0) \cdot (\frac{1-t}{2}, \frac{1-t}{2}) & \text{if } t < 0 \text{ is odd}. \end{cases}$$

(c) *Suppose* $\star' \in \{C^*, D^*\}$. *Note that* l *and* k *are even integers in this case. Then* $\mathcal{O}''$ *satisfies* $\mathbf{c}_i(\mathcal{O}'') = \mathbf{c}_{i-2}(\mathcal{O}')$ *for* $i \geqslant 4$ *and*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (l, l, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0, \\ (\mathbf{c}_1(\mathcal{O}'), l, l) & \text{if } t < 0. \end{cases}$$

When $\star' = D^*$, *we have*

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \sum_{p_0+q_0=\frac{t}{2}} \mathscr{O}' \cdot (2p_0, 2q_0) \cdot (0,0) & \text{if } t \geqslant 0, \\ \left(\Lambda_{(-\frac{t}{2}, -\frac{t}{2})}\mathscr{O}'\right) \cdot (0,0) \cdot (-\frac{t}{2}, -\frac{t}{2}) & \text{if } t < 0. \end{cases}$$

When $\star' = C^*$, *we have*

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \mathscr{O}' \cdot (\frac{t}{2}, \frac{t}{2}) \cdot (0,0) & \text{if } t \geqslant 0, \\ \sum_{p_0+q_0=-\frac{t}{2}} \left(\Lambda_{(2p_0, 2q_0)}\mathscr{O}'\right) \cdot (0,0) \cdot (2p_0, 2q_0) & \text{if } t < 0. \end{cases}$$

□

Retain the notation in Lemma 7.1. Let

$$\mathscr{F} \colon \bigsqcup_{\mathsf{s}, \mathcal{O}} \mathcal{K}_{\mathsf{s}}(\mathcal{O}) \longrightarrow \bigsqcup_{\mathsf{s}, \mathcal{O}} \mathbb{Z}[\mathrm{Nil}_{\mathsf{s}}(\mathcal{O})]$$

be the map defined by $\mathcal{E} \mapsto \dim(\mathcal{E}_{\mathbf{e}''})\mathscr{O}''$ where $\mathbf{e}'' \in \mathscr{O}'' \subset \mathcal{O}'' \cap \mathfrak{p}_{\mathsf{s}''}$ and $\mathcal{E} \in \mathcal{K}_{\mathsf{s}''}(\mathcal{O}'')$.

Lemma 7.2. *Let* $\mathcal{E}' \in \mathcal{K}_{\mathsf{s}'}(\mathcal{O})$.

(a) *Suppose* $\star' \in \{B, D\}$ *and* $2n = |\nabla_{\mathrm{naive}}(\mathcal{O}'')|$. *Let* $\mathsf{s} = (\star, n, n)$ *and*

$$\check{\vartheta}^2(\mathcal{E}') := \check{\vartheta}_{\mathsf{s}}^{\mathsf{s}''}(\check{\vartheta}_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{E}')) + \check{\vartheta}_{\mathsf{s}}^{\mathsf{s}''}(\check{\vartheta}_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{E}' \otimes \det)) \otimes \det.$$

Then

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = \begin{cases} 2\,\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')) & \text{if } t \geqslant 0 \text{ is even}, \\ \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')) & \text{if } t \geqslant -1 \text{ is odd}. \end{cases}$$

When $t < -1$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) \preceq \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality can be strict in general.*

(b) *Suppose* $\star' \in \{C, \widetilde{C}\}$. *Let* $d \in \mathbb{N}$ *and* $m = |\nabla_{\mathrm{naive}}(\mathcal{O}'')| + d$. *When* $t \geqslant 0$ *or* $(-t, d) \in (2\mathbb{N}) \times \{0\}$, *define*

$$\check{\vartheta}^2(\mathcal{E}') := \sum_{\substack{\mathsf{s}=(\star,p,q)\\ p+q=m}} \check{\vartheta}^{\mathsf{s}''}_{\mathsf{s}}(\check{\vartheta}^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{E}')).$$

When $t \geqslant 0$ *or* $(t, d) = (-2, 0)$,

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = (d+1)\operatorname{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')).$$

When $t < -2$ *is even and* $d = 0$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) \preceq \operatorname{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality can be strict in general*

(c) *Suppose* $\star' = C^*$ *and* $2n = |\nabla_{\mathrm{naive}}(\mathcal{O}'')|$. *Let* $\mathsf{s} = (\star, n, n)$ *and*

$$\check{\vartheta}^2(\mathcal{E}') := \check{\vartheta}^{\mathsf{s}''}_{\mathsf{s}}(\check{\vartheta}^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{E}')).$$

When $t \geqslant -2$,

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = \operatorname{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')).$$

When $t < -2$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) \preceq \operatorname{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality is strict in general.*

(d) *Suppose* $\star' = D^*$. *Let* $d \in \mathbb{N}$ *and* $2m = |\nabla_{\mathrm{naive}}(\mathcal{O}'')| + 2d$. *When* $t \geqslant 0$ *or* $(-t, d) \in (2\mathbb{N}) \times \{0\}$, *define*

$$\check{\vartheta}^2(\mathcal{E}') := \sum_{\substack{\mathsf{s}=(\star,2p,2q)\\ 2p+2q=2m}} \check{\vartheta}^{\mathsf{s}''}_{\mathsf{s}}(\check{\vartheta}^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{E}')).$$

When $t \geqslant 0$,

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = (d+1)\operatorname{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')).$$

When $t < 0$ *and* $d = 0$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = \operatorname{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality can be strict in general.* □

7.0.1. *On certain induced orbits.* The main step in the proof of **??** consists of comparison of bound of the associated cycle of $\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))$ with the formula (due to Barbasch) of the wavefront cycle of a certain parabolically induced representation.

In this section, let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^J$ be the Lie algebra of G, which is identified with its dual space $\mathfrak{g}_{\mathbb{R}}^*$ under the trace from. Let $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}})$ denote the set of nilpotent G-orbits in $\mathbf{i}\mathfrak{g}_{\mathbb{R}}$. Similar notation will be used without further explanation. For example, for every Levi subgroup M of G, $\mathfrak{m}_{\mathbb{R}}$ denotes its Lie algebra, which is identified with the dual space $\mathfrak{m}_{\mathbb{R}}^*$ by using the trace form on $\mathfrak{g}$.

Let

$$\mathsf{KS}\colon \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}}) \to \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$$

be the natural bijection given by the Kostant-Sekiguchi correspondence (*cf.* [76, Equation (6.7)]). By abuse of notation, we also let $\ddot{\mathbf{D}}$ denote the map $\ddot{\mathbf{D}} \circ \mathsf{KS}\colon \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}}) \to \ddot{\mathcal{P}}$. See **??** for the parametrization map $\ddot{\mathbf{D}}\colon \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p}) \to \ddot{\mathcal{P}}$.

Theorem 7.3 (*cf.* Barbasch [10, Corollary 5.0.10])**.** *Let* G_1 *be an arbitrary real reductive group and let* P_1 *be a real parabolic subgroup of* G_1, *namely a closed subgroup of* G_1 *whose Lie algebra is a parabolic subalgebra of* $\mathfrak{g}_{1,\mathbb{R}}$. *Let* N_1 *be the unipotent radical of* P_1 *and* $M_1 := P_1/N_1$. *Write*

$$r_1 : \mathbf{i}(\mathfrak{g}_{1,\mathbb{R}}/\mathfrak{n}_{1,\mathbb{R}})^* \longrightarrow \mathbf{i}\,\mathfrak{m}_{1,\mathbb{R}}^*$$

for the natural map. Let π_1 *be a Casselman-Wallach representation of* M_1 *with the wavefront cycle*

$$\mathrm{WF}(\pi_1) = \sum_{\mathscr{O}_{\mathbb{R}} \in \mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}_{1,\mathbb{R}}^*)} \mathrm{c}_{\mathscr{O}_{\mathbb{R}}}[\mathscr{O}_{\mathbb{R}}].$$

Then

$$\mathrm{WF}(\mathrm{Ind}_{P_1}^{G_1}\pi) = \sum_{(\mathscr{O}_{\mathbb{R}},\mathscr{O}'_{\mathbb{R}})} \mathrm{c}_{\mathscr{O}_{\mathbb{R}}} \frac{\#C_{G_1}(v')}{\#C_{P_1}(v')}[\mathscr{O}'_{\mathbb{R}}],$$

where the summation runs over all pairs $(\mathscr{O}_{\mathbb{R}}, \mathscr{O}'_{\mathbb{R}})$ *such that* $\mathscr{O}_{\mathbb{R}} \in \mathrm{Nil}_{M_1}(\mathfrak{im}^*_{1,\mathbb{R}})$ *and* $\mathscr{O}'_{\mathbb{R}} \in \mathrm{Nil}_{G_1}(\mathfrak{ig}^*_{1,\mathbb{R}})$ *is an induced orbit of* $\mathscr{O}_{\mathbb{R}}$, v' *is an element of* $\mathscr{O}'_{\mathbb{R}} \cap r_1^{-1}(\mathscr{O}_{\mathbb{R}})$, *and* $C_{G_1}(v')$ *and* $C_{P_1}(v')$ *are the component groups of the centralizers of* v' *in* G_1 *and* P_1, *respectively.*

Remarks. 1. While Barbasch proved the theorem when G_1 is the real points of a connected reductive algebraic group, his proof still works in the slightly more general setting of Theorem 7.3.
2. The notion of induced nilpotent orbits was introduced by Lusztig and Spaltenstein for complex reductive groups [54]. For a real reductive group G_1, a nilpotent orbit $\mathscr{O}'_{\mathbb{R}} \in \mathrm{Nil}_G(\mathfrak{ig}^*_{1,\mathbb{R}})$ is called an induced orbit of a nilpotent orbit $\mathscr{O}_{\mathbb{R}} \in \mathrm{Nil}_{M_1}(\mathfrak{im}^*_{1,\mathbb{R}})$ if $\mathscr{O}'_{\mathbb{R}} \cap r_1^{-1}(\mathscr{O}_{\mathbb{R}})$ is open in $r_1^{-1}(\mathscr{O}_{\mathbb{R}})$ (see [10, Definition 5.0.7] or [66]). We write $\mathrm{Ind}_{P_1}^{G_1}\mathscr{O}_{\mathbb{R}}$ for the set of all induced orbits of $\mathscr{O}_{\mathbb{R}}$. Similar notation applies for a complex reductive group.
3. According to the fundamental result of Schimd-Vilonen [76], the wave front cycle and the associated cycle agree under the Kostant-Sekiguchi correspondence. We thank Professor Vilonen for confirming that their result extends to nonlinear groups. Therefore the associated cycle of a parabolically induced representation as in the above theorem of Barbasch is also determined.

We now consider induced orbits appearing in our cases. Retain the setting in Section 5.2 (see **??** and onwards), where

$$\mathbf{V}^{\perp} = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}'_0,$$

$M_{\mathbf{E}_0} = G \times \mathrm{GL}_{\mathbf{E}_0}$ and the parabolic subgroup of $G_{\mathbf{V}^{\perp}}$ stabilizing $\mathbf{E}_0$ is $P_{\mathbf{E}_0} = M_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$.

Recall that $\mathbf{D}(\mathcal{O}') = [c_0, \cdots, c_k]$ and $\mathbf{D}(\mathcal{O}) = [c_1, \cdots, c_k]$. Put $l := \dim \mathbf{E}_0$. In the notation of (**??**), we have

$$l = c_0 + \delta \geqslant c_1. \tag{7.1}$$

Note that if $\mathbf{G}$ is an orthogonal group, then $c_0 - c_1$ is even. Hence $l - c_1$ is odd if G is real orthogonal and $l - c_1$ is even if G is quaternionic orthogonal. View $\mathcal{O}$ as a nilpotent orbit in $\mathfrak{m}_{\mathbf{E}_0}$ via inclusion. The following lemma is clear (*cf.* [23, Section 7.3]).

Lemma 7.4. *(i) If* G *is a real orthogonal group, then*

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^{\perp}}}\mathcal{O}) = [l+1, l-1, c_1, \cdots, c_k].$$

(ii) Otherwise,

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^{\perp}}}\mathcal{O}) = [l, l, c_1, \cdots, c_k].$$

By [25, Theorem 5.2 and 5.6], the complex nilpotent orbit

$$\mathcal{O}^{\perp} := \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}^{\perp}}(\mathcal{O}') = \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}^{\perp}}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'}(\mathcal{O}))$$

is given by

$$\mathbf{D}(\mathcal{O}^{\perp}) = \begin{cases} [c_0+2, c_0, c_1, \cdots, c_k], & \text{if } G \text{ is a real orthogonal group;} \\ [c_0-1, c_0-1, c_1, \cdots, c_k], & \text{if } G \text{ is a real symplectic group;} \\ [c_0, c_0, c_1, \cdots, c_k], & \text{otherwise.} \end{cases}$$

This implies that

$$\mathcal{O}^{\perp} = \mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^{\perp}}}\mathcal{O}.$$

View each orbit in $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}})$ as an orbit in $\mathrm{Nil}_{M_{\mathbf{E}_0}}(\mathbf{i}\,\mathfrak{m}^J_{\mathbf{E}_0})$ via inclusion. We state the result for the induction of real nilpotent orbits in the following lemma. The proof will be given in Appendix B.2, which is by elementary matrix manipulations.

Lemma 7.5. *Let $\mathscr{O}_{\mathbb{R}} \in \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}})$ be a real nilpotent orbit in $\mathcal{O}$ with $\ddot{\mathbf{D}}(\mathscr{O}_{\mathbb{R}}) = [d_1, \cdots, d_k]$.*

(i) Suppose G is a real orthogonal group. Then $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}}$ consists of a single orbit $\mathscr{O}_{\mathbb{R}}^\perp$ with

$$\ddot{\mathbf{D}}(\mathscr{O}_{\mathbb{R}}^\perp) = [d_1 + s + (1,1), \check{d}_1 + \check{s}, d_1, \cdots, d_k],$$

where $s := (\frac{l-c_1-1}{2}, \frac{l-c_1-1}{2}) \in \mathbb{Z}^2_{\geqslant 0}$. Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is injective and its image has index 2 in $C_{G_{\mathbf{V}^\perp}}(X)$ for $X \in \mathscr{O}_{\mathbb{R}}^\perp \cap (\mathscr{O}_{\mathbb{R}} + \mathbf{i}\,\mathfrak{n}^J_{\mathbf{E}_0})$.

(ii) Otherwise, $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}}$ equals the set

$$\left\{ \mathscr{O}_{\mathbb{R}}^\perp \in \mathrm{Nil}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\mathfrak{g}^J_{\mathbf{V}^\perp}) \;\middle|\; \begin{array}{l} \ddot{\mathbf{D}}(\mathscr{O}_{\mathbb{R}}^\perp) = [d_1 + s, \check{d}_1 + \check{s}, d_1, \cdots, d_k] \text{ for a signature} \\ s \text{ of a } (-\epsilon, -\dot{\epsilon})\text{-space of dimension } l - c_1 \end{array} \right\}.$$

Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is an isomorphism for $X \in \mathscr{O}_{\mathbb{R}}^\perp \cap (\mathscr{O}_{\mathbb{R}} + \mathbf{i}\,\mathfrak{n}^J_{\mathbf{E}_0})$.

Here $C_{P_{\mathbf{E}_0}}(X)$ and $C_{G_{\mathbf{V}^\perp}}(X)$ are the component groups as in Theorem 7.3. Under the setting of Lemma 7.5, we see that $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}}$ consists of a single orbit when G is a real orthogonal group or a quaternionic symplectic group.

For each $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, let

$$\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O} := \left\{ \mathsf{KS}(\mathscr{O}_{\mathbb{R}}) \;\middle|\; \mathscr{O}_{\mathbb{R}} \in \mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}}(\mathsf{KS}^{-1}(\mathscr{O})) \right\}.$$

7.0.2. *Finishing the proof when G is a real orthogonal group.* Let $\mathrm{sgn}_{\mathbf{V}}$ and $\mathrm{sgn}_{\mathbf{V}^\perp}$ be the sign character of the orthogonal groups G and $G_{\mathbf{V}^\perp}$ respectively. By **??**, Lemma 5.9 **??** and Lemma 5.10, we have

$$\begin{aligned} &\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}) \\ &\cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \cong \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})}(\pi) \oplus \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})\otimes \mathrm{sgn}_{\mathbf{U}}}(\pi) \\ &\cong \bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \oplus (\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi \otimes \mathrm{sgn}_{\mathbf{V}}))) \otimes \mathrm{sgn}_{\mathbf{V}^\perp}. \end{aligned} \tag{7.2}$$

By Theorem 7.3, Lemma 7.4, Lemma 7.5 (i) and [76, Theorem 1.4], the representation $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}))$ is $\mathcal{O}^\perp$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})) = 2 \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{7.3}$$

where the summation runs over all $\mathbf{K}$-orbits $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^\perp$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}$.

On the other hand, by the explicit formula for the descents of nilpotent orbits, we have

$$\nabla_{\mathbf{V}',\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'}(\mathscr{O}^\perp)) = \mathscr{O}.$$

Applying Theorem 6.17 twice, we have

$$\begin{aligned} \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))) &\preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and} \\ \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi \otimes \mathrm{sgn}_{\mathbf{V}}))) \otimes \mathrm{sgn}_{\mathbf{V}^\perp}) &\preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp] \end{aligned} \tag{7.4}$$

where the summations run over the same set as the right hand side of (7.3).

In view of (7.2) to (7.4), we conclude that both inequalities in (7.4) are equalities. Thus (**??**) follows.

7.0.3. *Finishing the proof when G is a real symplectic group.* Let $\mathbf{V}'_1, \mathbf{V}'_2, \cdots, \mathbf{V}'_s$ be a list of representatives of the isomorphic classes of all $(1,1)$-spaces with dimension $\dim \mathbf{V}'$.

By **??**, Lemma 5.9 **??** and Lemma 5.10, we have

$$\begin{aligned} &\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})\oplus \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}\left(\pi\otimes\chi'_{\mathbf{E}_0}\right)\\ &\cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi)\oplus\mathcal{R}_{\mathrm{I}(\chi'_{\mathbf{E}})}(\pi)\\ &\cong \bigoplus_{i=1}^{s}\mathcal{R}_{\bar\Theta_{\mathbf{V}'_i,\mathbf{U}}(1_{\mathbf{V}'_i})}(\pi)\cong\bigoplus_{i=1}^{s}\bar\Theta_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'_i}(\pi)). \end{aligned}\tag{7.5}$$

By Theorem 7.3, Lemma 7.4, Lemma 7.5 (ii) and [76, Theorem 1.4], $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})$ and $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}\left(\pi\otimes\chi'_{\mathbf{E}_0}\right)$ are $\mathcal{O}^\perp$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}))=\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}\left(\pi\otimes\chi'_{\mathbf{E}_0}\right))=\sum_{\mathscr{O},\mathscr{O}^\perp}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],\tag{7.6}$$

where the summation runs over all pairs $(\mathscr{O},\mathscr{O}^\perp)$, where $\mathscr{O}$ is a $\mathbf{K}$-orbit in $\mathcal{O}\cap\mathfrak{p}_{\mathbf{V}}$ and $\mathscr{O}^\perp\in\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}}\mathscr{O}$.

Applying Theorem 6.17 twice, we see that $\bar\Theta_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'_i}(\pi))$ is $\mathcal{O}^\perp$-bounded ($1\leqslant i\leqslant s$), and

$$\sum_{i=1}^{s}\mathrm{AC}_{\mathcal{O}^\perp}(\bar\Theta_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'_i}(\pi)))\preceq\sum_{i=1}^{s}\sum_{\mathscr{O},\mathscr{O}^\perp}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],$$

where the inner summation runs over all pairs of orbits $(\mathscr{O},\mathscr{O}^\perp)$ such that

$$\nabla_{\mathbf{V}'_i,\mathbf{V}}(\nabla^{\mathrm{gen}}_{\mathbf{V}^\perp,\mathbf{V}'_i}(\mathscr{O}^\perp))=\mathscr{O}.\tag{7.7}$$

By Lemma 7.5 (ii) and **??**, such kind of pairs $(\mathscr{O},\mathscr{O}^\perp)$ are the same as those of the right hand side of (7.6). Suppose $\ddot{\mathbf{D}}(\mathscr{O}^\perp)=[d,\check{d},d_1,\cdots,d_k]$, then $\ddot{\mathbf{D}}(\mathscr{O})=[d_1,\cdots,d_k]$ and (7.7) holds for exactly two $\mathbf{V}'_i$, having signature $\check{d}+\mathrm{Sign}(\mathbf{V})+(1,0)$ and $\check{d}+\mathrm{Sign}(\mathbf{V})+(0,1)$ respectively. Hence we have

$$\sum_{i=1}^{s}\mathrm{AC}_{\mathcal{O}^\perp}(\bar\Theta_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'_i}(\pi)))\preceq\sum_{\mathscr{O},\mathscr{O}^\perp}2\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],\tag{7.8}$$

where the summation is as in the right hand side of (7.6).

In view of (7.5), (7.6) and (7.8), the equality holds in (7.8). Thus **??** holds.

7.0.4. *Finishing the proof when G is a quaternionic symplectic group.* Using **??**, Lemma 5.9 **??**, Lemma 5.10, Theorem 7.3, Lemma 7.5 (ii), [76, Theorem 1.4] and Theorem 6.17, and a similar argument as in Section 7.0.2 shows that

$$\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})\cong\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi)\cong\bar\Theta_{\mathbf{V}',\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi))\tag{7.9}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}))=\sum_{\mathscr{O}}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],\quad\text{and}\tag{7.10}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\bar\Theta_{\mathbf{V}',\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi)))\preceq\sum_{\mathscr{O}}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],\tag{7.11}$$

Here the summations run over all $\mathbf{K}_{\mathbf{V}}$-orbit $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^{\perp}$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^{\perp}}} \mathscr{O}$. Note that

$$\nabla_{\mathbf{V}',\mathbf{V}}(\nabla_{\mathbf{V}^{\perp},\mathbf{V}'}(\mathscr{O}^{\perp})) = \mathscr{O}.$$

Now (7.9), (7.10) and (7.11) imply that (7.11) is an equality and so **??** holds.

7.0.5. *Finishing the proof when G is a quaternionic orthogonal group.* Let $\mathbf{V}'_1, \mathbf{V}'_2, \cdots, \mathbf{V}'_s$ be a list of representatives of the isomorphic classes of all $(-1,1)$-spaces with dimension $\dim \mathbf{V}'$.

Lemma 7.6. *One has that*

$$\mathrm{Ch}_{\mathcal{O}^{\perp}}(\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi)) = \sum_{i=1}^{s} \mathrm{Ch}_{\mathcal{O}^{\perp}}(\mathcal{R}_{\bar{\Theta}_{\mathbf{V}'_i,\mathbf{U}}(1_{\mathbf{V}'_i})}(\pi)).$$

Proof. For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in Lemma 5.9 **??**. Note that $\mathcal{R}_{I_2}(\pi)$ is $\mathcal{O}^{\perp}$-bounded by **??**, Theorem 7.3 and [76, Theorem 1.4]. By the definition in **??**, we could view $\mathcal{R}_{I_1}(\pi)$ as a subrepresentation of $\mathcal{R}_{I_2}(\pi)$ which is also $\mathcal{O}^{\perp}$-bounded.

Since G acts on $\mathcal{R}_{I_2}(\pi)$ trivially, the natural homomorphism

$$\pi\widehat{\otimes} I_3 \cong \pi\widehat{\otimes} I_2/\pi\widehat{\otimes} I_1 \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)$$

descents to a surjective homomorphism

$$(\pi\widehat{\otimes} I_3)_G \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi).$$

For each $(-1,1)$-space $\mathbf{V}'''$ of dimension $\dim \mathbf{V}' - 2$, by applying **??** twice, we see that

$$(\pi\widehat{\otimes}(\omega_{\mathbf{V}''',\mathbf{U}})_{G_{\mathbf{V}'''}})_G \cong ((\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'''})_G\widehat{\otimes}\omega_{\mathbf{V}''',\mathbf{V}^{\perp}})_{G_{\mathbf{V}'''}}$$

is bounded by $\boldsymbol{\vartheta}_{\mathbf{V}''',\mathbf{V}^{\perp}}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'''}(\mathcal{O}))$. Using formulas in [25, Theorem 5.2 and 5.6], one checks that the latter set is contained in the boundary of $\mathcal{O}^{\perp}$. Hence

$$\mathrm{Ch}_{\mathcal{O}^{\perp}}(\mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)) = 0$$

and the lemma follows. □

Using Theorem 7.3, Lemma 7.5 (ii), [76, Theorem 1.4], Theorem 6.17 and a similar argument as in Section 7.0.3, we have

$$\mathrm{AC}_{\mathcal{O}^{\perp}}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^{\perp}}}(\pi \otimes \chi_{\mathbf{E}_0})) = \sum_{\mathscr{O},\mathscr{O}^{\perp}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^{\perp}], \quad \text{and} \tag{7.12}$$

$$\bigoplus_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^{\perp}}(\bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^{\perp}}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi))) \preceq \sum_{\mathscr{O},\mathscr{O}^{\perp}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^{\perp}]. \tag{7.13}$$

Here the summations run over all pairs $(\mathscr{O},\mathscr{O}^{\perp})$ such that $\mathscr{O}^{\perp}$ is a $\mathbf{K}_{\mathbf{V}^{\perp}}$-orbit in $\mathcal{O}^{\perp} \cap \mathfrak{p}_{\mathbf{V}^{\perp}}$ and $\mathscr{O} = \nabla_{\mathbf{V}'_i,\mathbf{V}}(\nabla_{\mathbf{V}^{\perp},\mathbf{V}'_i}(\mathscr{O}^{\perp}))$ for a unique $(-1,1)$-space $\mathbf{V}'_i$.

In view of **??**, Lemma 7.6, (7.12) and (7.13), we see that the equality holds in (7.13). Thus **??** holds.

8. The unipotent representations construction and iterated theta lifting

In this section, we will prove **??**. Recall that $\mathbf{V}$ is an $(\epsilon,\dot{\epsilon})$-space and $\mathcal{O} \in \mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{g})$. Since **??** is obvious when $\mathcal{O}$ is the zero orbit, we assume without loss of generality that $\mathcal{O}$ is not the zero orbit.

8.1. **The construction.** Suppose there is a $\mathbf{K}$-orbit $\mathscr{O} \in \mathcal{O} \cap \mathfrak{p}$. Define a sequence $(\mathbf{V}_0, \mathscr{O}_0), (\mathbf{V}_1, \mathscr{O}_1), \cdots, (\mathbf{V}_k, \mathscr{O}_k)$ $(k \geqslant 1)$ such that

- $\mathbf{V}_j$ is a nonzero $((-1)^j\epsilon, (-1)^j\dot{\epsilon})$-space for all $0 \leqslant j \leqslant k$;
- $(\mathbf{V}_0, \mathscr{O}_0) = (\mathbf{V}, \mathscr{O})$, and $\mathscr{O}_j = \nabla_{\mathbf{V}_{j-1}, \mathbf{V}_j}(\mathscr{O}_{j-1}) \in \mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_j}}(\mathfrak{p}_{\mathbf{V}_j})$ for all $1 \leqslant j \leqslant k$;
- $\mathscr{O}_k$ is the zero orbit.

Let $\mathcal{O}_j$ denote the nilpotent $G_{\mathbf{V}_j}$-orbit containing $\mathscr{O}_j$ $(0 \leqslant j \leqslant k)$. To ease the notation, we also let $\mathbf{V}_{k+1}$ be the zero $((-1)^{k+1}\epsilon, (-1)^{k+1}\dot{\epsilon})$-space and let $\mathscr{O}_{k+1} \in \mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_{k+1}}}(\mathfrak{p}_{\mathbf{V}_{k+1}})$ and $\mathcal{O}_{k+1} \in \mathrm{Nil}_{\mathbf{G}_{\mathbf{V}_{k+1}}}(\mathfrak{g}_{\mathbf{V}_{k+1}})$ be $\{0\}$.

Let

$$\eta = \chi_0 \boxtimes \chi_1 \boxtimes \cdots \boxtimes \chi_k \tag{8.1}$$

be a character of $G_{\mathbf{V}_0} \times G_{\mathbf{V}_1} \times \cdots \times G_{\mathbf{V}_k}$. For $0 \leqslant j \leqslant k$, put

$$\eta_j := \chi_j \boxtimes \chi_{j+1} \boxtimes \cdots \boxtimes \chi_k.$$

For $0 \leqslant j < k$, write

$$\omega_{\mathscr{O}_j} := \omega_{\mathbf{V}_j, \mathbf{V}_{j+1}} \widehat{\otimes} \omega_{\mathbf{V}_{j+1}, \mathbf{V}_{j+2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{\mathbf{V}_{k-1}, \mathbf{V}_k},$$

and one checks that the integrals in

$$\begin{aligned} (\omega_{\mathscr{O}_j} \otimes \eta_j) \times (\omega^\vee_{\mathscr{O}_j} \otimes \eta_j^{-1}) &\longrightarrow \mathbb{C}, \\ (u, v) &\longmapsto \int_{\widetilde{G}_{\mathbf{V}_{j+1}} \times \widetilde{G}_{\mathbf{V}_{j+2}} \times \cdots \times \widetilde{G}_{\mathbf{V}_k}} \langle g \cdot u, v \rangle \, \mathrm{d}g, \end{aligned} \tag{8.2}$$

are absolutely convergent and define a continuous bilinear map using **??**. Define

$$\pi_{\mathscr{O}_j, \eta_j} := \frac{\omega_{\mathscr{O}_j} \otimes \eta_j}{\text{the left kernel of (8.2)}},$$

which is a Casselman-Wallach representation of $\widetilde{G}_{\mathbf{V}_j}$, as in (5.15). Set $\pi_{\mathscr{O}_k, \eta_k} := \chi_k$ by convention.

For $0 \leqslant j \leqslant k$, let $\chi_i|_{\widetilde{\mathbf{K}}_{\mathbf{V}_i}}$ denote the algebraic character whose restriction to $\widetilde{K}_{\mathbf{V}_i}$ equals the pullback of χ_i through the natural homomorphism $\widetilde{K}_{\mathbf{V}_i} \to G_{\mathbf{V}_i}$. Let $\mathcal{E}_{\mathscr{O}_k, \chi_k}$ denote the $\widetilde{\mathbf{K}}_{\mathbf{V}_k}$-equivariant algebraic line bundle on the zero orbit $\mathscr{O}_k$ corresponding to $\chi_k|_{\widetilde{\mathbf{K}}_{\mathbf{V}_k}}$. Inductively define

$$\mathcal{E}_{\mathscr{O}_j, \eta_j} := \chi_j|_{\widetilde{\mathbf{K}}_{\mathbf{V}_j}} \otimes \vartheta_{\mathscr{O}_{j+1}, \mathscr{O}_j}(\mathcal{E}_{\mathscr{O}_{j+1}, \eta_{j+1}}), \quad 0 \leqslant j < k.$$

This is an admissible orbit datum over $\mathscr{O}_i$ by **??**.

Theorem 8.1. *For each $0 \leqslant j \leqslant k$, $\pi_{\mathscr{O}_j, \eta_j}$ is an irreducible, unitarizable, $\mathcal{O}_j$-unipotent representation whose associated character*

$$\mathrm{Ch}_{\mathcal{O}_j}(\pi_{\mathscr{O}_j, \eta_j}) = \mathcal{E}_{\mathscr{O}_j, \eta_j}. \tag{8.3}$$

Moreover,

$$\bar{\Theta}_{\mathbf{V}_{j+1}, \mathbf{V}_j}(\pi_{\mathscr{O}_{j+1}, \eta_{j+1}}) \otimes \chi_j = \pi_{\mathscr{O}_j, \eta_j} \quad \textit{for all } 0 \leqslant j < k-1. \tag{8.4}$$

Proof. Since $\mathcal{O} \in \mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{p})$, one verifies that

- $\dim^\circ \mathbf{V}_j > 0$ for $0 \leqslant j < k$,
- $\dim \mathbf{V}_{j+1} + \dim \mathbf{V}_{j-1} > 2 \dim^\circ \mathbf{V}_j$ for $1 \leqslant j \leqslant k$, and
- $\pi_{\mathscr{O}_j, \eta_j}$ is $\mathbb{P}$-genuine for $0 \leqslant j \leqslant k$.

The representation $\pi_{\mathcal{O}_k,\eta_k}$ clearly satisfies all claims in the theorem. For $j = k-1$, $\pi_{\mathscr{O}_{k-1},\eta_{k-1}}$ is the twist by χ_{k-1} of the theta lift of character χ_k in the stable range. It is known that the statement of Theorem 8.1 holds in this case (*cf.* [50, Section 2] and [53, Section 1.8]).

We now prove the theorem by induction. Assume the theorem holds for $j+1$ with $1 \leqslant j+1 \leqslant k-1$. Applying **??** and **??**, we see that $(\pi_{\mathscr{O}_{j+1},\eta_{j+1}}, \mathbf{V}_j)$ is in the convergent range. Thus (8.4) holds by Fubini's theorem. By [70, Theorem 1.19], $\mathrm{U}(\mathfrak{g}_{\mathbf{V}_j})^{\mathbf{G}_{\mathbf{V}_j}}$ acts on $\pi_{\mathscr{O}_j,\eta_j}$ through the character $\lambda_{\mathcal{O}_j}$. By **??**, $\pi_{\mathscr{O}_j,\eta_j}$ is $\mathcal{O}_j$-bounded and (8.3) holds. The unitarity and irreducibility of $\pi_{\mathscr{O}_j,\eta_j}$ follows from **??**.

By **??**, $\pi_{\mathscr{O}_j,\eta_j}$ is thus $\mathcal{O}_j$-unipotent. This finishes the proof of the theorem. □

9. Concluding remarks

We record the following result, which is a form of automatic continuity.

Proposition 9.1. *Retain the setting in Section 8.1. The representation $\pi_{\mathscr{O},\eta}$ is isomorphic to*

$$(\omega_{\mathbf{V}_0,\mathbf{V}_1} \widehat{\otimes} \omega_{\mathbf{V}_1,\mathbf{V}_2} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{\mathbf{V}_{k-1},\mathbf{V}_k} \otimes \eta)_{\tilde{G}_{\mathbf{V}_1} \times \tilde{G}_{\mathbf{V}_2} \times \cdots \times \tilde{G}_{\mathbf{V}_k}}, \tag{9.1}$$

and its underlying Harish-Chandra module is isomorphic to

$$(\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1} \otimes \mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2} \otimes \cdots \otimes \mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k} \otimes \eta)_{(\mathfrak{g}_{\mathbf{V}_1} \times \mathfrak{g}_{\mathbf{V}_2} \times \cdots \times \mathfrak{g}_{\mathbf{V}_k}, \tilde{\mathbf{K}}_{\mathbf{V}_1} \times \tilde{\mathbf{K}}_{\mathbf{V}_2} \times \cdots \times \tilde{\mathbf{K}}_{\mathbf{V}_k})}. \tag{9.2}$$

Proof. Note that the representation $\pi_{\mathscr{O},\eta}$ is a quotient of the representation (9.1), and the underlying Harish-Chandra module of (9.1) is a quotient of (9.2). Thus it suffices to show that (9.2) is irreducible. We prove by induction on k. Assume that $k \geqslant 1$ and

$$\pi_1 := (\mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2} \otimes \cdots \otimes \mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k} \otimes \eta_1)_{(\mathfrak{g}_{\mathbf{V}_2} \times \cdots \times \mathfrak{g}_{\mathbf{V}_k}, \tilde{\mathbf{K}}_{\mathbf{V}_2} \times \cdots \times \tilde{\mathbf{K}}_{\mathbf{V}_k})}$$

is irreducible. Then π_1 is isomorphic to the Harish-Chandra module of $\pi_{\mathscr{O}_1,\eta_1}$. The representation (9.2) is isomorphic to

$$\pi_0 := \chi_0 \otimes (\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1} \otimes \pi_1)_{(\mathfrak{g}_{\mathbf{V}_1}, \tilde{\mathbf{K}}_{\mathbf{V}_1})}.$$

We know that $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ acts on $\pi_{\mathscr{O},\eta}$ through the character $\lambda_{\mathcal{O}}$ (*cf.* [70]), and by **??**, π_0 is $\mathcal{O}$-bounded. Thus every irreducible subquotient of π_0 is $\mathcal{O}$-unipotent. Theorem 6.17 implies that $\mathrm{AC}_{\mathcal{O}}(\pi_0)$ is bounded by $1 \cdot [\mathscr{O}]$. Now **??** implies that π_0 must be irreducible. □

Finally, we remark that the Whittaker cycles (attached to G-orbits in $\mathcal{O} \cap \mathfrak{ig}_{\mathbb{R}}$) of all unipotent representations in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ can be calculated by using [29, Theorem 1.1]. These agree with the associated cycles under the Kostant-Sekiguchi correspondence.

Appendix A. Combinatorial parameterization of special unipotent representations

A.1. **Parameterize of Unipotent representations.** We fix an abstract complex Cartan subgroup $\mathbf{H}_a$ and $\mathfrak{h}_a$ in $\mathbf{G}$ and a set of simple roots Π_a. Let $\mathcal{P}(\mathbf{G})$ be the set of all Langlands parameters of G-modules with character ρ (i.e. the infinitesimal character of the trivial representation). For $\gamma \in \mathcal{P}(\mathbf{G})$, let $\mathcal{L}(\gamma)$, $\mathcal{S}(\gamma)$ and Φ_γ be the corresponding Langlands quotient, standard module and coherent family such that $\Phi_\gamma(\rho) = \mathcal{L}(\gamma)$. Let $\mathcal{M}(\mathbf{G})$ be the span of $\mathcal{L}(\gamma)$. Let $\{\mathbb{B}\}$ be the set of all blocks. Then $\mathcal{P}(\mathbf{G}) = \bigsqcup_{\mathcal{B}} \mathcal{B}$. The Weyl group $W = W(G)$ acts on $\mathcal{M}(\mathbf{G})$ by coherent continuation. Let $\mathcal{M}_{\mathcal{B}}$ be the submodule of $\mathcal{M}(\mathbf{G})$ spand by $\gamma \in \mathcal{B}$, then

$$\mathcal{M}(\mathbf{G}) = \bigoplus_{\mathcal{B}} \mathcal{M}_{\mathcal{B}}$$

Let $\tau(\gamma) \subset \Pi_a$ be the τ-invariant of γ.

Let $\check{\mathcal{O}}$ be even orbit. $\lambda = \frac{1}{2}\check{h}$. Define

$$S(\lambda) = \{\, \alpha \in \Pi_a \mid \langle \alpha, \lambda \rangle = 0 \,\}.$$

Let $\mathcal{P}_\lambda(\mathbf{G})$ be the set of all Langlands parameters with infinitesimal character λ. Let $T_{\lambda,\rho}$ be the translation functor. Let

$$\mathcal{B}(S) = \{\, \gamma \in \mathcal{B} \mid S \cap \tau(\gamma) = \varnothing \,\}$$

and

$$\mathcal{P}(\mathbf{G}, S) = \bigsqcup_{\mathcal{B}} \mathcal{B}(S)$$

Then

$$\begin{aligned} \mathcal{P}(\mathbf{G}, S) &\longrightarrow \mathcal{P}_\lambda(\mathbf{G}) \\ \gamma &\longmapsto T_{\lambda,\rho}(\gamma) \end{aligned}$$

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Let

$$\mathcal{B}(S, \mathcal{O}) = \{\, \gamma \in \mathcal{B}(S) \mid \mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}} \,\}$$

Let

$$\begin{aligned} m_S(\sigma) &= [\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] \\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{M}_{\mathcal{B}}] \end{aligned}$$

Barbasch [11, Theorem 9.1] established the following theorem.

Theorem A.1.

$$|\mathcal{B}(S, \mathcal{O})| = \sum_{\sigma} m_{\mathcal{B}}(\sigma) m_S(\sigma)$$

Here $\sigma \times \sigma$ running over the $W \times W$ appears in the double cell $\mathcal{C}(\mathcal{O})$.

Proof. We need to take the graded module of $\mathcal{M}(\mathbf{G})$ with respect to the $\overset{LR}{\leqslant}$. By abuse of notation, we identify the basis $\mathcal{P}(\mathbf{G})$ with its image in the graded module. Note that $S \cap \tau(\lambda) = \varnothing$ if and only if $W(S)$ acts on γ trivially by [81, Lemma 14.7]. On the other hand, by [81, Theorem 14.10, and page 58], $\mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}}$ only if γ generate a W-module in the double cell of $\mathcal{O}$. □

Now assume $S = S(\lambda)$. By [16, Cor 5.30 b) and c)], $[\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] = [\mathbf{1}|_{W(S)} : \sigma] \leqslant 1$.

A.2. **Combinatorics of Weyl group representations.** The irreducible representations of S_n are parameterized by Young diagrams. We use the notation that the trivial representation corresponds to a row and the sign representation corresponds to a column.

$$triv \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|} \hline n & & b & o & x & e & s \\ \hline \end{array} \qquad \mathrm{sgn} \leftrightarrow \begin{array}{|c|} \hline n \\ \hline \\ \hline b \\ \hline o \\ \hline x \\ \hline e \\ \hline s \\ \hline \end{array}$$

[This notation coincide with the Springer correspondence, where a row of boxes represents the regular nilpotent orbit and the corresponding representation is the trivial representation. The trivial orbit yeilds the sign representation.]

Under the above paramterization, $\tau \mapsto \tau \otimes \mathrm{sgn}$ is described by the transpose of Young diagram. The branching rule is given by Littlewoods-Richardson. Let $\mathcal{K} = \bigoplus_n \mathrm{Groth}(S_n)$

be the graded ring of the Grothendieck groups of S_n. This yeilds well defined ring structure on the graded algebra

$$\mu\nu := \mathrm{Ind}_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu\otimes\nu.$$

Notation, we will use $[r_1, r_2, \cdots, r_k]$ or $(c_1, c_2, \cdots, c_k)$ denote the Young diagram, where $\{ r_i \}$ denote the lengthes of its rows and $\{ c_i \}$ denote the lengthes of its columns.

Let W_n denote the Weyl group of type BC: $W_n = S_n \ltimes \{ \pm 1 \}^n$. Now $\mathrm{Irr}(W_n)$ is paramterized by a pair of Young diagram, we use the notation $\mu\times\nu$.

As the tensor algebra, $\bigoplus_n \mathrm{Groth}(W_n) \cong \mathcal{K}\otimes\mathcal{K}$ via $\mu\times\nu \mapsto \mu\otimes\nu$.

The branching rule is given by: $(\mu_1\times\nu_1)(\mu_2\times\nu_2) = (\mu_1\mu_2)\times(\nu_1\nu_2)$. The sgn repersentation of W_n is parameterized by $\varnothing\times(n)$. From the definitioin of the identification of bipartition with $\mathrm{Irr}(W_n)$, we also have

$$(\mu\times\nu)\otimes\mathrm{sgn} = \nu^t\times\mu^t.$$

We also will use the following branching formula:

$$\mathrm{Ind}_{S_n}^{W_n}\mathrm{triv} = \sum_{\substack{a,b,\\ a+b=n}} [a]\times[b],$$

and

$$\mathrm{Ind}_{S_n}^{W_n}\mathrm{sgn} = \sum_{\substack{a,b,\\ a+b=n}} (a)\times(b). \tag{A.1}$$

[Sketch of the proof of the first formula, the dimension of LHS is $n!2^n/n! = 2^n$. On the other hand, by Mackey theory, $[LHS : \mathrm{Ind}_{W_a\times W_b}^{W_n}\mathrm{triv}\otimes\chi] \geqslant 1$ since $S_n\cap W_a\times W_b = S_a\times S_b$. Therefore, $LHS \supset RHS$. On the other hand, dimension of RHS is $\sum_a n!/a!b! = 2^n$. This finished the proof. The proof of the second formula is similar. One also can obtain it by tensoring with sgn of the first formula.]

A.2.1. *Coherent continuation representations.* TBA

Appendix B. A few geometric facts

B.1. **Geometry of moment maps.** In this section, let $(\mathbf{V}, \mathbf{V}')$ be a rational dual pair. We use the notation of **??**. Write $\breve{\mathfrak{p}} := \mathfrak{p}\oplus\mathfrak{p}'$ and

$$\breve{M} := M\times M' \colon \mathcal{X} \longrightarrow \breve{\mathfrak{p}} = \mathfrak{p}\times\mathfrak{p}'.$$

When there is a morphism $f\colon A\to B$ between smooth algebraic varieties, let $\mathrm{d}f_a\colon \mathrm{T}_aA \to \mathrm{T}_{f(a)}B$ denote the tangent map at a closed point $a\in A$, where T_aA and $\mathrm{T}_{f(a)}B$ are the tangent spaces. For a sub-scheme S of B, let $A\times_f S$ denote the scheme theoretical inverse image of S under f, which is a subscheme of A. Along the proof, we will use the Jacobian criterion for regularity in various places (see [52, Theorem 2.19]).

B.1.1. *Scheme theoretical results on descents.*

Lemma B.1 (*cf.* [65, Lemma 13 and Lemma 14])**.** *Suppose $\mathscr{O}'\in\mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the descent of $\mathscr{O}\in\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. Fix $X\in\mathscr{O}$ and let*

$$\mathcal{Z}_X := \mathcal{X}\times_M \{ X \}$$

be the scheme theoretical inverse image of X under the moment map $M:\mathcal{X}\to\mathfrak{p}$. Then

(i) $\mathcal{Z}_X$ is smooth (and hence reduced);
(ii) $\mathcal{Z}_X$ is a single free $\mathbf{K}'$-orbit in $\mathcal{X}^\circ$;
(iii) $M'(\mathcal{Z}_X) = \mathscr{O}'$ and $\mathcal{Z}_X = \mathcal{X}\times_{\breve{M}} (\{ X \}\times\overline{\mathscr{O}'})$.

Proof. Fix a point $w \in \mathcal{X}^\circ$ such that $M(w) = X$. It is elementary to see that

$$M^{-1}(X) = \mathbf{K}' \cdot w$$

is a single $\mathbf{K}'$-orbit. This orbit is clearly Zariski closed and the stabilizer $\mathrm{Stab}_{\mathbf{K}'}(w)$ of w under the $\mathbf{K}'$-action is trivial. This proves part (ii) and the first assertion of part (iii).

We identify the tangent space $\mathrm{T}_w\mathcal{X}$ with $\mathcal{X}$. Let $N \subset \mathcal{X}$ be a complement of $\mathrm{T}_w(\mathbf{K}' \cdot w)$ in $\mathrm{T}_w\mathcal{X}$. By [69, Section 6.3], there is an affine open subvariety N° of N containing 0 such that the diagram

$$\begin{tikzcd} \mathbf{K}' \times N^\circ \arrow[r, "(k',n)\mapsto k'\cdot(w+n)"] \arrow[d, "\text{projection}"'] & \mathcal{X} \arrow[d, "M"] \\ N^\circ \arrow[r, "n\mapsto M(w+n)"] & M(\mathcal{X}) \end{tikzcd}$$

is Cartesian and the horizontal morphisms are étale. Therefore the scheme $\mathcal{X} \times_M \{X\}$ is a smooth algebraic variety. This proves part (i), and the second assertion of (iii) then easily follows. □

Remark. Retain the setting of Lemma B.1. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from X to $X' \in \mathfrak{p}'$. Let $\check{X} := (X, X') \in \mathfrak{p} \oplus \mathfrak{p}'$ and $\mathbf{K}'_{X'}$ be the stabilizer of X' under the $\mathbf{K}'$-action. By the $\mathbf{K}'$-equivariance, the map $\mathcal{Z}_X \to \mathscr{O}'$ is a smooth morphism between smooth schemes. Hence the scheme theoretical fibre

$$\mathcal{Z}_{\check{X}} := \mathcal{X} \times_{\check{M}} \{\check{X}\} = \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{X'\}$$

is smooth and equals the orbit $\mathbf{K}'_{X'} \cdot w$. Consequently, we have an exact sequence by the Jacobian criterion for regularity:

$$0 \longrightarrow \mathrm{T}_w\mathcal{Z}_{\check{X}} \longrightarrow \mathrm{T}_w\mathcal{X} \xrightarrow{\mathrm{d}\check{M}_w} \mathrm{T}_X\mathfrak{p} \oplus \mathrm{T}_{X'}\mathfrak{p}'. \tag{B.1}$$

Lemma B.2. *Retain the setup in Lemma B.1. Let*

$$U := \mathfrak{p} \backslash ((\overline{\mathcal{O}} \cap \mathfrak{p}) \backslash \mathscr{O})$$

and $\mathcal{Z}_{U,\overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$. *Then*

(i) U is a Zariski open subset of $\mathfrak{p}$ such that $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$;
(ii) $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is smooth and it is a single $\mathbf{K} \times \mathbf{K}'$-orbit.

Proof. Part (i) is clear by **??**. It is also clear that the underlying set of $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit by Lemma B.1 (iii).

Note that $M'|_{\mathcal{X}^\circ}$ is smooth because $\mathrm{d}M'_w \colon \mathrm{T}_w\mathcal{X}^\circ \longrightarrow \mathrm{T}_{M'(w)}\mathfrak{p}'$ is surjective for each $w \in \mathcal{X}^\circ$ (*cf.* [33, Proposition 10.4]). Since smooth morphisms are stable under base changes and compositions, the scheme $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$ is a smooth algebraic variety. Also note that $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is an open subscheme of $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$. Thus it is smooth. □

Remark. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from $X \in \mathscr{O}$ to $X' \in \mathscr{O}' \subset \mathfrak{p}'$. By the proof of Lemma B.2,

$$\mathcal{Z}_{U,X'} := \mathcal{X} \times_{\check{M}} (U \times \{X'\}) = \mathbf{K}\mathbf{K}'_{X'} \cdot w$$

is smooth. Similar to the remark of Lemma B.1, this yields an exact sequence:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A,A')\mapsto A'w - wA]{} \mathcal{X} \xrightarrow[b\mapsto bw^\star + wb^\star]{\mathrm{d}M'_w} \mathfrak{p}'. \tag{B.2}$$

B.1.2. *Scheme theoretical results on generalized descents of good orbits.*

Lemma B.3. *Suppose $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is good for generalized descent (see* **??***) and $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Fix $X \in \mathscr{O}$ and let $\mathcal{Z}_X := \mathcal{X} \times_{\breve{M}} (\{ X \} \times \overline{\mathscr{O}'})$. Then*

(i) $\mathcal{Z}_X$ is smooth;
(ii) $\mathcal{Z}_X$ is a single $\mathbf{K}'$-orbit;
(iii) $\mathcal{Z}_X$ is contained in $\mathcal{X}^{\mathrm{gen}}$ and $M'(\mathcal{Z}_X) = \mathscr{O}'$.

Proof. Part (ii) follows from [26, Table 4]. Part (iii) follows from part (ii). By the generic smoothness, in order to prove part (i), it suffices to show that $\mathcal{Z}_X$ is reduced. Note that the $\mathbf{K}'$-equivariant morphism $\mathcal{Z}_X \xrightarrow{M'} \mathscr{O}'$ is flat by generic flatness [30, Théorème 6.9.1]. By [31, Proposition 11.3.13], to show that $\mathcal{Z}_X$ is reduced, it suffices to show that

$$\mathcal{Z}_{\breve{X}} := \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{ X' \} = \mathcal{X} \times_{\breve{M}} \{ \breve{X} \}$$

is reduced, where $X' \in \mathscr{O}'$ and $\breve{X} := (X, X')$. Let $w \in \mathcal{Z}_{\breve{X}}$ be a closed point. Then $\mathbf{K}' \cdot w$ is the underlying set of $\mathcal{Z}_X$, and $Z_{\breve{X}} := \mathbf{K}'_{X'} \cdot w$ is the underlying set of $\mathcal{Z}_{\breve{X}}$. By the Jacobian criterion for regularity, to complete the proof of the lemma, it remains to prove the following claim.

Claim. *The following sequence is exact:*

$$0 \longrightarrow \mathrm{T}_w Z_{\breve{X}} \longrightarrow \mathrm{T}_w \mathcal{X} \xrightarrow{\mathrm{d}\breve{M}} \mathrm{T}_X \mathfrak{p} \oplus \mathrm{T}_{X'} \mathfrak{p}'.$$

We prove the above claim in what follows. Identify $\mathrm{T}_w \mathcal{X}$, $\mathrm{T}_X \mathfrak{p}$ and $\mathrm{T}_{X'} \mathfrak{p}'$ with $\mathcal{X}, \mathfrak{p}$ and $\mathfrak{p}'$ respectively and view $\mathrm{T}_w Z_{\breve{X}}$ as a quotient of $\mathfrak{k}'_{X'}$. It suffices to show that the following sequence is exact:

$$\mathfrak{k}'_{X'} \xrightarrow[A' \mapsto A'w]{} \mathcal{X} \xrightarrow[b \mapsto (b^\star w + w^\star b, bw^\star + wb^\star)]{\mathrm{d}\breve{M}_w} \mathfrak{p} \oplus \mathfrak{p}' = \breve{\mathfrak{p}}. \tag{B.3}$$

In fact, we will show that the following sequence

$$\mathfrak{g}'_{X'} \longrightarrow \mathbf{W} \longrightarrow \mathfrak{g} \oplus \mathfrak{g}' \tag{B.4}$$

is exact, where $\mathfrak{g}'_{X'}$ is the Lie algebra of the stabilizer group $\mathrm{Stab}_{\mathbf{G}'}(X')$ and the arrows are defined by the same formulas in (B.3). Then the exactness of (B.3) will follow since (B.4) is compatible with the natural (L, L')-actions.

We now prove the exactness of (B.4). Let $\mathbf{V}'_1 := w\,\mathbf{V}$ and $\mathbf{V}'_2$ be the orthogonal complement of $\mathbf{V}'_1$ so that

$$\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2.$$

Let $\mathbf{W}_i := \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_i)$ and $\mathfrak{g}'_i := \mathfrak{g}_{\mathbf{V}'_i}$ for $i = 1, 2$. Let $*\colon \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \to \mathrm{Hom}(\mathbf{V}'_2, \mathbf{V}'_1)$ denote the adjoint map. We have $\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2$ and $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxslash}$, where

$$\mathfrak{g}^{\boxslash} := \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \right\}.$$

Now w and X' are naturally identified with an element w_1 in $\mathbf{W}_1$ and an element X'_1 in $\mathfrak{g}'_1$, respectively. We have $\mathfrak{g}'_{X'} = \mathfrak{g}'_{1,X'_1} \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxslash}_{X'_1}$, where

$$\mathfrak{g}^{\boxslash}_{X'_1} = \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2),\ EX'_1 = 0 \right\}.$$

Now maps in (B.4) have the following forms respectively:

$$\mathfrak{g}'_{X'} \ni A' = (A'_1, A'_2, \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix}) \mapsto A'w = (A'_1 w_1, E w_1) \in \mathbf{W} \qquad \text{and}$$

$$\mathbf{W} \ni (b_1, b_2) \mapsto \left(b_1^\star w_1 + w_1^\star b_1, \begin{pmatrix} b_1 w_1^\star + w_1 b_1^\star & w_1 b_2^\star \\ b_2 w_1^\star & 0 \end{pmatrix} \right) \in \mathfrak{g} \oplus \mathfrak{g}'.$$

Applying (B.1) to the complex dual pair $(\mathbf{V}, \mathbf{V}'_1)$, we see that the sequence

$$\mathfrak{g}'_{1,X'_1} \xrightarrow{A'_1 \mapsto A'_1 w_1} \mathbf{W}_1 \xrightarrow{b_1 \mapsto (b_1^\star w_1 + w_1^\star b_1, b_1 w_1^\star + w_1 b_1^\star)} \mathfrak{g} \oplus \mathfrak{g}'_1$$

is exact. The task is then reduced to show that the following sequence is exact:

$$\mathfrak{g}^{\boxslash}_{X'_1} \xrightarrow[①]{\begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \mapsto E w_1} \mathbf{W}_2 \xrightarrow[②]{b_2 \mapsto b_2 w_1^\star} \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2). \tag{B.5}$$

Note that w_1 is a surjection, hence ① is an injection. We have

$$\dim \mathfrak{g}^{\boxslash}_{X'_1} = \dim \mathbf{V}'_2 \cdot \dim \mathrm{Ker}(X'_1)$$

and

$$\dim \mathrm{Ker}(②) = \dim \mathbf{V}'_2 \cdot (\dim \mathbf{V} - \dim \mathrm{Im}(w_1)) = \dim \mathbf{V}'_2 \cdot (\dim \mathbf{V} - \dim \mathbf{V}'_1).$$

Note that $\dim \mathrm{Ker}(X'_1) = c_1$, and $\dim \mathbf{V} - \dim \mathbf{V}'_1 = c_0$. Since we are in the setting of good generalized descent (**??**), we have $c_0 = c_1$. Now the exactness of (B.5) follows by dimension counting. □

Remark. Suppose $\mathscr{O}$ is not good for generalized descent. Then the underlying set of $\mathcal{Z}_X$ may not be a single $\mathbf{K}'$-orbit. Even if it is a single orbit, the scheme $\mathcal{Z}_X$ may not be reduced.

Lemma B.4. *Retain the notation in Lemma B.3. Let*

$$U := \mathfrak{p} \backslash ((\overline{\mathcal{O}} \cap \mathfrak{p}) \backslash \mathscr{O})$$

and $\mathcal{Z}_{U,\overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$. *Then*

(i) U is a Zariski open subset of $\mathfrak{p}$ such that $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$;
(ii) $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is smooth and it is a single $\mathbf{K} \times \mathbf{K}'$-orbit.

Proof. Part (i) is follows from **??**. By Lemma B.3 (ii), the underlying set of $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit whose image under M' is contained in $\mathscr{O}'$. By generic flatness [30, Théorème 6.9.1], the morphism $\mathcal{Z}_{U,\overline{\mathscr{O}'}} \longrightarrow \mathscr{O}'$ is flat. To prove the scheme theoretical claim in part (ii), it suffices to show that

$$\mathcal{Z}_{U,X'} := \mathcal{X} \times_{\check{M}} (U \times \{ X' \})$$

is reduced, where $X' \in \mathscr{O}'$.

Let $w \in \mathcal{Z}_{U,X'}$ be a closed point. As in the proof of Lemma B.3, by the Jacobian criterion for regularity, it suffices to show that the following sequence is the exact:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A,A') \mapsto A'w - wA]{} \mathcal{X} \xrightarrow[b \mapsto bw^\star + wb^\star]{\mathrm{d}M'_w} \mathfrak{p}'.$$

The proof of the exactness follows the same line as the proof of Lemma B.3 utilizing (B.5) and the established exact sequence (B.2) in the descent case. We leave the details to the reader. □

B.2. **Proof of Lemma 7.5.** We will use notations of Section **??** and Section 7.0.1. Let Hom_J denote the space of homomorphisms commuting with the $\dot{\epsilon}$-real from J, and $\mathfrak{n}_{\mathbb{R}}$ denote the Lie algebra of $N_{\mathbf{E}_0}$.

We first explicitly construct some elements in the induced orbits in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}}$. Fix an element $\mathbf{i}\,X \in \mathscr{O}_{\mathbb{R}}$ and a J-invariant complement $\mathbf{V}_+$ of $\mathrm{Im}(X)$ in the vector space $\mathbf{V}^- = \mathbf{V}$ so that

$$\mathbf{V}^- = \mathbf{V}_+ \oplus \mathrm{Im}(X).$$

Fix any J-invariant decompositions

$$\mathbf{E}_0 = \mathbf{L}_1 \oplus \mathbf{L}_0 \quad \text{and} \quad \mathbf{E}_0' = \mathbf{L}_1' \oplus \mathbf{L}_0'$$

such that $\dim \mathbf{L}_1 = \dim \mathbf{V}_+$, and $(\mathbf{L}_1 \oplus \mathbf{L}_1') \perp (\mathbf{L}_0 \oplus \mathbf{L}_0')$. (This is possible due to the dimension inequality in (7.1).)

Fix a linear isomorphism

$$S \in \mathrm{Hom}_J(\mathbf{L}_1',\, \mathbf{V}_+),$$

and view it as an element of $\mathrm{Hom}_J(\mathbf{E}_0',\, \mathbf{V}^-)$ via the aforementioned decompositions. Put

$$\mathscr{T} := \{\, T \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0) \mid T^* + T = 0 \,\} \quad \text{and}$$
$$\mathscr{T}^\circ := \{\, T \in \mathscr{T} \mid T \text{ has maximal possible rank} \,\}$$

to be viewed as subsets of $\mathrm{Hom}_J(\mathbf{E}_0', \mathbf{E}_0)$ as before, where $T^* \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0)$ is specified by requiring that

$$\langle T \cdot u, v\rangle_{\mathbf{V}^\perp} = \langle u, T^* v\rangle_{\mathbf{V}^\perp}, \quad \text{for all } u, v \in \mathbf{L}_0', \tag{B.6}$$

and $\langle\ ,\ \rangle_{\mathbf{V}^\perp}$ denotes the ϵ-symmetric bilinear form on $\mathbf{V}^\perp$.

Each $T \in \mathscr{T}$ defines a $(-\epsilon)$-symmetric bilinear form $\langle\ ,\ \rangle_T$ on $\mathbf{L}_0'$ where

$$\langle v_1, v_2\rangle_T := -\langle v_1, T v_2\rangle, \quad \text{for all } v_1, v_2 \in \mathbf{L}_0'.$$

Put

$$X_{S,T} := \begin{pmatrix} 0 & S^* & T \\ & X & S \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_{\mathbb{R}} \tag{B.7}$$

according to the decomposition $\mathbf{V}^\perp = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}_0'$, where $S^* \in \mathrm{Hom}_J(\mathbf{V}^-, \mathbf{E}_0)$ is similarly defined as in (B.6).

Now suppose that $T \in \mathscr{T}^\circ$.

In case (ii) of Lemma 7.5, the form $\langle\ ,\ \rangle_T$ must be non-degenerate and $((\mathbf{L}_0', \langle\ ,\ \rangle_T), J|_{\mathbf{L}_0'})$ becomes a $(-\epsilon, -\dot{\epsilon})$-space. Conversely, up to isomorphism, every $(-\epsilon, -\dot{\epsilon})$-space of dimension $l - c_1$ is isomorphic to $((\mathbf{L}_0', \langle\ ,\ \rangle_T), J|_{\mathbf{L}_0'})$ for some $T \in \mathscr{T}$.

Let s be the signature of $((\mathbf{L}_0', \langle\ ,\ \rangle_T), J|_{\mathbf{L}_0'})$ and let $G_T := \mathbf{G}^J_{\mathbf{L}_0'}$ denote the corresponding real group. Then $\mathbf{i}\,X_{S,T}$ generates an induced orbit of $\mathscr{O}_{\mathbb{R}}$ with signed Young diagram $[d_1 + s, \check{d}_1 + \check{s}, d_1, \cdots, d_k]$.[1] One checks that the reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i}\,X_{S,T})$ and $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\,X_{S,T})$ are both canonically isomorphic to $R \times G_T$, where R is the reductive quotient of $\mathrm{Stab}_G(X)$. Hence $C_{P_{\mathbf{E}_0}}(\mathbf{i}\,X_{S,T}) \cong C_{G_{\mathbf{V}^\perp}}(\mathbf{i}\,X_{S,T})$.

In case (i) of Lemma 7.5, $\langle\ ,\ \rangle_T$ is skew symmetric, and T has rank $\dim \mathbf{L}_0 - 1 = l - c_1 - 1$ since $l - c_1$ is odd. One checks that $\mathbf{i}\,X_{S,T}$ generates an induced orbit of $\mathscr{O}_{\mathbb{R}}$ with the signed Young diagram prescribed in (i) of Lemma 7.5.

Fix decompositions

$$\mathbf{L}_0 = \mathbf{L}_2 \oplus \mathbf{L}_3 \quad \text{and} \quad \mathbf{L}_0' = \mathbf{L}_2' \oplus \mathbf{L}_3'$$

[1]The signed Young diagram may be computed using the explicit description of Kostant-Sekiguchi correspondence in [26, Propositions 6.2 and 6.4].

which are dual to each other such that $\mathrm{Ker}(T) = \mathbf{L}_3'$ and $\langle\ ,\ \rangle_T|_{\mathbf{L}_2'\times\mathbf{L}_2'}$ is a non-degenerate skew symmetric bilinear form on $\mathbf{L}_2'$. Let $G_T := \mathbf{G}^J_{\mathbf{L}_2'}$. Then G_T is a real symplectic group. The reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i}X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathrm{GL}(\mathbf{L}_3)^J$ and the reductive quotient of $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i}X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathbf{G}^J_{\mathbf{L}_3\oplus\mathbf{L}_3'}$, where R is the reductive quotient of $\mathrm{Stab}_G(\mathbf{i}X)$ and $\mathbf{G}^J_{\mathbf{L}_3\oplus\mathbf{L}_3'} \cong \mathrm{O}(1,1)$. The homomorphism

$$C_{P_{\mathbf{E}_0}}(\mathbf{i}\,X_{S,T}) \longrightarrow C_{G_{\mathbf{V}^\perp}}(\mathbf{i}X_{S,T})$$

is therefore an injection whose image has index 2.

To finish the proof of the lemma, it suffices to show that

$$\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}} = \{\, G_{\mathbf{V}^\perp} \cdot \mathbf{i}X_{S,T} \mid T \in \mathscr{T}^\circ \,\}\,. \tag{B.8}$$

Consider the set

$$\mathfrak{A} := \left\{ \begin{pmatrix} 0 & B^* & C \\ & X & B \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_{\mathbb{R}} \;\middle|\; X \oplus B \in \mathrm{Hom}_J(\,\mathbf{V}^- \oplus \mathbf{E}_0',\, \mathbf{V}^-) \text{ is surjective} \right\}.$$

Clearly $P'_{\mathbf{E}_0} := \mathrm{GL}_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$ acts on $\mathfrak{A}$ (*cf.* **??**). By suitable matrix manipulations, one sees that every element in $\mathfrak{A}$ is conjugated to an element in $\{\, X_{S,T} \mid T \in \mathscr{T} \,\}$ under the $P'_{\mathbf{E}_0}$-action. Hence $P'_{\mathbf{E}_0} \cdot \{\, X_{S,T} \mid T \in \mathscr{T}^\circ \,\}$ is open dense in $\mathfrak{A}$. Now (B.8) follows since $P_{\mathbf{E}_0} \cdot \mathbf{i}\mathfrak{A}$ is open dense in $\mathscr{O}_{\mathbb{R}} + \mathbf{i}\mathfrak{n}_{\mathbb{R}}$, □

References

[1] J. Adams, *Discrete spectrum of the reductive dual pair* $(O(p,q), Sp(2m))$, Invent. Math. **74** (1983), no. 3, 449–475.

[2] J. Adams, B. Barbasch, and D. A. Vogan, *The Langlands classification and irreducible characters for real reductive groups*, Progress in Math., vol. 104, Birkhauser, 1991.

[3] Jeffrey Adams and Fokko du Cloux, *Algorithms for representation theory of real reductive groups*, Journal of the Institute of Mathematics of Jussieu **8** (2009), no. 2, 209-259, DOI 10.1017/S1474748008000352.

[4] J. Arthur, *On some problems suggested by the trace formula*, Lie group representations, II (College Park, Md.), Lecture Notes in Math. 1041 (1984), 1–49.

[5] ———, *Unipotent automorphic representations: conjectures*, Orbites unipotentes et représentations, II, Astérisque **171-172** (1989), 13–71.

[6] L. Auslander and B. Kostant, *Polarizations and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

[7] D. Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. **96** (1989), no. 1, 103–176.

[8] ———, *Unipotent representations for real reductive groups*, Proceedings of ICM (1990), Kyoto (2000), 769–777.

[9] D. Barbasch and David Vogan, *Weyl Group Representations and Nilpotent Orbits*, Representation Theory of Reductive Groups: Proceedings of the University of Utah Conference 1982, 1983, pp. 21–33.

[10] D. Barbasch, *Orbital integrals of nilpotent orbits*, The mathematical legacy of Harish-Chandra, Proc. Sympos. Pure Math. **68** (2000), 97–110.

[11] ———, *The unitary spherical spectrum for split classical groups*, J. Inst. Math. Jussieu **9** (2010), 265–356.

[12] ———, *Unipotent representations and the dual pair correspondence*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 47–85.

[13] D. Barbasch, J.-J. Ma, B.-Y. Sun, and C.-B. Zhu, *On the notion of metaplectic Barbasch-Vogan duality*, arXiv:2010.16089.

[14] ———, *Counting special unipotent representations: orthogonal and symplectic groups*, in preparation.

[15] D. Barbasch and D. A. Vogan, *Weyl group representations and nilpotent orbits*, in Representation theory of reductive groups (Park City, Utah, 1982), Progress in Math. **40** (1983), 21–33.

[16] _____, *Unipotent representations of complex semisimple groups*, Annals of Math. **121** (1985), no. 1, 41–110.

[17] R. Brylinski, *Dixmier algebras for classical complex nilpotent orbits via Kraft-Procesi models. I*, The orbit method in geometry and physics (Marseille, 2000). Progress in Math. **213** (2003), 49–67.

[18] W. Borho, *Recent advances in enveloping algebras of semisimple Lie-algebras*, Séminaire Bourbaki, Exp. No. 489 (1976/77), 1–18.

[19] R. W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993.

[20] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. **41** (1989), 385–438.

[21] Y. Chen and B. Sun, *Schwartz homologies of representations of almost linear Nash groups*, J. Funct. Anal. **280** (2021), 108817.

[22] F. Du Cloux, *Sur les représentations différentiables des groupes de Lie algébriques*, Ann. Sci. École Norm. Sup. **24** (1991), no. 3, 257–318.

[23] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, Van Nostrand Reinhold Co., 1993.

[24] M. Cowling, U. Haagerup, and R. Howe, *Almost L2 matrix coefficients*, J. Reine Angew. Math. **387** (1988), 97–110.

[25] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda, *Nilpotent orbits and complex dual pairs*, J. Algebra **190** (1997), no. 2, 518 - 539.

[26] _____, *Dual pairs and Kostant-Sekiguchi correspondence. II. Classification of nilpotent elements*, Central European J. Math. **3** (2005), 430–474.

[27] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) **102** (1978), 307–330.

[28] M. Duflo, *Théorie de Mackey pour les groupes de Lie algébriques*, Acta Math. **149** (1982), no. 3-4, 153–213.

[29] R. Gomez and C.-B. Zhu, *Local theta lifting of generalized Whittaker models associated to nilpotent orbits*, Geom. Funct. Anal. **24** (2014), no. 3, 796–853.

[30] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algb́rique IV: Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[31] _____, *Éléments de géométrie algb́rique IV: Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966).

[32] M. Harris, J.-S. Li, and B. Sun, *Theta correspondences for close unitary groups*, Arithmetic Geometry and Automorphic Forms, Adv. Lect. Math. (ALM) **19** (2011), 265–307.

[33] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag, 1983.

[34] H. He, *Unipotent representations and quantum induction*, arXiv:math/0210372 (2002).

[35] _____, *Unitary representations and theta correspondence for type I classical groups*, J. Funct. Anal. **199** (2003), no. 1, 92–121.

[36] J.-S. Huang and J.-S. Li, *Unipotent representations attached to spherical nilpotent orbits*, Amer. J. Math. **121** (1999), no. 3, 497–517.

[37] J.-S. Huang and C.-B. Zhu, *On certain small representations of indefinite orthogonal groups*, Represent. Theory **1** (1997), 190–206.

[38] R. Howe, *θ-series and invariant theory*, Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math, vol. 33, 1979, pp. 275-285.

[39] _____, *On a notion of rank for unitary representations of the classical groups*, Harmonic analysis and group representations, Liguori, Naples (1982), 223-331.

[40] _____, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

[41] _____, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Piatetski-Shapiro, I. et al. (eds.), The Schur lectures (1992). Ramat-Gan: Bar-Ilan University, Isr. Math. Conf. Proc. 8, (1995), 1-182.

[42] D. Jiang, B. Liu, and G. Savin, *Raising nilpotent orbits in wave-front sets*, Represent. Theory **20** (2016), 419–450.

[43] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.

[44] B. Kostant, *Quantization and unitary representations*, Lectures in Modern Analysis and Applications III, Lecture Notes in Math. **170** (1970), 87–208.

[45] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.

[46] S. S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.

[47] S. S. Kudla, *Some extensions of the Siegel-Weil formula*, In: Gan W., Kudla S., Tschinkel Y. (eds) Eisenstein Series and Applications. Progress in Mathematics, vol 258. Birkhäuser Boston (2008), 205–237.

[48] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence II*, Israel J. Math. **100** (1997), 29–59.

[49] ———, *Degenerate principal series of metaplectic groups and Howe correspondence*, D. Prasad at al. (eds.), Automorphic Representations and L-Functions, Tata Institute of Fundamental Research, India, (2013), 379–408.

[50] J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), no. 2, 237–255.

[51] ———, *Theta lifting for unitary representations with nonzero cohomology*, Duke Math. J. **61** (1990), no. 3, 913–937.

[52] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2006.

[53] H. Y. Loke and J. Ma, *Invariants and K-spectrums of local theta lifts*, Compositio Math. **151** (2015), 179–206.

[54] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, j. London Math. Soc. **19** (1979), 41–52.

[55] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272, DOI 10.1007/BF01388564. MR732546

[56] G. W. Mackey, *Unitary representations of group extentions*, Acta Math. **99** (1958), 265–311.

[57] W. M McGovern, *Cells of Harish-Chandra modules for real classical groups*, Amer. J. of Math. **120** (1998), 211–228.

[58] C. Mœglin, *Front d'onde des représentations des groupes classiques p-adiques*, Amer. J. Math. **118** (1996), 1313–1346.

[59] ———, *Paquets d'Arthur Spéciaux Unipotents aux Places Archimédiennes et Correspondance de Howe*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 469–502.

[60] C. Mœglin and D. Renard, *Sur les paquets d'Arthur des groupes classiques réels*, J. Eur. Math. Soc. **22** (2020), 1827–1892.

[61] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer, 1987.

[62] K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*, Astérisque **273** (2001), 1–163.

[63] K. Nishiyama, H. Ochiai, and C.-B. Zhu, *Theta lifting of nilpotent orbits for symmetric pairs*, Trans. Amer. Math. Soc. **358** (2006), 2713–2734.

[64] K. Nishiyama and C.-B. Zhu, *Theta lifting of unitary lowest weight modules and their associated cycles*, Duke Math. J. **125** (2004), 415–465.

[65] T. Ohta, *The closures of nilpotent orbits in the classical symmetric pairs and their singularities*, Tohoku Math. J. **43** (1991), no. 2, 161–211.

[66] ———, *Induction of nilpotent orbits for real reductive groups and associated varieties of standard representations*, Hiroshima Math. J. **29** (1999), no. 2, 347–360.

[67] ———, *Nilpotent orbits of $\mathbb{Z}_4$-graded Lie algebra and geometry of moment maps associated to the dual pair* $(\mathrm{U}(p,q),\mathrm{U}(r,s))$, Publ. RIMS **41** (2005), no. 3, 723–756.

[68] A. Paul and P. Trapa, *Some small unipotent representations of indefinite orthogonal groups and the theta correspondence*, University of Aarhus Publ. Series **48** (2007), 103–125.

[69] V. L. Popov and E. B. Vinberg, *Invariant Theory*, Algebraic Geometry IV: Linear Algebraic Groups, Invariant Theory, Encyclopedia of Mathematical Sciences, vol. 55, Springer, 1994.

[70] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloq. Math. **70** (1996), 93–102.

[71] ———, *Characters, dual pairs, and unitary representations*, Duke Math. J. **69** (1993), no. 3, 547–592.

[72] S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.

[73] D. Renard and P. Trapa, *Irreducible genuine characters of the metaplectic group: Kazhdan-Lusztig algorithm and Vogan duality*, Represent. Theory **4** (2000), 245–295, DOI 10.1090/S1088-4165-00-00105-9.

[74] S. Sahi, *Explicit Hilbert spaces for certain unipotent representations*, Invent. Math. **110** (1992), no. 2, 409–418.

[75] J. Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138.
[76] W. Schmid and K. Vilonen, *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Annals of Math. **151** (2000), no. 3, 1071–1118.
[77] E. Sommers, *Lusztig's canonical quotient and generalized duality*, J. Algebra **243** (2001), no. 2, 790–812.
[78] T. A. Springer and R. Steinberg, *Seminar on algebraic groups and related finite groups; Conjugate classes*, Lecture Notes in Math., vol. 131, Springer, 1970.
[79] B. Sun and C.-B. Zhu, *A general form of Gelfand-Kazhdan criterion*, Manuscripta Math. **136** (2011), 185–197.
[80] P. Trapa, *Special unipotent representations and the Howe correspondence*, University of Aarhus Publication Series **47** (2004), 210–230.
[81] D. A. Vogan, *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. **49** (1982), no. 4, 943–1073. MR683010
[82] ———, *Unitary representations of reductive Lie groups*, Ann. of Math. Stud., vol. 118, Princeton University Press, 1987.
[83] ———, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups, Proc. Conf., Brunswick/ME (USA) 1989, Prog. Math. **101** (1991), 315–388.
[84] ———, *The method of coadjoint orbits for real reductive groups*, Representation theory of Lie groups (Park City, UT, 1998). IAS/Park City Math. Ser. **8** (2000), 179–238.
[85] ———, *Unitary representations of reductive Lie groups*, Mathematics towards the Third Millennium (Rome, 1999). Accademia Nazionale dei Lincei, (2000), 147–167.
[86] N. R. Wallach, *Real reductive groups I*, Academic Press Inc., 1988.
[87] ———, *Real reductive groups II*, Academic Press Inc., 1992.
[88] A. Weil, *Sur certain group d'operateurs unitaires*, Acta Math. **111** (1964), 143–211.
[89] H. Weyl, *The classical groups: their invariants and representations*, Princeton University Press, 1947.
[90] S. Yamana, *Degenerate principal series representations for quaternionic unitary groups*, Israel J. Math. **185** (2011), 77–124.
[91] C.-B. Zhu, *Local theta correspondence and nilpotent invariants*, Proceedings of Symposia in Pure Mathematics **101** (2019), 427–450.

the Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, New York 14853
Email address: dmb14@cornell.edu

School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China
Email address: hoxide@sjtu.edu.cn

Institute for Advanced Study in Mathematics, Zhejiang University, Hangzhou, 310058, China
Email address: sunbinyong@zju.edu.cn

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
Email address: matzhucb@nus.edu.sg