

Special unipotent representations
of real classical groups
and theta correspondence

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Classical groups and special unipotent representations

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D_n	$O(p, 2n - p)$	$O(2n, \mathbb{C})$	$O(2n, \mathbb{C})$	D_n
C_n	$Sp(2n, \mathbb{R})$	$Sp(2n, \mathbb{C})$	$SO(2n + 1, \mathbb{C})$	B_n
B_n	$O(p, 2n + 1 - p)$	$O(2n + 1, \mathbb{C})$	$Sp(2n, \mathbb{C})$	C_n
\tilde{C}_n	$Mp(2n, \mathbb{R})$	$Sp(2n, \mathbb{C})$	$Sp(2n, \mathbb{C})$	C_n
D_n	$O^*(n)$	$SO(2n, \mathbb{C})$	$SO(2n, \mathbb{C})$	D_n
C_n	$Sp(p, n - p)$	$Sp(2n, \mathbb{C})$	$SO(2n + 1, \mathbb{C})$	B_n
A_n	$U(p, n - p)$	$GL(n, \mathbb{C})$	$GL(n, \mathbb{C})$	A_n
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 All *special unipotent representations* of G are *unitarizable*.

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- *Atlas of Lie group:* \rightsquigarrow complete answer for exceptional groups.

Example of Atlas' computation $G = \mathrm{Sp}(2, 2)$

```
atlas> set G = Sp(2,2)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> void: for v in up do for r in v do print(r,infinitesimal_character(r)) od od
(final parameter(x=12,lambda=[1,0,1,0]/1,nu=[0,0,0,0]/1),[ 1, 1, 1, 1 ]/2)
(final parameter(x=25,lambda=[2,1,0,-1]/1,nu=[1,0,0,-1]/2),[ 2, 1, 1, 0 ]/2)
(final parameter(x=36,lambda=[3,3,2,2]/1,nu=[1,1,1,1]/1),[ 3, 3, 1, 1 ]/2)
(final parameter(x=39,lambda=[4,3,1,0]/1,nu=[3,3,0,0]/2),[ 4, 2, 1, 1 ]/2)
(final parameter(x=35,lambda=[4,2,3,-1]/1,nu=[3,1,3,-1]/2),[ 2, 1, 1, 0 ]/1)
(final parameter(x=39,lambda=[4,3,1,0]/1,nu=[5,5,1,-1]/2),[ 3, 2, 1, 0 ]/1)
(final parameter(x=41,lambda=[4,3,2,1]/1,nu=[7,7,3,3]/2),[ 4, 3, 2, 1 ]/1)
atlas>
```

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- **Lemma:**

$$\begin{aligned} & \# \{ \pi \in \text{Irr}_\mu(\mathfrak{g}, K)(G) \mid \text{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \} \\ &= \sum_{\substack{\tau \in \mathcal{D} \\ \mathcal{D} \rightsquigarrow \mathcal{O}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathcal{G}_\lambda(\mathfrak{g}, K)] \end{aligned}$$

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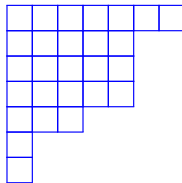
$$W(\mathrm{Sp}(2n)) = S_n \ltimes \{\pm 1\}^n,$$

$$\mathcal{G}_\rho(\mathrm{Sp}(2n, \mathbb{R})) = \sum_{\substack{p, q, t, s, \\ \sigma \in \widehat{S}_s}} \mathrm{Ind}_{S_t \times W_{2s} \times W_p \times W_q}^{W_n} \mathrm{sgn} \otimes (\sigma \times \sigma) \otimes \mathbf{1} \otimes \mathbf{1}.$$

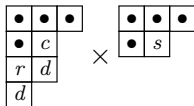
- $[\tau : 1_{W_{\lambda_{\check{\mathcal{O}}}}}] = 1.$
- $[\tau : \mathcal{G}_\rho(G)]$ is counted by painted bi-partitions $\mathrm{PBP}(\check{\mathcal{O}}).$

Example of PBP

$$\check{O} = [7, 5, 5, 5, 3, 1, 1] =$$



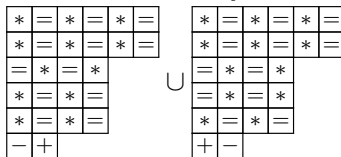
$\text{PBP}_{\check{O}}(\text{Sp}(2n, \mathbb{R}))$



⋮

\mapsto

Associated cycle



⋮

Nilpotent orbits with “good/bad parity”

- Bad parity (must occur with even multiplicity in $\check{\mathcal{O}}$):

$$\begin{cases} \text{even number,} & \text{when } \mathbf{G}^\vee \text{ is type } B \text{ or } D \\ \text{odd number,} & \text{when } \mathbf{G}^\vee \text{ is type } C \end{cases}$$

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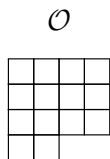
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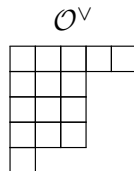
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- Example of good parity:



$\mathrm{Sp}(14, \mathbb{C})$



$\mathrm{SO}(15, \mathbb{C})$

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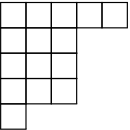
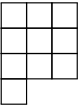


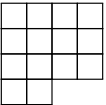
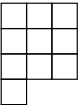


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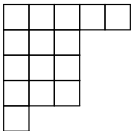
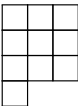


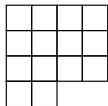
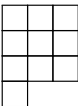


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- We assume $\check{\mathcal{O}}$ has **good parity** from now on.

Example of descent sequences

\mathbf{G}_i^\vee	$\mathrm{SO}(15, \mathbb{C})$	$\mathrm{O}(10, \mathbb{C})$	$\mathrm{SO}(7, \mathbb{C})$	$\mathrm{O}(4, \mathbb{C})$
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[Kraft-Procesi's](#) resolution of singularities of the closure of complex nilpotent orbits.

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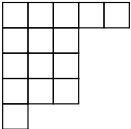
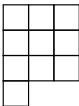


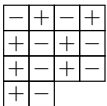
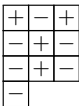


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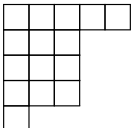
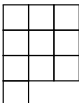


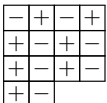
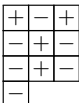


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Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

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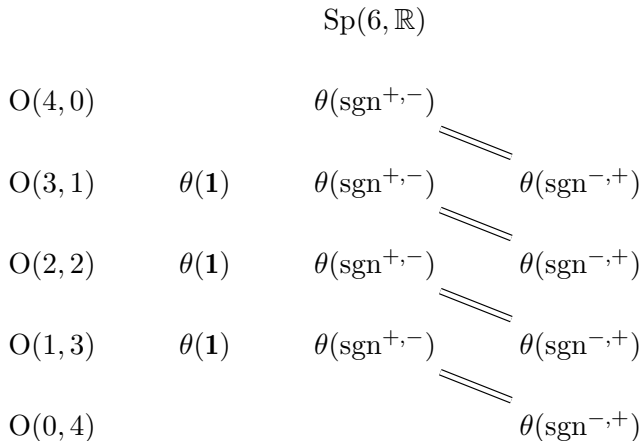
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Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.

$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.



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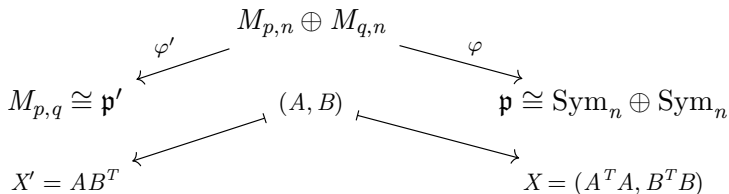
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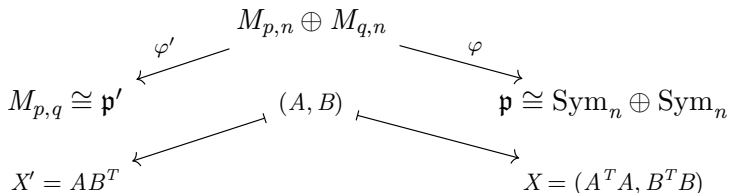
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such that

$$\mathrm{AC}(\Theta(\pi')) \preceq \vartheta^{\mathrm{geo}}(\mathrm{AC}(\pi')),$$

for any π' with $\mathrm{AV}(\pi') \subset \overline{\mathcal{O}'}$

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- Stable range lifting trick: Suppose $n > p + q$.

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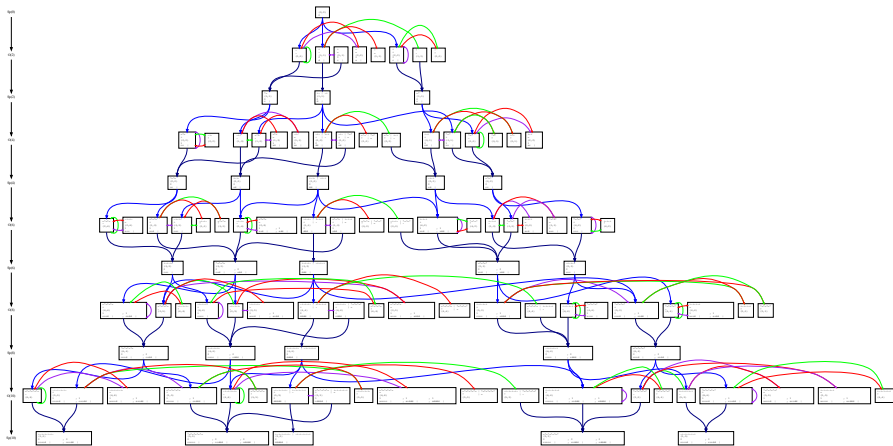
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- The **injectivity** of theta lifting is crucial!

Example: $\mathcal{O} =$ regular nilpotent orbit



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They are unitary by definition!

- **Unipotent Arthur parameter:** $\psi|_{\mathbb{C}^{\times}}$ is trivial.

Moeglin: $\pi_{\psi, \eta}$ is zero or multiplicity free ($\eta \in \mathrm{Irr}(\pi_1(Z_{\mathbf{G}^{\vee}}(\psi)))$).

Warning: $\Pi_{\psi}^A(G) \cap \Pi_{\psi'}^A(G) \neq \emptyset$ in general.

- “Corollary”:

$$\Pi_{\psi}^A(G) = \Pi_{\psi}^{ABV}(G)$$

- **Question:** How to describe $\pi_{\psi, \eta}$ explicitly?

Dan Barabasch, M. , Binyong Sun and Chen-Bo Zhu
Special unipotent representations: orthogonal and symplectic groups
ArXiv e-prints: <https://arxiv.org/abs/1712.05552v2>

Thank you for your attention!

