COUNTING SPECIAL UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS

DAN M. BARBASCH, JIA-JUN MA, BINYONG SUN, AND CHEN-BO ZHU

Contents

1.	Introduction	1
2.	Counting formula	2
3.	Primitive ideals and Weyl group representations	6
4.	Harish-Chandra cells	11
5.	Counting in type A	11
6.	Counting in type BCD	15
7.	Type C	16
8.	Type $\widetilde{C}$	23
9.	Type B	24
10.	Type D	27
Appendix A.	Combinatorics of Weyl group representations in the classical types	29
Appendix B.	Remarks on the Counting theorem of unipotent representations	30
Appendix C.	Matching specail shape and non-special shape painted bipartitions	40
References		40

1. Introduction

Barbasch-Vogan [12, 16] developed a formula counting the number of special unipotent representations.

In this paper, we will calculate the number of special unipotent representations for real classical groups explicitly.

1.1. **Counting theorem for the real classical groups of type BCD.** In this section, we assume $\star \in \{ B, \widetilde{C}, C, C^*, D, D^* \}$.

Let

$$G_{\mathbb{C}} = \begin{cases} \mathrm{SO}(2n+1,\mathbb{C}) & \text{if } \star \in \{ B \}, \\ \mathrm{Sp}(2n,\mathbb{C}) & \text{if } \star \in \{ \widetilde{C}, C, C^* \}, \\ \mathrm{SO}(2n,\mathbb{C}) & \text{if } \star \in \{ D, D^* \}. \end{cases}$$

and the dual group of $G_{\mathbb{C}}$ is defined to be

$$\check{G}_{\mathbb{C}} = \begin{cases} \mathrm{Sp}(2n,\mathbb{C}) & \text{if } \star \in \{ B, \widetilde{C} \}, \\ \mathrm{SO}(2n+1,\mathbb{C}) & \text{if } \star \in \{ C, C^* \}, \\ \mathrm{SO}(2n,\mathbb{C}) & \text{if } \star \in \{ D, D^* \}. \end{cases}$$

Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G}_{\mathbb{C}})$. Let $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$ be the decomposition of $\check{\mathcal{O}}$ into good and bad parity parts.

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Let $\mathfrak{I}_\star(n)$ be the set of real groups given by

$$\mathfrak{I}_\star(n) := \begin{cases} \{\, \mathrm{SO}(2p+1,2q) \mid p,q\in\mathbb{N} \text{ and } p+q=n \,\} & \text{if } \star = B\\ \{\, \mathrm{Sp}(2n,\mathbb{R}) \,\} & \text{if } \star = C\\ \{\, \mathrm{Mp}(2n,\mathbb{R}) \,\} & \text{if } \star = \widetilde{C}\\ \{\, \mathrm{Sp}(p,q) \mid p,q\in\mathbb{N} \text{ and } p+q=n \,\} & \text{if } \star = C^*\\ \{\, \mathrm{SO}(p,q) \mid p,q\in\mathbb{N} \text{ and } p+q=2n \,\} & \text{if } \star = D\\ \{\, \mathrm{O}^*(2n) \,\} & \text{if } \star = D^* \end{cases}$$

and $\mathfrak{I}_\star = \bigcup_{n\in\mathbb{N}} \mathfrak{I}_\star(n)$.

Let $\mathrm{unip}_\star(\check{\mathcal{O}})$ be the set of special unipotent representations of groups in $\mathfrak{I}_\star$ attached to $\check{\mathcal{O}}$.

Theorem 1.1. *Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Then*

$$\left|\mathrm{unip}_\star(\check{\mathcal{O}})\right| \leqslant \left|\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})\right|.$$

Theorem 1.2. *Let $\check{\mathcal{O}}'_b$ be the partition such that $\check{\mathcal{O}}'_b \overset{r}{\sqcup} \check{\mathcal{O}}'_b = \check{\mathcal{O}}_b$ and*

$$\star' = \begin{cases} A^{\mathbb{R}} & \star\in\{\, B,\widetilde{C},C,D \,\}\\ A^{\mathbb{H}} & \star\in\{\, C^*,D^* \,\} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \mathrm{Unip}_{\star'}(\check{\mathcal{O}}'_b)\times \mathrm{Unip}_\star(\check{\mathcal{O}}_g) & \longrightarrow & \mathrm{Unip}_\star(\check{\mathcal{O}})\\ (\pi',\pi_0) & \mapsto & \pi'\rtimes\pi_0. \end{array}$$

2. Counting formula

Let G be a real reductive group in the Harish-Chandra class. Here G can be a non-linear group.

Fixing a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, we identify the set of infinitesimal character with $\mathfrak{h}^*/W$ where W is the Weyl group acting on $\mathfrak{h}$.

Let $\mathcal{I}_\mu$ be the maximal primitive ideal with infinitesimal character $W\cdot\mu$. Then there is a unique double cell $\mathcal{D}_\mu$ of $\mathrm{Irr}(W_{[\mu]})$ attached to $\mathcal{I}_\mu$ having associated variety $\mathcal{O}_\mu$.

Let

$$\Pi_{\mathcal{I}_\mu}(G) := \left\{\, \pi\in\mathrm{Irr}(G) \;\middle|\; \mathcal{I}_\mu \subseteq \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) \,\right\}.$$

Let $Q\subset\mathfrak{h}^*$ be the root lattice and write $[\mu] := \mu+Q$ for the coset $\mathfrak{h}^*/Q$ containing μ. Denote $\mathrm{Coh}_{[\mu]}(G)$ the space of coherent families based on $[\mu]$ which has a $W_{[\mu]}$-action.

For each $G_{\mathbb{C}}$-invariant closed subset S in the nilpotent cone of $\mathfrak{g}$, let

$$\mathrm{Coh}_{[\mu],\mathsf{S}}(G) := \left\{\, \Theta\in\mathrm{Coh}_{[\mu]} \;\middle|\; \mathrm{AV}_{\mathbb{C}}(\Theta(\mu'))\subset\mathsf{S} \text{ for } \mu'\in[\mu] \,\right\}.$$

The following theorem gives an upper bound of $|\Pi_{\mathcal{I}_{mu}}(G)|$.

Theorem 2.1 (Barbasch-Vogan)**.** *Let $\Pi_{\mathcal{O},\mu}(G)$ denote the set of irreducible admissible G-module with complex associated variety $\overline{\mathcal{O}}$. Then*

$$\begin{aligned} \left|\Pi_{\mathcal{I}_\mu}(G)\right| &= \sum_{\sigma\in\mathcal{D}_\mu} [\sigma : \mathrm{Coh}_{[\mu],\overline{\mathcal{O}_\mu}}(G)]\cdot[1_{W_\mu},\sigma|_{W_\mu}]\\ &\leqslant \sum_{\sigma\in\mathcal{D}_\mu} [\sigma : \mathrm{Coh}_{[\mu]}(G)]\cdot[1_{W_\mu},\sigma|_{W_\mu}]. \end{aligned}$$

Lemma 2.2 ([16, (5.26), Proposition 5.28])**.** *Let $\check{\mathcal{O}}$ be a nilpotent orbit in $\check{G}_{\mathbb{C}}$ and $\lambda_{\check{\mathcal{O}}}$ be the infinitesimal character attached to $\check{\mathcal{O}}$. Then the set*

$$ {}^{L}\mathscr{C}(\check{\mathcal{O}}) := \left\{ \sigma \in \mathcal{D}_{\lambda_{\check{\mathcal{O}}}} \;\middle|\; [1_{W_\mu} : \sigma] \neq 0 \right\} $$

is a left cell in $\mathcal{D}_{\lambda_{\check{\mathcal{O}}}}$ given by

$$ (J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \operatorname{sgn}) \otimes \operatorname{sgn}. $$

Moreover, the multiplicity $[1_{W_\mu} : \sigma]$ is one when $\sigma \in {}^{L}\mathscr{C}(\check{\mathcal{O}})$.

Corollary 2.3. *Under the notation of Lemma 2.2, we have*

$$ \left|\Pi_{\mathcal{I}_\mu}(G)\right| \leqslant \sum_{\sigma \in {}^{L}\mathscr{C}(\check{\mathcal{O}})} [\sigma : \operatorname{Coh}_{[\mu]}(G)] $$

2.1. **Coherent family.** Let Q be the $\mathfrak{h}$ root lattice of $\mathfrak{g}$. Let $\mathcal{G}$ be the Grothendieck group of finite dimensional $\mathfrak{g}$-modules occur in $S(\mathfrak{g})$. Note that $\mathcal{G}$ is also an algebra under tensor product. For each finite dimensional $\mathfrak{h}$-module F, let $\Delta(F)$ denote the multi-set of $\mathfrak{h}$-weights in F.

Via the highest weight theory, every W-orbit $W\cdot\mu$ in ${}^{a}Q$ corresponds with the irreducible finite dimensional representation F_μ with extremal weight μ.

For any $\lambda \in {}^{a}\mathfrak{h}^*$, we define the lattice

$$ \Lambda := [\lambda] := \lambda + Q \subset \mathfrak{h}^* $$

and define

$$ \begin{aligned} R_{[\lambda]} &:= \{\, \alpha \in R \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \,\}, \\ W_{[\lambda]} &:= \{\, w \in W \mid w \cdot \lambda - \lambda \in Q \,\} \\ R_{\lambda} &:= \{\, \alpha \in {}^{a}R \mid \langle \lambda, \check{\alpha} \rangle = 0 \,\}, \quad \text{and} \\ W_{\lambda} &:= \langle s_\alpha | \alpha \in R_\lambda \rangle = \langle w \in W | w \cdot \lambda = \lambda \rangle \subseteq W. \end{aligned} \tag{2.1} $$

It is known that $R_{[\lambda]}$ is a root system and

$$ \begin{aligned} W_{[\lambda]} &= \operatorname{Stab}_W([\lambda]) = \langle s_\alpha | \alpha \in R_{[\lambda]} \rangle \subseteq W \quad \text{and} \\ W_{\lambda} &= \langle s_\alpha | \alpha \in R_\lambda \rangle \subseteq W_{[\lambda]}. \end{aligned} $$

In fact W_λ is a parabolic subgroup of $W_{[\lambda]}$ by Chevalley's theorem [93, Lemma 6.3.28].

When λ is regular, let

$$ R^+_{[\lambda]} := \left\{ \alpha \in R_{[\lambda]} \;\middle|\; \langle \check{\alpha}, \lambda \rangle > 0 \right\} $$

be the fixed positive root system.

In the following we define the notion of coherent family based on the lattice $[\lambda]$ in a quite general setting.

Definition 2.4. *Suppose that $\mathcal{M}$ is be an abelian group with $\mathcal{G}$-action:*

$$ \mathcal{G} \times \mathcal{M} \ni (F, m) \mapsto F \otimes m. $$

In addition, a subgroup $\mathcal{M}_\mu$ of $\mathcal{M}$ is fixed for each $\mu \in [\lambda]$ such that $\mathcal{M}_\mu = \mathcal{M}_{w\cdot\mu}$ for any $\mu \in [\lambda]$ and $w \in W_{[\lambda]}$.

A function $f \colon \Lambda \to \mathcal{M}$ is called a coherent family based on Λ if it satisfies $f(\mu) \in \mathcal{M}_\mu$ and

$$ F \otimes f(\mu) = \sum_{\nu \in \Delta(F)} f(\mu + \nu) \qquad \forall \mu \in \Lambda, F \in \mathcal{G}. $$

Let $\operatorname{Coh}_\Lambda(\mathcal{M})$ be the abelian group of all coherent families based on Λ and taking value in $\mathcal{M}$. We can define $W_{[\lambda]}$ action on $\operatorname{Coh}_[\lambda](\mathcal{M})$ by

$$ w \cdot f(\mu) = f(w^{-1} \cdot \mu) \qquad \forall \mu \in \Lambda, w \in W_\Lambda. $$

In this paper, we will consider the following cases.

Example 2.5. *Suppose $\mathcal{M}$ is a field of characteristic zero and*

$$F \otimes m := \dim(F) \cdot m \quad \text{for all } F \in \mathcal{G} \text{ and } m \in \mathcal{M}.$$

We let $\mathcal{M}_\mu = \mathcal{M}$ for every $\mu \in \Lambda$. Then the set of W-harmonic polynomials on $\mathfrak{h}$ is naturally identified with $\mathrm{Coh}_{[\lambda]}(\mathcal{M})$ via the restriction on $[\lambda]$ by Vogan [92, Lemma 4.3].

[*Note that the polynomials are W-harmonic not necessary $W_{[\lambda]}$-harmonic. ($W_{[\lambda]}$-invariant differential operators are more than W-invariant differential operators.)*]

Example 2.6. *Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$, let $\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ be the Grothendieck group of the category $\mathcal{O}$ with coefficients in $\mathbb{C}$, i.e. the category of finitely generated $\mathcal{U}(\mathfrak{g})$-modules with semisimple $\mathfrak{h}$-action and locally finite $\mathfrak{n}$-action. For $\lambda \in \mathfrak{h}^*$, let $\mathcal{G}_{W\cdot\lambda}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ be the subgroup spanned by $\mathfrak{g}$-modules with infinitesimal character $W \cdot \lambda$. Here $\mathcal{G}$ acts on $\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ via the tensor product of $\mathfrak{g}$-modules.*

To ease the notation, we write

$$\mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) := \mathrm{Coh}_{[\lambda]}(\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})).$$

For $\lambda \in \mathfrak{h}^*$, let $\rho := \sum_{\alpha\in\Delta(\mathfrak{n})} \alpha$,

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

be the Verma module with highest weight $\lambda - \rho$ and $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$.

Each $w \in W$ defines a coherent family

$$M_w(\mu) := M(w \cdot \mu) \quad \forall \mu \in [\lambda].$$

The map

$$\begin{array}{ccc} \mathbb{C}[W] & \longrightarrow & \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) \\ w & \mapsto & M_w \end{array}$$

is $W_{[\lambda]}$-equivariant isomorphism where $W_{[\lambda]}$ acts on $\mathbb{C}[W]$ by right translation.

One of the crucial property is that each irreducible module can be fitted into a coherent family. More precisely, the evaluation map descents to yields an isomorphism $\overline{\mathrm{ev}}_\mu$ in the following diagram.

$$\begin{tikzcd} \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) \arrow[r, "\Theta\mapsto\Theta(\mu)"] \arrow[d] & \mathcal{G}_{W\cdot\mu}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) \\ \left(\mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})\right)_{W_\mu} \arrow[ur, hook, two heads, "\overline{\mathrm{ev}}_\mu"'] & \end{tikzcd} \tag{2.2}$$

[*The subjectivity is because of Verma modules form a basis of the category $\mathcal{O}$. The LHS of $\overline{\mathrm{ev}}_\mu$ has dimension $|W/W_\mu|$, the RHS has dimension $W \cdot \mu$. Now the isomorphism follows by dimension counting.*]

Example 2.7. *Suppose G is a reductive Lie group in the Harish-Chandra class. Let $\mathcal{G}(G)$ be the Grothendieck group of finite length admissible G-modules and $\mathcal{G}_\mu(G)$ be the subgroup of $\mathcal{G}(\mathfrak{g}, K)$ generated by the set of irreducible G-modules with infinitesimal character μ.*

By [93, 0.4.6], *we can naturally identify Q with the set of H^s-weights consisting the characters occurs in $S(\mathfrak{g})$ where H^s is a maximally split Cartan in G. Therefore, the set of irreducible G-submodules occur in $S(\mathfrak{g})$ is also naturally identified with $Q/W = \mathcal{G}$. We let $\mathcal{G}$ acts on $\mathcal{G}(G)$ by the tensor product of G-modules.*

[*Note that by the assumption that G is in the Harish-Chandra class, each irreducible $\mathfrak{g}$-submodule F embeds in $S(\mathfrak{g})$ is automatically globalized to a G-module. The point is that the globalization is independent of the embedding of F in $S(\mathfrak{g})$!*]

We write

$$\mathrm{Coh}_{[\lambda]}(G) := \mathrm{Coh}_{[\lambda]}(\mathcal{G}(G))$$

for the space of coherent family of G-modules.

Example 2.8. *Fix a $G_{\mathbb{C}}$-invariant closed subset S in the nilpotent cone of $\mathfrak{g}$. Let $\mathcal{G}_{\mathsf{S}}(G)$ be the Grothendieck group of finite length admissible G-modules whose complex associated varieties are contained in S. Define*

$$\mathcal{G}_{\mu,\mathsf{S}}(G) := \mathcal{G}_{\mu}(G) \cap \mathcal{G}_{\mathsf{S}}(G).$$

and write

$$\mathrm{Coh}_{[\lambda],\mathsf{S}}(G) := \mathrm{Coh}_{[\lambda]}(\mathcal{G}_{\mathsf{S}}(\mathfrak{g}, K)).$$

Note that

$$\mathrm{AV}_{\mathbb{C}}(\pi \otimes F) = \mathrm{AV}_{\mathbb{C}}(\pi).$$

for each finite length G-module π and finite dimensional G-module F. Therefore the $W_{[\lambda]}$-module

$$\mathrm{Coh}_{[\lambda],\mathsf{S}}(G) = (\mathrm{ev}_{\mu})^{-1}(\mathcal{G}_{\mu,\mathsf{S}}(G))$$

for any regular $\mu \in [\lambda]$.

Similarly, we define the space $\mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ of coherent families in category $\mathcal{O}$ whose associated variety are contained in S. In particular,

$$\mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) = (\mathrm{ev}_{\mu})^{-1}(\mathcal{G}_{\mu,\mathsf{S}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})).$$

A remarkable fact that the diagram Equation (2.2) still holds for $\mathrm{Coh}_{[\lambda],\mathsf{S}}(G)$. This is one of the first step towards the counting of the set

$$\mathrm{Irr}_{\mu,\mathsf{S}}(G) := \{\, \pi \in \mathrm{Irr}_{\mu}(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) \subset \mathsf{S} \,\}.$$

where

$$\mathrm{Irr}_{\mu}(G) := \{\, \pi \in \mathrm{Irr}(G) \mid \pi \text{ has infinitesimal character } \mu \,\}.$$

Lemma 2.9. *For each μ and closed $G_{\mathbb{C}}$-invariant subset S in the nilpotent cone of $\mathfrak{g}$, we have an isomorphism*

$$\overline{\mathrm{ev}}_{\mu} \colon \left(\mathrm{Coh}_{[\mu],\mathsf{S}}(G)\right)_{W_{\mu}} \longrightarrow \mathcal{G}_{\mu,\mathsf{S}}(G).$$

In particular,

$$|\mathrm{Irr}_{\mu,\mathsf{S}}(G)| = [1_{W_{\mu}} : \mathrm{Coh}_{[\mu],\mathsf{S}}(G)]$$

Proof. This is a consequence of the formal properties of the translation functor, especially the theory of τ-invariant.

The properties of the coherent family and translation principal: (We refers to [93, Section 7] for the proofs which also work in our (possibly non-linear) setting.)

(a) The evaluation map

$$\mathrm{ev}_{\mu} \colon \mathrm{Coh}_{[\mu]}(G) \to \mathcal{G}_{\mu}(G)$$

is surjective for any $\mu \in [\mu]$, see [93, Theorem 7.2.7].

From now on we fix a regular element $\lambda \in [\mu]$ such that μ is dominant with respect to $R^{+}_{[\lambda]}$

(b) The evaluation map ev_{λ} at λ is an isomorphism [93, Proposition 7.2.27].

For each $\pi \in \mathcal{G}_{\lambda}(G)$, let

$$\Theta_{\pi} := (\mathrm{ev}_{\lambda})^{-1}(\pi)$$

be the unique coherent family such that $\Theta_{\pi}(\lambda) = \pi$,

(c) If $\pi \in \mathrm{Irr}_\lambda(G)$, then $\Theta_\pi(\mu)$ is either zero or an irreducible G-module [93, Proposition 7.3.10, Corollary 7.3.23].

(d) For $\pi \in \mathrm{Irr}_\lambda(G)$,

$$\mathrm{AV}(\Theta_\pi(\mu)) = \mathrm{AV}(\pi)$$

whenever μ is dominant and $\Theta_\pi(\mu)$ non-zero. [This because $\pi = \psi_\mu^\lambda(\Theta_\pi(\mu))$ and $\Theta_\pi(\mu) = \psi_\lambda^\mu(\pi)$. Here is the translation functor from λ to μ see [93, Definition 4.5.7]. Translation dose not increase the associated variety.]

(e) If π and π' are in $\mathrm{Irr}_\lambda(G)$ such that $\Theta_\pi(\mu) = \Theta_{\pi'}(\mu)$ is non-zero, then $\pi = \pi'$.

For $\pi \in \mathrm{Irr}_\lambda(G)$, define the τ-invariant of π to be

$$\tau(\pi) := \left\{ \alpha \in R^+_{[\lambda]} \;\middle|\; \begin{array}{l} \alpha \text{ is simple and} \\ s_\alpha \cdot \Theta_\pi(\lambda) = -\Theta_\pi(\lambda) \end{array} \right\} \tag{2.3}$$

(f) $\Theta_\pi(\mu) = 0$ if and only if $\tau(\gamma) \cap R_\mu \neq \varnothing$ [93, Corollary 7.3.23 (c)].

Now we start to prove the lemma. By the translation principle,

$$\begin{aligned} &\{\, \Theta_\pi(\mu) \mid \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ s.t. } \Theta_\pi(\mu) \neq 0 \,\} \\ =&\{\, \Theta_\pi(\mu) \mid \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ s.t. } \tau(\pi) \cap R_\mu = \varnothing \,\}. \end{aligned}$$

forms a basis of $\mathcal{G}_{\mu,\mathsf{S}}(G)$. [The set consists of distinct (so linearly independent) irreducible G-modules by (c) and (e). They are spanning set by (a). For the support condition, see (d). The τ-invariant condition is by (f).] Hence

$$\begin{aligned} \ker \mathrm{ev}_\mu =& \mathrm{Span}\{\, \Theta_\pi \mid \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ s.t. } \tau(\pi) \cap R_\mu \neq \varnothing \,\} \\ \subseteq& \mathrm{Span}\left\{\, \tfrac{1}{2}(\Theta_\pi - s_\alpha \cdot \Theta_\pi) \;\middle|\; \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ and } \alpha \in \tau(\pi) \cap R_\mu \,\right\} \\ & \quad (\text{by the definition of } \tau(\pi) \text{ in (2.3).}) \\ \subseteq& \mathrm{Span}\left\{\, \Theta - w \cdot \Theta \;\middle|\; \Theta \in \mathrm{Coh}_{[\mu],\mathsf{S}}(G) \,\right\} \\ \subseteq& \ker \mathrm{ev}_\mu. \\ & \quad (\text{by } w \cdot \Theta(\mu) = \Theta(w^{-1} \cdot \mu) = \Theta_\pi(\mu)) \end{aligned}$$

Since $\left(\mathrm{Coh}_{[\mu],\mathsf{S}}(G)\right)_{W_\mu} = \mathrm{Coh}_{[\mu],\mathsf{S}}(G)/\,\mathrm{Span}\{\, \Theta - w \cdot \Theta \mid \Theta \in \mathrm{Coh}_{[\mu],\mathsf{S}}(G) \,\}$, the lemma follows. □

3. Primitive ideals and Weyl group representations

3.1. Associated varieties of a primitive ideals and double cells in $W_{[\lambda]}$. In this section, we review the notion of double cells and its relation with the associated varieties of primitive ideals, see [11, 46]. We retain the notation in Example 2.6.

Let $\mathrm{Prim}_\lambda(\mathfrak{g})$ be the set of primitive ideals in $\mathcal{U}(\mathfrak{g})$ with infinitesimal character λ. Let $\lambda \in \mathfrak{h}^*$, each primitive ideal is the annihilator of a highest weight module by Duflo [28]. In other words, the following map is surjective

$$\begin{array}{ccc} W_{[\lambda]} & \longrightarrow & \mathrm{Prim}_\lambda(\mathfrak{g}) \\ w & \mapsto & I(w \cdot \lambda) := \mathrm{Ann}\, L(w \cdot \lambda). \end{array}$$

By the translation principal, we concentrate the discussion in the regular infinitesimal character case. From now on, we follows the convention in [11]. Let λ be a regular element in $\mathfrak{h}^*$ such that $R^+_{[\lambda]} \subset -\Delta(\mathfrak{n})$. [Here λ is regular anti-dominant ($\langle \lambda, \check{\alpha} \rangle \notin \mathbb{N}$ for each $\alpha \in \Delta(\mathfrak{n})$) with respect to the root system defining highest weight modules, but it is dominant with respect to $R^+_{[\lambda]}$.] For each $w \in W_{[\lambda]}$, define

$$a(w) := |\Delta(\mathfrak{n})| - \text{GK-}\dim(L(w\lambda)).$$

[Suppose λ is integral, then Under this definition, $a_{w_0} = |\Delta(\mathfrak{n})|$ and $a_e = 0$.] For each w one can attach a polynomial $\widetilde{p}_w$ such that $\widetilde{p}_w(\mu) = \operatorname{rank}(\mathcal{U}(\mathfrak{g})/\operatorname{Ann}(L(w\mu)))$ when $\mu \in [\lambda]$ is dominant (i.e. $-\langle \mu, \check{\alpha}\rangle \notin \mathbb{N}^+$ for all $\alpha \in R^+_{[\lambda]}$). $\widetilde{p}_w$ is called the Goldie-rank polynomial attached to the primitive ideal $\operatorname{Ann}(L(w\lambda))$. Fix a dominant regular element δ in $\mathfrak{h}$ (i.e. $\langle \delta, \alpha\rangle > 0$ for each $\alpha \in \Delta(\mathfrak{n})$). Let

$$r_w = \sum_{y\in W_{[\lambda]}} a_{y,w}(y^{-1}\delta)^{a(w)} \in S(\mathfrak{h})$$

where $a_{y,w}$ is determined by the equation

$$L(w\lambda) = \sum_{y\in W_{[\lambda]}} a_{y,w} M(w\lambda)$$

in $\mathcal{G}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$. Then r_w is a positive multiple of $\widetilde{p}_w$ [44, Section 1.4].

A partial order $\underset{L}{\leqslant}$ can be define on $W_{[\lambda]}$ by the following condition [11, Proposition 2.9]

$$\begin{aligned} w_1 \underset{L}{\leqslant} w_2 &\Leftrightarrow I(w_1\lambda) \subseteq I(w_2\lambda) \\ &\Leftrightarrow L(w_2^{-1}\lambda) \text{ is a subquotient of } L(w_1^{-1}\lambda)\otimes S(\mathfrak{g}). \end{aligned} \tag{3.1}$$

We say $w_1 \underset{L}{\approx} w_2$ if and only if $w_1 \underset{L}{\leqslant} w_2 \underset{L}{\leqslant} w_1$. For $w \in W_{[\lambda]}$, we call

$$\mathscr{C}^L_w := \left\{\, w' \in W_{[\lambda]} \,\middle|\, w \underset{L}{\approx} w' \,\right\}$$

the left cell in $W_{[\lambda]}$ containing w.

In summary, we have a bijection

$$\begin{array}{ccc} W_{[\lambda]}/\underset{L}{\approx} & \longrightarrow & \operatorname{Prim}_\lambda(\mathfrak{g}) \\ \mathscr{C}^L_w & \mapsto & \operatorname{Ann} L(w\lambda). \end{array}$$

The partial order $\underset{R}{\leqslant}$ is defined by

$$w_1 \underset{R}{\leqslant} w_2 \Leftrightarrow w_1^{-1} \underset{L}{\leqslant} w_2^{-1}.$$

The partial order $\underset{LR}{\leqslant}$ is defined to be the minimal partial order containing $\underset{L}{\leqslant}$ and $\underset{R}{\leqslant}$. The relation $\underset{R}{\approx}$, $\underset{LR}{\approx}$, right cell $\mathscr{C}^R_w$ and double cells $\mathscr{C}^{LR}_w$ are defined similarly.

Since the Kazhdan-Lusztig conjecture has been proven (for the integral infinitesimal character case by [17, 19] and reduced to the integral infinitesimal character case by [87] (see [42, Section 13.13])), the definition of the order $\underset{L}{\leqslant}$ is the same as the partial order defined by Kazhdan-Lusztig [61] which only depends on the Coxeter group structure of $W_{[\lambda]}$, see [11, Corollary 2.3]. [Note that $x \underset{L}{<} y$ implies $a(x) < a(y)$ and $x \underset{LR}{\approx} y$ implies $a(x) = a(y)$.]

Note that the left cells are exactly the fibers of the map $w \mapsto \widetilde{p}_w$. Take a double cell $\mathscr{C}^{LR}_w$ in $W_{[\lambda]}$ and a set of representatives $\{\, w_1, w_2, \cdots, w_k \,\}$ of the left cells in $\mathscr{C}^{LR}_w$. Due to Barbasch-Vogan[10, 11] and Joseph[43–46], the following statements holds:

- the set of Goldie rank polynomials $\{\, \widetilde{p}_{w_i} \mid i = 1, 2, \cdots, k \,\}$ form a basis of a special representation σ_w of $W_{[\lambda]}$ realized in $S^{a(w)}(\mathfrak{h})$;
- the multiplicity of σ_w in $S^{a(w)}(\mathfrak{h})$ is one,
- $a(w)$ is the minimal degree m such that σ_w occurs in $S^m(\mathfrak{h})$ which is the fake degree of σ_w; [When $W_{[\lambda]} = W$, this is the definition of the fake degree. Otherwise, $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}^{W_\lambda}$ where $\mathfrak{h}_0$ is the span of coroots of $W_{[\lambda]}$. Then $S(\mathfrak{h}_0)$ is embeds in $S(\mathfrak{h})$.]

- the map $W_{[\lambda]}/\underset{LR}{\approx} \ni \mathscr{C}^{LR}_w \mapsto \sigma_w \in \mathrm{Irr}(W_{[\lambda]})$ yields a bijection between the set of double cells and the set $\mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]})$ of special representations of $W_{[\lambda]}$.
- Under the W action, the $W_{[\lambda]}$-module $\sigma_w \subset S^{a(w)}(\mathfrak{h})$ generates an irreducible W-module $\widetilde{\sigma}_w := j^W_{W_{[\lambda]}}\sigma_w$. The W-module $\widetilde{\sigma}_w$ corresponds to the a nilpotent orbit $\mathcal{O}_{\widetilde{\sigma}_w}$ with trivial local system. Now the complex associated variety
$$G^{ad}_{\mathbb{C}}\mathrm{AV}(L(w\lambda)) = \mathrm{AV}(\mathrm{Ann}(L(w\lambda))) = \overline{\mathcal{O}_{\widetilde{\sigma}_w}},$$
where $G^{ad}_{\mathbb{C}}$ is the adjoint group of $\mathfrak{g}$, see [46, Section 2.10].

[Note that the j-induction is not injective in general. For example, $j^{S_{a+b}}_{S_a\times S_b}\tau_a\otimes\tau_b = \tau_a \overset{c}{\sqcup} \tau_b$ where τ_a, τ_b are partitions.]

Now we recall the relationships between cells and the coherent continuation representations.

For each $w \in W$, we define a coherent family L_w by the condition $L_w(\lambda) = L(w\lambda)$.

For each $\mu \in \mathfrak{h}^*$, let $\mathcal{O}_{[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$ be the subcategory of the category $\mathcal{O}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$ consists of modules whose $\mathfrak{h}$-weights are contained in $[\mu]$,
$$\mathcal{G}_{W\cdot\lambda,[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) = \mathcal{G}_{[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) \cap \mathcal{G}_{W\cdot\lambda}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$$
which is spanned by a block in the category $\mathcal{O}$ if $W\cdot\lambda \cap [\mu] \neq \varnothing$.

Consider the following subgroup of coherent continuation
$$\mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n};[\mu]) = \left\{\, \Theta \in \mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) \;\middle|\; \Theta(\lambda) \in \mathcal{G}_{W\cdot\lambda,[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) \,\right\}.$$

We identify $\mathbb{C}[W_{[\lambda]}]$ with $\mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n},[\lambda])$ via $w \mapsto M_w$.

For each $w \in W_{[\lambda]}$, let
$$\overline{\mathscr{V}}^R_w := \mathrm{Span}\left\{\, L_{w'} \;\middle|\; w \underset{R}{\leqslant} w' \,\right\}$$
$$\mathscr{V}^R_w = \overline{\mathscr{V}}^R_w \Big/ \sum_{w \underset{L}{<} w'} \overline{\mathscr{V}}^R_{w'}$$

By (3.1), $\overline{\mathscr{V}}^R_w$ and $\mathscr{V}^R_w$ are $W_{[\lambda]}$-modules under right translation/coherent continuation action.

We define left $W_{[\lambda]}$-module $\overline{\mathscr{V}}^L_w$ and $\mathscr{V}^L_w$ using $\underset{L}{\leqslant}$ and $W_{[\lambda]}\times W_{[\lambda]}$-module $\overline{\mathscr{V}}^{LR}_w$ and $\mathscr{V}^{LR}_w$ using $\underset{LR}{\leqslant}$ similarly.

Suppose $\sigma_1,\sigma_2 \in \mathrm{Irr}(W_{[\lambda]})$. We define
$$\sigma_1 \underset{LR}{\leqslant} \sigma_2 \Leftrightarrow \exists w \in W_{[\lambda]} \text{ such that } \begin{cases} \sigma_1\otimes\sigma_1 \text{ occurs in } \mathscr{V}^{LR}_w \text{ and} \\ \sigma_2\otimes\sigma_2 \text{ occurs in } \overline{\mathscr{V}}^{LR}_w. \end{cases}$$

Now $\sigma_1 \underset{LR}{\approx} \sigma_2$ if and only if there exists a $w \in W_{[\lambda]}$ such that $\sigma_1\otimes\sigma_1$ and $\sigma_2\otimes\sigma_2$ both occur in $\mathscr{V}^{LR}_w$. Now $\underset{LR}{\leqslant}$ is a well defined partial order and $\underset{LR}{\approx}$ is an equivalent relation on $\mathrm{Irr}(W_{[\lambda]})$ respectively. We write ${}^{LR}\mathscr{C}_\sigma \subseteq \mathrm{Irr}(W_{[\lambda]})$ for the double cell containing σ. [A priori $\sigma_1 \underset{LR}{\approx} \sigma_2 \Leftrightarrow \sigma_1 \underset{LR}{\leqslant} \sigma_2 \underset{LR}{\leqslant} \sigma_1$.

But note that $\bigoplus_{w\in W_{[\lambda]}/\underset{LR}{\approx}} \mathscr{V}^{LR}_w \cong \mathbb{C}[W_{[\lambda]}]$ and $\sigma\otimes\sigma$ has multiplicity one in $\mathbb{C}[W_{[\lambda]}]$ which implies the claim.]

A left (resp. right cell) in $\mathrm{Irr}(W_{[\lambda]})$ is the multiset of the irreducible constituents in $\mathscr{V}^L_w$ (resp. left cell) for some $w \in W_{[\lambda]}$.

The equivalence of Barbasch-Vogan's definition and Lusztig's definition of cells in $\mathrm{Irr}(W_{[\lambda]})$ is a consequence of Kazadan-Lusztig conjecture, see [11, remarks after Corollary 2.16].

The structure of double and left cells are explicitly described in [62, Section 4]. In particular, σ_w is the unique special representation occurs in the double cell

$$^{LR}\mathscr{C}_w := \{ \sigma \mid \sigma \otimes \sigma \text{ occurs in } \mathscr{V}_w^{LR} \} \subseteq \mathrm{Irr}(W_{[\lambda]}).$$

For this reason, we also write

$$^{LR}\mathscr{C}_\sigma := {}^{LR}\mathscr{C}_w$$

where $\sigma = \sigma_w$ is the unique special representation in $^{LR}\mathscr{C}_w$.

In summary, we have bijections

$$W_{[\lambda]}/\underset{LR}{\approx} \longleftrightarrow \mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]}) \longleftrightarrow \mathrm{Irr}(W_{[\lambda]})/\underset{LR}{\approx}.$$

We write $\mathscr{V}_\sigma^{LR}$ to be the unique double cell representation containing σ and $\overline{\mathscr{V}}_\sigma^{LR}$ to be the unique upper cone representation which is isomorphic to $\bigoplus_{\sigma \underset{LR}{\leqslant} \sigma'} \sigma' \otimes \sigma'$.

3.2. **Compare blocks.** Now we compare different blocks. Without of loss of generality, we assume λ is in the anti-dominant cone of $\Delta(\mathfrak{n})$, i.e $\langle \lambda, \check{\alpha} \rangle < 0$ for all $\alpha \in \Delta(\mathfrak{n})$.

Let $\{ r_i W_{[\lambda]} \mid i = 1, 2, \cdots, k \}$ be the list of right cosets of $W/W_{[\lambda]}$Michael Kovrig. We can choose r_i to be the unique element in the coset $r_i W_{[\lambda]}$ such that $r_i\lambda$ is anti-dominant, i.e. $R^+_{[r_i\lambda]} \subseteq -\Delta(\mathfrak{n})$.

Now the map $L(w\lambda) \mapsto L(r_i w\lambda)$ with w running over $w \in W_{[\lambda]}$ induces an equivalence of category from $\mathcal{O}_{W\cdot\lambda,[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})\cap$ to $\mathcal{O}_{W\cdot\lambda,[r_i\lambda]}$ by Soegel's theorem [42, 13.13]. In particular $w \mapsto r_i w r_i^{-1}$ induces isomorphism $W_{[\lambda]} \to W_{[r_i\lambda]}$ and preserves the cell structures. In other words, the following $W_{[\lambda]}$-module isomorphism

$$\begin{tikzcd} & \mathbb{C}[W] \arrow[dl, "w\mapsto M_w"'] \arrow[dr, "w\mapsto M_{r_i w}"] & \\ \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}; [\lambda]) \arrow[rr] & & \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}; [r_i\lambda]) \end{tikzcd}$$

maps cell representations to cell representations.

[Let $C = \{ x \in \mathfrak{h} \mid \langle x, \alpha \rangle > 0 \ \forall \alpha \in \Delta(\mathfrak{n}) \}$ and $D_\lambda = \{ x \in \mathfrak{h} \mid -\langle x, \beta \rangle > 0 \ \forall \beta \in R^+_{[\lambda]} \}$. C and D_λ are fundamental domains of $\mathfrak{h}$ under W and $W_{[\lambda]}$-actions. Clearly, $D_{w\lambda} = wD_\lambda$. The condition that $R^+_{[\mu]} \subset -\Delta(\mathfrak{n})$ is equivalent to $D_\mu \supset C$.

Now it is clear D_λ is the union of $r_i^{-1}C$ when r_i running over the preferred coset representatives of $W/W_{[\lambda]}$.]

The above discussion yields the following.

Lemma 3.1. *Fix a $G_{\mathbb{C}}^{ad}$-invariant subset S in the nilpotent cone of $\mathfrak{g}$.*

Let

$$\mathsf{C}_{\mathsf{S}}^{sp} := \left\{ \sigma \in \mathrm{Irr}^{sp}(W_{[\lambda]}) \;\middle|\; \mathrm{Springer}(j_{W_{[\lambda]}}^W \sigma) \subseteq \mathsf{S} \right\},$$

$$\mathsf{C}_{\mathsf{S}} := \left\{ \sigma' \in \mathrm{Irr}(W_{[\lambda]}) \;\middle|\; \exists \sigma \in \mathcal{D}_{\mathsf{S}}^{sp} \ s.\ t.\ \sigma \underset{LR}{\leqslant} \sigma' \right\}$$

Then, as an $W_{[\lambda]}$-module

$$\begin{aligned}\mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) &= \bigoplus_{i=1}^{k} \mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g},\mathfrak{h},\mathfrak{n};[r_i\lambda]) \\ &\cong \bigoplus_{i=1}^{k} \sum_{\sigma\in \mathsf{C}_{\mathsf{S}}^{sp}} \mathscr{V}_{\sigma}^{LR} \\ &\cong \bigoplus_{i=1}^{k} \bigoplus_{\sigma\in\mathsf{C}_{\mathsf{S}}} (\dim\sigma)\sigma\end{aligned}$$

As a baby case of the counting theorem for special unipotent representation of real reductive groups, we have the following counting theorem in the category $\mathcal{O}$.

We fix an regular element $\lambda \in [\mu]$ such that μ is dominant with respect to $R^+_{[\lambda]}$. Let $S_{[\lambda]}$ be the set of simple roots in $R^+_{[\lambda]}$. Let S_μ be the subset of simple roots in $R^+_{[\lambda]}$ orthogonal to μ. Observe that W_μ is always a parabolic subgroup attached to S_μ in $W_{[\lambda]} = W_{[\mu]}$. Let $\mathsf{D}_{\mathsf{S}_\mu}$ be the set of distinguished right coset representatives of $W_{[\lambda]}/W_\mu$. [$r \in \mathsf{D}_{\mathsf{S}_\mu}$ is the element with minimal lenght in rW_μ. Recall that $\tau(w) = \{\, \alpha \in S_{[\lambda]} \mid w\alpha \notin R^+_{[\lambda]} \,\}$ Note that $\tau(w) \cap R_\mu \varnothing$ is equivalent to require that $w\mathsf{S}_\mu \subseteq R^+_{[\lambda]}$, i.e. w is a minimal length element. See for example, Carter, Simple groups of Lie type, Theorem 2.5.8.]

Theorem 3.2. *Let $\mathcal{O}$ be an nilpotent orbit in $\mathfrak{g}$ and $\mu \in \mathfrak{h}$. Let $\Pi_{W\cdot\mu,\mathcal{O}}$ be the set of irreducible highest weight modules π such that $\mathrm{AV}_{\mathbb{C}}(\pi) = \mathcal{O}$. Let $\mathcal{D}^{sp}_{\mathcal{O}}$ be the set of set of special representations such that $\mathrm{Springer}(j_{W_{[\mu]}}^{W}\sigma) = \mathcal{O}$ and $\mathsf{D}_{\mathcal{O}} = \bigcup_{\sigma\in\mathcal{D}^{sp}_{\mathcal{O}}} {}^{LR}\mathscr{C}_\sigma$. Then*

$$|\Pi_{W\cdot\mu,\mathcal{O}}| = \left|W/W_{[\mu]}\right| \cdot \sum_{\sigma\in\mathsf{D}_{\mathcal{O}}} \left(\dim\sigma\cdot[1_{W_\mu}:\sigma]\right).$$

Let

$$\mathscr{C}^{LR}_{\sigma,\mu} = \mathscr{C}^{LR}_{\sigma} \cap \mathsf{D}_{\mathsf{S}_\mu}$$

and $\mathscr{C}^{LR}_{\mathcal{O},\mu} = \bigcup_{\sigma\in\mathcal{D}^{sp}_{\mathcal{O}}} \mathscr{C}^{LR}_{\sigma,\mu}$. Then

$$\Pi_{W\cdot\mu,\mathcal{O}} = \left\{\, L(r_i w) \;\middle|\; w \in \mathscr{C}^{LR}_{\mathcal{O},\mu} \text{ and } i = 1,2,\cdots,k \,\right\}.$$

Here $\mathscr{V}_{\sigma}^{LR}$ is understood as a submodule of $\mathbb{C}[W_{[\lambda]}]$.

Fix $\mu \in \mathfrak{h}$ and let $\mathcal{I}_\mu$ be the maximal primitive ideal with infinitesimal character μ. Let $\Pi_{W\cdot\mu}$ be the set of irreducible highest weight modules π such that $\mathrm{Ann}(\pi) = \mathcal{I}_\mu$.

Combine the above theorem with Lemma 2.2, we have the following counting theorem.

Theorem 3.3. *Let*

$${}^{L}\mathscr{C}_\mu = \left(J_{W_\mu}^{W_{[\mu]}}\,\mathrm{sgn}\right)\otimes \mathrm{sgn}$$

be the left cell attached to μ and

$$\sigma_\mu = \left(j_{W_\mu}^{W_{[\mu]}}\,\mathrm{sgn}\right)\otimes \mathrm{sgn}\,.$$

Then $_\mu$ is the unique special representation containing in ${}^{L}\mathscr{C}_\mu$ and

$$|\Pi_{W\cdot\mu}| = \left|W/W_{[\mu]}\right| \cdot \dim {}^{L}\mathscr{C}_\mu.$$

Moreover,

$$\Pi_{W\cdot\mu,\mathcal{O}} = \left\{\, L(r_i w) \;\middle|\; w \in \mathscr{C}^{LR}_{\sigma_\mu} \cap \mathsf{D}_{\mathsf{S}_\mu} \text{ and } i = 1,2,\cdots,k \,\right\}$$

□

4. Harish-Chandra cells

4.1. **Counting special unipotent representations.** Now let $\check{\mathcal{O}}$ be a nilpotent orbit in $\check{\mathcal{G}}$. It determine an infinitesimal character $\lambda_{\check{\mathcal{O}}}$.

Let $W_{[\lambda_{\check{\mathcal{O}}}]}$ be the integral Weyl group and $W_{\lambda_{\check{\mathcal{O}}}}$ be the stabilizer of $\lambda_{\check{\mathcal{O}}}$.

Special unipotent representations are defined to be the set of Harish-Chandra modules with minimal GK-dimension.

Let $a(\sigma)$ be the generic degree of a Weyl group representation σ.

In view of Theorem 2.1, we need the following lemma by Barabsch-Vogan.

Lemma 4.1 ([16, (5.26)]). *Let*

$$a_\mu = \max\{\, a(\sigma) \mid \sigma \in \mathrm{Irr}(W_{[\mu]}) \text{ and } [1_{W_\mu} : \sigma] \neq 0 \,\}.$$

Let

$${}^L\mathscr{C}_\mu = \{\, \sigma \in \mathrm{Irr}(W_{[\mu]}) \mid a(\sigma) = a_\mu \text{ and } [1_{W_\mu} : \sigma] \neq 0 \,\}.$$

Then ${}^L\mathscr{C}_\mu$ is a left cell of $W_{[\mu]}$ given by

$$(J_{W_\mu}^{W_{[\mu]}} \mathrm{sgn}) \otimes \mathrm{sgn}$$

which contains a unique special representation

$$\sigma = (j_{W_\mu}^{W_{[\mu]}} \mathrm{sgn}) \otimes \mathrm{sgn}$$

[WLOG, we can assume $\lambda_{\check{\mathcal{O}}}$ is integral. Now this is essentially contained in [16].

We adapt the notation in [16]: two special representations $\sigma \overset{LR}{\leqslant} \sigma'$ if and only if $\mathcal{O}_\sigma \supseteq \mathcal{O}_{\sigma'}$ where $\mathcal{O}_\sigma := \mathrm{Springer}(\sigma)$. The generic degree of σ is denoted by $a(\sigma)$. Note that the ordering of double cells/special representation is the same as the closure relation on special nilpotent orbits, see [16, Prop 3.23].

Note that induction maps left cone representation to a left cone representation [16, Prop 4.14 (a)]. Therefore $\mathrm{Ind}_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}$ is a left cone representation. $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}$ consists the constituents in the induced representation with the minimal generic degree. In particular, $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}$ is the set of minimal representations under the LR-order and it is contained in a unique double cell $\mathcal{D}$.

Recall that tensoring with sign (or rather twisting w_0) is an order reversing bijection of left cells in $W_{[\lambda_{\check{\mathcal{O}}}]}$ and induces a LR-order reversing bijection on $\mathrm{Irr}(W_{\lambda_{\check{\mathcal{O}}}})$, see [11, Prop. 2.25]. Therefore $\mathrm{Ind}_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} 1 = \left(\mathrm{Ind}_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}\right) \otimes \mathrm{sgn}$ has a set of constituents which is maximal under the LR-order, in particular the generic degree takes maximal value on these representations.

Hence we conclude that $\mathcal{C}_{\check{\mathcal{O}}} \otimes \mathrm{sgn} = J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}$.]

5. Counting in type A

The results in this section are well known to the experts.

Let YD be the set of Young diagrams viewed as a finite multiset of positive integers. The set of nilpotent orbits in $\mathrm{GL}_n(\mathbb{C})$ is identified with Young diagram of n boxes.

Let $\check{G}_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$. Fix an orbit $\check{\mathcal{O}} \in \mathrm{Nil}(ckG)$, let $\check{\mathcal{O}}_e$ (resp. $\check{\mathcal{O}}_o$) be the partition consists of all even (resp. odd) rows in $\check{\mathcal{O}}$.

Let S_n denote the Weyl group of $\mathrm{GL}_n(\mathbb{C})$. Let $W_n := S_n \ltimes \{\pm 1\}^n$ denote the Weyl group of type B_n or C_n. Let sgn denote the sign representation of the Weyl group. The group W_n is naturally embedded in S_{2n}. For W_n, let ϵ denote the unique non-trivial character which is trivial on S_n. Note that ϵ is also the restriction of the sgn of S_{2n} on W_n.

5.1. **Special unipotent representations of $G = \mathrm{GL}_n(\mathbb{C})$.** By [16], the set of unipotent representations of $G = \mathrm{GL}_n(\mathbb{C})$ one-one corresponds to nilpotent orbits in $\mathrm{Nil}(\check{G}_{\mathbb{C}})$. Suppose $\check{\mathcal{O}}$ has rows

$$\mathbf{r}_1(\check{\mathcal{O}}) \geqslant \mathbf{r}_2(\check{\mathcal{O}}) \geqslant \cdots \geqslant \mathbf{r}_k(\check{\mathcal{O}}) > 0.$$

Then $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}})$ has columns $\mathbf{c}_i(\mathcal{O}) = \mathbf{r}_i(\check{\mathcal{O}})$ for all $i \in \mathbb{N}^+$. The map $\check{\mathcal{O}} \mapsto \mathcal{O}$ is a bijection.

We set $\mathsf{CP} = \mathsf{YD}$ be the set of Young diagrams. For $\tau \in \mathsf{CP}$ which has k columns, let 1_c be the trivial representations of $\mathrm{GL}_c(\mathbb{C})$.

$$\pi_\tau = 1_{\mathbf{c}_1(\tau)} \times 1_{\mathbf{c}_2(\tau)} \times \cdots \times 1_{\mathbf{c}_k(\tau)}.$$

The Vogan duality gives a duality between Harish-Chandra cells. In this case, Harish-Chandra cells is the double cell of Lusztig. Now we have a duality

$$\pi_\tau \leftrightarrow \pi_{\tau^t}.$$

Let $\tau' := \nabla(\tau)$ be the partition obtained by deleting the first column of τ. Let $\theta_{a,b}$ (resp. $\Theta_{a,b}$) be the theta lift (resp. big theta lift) from $\mathrm{GL}_a(\mathbb{C})$ to $\mathrm{GL}_b(\mathbb{C})$. Then we have

$$\pi_\tau = \theta_{|\tau'|,|\tau|}(\pi_\tau).$$

5.2. **Counting unipotent representations of $\mathrm{GL}_n(\mathbb{R})$.** Now let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G}_{\mathbb{C}})$. Recall the decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_e \cup \check{\mathcal{O}}_o$. Let $n_e = \left|\check{\mathcal{O}}_e\right|, n_o = \left|\check{\mathcal{O}}_o\right|$ and $\lambda_{\check{\mathcal{O}}} = \frac{1}{2}\check{h}$.

Then

$$W_{\lambda_{\check{\mathcal{O}}}} \cong S_{\left|\check{\mathcal{O}}_e\right|} \times S_{\left|\check{\mathcal{O}}_o\right|}.$$

$$W_{\lambda_{\check{\mathcal{O}}}} = \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_e)} \times \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_o)}$$

By the formula of a-function, one can easily see that The cell in $W(\lambda_{\check{\mathcal{O}}})$ consists of the unique representation $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W[\lambda_{\check{\mathcal{O}}}]}(1)$. Now the W-cell $(J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W}\mathrm{sgn}) \otimes \mathrm{sgn}$ consists a single representation

$$\tau_{\check{\mathcal{O}}} = \check{\mathcal{O}}_e^t \boxtimes \check{\mathcal{O}}_o^t.$$

The representation $j_{W_{\lambda_{\check{\mathcal{O}}}}}^{S_n}\tau_{\check{\mathcal{O}}}$ corresponds to the orbit $\mathcal{O} = \check{\mathcal{O}}^t$ under the Springer correspondence. [WLOG, we assume $\check{\mathcal{O}} = \check{\mathcal{O}}_o$.

Let $\sigma \in \widehat{S_n}$. We identify σ with a Young diagram. Let $c_i = \mathbf{c}_i(\sigma)$. Then $\sigma = J_{W'}^{S_n}\epsilon_{W'}$ where $W' = \prod S_{c_i}$ (see Carter's book). This implies Lusztig's a-function takes value

$$a(\sigma) = \sum_i c_i(c_i-1)/2$$

Comparing the above with the dimension formula of nilpotent Orbits [24, Collary 6.1.4], we get (for the formula, see Bai ZQ-Xie Xun's paper on GK dimension of $SU(p,q)$)

$$\tfrac{1}{2}\dim(\sigma) = \dim(L(\lambda)) = n(n-1)/2 - a(\sigma).$$

Here $\dim(\sigma)$ is the dimension of nilpotent orbit attached to the Young diagram of σ (it is the Springer correspondence, regular orbit maps to trivial representation, note that $a(\mathrm{triv}) = 0$), $L(\lambda)$ is any highest weight module in the cell of σ.

Return to our question, let $S' = \prod_i S_{\mathbf{c}_i(\check{\mathcal{O}})}$. We want to find the component σ_0 in $\mathrm{Ind}_{S'}^{S_n} 1$ whose $a(\sigma_0)$ is maximal, i.e. the Young diagram of σ_0 is minimal.

By the branaching rule, $\sigma \subset \mathrm{Ind}_{S'}^{S_n} 1$ is given by adding rows of lenght $\mathbf{c}_i(\check{\mathcal{O}})$ repeatly (Each time add at most one box in each column). Now it is clear that $\sigma_0 = \check{\mathcal{O}}^t$ is desired.

This agrees with the Barbasch-Vogan duality d_{BV} given by

$$\check{\mathcal{O}} \xrightarrow{Springer} \check{\mathcal{O}} \xrightarrow{\otimes \mathrm{sgn}} \check{\mathcal{O}}^t \xrightarrow{Springer} \check{\mathcal{O}}^t.$$

]

The $W_{[\lambda_{\check{\mathcal{O}}}]}$-module $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}$ is given by the following formula:

$$\begin{aligned} \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]} &\cong \mathcal{C}_{n_e} \otimes \mathcal{C}_{n_o} \quad \text{with} \\ \mathcal{C}_n &:= \bigoplus_{\substack{s,a,b \\ 2s+a+b=n}} \mathrm{Ind}_{W_s \times S_a \times S_b}^{S_n} \epsilon \otimes 1 \otimes 1. \end{aligned}$$

According to Vogan duality, we can obtain the above formula by tensoring sgn on the forumla of the unitary groups in [12, Section 4].

By branching rules of the symmetric groups, $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ can be parameterized by painted partition.

(5.1)
$$\mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}) = \left\{ \uptau := (\tau, \mathcal{P}) \;\middle|\; \begin{array}{l} \tau = \check{\mathcal{O}}^t \\ \mathrm{Im}(\mathcal{P}) \subset \{ \bullet, c, d \} \\ \#\{ i \mid \mathcal{P}(i,j) = \bullet \} \text{ is even} \end{array} \right\}.$$

For $\uptau := (\tau, \mathcal{P}) \in \mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}})$, we write $\mathcal{P}_\uptau := \mathcal{P}$.

[The typical diagram of all columns with even length $2c$ are

$$\begin{array}{|c|c|c|c|c|c|} \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bullet & \cdots & \bullet & c & \cdots & c \\ \hline \bullet & \cdots & \bullet & d & \cdots & d \\ \hline \end{array}$$

The typical diagram of all columns with odd length $2c+1$ are

$$\begin{array}{|c|c|c|c|c|c|} \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline c & \cdots & c & d & \cdots & d \\ \hline \end{array}$$

]

Let $\mathrm{sgn}_n \colon \mathrm{GL}_n(\mathbb{R}) \to \{ \pm 1 \}$ be the sign of determinant. Let 1_n be the trivial representation of $\mathrm{GL}_n(\mathbb{R})$. For $\uptau \in \mathrm{PP}\check{\mathcal{O}}$, we attache the representation

(5.2)
$$\pi_\uptau := \bigtimes_j \underbrace{1_j \times \cdots \times 1_j}_{c_j\text{-terms}} \times \underbrace{\mathrm{sgn}_j \times \cdots \times \mathrm{sgn}_j}_{d_j\text{-terms}}.$$

Here

- j running over all column lengths in $\check{\mathcal{O}}^t$,
- d_j is the number of columns of length j ending with the symbol "d",
- c_j is the number of columns of length j ending with the symbol "$\bullet$" or "c", and
- "$\times$" denote the parabolic induction.

5.3. **Special Unipotent representations of $G = \mathrm{GL}_m(\mathbb{H})$.** Suppose that $\mathcal{O}$ is the complexificiation of a rational nilpotent $\mathrm{GL}_m(\mathbb{H})$-orbit. Then $\mathcal{O}$ has only even length columns. Therefore, $\mathrm{Unip}_{\check{\mathcal{O}}}(G) \neq \varnothing$ only if $\check{\mathcal{O}} = \check{\mathcal{O}}_e$.

In this case the coherent continuation representation is given by

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) = \mathrm{Ind}_{W_m}^{S_{2m}} \epsilon$$

and $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is a singleton. For each partition τ only having even columns, we define

$$\pi_\tau := \bigtimes_i 1_{\mathbf{c}_i(\tau)/2}.$$

5.4. **Counting special unipotent repesentations of $\mathrm{U}(p,q)$.** We call the parity of $|\check{\mathcal{O}}|$ the "good pairity". The other pairity is called the "bad parity". We write $\check{\mathcal{O}} = \check{\mathcal{O}}_g \cup \check{\mathcal{O}}_b$ where $\check{\mathcal{O}}_g$ and $\check{\mathcal{O}}_b$ consist of good parity length rows and bad parity rows respectively.

Let $(n_g, n_b) = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$. Now as the $S_{n_g} \times S_{n_b}$

$$\bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=n}} \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathrm{U}(p,q)) = \mathcal{C}_g \otimes \mathcal{C}_b$$

where

$$\mathcal{C}_g = \bigoplus_{\substack{s,a,b\in\mathbb{N}\\ 2s+a+b=n_g}} \mathrm{Ind}_{W_s\times S_a\times S_b}^{S_{n_g}} 1\otimes \mathrm{sgn}\otimes\mathrm{sgn}$$

$$\mathcal{C}_b = \begin{cases} \mathrm{Ind}_{W_{\frac{n_b}{2}}}^{S_{n_b}} 1 & \text{if } n_b \text{ is even} \\ 0 & \text{otherwise.}\end{cases}$$

By the above formula, we have

Lemma 5.1. *(i) The set $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,q)) \neq \varnothing$ if and only if $p = q$ and each row lenght in $\check{\mathcal{O}}$ has even multiplicity.*

(ii) Suppose $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \varnothing$, let $\check{\mathcal{O}}'$ be the Young diagram such that $\mathbf{r}_i(\check{\mathcal{O}}') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ and π' be the unique special uinpotent representation in $\mathrm{Unip}_{\check{\mathcal{O}}'}(\mathrm{GL}_p(\mathbb{C}))$. Then the unique element in $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p))$ is given by

$$\pi := \mathrm{Ind}_P^{\mathrm{U}(p,p)} \pi'$$

where P is a parabolic subgroup in $\mathrm{U}(p,p)$ with Levi factor equals to $\mathrm{GL}_p(\mathbb{C})$.

(iii) In general, when $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \varnothing$, we have a natural bijection

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}(n_1+p,n_2+p)) \\ \pi_0 & \mapsto & \mathrm{Ind}_P^{\mathrm{U}(n_1+p,n_2+p)} \pi'\otimes\pi_0 \end{array}$$

where P is a parabolic subgroup with Levi factor $\mathrm{GL}_p(\mathbb{C})\times\mathrm{U}(n_1,n_2)$.

The above lemma ensure us to reduce the problem to the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Now assume $\check{\mathcal{O}} = \check{\mathcal{O}}_g$ and so $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ corresponds to the blocks of the infinitesimal character of the trivial representation.

By [12, Theorem 4.2], Harish-Chandra cells in $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ are in one-one correspondence to real nilpotent orbits in $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t$.

[From the branching rule, the cell is parametered by painted partition

$$\mathrm{PP(U)} := \left\{\, \tau\in\mathrm{PP} \;\middle|\; \begin{array}{l}\mathrm{Im}(\tau)\subseteq\{\bullet, s, r\}\\ \text{“}\bullet\text{” occures even times in each row}\end{array} \right\}.$$

The bijection $\mathrm{PP(U)} \to \mathsf{SYD}, \tau\mapsto\mathscr{O}$ is given by the following recipe: The shape of $\mathscr{O}$ is the same as that of τ. $\mathscr{O}$ is the unique (upto row switching) signed Young diagram such that

$$\mathscr{O}(i,\mathbf{r}_i(\tau)) := \begin{cases} +, & \text{when } \tau(i,\mathbf{r}_i(\tau)) = r; \\ -, & \text{otherwise, i.e. } \tau(i,\mathbf{r}_i(\tau))\in\{\bullet, s\}. \end{cases}$$

Example 5.2.

$$\begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet & r \\ \hline \bullet & \bullet \\ \cline{1-2} s & r \\ \cline{1-2} s \\ \cline{1-1} r \\ \cline{1-1} \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|} \hline + & - & + & - & + \\ \hline + & - \\ \cline{1-2} - & + \\ \cline{1-2} - \\ \cline{1-1} + \\ \cline{1-1} \end{array}$$

]
Now the following lemma is clear.

Lemma 5.3. *When $\check{\mathcal{O}} = \check{\mathcal{O}}_g$, the associated varity of every special unipotent representations in* $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U})$ *is irreducible. Moreover, the following map is a bijection.*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1, n_2)) & \longrightarrow & \{\, \textit{rational forms of } \check{\mathcal{O}}^t \,\} \\ \pi_0 & \mapsto & \mathrm{AV}^{\mathrm{weak}}(\pi_0). \end{array}$$

□

Remark. Note that the parabolic induction of an rational nilpotent orbit can be reducible. Therefore, when $\check{\mathcal{O}}_b \neq \varnothing$, the special unipotent representations can have reducible associated variety. Meanwhile, it is easy to see that the map $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) \ni \pi \mapsto \mathrm{AV}^{\mathrm{weak}}(\pi)$ is still injective.

We will show that every elements in $\mathrm{Unip}_{\check{\mathcal{O}}_g}$ can be constructed by iterated theta lifting. For each τ, let $\mathscr{O}$ be the corresponding real nilpotent orbit. Let $\mathrm{Sign}(\mathscr{O})$ be the signature of $\mathscr{O}$, $\nabla(\mathscr{O})$ be the signed Young diagram obtained by deleteing the first column of $\mathscr{O}$. Suppose $\mathscr{O}$ has k-columns. Inductively we have a sequence of unitary groups $\mathrm{U}(p_i, q_i)$ with $(p_i, q_i) = \mathrm{Sign}(\nabla^i(\mathscr{O}))$ for $i = 0, \cdots, k$. Then

$$\pi_\tau = \theta_{\mathrm{U}(p_1,q_1)}^{\mathrm{U}(p_0,q_0)} \theta_{\mathrm{U}(p_2,q_2)}^{\mathrm{U}(p_1,q_1)} \cdots \theta_{\mathrm{U}(p_k,q_k)}^{\mathrm{U}(p_{k-1},q_{k-1})}(1) \tag{5.3}$$

where 1 is the trivial representation of $\mathrm{U}(p_k, q_k)$.

Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Form the duality between cells of $\mathrm{U}(p, q)$ and $\mathrm{GL}(n, \mathbb{R})$. We have an ad-hoc (bijective) duality between unipotent representations:

$$\begin{array}{rcl} d_{\mathrm{BV}} \colon \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) & \to & \mathrm{Unip}_{\check{\mathcal{O}}^t}(\mathrm{GL}(\mathbb{R})) \\ \pi_\tau & \mapsto & \pi_{d_{\mathrm{BV}}(\tau)} \end{array}$$

Here $\check{\mathcal{O}}^t = d_{\mathrm{BV}}(\check{\mathcal{O}})$ and $d_{\mathrm{BV}}(\tau)$ is the pained bipartition obtained by transposeing τ and replace s and r by c and d respectively. See (5.3) and (5.2) for the definition of special unipotent representations on the two sides.

6. Counting in type BCD

In this section, we consider the case when $\check{\star} \in \{ B, C, D \}$, i.e $\star \in \{ B, \widetilde{C}, C, D, C^*, D^* \}$.

We identify $\mathfrak{h}^*$ with $\mathbb{Z}^n$ where $n = \mathrm{rank}(G_{\mathbb{C}})$ and let ρ be the half sum of all positive roots.

Recall that

$$\text{good pairity} = \begin{cases} \text{odd} & \text{when } \star \in \{ C, C^*, D, D^* \} \\ \text{even} & \text{when } \star \in \{ B, \widetilde{C} \} \end{cases}$$

Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathcal{G}})$ with decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$. Then $\check{\mathcal{G}}_{\lambda_{\check{\mathcal{O}}}} = \check{\mathcal{G}}_b \times \check{\mathcal{G}}_g$. Let n_b and n_g be the rank of $\check{\mathcal{G}}_b$ and $\check{\mathcal{G}}_g$ respectively. We have

$$(n_b, n_g) = \begin{cases} (\frac{1}{2}\left|\check{\mathcal{O}}_b\right|, \frac{1}{2}(\left|\check{\mathcal{O}}_g\right| - 1)) & \text{when } \star \in \{ C, C^* \} \\ (\frac{1}{2}\left|\check{\mathcal{O}}_b\right|, \frac{1}{2}\left|\check{\mathcal{O}}_g\right|) & \text{when } \star \in \{ B, \widetilde{C}, D, D^* \} \end{cases}$$

and integral Weyl group is a product of two factors

$$W_{[\lambda_{\check{\mathcal{O}}}]} = W_b \times W_g$$

where

$$W_b := \begin{cases} \mathsf{W}_{n_b} & \text{when } \star \in \{ B, \widetilde{C} \} \\ \mathsf{W}'_{n_b} & \text{when } \star \in \{ C, C^*, D, D^* \} \end{cases}$$

$$W_g := \begin{cases} \mathsf{W}_{n_g} & \text{when } \star \in \{ B, C, C^* \} \\ \mathsf{W}'_{n_g} & \text{when } \star \in \{ \widetilde{C}, D, D^* \} \end{cases}$$

6.1. **The left cell.**

Lemma 6.1. *In all the cases there is a irreducible W_b-representation τ_b such that*

$$\begin{array}{ccc} {}^L\mathscr{C}(\check{\mathcal{O}}_g) & \longrightarrow & {}^L\mathscr{C}(\check{\mathcal{O}}) \\ \tau_g & \mapsto & \tau_b \otimes \tau_g \end{array}$$

is a bijection. Here

$$\tau_b = \begin{cases} \Big(\big(\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)+1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)+1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)+1}{2}\big), \\ \quad \big(\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)-1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)-1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)-1}{2}\big)\Big) & \text{if } \star \in \{ B, \widetilde{C} \}, \\ \Big(\big(\frac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \frac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \frac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b)\big), \\ \quad \big(\frac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \frac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \frac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b)\big)\Big)_I & \text{if } \star \in \{ C, C^*, D, D^* \}. \end{cases}$$

Set

$$\mathfrak{P}_\star(\check{\mathcal{O}}_g) = \Big\{ \{ (2i-1, 2i) \mid \mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i}(\check{\mathcal{O}}_g) > 0, \text{ and } i \in \mathbb{N}^+ \}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

6.2. **The left cell.**

6.3. **Coherent continuation representations.** Let Q be the root lattice in $\mathfrak{h}^*$ which is

$$Q = \begin{cases} \mathbb{Z}^n & \text{if } \star = B \\ \{ (a_i) \in \mathbb{Z}^n \mid \sum_{i=1}^n a_i \text{ is even} \} & \text{if } \star \in \{ C, \widetilde{C}, C^*, D, D^* \} \end{cases}$$

We consider the lattice

$$\Lambda_{n_b,n_g} = (\underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_b\text{-terms}}, \underbrace{0, \cdots, 0}_{n_g\text{-terms}}) + Q. =$$

7. Type C

The dual group of $G_{\mathbb{C}} = \mathrm{Sp}(2n, \mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{SO}(2n+1, \mathbb{C})$. The even is the bad parity, and odd is the good parity.

Fix n_b, n_g such that $n_b + n_g = n$. Let

$$\Lambda_{n_b,n_g} = (\underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_b\text{-terms}}, \underbrace{0, \cdots, 0}_{n_g\text{-terms}}) + Q. \tag{7.1}$$

Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\mathrm{SO}(2n+1, \mathbb{C}))$ with decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$.

Let $\check{\mathcal{O}}'_b$ be the Young diagram such that $\mathbf{r}_i(\check{\mathcal{O}}'_b) = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ and $\mathcal{O}'_b$ be the transpose of $\check{\mathcal{O}}'_b$.

$$\tau_b = \Big(\left(\tfrac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \tfrac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \tfrac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b)\right), \left(\tfrac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \tfrac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \tfrac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b)\right) \Big)_I \in \mathrm{Irr}(W'_b).$$

Set

$$\mathfrak{P}(\check{\mathcal{O}}_g) = \{\, (2i-1, 2i) \mid \mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i}(\check{\mathcal{O}}_g) > 0, \text{ and } i \in \mathbb{N}^+ \,\}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

For $\mathscr{P} \in \overline{\mathsf{A}}(\check{\mathcal{O}})$, let

$$\tau_{\mathscr{P}} := (\imath, \jmath) \in \mathrm{Irr}(W_g)$$

such that

$$(\mathbf{c}_{l+1}(\imath), \mathbf{c}_{l+1}(\jmath)) := (0, \tfrac{1}{2}(\mathbf{r}_{2l+1}(\check{\mathcal{O}}_g) - 1))$$

and for all $1 \leqslant i \leqslant l$

$$(\mathbf{c}_i(\imath), \mathbf{c}_i(\jmath)) := \begin{cases} (\frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) - 1)) & \text{if } (2i-1, 2i) \notin \mathscr{P}, \\ (\frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) - 1)) & \text{otherwise.} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \overline{\mathsf{A}}(\check{\mathcal{O}}) & \longrightarrow & {}^L\mathscr{C}(\check{\mathcal{O}}) \\ \mathscr{P} & \mapsto & \tau_b \otimes \tau_{\mathscr{P}}. \end{array}$$

such that $\tau_b \otimes \tau_\varnothing$ is the special representation in ${}^L\mathscr{C}(\check{\mathcal{O}})$.

Moreover $\mathrm{Springer}(j_{W'_b \times W_g}^{W_n} \tau_b \otimes \tau_\varnothing) = \mathcal{O}_b \overset{c}{\sqcup} \mathcal{O}_g$ where $\mathcal{O}_g = d_{\mathrm{BV}}(\check{\mathcal{O}}_g)$ and $\mathcal{O}_b = \check{\mathcal{O}}_b^t$.

Lemma 7.1. *Suppose $\star \in \{\, C, C^* \,\}$, $\check{\mathcal{O}}_b$ has $2k$ rows and $\check{\mathcal{O}}_g$ has $2l+1$-rows. Here each row in $\check{\mathcal{O}}_b$ has even length and each row in $\check{\mathcal{O}}_g$ has odd length, and $W_{[\lambda_{\check{\mathcal{O}}}]} = W'_b \times W_g$ where $b = \frac{|\check{\mathcal{O}}_b|}{2}$ and $g = \frac{|\check{\mathcal{O}}_g| - 1}{2}$. Let*

$$\tau_b = \Big(\left(\tfrac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \tfrac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \tfrac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b)\right), \left(\tfrac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \tfrac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \tfrac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b)\right) \Big)_I \in \mathrm{Irr}(W'_b).$$

Set

$$\mathfrak{P}(\check{\mathcal{O}}_g) = \{\, (2i-1, 2i) \mid \mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i}(\check{\mathcal{O}}_g) > 0, \textit{ and } i \in \mathbb{N}^+ \,\}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

For $\mathscr{P} \in \overline{\mathsf{A}}(\check{\mathcal{O}})$, let

$$\tau_{\mathscr{P}} := (\imath, \jmath) \in \mathrm{Irr}(W_g)$$

such that

$$(\mathbf{c}_{l+1}(\imath), \mathbf{c}_{l+1}(\jmath)) := (0, \tfrac{1}{2}(\mathbf{r}_{2l+1}(\check{\mathcal{O}}_g) - 1))$$

and for all $1 \leqslant i \leqslant l$

$$(\mathbf{c}_i(\imath), \mathbf{c}_i(\jmath)) := \begin{cases} (\frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) - 1)) & \textit{if } (2i-1, 2i) \notin \mathscr{P}, \\ (\frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) - 1)) & \textit{otherwise.} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \overline{\mathsf{A}}(\check{\mathcal{O}}) & \longrightarrow & {}^L\mathscr{C}(\check{\mathcal{O}}) \\ \mathscr{P} & \mapsto & \tau_b \otimes \tau_{\mathscr{P}}. \end{array}$$

such that $\tau_b \otimes \tau_\varnothing$ is the special representation in ${}^L\mathscr{C}(\check{\mathcal{O}})$.

Moreover $\mathrm{Springer}(j_{W'_b \times W_g}^{W_n} \tau_b \otimes \tau_\varnothing) = \mathcal{O}_b \overset{c}{\sqcup} \mathcal{O}_g$ *where $\mathcal{O}_g = d_{\mathrm{BV}}(\check{\mathcal{O}}_g)$ and $\mathcal{O}_b = \check{\mathcal{O}}_b^t$.*

[In this case, bad parity is even and each row length occur with even multiplicity. Suppose $\check{\mathcal{O}}_b = (C_1, C_1, C_2, C_2, \cdots, C_{k'}, C_{k'})$ with $c_1 = 2k$ and $k' = \mathbf{r}_1(\check{\mathcal{O}}_b)$.

$$W_{\lambda_{\check{\mathcal{O}}_b}} = S_{C_1} \times S_{C_2} \times \cdots S_{C_{k'}}.$$

The symbol of trivial representation of trivial group of type D is

$$\begin{pmatrix} 0, 1, \cdots, k-1 \\ 0, 1, \cdots, k-1 \end{pmatrix}.$$

Now it is easy to see that

$$J^{W_b}_{W_{\lambda_{\check{\mathcal{O}}_b}}} \operatorname{sgn} = ((\tfrac{1}{2}C_1, \tfrac{1}{2}C_2, \cdots, \tfrac{1}{2}C_{k'}), (\tfrac{1}{2}C_1, \tfrac{1}{2}C_2, \cdots, \tfrac{1}{2}C_{k'})).$$

For the good parity part. Suppose $\check{\mathcal{O}}_g = (2c_1 + 1, C_2, C_2, C_3, C_3, \cdots, C_{k'}, C_{k'})$ with $2c_1 + 1 = 2l + 1$ and $2k' + 1 = \mathbf{r}_1(\check{\mathcal{O}}_g)$.

$$W_{\lambda_{\check{\mathcal{O}}_g}} = W_{c_1} \times S_{C_2} \times \cdots \times S_{C_{k'}}.$$

The symbol of sign representation of W_{c_1} is

$$\begin{pmatrix} 0, 1, 2, \cdots, c_1 \\ 1, 2, \cdots, c_1 \end{pmatrix}.$$

By induction on number of columns, we see that when even column of length $2c$ occurs, it adds length c columns on the both sides of the bipartition; when odd column $C_i = 2c_i+1$ with $i > 1$ and multiplicity $2r'$ occur, the bifurcation happens: one can attach r' columns of length c_i+1 on the right and r' columns of length c_i on the left (special representations) or attach r' columns of length $c_i + 1$ on the left and r' columns of length c_i on the right.

Therefore,

$$\begin{array}{ccc} J^{W_g}_{W_{\lambda_{\check{\mathcal{O}}_g}}} \operatorname{sgn} & \leftrightarrow & \mathbb{F}_2(\mathfrak{P}(\check{\mathcal{O}}_g)) \\ \check{\tau}_{\mathscr{P}} & \leftrightarrow & \mathscr{P} \end{array}$$

where

$$\mathbf{r}_{l+1}(\check{\tau}_L) = \tfrac{1}{2}(\mathbf{r}_{2l+1}(\check{\mathcal{O}}_g) - 1)$$
$$(\mathbf{r}_i(\check{\tau}_L), \mathbf{r}_i(\check{\tau}_R)) = \begin{cases} (\frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) - 1), \frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) + 1)) & (2i-1, 2i) \notin \mathscr{P} \\ (\frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) - 1), \frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) + 1)) & (2i-1, 2i) \in \mathscr{P} \end{cases}$$

Now tensor with sign yields the result.

We adopt the convention that

$$S_{\mathcal{O}} := \prod_{i \in \mathbb{N}^+} S_{\mathbf{r}_i(\mathcal{O})}$$

so that $j_{S_{\mathcal{O}}} \operatorname{sgn} = \mathcal{O}$ for each partition $\mathcal{O}$.

Consider the orbit under the Springer correspondence. Let $\mathcal{O}'_b = (r_2(\check{\mathcal{O}}_b), r_4(\check{\mathcal{O}}_b), \cdots, r_{2k}(\check{\mathcal{O}}_b))$. Note that $\tau_b = j^{W'_b}_{S_{\mathcal{O}'_b}} \operatorname{sgn}$. So

$$\widetilde{\tau} := j^{W_n}_{W'_b \times W_g} \tau_b \otimes \tau_\varnothing) = j^{W_n}_{S_{\mathcal{O}'_b} \times W_g} \operatorname{sgn} \otimes \tau_{\mathscr{P}}.$$

Since the Springer correspondence commutes with parabolic induction, we get $\mathrm{Springer}(\widetilde{\tau}) = \mathrm{Ind}^{\mathrm{Sp}(2n)}_{\mathrm{GL}_{\mathcal{O}'_b} \times \mathrm{Sp}(2g)} 0 \times \mathcal{O}_g = \mathcal{O}_b \overset{c}{\sqcup} \mathcal{O}_g$]

7.1. **Counting special unipotent representation of $G = \mathrm{Sp}(p,q)$.** .

The dual group of $G_{\mathbb{C}} = \mathrm{Sp}(2n,\mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{SO}(2n+1,\mathbb{C})$.

We set $(2m+1, 2m') = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$.

We view W_{2t} as the reflection group acts on $\mathbb{C}^t$ as usual. Let $H_t \cong W_t \ltimes \{\pm 1\}^t$ be the subgroup in W_{2t} such that

- the first factor W_t sits in S_{2t} commuting with the involution $(12)(34)\cdots((2t-1)(2t))$.
- The element $(1,\cdots,1,\underbrace{-1}_{i\text{-th term}},1,\cdots,1) \in \{\pm 1\}^t$ acts on $\mathbb{C}^{2t}$ by

$$(x_1, x_2, \cdots, x_{2t}) \mapsto (x_1, \cdots, x_{2i-2}, -x_{2i}, -x_{2i-1}, x_{2i+1}, \cdots, x_{2t}).$$

Note that H_t is also a subgroup of W'_{2t}. Define the quadratic character

$$\widetilde{\mathrm{sgn}} := q \otimes \mathrm{sgn}\colon \begin{array}{ccc} H_t = W_t \ltimes \{\pm 1\}^t & \longrightarrow & \{\pm 1\} \\ (g,(a_1,a_2,\cdots,a_t)) & \mapsto & a_1 a_2 \cdots a_t. \end{array}$$

The most important formula is

$$\mathrm{Ind}_{H_t}^{W_{2t}} \widetilde{\mathrm{sgn}} = \sum_{\sigma \in \mathrm{Irr}(S_t)} (\sigma, \sigma). \tag{7.2}$$

Note that we have chosen the embedding $W_t \subset S_{2t}$ in W_{2t}. We have

$$\mathrm{Ind}_{H_t}^{W'_{2t}} \widetilde{\mathrm{sgn}} = \sum_{\sigma \in \mathrm{Irr}(S_t)} (\sigma, \sigma)_I.$$

[In McGovern's paper, the coherent continuation representation is described as:

$$\sum_{t,s,a,b} \mathrm{Ind}_{W_t \times (W_s \ltimes W(A_1)^s) \times W_a \times W_b}^{W_{t+2s+a+b}} \mathrm{sgn} \otimes (\mathrm{triv} \otimes \mathrm{sgn}) \otimes \mathrm{triv} \otimes \mathrm{triv}$$

Now (7.2) was obtained by the following branching formula: [66, p220 (6)]

$$I_n := \mathrm{Ind}_{(W_s \ltimes W(A_1)^s)}^{W_{2s}} \mathrm{triv} \otimes \mathrm{sgn} = \sum \lambda \times \lambda$$

where λ running over all Young diagrams of size s. As McGovern claimed the proof of the above formula is similar to Barbasch's proof of [?B.W, Lemma 4.1]:

$$\mathrm{Ind}_{W_n}^{S_{2n}} \mathrm{triv} = \sum \sigma \quad \text{where } \sigma \text{ has even rows only.}$$

Sketch of the proof (use branching rule and dimension counting): Note that $\dim I_n = \frac{(2p)!2^{2p}}{p!2^{2p}} = (2p)!/p! = \sum_\lambda \dim \lambda \times \lambda$ (For the last equality: $\dim \lambda \times \lambda = (2p)!(\dim \lambda)^2/(p!)^2$ where $\dim \lambda$ is the dimension of S_n representation determined by λ; But $\sum (\dim \lambda)^2 = p!$). On the other hand, $H := W_s \ltimes W(A_1)^s \cap W_s \times W_s = \Delta W_s \subset W_{2s}$. $\mathrm{triv} \otimes \mathrm{sgn}\,|_H = \mathrm{sgn}$ of ΔW_s Therefore, $\lambda \times \lambda$ appears in I_n by Mackey formula. Now by dimension counting, we get the formula.]

Lemma 7.2. *Recall* (9.1) *for the definition of lattice Λ_{n_b,n_g}. We have the following formula on the coherent continuation representations based on Λ_{n_b,n_g}:*

$$\bigoplus_{p+q=n} \mathrm{Coh}_{\Lambda_{n_b,n_g}}(\mathrm{Sp}(p,q)) \cong \mathcal{C}_b \times \mathcal{C}_g.$$

with

$$\begin{aligned}\mathcal{C}_g &= \bigoplus_{\substack{t,s,r\in\mathbb{N}\\ 2t+s+r=n_g}} \mathrm{Ind}_{H_t\times W_s\times W_t}^{W_{n_g}} \widetilde{\mathrm{sgn}}\otimes\mathrm{sgn}\otimes\mathrm{sgn}\\ &= \bigoplus_{\substack{t,s,r\in\mathbb{N}\\ 2t+s+r=n_g}} \mathrm{Ind}_{W_{2t}\times W_s\times W_r}^{W_{n_g}} (\sigma,\sigma)\otimes\mathrm{sgn}\otimes\mathrm{sgn}\end{aligned}$$

$$\mathcal{C}_b = \begin{cases}\mathrm{Ind}_{H_t}^{W'_{n_b}}\mathrm{sgn} & \text{if } n_b = 2t' \text{ is even}\\ 0 & \text{otherwise}\end{cases}$$

Let

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_b) = \left\{ (\imath,\jmath,\mathcal{P},\mathcal{Q}) \;\middle|\; \begin{array}{l}(\imath,\jmath)=\tau_b\\ \mathrm{Im}\mathcal{P}\subset\{\bullet\},\mathrm{Im}\mathcal{Q}\subset\{\bullet\}\end{array}\right\}$$

Lemma 7.3. *The set $\mathsf{PBP}_\star(\check{\mathcal{O}}_b)$ is a singleton. Let $\check{\mathcal{O}}_1$ be the partition with only one box. Then there only one special unipotent representation attached to $\check{\mathcal{O}}_b\overset{r}{\sqcup}\check{\mathcal{O}}_1$ which is given by*

$$\pi_{\star,\check{\mathcal{O}}_b} := \mathrm{Ind}_{\mathrm{GL}_b(\mathbb{H})}^{\mathrm{Sp}(b,b)} \pi_{\check{\mathcal{O}}'_b}$$

where $\pi_{\check{\mathcal{O}}'_b}$ is the unique special unipotent representation attached to $\check{\mathcal{O}}'_b$.

Proof. The claim of the size of $\mathsf{PBP}_\star$ is clear, i.e. there is only one special unipotent representation attached to $\check{\mathcal{O}}'_b$. A prior, $\mathrm{Ind}_{\mathrm{GL}_b(\mathbb{H})}^{\mathrm{Sp}(b,b)} \pi_{\check{\mathcal{O}}'_b}$ maybe a finite copies of the unique special unipotent representation. Meanwhile its associated variety is multiplicity free. hence $\pi_{\star,\check{\mathcal{O}}_b}$ must be irreducible. □

For each $\mathscr{P}\subset\mathfrak{P}(\check{\mathcal{O}}_g)$, let

$$\mathsf{PBP}_{\star,\mathscr{P}}(\check{\mathcal{O}}_g) = \left\{ (\imath,\jmath,\mathcal{P},\mathcal{Q}) \;\middle|\; \begin{array}{l}(\imath,\jmath)=\tau_{\mathscr{P}}\\ \mathrm{Im}\mathcal{P}\subset\{\bullet\},\mathrm{Im}\mathcal{Q}\subset\{\bullet,s,r\}\end{array}\right\}$$

where $\tau_{\mathscr{P}}$ is defined in Lemma 7.1. By abuse of notation, we write

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g) := \mathsf{PBP}_{\star,\varnothing}(\check{\mathcal{O}}_g)$$

Proposition 7.4. *Suppose $\check{\mathcal{O}}\in\mathrm{Nil}(\mathrm{SO}(2n+1,\mathbb{C}))$ with decomposition $\check{\mathcal{O}}=\check{\mathcal{O}}_b\overset{r}{\sqcup}\check{\mathcal{O}}_g$.*

(i) The size of $\mathrm{Unip}_\star(\check{\mathcal{O}})$ is counted by $\mathsf{PBP}_\star(\check{\mathcal{O}}_g)$.

(ii) The set $\mathsf{PBP}_\star(\check{\mathcal{O}}_g)$ is non-empty only if

$$\mathbf{r}_{2i-1}(\check{\mathcal{O}}) > \mathbf{r}_{2i}(\check{\mathcal{O}})$$

for each $i\in\mathbb{N}^+$ with $\mathbf{r}_{2i}(\check{\mathcal{O}})>0$.

(iii) We have

$$\left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g)\right| = |\mathrm{Nil}_{C^*}(\mathcal{O}_g)|$$

with $\mathcal{O}_g := d_{\mathrm{BV}}(\check{\mathcal{O}}_g)$.

(iv) When $\mathrm{Unip}_{C^*}(\check{\mathcal{O}})\neq\varnothing$, there is a bijection

$$\begin{array}{ccccc}\mathrm{Nil}_{C^*}(\mathcal{O}_g) & \longrightarrow & \mathrm{Unip}_{C^*}(\check{\mathcal{O}}_g) & \longrightarrow & \mathrm{Unip}_{C^*}(\check{\mathcal{O}})\\ \mathscr{O}_g & \mapsto & \pi_{\mathscr{O}_g} & \mapsto & \pi'\rtimes\pi_{\mathscr{O}_g}.\end{array}$$

Here π' is the unique special unipotent representation of $\mathrm{GL}_p(\mathbb{H})$ with associated variety $\mathcal{O}_b$, $\pi_{\mathscr{O}_g}$ is the unique special unipotent representation of $\mathrm{Sp}(p,q)$ with associated variety $\mathscr{O}_g$ such that $\mathrm{Sign}(\mathscr{O}_g)=(2p,2q)$, and $\pi'\rtimes\pi_{\mathscr{O}_g}$ denote the parabolic induction.

Proof. (i) Suppose $(2i-1,2i)\in\mathscr{P}\subset\mathfrak{P}(\check{\mathcal{O}}_g)$. Let $(\imath,\jmath):=\tau_{\mathscr{P}}$. Then $\mathbf{c}_i(\imath)>\mathbf{c}_i(\jmath)$ which implies that $\mathsf{PBP}_{\star,\mathscr{P}}(\check{\mathcal{O}}_g)=\varnothing$.

(ii) The first claim is similar to part (i), see [9, Proposition 10.1].
(iii) This is [66, Theorem 6], also see [9].
(iv) It follows from Lemma 7.3 and the Vogan duality. See appendix.

$\square$

[Let $W = W(G_{\mathbb{C}})$ where $G_{\mathbb{C}}$ is natrualy embeded in $\mathrm{GL}(n,\mathbb{C})$. Let $s_{\varepsilon_i i,\varepsilon_j j}$ be the permutation matrix of index i,j, where the (i,j)-th entry is ε_i, the (j,i)-th entry is ε_j and the other place is the identity matrix. Let $w^{\epsilon}_{i,\pm j}$ be the element such that

- it is in $G_{\mathbb{C}}$ and entries in $\{0\} \cup \mu_4$
- it lifts the element $e_i \leftrightarrow \pm e_j$ in the Weyl group.
- $(w^{\epsilon}_{i,\pm j})^2 = \epsilon 1 \in \{\pm 1\}$.

Let

$$h^{+}_{\pm i} = \mathrm{diag}(1,\cdots,1,\pm 1,1,\cdots,1)$$

where ± 1 is the i-th place. Let

$$h^{-}_{\pm i} = \mathrm{diag}(1,\cdots,1,\pm\sqrt{-1},1,\cdots,1)$$

where $\pm\sqrt{-1}$ is the i-th place. Let $e_{\pm i} = s_{\pm i,\pm(n-i+1)}$.

Let $\breve{w}^{\pm}_{i,\pm j}$ be the lift of $e_i \leftrightarrow \pm e_j$ such that

- it is in $G_{\mathbb{C}}$ and entries in $\{0\} \cup \mu_4$
- it lifts the element $e_i \leftrightarrow \pm e_j$ in the Weyl group.
- $(w^{\epsilon}_{i,\pm j})^2|_{[i,j]} = \epsilon 1 \in \{\pm 1\}$ here "$|_{[i,j]}$" means restricts on the $e_i, e_j, -e_i, -e_j$-weights space.

Let

$$\begin{aligned} x_{b,s,r} &= w^{+}_{1,2}\cdots w^{+}_{2b-1,2b}\, h^{+}_{2b+1}\cdots h^{+}_{2b+s} \\ y^{+}_{b,s,r} &= \breve{w}^{+}_{1,-2}\cdots \breve{w}^{+}_{2b-1,-2b}\, e_{2b+1}\cdots e_{2b+s+r} \\ y^{-}_{b,s,r} &= \breve{w}^{-}_{1,-2}\cdots \breve{w}^{-}_{2b-1,-2b} \end{aligned}$$

We compute the parameter space

$$\begin{aligned} \mathcal{Z}_g &= \bigcup_{2b+s+r=m} W_m \cdot (x^{+}_{b,s,r}, y^{+}_{b,s,r}) \\ \mathcal{Z}_b &= \bigcup_{2b=m} W_m \cdot (x^{+}_{b,0,0}, y^{-}_{b,0,0}) \end{aligned}$$

]

7.2. **Counting special unipotent representation of $G = \mathrm{Sp}(2n,\mathbb{R})$.** In this section, we consider the case where $\star = C$.

Recall (9.1) for the definition of the lattice Λ_{n_b,n_g}.

Lemma 7.5. *We have the following formula on the coherent continuation representations based on Λ_{n_b,n_g}:*

$$\mathrm{Coh}_{\Lambda_{n_b,n_g}}(\mathrm{Sp}(2n,\mathbb{R})) = \mathcal{C}_g \otimes \mathcal{C}_b$$

with

$$\begin{aligned} \mathcal{C}_g &= \bigoplus_{2t+a+c+d=n_g} \mathrm{Ind}^{W_{n_g}}_{H_t\times S_a\times W_c\times W_d} 1\otimes \mathrm{sgn}\otimes 1\otimes 1 \\ \mathcal{C}_b &= \mathrm{Res}^{W'_{n_b}}_{W_{n_b}}\left(\bigoplus_{2t+a=n_b} \mathrm{Ind}^{W_{n_b}}_{H_t\times S_a} \widetilde{\mathrm{sgn}}\otimes 1\right) \end{aligned}$$

$\square$

[Consider $\mathcal{C}_b$. The real Cartan must be $\mathbb{C}^{\times t} \times \mathbb{R}^{\times t}$ In real Weyl group is $H_t \times W_a$ generated by The H_t action is known. W_a is generated by the reflections of $e_i \pm e_j$ $n-2t < i \neq j \leqslant n$. We consider the regular character $\gamma = * \otimes \underbrace{||^{\frac{1}{2}} \otimes \cdots \otimes ||^{\frac{1}{2}}}_{a\text{-terms}}$. The cross action of $s_{e_{n-1}+e_n}$ is given by

$$
\begin{aligned}
s_{e_{n-1}+e_n} \times \gamma &= * \otimes \underbrace{||^{\frac{1}{2}} \otimes \cdots \otimes ||^{\frac{1}{2}} \otimes \mathrm{sgn}\, ||^{-\frac{1}{2}} \otimes \mathrm{sgn}\, ||^{-\frac{1}{2}}}_{a\text{-terms}} \\
&\neq s_{e_{n-1}+e_n} \cdot \gamma \\
&= * \otimes \underbrace{||^{\frac{1}{2}} \otimes \cdots \otimes ||^{\frac{1}{2}} \otimes ||^{-\frac{1}{2}} \otimes ||^{-\frac{1}{2}}}_{a\text{-terms}}.
\end{aligned}
$$

Now we see that the cross stabilizer should be $H_t \times S_a$.]

Now $[\tau_b : \mathcal{C}_b]$ is counted by the size of the following set

$$
\mathsf{PBP}_\star(\check{\mathcal{O}}_b) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath, \jmath) = \tau_b \\ \mathrm{Im}\mathcal{P} \subset \{ \bullet, c \}, \mathrm{Im}\mathcal{Q} \subset \{ \bullet, d \} \end{array} \right\}
$$

and for each $\mathscr{P} \subset \mathfrak{P}(\check{\mathcal{O}}_g)$, the multiplicity $[\tau_{\mathscr{P}} : \mathcal{C}_g]$ is counted by the size of

$$
\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \mathscr{P}) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath, \jmath) = \tau_{\mathscr{P}} \\ \mathrm{Im}\mathcal{P} \subset \{ \bullet, r, c, d \}, \mathrm{Im}\mathcal{Q} \subset \{ \bullet, s \} \end{array} \right\}
$$

and for each $\mathscr{P}$

Recall the definition of $\mathrm{PP}_{A^\mathbb{R}}(\check{\mathcal{O}}'_b)$ in (5.1).

Lemma 7.6. *For each $\tau' \in \mathrm{PP}_{A^\mathbb{R}}(\check{\mathcal{O}}'_b)$, there is a unique painted bipartition $\tau \in \mathsf{PBP}_\star(\check{\mathcal{O}}_b)$ such that*

$$
\mathcal{P}_{\tau'}(i,j) = d \quad \Leftrightarrow \quad \mathcal{Q}_\tau(i,j) = d \quad \forall (i,j) \in \mathrm{Box}(\imath)
$$

The map gives a bijection

$$
\begin{array}{ccccccc}
\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}_b(\mathbb{R})) & \longleftarrow & \mathrm{PP}_{A^\mathbb{R}}(\check{\mathcal{O}}'_b) & \longleftrightarrow & \mathsf{PBP}_\star(\check{\mathcal{O}}_b) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}_b \stackrel{r}{\sqcup} \check{\mathcal{O}}_1}(\mathrm{Sp}(2b,\mathbb{R})) \\
\pi_{\tau'} & \leftarrow\!\shortmid & \tau' & \mapsto & \tau & \mapsto & \pi_\tau.
\end{array}
$$

Here $\pi_{\tau'}$ is defined in (5.2) and

$$
\pi_\tau := \mathrm{Ind}_{\mathrm{GL}_b(\mathbb{R})}^{\mathrm{Sp}_{2b}(\mathbb{R})} \pi_{\tau'}.
$$

Proof. The claims about painted bipartitions are clear. The irreduciblity of π_τ follows from the multiplicity one of its wavefront cycle.

The representations $\pi_{\tau_1} \neq \pi_{\tau_2}$ if $\tau_1 \neq \tau_2$ since they have different cuspidal data by the construction. □

We define

$$
\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}_g) := \mathsf{PBP}_\star(\check{\mathcal{O}}_g) \times \overline{\mathsf{A}}(\check{\mathcal{O}}_g).
$$

Now we consider the good parity part. We defer the proof of the following lemma in the Appendix C

Lemma 7.7. *For each $\mathscr{P} \in\subset \mathfrak{P}(\check{\mathcal{O}}_g)$, we have*

$$
\left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \mathscr{P})\right| = \left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \varnothing)\right|.
$$

In particular, we have

$$
\mathrm{Unip}_\star(\check{\mathcal{O}}_g) = \left|\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}_g)\right|.
$$

8. Type $\widetilde{C}$

The left cell.

Lemma 8.1. *Suppose $\star = \widetilde{C}$, $\check{\mathcal{O}}_b$ has $2k$ rows. Here each row in $\check{\mathcal{O}}_b$ has odd length and each row in $\check{\mathcal{O}}_g$ has even length, and $W_{[\lambda_{\check{\mathcal{O}}}]} = W_b \times W'_g$ where $b = \frac{1}{2}\left|\check{\mathcal{O}}_b\right|$ and $g = \frac{1}{2}\left|\check{\mathcal{O}}_g\right|$. Let*

$$\tau_b = \Big((\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)+1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)+1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)+1}{2}), \\ (\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)-1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)-1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)-1}{2}) \Big) \in \mathrm{Irr}(W_b).$$

Set

$$\mathfrak{P}(\check{\mathcal{O}}_g) = \{\, (2i-1, 2i) \mid \mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i}(\check{\mathcal{O}}_g), \text{ and } i \in \mathbb{N}^+ \,\}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

For $\mathscr{P} \in \overline{\mathsf{A}}(\check{\mathcal{O}})$, let

$$\tau_{\mathscr{P}} := (\imath, \jmath) \in \mathrm{Irr}(W'_g)$$

such that for all $i \geqslant 1$

$$(\mathbf{c}_i(\imath), \mathbf{c}_i(\jmath)) := \begin{cases} (\frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g)) & \text{if } (2i-1, 2i) \notin \mathscr{P}, \\ (\frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g)) & \text{otherwise.} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \overline{\mathsf{A}}(\check{\mathcal{O}}) & \longrightarrow & {}^L\mathscr{C}(\check{\mathcal{O}}) \\ \mathscr{P} & \mapsto & \tau_b \otimes \tau_{\mathscr{P}}, \end{array}$$

such that $\tau_b \otimes \tau_{\mathscr{P}}$ is the special representation in ${}^L\mathscr{C}(\check{\mathcal{O}})$.

We remark that if $\mathfrak{P}(\check{\mathcal{O}}_g) = \varnothing$ the the representation $\tau_{\varnothing}$ has label I.

[The bad parity part is the same as the case when $\star = B$.

For the good parity part, note that the trivial representation of the trivial group has symbol

$$\begin{pmatrix} 0, 1, \cdots, r \\ 0, 1, \cdots, r \end{pmatrix}.$$

Here we assume $\check{\mathcal{O}}_g$ has at most $2r$ rows.

Now the bifurcation happens for the odd length column.]

The coherent continuation representation.

Fix n_b, n_g such that $n_b + n_g = n$. Let

$$\Lambda_{n_b,n_g} = (\underbrace{0, \cdots, 0}_{n_b\text{-terms}}, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_g\text{-terms}}) + Q. \tag{8.1}$$

We use Renard-Trapa's result.

Lemma 8.2. *We have the following formula on the coherent continuation representations based on Λ_{n_b,n_g}:*

$$\bigoplus_{p+q=n} \mathrm{Coh}_{\Lambda_{n_b,n_g}}(\mathrm{Mp}(2n,\mathbb{R})) \cong \mathcal{C}_b \otimes \mathcal{C}_g.$$

with

$$\mathcal{C}_g = \mathrm{Res}^{W'_{n_g}}_{W_{n_g}} \left(\bigoplus_{\substack{t,a,a'\in\mathbb{N}\\ 2t+a+a'=n_g}} \mathrm{Ind}^{W_{n_b}}_{H_t\times S_a\times S_{a'}} \widetilde{\mathrm{sgn}}\otimes 1\otimes \mathrm{sgn} \right)$$

$$\mathcal{C}_b = \bigoplus_{\substack{t,c,d\in\mathbb{N}\\ 2t+c+d=n_b}} \mathrm{Ind}^{W_{n_b}}_{H_t\times W_c\times W_d} \widetilde{\mathrm{sgn}}\otimes 1\otimes 1$$

Proof. This is contained in [82, 83]. By [83, Theorem 5.2] and [83, Corollary 4.5 (2)], $\mathrm{Coh}_{\Lambda_{n_b,n_g}}(\mathrm{Mp}(2n,\mathbb{R}))$ is dual to $\mathcal{C}_g\otimes\check{\mathcal{C}}_b$ where $\mathcal{C}_g$ is the coherent continuation representation based on the lattice Λ_{0,n_g} and

$$\begin{aligned}\check{\mathcal{C}}_{2n_b,\mathbb{R}} &\cong \bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=n_b}} \mathrm{Coh}_{\Lambda_{n_b,0}}(\mathrm{Sp}(p,q))\\ &= \bigoplus_{\substack{t,c,d\in\mathbb{N}\\ 2t+c+d=n_b}} \mathrm{Ind}^{W_{n_b}}_{H_t\times W_c\times W_d} \widetilde{\mathrm{sgn}}\otimes \mathrm{sgn}\otimes \mathrm{sgn}\end{aligned}$$

The main result in [82] implies that

$$\mathcal{C}_g = \mathrm{Res}^{W'_{n_g}}_{W_{n_g}} \left(\bigoplus_{\substack{t,a,a'\in\mathbb{N}\\ 2t+a+a'=n_g}} \mathrm{Ind}^{W_{n_b}}_{H_t\times S_a\times S_{a'}} \widetilde{\mathrm{sgn}}\otimes 1\otimes \mathrm{sgn} \right)$$

is self-dual.

Tensor with the sign representation yields the lemma. □

9. Type B

The dual group of $G_{\mathbb{C}} = \mathrm{SO}(2n+1,\mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{Sp}(2n,\mathbb{C})$. The odd is the bad parity, and even is the good parity.

Fix n_b, n_g such that $n_b + n_g = n$. Let

$$\Lambda_{n_b,n_g} = (\underbrace{\tfrac{1}{2},\cdots,\tfrac{1}{2}}_{n_b\text{-terms}},\underbrace{0,\cdots,0}_{n_g\text{-terms}}) + Q. \tag{9.1}$$

Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\mathrm{SO}(2n+1,\mathbb{C}))$ with decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$.

Let $\check{\mathcal{O}}'_b$ be the Young diagram such that $\mathbf{r}_i(\check{\mathcal{O}}'_b) = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ and $\mathcal{O}'_b$ be the transpose of $\check{\mathcal{O}}'_b$.

9.1. The left cell.

Lemma 9.1. *Suppose $\star = B$, $\check{\mathcal{O}}_b$ has $2k$ rows. Here each row in $\check{\mathcal{O}}_b$ has odd length and each row in $\check{\mathcal{O}}_g$ has even length, and $W_{[\lambda_{\check{\mathcal{O}}}]} = W_b \times W_g$ where $b = \frac{1}{2}\left|\check{\mathcal{O}}_b\right|$ and $g = \frac{1}{2}\left|\check{\mathcal{O}}_g\right|$. Let*

$$\begin{aligned}\tau_b = \Bigg(&(\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)+1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)+1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)+1}{2}),\\ &(\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)-1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)-1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)-1}{2}) \Bigg) \in \mathrm{Irr}(W_b).\end{aligned}$$

Set

$$\mathfrak{P}(\check{\mathcal{O}}_g) = \{\, (2i, 2i+1) \mid \mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i}(\check{\mathcal{O}}_g),\ \text{and } i \in \mathbb{N}^+ \,\}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

For $\mathscr{P} \in \overline{\mathsf{A}}(\check{\mathcal{O}})$, let

$$\tau_{\mathscr{P}} := (\imath, \jmath) \in \mathrm{Irr}(W_b)$$

such that

$$\mathbf{c}_1(\jmath) := \tfrac{1}{2}\mathbf{r}_1(\check{\mathcal{O}}_g)$$

and for all $i \geqslant 1$

$$(\mathbf{c}_i(\imath), \mathbf{c}_{i+1}(\jmath)) := \begin{cases} (\frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g)) & \text{if } (2i, 2i+1) \notin \mathscr{P}, \\ (\frac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g)) & \text{otherwise.} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \overline{\mathsf{A}}(\check{\mathcal{O}}) & \longrightarrow & {}^{L}\mathscr{C}(\check{\mathcal{O}}) \\ \mathscr{P} & \mapsto & \tau_b \otimes \tau_{\mathscr{P}}, \end{array}$$

such that $\tau_b \otimes \tau_{\mathscr{P}}$ is the special representation in ${}^{L}\mathscr{C}(\check{\mathcal{O}})$.

[In this case, bad parity is odd and every odd row occurs with with even times. We take the convention that $\dagger\mathcal{O} = [r_i + 1]$. By abuse of notation, let $\dagger_n\sigma$ denote the $j_{S_n \times W_{|\sigma|}}^{W_{n+|\sigma|}} \mathrm{sgn} \otimes \sigma$. We can write

$$\check{\mathcal{O}}_b = [2r_1+1, 2r_1+1, \cdots, 2r_k+1, 2r_k+1] = (2c_0, 2c_1, 2c_1, \cdots, 2c_l, 2c_l)$$

with $k = c_0$ and $l = r_1$.

$$\begin{aligned} W_{\lambda_{\check{\mathcal{O}}_b}} &= W_{c_0} \times S_{2c_1} \times S_{2c_2} \times \cdots \times S_{2c_l} \\ \check{\sigma}_b := \sigma_b \otimes \mathrm{sgn} &= j_{W_{\lambda_{\check{\mathcal{O}}_b}}}^{W_b} \mathrm{sgn} \\ &= \dagger_{2c_l} \cdots \dagger_{2c_1} \begin{pmatrix} 0, 1, \cdots, c_0 \\ 1, \cdots, c_0 \end{pmatrix} \\ &= \begin{pmatrix} 0, 1+r_k, 2+r_{k-1} \cdots, c_0 + r_1 \\ 1+r_k, 2+r_{k-1}, \cdots, c_0 + r_1 \end{pmatrix} \\ &= ([r_1, r_2, \cdots, r_k], [r_1+1, r_2+1, \cdots, r_k+1]) \\ &= ((c_1, c_2, \cdots, c_k), (c_0, c_1, \cdots, c_l)) \end{aligned}$$

Therefore

$$\begin{aligned} \sigma_b = \check{\sigma}_b \otimes \mathrm{sgn} &= ((r_1+1, r_2+1, \cdots, r_k+1), (r_1, r_2, \cdots, r_k)) \\ &= j_{S_{2r_1+1} \times \cdots S_{2r_k+1}}^{W_b} \mathrm{sgn} \\ &= j_{S_b}^{W_b}(2r_1+1, 2r_2+1, \cdots, 2r_k+1) \end{aligned}$$

which corresponds to the orbit

$$\mathcal{O}_b = (2r_1+1, 2r_1+1, 2r_2+1, 2r_2+1, \cdots, 2r_k+1, 2r_k+1) = \check{\mathcal{O}}_b^t.$$

(Note that $\mathcal{O}'_b = (2r_1+1, 2r_2+1, \cdots, 2r_k+1)$ which corresponds to $j_{W_{L_b}}^{S_b} \mathrm{sgn}$ and $\mathrm{ind}_L^G \mathcal{O}'_b = \mathcal{O}_b$.) The J-induction is calculated by [62, (4.5.4)]. It is easy to see that in our case $J_{W_{\lambda_{\check{\mathcal{O}}_b}}}^{W_b} \mathrm{sgn}$ consists of the single special representation by induction.

Now we consider the good parity parts.

Consider

$$\mathcal{O}_g = [2r_1, 2r_2, \cdots, 2r_{2k-1}, 2r_{2k}] = (C_1, C_1, C_2, C_2, \cdots, C_l, C_l).$$

with $l = r_1$ and $k = [C_1/2]$. Write ${}^{L}\check{\mathscr{C}}_{\check{\mathcal{O}}} = J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}$.

Note that the trivial representation of the trivial group has symbol

$$\begin{pmatrix} 0, 1, 2, \cdots, k \\ 0, 1, \cdots, k-1 \end{pmatrix}.$$

Now it easy to deduce that

$$\check{\sigma}_{[\underbrace{2r, 2r, \cdots, 2r}_{2k+1}]} = ([\underbrace{r, r, \cdots, r}_{k+1}], [\underbrace{r, r, \cdots, r}_{k}]), \text{ and}$$

$$ {}^{L}\check{\mathscr{C}}_{[\underbrace{2r', 2r', \cdots, 2r', 2r}_{2k+1 \text{ terms}}]} = \begin{cases} ([\underbrace{r', r', \cdots, r', r'}_{k+1}], [\underbrace{r', r', \cdots, r', r'}_{k+1}]) & \text{if } r' = r \geqslant 0 \\ ([\underbrace{r', r', \cdots, r', r}_{k+1}], [\underbrace{r', r', \cdots, r', r'}_{k}]) & \\ +([\underbrace{r', r', \cdots, r', r'}_{k+1}], [\underbrace{r', r', \cdots, r', r}_{k}]) & \text{if } r' > r \geqslant 0 \\ \text{(the first term is special)} & \end{cases}$$

Now

$$\begin{aligned} \check{\sigma}_{\check{\mathcal{O}}_g} &= J^{W_a}_{S_{C_1} \times \cdots \times S_{C_l}} \operatorname{sgn} \\ &= {}^{L}\check{\mathscr{C}}_{[\underbrace{2r_{2k}, 2r_{2k}, \cdots, 2r_{2k}, 2r_{2k+1}}_{2k+1\text{terms}}]} \\ &\quad \overset{r}{\sqcup} {}^{L}\mathscr{C}_{[2(r_1 - r_{2k}), 2(r_2 - r_{2k}), \cdots, 2(r_{2k-1} - r_{2k})]}. \end{aligned}$$

By induction on the number of columns, we conclude that ${}^{L}\check{\mathscr{C}}_{\check{\mathcal{O}}_g}$ is in one-one corresponds to the subsets of

$$\mathfrak{P}(\check{\mathcal{O}}_g) = \{\, (2i, 2i+1) \mid \mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i}(\check{\mathcal{O}}_g), \text{ and } i \in \mathbb{N}^+ \,\}.$$

For $\mathscr{P} \in \overline{\mathsf{A}}(\check{\mathcal{O}}_g)$, let $\sigma_{\mathscr{P}} = (\imath, \jmath)$ such that

$$\begin{aligned} \mathbf{r}_1(\imath) &:= \tfrac{1}{2}\mathbf{r}_1(\check{\mathcal{O}}_g) \\ (\mathbf{r}_{l+1}(\imath), \mathbf{r}_l(\jmath)) &:= \begin{cases} (\frac{1}{2}\mathbf{r}_{2l+1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2l}(\check{\mathcal{O}}_g)) & \text{if } (2l, 2l+1) \notin \mathscr{P} \\ (\frac{1}{2}\mathbf{r}_{2l}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2l+1}(\check{\mathcal{O}}_g)) & \text{otherwise} \end{cases} \end{aligned}$$

Note that according to Lusztig and BV, the subsets of $\mathfrak{P}(\check{\mathcal{O}})$ is in one-one correspondence to the canonical quotient $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}})]$.]

9.2. Counting special unipotent representation of $G = \mathrm{SO}(2p+1, 2q)$ with $p+q = n$. .

The dual group of $G_{\mathbb{C}} = \mathrm{SO}(2n, \mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{SO}(2n+1, \mathbb{C})$.

Let

$$\Lambda_{n_1, n_2} = (\underbrace{0, \cdots, 0}_{n_1\text{-terms}}, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_2\text{-terms}},)$$

We set $(2n_g, 2n_b) = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$.

Lemma 9.2. *We have the following formula on the coherent continuation representations based on Λ_{n_1,n_2}:*

$$\bigoplus_{p+q=n} \mathrm{Coh}_{\Lambda_{n_1,n_2}}(\mathrm{SO}(2p+1, 2q)) \cong \mathcal{C}_b \otimes \mathcal{C}_g.$$

with

$$\mathcal{C}_g = \bigoplus_{\substack{t,s,r\in\mathbb{N}\\ 2t+s+r=n_g}} \operatorname{Ind}_{H_t\times S_a\times W_s\times W_r}^{W_{n_g}} \widetilde{\operatorname{sgn}}\otimes 1\otimes \operatorname{sgn}\otimes\operatorname{sgn}$$

$$\mathcal{C}_b = \bigoplus_{\substack{t,c,d\in\mathbb{N}\\ 2t+c+d=n_b}} \operatorname{Ind}_{H_t\times W_c\times W_d}^{W_{n_b}} \widetilde{\operatorname{sgn}}\otimes 1\otimes 1$$

From now on, we set $(2n_1, 2n_2) := (2g, 2b) := (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$.

Clearly $[\tau_b : \mathcal{C}_b]$ is counted by the size of the following set

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_b) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \,\middle|\, \begin{array}{l} (\imath,\jmath) = \tau_b \\ \operatorname{Im}\mathcal{P} \subset \{\bullet, c, d\}, \operatorname{Im}\mathcal{Q}\subset\{\bullet\} \end{array} \right\}$$

and for each $\mathscr{P} \subset \mathfrak{P}(\check{\mathcal{O}}_g)$, the multiplicity $[\tau_{\mathscr{P}} : \mathcal{C}_g]$ is counted by the size of

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \mathscr{P}) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \,\middle|\, \begin{array}{l} (\imath,\jmath) = \tau_{\mathscr{P}} \\ \operatorname{Im}\mathcal{P} \subset \{\bullet, c\}, \operatorname{Im}\mathcal{Q}\subset\{\bullet, s, r, d\} \end{array} \right\}$$

and for each $\mathscr{P}$.

Recall the definition of $\mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b)$ in (5.1).

Lemma 9.3. *For each $\tau' \in \mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b)$, there is a unique painted bipartition $\tau \in \mathsf{PBP}_\star(\check{\mathcal{O}}_b)$ such that*

$$\mathcal{P}_{\tau'}(i,j) = d \quad\Leftrightarrow\quad \mathcal{P}_\tau(i,j) = d \quad \forall (i,j)\in \operatorname{Box}(\imath)$$

The map gives a bijection

$$\begin{array}{ccccccc} \operatorname{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}_b(\mathbb{R})) & \longleftarrow & \mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b) & \longleftrightarrow & \mathsf{PBP}_\star(\check{\mathcal{O}}_b) & \longrightarrow & \operatorname{Unip}_{\check{\mathcal{O}}_b}(\mathrm{SO}(2b+1,2b)) \\ \pi_{\tau'} & \longleftarrow\!\shortmid & \tau' & \mapsto & \tau & \mapsto & \pi_\tau. \end{array}$$

Here $\pi_{\tau'}$ is defined in (5.2) *and*

$$\pi_\tau := \operatorname{Ind}_{\mathrm{GL}_b(\mathbb{R})\times\mathrm{SO}(1,0)}^{\mathrm{SO}(2b+1,2b)} \pi_{\tau'}.$$

Proof. The claims about painted bipartitions are clear. The irreduciblity of π_τ follows from the multiplicity one of its wavefront cycle.

The representations $\pi_{\tau_1} \neq \pi_{\tau_2}$ if $\tau_1 \neq \tau_2$ since they have different cuspidal data by the construction. $\square$

Now we consider the good parity part. We defer the proof of the following lemma in the Appendix C

Lemma 9.4. *For each $\mathscr{P} \in\subset \mathfrak{P}(\check{\mathcal{O}}_g)$, we have*

$$|\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \mathscr{P})| = |\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \emptyset)|.$$

In particular, we have

$$\operatorname{Unip}_\star(\check{\mathcal{O}}_g) = |\mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}_g)|.$$

Here $\operatorname{Unip}_\star(\check{\mathcal{O}}_g)$ denote the set of all unipotent representations attached to the inner class $\mathrm{SO}(2p+1,2q)$.

10. Type D

Let $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$.

10.1. The left cell ${}^{L}\mathscr{C}_{\check{\mathcal{O}}}$.

Lemma 10.1. *Suppose $\star \in \{ D, D^* \}$, $\check{\mathcal{O}}_b$ has $2k$ rows and $\check{\mathcal{O}}_g$ has $2l$-rows. Here each row in $\check{\mathcal{O}}_b$ has even length and each row in $\check{\mathcal{O}}_g$ has odd length, and $W_{[\lambda_{\check{\mathcal{O}}}]} = W'_b \times W'_g$ where $b = \frac{|\check{\mathcal{O}}_b|}{2}$ and $g = \frac{|\check{\mathcal{O}}_g|}{2}$. Let*

$$\tau_b = \left((\tfrac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \tfrac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \tfrac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b), (\tfrac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \tfrac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \tfrac{1}{2}\mathbf{r}_{2k}(\check{\mathcal{O}}_b)\right)_I \in \mathrm{Irr}(W'_b).$$

Set

$$\mathfrak{P}(\check{\mathcal{O}}_g) = \{ (2i, 2i+1) \mid \mathbf{r}_{2i}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) > 0, \text{ and } i \in \mathbb{N}^+ \}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

For $\mathscr{P} \in \overline{\mathsf{A}}(\check{\mathcal{O}})$, let

$$\tau_{\mathscr{P}} := (\imath, \jmath) \in \mathrm{Irr}(W'_g)$$

such that

$$\mathbf{c}_1(\imath) := \tfrac{1}{2}(\mathbf{r}_1(\check{\mathcal{O}}_g) + 1)$$
$$(\mathbf{c}_{l+1}(\imath), \mathbf{c}_l(\jmath)) := (0, \tfrac{1}{2}(\mathbf{r}_{2l}(\check{\mathcal{O}}_g) - 1))$$

and for all $1 \leqslant i < l$

$$(\mathbf{c}_{i+1}(\imath), \mathbf{c}_i(\jmath)) := \begin{cases} (\frac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) - 1)) & \text{if } (2i, 2i+1) \notin \mathscr{P}, \\ (\frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) - 1)) & \text{otherwise.} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \overline{\mathsf{A}}(\check{\mathcal{O}}) & \longrightarrow & {}^{L}\mathscr{C}(\check{\mathcal{O}}) \\ \mathscr{P} & \mapsto & \tau_b \otimes \tau_{\mathscr{P}}. \end{array}$$

such that $\tau_b \otimes \tau_{\mathscr{P}}$ is the special representation in ${}^{L}\mathscr{C}(\check{\mathcal{O}})$.

[The bad parity part is the same as that of the case when $\star = C$.

For the good parity part. Suppose $\check{\mathcal{O}}_g = (2c_1, C_2, C_2, C_3, C_3, \cdots, C_{k'}, C_{k'}, C_{k'+1})$ with $2c_1 = 2l$ and $2k' + 2 = \mathbf{r}_1(\check{\mathcal{O}}_g)$.

We use the two facts:

$$W_{\lambda_{\check{\mathcal{O}}_g}} = W_{c_1} \times S_{C_2} \times \cdots \times S_{C_{k'}}.$$

The symbol of sign representation of W'_{c_1} is

$$\begin{pmatrix} 0, 1, \cdots, c_1 - 1 \\ 1, 2, \cdots, c_1 \end{pmatrix}.$$

The bifurcation happens for even length columns.

]

Let

$$\Lambda_{n_1,n_2} = (\underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_1\text{-terms}}, \underbrace{0, \cdots, 0}_{n_2\text{-terms}},)$$

We set $(2n_g, 2n_b) = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$.

Lemma 10.2. *We have the following formula on the coherent continuation representations based on Λ_{n_1,n_2}:*

$$\bigoplus_{p+q=2n} \mathrm{Coh}_{\Lambda_{n_1,n_2}}(\mathrm{SO}(p,q)) \cong \mathcal{C}_b \otimes \mathcal{C}_g.$$

with

$$\mathcal{C}_g = \bigoplus_{\substack{t,c,d,s,r\in\mathbb{N}\\ 2t+c+d+s+r=n_g}} \mathrm{Ind}_{H_t\times W_s\times W_r\times W_c\times W_d}^{W_{n_g}} \widetilde{\mathrm{sgn}}\otimes\mathrm{sgn}\otimes\mathrm{sgn}\otimes 1\otimes 1$$

$$\mathcal{C}_b = \bigoplus_{\substack{t,a\in\mathbb{N}\\ 2t+a=n_1}} \mathrm{Ind}_{H_t\times S_a}^{W_{n_1}} \widetilde{\mathrm{sgn}}\otimes 1$$

Clearly $[\tau_b : \mathcal{C}_b]$ is counted by the size of the following set

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_b) := \left\{ (\imath,\jmath,\mathcal{P},\mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath,\jmath)=\tau_b \\ \mathrm{Im}\mathcal{P}\subset\{\bullet,c,d\}, \mathrm{Im}\mathcal{Q}\subset\{\bullet\} \end{array} \right\}$$

and for each $\mathscr{P}\subset\mathfrak{P}(\check{\mathcal{O}}_g)$, the multiplicity $[\tau_{\mathscr{P}} : \mathcal{C}_g]$ is counted by the size of

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g,\mathscr{P}) := \left\{ (\imath,\jmath,\mathcal{P},\mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath,\jmath)=\tau_{\mathscr{P}} \\ \mathrm{Im}\mathcal{P}\subset\{\bullet,s,r,c,d\}, \mathrm{Im}\mathcal{Q}\subset\{\bullet\} \end{array} \right\}$$

and for each $\mathscr{P}$.

Lemma 10.3. *For each $\tau'\in\mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b)$, there is a unique painted bipartition $\tau\in\mathsf{PBP}_\star(\check{\mathcal{O}}_b)$ such that*

$$\mathcal{P}_{\tau'}(i,j)=d \quad\Leftrightarrow\quad \mathcal{P}_\tau(i,j)=d \quad \forall (i,j)\in\mathrm{Box}(\imath)$$

The map gives a bijection

$$\begin{array}{ccccccc} \mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}_b(\mathbb{R})) & \longleftarrow & \mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b) & \longleftrightarrow & \mathsf{PBP}_\star(\check{\mathcal{O}}_b) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{SO}(2b+1,2b)) \\ \pi_{\tau'} & \longleftarrow\!\shortmid & \tau' & \mapsto & \tau & \mapsto & \pi_\tau. \end{array}$$

Here $\pi_{\tau'}$ is defined in (5.2) *and*

$$\pi_\tau := \mathrm{Ind}_{\mathrm{GL}_b(\mathbb{R})}^{\mathrm{SO}(2b,2b)}\pi_{\tau'}.$$

Proof. The claims about painted bipartitions are clear. The irreduciblity of π_τ follows from the multiplicity one of its wavefront cycle.

The representations $\pi_{\tau_1}\neq\pi_{\tau_2}$ if $\tau_1\neq\tau_2$ since they have different cuspidal data by the construction. □

Appendix A. Combinatorics of Weyl group representations in the classical types

A.1. **The $\jmath$-induction.** If μ and ν are two partitions representing two symmetric groups representations. Then

$$\jmath_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}}\mu\boxtimes\nu = \mu\cup\nu,$$

where $\mu\cup\nu$ is the partition such that

$$\{\, \mathbf{c}_i(\mu\cup\nu) \mid i\in\mathbb{N}^+ \,\} = \{\, \mathbf{c}_i(\mu) \mid i\in\mathbb{N}^+ \,\}\cup\{\, \mathbf{c}_i(\nu) \mid i\in\mathbb{N}^+ \,\}$$

as multisets.

[Use the inductive by stage of $\jmath$-induction

$$\begin{aligned} &\jmath_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}}\mu\boxtimes\nu \\ &= \jmath_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}}\jmath_{\prod_i S_{\mathbf{c}_i(\mu)}\times\prod_j S_{\mathbf{c}_j(\nu)}}^{S_{|\mu|}\times S_{|\nu|}}\mathrm{sgn} \\ &= \jmath_{\prod_i S_{\mathbf{c}_i(\mu\cup\nu)}}^{S_{|\mu|+|\nu|}}\mathrm{sgn} \\ &= \mu\cup\nu. \end{aligned}$$

]

We have

$$j_{S_n}^{W_n}\,\mathrm{sgn} = \begin{cases} (\square_k, \square_k) & \text{if } n = 2k \text{ is even,} \\ (\square_{k+1}, \square_k) & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

[The symbol of trivial of trivial group is

$$\binom{0,1,\cdots,k}{0,\cdots,k-1}.$$

Apply the formula [62, 4.5.4], When n is even, the symbol of the induce is

$$\binom{0,2,\cdots,k+1}{1,\cdots,k}.$$

corresponds to $(\square_k, \square_k)$.

When n is even, the symbol of the induce is

$$\binom{1,2,\cdots,k+1}{1,\cdots,k}.$$

corresponds to $(\square_{k+1}, \square_k)$.]

If $\tau = (\tau_L, \tau_R)$ and $\sigma = (\sigma_L, \sigma_R)$ be two bipartition. Then

$$j_{W_{|\tau|}\times W_{|\sigma|}}^{W_{|\tau|+|\sigma|}} \tau \boxtimes \sigma = (\tau_L \cup \sigma_L, \tau_R \cup \sigma_R)$$

Appendix B. Remarks on the Counting theorem of unipotent representations

B.1. **Coherent family.** For each finite dimensional $\mathfrak{g}$-module or $G_{\mathbb{C}}$-module F, let F^* be its contragredient representation and let $\Delta(F) \subseteq {}^aX$ denote the multi-set of weights in F.

Let $\Pi_{\Lambda_0}(G_{\mathbb{C}})$ be the set of irreducible finite dimensional representations of $G_{\mathbb{C}}$ with extreme weight in Λ_0 and $\mathcal{G}_\Lambda(G_{\mathbb{C}})$ be the subgroup generated by $\Pi_{\Lambda_0}(G_{\mathbb{C}})$. Let

$${}^aP := \{\, \mu \in {}^aX \mid \mu \text{ is a } {}^a\mathfrak{h}\text{-weight of an } F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}}) \,\}.$$

Via the highest weight theory, every W-orbit $W \cdot \mu$ in ${}^aP$ corresponds with the irreducible finite dimensional representation $F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}})$ with extremal weight μ.

Now the Grothendieck group $\mathcal{G}(G_{\mathbb{C}})$ of finite dimensional representation of $G_{\mathbb{C}}$ is identified with $\mathbb{Z}[{}^aP/W]$. In fact $\mathcal{G}(G_{\mathbb{C}})$ is a $\mathbb{Z}$-algebra under the tensor product and equipped with the involution $F \mapsto F^*$.

Fix a W-invariant sub-lattice $\Lambda_0 \subset {}^aX$ containing ${}^aQ$.

For any $\lambda \in {}^a\mathfrak{h}^*$, we define

$$\begin{aligned} [\lambda] &:= \lambda + {}^aQ, \\ R_{[\lambda]} &:= \{\, \alpha \in {}^aR \mid \langle \lambda, \check\alpha \rangle \in \mathbb{Z} \,\}, \\ W_{[\lambda]} &:= \langle s_\alpha | \alpha \in R_{[\lambda]} \rangle \subseteq W, \\ R_\lambda &:= \{\, \alpha \in {}^aR \mid \langle \lambda, \check\alpha \rangle = 0 \,\}, \quad \text{and} \\ W_\lambda &:= \langle s_\alpha | \alpha \in R_\lambda \rangle = \langle w \in W | w \cdot \lambda = \lambda \rangle \subseteq W. \end{aligned} \tag{B.1}$$

For any lattice $\Lambda = \lambda + \Lambda_0 \in \mathfrak{h}^*/\Lambda_0$ with $\lambda \in \overline{\mathsf{C}}$, we define

$$W_\Lambda := \{\, w \in W \mid w \cdot \Lambda = \Lambda \,\}.$$

Clearly, we have

$$W_{[\lambda]} < W_\Lambda, \quad \forall\ \lambda \in \Lambda.$$

Definition B.1. *Suppose $\mathcal{M}$ is an abelian group with $\mathcal{G}_\Lambda(G_{\mathbb{C}})$-action*

$$\mathcal{G}_\Lambda(G_{\mathbb{C}}) \times \mathcal{M} \ni (F, m) \mapsto F \otimes m.$$

In addition, we fix a subgroup $\mathcal{M}_{\bar{\mu}}$ of $\mathcal{M}$ for each for each W_Λ-orbit $\bar{\mu} = W_\Lambda \cdot \mu \in \Lambda/W_\Lambda$.

A function $f\colon \Lambda \to \mathcal{M}$ is called a coherent family based on Λ if it satisfies $f(\mu) \in \mathcal{M}_\mu$ and

$$F \otimes f(\mu) = \sum_{\nu \in \Delta(F)} f(\mu + \nu) \qquad \forall \mu \in \Lambda, F \in \Pi_{\Lambda_0}(G_{\mathbb{C}}).$$

Let $\mathrm{Coh}_\Lambda(\mathcal{M})$ be the abelian group of all coherent families based on Λ and value in $\mathcal{M}$.

In this paper, we will consider the following cases.

Example B.2. *Suppose $\mathcal{M} = \mathbb{Q}$ and $F \otimes m = \dim(F) \cdot m$ for $F \in \Pi_{\Lambda_0}(G_{\mathbb{C}})$ and $m \in \mathcal{M}$. We let $\mathcal{M}_{\bar{\mu}} = \mathcal{M}$ for every $\mu \in \Lambda$. When $\Lambda = \Lambda_0$, the set of W-harmonic polynomials on ${}^a\mathfrak{h}^*$ is naturally identified with $\mathrm{Coh}_\Lambda(\mathcal{M})$ via restriction (Vogan's result)*

Example B.3. *Let $\mathcal{G}(\mathfrak{g}, K)$ be the Grothendieck group of finite length $(\mathfrak{g}, K)$-modules and $\mathcal{G}_\chi(\mathfrak{g}, K)$ be the subgroup of $\mathcal{G}(\mathfrak{g}, K)$ generated by the set of irreducible $(\mathfrak{g}, K)$-modules with infinitesimal character χ.*

Then $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$ is the group of coherent families of Harish-Chandra modules. The space $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$ is equipped with a W_Λ-action by

$$w \cdot f(\mu) = f(w^{-1}\mu) \qquad \forall \mu \in \Lambda, w \in W_\Lambda, f \in \mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K)).$$

Example B.4. *Fix a $G_{\mathbb{C}}$-invariant closed subset $\mathcal{Z}$ in the nilpotent cone of $\mathfrak{g}$. Let $\mathcal{G}_{\mathcal{Z}}(\mathfrak{g}, K)$ be the Grothendieck group of $(\mathfrak{g}, K)$-modules whose complex associated varieties are contained in $\mathcal{Z}$. We define*

$$\mathcal{G}_{\chi,\mathcal{Z}}(\mathfrak{g}, K) := \mathcal{G}_\chi(\mathfrak{g}, K) \cap \mathcal{G}_{\mathcal{Z}}(\mathfrak{g}, k).$$

Now $\mathrm{Coh}_\Lambda(\mathcal{G}_{\mathcal{Z}}(\mathfrak{g}, K))$ is also a W_Λ-submodule of $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$.

Example B.5. *Fixing a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$, let $\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ be the Grothendieck group of the category $\mathcal{O}$. The space $\mathcal{G}_{\mathcal{Z}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ is defined similarly. The space $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}))$ and $\mathrm{Coh}_{\Lambda(\mathcal{G})}$ defined similarly.*

We can define W_Λ action on $\mathrm{Coh}_\Lambda(\mathcal{M})$ by

$$w \cdot f(\mu) = f(w^{-1}\mu) \qquad \forall \mu \in \Lambda, w \in W_\Lambda.$$

Example B.6. *For each infinitesimal character χ and a close G-invariant set $\mathcal{Z} \in \mathcal{N}_{\mathfrak{g}}$. Let $\mathcal{G}_{\chi,\mathcal{Z}}(\mathfrak{g}, K)$ be the Grothendieck group of $(\mathfrak{g}, K)$-module with infinitesimal character χ and complex associated variety contained $\mathcal{Z}$. Similarly, let $\mathcal{G}_{\chi,\mathcal{Z}}()$*

Translation principal assumption. Recall that we have fixed a set of simple roots of W_Λ.

We make the following assumption for $\mathrm{Coh}_\Lambda(\mathcal{M})$.

- There is a basis $\{\, \Theta_\gamma \mid \gamma \in \mathrm{Parm} \,\}$ of $\mathrm{Coh}_\Lambda(\mathcal{M})$ where Parm is a parameter set;
- For every $\mu \in \Lambda$, the evaluation at μ is surjective;

  $$\mathrm{ev}_\mu\colon \mathrm{Coh}_\Lambda(\mathcal{M}) \to \mathcal{M}_{\bar{\mu}} \qquad f \mapsto f(\mu)$$

  is surjective;
- a subset $\tau(\gamma)$ of the simple roots of W_Λ is attached to each $\gamma \in \mathrm{Parm}$ such that $s \cdot \Theta_\gamma = -\Theta_\gamma$;
- $\Theta_\gamma(\mu) = 0$ if and only if $\tau(\gamma) \cap R_\mu \neq \varnothing$.
- $\{\, \Theta_\gamma(\mu) \mid \tau(\gamma) \cap R_\mu = \varnothing \,\}$ form a basis of $\mathcal{M}_{\bar{\gamma}}$.

The translation principle assumption implies

$$\operatorname{Ker}\operatorname{ev}_\mu = \operatorname{span}\{\,\Theta_\gamma \mid \tau(\gamma)\cap R_\mu \neq \varnothing\,\} \tag{B.2}$$

Now we have the following counting lemma.

Lemma B.7. *For each μ, we have*

$$\dim \mathcal{M}_{\bar\mu} = \dim(\operatorname{Coh}_\Lambda(\mathcal{M}))_{W_\mu} = [\operatorname{Coh}_\Lambda(\mathcal{M}), 1_{W_\mu}].$$

Proof. Clearly

$$\operatorname{span}\{\,\Theta_\gamma - w\Theta_\gamma \mid \gamma\in\operatorname{Parm}, w\in W_\Lambda\,\}\subseteq \ker\operatorname{ev}_\mu$$

since $w\cdot\Theta_\gamma(\mu) = \Theta_\gamma(w^{-1}\cdot\mu) = \Theta_\gamma(\mu)$ for $w\in W_\mu$. Combine this with (B.2), we conclude that

$$\operatorname{span}\{\,\Theta_\gamma - w\Theta_\gamma \mid \gamma\in\operatorname{Parm}, w\in W_\Lambda\,\} = \ker\operatorname{ev}_\mu.$$

Therefore, ev_μ induces an isomorphism $(\operatorname{Coh}_\Lambda(\mathcal{M}))_{W_\mu}\to\mathcal{M}_{\bar\mu}$. Now the dimension equality follows. □

Example B.8. *For the case of category $\mathcal{O}$. We can take* Parm $= W$. *Let $\Theta_w(\mu) = L(w\mu)$ for each $w\in W_\Lambda$ with $\tau(w) = \{\, s_\alpha \mid \alpha\in\Delta^+, w\alpha\notin R^+\,\}$.*

Example B.9. *In the Harish-Chandra module case,* Parm *consists of certain irreducible K-equivariant local systems on a K-orbit of the flag variety of G. $\tau(\gamma)$ is the τ-invariants of the parameter γ.*

B.2. **Primitive ideals and left cells.** In this section, we recall some results about the primitive ideals and cells developed by Barbasch Vogan, Joseph and Lusztig etc.

B.3. **Highest weight module.** Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{b} = \mathfrak{h}\oplus\mathfrak{n}$ is a fixed Borel. Let

$$M(\lambda) := \mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}\mathbb{C}_{\lambda-\rho}$$

and $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$.

In the rest of this section, we fix a lattice $\Lambda\in\mathfrak{h}^*/X^*$. Let R_Λ and R_Λ^+ be the set of integral root system and the set of positive integral roots. Write W_Λ for the integral Weyl group.

For $\mu\in\Lambda$,

$$\begin{array}{ccccc} \mu\geq 0 & \Leftrightarrow & \mu \text{ is dominant} & \Leftrightarrow & \langle\lambda,\check\alpha\rangle\geqslant 0 \quad \forall\alpha\in R_\Lambda^+ \\ \mu\not> 0 & \Leftrightarrow & \mu \text{ is regular} & \Leftrightarrow & \langle\lambda,\check\alpha\rangle\neq 0 \quad \forall\alpha\in R_\Lambda^+ \end{array}$$

regular if $\langle\lambda,\check\alpha\rangle\neq 0$.

For each $w\in W$, it give a coherent family such that

$$M_w(\mu) = M(w\mu)\quad\forall\mu\in\Lambda$$

Let L_w be the unique coherent family such that $L_w(\mu) = L(w\mu)$ for any regular dominant μ in Λ. [Note that the Grothendieck group $\mathcal{K}(\mathcal{O})$ of category $\mathcal{O}$ is naturally embedded in the space of formal character. M_w is a coherent family: the formal character $\operatorname{ch} M_w(\mu)$ of $M_w(\mu)$ is

$$\operatorname{ch} M_w(\mu) = \frac{e^{w\mu-\rho}}{\prod_{\alpha\in R^+}(1-e^{-\alpha})} = \frac{e^{w\mu}}{\prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

It is clear that $\operatorname{ch} M_w$ satisfies the condition for coherent continuation.

From now on, we fix a regular dominant weight $\lambda\in\mathfrak{h}^*$. Then $w[\lambda] = [w\cdot\lambda]$ for any $w\in W$.

Now $W_{w[\lambda]} = w\,W_{[\lambda]}\,w^{-1}$.

As $W_{[\lambda]}$-module, we have the following decomposition

$$\mathrm{Coh}_{[\lambda]} = \bigoplus_{r \in W/W_{[\lambda]}} \mathrm{Coh}_r \quad \text{with}$$

$$\mathrm{Coh}_r = \mathrm{Coh}_{r\lambda} := \{\, \Theta \in \mathrm{Coh}_{[\lambda]} \mid \Theta(\lambda) \in \mathcal{O}'_{[r\cdot\lambda]} \,\}$$

Here $\mathcal{O}'_S$ is the set of highest weight module whose $\mathfrak{h}$-weights are in $S \subset \mathfrak{h}^*$.
Note that the following map

$$\begin{array}{ccccc} \mathbb{C}[W] & \longrightarrow & \mathrm{Coh}_{[\lambda]} & \longrightarrow & \mathbb{C}[S] \\ w & \mapsto & (\mu \mapsto M_w(\mu)) & \mapsto & w \cdot \lambda \end{array}$$

is $W_{[\lambda]}$-equivariant where $W_{[\lambda]}$ acts by right translation on $\mathbb{C}[W]$ and $S = W \cdot \lambda$. The action of $W_{[\lambda]}$ on S is by transport of structure and so $(a, w\,\lambda) \mapsto wa^{-1}\,\lambda$.
Now $\mathbb{C}[rW_{[\lambda]}] \subset \mathbb{C}[W]$ and $\mathbb{C}[rW_{[\lambda]} \cdot \lambda]$ are identified with Coh_r.
]
For each $w \in W_\Lambda$, the function

$$\widetilde{p}_w(\mu) := \mathrm{rank}(\mathcal{U}(\mathfrak{g})/\operatorname{Ann}(L(w\mu))) \qquad \forall \mu \in \Lambda \text{ dominant.}$$

extends to a Harmonic polynomial on $\mathfrak{h}^*$. In particular, $\widetilde{p}_w \in P(\mathfrak{h}^*) = S(\mathfrak{h})$. [Let Λ^+ be the set of dominant weights in Λ. Assume $\lambda > 0$, i.e. dominant regular. Then $\mathbb{R}^0_\lambda =$, $w^{w\lambda} = w^{-1}$. The set

$$\hat{F}_{w\lambda} = \{\, \mu \in \Lambda \mid w^{-1}\mu \geq 0 \,\} = w\Lambda^+.$$

Joseph's $p_w(\mu) = \mathrm{rank}\,\mathcal{U}(\mathfrak{g})/(\operatorname{Ann} L(w\mu))$ is defined on $\hat{F}_{w\lambda} = w\Lambda^+$. Now his $\widetilde{p}_w(\mu) = w^{-1}p_w$ is defined on Λ^+ and given by $\widetilde{p}_w(\mu) = \mathrm{rank}\,\mathcal{U}(\mathfrak{g})/(\operatorname{Ann} L(w\mu))$.]
There is a unique function $a_\Lambda \colon W_\Lambda \times W_\Lambda \to \mathbb{Q}$ such that

$$L_w = \sum_{w' \in W_\Lambda} a_\Lambda(w, w') M_{w'}$$

Suppose V is a module in $\mathcal{O}$ whose Gelfand-Kirillov dimension $\leqslant d$ Let $c(V)$ be the Bernstein degree. Let $\mathrm{Coh}^d_{[\Lambda]}$ be the submodule of $\mathrm{Coh}_{[\Lambda]}$ generated by L_w such that the $\dim L(w\mu) \leqslant d$. Then the map

$$\mathcal{P}^d : \mathrm{Coh}^d_\Lambda \to S(\mathfrak{h}) \qquad \Theta \mapsto (\mu \mapsto c(\Theta(\mu)))$$

is W_Λ-equivariant. Let $c_w := c \circ (L_w(\mu)) \in S(\mathfrak{h})$.
The following result of Joseph implies that the set of primitive ideals can be parameterized by Goldie rank polynomials.

Theorem B.10 (Joseph)**.** *correct formulation? For μ dominant, the primitive ideal* $\operatorname{Ann} L(w\mu) = \operatorname{Ann} L(w'\mu)$ *if and only if* $p_w = p'_w$. *(*[43, ref?]*)*

Theorem B.11. *Let* $r = \dim \mathfrak{n}$. *Suppose* $\dim L(w\nu) = d$.

(i) Let $x \in \mathfrak{h}$ such that $\alpha(x) = 1$ for every simple roots α. Then there are non-zero rational numbers a, a' such that [?J12, Theorem 5.1 and 5.7]

$$a\,\widetilde{p}_w(\mu) = a'\,c_w(\mu) = \sum_{w' \in W_\Lambda} a_\Lambda(w, w')\,\langle \mu, w'^{-1}x \rangle^{r-d}.$$

(ii) The submodule $\mathbb{C}W_\Lambda\widetilde{p}_w$ in $S(\mathfrak{h})$ is a special irreducible W_Λ-representation. Moreover all special representations of W_Λ occurs as a certain $\sigma(w)$. See [10, Theorem D], [11, Theorem 1.1].

(iii) Let $\sigma(w)$ be the submodule of $S(\mathfrak{h})$ generated by $\widetilde{p}_w$. Then $\sigma(w)$ is irreducible and it occurs in $S^{r-d}(\mathfrak{h})$ with multiplicity 1*, see* [48, 5.3],

By Theorem B.10, the module $\sigma(w)$ only depends on the primitive ideal $\mathcal{J} := \operatorname{Ann} L(w\mu)$. So we can also write $\sigma(\mathcal{J}) := \sigma(w)$.

From the above theorem, we see that the map $\mathcal{P}^d$ factor through $\mathrm{Coh}^d_\Lambda/\mathrm{Coh}^{d-1}_\Lambda$ and its image is contained in the space $\mathcal{H}^d$ of degree d-Harmonic polynomials.

Note that the associated variety of a primitive ideal $\mathcal{J}$ is the closure of a nilpotent orbit in $\mathfrak{g}$. Now the associated variety of $\mathcal{J}$ is computed by the following result of Joseph (and include Hotta ?).

Theorem B.12. *Let $\mathcal{J}$ be a primitive ideal in $\mathcal{U}(\mathfrak{g})$. Then $\sigma(\mathcal{J})$ is in the image of Springer correspondence. Moreover, the associated variety of $\mathcal{J}$ is the orbit $\mathcal{O}$ corresponding to $\sigma(\mathcal{J})$. See* [46, Proposition 2.10].

Lemma B.13. *Let $\mu \in \Lambda$, $W_\mu = \{\, w \in W_\Lambda \mid w\cdot\mu = \mu \,\}$ and*

$$a_\mu = \max\{\, a(\sigma) \mid \widehat{W_\Lambda} \ni \sigma \text{ occurs in } \operatorname{Ind}_{W_\mu}^{W_\Lambda} 1 \,\}.$$

Suppose W_μ is a parabolic subgroup of W_Λ. Then the set of all irreducible representations σ occurs in $\operatorname{Ind}_{W_\mu}^{W_\Lambda} 1$ such that $a(\sigma) = a_\mu$ forms a left cell given by

$$\left(J_{W_\mu}^{W_\Lambda} \operatorname{sgn}\right) \otimes \operatorname{sgn}.$$

Now the maximal primitive ideal having infinitesimal character μ has the associated variety $\mathcal{O}$.

Recall that $W_\Lambda = W_b \times W_g$. Let

$$\sigma_b = (j_{W_{\check{\mathcal{O}}_b}}^{W_b} \operatorname{sgn}) \otimes \operatorname{sgn}$$
$$\sigma_g = (j_{W_{\check{\mathcal{O}}_g}}^{W_g} \operatorname{sgn}) \otimes \operatorname{sgn}$$

Then $\sigma_b \otimes \sigma_g$ is a special representation of W_Λ. Moreover $\mathcal{O}$ corresponds to the representation

$$\sigma := j_{W_b\times W_g}^{W} \sigma_b \otimes \sigma_g$$

[We recall some facts about the j-induction and J-induction. If W' is a parabolic subgroup in W then $j_{W'}^W$ maps special representation to special representation. It maps irreducible representation to irreducible ones. The j induction has induction by stage [20, 11.2.4].

For classical group, the representations satisfies property B. When W' is parabolic, the orbit $\mathcal{O}$ corresponds to $j_{W'}^W$ is the orbit $\operatorname{Ind}_L^G \mathcal{O}$.

The J-induction maps left cell to left cell, which is only defined for parabolic subgroups.

The j-induction for classical group is in [20, Section 11.4]. Basically, W_n has D_{c_i}, B/C_{c_i} factors,

$$j_{\prod_i D_{c_i}\times \prod_j B/C_{c_j}} \operatorname{sgn} = ((c_i),(c_j)).$$

Here (c_i) denote the Young diagram with columns c_i etc.

Moreover, we have the following formula for j-induction.

$$\operatorname{ind}_{A_{2m}}^{W_{2m}} = ((m),(m)) \qquad \operatorname{ind}_{A_{2m+1}}^{W_{2m+1}} = ((m),(m+1)).$$

]

B.4. **Harish-Chandra cells and primitive ideals.** For simplicity we assume G is connected in this section.

We recall McGovern and Casian's works on the Harish-Chandra cells.

Fix a θ stable Cartan subgroup $H = TA$ of G such that T is the maximal compact subgroup of H and A is the split part of H. Let $\widehat{H}$ denote the set of irreducible representations of $\widehat{H}$ (also viewed as an $(\mathfrak{h}, T)$-module). Let $C(H)$ be component group of H, which is also the component group of T. Let $\mathrm{d}\colon \widehat{H} \longrightarrow \mathfrak{h}^*$ be the map takes $\phi \in \widehat{H}$ to its derivative. Then $\widehat{H}$ is a $\widehat{C(H)}$-torsor over $\mathfrak{h}^*$. [This means for each $\lambda \in \mathfrak{h}^*$, its fiber $\mathrm{d}^{-1}(\lambda)$ is either empty or with a set with free transitive $\widehat{C(H)}$-action (the tensor product action).]

[The classification of Primitive ideals.

$$\begin{array}{ccll} W_{[\lambda]} & \longrightarrow & \mathrm{Prim}_\lambda & \\ w & \mapsto & I(w\lambda) & := \mathrm{Ann}\, L(w\lambda). \end{array}$$

We now recall some results about the blocks in category $\mathcal{O}$.

Assume $\lambda \in \mathsf{C}$ where $\mathsf{C} = \{\, \nu \in \mathfrak{h}^* \mid \langle \nu, \check{\alpha} \rangle > 0 \,\}$ is the positive cone. Let $\lambda' = w_0\lambda$ which is negative.

Let w_0 (resp. $w_{[\lambda]}$)be the longest element in W (resp. $W_{[\lambda]}$).

Define (We adapt the convention that $e \underset{L}{\leqslant} w_{[\lambda]}$.)

$$\begin{aligned} w_1 \underset{L}{\leqslant} w_2 &\Leftrightarrow I(w_1\lambda') \subseteq I(w_2\lambda') \\ &\Leftrightarrow I(w_1w_0\lambda) \subseteq I(w_2w_0\lambda) \\ &\Leftrightarrow [L(w_2^{-1}\lambda'), L(w_1^{-1}\lambda') \otimes S(\mathfrak{g})] \neq 0 \end{aligned}$$

Note that

$$w_1 \underset{L}{\leqslant} w_2 \Leftrightarrow w_2^{-1} \underset{R}{\leqslant} w_1^{-1}.$$

Therefore, the last condition implies that

$$\begin{aligned} \mathscr{V}^R(w) &:= \mathrm{span}\,\{\, L(w'\lambda) \mid w' \underset{R}{\leqslant} w \,\} \\ &= \mathrm{span}\,\{\, L(w'\lambda) \mid w^{-1} \underset{L}{\leqslant} w'^{-1} \,\} \end{aligned}$$

]

We also fix a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ and let $\mathfrak{n}$ be the span of positive roots. Let

$$Q := \mathbb{Z}[\Delta(\mathfrak{g}, \mathfrak{h})] \subset \mathfrak{h}^*$$

be the root lattice of $\mathfrak{g}$, which is exactly the set of $\mathfrak{h}$-weights occur in $S(\mathfrak{g})$.

For each $\lambda \in \mathfrak{h}^*$, let $\langle \lambda \rangle := W_{[\lambda]} \cdot \lambda$. A set $S \subset \mathfrak{h}^*$ is called a *type* if

$$S = \bigcup_{\lambda \in S} \langle \lambda \rangle.$$

Clearly, $\langle \lambda \rangle$ is the smallest type containing λ.

Suppose S is a type, we define the category $\mathcal{O}'_S$ to be the category of $\mathfrak{g}$-modules such that $M \in \mathcal{O}'_S$ if and only if

- the $\mathfrak{b}$-action on M is locally finite;
- M is finitely generated $\mathcal{U}(\mathfrak{g})$-module,
- $M = \sum_{\mu \in S} M_\mu$ where M_μ is the μ-isotypic component of M.

Clearly, $\mathcal{O}'_S$ is a union of blocks in $\mathcal{O}'$.

For each set $\tilde{S} \subset \widehat{H}$, let $[\tilde{S}] := \{\, \phi \otimes \alpha \mid \phi \in \tilde{S}, \alpha \in Q \,\}$ be the Q-translations of $\tilde{S}$. We say $\tilde{S}$ is Q-saturated if $\tilde{S} = \mathrm{d}^{-1}(\mathrm{d}(\tilde{S})) \cap [\tilde{S}]$. We say $\tilde{S}$ is a type if $\mathrm{d}(S)$ is a type. Suppose $\tilde{S}$ is Q-saturated and

For $\phi \in \widehat{H}$, we write

$$\langle \phi \rangle := [\phi] \cap \phi^{-1}(\langle \mathrm{d}\phi \rangle)$$

which is the smallest W-invariant Q-saturated subset in $\widehat{H}$ containing ϕ. Let $M(\phi) := \mathrm{U}(\mathfrak{g}) \otimes_{\mathfrak{b},T} \phi$ where ϕ is inflated to a $(\mathfrak{b}, T)$-module.

Suppose $\tilde{S}$ is Q-saturated type, we define the category $\mathcal{O}'_S$ to be the category of $(\mathfrak{g}, T)$-module such that $M \in \mathcal{O}'_S$ if and only if

- the $\mathfrak{b}$-action is locally finite;
- M is finitely generated $\mathcal{U}(\mathfrak{g})$-module,
- $M = \sum_{\mu \in \tilde{S}} M_\mu$ where M_μ is the μ-isotypic component of M.

In particular, $\mathcal{O}'_{\widehat{H}}$ is the category of finitely generated $(\mathfrak{g}, T)$-module such that $\mathfrak{b}$-acts locally finite.

Let $\mathcal{F}\colon \mathcal{O}'_{\widehat{H}} \to \mathcal{O}'_{\mathfrak{h}}$ be the forgetful functor of the T-module structure. The following lemma is clear.

Lemma B.14. *The forgetful functor $\mathcal{F}$ yields an equivalence of category*

$$\mathcal{O}'_{\langle \phi \rangle} \to \mathcal{O}'_{\langle \mathrm{d}\phi \rangle}$$

which induces an isomorphism of $W_{[\mathrm{d}\phi]}$-module

$$\mathrm{Coh}_\phi \to \mathrm{Coh}_{\mathrm{d}\phi}.$$

Suppose $\gamma \in \widehat{C(H)}$. By Lemma B.14, we have an equivalent of category

$$\mathcal{O}'_{\langle \phi \rangle} \cong \mathcal{O}'_{\langle \mathrm{d}\phi \rangle} \cong \mathcal{O}'_{\langle \phi \otimes \gamma \rangle}$$

which sends $M(\phi)$ to $M(\phi \otimes \gamma)$. This isomorphism induces a $W_{[\mathrm{d}\phi]}$-module isomorphism

$$\mathrm{Coh}_\phi \xrightarrow{\otimes\gamma} \mathrm{Coh}_{\phi\otimes\gamma}.$$

Let $\widetilde{\mathrm{Coh}}_{[\lambda]} := \mathcal{F}^{-1}(\mathrm{Coh}_{[\lambda]})$.

In summary, we conclude that $\widetilde{\mathrm{Coh}}_{[\lambda]} \xrightarrow{\mathcal{F}} \mathrm{Coh}_{[\lambda]}$ is a $\widehat{C(H)}$-torsor over $\mathrm{Coh}_{[\lambda]}$.

Fix a set $\{ r_1, \cdots, r_k \}$ of representatives of $W/W_{[\lambda]}$ and fix an element $\phi_i \in \mathrm{d}^{-1}(r_i\lambda)$ for each r_i. Let $\Phi := \{ \phi_i \mid i = 1, 2, \cdots, k \}$.

Now we have an isomorphism of $W_{[\lambda]}$-module

$$\mathcal{F}_\Phi \colon \bigoplus_i \mathrm{Coh}_{\phi_i} \longrightarrow \bigoplus_i \mathrm{Coh}_{r_i\lambda} = \mathrm{Coh}_{[\lambda]}.$$

Now

$$\mathrm{Coh}_{[\lambda]} \otimes \mathbb{Z}[\widehat{C(H)}] \xrightarrow{\ \cong\ } \widetilde{\mathrm{Coh}}_{[\lambda]}.$$

given by $\Theta \otimes \alpha \mapsto \mathcal{F}_\Phi^{-1}(\Theta) \otimes \alpha$ is a $W_{[\lambda]} \times \widehat{C(H)}$-module isomorphism.

Let $K = G^\theta$.

Recall Casian's result:

Theorem B.15 ([21, Proposition 2.10, Theorem 3.1])**.** *Let H be a θ-stable Cartan subgroup of G. Fix a positive system of real roots $\Delta^+_{\mathbb{R}}$ and a positive system Δ^+ of roots such that $\Delta^+_{\mathbb{R}} \subseteq \Delta^+$. Let* $\mathbf{n}$ *be the maximal nilptent Lie subalgebra of $\mathfrak{g}$ with spanned by roots in Δ^+. Let M be a Harish-Chandra $(\mathfrak{g}, K)$-module, then*

(i) the Lie algebra cohomology $H^q(\mathfrak{n}, M)$ is finite dimensional;
(ii) the localization $\gamma^q_{\mathfrak{n}} M$ is in the category $\mathcal{O}'_{\widehat{H}}$; (see [21, Section 1]*)*
(iii) $\mathrm{Ann}\, M \subseteq \mathrm{Ann}(\gamma^q_{\mathfrak{n}} M)$.
(iv) Then the localization functor induces a homomorphism

$$\begin{array}{cccc} \gamma_{\mathfrak{n}}: & \mathcal{G}_{\chi,\mathcal{Z}}(\mathfrak{g}, K) & \longrightarrow & \mathcal{G}_{\chi,\mathcal{Z}}(\mathcal{O}_{\widehat{H}}) \\ & M & \mapsto & \sum_q (-1)^q \gamma^q_{\mathfrak{n}} M \end{array}$$

□

Theorem B.16 ([21, Theorem 3.1]). *Let $H_1, H_2, \cdots, H_s$ form a set of representatives of the conjugacy class of θ-stable Cartan subgroup of G. Fix maximal nilpotent Lie subalgebra $\mathfrak{n}_i$ for each H_i as in Theorem B.15. Then*

$$\gamma := \oplus_i \gamma_{\mathfrak{n}_i} : \ \mathcal{G}_\chi(\mathfrak{g}, K) \longrightarrow \bigoplus_i \mathcal{G}_\chi(\mathcal{O}_{\widehat{H}_i})$$

is an embedding of W_Λ-module.

Proof. We retain the notation in [21]. Let H_i^{rs} be the set of regular semisimple elements in H_i. By Harish-Chandra, taking the character of the elements induces an embedding of $\mathcal{G}_\chi(\mathfrak{g}, K)$ into the space of analytic functions on $\bigsqcup_i H_i^{rs}$. Now [21, Theorem 3.1] implies that the global character ΘM of an element $M \in \mathcal{G}_\chi(\mathfrak{g}, K)$ is completely determined by the formal character $\mathrm{ch}(\gamma(M))$. ☐

Corollary B.17. *Fix an irreducible $(\mathfrak{g}, K)$-module π with infinitesimal character χ_λ. Let $\mathscr{V}^{\mathrm{HC}}(\pi)$ be the Harish-Chandra cell representation containing π and $\mathcal{D}$ be the double cell in $\widehat{W_{[\lambda]}}$ containing the special representation $\sigma(\pi)$attached to* $\mathrm{Ann}(\pi)$*. Then $[\sigma, \mathscr{V}^{\mathrm{HC}}(\pi)] \neq 0$ only if $\sigma \in \mathcal{D}$. Moreover, $\sigma(\pi)$ always occures in $\mathscr{V}^{\mathrm{HC}}(\pi)$*

Proof. The occurrence of $\sigma(\pi)$ is a result of King.

Note that we have an embedding

$$\gamma \colon \mathcal{G}_\chi(\mathfrak{g}, K) \longrightarrow \bigoplus_i \mathcal{G}(\mathfrak{g}, \mathfrak{b}, \lambda).$$

where the left hand sides is identified with a finite copies of $\mathbb{C}[W_{[\lambda]}]$.

Since $\mathrm{Ann}(\pi) \subseteq \mathrm{Ann}(\gamma_{\mathfrak{n}}^q(\pi))$, we conclude that $[\sigma, \mathscr{V}^{\mathrm{HC}}(\pi)] \neq 0$ implies that $\sigma(\pi) \underset{LR}{\leqslant} \sigma$.

By the Vogan duality, $\mathcal{D} \otimes \mathrm{sgn}$ is also a Harish-Chandra cell. So we have $\sigma(\pi) \otimes \mathrm{sgn} \underset{LR}{\leqslant} \sigma \otimes \mathrm{sgn}$.

Therefore, $\sigma(\pi) \underset{LR}{\approx}$ ☐

B.5. **Primitive ideals for $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ at half integral infinitesimal character.** Let $\check{\mathcal{O}}$ be a metaplectic "good" nilpotent orbit in $\mathfrak{sp}(2n, \mathbb{C})$, i.e. $\mathbf{r}_i(\check{\mathcal{O}})$ is even for each $i \in \mathbb{N}^+$.

Then $W_{[\lambda_{\check{\mathcal{O}}}]} = W_n'$ and the left cell is given by

$$(J_{W_{\check{\mathcal{O}}}}^{W_n'} \mathrm{sgn}) \otimes \mathrm{sgn} \qquad \text{with} \qquad W_{\check{\mathcal{O}}} = \prod_{i \in \mathbb{N}^+} S_{\mathbf{c}_{2i}(\check{\mathcal{O}})}.$$

Here $W_{\check{\mathcal{O}}}$ is an subgroup of S_n and the embedding of S_n in W_n' is fixed. When n is even,we the symbol of $J_{S_n}^{W_n'} \mathrm{sgn}$ is degenerate and we label it by "I".

Let

$$\mathrm{PP}\tilde{C}(\check{\mathcal{O}}) = \{\, (2i-1, 2i+2) \mid i \in \mathbb{N}^+, \mathbf{r}_{2i-1}(\check{\mathcal{O}}) \neq \mathbf{r}_{2i}(\check{\mathcal{O}}) \,\}.$$

For each $\wp \subset \mathrm{PP}\tilde{C}(\check{\mathcal{O}})$, let $\tau_\wp := (\imath_\wp, \jmath_\wp)$ be the bipartition given by

$$(\mathbf{r}_i(\imath_\wp), \mathbf{r}_i(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}), & \text{if } (2i-1, 2i) \notin \wp, \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}), & \text{otherwise.} \end{cases}$$

If $\tau_\wp$ is degenerate, it represent the element in $\widehat{W_n'}$ with label I. Let $\wp^c$ be the complement of $\wp$ in $\mathrm{PP}\tilde{C}(\check{\mathcal{O}})$. Then $\tau_\wp$ and $\tau_{\wp^c}$ represent the same irreducible W_n'-module.

Using the induction formula in [?L, (4.6.6) (4.6.7)], we have the following lemma.

Lemma B.18. *The representation*

$${}^L\mathscr{C}'(\check{\mathcal{O}}) := J_{W_{\check{\mathcal{O}}}}^{W_n'} \mathrm{sgn}$$

is multiplicity free. The map

$$\mathbb{Z}[\mathrm{PP}\tilde{C}(\check{\mathcal{O}})]/\sim \longrightarrow \{\, \tau \in \widehat{W'_n} \mid \tau \subset {}^L\mathscr{C}'(\check{\mathcal{O}}) \,\} \qquad \wp \mapsto \tau_\wp$$

is a bijection where $\sim$ is the equivalent relation identifying $\wp$ with $\wp^c$. Moreover, the special representation in ${}^L\mathscr{C}'(\check{\mathcal{O}})$ is $\tau_\varnothing$. □

In the following, we write

$$\tau_{\check{\mathcal{O}}} := \tau_\varnothing$$

for the unique special representation in ${}^L\mathscr{C}'(\check{\mathcal{O}})$.

Let $\tau_{\check{\mathcal{O}}} = (\imath, \jmath)$ such that $\mathbf{r}_i(\imath) \leqslant \mathbf{r}_i(\jmath)$ for all $i \in \mathbb{N}^+$. Then the unique special representation in $\mathrm{Ind}_{W_{\check{\mathcal{O}}}}^{W'_n} 1$ where the a function take maximal value is given by the bipartition

$$\tau_{\mathcal{O}} := (\jmath^t, \imath^t).$$

Let $\sigma'(\mathcal{O})$ denote the $\tau_{\mathcal{O}}$-isotypic component of W'_n-harmonic polynomials in $S(\mathfrak{h})$. Comparing the fake degree formulas of type C and D (see [20, Proposition 11.4.3, 11.4.4]), we conclude that the W_n-representation

$$\sigma(\mathcal{O}) := \mathbb{C}[W_n] \cdot \sigma'(\mathcal{O}) \tag{B.3}$$

is irreducible whose type is also given by the bipartition $\tau_{\mathcal{O}}$. The type of $\sigma(\mathcal{O})$ is $j_{W'}^{W}\sigma'(\mathcal{O})$.

Lemma B.19. *Suppose σ' is a W'_n-special representation.Let $\check{\mathcal{O}}'$ be the partition of type D corresponds to the W'_n-special representation $\sigma' \otimes \mathrm{sgn}_D$. Then $\sigma := j_{W'_n}^{W_n}\sigma'$ corresponds to the partition $\check{\mathcal{O}}'^t$ under the Springer correspondence of type C.*

Proof. First note that $\check{\mathcal{O}}'$ is a specail nilpotent orbit of type D_n, therefore $\check{\mathcal{O}}'^t$ is a partition of type C_n (see [24, Proposition 6.3.7]).

By the explicitly formula of the Lusztig-Spaltenstein duality, we known that the D-collapsing $(\check{\mathcal{O}}'^t)_D$ of $\check{\mathcal{O}}'^t$ corresponds to the special representation σ' of type D.

The Springer correspondence algorithm for classical groups can be naturally extends to all partitions. Sommers showed that two partitions are mapped to the same Weyl group representations if and only if they has the same D-collapsing [88, Lemma 9]. Note that the algorithms computing the Springer correspondence for type C and D are essentially the same (see [20, Section 13.3] or [88, Section 7]). By the injectivity of the Springer correspondence, we conclude that the type C partition $\check{\mathcal{O}}'^t$ must corresponds to σ. □

The following lemma is a direct consequence of Lemma B.19.

Lemma B.20. *Suppose $\check{\mathcal{O}}$ has good parity of type $\tilde{C}$. Under the Springer correspondence of type C, the representation $\sigma(\check{\mathcal{O}})$ (see* (B.3)*) corresponds to the nilpotent orbit $\mathcal{O}$ defined by*

$$(\mathbf{c}_{2i-1}(\mathcal{O}), \mathbf{c}_{2i}(\mathcal{O})) = \begin{cases} (\mathbf{r}_{2i-1}(\check{\mathcal{O}}), \mathbf{r}_{2i}(\check{\mathcal{O}})), & \textit{if } \mathbf{r}_{2i-1}(\check{\mathcal{O}}), \mathbf{r}_{2i}(\check{\mathcal{O}}) \\ (\mathbf{r}_{2i-1}(\check{\mathcal{O}}) - 1, \mathbf{r}_{2i}(\check{\mathcal{O}}) + 1), & \textit{otherwise} \end{cases}$$

for all $i \in \mathbb{N}^+$.

Proof. Apply the algorithm of Springer correspondence of type D to the representation $\sigma'(\check{\mathcal{O}})$ gives the above formula. □

[A technical point, the representation of W'_n is given by symbol $\binom{\xi}{\mu}$ or $\binom{\mu}{\xi}$. However, using the Springer correspondence formula, only one of the arrangement can give a valid type D partition.

It is a interesting fact that a type D orbit $\mathcal{O}$ is special if and only if $\mathcal{O}^t$ is of type C.
]

Lemma B.21. *Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_b \cup \check{\mathcal{O}}_g$. Then $\mathcal{O} = \mathcal{O}_b \cup \mathcal{O}_g$ where $\mathcal{O}_b = \check{\mathcal{O}}_b^t$ and $\mathcal{O}_g = \tilde{d}_{BV}(\check{\mathcal{O}}_g)$.*

[We take the convention that $2\mathcal{O} = [2r_i]$ if $\mathcal{O} = [r_i]$. We also write $[r_i] \cup [r_j] = [r_i, r_j]$. $\dagger\mathcal{O} = [r_i + 1]$.

We suppose

$$\check{\mathcal{O}}_b = [2r_1 + 1, 2r_1 + 1, \cdots, 2r_k + 1, 2r_k + 1] = (2c_0, 2c_1, 2c_1, \cdots, 2c_l, 2c_l)$$

where $l = r_1$.

Now

$$\begin{aligned} W_{\check{\mathcal{O}}_b} &= W_{c_0} \times S_{2c_1} \times S_{2c_2} \times \cdots \times S_{2c_l} \\ \check{\sigma}_b := j_{W_{\check{\mathcal{O}}_b}}^{W_b} \operatorname{sgn} &= ((c_1, c_2, \cdots, c_k), (c_0, c_1, \cdots, c_l)) \\ &= ([r_1, r_2, \cdots, r_k], [r_1 + 1, r_2 + 1, \cdots, r_k + 1]) \end{aligned}$$

Therefore

$$\sigma_b = \check{\sigma}_b \otimes \operatorname{sgn} = ((r_1 + 1, r_2 + 1, \cdots, r_k + 1), (r_1, r_2, \cdots, r_k))$$

which corresponds to the orbit

$$\mathcal{O}_b = (2r_1 + 1, 2r_1 + 1, 2r_2 + 1, 2r_2 + 1, \cdots, 2r_k + 1, 2r_k + 1) = \check{\mathcal{O}}_b^t.$$

This implies

$$\sigma_b = j_{W_{L_b}}^{W_b} \operatorname{sgn}, \quad \text{where } W_{L,b} = \prod_{i=1}^{k} S_{2r_i+1}.$$

(Note that $\mathcal{O}'_b = (2r_1+1, 2r_2+1, \cdots, 2r_k+1)$ which corresponds to $j_{W_{L_b}}^{S_b} \operatorname{sgn}$ and $\operatorname{ind}_L^G \mathcal{O}'_b = \mathcal{O}_b$.)

Now we deduce that

$$\begin{aligned} \sigma &:= j_{W_b \times W_g}^{W_n} \sigma_b \otimes \sigma_g \\ &= j_{W_{L_b} \times W_g}^{W_n} \operatorname{sgn} \otimes \sigma_g \end{aligned}$$

where $W_{L_b} \times W_g$ is a parabolic subgroup of W_n corresponds to the Levi factor L of type

$$A_{2r_1+1} \times A_{2r_2+1} \times \cdots \times A_{2r_k+1} \times W_g.$$

Therefore

$$\mathcal{O} = \operatorname{ind}_L^G \operatorname{triv} \times \mathcal{O}_g = \mathcal{O}_b \cup \mathcal{O}_g.$$

We claim that the map $\tilde{d}\colon \check{\mathcal{O}} \mapsto \mathcal{O}$ defined here coincide with our Metaplectic BV duality paper.

Note that $\tilde{d}_{\mathrm{BV}}$ is compatible with parabolic induction. Suppose $\check{\mathfrak{l}}$ is a Levi subgroup of $\check{\mathfrak{g}}$ and $\check{\mathcal{O}}_{\check{\mathfrak{l}}} := \check{\mathcal{O}} \cap \check{\mathfrak{l}} \neq \emptyset$. Then

$$\tilde{d}_{\mathrm{BV}}(\check{\mathcal{O}}) = \operatorname{ind}_{\mathfrak{l}}^{\mathfrak{g}} \tilde{d}_{\mathrm{BV}}(\check{\mathcal{O}}_{\check{\mathfrak{l}}}).$$

(Since $\tilde{d}_{\mathrm{BV}}$ commute with the descent map, the claim follows from the prosperity of d_{BV})

The map $\tilde{d}$ also compatible with parabolic induction. It suffice to consider the case where $\check{\mathfrak{l}}$ is a maximal parabolic of type $A_l \times C_n$ and the orbit is trivial on the A_l factor. Suppose l is the bad parity, the claim is clear by our computation for the bad parity case.

Now suppose $l = 2m$ has good parity. If there is a i such that $\mathbf{r}_{2i+1}(\check{\mathcal{O}}) = \mathbf{r}_{2i+2}(\check{\mathcal{O}}) = 2m$. Then $\mathbf{c}_{2i+1}(\mathcal{O}) = \mathbf{c}_{2i+2}(\mathcal{O}) = 2m$ and the claim follows.

Otherwise, we can assume

$$R_{2i-1} := \mathbf{r}_{2i-1}(\check{\mathcal{O}}) > \mathbf{r}_{2i}(\check{\mathcal{O}}) = \mathbf{r}_{2i+1}(\check{\mathcal{O}}) > \mathbf{r}_{2i+2}(\check{\mathcal{O}}) =: R_{2i+2}.$$

and then

$$\mathcal{O} = (\cdots, R_{2i-1} - 1, 2m + 1, 2m - 1, R_{2i+2}, \cdots)$$

One check again that $\mathcal{O} = \mathrm{ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{l}}$. (Note that the induction operation is add two length $2m$ columns and then apply C-collapsing.)

Now it suffice to check that $\tilde{d}(\check{\mathcal{O}}) = \tilde{d}_{\mathrm{BV}}(\check{\mathcal{O}})$ for every orbits $\check{\mathcal{O}}$ whose rows are multiplicity free) (i.e. $\check{\mathcal{O}}$ is distinguished). This is clear by the explicit formula for the both sides.]

Appendix C. Matching specail shape and non-special shape painted bipartitions

C.1. **Proof of ...**

Proof. Let i be the minimal integer such that $(2i-1, 2i) \in \mathscr{P}$. We define $\mathscr{P}' = \mathscr{P} - \{\,(2i-1, 2i)\,\}$. We will establish a bijection

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \mathscr{P}') \longrightarrow \mathsf{PBP}_\star(\check{\mathcal{O}}_g, \mathscr{P}).$$

□

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the Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, New York 14853

Email address: dmb14@cornell.edu

School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China

Email address: hoxide@sjtu.edu.cn

Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China

Email address: sun@math.ac.cn

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076

Email address: matzhucb@nus.edu.sg