# Special unipotent representations of real classical groups and theta correspondence


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# Classical groups and special unipotent representations

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All *specail unipotent representations* of $G$ are *unitarizable*.

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- $\mathrm{Unip}_{\check{\mathcal{O}}}(G) :=$ { special unipotent repn. attached to $\check{\mathcal{O}}$}.

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- *Altas of Lie group:* $\rightsquigarrow$ complete answer for exceptional groups.

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- Lemma:

$$\begin{aligned} &\# \{ \pi \in \mathrm{Irr}_\mu(\mathfrak{g}, K)(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \} \\ &= \sum_{\substack{\tau \in \mathcal{D} \\ \mathcal{D} \rightsquigarrow \mathcal{O}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathscr{G}_\lambda(\mathfrak{g}, K)] \end{aligned}$$

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$$\mathscr{G}_\rho(\mathrm{Sp}(2n, \mathbb{R})) = \sum_{\substack{p,q,t,s,\\ \sigma \in \widehat{S}_s}} \mathrm{Ind}_{S_t \times W_{2s} \times W_p \times W_q}^{W_n} \mathrm{sgn} \otimes (\sigma \times \sigma) \otimes \mathbf{1} \otimes \mathbf{1}.$$

- $[\tau : 1_{W_{\lambda_{\check{\mathcal{O}}}}}] = 1.$
- $[\tau : \mathscr{G}_\rho(G)]$ is counted by painted bi-partitions $\mathrm{PBP}(\check{\mathcal{O}})$.

# Example of PBP

$$\check{\mathcal{O}} = [7,5,5,5,3,1,1] = \ydiagram{7,5,5,5,3,1,1}$$

$\mathrm{PBP}_{\check{\mathcal{O}}}(\mathrm{Sp}(2n,\mathbb{R}))$ Associated cycle

$$\begin{ytableau} \bullet & \bullet & \bullet \\ \bullet & c \\ r & d \\ d \end{ytableau} \times \begin{ytableau} \bullet & \bullet & \bullet \\ \bullet & s \end{ytableau} \quad \mapsto \quad \begin{ytableau} * & = & * & = & * & = \\ * & = & * & = & * & = \\ = & * & = & * \\ * & = & * & = \\ * & = & * & = \\ - & + \end{ytableau} \cup \begin{ytableau} * & = & * & = & * & = \\ * & = & * & = & * & = \\ = & * & = & * \\ = & * & = & * \\ * & = & * & = \\ + & - \end{ytableau}$$

$$\vdots \qquad\qquad\qquad \vdots$$

# Nilpotent orbits with “good/bad parity”

- Bad parity (must occurs with even multiplicity in $\check{\mathcal{O}}$):

$$\begin{cases} \text{even number}, & \text{when } \mathbf{G}^\vee \text{ is type } B \text{ or } D \\ \text{odd number}, & \text{when } \mathbf{G}^\vee \text{ is type } C \end{cases}$$

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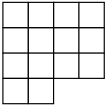
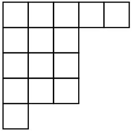
- Example of good parity:

$\mathcal{O}$



$\mathrm{Sp}(14, \mathbb{C})$

$\mathcal{O}^\vee$



$\mathrm{SO}(15, \mathbb{C})$

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(\pi', \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}(|\check{\mathcal{O}}'_b|, \mathbb{R}) \times \mathrm{Sp}(2n_0, \mathbb{R}) \ltimes U}^{\mathrm{Sp}(2n, \mathbb{R})} \pi' \otimes \pi_0
\end{array}
$$

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\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}) = \left\{ \mathrm{Ind} \bigotimes_{j=1}^{k} \mathrm{sgn}_{\mathrm{GL}(r_j, \mathbb{R})}^{\epsilon_j} \,\middle|\, \epsilon_j \in \mathbb{Z}/2\mathbb{Z} \right\}
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- Use *theta correspondence* to construct $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
- We assume $\check{\mathcal{O}}$ has good parity from now on.

# Example of descent sequences

| | | | | |
|---|---|---|---|---|
| $\mathbf{G}_i^\vee$ | $\mathrm{SO}(15,\mathbb{C})$ | $\mathrm{O}(10,\mathbb{C})$ | $\mathrm{SO}(7,\mathbb{C})$ | $\mathrm{O}(4,\mathbb{C})$ |
| $\mathcal{O}_i^\vee$ | $\ydiagram{5,3,3,3,1}$ | $\ydiagram{3,3,3,1}$ | $\ydiagram{3,3,1}$ | $\ydiagram{3,1}$ |
| $\mathcal{O}_i$ | $\ydiagram{4,4,4,2}$ | $\ydiagram{3,3,3,1}$ | $\ydiagram{2,2,2}$ | $\ydiagram{1,1,1,1}$ |
| $\mathbf{G}_i$ | $\mathrm{Sp}(14,\mathbb{C})$ | $\mathrm{O}(10,\mathbb{C})$ | $\mathrm{Sp}(6,\mathbb{C})$ | $\mathrm{O}(4,\mathbb{C})$ |

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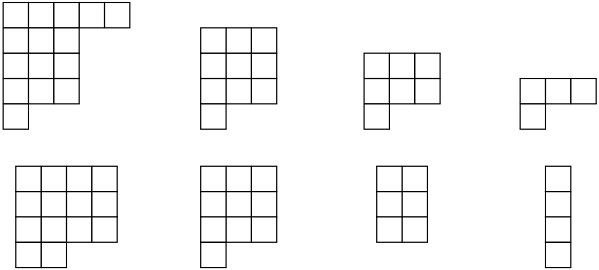
$\mathcal{O}_i^\vee$

$\mathcal{O}_i$



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Kraft-Procesi's resolution of singularities of the closure of complex nilpotent orbits.

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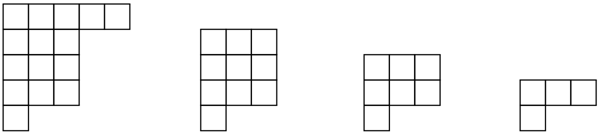

| | | | | |
|---|---|---|---|---|
| $\mathcal{O}_i$ | $\begin{array}{\|c\|c\|c\|c\|}\hline - & + & - & + \\ \hline + & - & + & - \\ \hline + & - & + & - \\ \hline + & - & & \\ \cline{1-2}\end{array}$ | $\begin{array}{\|c\|c\|c\|}\hline + & - & + \\ \hline - & + & - \\ \hline - & + & - \\ \hline - & & \\ \cline{1-1}\end{array}$ | $\begin{array}{\|c\|c\|}\hline - & + \\ \hline + & - \\ \hline + & - \\ \hline\end{array}$ | $\begin{array}{\|c\|}\hline + \\ \hline + \\ \hline - \\ \hline - \\ \hline\end{array}$ |
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Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

## Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

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# Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

- $\chi = \bigotimes_{j=0}^{2a} \chi_j$, a 1-dim repn. of $\prod_{j=0}^{2a} G_j$.
- $\chi_j \in \{\mathbf{1}, \mathrm{sgn}^{+,-}, \mathrm{sgn}^{-,+}, \det\}$
- Define a smooth repn. of $G = G_{2a}$ (the symplectic group).

$$\pi_\chi := (\omega_{G_{2a}, G_{2a-1}} \widehat{\otimes} \omega_{G_{2a-1}, G_{2a-2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1, G_0} \otimes \chi)_{G_{2a-1} \times G_{2a-2} \times \cdots \times G_0}$$

**Theorem (Barbasch-M.-Sun-Zhu)**

*Let $\check{\mathcal{O}}^\vee$ be an orbit with good parity. Then*

- *either* $\pi_\chi = 0$ *or*
- $\pi_\chi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$ *and unitarizable.*
- *Moreover,*

$$\mathrm{Unip}_{\mathcal{O}^\vee}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}.$$

# Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6,\mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4,\mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

$$\begin{array}{ccccc}
 & & \mathrm{Sp}(6,\mathbb{R}) & & \\
\mathrm{O}(4,0) & & \theta(\mathrm{sgn}^{+,-}) & & \\
 & & & = & \\
\mathrm{O}(3,1) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & & \theta(\mathrm{sgn}^{-,+}) \\
 & & & = & \\
\mathrm{O}(2,2) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & & \theta(\mathrm{sgn}^{-,+}) \\
 & & & = & \\
\mathrm{O}(1,3) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & & \theta(\mathrm{sgn}^{-,+}) \\
 & & & = & \\
\mathrm{O}(0,4) & & & & \theta(\mathrm{sgn}^{-,+})
\end{array}$$

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- Corollary: (using [Gomez-Zhu]) For $\pi_\chi$,

$$\text{Whittaker cycle} = \text{Wavefront cycle}.$$

# Associated cycle formula I

- Example $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$

$$\begin{tikzcd}
& M_{p,n} \oplus M_{q,n} \arrow[dl, "\varphi'"'] \arrow[dr, "\varphi"] & \\
M_{p,q} \cong \mathfrak{p}' & (A, B) \arrow[dl, mapsto] \arrow[dr, mapsto] & \mathfrak{p} \cong \mathrm{Sym}_n \oplus \mathrm{Sym}_n \\
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such that

$$\mathrm{AC}(\Theta(\pi')) \preceq \vartheta^{\mathrm{geo}}(\mathrm{AC}(\pi')),$$

for any $\pi'$ with $\mathrm{AV}(\pi') \subset \overline{\mathcal{O}'}$

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- Stable range lifting trick: Suppose $n > p + q$.

$$\bigcup_{p,q} \mathrm{Unip}_{\mathcal{O}'^\vee}(\mathrm{O}(p, q)) \hookrightarrow \mathrm{Unip}_{\mathcal{O}^\vee}(\mathrm{Sp}(2n, \mathbb{R}))$$

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- Question: How to describe $\pi_{\psi,\eta}$ explicitly?

Dan Barabasch, M. , Binyong Sun and Chen-Bo Zhu
Special unipotent representations: orthogonal and symplectic groups
ArXiv e-prints: https://arxiv.org/abs/1712.05552v2

**Thank you for your attention!**