Unipotent representations and theta correspondence

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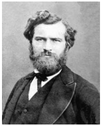
-  

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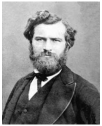
-  

- Classification of conjugation classes in G:

$$G/\sim = \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \{\, su \mid u \in G^s \,\} / \sim \xleftrightarrow{bij.} \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \mathrm{unip}(G^s)$$

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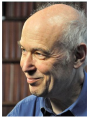

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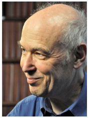

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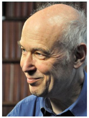

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Preserve cuspidality.

Unipotent cuspidal representations

Let q be an odd prime power.

Unipotent cuspidal representations (for classical groups) exist only in the following cases:

- $\mathrm{U}_n(\mathbb{F}_q)$ has one π_k, when $n = k(k+1)/2$;
- $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ has one π_k^{Sp}, when $n = k(k+1)$
- $\mathrm{O}_{2n}^{\epsilon}(\mathbb{F}_q)$ has two $\pi_{k,a/b}^{e}$, when $n = k^2$ and $\epsilon = (-1)^k$.
- $\mathrm{O}_{2n+1}^{\epsilon}(\mathbb{F}_q)$ has two $\pi_{k,a/b}^{o}$, when $n = k(k+1)$.

By Adams-Moy, all the unipotent cuspidal representations can be consturced by theta lifting.

Rational nilpotent orbits of orthogonal/symplectic groups

- $\mathrm{Quad}(k)$ be the isometric classes of quadratic spaces over k.
- $\mathrm{Witt}(k)$ be the Witt group of a field k.

  We identify $\mathrm{Witt}(k) \times \mathbb{N} = \mathrm{Quad}(k)$ via

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- The rational nilpotent oribts are parameterized by formed Yong-diagram :

$$[(Q_1, r_1), (Q_2, r_2), \cdots (Q_l, r_l)]$$

  such that
  - $Q_i \in \mathrm{Quad}(k)$,
  - $r_1 > r_2 > \cdots r_l > 0$
  - and Q_i is split if r_i is even for orthogonal group
    r_i is odd for symplectic group.

The chain/descent sequence of unipotent cuspdial representations

$$\begin{tikzcd}
\emptyset & \pi^{e}_{0,a} \arrow[r] & \pi^{\mathrm{Sp}}_{0} \arrow[dl] & \emptyset \\
\ytableaushort{Q,Q} & \pi^{e}_{1,a} \arrow[d, "\det\otimes"] & & \\
 & \pi^{e}_{1,b} \arrow[r] & \pi^{\mathrm{Sp}}_{1} \arrow[dl] & \ytableaushort{Q{},Q{}} \\
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(The complete result on the rational Wavefront of finite classical group has been worked out by Z.-C. Wang)

Lifting of cycles

- Example $(G, G') = (\mathrm{Sp}_{2n}, \mathrm{O}_m)$

$$\begin{tikzcd}
 & M_{m,2n} \arrow[dl, "\varphi'"'] \arrow[dr, "\varphi"] & \\
\mathrm{Alt}^2_m \cong \mathfrak{g}' & A \arrow[dl, mapsto] \arrow[dr, mapsto] & \mathfrak{g} \cong \mathrm{Sym}^2_{2n} \\
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- One can define lifting of cycles use the geometry of moment maps

$$\vartheta^{\mathrm{geo}} \colon \mathcal{K}_{\mathcal{O}'}(G') \longrightarrow \mathcal{K}_{\mathcal{O}}(G)$$

Theorem (Gomez-Zhu) Theta lift of generalized Whittaker models

$$\mathrm{Wh}_{\mathcal{O}}(\Theta(\pi')) = \vartheta^{\mathrm{geo}}(\mathrm{Wh}_{\mathcal{O}'}(\pi')),$$

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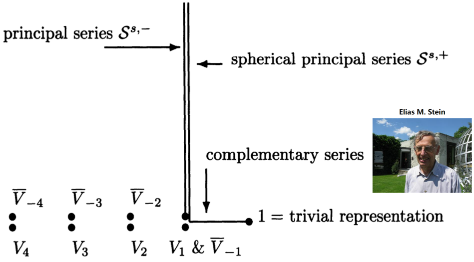

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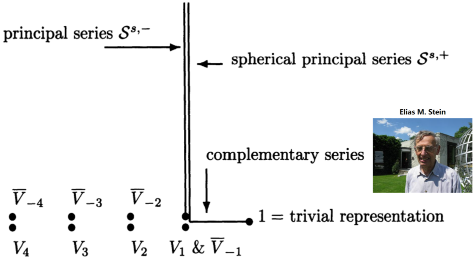

- *Open problem:* Structure of the unitary dual!

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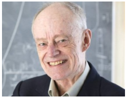
-  *Conjecture:* $\exists$ unipotent Arthur packets

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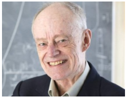
-  *Conjecture:* $\exists$ unipotent Arthur packets
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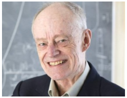
-  *Conjecture:* $\exists$ unipotent Arthur packets
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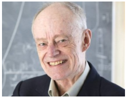
-  *Conjecture:* $\exists$ unipotent Arthur packets
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- Conjecture 2: $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is the union of certain Arthur packets.

Special unip. repn. of simply conn. classical groups

Theorem (Barbasch-M.-Sun-Zhu)

Suppose G is a simply connected real classical group, i.e. one of the following groups

$$\mathrm{SU}(p,q), \mathrm{Spin}(p,q), \mathrm{Spin}(2n,\mathbb{H}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{Mp}(2n,\mathbb{R}), \mathrm{Sp}(p,q)$$

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All *special unipotent repn.* of G are *unitarizable*.

The Arthur parameter of these representations are determined by recent work of Sun-Xu.

Reduction to the "good parity"

- Consider $G = \mathrm{Sp}(2n, \mathbb{R})$ for example.
- $\check{\mathcal{O}}$ decompose into two parts $\check{\mathcal{O}}_g$ (good parity) and $\check{\mathcal{O}}_b$ (bad parity).
- Assume $\check{\mathcal{O}}_b = \{r_1, r_1, \cdots, r_k, r_k\}$ and
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(\pi', \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}_{|\check{\mathcal{O}}'_b|} \times \mathrm{Sp}(2n_0, \mathbb{R}) \ltimes U}^{\mathrm{Sp}(2n,\mathbb{R})} \pi' \otimes \pi_0
\end{array}$$

$$\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}) = \left\{ \mathrm{Ind} \bigotimes_{j=1}^{k} \mathrm{sgn}_{\mathrm{GL}(r_j,\mathbb{R})}^{\epsilon_j} \;\middle|\; \epsilon_j \in \mathbb{Z}/2\mathbb{Z} \right\}$$

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$$\begin{array}{ccc} \mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}) \times \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{Sp}) & \xrightarrow{bij.} & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}) \\ (\pi', \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}_{|\check{\mathcal{O}}'_b|} \times \mathrm{Sp}(2n_0, \mathbb{R}) \ltimes U}^{\mathrm{Sp}(2n, \mathbb{R})} \pi' \otimes \pi_0 \end{array}$$

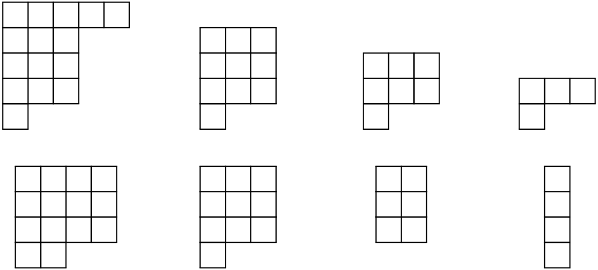
$$\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}) = \left\{ \mathrm{Ind} \bigotimes_{j=1}^{k} \mathrm{sgn}_{\mathrm{GL}(r_j, \mathbb{R})}^{\epsilon_j} \;\middle|\; \epsilon_j \in \mathbb{Z}/2\mathbb{Z} \right\}$$

- Use *theta correspondence* to study $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
- We assume $\check{\mathcal{O}}$ has good parity from now on.

Inductive structure of nilpotent orbits

$\check{\mathbf{G}}_i$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

$\check{\mathcal{O}}_i$



$\mathcal{O}_i$

$\mathbf{G}_i$	$\mathrm{Sp}(14,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{Sp}(6,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

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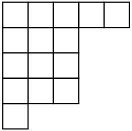
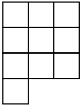
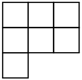
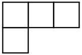
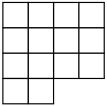
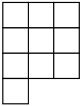
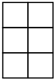
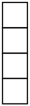
$\check{\mathcal{O}}_i$				
$\mathcal{O}_i$				
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Relate to Kraft-Procesi and Ohta's study of singularities of a nilpotent orbit closure.

Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

- $\chi = \bigotimes_{j=0}^{a} \chi_j$, a character of $\prod_{j=0}^{a} G_j$.
- $\chi_j \in \{\mathbf{1}, \mathrm{sgn}^{+,-}, \mathrm{sgn}^{-,+}, \det\}$ when G_j is an orthogonal group.
- Define a smooth repn. of $G = G_a$

  $$\pi_\chi := (\omega_{G_a, G_{a-1}} \widehat{\otimes} \omega_{G_{a-1}, G_{a-2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1, G_0} \otimes \chi)_{G_{a-1} \times G_{a-2} \times \cdots \times G_0}$$

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Theorem (Barbasch-M.-Sun-Zhu)

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- *either $\pi_\chi = 0$ or $\pi_\chi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$ and unitarizable.*
- *Moreover,*

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}.$$

Example 1:

Lift to $G = \mathrm{Sp}(8, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.
$\check{\mathcal{O}} = 531$ and $\mathcal{O} = 2222$. Then

$$\begin{aligned} &\mathrm{Unip}_{\check{\mathcal{O}}}(G) \\ &= \left\{\, \pi^{\pm}_{p,q} := \text{ theta lift of trivial and sign of } \mathrm{O}(p,q) \;\middle|\; p+q=4 \,\right\} \end{aligned}$$

Then $\mathrm{WF}(\pi^{\pm}_{p,q}) = \mathrm{Wh}(\pi^{\pm}_{p,q})$ consists of the single orbit:

$$\begin{array}{|c|c|}\hline \pm & \mp \\ \hline \pm & \mp \\ \vdots & \vdots \\ \hline \pm & \mp \\ \hline \end{array}$$

Example 2: Coincidences of theta liftings

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

		$\mathrm{Sp}(6, \mathbb{R})$	
$\mathrm{O}(4,0)$		$\theta(\mathrm{sgn}^{+,-})$	
$\mathrm{O}(3,1)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(2,2)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(1,3)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(0,4)$			$\theta(\mathrm{sgn}^{-,+})$

Example 2 (cont.)

All $\theta(\mathbf{1})$ has reducible associated cycle.

$$\mathrm{WF}(\theta_{\mathrm{O}(3,1)}^{\mathrm{Sp}_6(\mathbb{R})}(\mathbf{1})) = \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline - & + \\ \hline \end{array} \cup \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline + & - \\ \hline \end{array}$$

$$\mathrm{WF}(\theta_{\mathrm{O}(2,2)}^{\mathrm{Sp}_6(\mathbb{R})}(\mathbf{1})) = \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline + & - \\ \hline \end{array} \cup \begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline + & - \\ \hline \end{array}$$

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Weak unipotent packet for p-adic group

- G : a split orthogonal group or symplectic group defined over a p-adic field.
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- Weak unipotent packet of $\check{\mathcal{O}}$ (Ciubotaru-Mason-Brown-Okada)

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) := \left\{ X := X(q^{\frac{1}{2}\check{h}}, n, \rho) \;\middle|\; \mathrm{WF}(X) \subseteq d_{BV}(\check{\mathcal{O}}) \right\}$$

Here

- $n \in \mathfrak{g}^\vee$ such that $[\check{h}, n] = 2n$;
- ρ is an irreducible character of $A^1_{\check{G}}(s, n)$; (more or less the componet group of $Z_{\check{G}}(\{\check{h}, n\})$)
- $X(q^{\frac{1}{2}\check{h}}, n, \rho)$ is Lusztig's unipotent. representation.

Elements in a weak unipotent packet

- By Ciubotaru-Mason-Brown-Okada,

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \left\{ AZ(X(q^{\frac{1}{2}\check{h}}, n, \rho)) \;\middle|\; n \in \text{Special piece of } \check{\mathcal{O}} \right\}.$$

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- *Question:* Can we use theta lifting to construct all elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$?

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- descent sequence of $\pi_{k_1}^{\mathrm{Sp}}$ and $\pi_{k_2}^{\mathrm{Sp}} \rightsquigarrow$ descent sequence of π_{k_1,k_2}.

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- Apply Gomez-Zhu $\rightsquigarrow$ the wavefront set of π_{k_1,k_2}.
- $\mathrm{WF}(\pi_{k_1,k_2})$ contains a single orbit of “triangular shape”

Example 3: $\check{\mathcal{O}} = \ydiagram{3,3,1}$ $\quad d_{BV}(\check{\mathcal{O}}) = \ydiagram{2,2,2}$

- $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_6)$ has two elements.
- They are the theta lifts.

  $\theta_{\mathrm{O}_4^+}(1)$

  $\theta_{\mathrm{O}_4^-}(1)$
- they have reducible wavefront set.

Example 4: $\check{\mathcal{O}} = \ydiagram{5,3,1}$ $\quad d_{BV}(\check{\mathcal{O}}) = \ydiagram{2,2,2,2}$

- $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_8)$ has 5 elements.
- It is the union of two Arthur packets.
- The anti-tempered packet

$$\left\{ \delta = \theta_{\mathrm{O}_4^+}(1), \pi_1 = \theta_{\mathrm{O}_4^+}(\det), \pi_2 = \theta_{\mathrm{O}_4^-}(1), \pi_{sc} = \theta_{\mathrm{O}_4^-}(\det) \right\}$$

- They has irreducible wavefront set.
- One non-anti-tempered packet

$$\left\{ \pi_1, \pi_{sc}, \tau^t \right\}.$$

Thank you for your attention!