COUNTING SPECIAL UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS

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Contents

1. Introduction

Barbasch-Vogan [10, 14] developed a formula counting the number of special unipotent representations.

In this paper, we will calculate the number of special unipotent representations for real classical groups explicitly.

2. Counting formula

Let G be a real reductive group in the Harish-Chandra class. Fixing a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, we identify the set of infinitesimal character with $\mathfrak{h}^*/W$ where W is the Weyl group acting on $\mathfrak{h}$. Let Q be the root lattice and write $[\mu] := \mu + Q$ for the coset $\mathfrak{h}^*/Q$ containing μ. Denote $\mathrm{Coh}_{[\mu]}(G)$ the space of coherent families based on $[\mu]$ which has a $W_{[\mu]}$-action.

Theorem 2.1 (Barbasch-Vogan). *For a complex nilpotent orbit $\mathcal{O}$, define*

$$S_{\mathcal{O}} = \left\{ \sigma \in \widehat{W_{[\mu]}} \middle| Springer(j_{W_{[\mu]}}^{W} \sigma_s) = \mathcal{O} \right\}$$

where σ_s is the special representation in the double cell containing σ.

Let $\Pi_{\mathcal{O},\mu}(G)$ denote the set of irreducible admissible G-module with complex associated variety $\overline{\mathcal{O}}$. Then

$$\#\Pi_{\mathcal{O},\mu}(G) = \sum_{\sigma \in S_{\mathcal{O}}} [\sigma : \mathrm{Coh}_{[\mu]}(G)] \cdot [1_{W_\mu}, \sigma|_{W_\mu}].$$

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2.1. **Counting special unipotent representations.** Now let $\check{\mathcal{O}}$ be a nilpotent orbit in $\check{\mathcal{G}}$. It determine an infinitesimal character $\lambda_{\check{\mathcal{O}}}$.

Let $W_{[\lambda_{\check{\mathcal{O}}}]}$ be the integral Weyl group and $W_{\lambda_{\check{\mathcal{O}}}}$ be the stabilizer of $\lambda_{\check{\mathcal{O}}}$.

Special unipotent representations are defined to be the set of Harish-Chandra modules with minimal GK-dimension.

Let $a(\sigma)$ be the generic degree of a Weyl group representation σ.

In view of Theorem 2.1, we need the following lemma in [14].

Lemma 2.2. *Let*

$$a_{\check{\mathcal{O}}} = \max \{ a(\sigma) \mid \sigma \in \mathrm{Irr}(W_{[\lambda_{\check{\mathcal{O}}}]}) \textit{ and } [1_{W_{\lambda_{\check{\mathcal{O}}}}} : \sigma] \neq 0 \}.$$

Let

$$\mathcal{C}_{\check{\mathcal{O}}} = \{ \sigma \in \widehat{W_{[\lambda\ \mathcal{O}]}} \mid a(\sigma) = a_{\check{\mathcal{O}}} \textit{ and } [1_{W_{\lambda_{\check{\mathcal{O}}}}} : \sigma] \neq 0 \}.$$

Then

- $W_{\lambda_{\check{\mathcal{O}}}}$ *is a Levi subgroup of* $W_{[\lambda_{\check{\mathcal{O}}}]}$, *and*
- $\mathcal{C}_{\check{\mathcal{O}}}$ *is a left cell of* $W_{[\lambda_{\check{\mathcal{O}}}]}$ *given by*

$$(J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \mathrm{sgn}) \otimes \mathrm{sgn}$$

3. Counting in type A

The results in this section are well known to the experts.

Let YD *be the set of Young diagrams viewed as a finite multiset of positive integers. The set of nilpotent orbits in* $\mathrm{GL}_n(\mathbb{C})$ *is identified with Young diagram of* n *boxes.*

Let $\check{G}_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$. *Fix an orbit* $\check{\mathcal{O}} \in \mathrm{Nil}(ckG)$, *let* $\check{\mathcal{O}}_e$ *(resp.* $\check{\mathcal{O}}_o$*) be the partition consists of all even (resp. odd) rows in* $\check{\mathcal{O}}$.

Let S_n *denote the Weyl group of* $\mathrm{GL}_n(\mathbb{C})$. *Let* $W_n := S_n \ltimes \{ \pm 1 \}^n$ *denote the Weyl group of type* B_n *or* C_n. *Let* sgn *denote the sign representation of the Weyl group. The group* W_n *is naturally embedded in* S_{2n}. *For* W_n, *let* ϵ *denote the unique non-trivial character which is trivial on* S_n. *Note that* ϵ *is also the restriction of the* sgn *of* S_{2n} *on* W_n.

3.1. **Special unipotent representations of** $G = \mathrm{GL}_n(\mathbb{C})$**.** *By* [14], *the set of unipotent representations of* $G = \mathrm{GL}_n(\mathbb{C})$ *one-one corresponds to nilpotent orbits in* $\mathrm{Nil}(\check{G}_{\mathbb{C}})$. *Suppose* $\check{\mathcal{O}}$ *has rows*

$$\mathbf{r}_1(\check{\mathcal{O}}) \geqslant \mathbf{r}_2(\check{\mathcal{O}}) \geqslant \cdots \geqslant \mathbf{r}_k(\check{\mathcal{O}}) > 0.$$

Then $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}})$ *has columns* $\mathbf{c}_i(\mathcal{O}) = \mathbf{r}_i(\check{\mathcal{O}})$ *for all* $i \in \mathbb{N}^+$. *The map* $\check{\mathcal{O}} \mapsto \mathcal{O}$ *is a bijection.*

We set $\mathsf{CP} = \mathsf{YD}$ *be the set of Young diagrams. For* $\tau \in \mathsf{CP}$ *which has* k *columns, let* 1_c *be the trivial representations of* $\mathrm{GL}_c(\mathbb{C})$.

$$\pi_\tau = 1_{\mathbf{c}_1(\tau)} \times 1_{\mathbf{c}_2(\tau)} \times \cdots \times 1_{\mathbf{c}_k(\tau)}.$$

The Vogan duality gives a duality between Harish-Chandra cells. In this case, Harish-Chandra cells is the double cell of Lusztig. Now we have a duality

$$\pi_\tau \leftrightarrow \pi_{\tau^t}.$$

Let $\tau' := \nabla(\tau)$ *be the partition obtained by deleting the first column of* τ. *Let* $\theta_{a,b}$ *(resp.* $\Theta_{a,b}$*) be the theta lift (resp. big theta lift) from* $\mathrm{GL}_a(\mathbb{C})$ *to* $\mathrm{GL}_b(\mathbb{C})$. *Then we have*

$$\pi_\tau = \theta_{|\tau'|,|\tau|}(\pi_\tau).$$

3.2. **Counting unipotent representations of $\mathrm{GL}_n(\mathbb{R})$.** *Now let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G}_{\mathbb{C}})$. Recall the decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_e \cup \check{\mathcal{O}}_o$. Let $n_e = |\check{\mathcal{O}}_e|, n_o = |\check{\mathcal{O}}_o|$ and $\lambda_{\check{\mathcal{O}}} = \frac{1}{2}\check{h}$.*

Then

$$W(\lambda_{\check{\mathcal{O}}}) \cong S_{|\check{\mathcal{O}}_e|} \times S_{|\check{\mathcal{O}}_o|}.$$

$$W_{\lambda_{\check{\mathcal{O}}}} = \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_e)} \times \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_o)}$$

By the formula of a-function, one can easily see that The cell in $W(\lambda_{\check{\mathcal{O}}})$ consists of the unique representation $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W[\lambda_{\check{\mathcal{O}}}]}(1)$. Now the W-cell is $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W} 1$ corresponds to the orbit $\mathcal{O} = \check{\mathcal{O}}^t$ under the Springer correspondence. [*WLOG, we assume $\check{\mathcal{O}} = \check{\mathcal{O}}_o$.*

Let $\sigma \in \widehat{S_n}$. We identify σ with a Young diagram. Let $c_i = \mathbf{c}_i(\sigma)$. Then $\sigma = J_{W'}^{S_n}\epsilon_{W'}$ where $W' = \prod S_{c_i}$ (see Carter's book). This implies Lusztig's a-function takes value

$$a(\sigma) = \sum_i c_i(c_i - 1)/2$$

Compairing the above with the dimension formula of nilpotent Orbits [20, Collary 6.1.4], *we get (for the formula, see Bai ZQ-Xie Xun's paper on GK dimension of $SU(p,q)$)*

$$\tfrac{1}{2}\dim(\sigma) = \dim(L(\lambda)) = n(n-1)/2 - a(\sigma).$$

Here $\dim(\sigma)$ is the dimension of nilpotent orbit attached to the Young diagram of σ (it is the Springer correspondence, regular orbit maps to trivial representation, note that $a(\mathrm{triv}) = 0$), $L(\lambda)$ is any highest weight module in the cell of σ.

Return to our question, let $S' = \prod_i S_{\mathbf{c}_i(\check{\mathcal{O}})}$. We want to find the component σ_0 in $\mathrm{Ind}_{S'}^{S_n} 1$ whose $a(\sigma_0)$ is maximal, i.e. the Young diagram of σ_0 is minimal.

By the branching rule, $\sigma \subset \mathrm{Ind}_{S'}^{S_n} 1$ is given by adding rows of lenght $\mathbf{c}_i(\check{\mathcal{O}})$ repeatly (Each time add at most one box in each column). Now it is clear that $\sigma_0 = \check{\mathcal{O}}^t$ is desired.

This agrees with the Barbasch-Vogan duality d_{BV} given by

$$\check{\mathcal{O}} \xrightarrow{\text{Springer}} \check{\mathcal{O}} \xrightarrow{\otimes \mathrm{sgn}} \check{\mathcal{O}}^t \xrightarrow{\text{Springer}} \check{\mathcal{O}}^t.$$

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The $W_{[\lambda_{\check{\mathcal{O}}}]}$-module $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}$ is given by the following formula:

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]} \cong \mathcal{C}_{n_e} \otimes \mathcal{C}_{n_o} \quad \textit{with}$$

$$\mathcal{C}_n := \bigoplus_{\substack{s,a,b \\ 2s+a+b=n}} \mathrm{Ind}_{W_s \times S_a \times S_b}^{S_n} \epsilon \otimes 1 \otimes 1.$$

According to Vogan duality, we can obtain the above formula by tensoring sgn *on the forumla of the unitary groups in* [10, Section 4].

By branching rules of the symmetric groups, $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ can be parameterized by painted partition.

$$\mathrm{PP}_{\check{\mathcal{O}}} = \left\{ \tau \colon \mathrm{Box}(\check{\mathcal{O}}^t) \to \{\bullet, c, d\} \;\middle|\; \begin{array}{l} \text{“}\bullet\text{” occures with even} \\ \text{mulitplicity in each column} \end{array} \right\}.$$

[*The typical diagram of all columns with even length $2c$ are*

$\bullet$	$\cdots$	$\bullet$	$\bullet$	$\cdots$	$\bullet$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bullet$	$\cdots$	$\bullet$	c	$\cdots$	c
$\bullet$	$\cdots$	$\bullet$	d	$\cdots$	d

The typical diagram of all columns with odd length $2c+1$ are

$$\begin{array}{|c|c|c|c|c|c|}\hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline c & \cdots & c & d & \cdots & d \\ \hline \end{array}$$

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Let $\mathrm{sgn}_n\colon \mathrm{GL}_n(\mathbb{R}) \to \{\pm 1\}$ *be the sign of determinant. Let* 1_n *be the trivial representation of* $\mathrm{GL}_n(\mathbb{R})$*. For* $\tau \in \mathrm{PP}_{\check{\mathcal{O}}}$*, we attache the representation*

$$\pi_\tau := \bigtimes_j \underbrace{1_j \times \cdots \times 1_j}_{c_j\text{-terms}} \times \underbrace{\mathrm{sgn}_j \times \cdots \times \mathrm{sgn}_j}_{d_j\text{-terms}}. \tag{3.1}$$

Here

- j *running over all column lengths in* $\check{\mathcal{O}}^t$,
- d_j *is the number of columns of length* j *ending with the symbol "d",*
- c_j *is the number of columns of length* j *ending with the symbol "•" or "c", and*
- *"×" denote the parabolic induction.*

3.3. **Special Unipotent representations of $G = \mathrm{GL}_m(\mathbb{H})$.** *Suppose that* $\mathcal{O}$ *is the complexificiation of a rational nilpotent* $\mathrm{GL}_m(\mathbb{H})$*-orbit. Then* $\mathcal{O}$ *has only even length columns. Therefore,* $\mathrm{Unip}_{\check{\mathcal{O}}}(G) \neq \emptyset$ *only if* $\check{\mathcal{O}} = \check{\mathcal{O}}_e$.

In this case the coherent continuation representation is given by

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) = \mathrm{Ind}_{W_m}^{S_{2m}} \epsilon$$

and $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ *is a singleton. For each partition* τ *only having even columns, we define*

$$\pi_\tau := \bigtimes_i 1_{\mathbf{c}_i(\tau)/2}.$$

3.4. **Counting special unipotent repesentations of $\mathrm{U}(p,q)$.** *We call the parity of* $|\check{\mathcal{O}}|$ *the "good pairity". The other pairity is called the "bad parity". We write* $\check{\mathcal{O}} = \check{\mathcal{O}}_g \cup \check{\mathcal{O}}_b$ *where* $\check{\mathcal{O}}_g$ *and* $\check{\mathcal{O}}_b$ *consist of good pairity length rows and bad pairity rows respectively.*

Let $(n_g, n_b) = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$*. Now as the* $S_{n_g} \times S_{n_b}$

$$\bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=n}} \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathrm{U}(p,q)) = \mathcal{C}_{gd} \otimes \mathcal{C}_{bd}$$

where

$$\mathcal{C}_{gd} = \bigoplus_{\substack{s,a,b\in\mathbb{N}\\ 2s+a+b=n_g}} \mathrm{Ind}_{W_s\times S_a\times S_b}^{S_{n_g}} 1\otimes \mathrm{sgn}\otimes\mathrm{sgn}$$

$$\mathcal{C}_{bd} = \begin{cases} \mathrm{Ind}_{W_{\frac{n_b}{2}}}^{S_{n_b}} 1 & \text{if } n_b \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

By the above formula, we have

Lemma 3.1. *(i) The set* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,q)) \neq \emptyset$ *if and only if* $p = q$ *and each row lenght in* $\check{\mathcal{O}}$ *has even multiplicity.*
(ii) Suppose $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$*, let* $\check{\mathcal{O}}'$ *be the Young diagram such that* $\mathbf{r}_i(\check{\mathcal{O}}') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ *and* π' *be the unique special uinpotent representation in* $\mathrm{Unip}_{\check{\mathcal{O}}'}(\mathrm{GL}_p(\mathbb{C}))$*. Then the unique element in* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p))$ *is given by*

$$\pi := \mathrm{Ind}_P^{\mathrm{U}(p,p)} \pi'$$

where P is a parabolic subgroup in $\mathrm{U}(p,p)$ with Levi factor equals to $\mathrm{GL}_p(\mathbb{C})$.
(iii) In general, when $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$, we have a natural bijection

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}(n_1+p,n_2+p)) \\ \pi_0 & \mapsto & \mathrm{Ind}_P^{\mathrm{U}(n_1+p,n_2+p)} \pi' \otimes \pi_0 \end{array}$$

where P is a parabolic subgroup with Levi factor $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{U}(n_1,n_2)$.

The above lemma ensure us to reduce the problem to the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Now assume $\check{\mathcal{O}} = \check{\mathcal{O}}_g$ and so $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ corresponds to the blocks of the infinitesimal character of the trivial representation.

By [10, Theorem 4.2], *Harish-Chandra cells in $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ are in one-one correspondence to real nilpotent orbits in $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t$.*

[*From the branching rule, the cell is parametered by painted partition*

$$\mathrm{PP}(\mathrm{U}) := \left\{\, \tau \in \mathrm{PP} \;\middle|\; \begin{array}{l} \mathrm{Im}(\tau) \subseteq \{\, \bullet, s, r \,\} \\ \text{“}\bullet\text{” occures even times in each row} \end{array} \right\}.$$

The bijection $\mathrm{PP}(\mathrm{U}) \to \mathsf{SYD}, \tau \mapsto \mathscr{O}$ is given by the following recipe: The shape of $\mathscr{O}$ is the same as that of τ. $\mathscr{O}$ is the unique (upto row switching) signed Young diagram such that

$$\mathscr{O}(i, \mathbf{r}_i(\tau)) := \begin{cases} +, & \text{when } \tau(i,\mathbf{r}_i(\tau)) = r; \\ -, & \text{otherwise, i.e. } \tau(i,\mathbf{r}_i(\tau)) \in \{\, \bullet, s \,\}. \end{cases}$$

Example 3.2.

$$\begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet & r \\ \hline \bullet & \bullet \\ \cline{1-2} s & r \\ \cline{1-2} s \\ \cline{1-1} r \\ \cline{1-1} \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|} \hline + & - & + & - & + \\ \hline + & - \\ \cline{1-2} - & + \\ \cline{1-2} - \\ \cline{1-1} + \\ \cline{1-1} \end{array}$$

]

Now the following lemma is clear.

Lemma 3.3. *When $\check{\mathcal{O}} = \check{\mathcal{O}}_g$, the associated varity of every special unipotent representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U})$ is irreducible. Moreover, the following map is a bijection.*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \{\, \text{rational forms of } \check{\mathcal{O}}^t \,\} \\ \pi_0 & \mapsto & \mathrm{AV}^{\mathrm{weak}}(\pi_0). \end{array}$$

□

Remark. Note that the parabolic induction of an rational nilpotent orbit can be reducible. Therefore, when $\check{\mathcal{O}}_b \neq \emptyset$, the special unipotent representations can have reducible associated variety. Meanwhile, it is easy to see that the map $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) \ni \pi \mapsto \mathrm{AV}^{\mathrm{weak}}(\pi)$ is still injective.

We will show that every elements in $\mathrm{Unip}_{\check{\mathcal{O}}_g}$ can be constructed by iterated theta lifting. For each τ, let $\mathscr{O}$ be the corresponding real nilpotent orbit. Let $\mathrm{Sign}(\mathscr{O})$ be the signature of $\mathscr{O}$, $\nabla(\mathscr{O})$ be the signed Young diagram obtained by deleteing the first column of $\mathscr{O}$. Suppose $\mathscr{O}$ has k-columns. Inductively we have a sequence of unitary groups $\mathrm{U}(p_i,q_i)$ with $(p_i,q_i) = \mathrm{Sign}(\nabla^i(\mathscr{O}))$ for $i = 0, \cdots, k$. Then

$$\pi_\tau = \theta_{\mathrm{U}(p_1,q_1)}^{\mathrm{U}(p_0,q_0)} \theta_{\mathrm{U}(p_2,q_2)}^{\mathrm{U}(p_1,q_1)} \cdots \theta_{\mathrm{U}(p_k,q_k)}^{\mathrm{U}(p_{k-1},q_{k-1})}(1) \tag{3.2}$$

where 1 is the trivial representation of $\mathrm{U}(p_k,q_k)$.

Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Form the duality between cells of $\mathrm{U}(p,q)$ *and* $\mathrm{GL}(n,\mathbb{R})$. *We have an ad-hoc (bijective) duality between unipotent representations:*

$$\begin{aligned} d_{\mathrm{BV}}\colon \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) &\;\to\; \mathrm{Unip}_{\check{\mathcal{O}}^t}(\mathrm{GL}(\mathbb{R})) \\ \pi_\tau &\;\mapsto\; \pi_{d_{\mathrm{BV}}(\tau)} \end{aligned}$$

Here $\check{\mathcal{O}}^t = d_{\mathrm{BV}}(\check{\mathcal{O}})$ and $d_{\mathrm{BV}}(\tau)$ is the pained bipartition obtained by transposeing τ and replace s and r by c and d respectively. See (3.2) *and* (3.1) *for the definition of special unipotent representations on the two sides.*

4. Counting in type BCD

In this section, we consider the case when $\check{\star} \in \{ B, C, D \}$, i.e $\star \in \{ B, \widetilde{C}, C, D, C^*, D^* \}$. Recall that

$$\textit{good pairity} = \begin{cases} \textit{odd} & \textit{when} \check{\star} \in \{ B, D \} \\ \textit{even} & \textit{when} \check{\star} = C \end{cases}$$

In this section, we count special unipotent representations of real orthogonal groups (type B,D), symplectic groups (type C) and metaplectic groups (type $\widetilde{C}$).

Recall that

$$\textit{good pairity} = \begin{cases} \textit{odd} & \textit{in type } C \textit{ and } D \\ \textit{even} & \textit{in type } C \textit{ and } D \end{cases}$$

We decompose $\check{\mathcal{O}} = \check{\mathcal{O}}_g \cup \check{\mathcal{O}}_b$ as before.

Let $\widetilde{\mathrm{sgn}}$ be the character of W_n inflated from the sign character of S_n via the natural map $W_n \to S_n$. Recall the following formula

$$\mathrm{Ind}_{W_b \ltimes \{ \pm 1 \}^b}^{W_{2b}} 1 \otimes \mathrm{sgn} = \sum_{\sigma} (\sigma, \sigma)$$

where σ running over all Young diagrams of size b.

4.1. **Counting special unipotent representation of $G = \mathrm{Sp}(p,q)$. .** *The dual group of $G_{\mathbb{C}} = \mathrm{Sp}(2n,\mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{SO}(2n+1,\mathbb{C})$. We set $(2m+1, 2m') = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_n|)$. By Vogan duality, the dual group for class $[\lambda_{\check{\mathcal{O}}}]$ is*

$$\mathrm{SO}(2m+1,\mathbb{C}) \times \mathrm{O}(2m',\mathbb{C}).$$

Now we have

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) \otimes \mathrm{sgn} = \mathcal{C}_g \otimes \mathcal{C}_b$$

with

$$\begin{aligned} \mathcal{C}_g &= \bigoplus_{p+q=m} \mathrm{Coh}_{[\rho]}(\mathrm{Sp}(p,q)) \\ &= \bigoplus_{\substack{b,s,r\in\mathbb{N} \\ 2b+s+r=m}} \mathrm{Ind}_{W_b \ltimes \{ \pm 1 \}^b \times W_s \times W_b}^{W_m} 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn} \otimes \mathrm{sgn} \\ &= \bigoplus_{\substack{b,s,r\in\mathbb{N} \\ 2b+s+r=m}} \mathrm{Ind}_{W_{2b} \times W_s \times W_b}^{W_m} (\sigma,\sigma) \otimes \mathrm{sgn} \otimes \mathrm{sgn} \\ \mathcal{C}_b &= \mathrm{Ind}_{W_{\frac{m'}{2}} \ltimes \{ \pm 1 \}^{\frac{m'}{2}}}^{W_{m'}} \mathrm{sgn} \end{aligned}$$

In $\mathrm{SO}(2m',\mathbb{C})$*, the dobule cell corresponds to $\check{\mathcal{O}}_b$ consists of a single representation*

$$\check{\tau}_b = (\check{\sigma}_b, \check{\sigma}_b) \textit{ such that } \mathbf{r}_i(\check{\sigma}_b) = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)/2$$

Let $\check{\mathcal{O}}_b'$ be the Young diagram such that $\mathbf{r}_i(\check{\mathcal{O}}_b') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ and π_b' be the unique special unipotent representation attached to $\mathrm{GL}_p(\mathbb{H})$

Lemma 4.1. *The set* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{Sp}(p,q)) \neq \emptyset$ *only if* $p = q = |\check{\sigma}_b|$ *and in this case it consists a single element*

$$\pi_b := \mathrm{Ind}_P^{\mathrm{Sp}(p,p)} \pi'_{\check{\sigma}_b}$$

where P *has the Levi subgroup* $\mathrm{GL}_p(\mathbb{H})$*.*

The special reprsentation corresponds to $\check{\mathcal{O}}_b$ *is*

$$aa$$

Let

$$\mathsf{PBP}_{\check{\mathcal{O}}_g}(C^*) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath, \jmath) = \check{\tau}_g \\ \mathrm{Im}\mathcal{P} \subset \{\bullet\}, \mathrm{Im}\mathcal{Q} \subset \{\bullet, s, r\} \end{array} \right\}.$$

Lemma 4.2. *The set of* $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q))$ *is paramterized by the set such that*

$$ss$$

[*Let* $W = W(G_{\mathbb{C}})$ *where* $G_{\mathbb{C}}$ *is natrualy embeded in* $\mathrm{GL}(n,\mathbb{C})$*. Let* $s_{\varepsilon_i i, \varepsilon_j j}$ *be the permutation matrix of index* i, j*, where the* (i,j)*-th entry is* ε_i*, the* (j,i)*-th entry is* ε_j *and the other place is the identity matrix. Let* $w^{\epsilon}_{i,\pm j}$ *be the element such that*

- *it is in* $G_{\mathbb{C}}$ *and entries in* $\{0\} \cup \mu_4$
- *it lifts the element* $e_i \leftrightarrow \pm e_j$ *in the Weyl group.*
- $(w^{\epsilon}_{i,\pm j})^2 = \epsilon 1 \in \{\pm 1\}$*.*

Let

$$h^+_{\pm i} = \mathrm{diag}(1, \cdots, 1, \pm 1, 1, \cdots, 1)$$

where ± 1 *is the* i*-th place. Let*

$$h^-_{\pm i} = \mathrm{diag}(1, \cdots, 1, \pm\sqrt{-1}, 1, \cdots, 1)$$

where $\pm\sqrt{-1}$ *is the* i*-th place. Let* $e_{\pm i} = s_{\pm i, \pm(n-i+1)}$*.*

Let $\check{w}^{\pm}_{i,\pm j}$ *be the lift of* $e_i \leftrightarrow \pm e_j$ *such that*

- *it is in* $G_{\mathbb{C}}$ *and entries in* $\{0\} \cup \mu_4$
- *it lifts the element* $e_i \leftrightarrow \pm e_j$ *in the Weyl group.*
- $(w^{\epsilon}_{i,\pm j})^2|_{[i,j]} = \epsilon 1 \in \{\pm 1\}$ *here "*$|_{[i,j]}$*" means restricts on the* $e_i, e_j, -e_i, -e_j$*-weights space.*

Let

$$\begin{aligned} x_{b,s,r} &= w^+_{1,2} \cdots w^+_{2b-1,2b}\, h^+_{2b+1} \cdots h^+_{2b+s} \\ y^+_{b,s,r} &= \check{w}^+_{1,-2} \cdots \check{w}^+_{2b-1,-2b}\, e_{2b+1} \cdots e_{2b+s+r} \\ y^-_{b,s,r} &= \check{w}^-_{1,-2} \cdots \check{w}^-_{2b-1,-2b} \end{aligned}$$

We compute the parameter space

$$\begin{aligned} \mathcal{Z}_g &= \bigcup_{2b+s+r=m} W_m \cdot (x^+_{b,s,r}, y^+_{b,s,r}) \\ \mathcal{Z}_b &= \bigcup_{2b=m} W_m \cdot (x^+_{b,0,0}, y^-_{b,0,0}) \end{aligned}$$

]

4.2. **Counting special unipotent representation of $G = \mathrm{Sp}(2n, \mathbb{R})$.** *Now we have*

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) \otimes \mathrm{sgn} = \mathcal{C}_g \otimes \mathcal{C}_b$$

with

$$\begin{aligned}
\mathcal{C}_g &= \bigoplus_{p+q=2m+1} \mathrm{Coh}_{[\rho]}(\mathrm{SO}(p,q)) \\
&= \bigoplus_{2b+s+r+c+d=m} \mathrm{Ind}^{W_m}_{W_b \ltimes \{-1\}^b \times W_s \times W_r \times W_c \times W_d} 1 \otimes \det \otimes \det \otimes 1 \otimes 1 \\
\mathcal{C}_b &= \mathrm{Coh}_{[\rho]}(\mathrm{SO}^*(2m)) = \bigoplus_{2s+b=m} \mathrm{Ind}^{W_m}_{W_s \ltimes \{-1\}^s \times S_b} 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn}
\end{aligned}$$

5. Examples

5.1. **Symplectic group.** *Let $G = \mathrm{Sp}(4, \mathbb{R})$. Then $\check{G}_{\mathbb{C}} = \mathrm{SO}(5, \mathbb{C})$. Then $\check{\mathcal{O}}$ has the following possibilities*

$$\check{\mathcal{O}}_1 := \begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline\end{array}, \quad \check{\mathcal{O}}_2 := \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1} \ & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \quad \check{\mathcal{O}}_3 := \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \quad \check{\mathcal{O}}_4 := \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$$

Here the orbit $\check{\mathcal{O}}_3$ is non-special.

$$\mathcal{O}_1 := \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}, \quad \mathcal{O}_2 := \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}, \quad \mathcal{O}_3 := \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}, \quad \mathcal{O}_4 := \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline\end{array}$$

For the real form of $\check{G}_{\mathbb{C}}$, we always consider the inner class $\{\mathrm{SO}(p,q) \mid q \text{ is even}\}$. Therefore, we have 3 groups $\mathrm{SO}(1,4)$, $\mathrm{SO}(3,2)$ and $\mathrm{SO}(5,0)$.

When $\check{\mathcal{O}} = \check{\mathcal{O}}_1$

Now

$$\mathrm{Unip}_{\check{\mathcal{O}}_1}(\mathrm{Sp}_4(\mathbb{R}) = \{\,\mathrm{triv}\,\}.$$

and

$$\mathsf{PBP}_C(\check{\mathcal{O}}) = \left\{ \varnothing \times \begin{array}{|c|}\hline s \\ \hline s \\ \hline\end{array} \right\}.$$

The orbit $\mathcal{O}_1$ has bad parity for SO_5. Now

$$\mathrm{Unip}_{\mathcal{O}_1}(\mathrm{SO}_5) = \{\, \mathrm{Ind}^{\mathrm{SO}(3,2)}_{\mathrm{GL}_1 \times \mathrm{GL}_1} \mathrm{sgn}^{\epsilon_1} \otimes \mathrm{sgn}^{\epsilon_2} \mid \epsilon_1, \epsilon_2 \in \{\,0,1\,\}\,\}$$

which has 3 elements.

When $\check{\mathcal{O}} = \check{\mathcal{O}}_2$

There are three unipotent Arthur parameters $\psi_{++}, \psi_{+-}, \psi_{--}$. These parameters corresponds to the three rational forms of $\check{\mathcal{O}}$ respectively

$$\begin{array}{|c|c|c|}\hline - & + & - \\ \hline + & \multicolumn{2}{c}{} \\ \cline{1-1} + & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline + & - & + \\ \hline + & \multicolumn{2}{c}{} \\ \cline{1-1} - & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline - & + & - \\ \hline - & \multicolumn{2}{c}{} \\ \cline{1-1} - & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}.$$

Here we fix the inner class of $\mathrm{SO}(p,q)$ such that q is even.

Now $\mathsf{PBP}_C(\check{\mathcal{O}})$ has 8 elements. The set $\mathsf{PBP}_C(\check{\mathcal{O}})$ one-one corresponds to the associated cycles.

The Arthur packet $\Pi_{\psi_{++}}$ contains two elements. We expect that they consists of $\mathrm{Sp}(4,\mathbb{R})$ *modules with associated variety*

$$\begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline\end{array}.$$

Similarly, the Arthur packet $\Pi_{\psi_{--}}$ contains two elements and they consists of $\mathrm{Sp}(4,\mathbb{R})$ *modules with associated variety*

$$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline\end{array}.$$

The Arthur packet $\Pi_{\psi_{+-}}$ has 4-elements, we expect that they consists of modules with associated variety

$$\begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline\end{array}.$$

The explicit matching of the packet can be down by compare the K-types in ABV's (27.17) with theta lifts.

When $\check{\mathcal{O}} = \check{\mathcal{O}}_3$ *The orbit has bad parity part. There are two special unipotent representations given by*

$$\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{R})}^{\mathrm{Sp}_4(\mathbb{R})} \mathrm{sgn}^{\epsilon} \qquad \epsilon \in \{\,0, 1\,\}\,.$$

When $\check{\mathcal{O}} = \check{\mathcal{O}}_4$ *In this case* $\mathrm{Unip}_{\mathrm{Sp}(2n,\mathbb{R})}(\check{\mathcal{O}})$ *has* 5 *elements which are distinguished by their associated cycles listed below.*

$$\begin{array}{|c|c|c|c|}\hline - & + & - & + \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline \times & / & \times & / \\ \hline\end{array}, \quad \begin{array}{|c|c|c|c|}\hline / & \times & / & \times \\ \hline\end{array},$$
$$\begin{array}{|c|c|c|c|}\hline - & + & - & + \\ \hline\end{array} \cup \begin{array}{|c|c|c|c|}\hline - & + & - & + \\ \hline\end{array},$$

Totally there are 3 *Arthur parameters corresponds to the trivial orbits of* $\mathrm{SO}(1,4)$, $\mathrm{SO}(3,2)$ *and* $\mathrm{SO}(5,0)$. *We expect the Arthur packets are disjoint with each other.*

The orbit $\mathcal{O}_4$ has good parity for SO_5. *So there must be a one-one correspondence between* $\mathrm{Unip}_{\mathrm{Sp}_{2n}(\mathbb{R})}(\check{\mathcal{O}})$ *In fact, there are totally* 5 *special unipotent representations of real forms in a inner class of* SO_5, 2 *for* $\mathrm{SO}(1,4)$, 2 *for* $\mathrm{SO}(3,2)$ *and* 1 *for* $\mathrm{SO}(5,0)$. *Theses representations are trivial when restricted on the connected component.*

Appendix A. Combinatorics of Weyl group representations in the classical types

A.1. **The j-induction.** *If μ and ν are two partitions representing two symmetric groups representations. Then*

$$j_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu \boxtimes \nu = \mu \cup \nu,$$

where $\mu \cup \nu$ is the partition such that

$$\{\,\mathbf{c}_i(\mu\cup\nu) \mid i \in \mathbb{N}^+\,\} = \{\,\mathbf{c}_i(\mu) \mid i \in \mathbb{N}^+\,\} \cup \{\,\mathbf{c}_i(\nu) \mid i \in \mathbb{N}^+\,\}$$

as multisets.

[*Use the inductive by stage of j-induction*

$$\begin{aligned} & j_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu \boxtimes \nu \\ &= j_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}} j_{\prod_i S_{\mathbf{c}_i(\mu)} \times \prod_j S_{\mathbf{c}_j(\nu)}}^{S_{|\mu|}\times S_{|\nu|}} \mathrm{sgn} \\ &= j_{\prod_i S_{\mathbf{c}_i(\mu\cup\nu)}}^{S_{|\mu|+|\nu|}} \mathrm{sgn} \\ &= \mu \cup \nu. \end{aligned}$$

]

We have

$$j_{S_n}^{W_n}\operatorname{sgn} = \begin{cases} (\square_k, \square_k) & \text{if } n = 2k \text{ is even,} \\ (\square_{k+1}, \square_k) & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

[*The symbol of trivial of trivial group is*

$$\binom{0,1,\cdots,k}{0,\cdots,k-1}.$$

Apply the formula [54, 4.5.4], *When n is even, the symbol of the induce is*

$$\binom{0,2,\cdots,k+1}{1,\cdots,k}.$$

corresponds to $(\square_k, \square_k)$.

When n is even, the symbol of the induce is

$$\binom{1,2,\cdots,k+1}{1,\cdots,k}.$$

corresponds to $(\square_{k+1}, \square_k)$.]

If $\tau = (\tau_L, \tau_R)$ and $\sigma = (\sigma_L, \sigma_R)$ be two bipartition. Then

$$j_{W_{|\tau|}\times W_{|\sigma|}}^{W_{|\tau|+|\sigma|}} \tau \boxtimes \sigma = (\tau_L \cup \sigma_L, \tau_R \cup \sigma_R)$$

Appendix B. Remarks on the Counting theorem of unipotent representations

In this section, let $G_{\mathbb{C}}$ be a connected complex reductive group and $\mathfrak{g}$ is its Lie algebra. Fix a antiholomorphic involution σ on $G_{\mathbb{C}}$ and a corresponding Cartan involution θ of $G_{\mathbb{C}}$. Let G be a finite central extension of a open subgroup os $G_{\mathbb{C}}^{\sigma}$ and

$$\operatorname{pr}\colon G \to G_{\mathbb{C}}^{\sigma}$$

be the canonical projection. Let $K = \operatorname{pr}^{-1}(G_{\mathbb{C}}^{\sigma})$ and $W = W(G_{\mathbb{C}})$.

Let ${}^a\mathfrak{h}$ be the abstract Cartan subalgebra of $\mathfrak{g}$ and ${}^aX$ be the lattice of abstract weight spaces. Let ${}^aR \subseteq {}^aX$, ${}^aR^+$ and ${}^aQ$ be the abstract root system, the set of positive roots and the root lattice. Let

$$\mathsf{C} = \left\{ \mu \in {}^a\mathfrak{h}^* \;\middle|\; \begin{array}{l} \text{either } \langle \operatorname{Re}(\mu), \check{\alpha}\rangle > 0 \text{ or} \\ \langle \operatorname{Re}(\mu), \check{\alpha}\rangle = 0 \text{ and } \sqrt{-1}\langle \operatorname{Im}(\mu), \check{\alpha}\rangle > 0 \end{array} \right\}$$

and $\overline{\mathsf{C}}$ be the closure of C in ${}^a\mathfrak{h}$.

We identify the set of infinitesimal characters with ${}^a\mathfrak{h}^*/W$.

B.1. **Coherent family.** *For each finite dimensional $\mathfrak{g}$-module or $G_{\mathbb{C}}$-module F, let F^* be its contragredient representation and let $\Delta(F) \subseteq {}^aX$ denote the multi-set of weights in F.*

Let $\Pi_{\Lambda_0}(G_{\mathbb{C}})$ be the set of irreducible finite dimensional representations of $G_{\mathbb{C}}$ with extreme weight in Λ_0 and $\mathcal{G}_{\Lambda}(G_{\mathbb{C}})$ be the subgroup generated by $\Pi_{\Lambda_0}(G_{\mathbb{C}})$. Let

$${}^aP := \{\, \mu \in {}^aX \mid \mu \text{ is a } {}^a\mathfrak{h}\text{-weight of an } F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}}) \,\}.$$

Via the highest weight theory, every W-orbit $W \cdot \mu$ in ${}^aP$ corresponds with the irreducible finite dimensional representation $F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}})$ with extremal weight μ.

Now the Grothendieck group $\mathcal{G}(G_{\mathbb{C}})$ of finite dimensional representation of $G_{\mathbb{C}}$ is identified with $\mathbb{Z}[{}^aP/W]$. In fact $\mathcal{G}(G_{\mathbb{C}})$ is a $\mathbb{Z}$-algebra under the tensor product and equipped with the involution $F \mapsto F^*$.

Fix a W-invariant sub-lattice $\Lambda_0 \subset {}^aX$ containing ${}^aQ$.

For any $\lambda \in {}^a\mathfrak{h}^*$, *we define*

$$
\begin{aligned}
[\lambda] &:= \lambda + {}^aQ, \\
R_{[\lambda]} &:= \{\, \alpha \in {}^aR \mid \langle \lambda, \check\alpha\rangle \in \mathbb{Z} \,\}, \\
W_{[\lambda]} &:= \langle s_\alpha | \alpha \in R_{[\lambda]}\rangle \subseteq W, \\
R_\lambda &:= \{\, \alpha \in {}^aR \mid \langle \lambda, \check\alpha\rangle = 0 \,\}, \quad \text{and} \\
W_\lambda &:= \langle s_\alpha | \alpha \in R_\lambda\rangle = \langle w \in W | w\cdot\lambda = \lambda\rangle \subseteq W.
\end{aligned} \tag{B.1}
$$

For any lattice $\Lambda = \lambda + \Lambda_0 \in \mathfrak{h}^*/\Lambda_0$ *with* $\lambda \in \overline{\mathsf{C}}$, *we define*

$$W_\Lambda := \{\, w \in W \mid w\cdot\Lambda = \Lambda \,\}.$$

Clearly, we have

$$W_{[\lambda]} < W_\Lambda, \quad \forall\ \lambda \in \Lambda.$$

Definition B.1. *Suppose* $\mathcal{M}$ *is an abelian group with* $\mathcal{G}_\Lambda(G_\mathbb{C})$*-action*

$$\mathcal{G}_\Lambda(G_\mathbb{C}) \times \mathcal{M} \ni (F, m) \mapsto F \otimes m.$$

In addition, we fix a subgroup $\mathcal{M}_{\bar\mu}$ *of* $\mathcal{M}$ *for each for each* W_Λ*-orbit* $\bar\mu = W_\Lambda\cdot\mu \in \Lambda/W_\Lambda$.

A function $f\colon \Lambda \to \mathcal{M}$ *is called a coherent family based on* Λ *if it satisfies* $f(\mu) \in \mathcal{M}_\mu$ *and*

$$F \otimes f(\mu) = \sum_{\nu \in \Delta(F)} f(\mu + \nu) \qquad \forall \mu \in \Lambda, F \in \Pi_{\Lambda_0}(G_\mathbb{C}).$$

Let $\mathrm{Coh}_\Lambda(\mathcal{M})$ *be the abelian group of all coherent families based on* Λ *and value in* $\mathcal{M}$.

In this paper, we will consider the following cases.

Example B.2. *Suppose* $\mathcal{M} = \mathbb{Q}$ *and* $F \otimes m = \dim(F)\cdot m$ *for* $F \in \Pi_{\Lambda_0}(G_\mathbb{C})$ *and* $m \in \mathcal{M}$. *We let* $\mathcal{M}_{\bar\mu} = \mathcal{M}$ *for every* $\mu \in \Lambda$. *When* $\Lambda = \Lambda_0$, *the set of* W*-harmonic polynomials on* ${}^a\mathfrak{h}^*$ *is naturally identified with* $\mathrm{Coh}_\Lambda(\mathcal{M})$ *via restriction (Vogan's result)*

Example B.3. *Let* $\mathcal{G}(\mathfrak{g}, K)$ *be the Grothendieck group of finite length* $(\mathfrak{g}, K)$*-modules and* $\mathcal{G}_\chi(\mathfrak{g}, K)$ *be the subgroup of* $\mathcal{G}(\mathfrak{g}, K)$ *generated by the set of irreducible* $(\mathfrak{g}, K)$*-modules with infinitesimal character* χ.

Then $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$ *is the group of coherent families of Harish-Chandra modules. The space* $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$ *is equipped with a* W_Λ*-action by*

$$w\cdot f(\mu) = f(w^{-1}\mu) \qquad \forall \mu \in \Lambda, w \in W_\Lambda, f \in \mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K)).$$

Example B.4. *Fix a* $G_\mathbb{C}$*-invariant closed subset* $\mathcal{Z}$ *in the nilpotent cone of* $\mathfrak{g}$. *Let* $\mathcal{G}_\mathcal{Z}(\mathfrak{g}, K)$ *be the Grothendieck group of* $(\mathfrak{g}, K)$*-modules whose complex associated varieties are contained in* $\mathcal{Z}$. *We define*

$$\mathcal{G}_{\chi,\mathcal{Z}}(\mathfrak{g}, K) := \mathcal{G}_\chi(\mathfrak{g}, K) \cap \mathcal{G}_\mathcal{Z}(\mathfrak{g}, k).$$

Now $\mathrm{Coh}_\Lambda(\mathcal{G}_\mathcal{Z}(\mathfrak{g}, K))$ *is also a* W_Λ*-submodule of* $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$.

Example B.5. *Fixing a Borel subalgebra* $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$, *let* $\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ *be the Grothendieck group of the category* $\mathcal{O}$. *The space* $\mathcal{G}_\mathcal{Z}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ *is defined similarly. The space* $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}))$ *and* $\mathrm{Coh}_{\Lambda(\mathcal{G})}$ *defined similarly.*

We can define W_Λ *action on* $\mathrm{Coh}_\Lambda(\mathcal{M})$ *by*

$$w\cdot f(\mu) = f(w^{-1}\mu) \qquad \forall \mu \in \Lambda, w \in W_\Lambda.$$

Example B.6. *For each infinitesimal character* χ *and a close* G*-invariant set* $\mathcal{Z} \in \mathcal{N}_\mathfrak{g}$. *Let* $\mathcal{G}_{\chi,\mathcal{Z}}(\mathfrak{g}, K)$ *be the Grothendieck group of* $(\mathfrak{g}, K)$*-module with infinitesimal character* χ *and complex associated variety contained* $\mathcal{Z}$. *Similarly, let* $\mathcal{G}_{\chi,\mathcal{Z}}()$

Translation principal assumption. *Recall that we have fixed a set of simple roots of W_Λ.*

We make the following assumption for $\mathrm{Coh}_\Lambda(\mathcal{M})$.

- *There is a basis* $\{\, \Theta_\gamma \mid \gamma \in \mathrm{Parm} \,\}$ *of* $\mathrm{Coh}_\Lambda(\mathcal{M})$ *where* Parm *is a parameter set;*
- *For every $\mu \in \Lambda$, the evaluation at μ is surjective;*
$$\mathrm{ev}_\mu \colon \mathrm{Coh}_\Lambda(\mathcal{M}) \to \mathcal{M}_{\bar{\mu}} \qquad f \mapsto f(\mu)$$
is surjective;
- *a subset $\tau(\gamma)$ of the simple roots of W_Λ is attached to each $\gamma \in$* Parm *such that $s \cdot \Theta_\gamma = -\Theta_\gamma$;*
- $\Theta_\gamma(\mu) = 0$ *if and only if $\tau(\gamma) \cap R_\mu \neq \varnothing$.*
- $\{\, \Theta_\gamma(\mu) \mid \tau(\gamma) \cap R_\mu = \varnothing \,\}$ *form a basis of $\mathcal{M}_{\bar{\gamma}}$.*

The translation principle assumption implies

$$\mathrm{Ker}\,\mathrm{ev}_\mu = \mathrm{span}\,\{\, \Theta_\gamma \mid \tau(\gamma) \cap R_\mu \neq \varnothing \,\} \tag{B.2}$$

Now we have the following counting lemma.

Lemma B.7. *For each μ, we have*
$$\dim \mathcal{M}_{\bar{\mu}} = \dim(\mathrm{Coh}_\Lambda(\mathcal{M}))_{W_\mu} = [\mathrm{Coh}_\Lambda(\mathcal{M}), 1_{W_\mu}].$$

Proof. Clearly
$$\mathrm{span}\,\{\, \Theta_\gamma - w\Theta_\gamma \mid \gamma \in \mathrm{Parm}, w \in W_\Lambda \,\} \subseteq \ker \mathrm{ev}_\mu$$
since $w \cdot \Theta_\gamma(\mu) = \Theta_\gamma(w^{-1} \cdot \mu) = \Theta_\gamma(\mu)$ for $w \in W_\mu$. Combine this with (B.2), we conclude that
$$\mathrm{span}\,\{\, \Theta_\gamma - w\Theta_\gamma \mid \gamma \in \mathrm{Parm}, w \in W_\Lambda \,\} = \ker \mathrm{ev}_\mu.$$
Therefore, ev_μ induces an isomorphism $(\mathrm{Coh}_\Lambda(\mathcal{M}))_{W_\mu} \to \mathcal{M}_{\bar{\mu}}$. Now the dimension equality follows. □

Example B.8. *For the case of category $\mathcal{O}$. We can take* $\mathrm{Parm} = W$. *Let $\Theta_w(\mu) = L(w\mu)$ for each $w \in W_\Lambda$ with $\tau(w) = \{\, s_\alpha \mid \alpha \in \Delta^+, w\alpha \notin R^+ \,\}$.*

Example B.9. *In the Harish-Chandra module case,* Parm *consists of certain irreducible K-equivariant local systems on a K-orbit of the flag variety of G. $\tau(\gamma)$ is the τ-invariants of the parameter γ.*

B.2. **Primitive ideals and left cells.** *In this section, we recall some results about the primitive ideals and cells developed by Barbasch Vogan, Joseph and Lusztig etc.*

B.3. **Highest weight module.** *Let $\mathfrak{g}$ be a reductive Lie algebra, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a fixed Borel. Let*
$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$
and $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$.

In the rest of this section, we fix a lattice $\Lambda \in \mathfrak{h}^*/X^*$. Let R_Λ and R_Λ^+ be the set of integral root system and the set of positive integral roots. Write W_Λ for the integral Weyl group.

For $\mu \in \Lambda$,
$$\begin{aligned} \mu \geq 0 \quad &\Leftrightarrow \quad \mu \text{ is dominant} \quad \Leftrightarrow \quad \langle \lambda, \check{\alpha} \rangle \geqslant 0 \quad \forall \alpha \in R_\Lambda^+ \\ \mu \not> 0 \quad &\Leftrightarrow \quad \mu \text{ is regular} \quad \Leftrightarrow \quad \langle \lambda, \check{\alpha} \rangle \neq 0 \quad \forall \alpha \in R_\Lambda^+ \end{aligned}$$
regular if $\langle \lambda, \check{\alpha} \rangle \neq 0$.

For each $w \in W$, it give a coherent family such that
$$M_w(\mu) = M(w\mu) \quad \forall \mu \in \Lambda$$

Let L_w be the unique coherent family such that $L_w(\mu) = L(w\mu)$ for any regular dominant μ in Λ. [*Note that the Grothendieck group $\mathcal{K}(\mathcal{O})$ of category $\mathcal{O}$ is naturally embedded in the space of formal character. M_w is a coherent family: the formal character $\operatorname{ch} M_w(\mu)$ of $M_w(\mu)$ is*

$$\operatorname{ch} M_w(\mu) = \frac{e^{w\mu-\rho}}{\prod_{\alpha\in R^+}(1-e^{-\alpha})} = \frac{e^{w\mu}}{\prod_{\alpha\in R^+}(e^{\alpha/2}-e^{-\alpha/2})}$$

It is clear that $\operatorname{ch} M_w$ satisfies the condition for coherent continuation.

From now on, we fix a regular dominant weight $\lambda \in \mathfrak{h}^*$. Then $w[\lambda] = [w\cdot\lambda]$ for any $w \in W$.

Now $W_{w[\lambda]} = w\, W_{[\lambda]}\, w^{-1}$.

As $W_{[\lambda]}$-module, we have the following decomposition

$$\operatorname{Coh}_{[\lambda]} = \bigoplus_{r\in W/W_{[\lambda]}} \operatorname{Coh}_r \quad \textit{with}$$

$$\operatorname{Coh}_r = \operatorname{Coh}_{r\lambda} := \{\, \Theta \in \operatorname{Coh}_{[\lambda]} \mid \Theta(\lambda) \in \mathcal{O}'_{[r\cdot\lambda]} \,\}$$

Here $\mathcal{O}'_S$ is the set of highest weight module whose $\mathfrak{h}$-weights are in $S \subset \mathfrak{h}^*$.

Note that the following map

$$\begin{array}{ccccc} \mathbb{C}[W] & \longrightarrow & \operatorname{Coh}_{[\lambda]} & \longrightarrow & \mathbb{C}[S] \\ w & \mapsto & (\mu \mapsto M_w(\mu)) & \mapsto & w\cdot\lambda \end{array}$$

is $W_{[\lambda]}$-equivariant where $W_{[\lambda]}$ acts by right translation on $\mathbb{C}[W]$ and $S = W\cdot\lambda$. The action of $W_{[\lambda]}$ on S is by transport of structure and so $(a, w\,\lambda) \mapsto wa^{-1}\,\lambda$.

Now $\mathbb{C}[rW_{[\lambda]}] \subset \mathbb{C}[W]$ and $\mathbb{C}[rW_{[\lambda]}\cdot\lambda]$ are identified with Coh_r.

]

For each $w \in W_\Lambda$, the function

$$\widetilde{p}_w(\mu) := \operatorname{rank}(\mathcal{U}(\mathfrak{g})/\operatorname{Ann}(L(w\mu))) \qquad \forall \mu \in \Lambda \ \textit{dominant}.$$

extends to a Harmonic polynomial on $\mathfrak{h}^*$. In particular, $\widetilde{p}_w \in P(\mathfrak{h}^*) = S(\mathfrak{h})$. [*Let Λ^+ be the set of dominant weights in Λ. Assume $\lambda > 0$, i.e. dominant regular. Then $\mathbb{R}^0_\lambda =$, $w^{w\lambda} = w^{-1}$. The set*

$$\hat{F}_{w\lambda} = \{\, \mu \in \Lambda \mid w^{-1}\mu \geq 0 \,\} = w\Lambda^+.$$

Joseph's $p_w(\mu) = \operatorname{rank}\mathcal{U}(\mathfrak{g})/(\operatorname{Ann} L(w\mu))$ is defined on $\hat{F}_{w\lambda} = w\Lambda^+$. Now his $\widetilde{p}_w(\mu) = w^{-1}p_w$ is defined on Λ^+ and given by $\widetilde{p}_w(\mu) = \operatorname{rank}\mathcal{U}(\mathfrak{g})/(\operatorname{Ann} L(w\mu))$.]

There is a unique function $a_\Lambda \colon W_\Lambda \times W_\Lambda \to \mathbb{Q}$ such that

$$L_w = \sum_{w'\in W_\Lambda} a_\Lambda(w, w') M_{w'}$$

Suppose V is a module in $\mathcal{O}$ whose Gelfand-Kirillov dimension $\leqslant d$ Let $c(V)$ be the Bernstein degree. Let $\operatorname{Coh}^d_{[\Lambda]}$ be the submodule of $\operatorname{Coh}_{[\Lambda]}$ generated by L_w such that the $\dim L(w\mu) \leqslant d$. *Then the map*

$$\mathcal{P}^d : \operatorname{Coh}^d_\Lambda \to S(\mathfrak{h}) \qquad \Theta \mapsto (\mu \mapsto c(\Theta(\mu)))$$

is W_Λ-equivariant. Let $c_w := c \circ (L_w(\mu)) \in S(\mathfrak{h})$.

The following result of Joseph implies that the set of primitive ideals can be parameterized by Goldie rank polynomials.

Theorem B.10 (Joseph). *correct formulation? For μ dominant, the primitive ideal* $\operatorname{Ann} L(w\mu) = \operatorname{Ann} L(w'\mu)$ *if and only if $p_w = p'_w$.* ([37, ref?])

Theorem B.11. *Let $r = \dim \mathfrak{n}$. Suppose $\dim L(w\nu) = d$.*

(i) Let $x \in \mathfrak{h}$ such that $\alpha(x) = 1$ for every simple roots α. Then there are non-zero rational numbers a, a' such that [37, Theorem 5.1 and 5.7]

$$a\, \widetilde{p}_w(\mu) = a'\, c_w(\mu) = \sum_{w' \in W_\Lambda} a_\Lambda(w, w') \left\langle \mu, w'^{-1} x \right\rangle^{r-d}.$$

(ii) The submodule $\mathbb{C}W_\Lambda \widetilde{p}_w$ in $S(\mathfrak{h})$ is a special irreducible W_Λ-representation. Moreover all special representations of W_Λ occurs as a certain $\sigma(w)$. See [8, Theorem D], [9, Theorem 1.1].

(iii) Let $\sigma(w)$ be the submodule of $S(\mathfrak{h})$ generated by $\widetilde{p}_w$. Then $\sigma(w)$ is irreducible and it occurs in $S^{r-d}(\mathfrak{h})$ with multiplicity 1, see [41, 5.3],

By Theorem B.10, the module $\sigma(w)$ only depends on the primitive ideal $\mathcal{J} := \operatorname{Ann} L(w\mu)$. So we can also write $\sigma(\mathcal{J}) := \sigma(w)$.

From the above theorem, we see that the map $\mathcal{P}^d$ factor through $\mathrm{Coh}^d_\Lambda / \mathrm{Coh}^{d-1}_\Lambda$ and its image is contained in the space $\mathcal{H}^d$ of degree d-Harmonic polynomials.

Note that the associated variety of a primitive ideal $\mathcal{J}$ is the closure of a nilpotent orbit in $\mathfrak{g}$. Now the associated variety of $\mathcal{J}$ is computed by the following result of Joseph (and include Hotta ?).

Theorem B.12. *Let $\mathcal{J}$ be a primitive ideal in $\mathcal{U}(\mathfrak{g})$. Then $\sigma(\mathcal{J})$ is in the image of Springer correspondence. Moreover, the associated variety of $\mathcal{J}$ is the orbit $\mathcal{O}$ corresponding to $\sigma(\mathcal{J})$. See* [39, Proposition 2.10].

Lemma B.13. *Let $\mu \in \Lambda$, $W_\mu = \{\, w \in W_\Lambda \mid w \cdot \mu = \mu \,\}$ and*

$$a_\mu = \max \{\, a(\sigma) \mid \widehat{W_\Lambda} \ni \sigma \text{ occurs in } \operatorname{Ind}_{W_\mu}^{W_\Lambda} 1 \,\}.$$

Suppose W_μ is a parabolic subgroup of W_Λ. Then the set of all irreducible representations σ occurs in $\operatorname{Ind}_{W_\mu}^{W_\Lambda} 1$ such that $a(\sigma) = a_\mu$ forms a left cell given by

$$\left(J_{W_\mu}^{W_\Lambda} \operatorname{sgn} \right) \otimes \operatorname{sgn}.$$

Now the maximal primitive ideal having infinitesimal character μ has the associated variety $\mathcal{O}$.

Recall that $W_\Lambda = W_b \times W_g$. Let

$$\sigma_b = (j_{W_{\check{\mathcal{O}}_b}}^{W_b} \operatorname{sgn}) \otimes \operatorname{sgn}$$
$$\sigma_g = (j_{W_{\check{\mathcal{O}}_g}}^{W_g} \operatorname{sgn}) \otimes \operatorname{sgn}$$

Then $\sigma_b \otimes \sigma_g$ is a special representation of W_Λ. Moreover $\mathcal{O}$ corresponds to the representation

$$\sigma := j_{W_b \times W_g}^{W} \sigma_b \otimes \sigma_g$$

[*We recall some facts about the j-induction and J-induction. If W' is a parabolic subgroup in W then $j_{W'}^{W}$ maps special representation to special representation. It maps irreducible representation to irreducible ones. The j induction has induction by stage* [16, 11.2.4].

For classical group, the representations satisfies property B. When W' is parabolic, the orbit $\mathcal{O}$ corresponds to $j_{W'}^{W}$ is the orbit $\operatorname{Ind}_L^G \mathcal{O}$.

The J-induction maps left cell to left cell, which is only defined for parabolic subgroups.

The j-induction for classical group is in [16, Section 11.4]. *Basically, W_n has D_{c_i}, B/C_{c_i} factors,*

$$j_{\prod_i D_{c_i} \times \prod_j B/C_{c_j}} \operatorname{sgn} = ((c_i), (c_j)).$$

Here (c_i) denote the Young diagram with columns c_i etc.

Moreover, we have the following formula for j-induction.

$$\mathrm{ind}_{A_{2m}}^{W_{2m}} = ((m),(m)) \qquad \mathrm{ind}_{A_{2m+1}}^{W_{2m+1}} = ((m),(m+1)).$$

]

B.4. **Harish-Chandra cells and primitive ideals.** *For simplicity we assume G is connected in this section.*

We recall McGovern and Casian's works on the Harish-Chandra cells.

Fix a θ stable Cartan subgroup $H = TA$ of G such that T is the maximal compact subgroup of H and A is the split part of H. Let $\widehat{H}$ denote the set of irreducible representations of $\widehat{H}$ (also viewed as an $(\mathfrak{h}, T)$-module). Let $C(H)$ be component group of H, which is also the component group of T. Let $\mathrm{d}\colon \widehat{H} \longrightarrow \mathfrak{h}^*$ be the map takes $\phi \in \widehat{H}$ to its derivative. Then $\widehat{H}$ is a $\widehat{C(H)}$-torsor over $\mathfrak{h}^*$. [*This means for each $\lambda \in \mathfrak{h}^*$, its fiber $\mathrm{d}^{-1}(\lambda)$ is either empty or with a set with free transitive $\widehat{C(H)}$-action (the tensor product action).*]

[*The classification of Primitive ideals.*

$$\begin{array}{rcll} W_{[\lambda]} & \longrightarrow & \mathrm{Prim}_\lambda & \\ w & \mapsto & I(w\lambda) & := \mathrm{Ann}\, L(w\lambda). \end{array}$$

We now recall some results about the blocks in category $\mathcal{O}$.

Assume $\lambda \in \mathsf{C}$ where $\mathsf{C} = \{\, \nu \in \mathfrak{h}^* \mid \langle \nu, \check{\alpha}\rangle > 0 \,\}$ is the positive cone. Let $\lambda' = w_0\lambda$ which is negative.

Let w_0 (resp. $w_{[\lambda]}$)be the longest element in W (resp. $W_{[\lambda]}$).

Define (We adapt the convention that $e \overset{L}{\leqslant} w_{[\lambda]}$.)

$$\begin{aligned} w_1 \overset{L}{\leqslant} w_2 &\Leftrightarrow I(w_1\lambda') \subseteq I(w_2\lambda') \\ &\Leftrightarrow I(w_1 w_0\lambda) \subseteq I(w_2 w_0 \lambda) \\ &\Leftrightarrow [L(w_2^{-1}\lambda'), L(w_1^{-1}\lambda') \otimes S(\mathfrak{g})] \neq 0 \end{aligned}$$

Note that

$$w_1 \overset{L}{\leqslant} w_2 \Leftrightarrow w_2^{-1} \overset{R}{\leqslant} w_1^{-1}.$$

Therefore, the last condition implies that

$$\begin{aligned} \mathscr{V}^R(w) &:= \mathrm{span}\,\{\, L(w'\lambda) \mid w' \overset{R}{\leqslant} w \,\} \\ &= \mathrm{span}\,\{\, L(w'\lambda) \mid w^{-1} \overset{L}{\leqslant} w'^{-1} \,\} \end{aligned}$$

]

We also fix a positive system $\Delta^+(\mathfrak{g},\mathfrak{h})$ and let $\mathfrak{n}$ be the span of positive roots. Let

$$Q := \mathbb{Z}[\Delta(\mathfrak{g},\mathfrak{h})] \subset \mathfrak{h}^*$$

be the root lattice of $\mathfrak{g}$, which is exactly the set of $\mathfrak{h}$-weights occur in $S(\mathfrak{g})$.

By [80, 0.4.6], *we can naturally identify Q with the subset of $\widehat{H}$ consisting the characters occurs in $S(\mathfrak{g})$.*

For each $\lambda \in \mathfrak{h}^*$, let $\langle\lambda\rangle := W_{[\lambda]} \cdot \lambda$. A set $S \subset \mathfrak{h}^*$ is called a type *if*

$$S = \bigcup_{\lambda \in S} \langle \lambda \rangle .$$

Clearly, $\langle\lambda\rangle$ is the smallest type containing λ.

Suppose S is a type, we define the category $\mathcal{O}'_S$ to be the category of $\mathfrak{g}$-modules such that $M \in \mathcal{O}'_S$ if and only if

- *the* $\mathfrak{b}$*-action on* M *is locally finite;*
- M *is finitely generated* $\mathcal{U}(\mathfrak{g})$*-module,*
- $M = \sum_{\mu \in S} M_\mu$ *where* M_μ *is the* μ*-isotypic component of* M*.*

Clearly, $\mathcal{O}'_S$ *is a union of blocks in* $\mathcal{O}'$*.*

For each set $\tilde{S} \subset \widehat{H}$*, let* $[\tilde{S}] := \{\, \phi \otimes \alpha \mid \phi \in \tilde{S}, \alpha \in Q \,\}$ *be the* Q*-translations of* $\tilde{S}$*. We say* $\tilde{S}$ *is* Q*-saturated if* $\tilde{S} = \mathrm{d}^{-1}(\mathrm{d}(\tilde{S})) \cap [\tilde{S}]$*. We say* $\tilde{S}$ *is a type if* $\mathrm{d}(S)$ *is a type. Suppose* $\tilde{S}$ *is* Q*-saturated and*

For $\phi \in \widehat{H}$*, we write*

$$\langle \phi \rangle := [\phi] \cap \phi^{-1}(\langle \mathrm{d}\phi \rangle)$$

which is the smallest W*-invariant* Q*-saturated subset in* $\widehat{H}$ *containing* ϕ*. Let* $M(\phi) := \mathrm{U}(\mathfrak{g}) \otimes_{\mathfrak{b},T} \phi$ *where* ϕ *is inflated to a* $(\mathfrak{b}, T)$*-module.*

Suppose $\tilde{S}$ *is* Q*-saturated type, we define the category* $\mathcal{O}'_S$ *to be the category of* $(\mathfrak{g}, T)$*-module such that* $M \in \mathcal{O}'_S$ *if and only if*

- *the* $\mathfrak{b}$*-action is locally finite;*
- M *is finitely generated* $\mathcal{U}(\mathfrak{g})$*-module,*
- $M = \sum_{\mu \in \tilde{S}} M_\mu$ *where* M_μ *is the* μ*-isotypic component of* M*.*

In particular, $\mathcal{O}'_{\widehat{H}}$ *is the category of finitely generated* $(\mathfrak{g}, T)$*-module such that* $\mathfrak{b}$*-acts locally finite.*

Let $\mathcal{F} \colon \mathcal{O}'_{\widehat{H}} \to \mathcal{O}'_{\mathfrak{h}}$ *be the forgetful functor of the* T*-module structure. The following lemma is clear.*

Lemma B.14. *The forgetful functor* $\mathcal{F}$ *yields an equivalence of category*

$$\mathcal{O}'_{\langle \phi \rangle} \to \mathcal{O}'_{\langle \mathrm{d}\phi \rangle}$$

which induces an isomorphism of $W_{[\mathrm{d}\phi]}$*-module*

$$\mathrm{Coh}_\phi \to \mathrm{Coh}_{\mathrm{d}\phi}.$$

Suppose $\gamma \in \widehat{C(H)}$*. By Lemma B.14, we have an equivalent of category*

$$\mathcal{O}'_{\langle \phi \rangle} \cong \mathcal{O}'_{\langle \mathrm{d}\phi \rangle} \cong \mathcal{O}'_{\langle \phi \otimes \gamma \rangle}$$

which sends $M(\phi)$ *to* $M(\phi \otimes \gamma)$*. This isomorphism induces a* $W_{[\mathrm{d}\phi]}$*-module isomorphism*

$$\mathrm{Coh}_\phi \xrightarrow{\otimes \gamma} \mathrm{Coh}_{\phi \otimes \gamma}.$$

Let $\widetilde{\mathrm{Coh}}_{[\lambda]} := \mathcal{F}^{-1}(\mathrm{Coh}_{[\lambda]})$*.*

In summary, we conclude that $\widetilde{\mathrm{Coh}}_{[\lambda]} \xrightarrow{\mathcal{F}} \mathrm{Coh}_{[\lambda]}$ *is a* $\widehat{C(H)}$*-torsor over* $\mathrm{Coh}_{[\lambda]}$*.*

Fix a set $\{\, r_1, \cdots, r_k \,\}$ *of representatives of* $W/W_{[\lambda]}$ *and fix an element* $\phi_i \in \mathrm{d}^{-1}(r_i \lambda)$ *for each* r_i*. Let* $\Phi := \{\, \phi_i \mid i = 1, 2, \cdots, k \,\}$*.*

Now we have an isomorphism of $W_{[\lambda]}$*-module*

$$\mathcal{F}_\Phi \colon \bigoplus_i \mathrm{Coh}_{\phi_i} \longrightarrow \bigoplus_i \mathrm{Coh}_{r_i \lambda} = \mathrm{Coh}_{[\lambda]}.$$

Now

$$\mathrm{Coh}_{[\lambda]} \otimes \mathbb{Z}[\widehat{C(H)}] \xrightarrow{\;\cong\;} \widetilde{\mathrm{Coh}}_{[\lambda]}.$$

given by $\Theta \otimes \alpha \mapsto \mathcal{F}_\Phi^{-1}(\Theta) \otimes \alpha$ *is a* $W_{[\lambda]} \times \widehat{C(H)}$*-module isomorphism.*

Let $K = G^\theta$*.*

Recall Casian's result:

Theorem B.15 ([17, Proposition 2.10, Theorem 3.1]). *Let H be a θ-stable Cartan subgroup of G. Fix a positive system of real roots $\Delta_{\mathbb{R}}^{+}$ and a positive system Δ^{+} of roots such that $\Delta_{\mathbb{R}}^{+} \subseteq \Delta^{+}$. Let $\mathbf{n}$ be the maximal nilptent Lie subalgebra of $\mathfrak{g}$ with spanned by roots in Δ^{+}. Let M be a Harish-Chandra $(\mathfrak{g}, K)$-module, then*
(i) the Lie algebra cohomology $H^{q}(\mathfrak{n}, M)$ is finite dimensional;
(ii) the localization $\gamma_{\mathfrak{n}}^{q} M$ is in the category $\mathcal{O}'_{\hat{H}}$; (see [17, Section 1]*)*
(iii) $\operatorname{Ann} M \subseteq \operatorname{Ann}(\gamma_{\mathfrak{n}}^{q} M)$.
(iv) Then the localization functor induces a homomorphism

$$\begin{array}{rccc} \gamma_{\mathfrak{n}}: & \mathcal{G}_{\chi,\mathcal{Z}}(\mathfrak{g}, K) & \longrightarrow & \mathcal{G}_{\chi,\mathcal{Z}}(\mathcal{O}_{\hat{H}}) \\ & M & \mapsto & \sum_{q} (-1)^{q} \gamma_{\mathfrak{n}}^{q} M \end{array}$$

□

Theorem B.16 ([17, Theorem 3.1]). *Let $H_1, H_2, \cdots, H_s$ form a set of representatives of the conjugacy class of θ-stable Cartan subgroup of G. Fix maximal nilpotent Lie subalgebra $\mathfrak{n}_i$ for each H_i as in Theorem B.15. Then*

$$\gamma := \oplus_i \gamma_{\mathfrak{n}_i}: \quad \mathcal{G}_{\chi}(\mathfrak{g}, K) \longrightarrow \bigoplus_i \mathcal{G}_{\chi}(\mathcal{O}_{\hat{H}_i})$$

is an embedding of W_{Λ}-module.

Proof. We retain the notation in [17]. Let H_i^{rs} be the set of regular semisimple elements in H_i. By Harish-Chandra, taking the character of the elements induces an embedding of $\mathcal{G}_{\chi}(\mathfrak{g}, K)$ into the space of analytic functions on $\bigsqcup_i H_i^{rs}$. Now [17, Theorem 3.1] implies that the global character ΘM of an element $M \in \mathcal{G}_{\chi}(\mathfrak{g}, K)$ is completely determined by the formal character $\operatorname{ch}(\gamma(M))$. □

Corollary B.17. *Fix an irreducible $(\mathfrak{g}, K)$-module π with infinitesimal character χ_{λ}. Let $\mathscr{V}^{\mathrm{HC}}(\pi)$ be the Harish-Chandra cell representation containing π and $\mathcal{D}$ be the double cell in $\widehat{W_{[\lambda]}}$ containing the special representation $\sigma(\pi)$ attached to* $\operatorname{Ann}(\pi)$. *Then $[\sigma, \mathscr{V}^{\mathrm{HC}}(\pi)] \neq 0$ only if $\sigma \in \mathcal{D}$. Moreover, $\sigma(\pi)$ always occures in $\mathscr{V}^{\mathrm{HC}}(\pi)$*

Proof. The occurrence of $\sigma(\pi)$ is a result of King.

Note that we have an embedding

$$\gamma\colon \mathcal{G}_{\chi}(\mathfrak{g}, K) \longrightarrow \bigoplus_i \mathcal{G}(\mathfrak{g}, \mathfrak{b}, \lambda).$$

where the left hand sides is identified with a finite copies of $\mathbb{C}[W_{[\lambda]}]$.

Since $\operatorname{Ann}(\pi) \subseteq \operatorname{Ann}(\gamma_{\mathfrak{n}}^{q}(\pi))$, we conclude that $[\sigma, \mathscr{V}^{\mathrm{HC}}(\pi)] \neq 0$ implies that $\sigma(\pi) \overset{LR}{\leqslant} \sigma$.

By the Vogan duality, $\mathcal{D} \otimes \operatorname{sgn}$ is also a Harish-Chandra cell. So we have $\sigma(\pi) \otimes \operatorname{sgn} \overset{LR}{\leqslant} \sigma \otimes \operatorname{sgn}$.

Therefore, $\sigma(\pi)_{LR}\approx$ □

B.5. **Primitive ideals for $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ at half integral infinitesimal character.** *Let $\check{\mathcal{O}}$ be a metaplectic "good" nilpotent orbit in $\mathfrak{sp}(2n, \mathbb{C})$, i.e. $\mathbf{r}_i(\check{\mathcal{O}})$ is even for each $i \in \mathbb{N}^{+}$.*

Then $W_{[\lambda_{\check{\mathcal{O}}}]} = W'_n$ and the left cell is given by

$$(J_{W_{\check{\mathcal{O}}}}^{W'_n} \operatorname{sgn}) \otimes \operatorname{sgn} \qquad \textit{with} \qquad W_{\check{\mathcal{O}}} = \prod_{i \in \mathbb{N}^{+}} S_{\mathbf{c}_{2i}(\check{\mathcal{O}})}.$$

Here $W_{\check{\mathcal{O}}}$ is an subgroup of S_n and the embedding of S_n in W'_n is fixed. When n is even,we the symbol of $J_{S_n}^{W'_n} \operatorname{sgn}$ is degenerate and we label it by "I".

Let

$$\mathrm{PP}_{\tilde{C}}(\check{\mathcal{O}}) = \{ (2i-1, 2i+2) \mid i \in \mathbb{N}^{+}, \mathbf{r}_{2i-1}(\check{\mathcal{O}}) \neq \mathbf{r}_{2i}(\check{\mathcal{O}}) \}.$$

For each $\wp \subset \mathrm{PP}_{\tilde{C}}(\check{\mathcal{O}})$, let $\tau_\wp := (\imath_\wp, \jmath_\wp)$ be the bipartition given by

$$(\mathbf{r}_i(\imath_\wp), \mathbf{r}_i(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}), & \text{if } (2i-1, 2i) \notin \wp, \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}), & \text{otherwise.} \end{cases}$$

If $\tau_\wp$ is degenerate, it represent the element in $\widehat{W'_n}$ with label I. Let $\wp^c$ be the complement of $\wp$ in $\mathrm{PP}_{\tilde{C}}(\check{\mathcal{O}})$. Then $\tau_\wp$ and $\tau_{\wp^c}$ represent the same irreducible W'_n-module.

Using the induction formula in [?L, (4.6.6) (4.6.7)], *we have the following lemma.*

Lemma B.18. *The representation*

$${}^L\mathscr{C}'(\check{\mathcal{O}}) := J_{W_{\check{\mathcal{O}}}}^{W'_n} \operatorname{sgn}$$

is multiplicity free. The map

$$\mathbb{Z}[\mathrm{PP}_{\tilde{C}}(\check{\mathcal{O}})]/\sim \longrightarrow \{\, \tau \in \widehat{W'_n} \mid \tau \subset {}^L\mathscr{C}'(\check{\mathcal{O}}) \,\} \qquad \wp \mapsto \tau_\wp$$

is a bijection where $\sim$ is the equivalent relation identifying $\wp$ with $\wp^c$. Moreover, the special representation in ${}^L\mathscr{C}'(\check{\mathcal{O}})$ is $\tau_\varnothing$. □

In the following, we write

$$\tau_{\check{\mathcal{O}}} := \tau_\varnothing$$

for the unique special representation in ${}^L\mathscr{C}'(\check{\mathcal{O}})$.

Let $\tau_{\check{\mathcal{O}}} = (\imath, \jmath)$ such that $\mathbf{r}_i(\imath) \leqslant \mathbf{r}_i(\jmath)$ for all $i \in \mathbb{N}^+$. Then the unique special representation in $\operatorname{Ind}_{W_{\check{\mathcal{O}}}}^{W'_n} 1$ where the a function take maximal value is given by the bipartition

$$\tau_{\mathcal{O}} := (\jmath^t, \imath^t).$$

Let $\sigma'(\mathcal{O})$ denote the $\tau_{\mathcal{O}}$-isotypic component of W'_n-harmonic polynomials in $S(\mathfrak{h})$. Comparing the fake degree formulas of type C and D (see [16, Proposition 11.4.3, 11.4.4]*), we conclude that the W_n-representation*

$$\sigma(\mathcal{O}) := \mathbb{C}[W_n] \cdot \sigma'(\mathcal{O}) \tag{B.3}$$

is irreducible whose type is also given by the bipartition $\tau_{\mathcal{O}}$. The type of $\sigma(\mathcal{O})$ is $j_{W'}^{W} \sigma'(\mathcal{O})$.

Lemma B.19. *Suppose σ' is a W'_n-special representation.Let $\check{\mathcal{O}}'$ be the partition of type D corresponds to the W'_n-special representation $\sigma' \otimes \operatorname{sgn}_D$. Then $\sigma := j_{W'_n}^{W_n} \sigma'$ corresponds to the partition $\check{\mathcal{O}}'^t$ under the Springer correspondence of type C.*

Proof. First note that $\check{\mathcal{O}}'$ is a specail nilpotent orbit of type D_n, therefore $\check{\mathcal{O}}'^t$ is a partition of type C_n (see [20, Proposition 6.3.7]).

By the explicitly formula of the Lusztig-Spaltenstein duality, we known that the D-collapsing $(\check{\mathcal{O}}'^t)_D$ of $\check{\mathcal{O}}'^t$ corresponds to the special representation σ' of type D.

The Springer correspondence algorithm for classical groups can be naturally extends to all partitions. Sommers showed that two partitions are mapped to the same Weyl group representations if and only if they has the same D-collapsing [76, Lemma 9]. Note that the algorithms computing the Springer correspondence for type C and D are essentially the same (see [16, Section 13.3] or [76, Section 7]). By the injectivity of the Springer correspondence, we conclude that the type C partition $\check{\mathcal{O}}'^t$ must corresponds to σ. □

The following lemma is a direct consequence of Lemma B.19.

Lemma B.20. *Suppose $\check{\mathcal{O}}$ has good parity of type $\tilde{C}$. Under the Springer correspondence of type C, the representation $\sigma(\check{\mathcal{O}})$ (see* (B.3)*) corresponds to the nilpotent orbit $\mathcal{O}$ defined by*

$$(\mathbf{c}_{2i-1}(\mathcal{O}), \mathbf{c}_{2i}(\mathcal{O})) = \begin{cases} (\mathbf{r}_{2i-1}(\check{\mathcal{O}}), \mathbf{r}_{2i}(\check{\mathcal{O}})), & \text{if } \mathbf{r}_{2i-1}(\check{\mathcal{O}}), \mathbf{r}_{2i}(\check{\mathcal{O}}) \\ (\mathbf{r}_{2i-1}(\check{\mathcal{O}}) - 1, \mathbf{r}_{2i}(\check{\mathcal{O}}) + 1), & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}^+$.

Proof. Apply the algorithm of Springer correspondence of type D to the representation $\sigma'(\check{\mathcal{O}})$ gives the above formula. $\square$

[*A technical point, the representation of* W_n' *is given by symbol* $\binom{\xi}{\mu}$ *or* $\binom{\mu}{\xi}$. *However, using the Springer correspondence formula, only one of the arrangement can give a valid type D partition.*

It is a interesting fact that a type D orbit $\mathcal{O}$ *is special if and only if* $\mathcal{O}^t$ *is of type C.*
]

Lemma B.21. *Suppose* $\check{\mathcal{O}} = \check{\mathcal{O}}_b \cup \check{\mathcal{O}}_g$. *Then* $\mathcal{O} = \mathcal{O}_b \cup \mathcal{O}_g$ *where* $\mathcal{O}_b = \check{\mathcal{O}}_b^t$ *and* $\mathcal{O}_g = \tilde{d}_{BV}(\check{\mathcal{O}}_g)$.

[*We take the convention that* $2\mathcal{O} = [2r_i]$ *if* $\mathcal{O} = [r_i]$. *We also write* $[r_i] \cup [r_j] = [r_i, r_j]$. $\dagger\mathcal{O} = [r_i + 1]$.

We suppose

$$\check{\mathcal{O}}_b = [2r_1+1, 2r_1+1, \cdots, 2r_k+1, 2r_k+1] = (2c_0, 2c_1, 2c_1, \cdots, 2c_l, 2c_l)$$

where $l = r_1$.

Now

$$\begin{aligned} W_{\check{\mathcal{O}}_b} &= W_{c_0} \times S_{2c_1} \times S_{2c_2} \times \cdots \times S_{2c_l} \\ \check{\sigma}_b &:= j_{W_{\check{\mathcal{O}}_b}}^{W_b} \operatorname{sgn} = ((c_1, c_2, \cdots, c_k), (c_0, c_1, \cdots, c_l)) \\ &= ([r_1, r_2, \cdots, r_k], [r_1+1, r_2+1, \cdots, r_k+1]) \end{aligned}$$

Therefore

$$\sigma_b = \check{\sigma}_b \otimes \operatorname{sgn} = ((r_1+1, r_2+1, \cdots, r_k+1), (r_1, r_2, \cdots, r_k))$$

which corresponds to the orbit

$$\mathcal{O}_b = (2r_1+1, 2r_1+1, 2r_2+1, 2r_2+1, \cdots, 2r_k+1, 2r_k+1) = \check{\mathcal{O}}_b^t.$$

This implies

$$\sigma_b = j_{W_{L_b}}^{W_b} \operatorname{sgn}, \quad \text{where } W_{L,b} = \prod_{i=1}^{k} S_{2r_i+1}.$$

(Note that $\mathcal{O}_b' = (2r_1+1, 2r_2+1, \cdots, 2r_k+1)$ *which corresponds to* $j_{W_{L_b}}^{S_b} \operatorname{sgn}$ *and* $\operatorname{ind}_L^G \mathcal{O}_b' = \mathcal{O}_b$. *)*

Now we deduce that

$$\begin{aligned} \sigma &:= j_{W_b \times W_g}^{W_n} \sigma_b \otimes \sigma_g \\ &= j_{W_{L_b} \times W_g}^{W_n} \operatorname{sgn} \otimes \sigma_g \end{aligned}$$

where $W_{L_b} \times W_g$ *is a parabolic subgroup of* W_n *corresponds to the Levi factor* L *of type*

$$A_{2r_1+1} \times A_{2r_2+1} \times \cdots \times A_{2r_k+1} \times W_g.$$

Therefore

$$\mathcal{O} = \operatorname{ind}_L^G \operatorname{triv} \times \mathcal{O}_g = \mathcal{O}_b \cup \mathcal{O}_g.$$

We claim that the map $\tilde{d} \colon \check{\mathcal{O}} \mapsto \mathcal{O}$ *defined here coincide with our Metaplectic BV duality paper.*

Note that $\tilde{d}_{BV}$ *is compatible with parabolic induction. Suppose* $\check{\mathfrak{l}}$ *is a Levi subgroup of* $\check{\mathfrak{g}}$ *and* $\check{\mathcal{O}}_{\check{\mathfrak{l}}} := \check{\mathcal{O}} \cap \check{\mathfrak{l}} \neq \emptyset$. *Then*

$$\tilde{d}_{BV}(\check{\mathcal{O}}) = \operatorname{ind}_{\mathfrak{l}}^{\mathfrak{g}} \tilde{d}_{BV}(\check{\mathcal{O}}_{\check{\mathfrak{l}}}).$$

(Since $\tilde{d}_{BV}$ *commute with the descent map, the claim follows from the prosperity of* d_{BV}*)*

The map $\tilde{d}$ also compatible with parabolic induction. It suffice to consider the case where $\check{\mathfrak{l}}$ is a maximal parabolic of type $A_l \times C_n$ and the orbit is trivial on the A_l factor. Suppose l is the bad parity, the claim is clear by our computation for the bad parity case.

Now suppose $l = 2m$ has good parity. If there is a i such that $\mathbf{r}_{2i+1}(\check{\mathcal{O}}) = \mathbf{r}_{2i+2}(\check{\mathcal{O}}) = 2m$. Then $\mathbf{c}_{2i+1}(\mathcal{O}) = \mathbf{c}_{2i+2}(\mathcal{O}) = 2m$ and the claim follows.

Otherwise, we can assume

$$R_{2i-1} := \mathbf{r}_{2i-1}(\check{\mathcal{O}}) > \mathbf{r}_{2i}(\check{\mathcal{O}}) = \mathbf{r}_{2i+1}(\check{\mathcal{O}}) > \mathbf{r}_{2i+2}(\check{\mathcal{O}}) =: R_{2i+2}.$$

and then

$$\mathcal{O} = (\cdots, R_{2i-1} - 1, 2m+1, 2m-1, R_{2i+2}, \cdots)$$

One check again that $\mathcal{O} = \mathrm{ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{l}}$. (Note that the induction operation is add two length $2m$ columns and then apply C-collapsing.)

Now it suffice to check that $\tilde{d}(\check{\mathcal{O}}) = \tilde{d}_{BV}(\check{\mathcal{O}})$ for every orbits $\check{\mathcal{O}}$ whose rows are multiplicity free) (i.e. $\check{\mathcal{O}}$ is distinguished). This is clear by the explicit formula for the both sides.]

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