SPECIAL UNIPOTENT REPRESENTATIONS : ORTHOGONAL AND SYMPLECTIC GROUPS

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ABSTRACT. Let G be a real classical group of type B, C, D (including the real metaplectic group). We consider a nilpotent adjoint orbit $\check{\mathcal{O}}$ of $\check{G}$, the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group). We classify all special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. When $\check{\mathcal{O}}$ is of good parity, we construct all such representations of G via the method of theta lifting. As a direct consequence of the construction and the classification, we conclude that all special unipotent representations of G are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture.

CONTENTS

1. INTRODUCTION AND THE MAIN RESULTS

1.1. **Unitary representations and the orbit method.** A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [43, 44]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [6, 43]. For more general Lie groups, work of Mackey and Duflo [28, 56] suggest that one should focus attention on reductive Lie groups. As expounded by Vogan in his writings (see for example [82, 84, 85]), the problem finally is to

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quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We highlight two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [4,5], Adams, Barbasch and Vogan established some important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [2]. The problem of classifying (integral) special unipotent representations for complex semisimple groups was solved earlier by Barbasch and Vogan [16] and the unitarity of these representations was established by Barbasch for complex classical groups [7, Section 10]. Shortly after, Barbasch outlined a proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [8]. The second major development is Vogan's theory of associated varieties [83] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a representation "attached" to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [39], the description of discrete spectrum by Adams [1] and Li [51], and the preservation of unitarity in stable range theta lifting by Li [50]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [12,17,34,36,37,68,71,74,80]. We would also like to mention the work of Przebinda [71] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [34] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [29,53,62,64], which among other things demonstrate the tight link between the orbit method and Howe's theory.

As mentioned earlier, there have been extensive investigations of unipotent representations for real reductive groups, by Vogan and his collaborators (see e.g. [2,82,83]). Nevertheless, the subject remains extraordinarily mysterious thus far.

In the present article we will demonstrate that Howe's (constructive) theory has immense implications for the orbit method, and in particular has near perfect synergy with special unipotent representations (in the case of classical groups). (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [12,34,59,80].) We will restrict our attention to a real classical group G of type B, C or D (which in our terminology includes a real metaplectic group), and we will classify all special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. Here $\check{\mathcal{O}}$ is a nilpotent adjoint orbit of $\check{G}$, the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group [13]). When $\check{\mathcal{O}}$ is of good parity [60], we will construct all special unipotent representations of G attached to $\check{\mathcal{O}}$ via the method of theta lifting. As a direct consequence of the construction and the classification, we conclude that all special unipotent representations of G are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([2, Introduction]).

1.2. **Special unipotent representations of classical groups of type B, C or D.** In this article, we aim to classify special unipotent representations of classical groups of type B, C or D. As the cases of complex orthogonal groups and complex symplectic groups are well-understood (see [16] and [12]), we will focus on the following groups:

$$\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p,q),\ \mathrm{O}^*(2n), \tag{1.1}$$

where $p,q,n \in \mathbb{N} := \{0,1,2,\cdots\}$. Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the real metaplectic group, namely the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n=0$.

Let G be one of the groups in (1.1). As usual, we view G as a real form of $G_{\mathbb{C}}$ (or a double cover of a real form of $G_{\mathbb{C}}$ in the metaplecitic case), where

$$G_{\mathbb{C}} := \begin{cases} \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q);\\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \text{ or } \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R});\\ \mathrm{Sp}_{2p+2q}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q);\\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ for the Lie algebras of G and $G_{\mathbb{C}}$, respectively, and view $\mathfrak{g}_{\mathbb{R}}$ as a real form of $\mathfrak{g}$.

Denote $r_{\mathfrak{g}}$ the rank of $\mathfrak{g}$. Let $W_{r_{\mathfrak{g}}}$ be the subgroup of $\mathrm{GL}_{r_{\mathfrak{g}}}(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. Then as usual, Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} = (\mathrm{S}(\mathbb{C}^{r_{\mathfrak{g}}}))^{W_{r_{\mathfrak{g}}}}. \tag{1.2}$$

Here and henceforth, "U" indicates the universal enveloping algebra of a Lie algebra, a superscript group indicate the space of invariant vectors under the group action, and "S" indicates the symmetric algebra. Unless $G_{\mathbb{C}}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. By (1.2), we have the following parameterization of characters of $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$:

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}, \mathbb{C}) = W_{r_{\mathfrak{g}}}\backslash(\mathbb{C}^{r_{\mathfrak{g}}})^* = W_{r_{\mathfrak{g}}}\backslash\mathbb{C}^{r_{\mathfrak{g}}}$$

Here "$\mathrm{Hom}_{\mathrm{alg}}$" indicates the set of $\mathbb{C}$-algebra homomorphisms, and a superscript "$*$" over a vector space indicates the dual space.

We define the Langlands dual of G to be the complex group

$$\check{G} := \begin{cases} \mathrm{Sp}_{p+q-1}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is odd};\\ \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is even};\\ \mathrm{O}_{2n+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R});\\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R});\\ \mathrm{O}_{2p+2q+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q);\\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$.

Remark. The authors have defined the notion of metaplectic dual for a real metaplectic group [13]. In this article, we have chosen to use the uniform terminology of Langlands dual (rather than metaplectic dual in the case of a real metaplectic group).

Denote by $\mathrm{Nil}(\check{\mathfrak{g}})$ the set of nilpotent $\check{G}$-orbits in $\check{\mathfrak{g}}$. When no confusion is possible, we will not distinguish a nilpotent orbit in $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ with its corresponding Young diagram. In particular, the zero orbit is represented by the Young diagram consisting of one nonempty column.

Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. It determines a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$ as in what follows. For every $a \in \mathbb{N}$, write

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd;} \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even;} \end{cases}$$

By convention, $\rho(1)$ and $\rho(0)$ are the empty sequence. Write $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ ($s \geqslant 0$) for the row lengths of $\check{\mathcal{O}}$. Define

$$\chi(\check{\mathcal{O}}) := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0), \tag{1.3}$$

to be viewed as a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$. Here the number of 0's is

$$\left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor.$$

Recall the following well-known result of Dixmier ([18, Section 3]): for every algebraic character χ of $\mathrm{Z}(\mathfrak{g})$, there exists a unique maximal ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of χ. As an easy consequence, there will be a unique maximal $G_{\mathbb{C}}$-stable ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$, for $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.

Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation ([20,87]) if its Harish-Chandra module has finite length. When $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a metaplectic group, write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then a representation of G is said to be genuine if ε_G acts via the scalar multiplication by -1. The notion of "genuine" will be used in similar situations without further explanation. Following Barbasch and Vogan ([2, 16]), we make the following definition.

Definition 1.1. *Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. An irreducible Casselman-Wallach representation π of G is attached to $\check{\mathcal{O}}$ if*

- *$I_{\check{\mathcal{O}}}$ annihilates π; and*
- *π is genuine if G is a metaplectic group.*

Write $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ for the set of isomorphism classes of irreducible Casselman-Wallach representations of G that are attached to $\check{\mathcal{O}}$. We say that an irreducible Casselman-Wallach representation of G is special unipotent if it is attached to $\check{\mathcal{O}}$, for some $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. As mentioned earlier, we will construct all special unipotent representations of G, and will show that all of them are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([2, Introduction]).

1.3. **Combinatorial construct: painted bipartitions.** We introduce a symbol $\star$, taking values in $\{B, C, D, \widetilde{C}, C^*, D^*\}$, to specify the type of the groups that we are considering as in (1.1), namely odd real orthogonal groups, real symplectic groups, even real orthogonal groups, real metaplectic groups, quaternionic symplectic groups and quaternionic orthogonal groups, respectively.

For a Young diagram $\imath$, write

$$\mathbf{r}_1(\imath) \geqslant \mathbf{r}_2(\imath) \geqslant \mathbf{r}_3(\imath) \geqslant \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty} \mathbf{r}_i(\imath)$ the total size of $\imath$.

For any Young diagram $\imath$, we introduce the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as the following subset of $\mathbb{N}^+ \times \mathbb{N}^+$ ($\mathbb{N}^+$ denotes the set of positive integers):

$$\mathrm{Box}(\imath) := \left\{ (i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \;\middle|\; j \leqslant \mathbf{r}_i(\imath) \right\}. \tag{1.4}$$

We also introduce five symbols $\bullet$, s, r, c and d, and make the following definition.

Definition 1.2. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *when $S = \{s\}$ or $\{r\}$, every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$;*
- *when $S = \{c\}$ or $\{d\}$, every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$.*

A painted Young diagram is then a pair $(\imath, \mathcal{P})$, consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$.

Example. Suppose that $\imath = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$, then there are $25 + 12 + 6 + 2 = 45$ paintings on $\imath$ in total as listed below.

$\begin{array}{|c|c|}\hline \bullet & \alpha \\ \hline \beta \\ \cline{1-1}\end{array}$ $\alpha, \beta \in \{\bullet, s, r, c, d\}$

$\begin{array}{|c|c|}\hline s & \alpha \\ \hline \beta \\ \cline{1-1}\end{array}$ $\alpha \in \{r, c, d\}, \beta \in \{s, r, c, d\}$

$\begin{array}{|c|c|}\hline r & \alpha \\ \hline \beta \\ \cline{1-1}\end{array}$ $\alpha \in \{c, d\}, \beta \in \{r, c, d\}$

$\begin{array}{|c|c|}\hline c & \alpha \\ \hline d \\ \cline{1-1}\end{array}$ $\alpha \in \{c, d\}$

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.3. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- *$\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$;*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, r, c, d\}, & \text{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \text{if } \alpha = D; \\ \{\bullet, s, c\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \text{if } \alpha = C^*; \\ \{\bullet, s\}, & \text{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, s\}, & \text{if } \alpha = C; \\ \{\bullet\}, & \text{if } \alpha = D; \\ \{\bullet, r, d\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \text{if } \alpha = C^*; \\ \{\bullet, r\}, & \text{if } \alpha = D^*. \end{cases}$$

For any painted bipartition τ as in Definition 1.3, we write

$$\imath_\tau := \imath, \ \mathcal{P}_\tau := \mathcal{P}, \ \jmath_\tau := \jmath, \ \mathcal{Q}_\tau := \mathcal{Q}, \ \alpha_\tau := \alpha,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

We further attach some objects to τ in what follows:

$$|\tau|, \ (p_\tau, \ q_\tau), \ G_\tau, \ \dim \tau, \ \varepsilon_\tau.$$

$|\tau|$: This is the natural number
$$|\tau| := |\imath| + |\jmath|.$$

(p_τ, q_τ): If $\star_\tau \in \{B, D, C^*\}$, this is a pair of natural numbers given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:
$$\begin{cases} p_\tau := \#\bullet + 2\#r + \#c + \#d + \#B^+; \\ q_\tau := \#\bullet + 2\#s + \#c + \#d + \#B^-. \end{cases}$$

If $\star_\tau \in \{C, \widetilde{C}, D^*\}$, we let $p_\tau := q_\tau := |\tau|$.

G_τ: This is a classical group given by
$$G_\tau := \begin{cases} \mathrm{O}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

$\dim \tau$: This is the dimension of the standard representation of the complexification of G_τ, or equivalently,
$$\dim \tau := \begin{cases} 2\,|\tau| + 1, & \text{if } \star_\tau = B; \\ 2\,|\tau|\,, & \text{otherwise.} \end{cases}$$

ε_τ: This is the element in $\mathbb{Z}/2\mathbb{Z}$ such that
$$\varepsilon_\tau = 0 \Leftrightarrow \text{the symbol } d \text{ occurs in the first column of } (\imath, \mathcal{P}) \text{ or } (\jmath, \mathcal{Q}).$$

The triple $\mathsf{s}_\tau = (\star_\tau, p_\tau, q_\tau) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ will also be referred to as the classical signature attached to τ.

Example. Suppose that
$$\tau = \begin{array}{|c|c|} \hline \bullet & c \\ \hline \bullet \\ \cline{1-1} c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & s & r \\ \hline \bullet \\ \cline{1-1} r \\ \cline{1-1} r \\ \cline{1-1} \end{array} \times B^+.$$

Then
$$\begin{cases} |\tau| = 10; \\ p_\tau = 4 + 6 + 2 + 0 + 1 = 13; \\ q_\tau = 4 + 2 + 2 + 0 + 0 = 8; \\ G_\tau = \mathrm{O}(13, 8); \\ \dim \tau = 21; \\ \varepsilon_\tau = 1. \end{cases}$$

1.4. **Counting special unipotent representations by painted bipartitions.** We fix a classical group G which has type $\star$. Following [60, Definition 4.1], we say that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity if
$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \check{G} \text{ is a complex symplectic group; and} \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \check{G} \text{ is a complex orthogonal group.} \end{cases}$$

In general, the study of the special unipotent representations attached to $\check{\mathcal{O}}$ will be reduced to the case when $\check{\mathcal{O}}$ has $\star$-good parity. We refer the reader to [14].

For the rest of this subsection, assume that $\check{\mathcal{O}}$ has $\star$-good parity. Equivalently we consider $\check{\mathcal{O}}$ as a Young diagram that has $\star$-good parity in the following sense:

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \star \in \{B, \widetilde{C}\}; \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \star \in \{C, D, C^*, D^*\}; \text{ and} \\ \text{the total size } |\check{\mathcal{O}}| \text{ is odd if and only if } \star \in \{C, C^*\}. \end{cases}$$

Definition 1.4. *A $\star$-pair is a pair $(i, i+1)$ of consecutive positive integers such that*

$$\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be

- *vacant in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*
- *tailed in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and even.*

Denote $\mathrm{PP}_\star(\check{\mathcal{O}})$ the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$.

For any $\check{\mathcal{O}}$, we attach a pair of Young diagrams

$$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) := (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})),$$

as follows.

The case when $\star = B$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}).$$

The case when $\star = \widetilde{C}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}).$$

The case when $\star \in \{C, C^*\}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

The case when $\star \in \{D, D^*\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0, \end{cases}$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

Define

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) := \left\{ \tau \text{ is a painted bipartition} \mid \star_\tau = \star \text{ and } (\imath_\tau, \jmath_\tau) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})) \right\}.$$

We also define the following extended parameter set:

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) := \begin{cases} \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})\}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\varnothing\}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

We use $\boldsymbol{\tau} = (\tau, \wp)$ to denote an element in $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$.

Put

$$\mathrm{Unip}_\star(\check{\mathcal{O}}) := \begin{cases} \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|+1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = B; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{|\check{\mathcal{O}}|-1}(\mathbb{R})), & \text{if } \star = C; \\ \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = D; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{Sp}}_{|\check{\mathcal{O}}|}(\mathbb{R})), & \text{if } \star = \widetilde{C}; \\ \bigsqcup_{p,q\in\mathbb{N}, 2p+2q=|\check{\mathcal{O}}|-1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q)), & \text{if } \star = C^*; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}^*(|\check{\mathcal{O}}|)), & \text{if } \star = D^*. \end{cases}$$

The main result of [14], on counting of special unipotent representations, is as follows.

Theorem 1.5. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then*

$$\#(\mathrm{Unip}_\star \check{\mathcal{O}}) = \begin{cases} 2\#(\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})), & \text{if } \star \in \{B, D\} \text{ and } (\star, \check{\mathcal{O}}) \neq (D, \varnothing); \\ \#(\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})), & \text{otherwise}. \end{cases}$$

1.5. **Descending painted bipartitions and constructing representations by induction.** Let $\star$ and $\check{\mathcal{O}}$ be as before. For every $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, in what follows we will construct a representation $\pi_{\boldsymbol{\tau}}$ of G_τ by the method of theta lifting. For the initial case when $\check{\mathcal{O}}$ is the empty Young diagram, define

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \text{the one dimensional genuine representation,} & \text{if } \star = \widetilde{C} \text{ so that } G_\tau = \widetilde{\mathrm{Sp}}_0(\mathbb{R}); \\ \text{the one dimensional trivial representation,} & \text{if } \star \neq \widetilde{C}. \end{cases}$$

Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or } \ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or } \ D^*.$$

We call $\star'$ the Howe dual of $\star$. Assume now that $\check{\mathcal{O}}$ is nonempty, and define its dual descent to be

$$\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}}) := \text{the Young diagram obtained from } \check{\mathcal{O}} \text{ by removing the first row.}$$

Note that $\check{\mathcal{O}}'$ has $\star'$-good parity, and

$$\mathrm{PP}_{\star'}(\check{\mathcal{O}}') = \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \mathrm{PP}_\star(\check{\mathcal{O}})\}.$$

Define the dual descent of $\wp$ to be

$$\wp' := \check{\nabla}(\wp) := \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \wp\} \subset \mathrm{PP}_{\star'}(\check{\mathcal{O}}'). \tag{1.5}$$

In Section 2, we will define the descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

We then define the descent of $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ to be the element

$$\boldsymbol{\tau}' := (\tau', \wp') := \nabla(\boldsymbol{\tau}) := (\nabla(\tau), \check{\nabla}(\wp)) \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

Let $(W_{\tau,\tau'}, \langle \cdot, \cdot \rangle_{\tau,\tau'})$ be a real symplectic space of dimension $\dim \tau \cdot \dim \tau'$. As usual, there are continuous homomorphisms $G_\tau \to \mathrm{Sp}(W_{\tau,\tau'})$ and $G_{\tau'} \to \mathrm{Sp}(W_{\tau,\tau'})$ whose images form a reductive dual pair in $\mathrm{Sp}(W_{\tau,\tau'})$. We form the semidirect product

$$J_{\tau,\tau'} := (G_\tau \times G_{\tau'}) \ltimes \mathrm{H}(W_{\tau,\tau'}),$$

where

$$\mathrm{H}(W_{\tau,\tau'}) := W_{\tau,\tau'} \times \mathbb{R}$$

is the Heisenberg group with group multiplication

$$(w,t)(w',t') := (w+w', t+t' + \langle w, w' \rangle_{\tau,\tau'}), \quad w, w' \in W_{\tau,\tau'}, \; t, t' \in \mathbb{R}.$$

Let $\omega_{\tau,\tau'}$ be a suitably normalized smooth oscillator representation of $J_{\tau,\tau'}$ such that every $t \in \mathbb{R} \subset J_{\tau,\tau'}$ acts on it through the scalar multiplication by $e^{2\pi\sqrt{-1}\,t}$ (the letter π often denotes a representation, but here it stands for the circumference ratio). See Section 4.1 for details.

For any Casselman-Wallach representation π' of $G_{\tau'}$, write

$$\check{\Theta}^{\tau}_{\tau'}(\pi') := (\omega_{\tau,\tau'} \widehat{\otimes} \pi')_{G_{\tau'}} \qquad \text{(the Hausdorff coinvariant space)},$$

where $\widehat{\otimes}$ indicates the complete projective tensor product. This representation is clearly Fréchet, smooth, and of moderate growth, and so a Casselman-Wallach representation by the fundamental result of Howe [40].

Now we define the representation $\pi_{\boldsymbol{\tau}}$ of G_τ by induction on the number of nonempty rows of $\check{\mathcal{O}}$:

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'}) \otimes (1^{+,-}_{p_\tau,q_\tau})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'} \otimes \det^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\Theta}^{\tau}_{\tau'}(\pi_{\boldsymbol{\tau}'}), & \text{if } \star = C^* \text{ or } D^*. \end{cases} \tag{1.6}$$

Here $1^{+,-}_{p_\tau,q_\tau}$ denotes the character of $\mathrm{O}(p_\tau, q_\tau)$ whose restriction to $\mathrm{O}(p_\tau) \times \mathrm{O}(q_\tau)$ equals $1 \otimes \det$ (1 stands for the trivial character), and $\varepsilon_\wp$ denote the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\wp := 1 \Leftrightarrow (1,2) \in \wp.$$

In the sequel, we will use $\varnothing$ to denote the empty set (in the usual way), as well as the empty Young diagram or the painted Young diagram whose underlying Young diagram is empty. As always, we let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$.

We are now ready to state our first main theorem.

Theorem 1.6. *Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For every $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, the representation $\pi_{\boldsymbol{\tau}}$ of G_τ in (1.6) *is irreducible and attached to $\check{\mathcal{O}}$.*
(b) If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ (\boldsymbol{\tau}, \epsilon) & \mapsto & \pi_{\boldsymbol{\tau}} \otimes \det^{\epsilon} \end{array}$$

is bijective.
(c) In all other cases, the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ \boldsymbol{\tau} & \mapsto & \pi_{\boldsymbol{\tau}} \end{array}$$

is bijective.

By the above theorem, we have explicitly constructed all special unipotent representations in $\mathrm{Unip}_\star(\check{\mathcal{O}})$, when $\check{\mathcal{O}}$ has $\star$-good parity. The method of matrix coefficient integrals (Section 4) will imply the following

Corollary 1.7. *Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then all special unipotent representations in $\mathrm{Unip}_\star(\check{\mathcal{O}})$ are unitarizable.*

The reduction of [14] allows us to classify all special unipotent presentations attached to a general $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ from those which have $\star$-good parity. We thus conclude

Theorem 1.8. *All special unipotent representations of the classical groups in* (1.1) *are unitarizable.*

1.6. **Computing associated cycles of the constructed representations.** Let $G, G_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}, \check{\mathfrak{g}}$ and $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ be as in Section 1.2. Fix a maximal compact subgroup K of G whose Lie algebra is denoted by $\mathfrak{k}_{\mathbb{R}}$. We equipping $\mathfrak{g}$ with the trace form. Then we have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the complexification of $\mathfrak{k}_{\mathbb{R}}$. By taking the dual spaces, we also have a decomposition

$$\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*,$$

Write $K_{\mathbb{C}}$ for the complexification of the compact group K. It is a complex algebraic group with an obvious algebraic action on $\mathfrak{p}^*$.

For any $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$, write $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}^*)$ for the Barbarsch-Vogan dual of $\check{\mathcal{O}}$ so that its Zariski closure in $\mathfrak{g}^*$ equals the associated variety of the ideal $I_{\check{\mathcal{O}}} \subset \mathrm{U}(\mathfrak{g})$. See [13, 16]. The algebraic variety $\mathcal{O} \cap \mathfrak{p}^*$ is a finite union of $K_{\mathbb{C}}$-orbits. Given any such orbit $\mathscr{O}$, write $\mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O})$ for the Grothendieck group of the category of $K_{\mathbb{C}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O}).$$

We say that a Casselman-Wallach representation of G is $\mathcal{O}$-bounded if the associated variety of its annihilator ideal is contained in the Zariski closure of $\mathcal{O}$. Note that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are $\mathcal{O}$-bounded. Write $\mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}}$ for the Grothendieck group of the category of all such representations. From [83, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{AC}_{\mathcal{O}} \colon \mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}} \longrightarrow \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}).$$

We call $\mathrm{AC}_{\mathcal{O}}(\pi)$ the associated cycle of π, where π is an $\mathcal{O}$-bounded Casselman-Wallach representation of G. This is a fundamental invariant attached to π.

Following Vogan [83, Section 8], we make the following definition.

Definition 1.9. *Let $\mathscr{O}$ be a $K_{\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}^*$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $K_{\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_X$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_X$;*
- *if $\star = \widetilde{C}$, then ε_G acts on $\mathcal{E}$ by the scalar multiplication by -1.*

Here $X \in \mathscr{O}$, $\mathcal{E}_X$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_X$ denotes the Lie algebra of the stabilizer of X in $K_{\mathbb{C}}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_X$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$.

Note that in the situation of the classical groups we consider in this article, all admissible orbit data are line bundles. Denote by $\mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$. Put

$$\mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O}) \subset \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}). \tag{1.7}$$

For the rest of this subsection, we suppose that G has type $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and $\check{\mathcal{O}}$ has $\star$-good parity. Recall that a nilpotent orbit in $\check{\mathfrak{g}}$ is said to be distinguished if it is has no nontrivial intersection with any proper Levi subalgebra of $\check{\mathfrak{g}}$. Combinatorially, this is equivalent to saying that no pair of rows of the Young diagram have equal nonzero length. Note that all distinguished nilpotent orbits in $\check{\mathfrak{g}}$ has $\star$-good parity.

Definition 1.10. *(a) The orbit $\check{\mathcal{O}}$ (which has $\star$-good parity) is said to be quasi-distinguished if there is no $\star$-pair that is balanced in $\check{\mathcal{O}}$.*
(b) If $\star \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be weakly-distinguished if there is no positive even integer i such that $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = \mathbf{r}_{i+2}(\check{\mathcal{O}}) = \mathbf{r}_{i+3}(\check{\mathcal{O}}) > 0$.
(c) If $\star \in \{C, \widetilde{C}, C^*\}$ so that its Howe dual $\star' \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be weakly-distinguished if either it is the empty Young diagram or it is nonempty and its dual descent $\check{\mathcal{O}}'$ (which is a Young diagram that has $\star'$-good parity) is weakly-distinguished.

We will compute the associated cycle of π_τ for every painted bipartition τ associated to $\check{\mathcal{O}}$ which has $\star$-good parity. The computation will be a key ingredient in the proof of Theorem 1.6. Our second main theorem is the following result concerning associated cycles of the special unipotent representations.

Theorem 1.11. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For any $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the associated cycle $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$ is a nonzero sum of pairwise distinct elements of $\mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O})$.
(b) If $\check{\mathcal{O}}$ is weakly-distinguished, then the map

$$\mathrm{AC}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$$

is injective.
(c) If $\check{\mathcal{O}}$ is quasi-distinguished, then the map $\mathrm{AC}_{\mathcal{O}}$ induces a bijection

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}).$$

Remark. Suppose that $\star \in \{C^*, D^*\}$ so that G is quaternionic. Then there is precisely one admissible orbit datum over $\mathscr{O}$ for each $K_{\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}^*$. Thus

$$\mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O}) = K_{\mathbb{C}} \backslash (\mathcal{O} \cap \mathfrak{p}^*).$$

If $\check{\mathcal{O}}$ is not quasi-distinghuished, then $\mathcal{O} \cap \mathfrak{p}^*$ is empty (see [23, Theorems 9.3.4 and 9.3.5]), and hence $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is also empty.

2. Descents of painted bipartitions: combinatorics

In this section, we define the descent of a painted bipartition, as alluded to in Section 1.5. As before, let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.

For a diagram $\imath$, its naive descent, which is denoted by $\nabla_{\mathrm{naive}}(\imath)$, is defined to be the Young diagram obtained from $\imath$ by removing the first column. By convention, $\nabla_{\mathrm{naive}}(\emptyset) = \emptyset$.

In the rest of this section, we assume that $\check{\mathcal{O}} \neq \emptyset$. Recall that we have its dual descent $\check{\mathcal{O}}'$, and $\check{\mathcal{O}}'$ has $\star'$-good parity, where $\star'$ is the Howe dual of $\star$.

2.1. Naive descent of a painted bipartition.

In this subsection, let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ be a painted bipartition such that $\star_\tau = \star$. Put

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ does not occur in the first column of } (\imath, \mathcal{P}); \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ occurs in the first column of } (\imath, \mathcal{P}); \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{2.1}$$

Lemma 2.1. *If $\star \in \{B, C, C^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\imath, \nabla_{\mathrm{naive}}(\jmath))$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,
$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}(i,j), & \text{if } \mathcal{P}(i,j) \notin \{\bullet, s\}; \end{cases}$$
- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,
$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j+1), & \text{if } \mathcal{Q}(i,j+1) \notin \{\bullet, s\}. \end{cases}$$

Proof. First assume that the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$. Then the image of $\mathcal{P}$ is in fact contained in $\{\bullet\}$, and $(\imath, \jmath)$ is right interlaced in the sense that
$$\mathbf{c}_1(\jmath) \geqslant \mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\jmath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\jmath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots.$$
Hence $(\imath', \jmath') := (\imath, \nabla(\jmath))$ is left interlaced in the sense that
$$\mathbf{c}_1(\imath') \geqslant \mathbf{c}_1(\jmath') \geqslant \mathbf{c}_2(\imath') \geqslant \mathbf{c}_2(\jmath') \geqslant \mathbf{c}_3(\imath') \geqslant \mathbf{c}_3(\jmath') \geqslant \cdots.$$
Then it is clear that there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ such that images of $\mathcal{P}'$ and $\mathcal{Q}'$ are both contained in $\{\bullet, s\}$. This proves the lemma in the special case when the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$.

The proof of the lemma in the general case is easily reduced to this special case. □

Lemma 2.2. *If $\star \in \{\widetilde{C}, D, D^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\nabla_{\mathrm{naive}}(\imath), \jmath)$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,
$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{P}(i,j+1), & \text{if } \mathcal{P}(i,j+1) \notin \{\bullet, s\}; \end{cases}$$
- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,
$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j), & \text{if } \mathcal{Q}(i,j) \notin \{\bullet, s\}. \end{cases}$$

Proof. The proof is similar to that of Lemma 2.1. □

Definition 2.3. *In the notation of Lemma 2.1 and 2.2, we call τ' the naive descent of τ, to be denoted by $\nabla_{\mathrm{naive}}(\tau)$.*

Example. If
$$\tau = \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & \bullet & c \\ \hline \bullet & s & c & \\ \cline{1-3} s & & & \\ \cline{1-1} c & & & \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & r & d \\ \hline d & d & \\ \cline{1-2} \end{array} \times \widetilde{C},$$
then
$$\nabla_{\mathrm{naive}}(\tau) = \begin{array}{|c|c|c|}\hline \bullet & \bullet & c \\ \hline \bullet & c & \\ \cline{1-2} \end{array} \times \begin{array}{|c|c|c|}\hline \bullet & \bullet & s \\ \hline \bullet & r & d \\ \hline d & d & \\ \cline{1-2} \end{array} \times B^-.$$

2.2. **Descents of painted bipartitions.** Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ and write

$$\tau'_{\mathrm{naive}} = (\imath', \mathcal{P}'_{\mathrm{naive}}) \times (\jmath', \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'$$

for the naive descent of τ. This is clearly an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$.

The following two lemmas are easily verified and we omit the proofs. We will give an example for each case.

Lemma 2.4. *Suppose that*

$$\begin{cases} \alpha = B^+; \\ \mathbf{r}_2(\check{\mathcal{O}}) > 0; \\ \mathcal{Q}(\mathbf{c}_1(\jmath), 1) \in \{ r, d \}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} s, & \textit{if } (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \textit{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline \bullet & r \\ \hline r & d \\ \hline \end{array} \times B^+,$$

then

$$\tau'_{\mathrm{naive}} = \begin{array}{|c|c|}\hline s & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C} \qquad \text{and} \qquad \tau' = \begin{array}{|c|c|}\hline s & c \\ \hline s & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C}.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $4, 4, 4, 2$.

Lemma 2.5. *Suppose that*

$$\begin{cases} \alpha = D; \\ \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0; \\ \big(\mathcal{P}(\mathbf{c}_2(\imath), 1), \mathcal{P}(\mathbf{c}_2(\imath), 2)\big) = (r, c); \\ \mathcal{P}(\mathbf{c}_1(\imath), 1) \in \{ r, d \}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} r, & \textit{if } \ (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \textit{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & s \\ \hline \bullet & s \\ \hline r & c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times D,$$

then

$$\tau'_{\mathrm{naive}} = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet \\ \cline{1-1} \bullet \\ \cline{1-1} \end{array} \times C, \qquad \text{and} \qquad \tau' = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline r \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet \\ \cline{1-1} \bullet \\ \cline{1-1} \end{array} \times C.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $7, 7, 7, 3$.

Definition 2.6. *We define the descent of τ to be*

$$\nabla(\tau) := \begin{cases} \tau', & \textit{if either of the condition of Lemma 2.4 or 2.5 holds};\\ \nabla_{\mathrm{naive}}(\tau), & \textit{otherwise}, \end{cases}$$

which is an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$. *Here τ' is as in Lemmas 2.4 and 2.5.*

In conclusion, we have by now a well-defined descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

The following injectivity result will be important for us.

Proposition 2.7. *If $\star \in \{B, D, C^*\}$, then the map*

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \to & \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array} \tag{2.2}$$

is injective. If $\star \in \{C, \widetilde{C}, D^*\}$, then the map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \quad \to \quad \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \tag{2.3}$$

is injective.

3. From painted bipartitions to associated cycles

3.1. **Classical spaces.** Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ as before. Put

$$(\epsilon, \dot\epsilon) := (\epsilon_\star, \dot\epsilon_\star) := \begin{cases} (1,1), & \text{if } \star \in \{B, D\};\\ (-1,-1), & \text{if } \star \in \{C, \widetilde{C}\};\\ (-1,1), & \text{if } \star = C^*;\\ (1,-1), & \text{if } \star = D^*. \end{cases}$$

A classical signature is defined to be a triple $\mathsf{s} = (\star, p, q) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ such that

$$\begin{cases} p+q \text{ is odd }, & \text{if } \star = B;\\ p+q \text{ is even }, & \text{if } \star = D;\\ p = q, & \text{if } \star \in \{C, \widetilde{C}, D^*\};\\ \text{both } p \text{ and } q \text{ are even}, & \text{if } \star = C^*. \end{cases}$$

Suppose that $\mathsf{s} = (\star, p, q)$ is a classical signature in the rest of this section.

We omit the proof of the following lemma (*cf.* [65, Section 1.3]).

Lemma 3.1. *Then there is quadruple $(V, \langle\, ,\, \rangle, J, L)$ satisfying the following conditions:*

- *V is a complex vector space of dimension $p+q$;*
- *$\langle\, ,\, \rangle$ is an ϵ-symmetric non-degenerate bilinear form on V;*
- *$J : V \to V$ is a conjugate linear automorphism of V such that $J^2 = \epsilon \cdot \dot\epsilon$;*
- *$L : V \to V$ is a linear automorphism of V such that $L^2 = \dot\epsilon$;*
- $\langle Ju, Jv\rangle = \overline{\langle u, v\rangle}$, *for all $u, v \in V$;*

- *$\langle Lu, Lv\rangle = \langle u, v\rangle$, for all $u, v \in V$;*
- *$LJ = JL$;*
- *the Hermitian form $(u, v) \mapsto \langle Lu, Jv\rangle$ on V is positive definite;*
- *if $\dot\epsilon = 1$, then $\dim\{v \in V \mid Lv = v\} = p$ and $\dim\{v \in V \mid Lv = -v\} = q$.*

Moreover, such a quadruple is unique in the following sense: if $(V', \langle\,,\,\rangle', J', L')$ is another quadruple satisfying the analogous conditions, then there is a linear isomorphism $\phi : V \to V'$ that respectively transforms $\langle\,,\,\rangle$, J and L to $\langle\,,\,\rangle'$, J' and L'.

In the notation of Lemma 3.1, we call $(V, \langle\,,\,\rangle, J, L)$ a classical space of signature s, and denote it by $(V_{\mathsf{s}}, \langle\,,\,\rangle_{\mathsf{s}}, J_{\mathsf{s}}, L_{\mathsf{s}})$.

Denote by $G_{\mathsf{s},\mathbb{C}}$ the isometry group of $(V_{\mathsf{s}}, \langle\,,\,\rangle_{\mathsf{s}})$, which is an complex orthogonal group if $\epsilon = 1$ and a complex symplectic group if $\epsilon = -1$. Respectively denote by $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ and $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ the centralizes of J_{s} and L_{s} in $G_{\mathsf{s},\mathbb{C}}$. Then $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ is a real classical group isomorphic with

$$\begin{cases} \mathrm{O}(p,q), & \text{if } \star = B \text{ or } D; \\ \mathrm{Sp}_{2p}(\mathbb{R}), & \text{if } \star \in \{C, \widetilde{C}\}; \\ \mathrm{Sp}(\frac{p}{2}, \frac{q}{2}), & \text{if } \star = C^*; \\ \mathrm{O}^*(2p), & \text{if } \star = D^*. \end{cases}$$

Put

$$G_{\mathsf{s}} := \begin{cases} \text{the metaplectic double cover of } G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{if } \star = \widetilde{C}; \\ G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{otherwise;}\,. \end{cases}$$

Denote by K_{s} the inverse image of $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ under the natural homomorphism $G_{\mathsf{s}} \to G_{\mathsf{s},\mathbb{C}}$, which is a maximal compact subgroup of G_{s}. Write $K_{\mathsf{s},\mathbb{C}}$ for the complexification of K_{s}, which is a reductive complex linear algebraic group.

For every $\lambda \in \mathbb{C}$, write $V_{\mathsf{s},\lambda}$ for the eigenspace of L_{s} with eigenvalue λ. Write

$$\det_\lambda : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s},\lambda}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times. \tag{3.1}$$

If $\star = \widetilde{C}$, then the natural homomorphism $K_{\mathsf{s},\mathbb{C}} \to \mathrm{GL}(V_{\mathsf{s},\sqrt{-1}})$ is a double cover, and there is a unique genuine algebraic character $\det_{\sqrt{-1}}^{\frac{1}{2}} : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$ whose square equals $\det_{\sqrt{-1}}$.

Write

$$\det : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s}}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times. \tag{3.2}$$

This is the trivial character unless $\star = B$ or D.

Denote by $\mathfrak{g}_{\mathsf{s}}$ the Lie algebra of $G_{\mathsf{s},\mathbb{C}}$. Then we have a decomposition

$$\mathfrak{g}_{\mathsf{s}} = \mathfrak{k}_{\mathsf{s}} \oplus \mathfrak{p}_{\mathsf{s}},$$

where $\mathfrak{k}_{\mathsf{s}}$ is the Lie algebra of $K_{\mathsf{s},\mathbb{C}}$, and $\mathfrak{p}_{\mathsf{s}}$ is the orthogonal complement of $\mathfrak{k}_{\mathsf{s}}$ in $\mathfrak{g}_{\mathsf{s}}$ under the trace form. Using the trace form, we also identify $\mathfrak{g}_{\mathsf{s}}^*$ with $\mathfrak{g}_{\mathsf{s}}$. Denote by $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ the set of nilpotent $G_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathsf{s}}$, and by $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$ the set of nilpotent $K_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathsf{s}}$.

Given a $K_{\mathsf{s},\mathbb{C}}$-orbit $\mathscr{O}$ in $\mathfrak{p}_{\mathsf{s}}$, write $\mathcal{K}_{\mathsf{s}}(\mathscr{O})$ for the Grothendieck group of the categogy of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundles over $\mathscr{O}$. Denote by $\mathcal{K}_{\mathsf{s}}^+(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}(\mathscr{O})$ the submonoid generated by all these equivariant algebraic vector bundles. The notion of admissible orbit datum over $\mathscr{O}$ is defined as in Definition 1.9. Write

$$\mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}^+(\mathscr{O})$$

for the set of isomorphism classes of admissible orbit data over $\mathscr{O}$.

Let $\mathcal{O}$ be a $G_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s},\mathbb{C}}$. It is well-known that $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ has only finitely many $K_{\mathsf{s},\mathbb{C}}$-orbits. Put

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}_{\mathsf{s}}(\mathscr{O}),$$

and

$$\mathcal{K}^+_{\mathsf{s}}(\mathcal{O}) := \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}^+_{\mathsf{s}}(\mathscr{O}).$$

Put

$$\mathrm{AOD}_{\mathsf{s}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}^+_{\mathsf{s}}(\mathcal{O}).$$

Define a partial order $\preceq$ on $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$ such that

$$\mathcal{E}_1 \preceq \mathcal{E}_2 \Leftrightarrow \mathcal{E}_2 - \mathcal{E}_1 \in \mathcal{K}^+_{\mathsf{s}}(\mathcal{O}) \qquad (\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})).$$

For every algebraic character χ of $K_{\mathsf{s},\mathbb{C}}$, the twisting map

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \qquad \mathcal{E} \mapsto \mathcal{E} \otimes \chi$$

is obviously defined.

3.2. **The moment maps.** Recall that $\star'$ is the Howe dual of $\star$. Suppose that $\mathsf{s}' = (\star', p', q')$ is another classical signature. Put

$$W_{\mathsf{s},\mathsf{s}'} := \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_{\mathsf{s}'}).$$

Then we have the adjoint map

$$W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s}',\mathsf{s}}, \qquad \phi \mapsto \phi^*$$

that is specified by requiring

$$\langle \phi v, v' \rangle_{\mathsf{s}'} = \langle v, \phi^* v' \rangle_{\mathsf{s}}, \qquad \text{for all } v \in V_{\mathsf{s}},\ v' \in V_{\mathsf{s}'},\ \phi \in W_{\mathsf{s},\mathsf{s}'}.$$

Define three maps

$$\langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \times W_{\mathsf{s},\mathsf{s}'} \to \mathbb{C}, \quad (\phi_1, \phi_2) \mapsto \mathrm{tr}(\phi_1^* \phi_2),$$
$$J_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto J_{\mathsf{s}'} \circ \phi \circ J_{\mathsf{s}}^{-1};$$

and

$$L_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto \dot{\epsilon} L_{\mathsf{s}'} \circ \phi \circ L_{\mathsf{s}}^{-1}.$$

It is routine to check that

$$(W_{\mathsf{s},\mathsf{s}'}, \langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'}, J_{\mathsf{s},\mathsf{s}'}, L_{\mathsf{s},\mathsf{s}'})$$

is a classical space of signature $(C, \frac{(p+q)(p'+q')}{2}, \frac{(p+q)(p'+q')}{2})$.

Write

$$W_{\mathsf{s},\mathsf{s}'} = \mathcal{X}_{\mathsf{s},\mathsf{s}'} \oplus \mathcal{Y}_{\mathsf{s},\mathsf{s}'}, \tag{3.3}$$

where $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ and $\mathcal{Y}_{\mathsf{s},\mathsf{s}'}$ are the eigenspaces of $L_{\mathsf{s},\mathsf{s}'}$ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Then we have the following two well-defined algebraic maps:

$$\begin{array}{ccccc} \mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array} \tag{3.4}$$

These two maps M_{s} and $M_{\mathsf{s}'}$ are called the moment maps. They are both $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-equivariant. Here $K_{\mathsf{s}',\mathbb{C}}$ acts trivially on $\mathfrak{p}_{\mathsf{s}}$, $K_{\mathsf{s},\mathbb{C}}$ acts trivially on $\mathfrak{p}'_{\mathsf{s}}$, and all the other actions are the obvious ones.

Put

$$W^{\circ}_{\mathsf{s},\mathsf{s}'} := \{\phi \in W_{\mathsf{s},\mathsf{s}'} \mid \text{the image of } \phi \text{ is non-degenerate with respect to } \langle\, ,\, \rangle_{\mathsf{s}'}\}$$

and

$$\mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W^{\circ}_{\mathsf{s},\mathsf{s}'}.$$

Lemma 3.2 (*cf.* [65, Lemma 13], [91, Lemma 3.4])**.** *Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}. Then the set*

$$M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} \tag{3.5}$$

is a single $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-orbit. Moreover, for any element ϕ in $M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'}$, there is an exact sequence of algebraic groups:

$$1 \to K_{\mathsf{s}_1,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \xrightarrow{\textit{the projection to the first factor}} (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1,$$

where $\mathbf{e} := M_{\mathsf{s}}(\phi) \in \mathscr{O}$, s_1 is a certain classical signature of form $(\star', p_1, q_1)$ $(p_1, q_1 \in \mathbb{N})$, $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$ is the stabilizer of ϕ in $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$, and $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$ is the stabilizer of $\mathbf{e}$ in $K_{\mathsf{s},\mathbb{C}}$.

In the notation of Lemma 3.2, write

$$\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) := \text{the image of the set (3.5) under the moment map } M_{\mathsf{s}'},$$

which is a $K_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}'}$. This is called the descent of $\mathscr{O}$. It is an element of $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}'})$ if $\mathscr{O} \in \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$.

3.3. **Geometric theta lift.** Let $\zeta_{\mathsf{s},\mathsf{s}'}$ denote the algebraic character on $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ such that

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s},\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{B, D, C^*\}; \\ (\det_{\sqrt{-1}})^{\frac{q'-p'}{2}}, & \text{if } \star \in \{C, D^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{q'-p'}, & \text{if } \star = \widetilde{C}, \end{cases}$$

and

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s}',\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{C, \widetilde{C}, D^*\}; \\ (\det_{\sqrt{-1}})^{\frac{p-q}{2}}, & \text{if } \star \in \{D, C^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{p-q}, & \text{if } \star = B. \end{cases}$$

Here and henceforth, when no confusion is possible, we use 1 to indicate the trivial representation of a group (we also use 1 to denote the identity element of a group). Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$ as before. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}, and write $\mathscr{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O})$. Let $\phi, \mathbf{e}$ be as in Lemma 3.2 and let $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. We have an exact sequence

$$1 \to K_{\mathsf{s}_1,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \to (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1$$

as in Lemma 3.2.

Let $\mathcal{E}'$ be a $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}'$. Its fibre $\mathcal{E}'_{\mathbf{e}}$ at $\mathbf{e}'$ is an algebraic representation of the stabilizer group $(K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}$. We also view it as a representation of the group $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$ by the pull-back through the homomorphism

$$(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \xrightarrow{\text{the projection to the second factor}} (K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}.$$

Then $\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'}$ is a representation of $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$, and the coinvariant space

$$(\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}_1,\mathbb{C}}}$$

is an algebraic representation of $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$. Write $\mathcal{E} := \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$ for the $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$ whose fibre at $\mathbf{e}$ equals this coinvariant space representation. In this way, we get an exact functor $\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}$ from the category of $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}'$ to the category of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$. This exact functor induces a homomorphism of the Grothendieck groups:

$$\check{\vartheta}_{\mathscr{O}'}^{\mathscr{O}} : \mathcal{K}_{\mathsf{s}'}(\mathscr{O}') \to \mathcal{K}_{\mathsf{s}}(\mathscr{O}).$$

The above homomorphism is independent of the choice of ϕ.

Similar to (3.4), we have the following two well-defined algebraic maps:

$$\begin{array}{ccccc} \mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array} \tag{3.6}$$

These two maps are also called the moment maps. Similar to the maps in (3.4), they are both $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-equivariant.

Now we suppose that $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Similar to the first assertion of Lemma 3.2, the set

$$\tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap W_{\mathsf{s},\mathsf{s}'}^{\circ} \tag{3.7}$$

is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit. Write

$$\mathcal{O}' := \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O}) := \text{the image of the set (3.7) under the moment map } \tilde{M}_{\mathsf{s}'},$$

which is a $G_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s}'}$. This is called the descent of $\mathcal{O}$. It is an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ if $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$.

Finally, we define the geometric theta lift to be the homomorphism

$$\check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}} : \mathcal{K}_{\mathsf{s}'}(\mathcal{O}') \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

such that

$$\check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathcal{E}') = \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}},\ \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathscr{O}) = \mathscr{O}'} \check{\vartheta}_{\mathscr{O}'}^{\mathscr{O}}(\mathcal{E}'),$$

for all $K_{\mathsf{s}',\mathbb{C}}$-orbit $\mathscr{O}'$ in $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$, and all $\mathcal{E}' \in \mathcal{K}_{\mathsf{s}'}(\mathscr{O}')$.

3.4. **Associated cycles of painted bipartitions.** As before, let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity. Suppose that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is its Barbasch-Vogan dual, where $\mathsf{s} = (\star, p, q)$ is a classical signature. Put

$$\mathrm{PBP}_{\star}^{\mathrm{ext}}(\check{\mathcal{O}}, \mathsf{s}) := \{(\tau, \wp) \in \mathrm{PBP}_{\star}^{\mathrm{ext}}(\check{\mathcal{O}}) \mid (p_\tau, q_\tau) = (p, q)\}.$$

Let $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}_{\star}^{\mathrm{ext}}(\check{\mathcal{O}}, \mathsf{s})$. In what follows we will define the associated cycle $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ of $\boldsymbol{\tau}$.

First assume that $\check{\mathcal{O}} = \emptyset$. Then $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ is a singleton. If $\star = \widetilde{C}$, we define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ to be the element that corresponds to the one dimensional genuine representation of $K_{\mathsf{s},\mathbb{C}} = \{\pm 1\}$. In all other cases, we define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ to be the element that corresponds to the one dimensional trivial representation of $K_{\mathsf{s},\mathbb{C}}$.

Now we assume that $\check{\mathcal{O}} \neq \emptyset$. As before, let $\check{\mathcal{O}}'$ be its dual descent. Write $\boldsymbol{\tau}' \in \mathrm{PBP}_{\star'}^{\mathrm{ext}}(\check{\mathcal{O}}')$ for the descent of $\boldsymbol{\tau}$, and assume that $\boldsymbol{\tau}' \in \mathrm{PBP}_{\star'}^{\mathrm{ext}}(\check{\mathcal{O}}', \mathsf{s}')$.

Lemma 3.3. *The orbit $\mathcal{O}$ is in the image of the moment map $\tilde{M}_{\mathsf{s}}$, and*

$$\nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O}) = \textit{the Barbasch-Vogan dual of } \check{\mathcal{O}}'.$$

By Lemma 3.3, $\mathcal{O}' := \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ equals the Barbasch-Vogan dual of $\check{\mathcal{O}}'$.

Similar to the definition of $\pi_{\boldsymbol{\tau}}$ in the introductory section, we inductively define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ by

$$\mathrm{AC}(\boldsymbol{\tau}) := \begin{cases} \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}(\boldsymbol{\tau}')) \otimes (\det_{-1})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}(\boldsymbol{\tau}') \otimes \det^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}(\boldsymbol{\tau}')), & \text{if } \star = C^* \text{ or } D^*. \end{cases}$$

We have the following three analogous results of Theorem 1.11.

Proposition 3.4. *For every* $\uptau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$*, the associated cycle* $\mathrm{AC}(\uptau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ *is a nonzero sum of pairwise distinct elements of* $\mathrm{AOD}_{\mathcal{O}}(K_{\mathsf{s},\mathbb{C}})$.

Proof. This follows from Lemma 6.22 and Lemma 10.2. □

Proposition 3.5. *Suppose that* $\check{\mathcal{O}}$ *is weakly-distinguished. If* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \emptyset)$*, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \quad (\uptau, \epsilon) \mapsto \mathrm{AC}(\uptau) \otimes \det{}^{\epsilon}$$

is injective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

is injective.

Proof. When $\star \in \{ B, D \}$, this follows from Lemma 10.4 (b). When $\star = C^*$, this follows from Lemma 10.5. When $\star \in \{ C, \widetilde{C}, D^* \}$, this follows from Lemma 10.3 (b). □

Proposition 3.6. *Suppose that* $\check{\mathcal{O}}$ *is quasi-distinguished. If* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \emptyset)$*, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}), \quad (\uptau, \epsilon) \mapsto \mathrm{AC}(\uptau) \otimes \det{}^{\epsilon}$$

is well-defined and bijective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

is well-defined and bijective.

Proof. When $\star \in \{ B, D \}$, this follows from Lemma 10.4 (c). When $\star = C^*$, this follows from Lemma 10.5. When $\star \in \{ C, \widetilde{C}, D^* \}$, this follows from Lemma 10.3 (c). □

The following two propositions will also be important for us.

Proposition 3.7. *Suppose that* $\star \in \{B, D\}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \emptyset)$*. Let* $\uptau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$ *and* $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$*. If*

$$\mathrm{AC}(\uptau_1) \otimes \det{}^{\epsilon_1} = \mathrm{AC}(\uptau_2) \otimes \det{}^{\epsilon_2},$$

then

$$\epsilon_1 = \epsilon_2 \qquad \text{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Proof. This follows from Lemma 10.4 (a). □

Proposition 3.8. *Suppose that* $\star \in \{C, \widetilde{C}, D^*\}$ *and* $\check{\mathcal{O}} \neq \emptyset$*. Let* $\uptau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$*, and write* $\uptau'_i = (\tau'_i, \wp'_i)$ *for its descent* $(i = 1, 2)$*. If*

$$\mathrm{AC}(\uptau_1) = \mathrm{AC}(\uptau_2),$$

then

$$(p_{\tau'_1}, q_{\tau'_1}) = (p_{\tau'_2}, q_{\tau'_2}) \qquad \text{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

Proof. This follows from Lemma 10.3 (a). □

3.5. **Distinguishing the constructed representations.** Let $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$, where $\mathsf{s} = (\star, p, q)$ is a classical signature. Recall that π_τ is the Casselman-Wallach representation of G_{s}, defined in the introductory section. We will prove the following theorem in Section **??**.

Theorem 3.9. *The representation π_τ is irreducible, unitarizable and attached to $\check{\mathcal{O}}$. Moreover,*

$$\mathrm{AC}_{\mathcal{O}}(\pi_\tau) = \mathrm{AC}(\tau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

Recall from the introductory section, the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}})$ of isomorphism classes of irreducible Casselman-Wallach representations of G_{s} that are attached to $\check{\mathcal{O}}$.

Theorem 3.10. *If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map*

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}}), \quad (\tau, \epsilon) \mapsto \pi_\tau \otimes \det{}^\epsilon \tag{3.8}$$

is bijective. In all other cases, the map

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}}), \quad \tau \mapsto \pi_\tau \tag{3.9}$$

is bijective.

Proof. In view of Theorem 1.5, we only need to show the injectivity of the two maps in the statement of the theorem. We prove by induction on the number of nonempty rows of $\check{\mathcal{O}}$. The theorem is trivially true in the case when $\check{\mathcal{O}} = \varnothing$. So we assume that $\check{\mathcal{O}} \neq \varnothing$ and the theorem has been proved for the dual descent $\check{\mathcal{O}}'$. Suppose that $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$, $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$. Write $\tau'_i = (\tau'_i, \wp'_i)$ for the descent of τ_i.

First assume that $\star \in \{B, D\}$, and

$$\pi_{\tau_1} \otimes \det{}^{\epsilon_1} \cong \pi_{\tau_2} \otimes \det{}^{\epsilon_2}.$$

Theorem 3.9 implies that

$$\mathrm{AC}(\tau_1) \otimes \det{}^{\epsilon_1} = \mathrm{AC}(\tau_2) \otimes \det{}^{\epsilon_2}.$$

By Proposition 3.7, we know that

$$\epsilon_1 = \epsilon_2 \qquad \text{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Then the definition of π_{τ_1} and π_{τ_2} implies that

$$\check{\Theta}^{\tau_1}_{\tau'_1}(\pi_{\tau'_1}) \cong \check{\Theta}^{\tau_2}_{\tau'_2}(\pi_{\tau'_2}).$$

Consequently,

$$\pi_{\tau'_1} \cong \pi_{\tau'_2}$$

by the injectivity property of the theta correspondence. Hence $\tau'_1 = \tau'_2$ by the induction hypothesis, and Proposition 2.7 finally implies $\tau_1 = \tau_2$. This proves that the map (3.8) is injective.

A slightly simplified argument shows that the map (3.9) is injective when $\star = C^*$.

Now assume that $\star \in \{C, \widetilde{C}, D^*\}$. Suppose that

$$\pi_{\tau_1} \cong \pi_{\tau_2}.$$

Theorem 3.9 implies that

$$\mathrm{AC}(\tau_1) = \mathrm{AC}(\tau_2).$$

By Proposition 3.8, we know that

$$(p_{\tau'_1}, q_{\tau'_1}) = (p_{\tau'_2}, q_{\tau'_2}) \qquad \text{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

Then the definition of π_{τ_1} and π_{τ_2} implies that

$$\check{\Theta}^{\tau_1}_{\tau'_1}(\pi_{\tau'_1} \otimes \det{}^{\varepsilon_{\wp_1}}) \cong \check{\Theta}^{\tau_2}_{\tau'_2}(\pi_{\tau'_2} \otimes \det{}^{\varepsilon_{\wp_1}}).$$

The injectivity property of the theta correspondence then implies that

$$\pi_{\tau_1'} \otimes \det^{\varepsilon_{\wp_1}} \cong \pi_{\tau_2'} \otimes \det^{\varepsilon_{\wp_2}},$$

which further implies that $\pi_{\tau_1'} \cong \pi_{\tau_2'}$. Hence $\tau_1' = \tau_2'$ by the induction hypothesis, and Proposition 2.7 finally implies $\tau_1 = \tau_2$. This proves that the map (3.9) is injective.

A slightly simplified argument shows that the map (3.9) is injective when $\star = D^*$. This finishes the proof of the theorem.

□

Finally, our first main theorem (Theorem 1.6) follows from Theorems 3.9 and 3.10. Our second main theorem (Theorem 1.11) follows from Propositions 3.4 and 3.5, and Theorems 3.9 and 3.10.

4. Theta lifts via matrix coefficient integrals

In this section, we study theta lift via matrix coefficient integrals. Let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$.

4.1. **The oscillator representation.** We use the notation of Section 3.2. Write $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \subset W_{\mathsf{s},\mathsf{s}'}$ for the fixed point set of $J_{\mathsf{s},\mathsf{s}'}$. It is a real symplectic space under the restriction of the form $\langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'}$. Let $H_{\mathsf{s},\mathsf{s}'} := W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \times \mathbb{R}$ denote the Heisenberg group attached to $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$, with group multiplication

$$(\phi, t) \cdot (\phi', t') := (\phi + \phi', t + t' + \langle \phi, \phi' \rangle_{\mathsf{s},\mathsf{s}'}), \qquad \phi, \phi' \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}, \quad t, t' \in \mathbb{R}.$$

Denote by $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$ the complexified Lie algebra of $H_{\mathsf{s},\mathsf{s}'}$. Then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ (as in (3.3)) is an abelian Lie subalgebra of $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$.

The following is the smooth version of the Stone-von Neumann Theorem.

Lemma 4.1. *Up to isomorphism, there exists a unique irreducible smooth Fréchet representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ of moderate growth with central character*

$$\mathbb{R} \to \mathbb{C}^\times, \; t \mapsto e^{\sqrt{-1}\,t}.$$

Moreover, the space

$$\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}} := \{ v \in \omega_{\mathsf{s},\mathsf{s}'} \mid x \cdot v = 0 \quad \text{for all } x \in \mathcal{X}_{\mathsf{s},\mathsf{s}'} \}$$

is one-dimensional.

The group $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ acts on $H_{\mathsf{s},\mathsf{s}'}$ as group automorphisms via the following natural action of $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ on $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$:

$$(g, g') \cdot \phi := g' \circ \phi \circ g^{-1}, \qquad (g, g') \in G_{\mathsf{s}} \times G_{\mathsf{s}'}, \; \phi \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}.$$

From this action, we form the semidirect product $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Lemma 4.2. *The representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ in Lemma 4.1 uniquely extends to a smooth representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ such that $K_{\mathsf{s}} \times K_{\mathsf{s}'}$ acts on $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$ through the scalar multiplication by $\zeta_{\mathsf{s},\mathsf{s}'}$.*

Let $\omega_{\mathsf{s},\mathsf{s}'}$ denote the representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ as in Lemma 4.2, henceforth called the smooth oscillator representation or simply the oscillator representation. As is well-known, this representation is unitarizable [88]. Fix an invariant continuous Hermitian inner product $\langle\, ,\, \rangle$ on it, which is unique up to a positive scalar multiplication. Denote by $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ the completion of $\omega_{\mathsf{s},\mathsf{s}'}$ with respect to this inner product, which is a unitary representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Let $\omega^\vee_{\mathsf{s},\mathsf{s}'}$ denote the contragredient of the oscillator representation $\omega_{\mathsf{s},\mathsf{s}'}$. It is the smooth Fréchet representation of $(G_\mathsf{s} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ of moderate growth specified by the following conditions:

- there is given a $(G_\mathsf{s} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$-invariant, non-degenerate, continuous bilinear form
$$\langle\, ,\, \rangle : \omega_{\mathsf{s},\mathsf{s}'} \times \omega^\vee_{\mathsf{s},\mathsf{s}'} \to \mathbb{C};$$
- $\omega^\vee_{\mathsf{s},\mathsf{s}'}$ is irreducible as a representation of $H_{\mathsf{s},\mathsf{s}'}$.

Since $\omega_{\mathsf{s},\mathsf{s}'}$ is contained in the unitary representation $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$, $\omega^\vee_{\mathsf{s},\mathsf{s}'}$ is identified with the complex conjugation of $\omega_{\mathsf{s},\mathsf{s}'}$.

Put
$$\mathsf{s}'^- := (\star', q', p'),$$
Fix a linear isomorphism
$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^-}, \tag{4.1}$$
such that $(-\langle\, ,\, \rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, -L_{\mathsf{s}'})$ corresponds to $(\langle\, ,\, \rangle_{\mathsf{s}'^-}, J_{\mathsf{s}'^-}, L_{\mathsf{s}'^-})$ under this isomorphism. This induces an isomorphism
$$G_{\mathsf{s}'} \to G_{\mathsf{s}'^-}, \qquad g' \mapsto g'^-. \tag{4.2}$$
Then we have a group isomorphism
$$\begin{array}{rcl} (G_\mathsf{s} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'} & \to & (G_{\mathsf{s}'^-} \times G_\mathsf{s}) \ltimes H_{\mathsf{s}'^-,\mathsf{s}}, \\ (g, g', (\phi, t)) & \mapsto & (g'^-, g, (\phi^* \circ \iota_{\mathsf{s}'}^{-1}, t)). \end{array} \tag{4.3}$$
It is easy to see that the irreducible representation $\omega_{\mathsf{s},\mathsf{s}'}$ corresponds to the irreducible representation $\omega_{\mathsf{s}'^-,\mathsf{s}}$ under this isomorphism.

For every Casselman-Wallach representation π' of $G_{\mathsf{s}'}$, put
$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi') := (\omega_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \pi')_{G_{\mathsf{s}'}} \qquad \text{(the Hausdorff coinvariant space).}$$
This is a Casselman-Wallach representation of G_s.

4.2. **Growth of Casselman-Wallach representations.** Write $\mathfrak{p}_\mathsf{s}^{J_\mathsf{s}}$ for the centralizer of J_s in $\mathfrak{p}_\mathsf{s}$, which is a real form of $\mathfrak{p}_\mathsf{s}$. The Cartan decomposition asserts that
$$G_\mathsf{s} = K_\mathsf{s} \cdot \exp(\mathfrak{p}_\mathsf{s}^{J_\mathsf{s}}).$$
Denote by Ψ_s the function of G_s satisfying the following conditions:

- it is both left and right K_s-invariant;
- for all $g \in \exp(\mathfrak{p}_\mathsf{s}^{J_\mathsf{s}})$,
$$\Psi_\mathsf{s} = \prod_a \left(\frac{1+a}{2}\right)^{-\frac{1}{2}},$$
where a runs over all eigenvalues of $g : V_\mathsf{s} \to V_\mathsf{s}$, counted with multiplicities.

Note that all the eigenvalues of $g \in \exp(\mathfrak{p}_\mathsf{s}^{J_\mathsf{s}})$ are positive real numbers, and $0 < \Psi_\mathsf{s}(g) \leqslant 1$ for all $g \in G_\mathsf{s}$.

Set
$$|\mathsf{s}| := p + q,$$
and
$$\nu_\mathsf{s} := \begin{cases} |\mathsf{s}|, & \text{if } \star \in \{C, \widetilde{C}\}; \\ |\mathsf{s}| - 1, & \text{if } \star = C^*; \\ |\mathsf{s}| - 2, & \text{if } \star \in \{B, D\}; \\ |\mathsf{s}| - 3, & \text{if } \star = D^*. \end{cases}$$

Denote by Ξ_s the bi-K_s-invariant Harish-Chandra's Ξ function on G_s.

Lemma 4.3. *There exists a real number $C_{\mathsf{s}} > 0$ such that*

$$\Psi_{\mathsf{s}}^{\nu_{\mathsf{s}}}(g) \leqslant C_{\mathsf{s}} \cdot \Xi_{\mathsf{s}}(g) \quad \text{for all } g \in G_{\mathsf{s}}.$$

Proof. This is implied by the well-known estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

Lemma 4.4. *Let f be a bi-K_{s}-invariant positive function on G_{s} such that*

$$\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}} \to \mathbb{R}, \qquad x \mapsto f(\exp(x))$$

is a polynomial function. Then for every real number $\nu > 0$, the function $f \cdot \Psi_{\mathsf{s}}^{\nu} \cdot \Xi_{\mathsf{s}}^{2}$ is integrable with respect to a Haar measure on G_{s}.

Proof. This follows from the integral formula for G_{s} under the Cartan decomposition (see [86, Lemma 2.4.2]), as well as the estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

For every Casselman-Wallach representation π of G_{s}, write $\pi^{\vee}$ for its contragredient representation, which is a Casselman-Wallach representation of G_{s} equipped with a G_{s}-invariant, non-degenerate, continuous bilinear form

$$\langle\, , \rangle : \pi \times \pi^{\vee} \to \mathbb{C}.$$

Definition 4.5. *Let $\nu \in \mathbb{R}$. A positive function Ψ on G_{s} is ν-bounded if there is a function f as in Lemma 4.4 such that*

$$\Psi(g) \leqslant f(g) \cdot \Psi_{\mathsf{s}}^{\nu}(g) \cdot \Xi_{\mathsf{s}}(g) \qquad \text{for all } g \in G_{\mathsf{s}}.$$

A Casselman-Wallach representation π of G_{s} is said to be ν-bounded if there exist a ν-bounded positive function Ψ on G_{s}, and continuous seminorms $|\cdot|_{\pi}$ and $|\cdot|_{\pi^{\vee}}$ on π and $\pi^{\vee}$ respectively such that

$$|\langle g \cdot u, v\rangle| \leqslant \Psi(g) \cdot |u|_{\pi} \cdot |v|_{\pi^{\vee}}$$

for all $u \in \pi$, $v \in \pi^{\vee}$, and $g \in G_{\mathsf{s}}$.

Let X_{s} be a maximal J_{s}-stable totally isotropic subspace of V_{s}, which is unique up to the action of K_{s}. Put $Y_{\mathsf{s}} := L_{\mathsf{s}}(X_{\mathsf{s}})$. Then $X_{\mathsf{s}} \cap Y_{\mathsf{s}} = \{0\}$. Write

$$P_{\mathsf{s}} = R_{\mathsf{s}} \ltimes N_{\mathsf{s}}$$

for the parabolic subgroup of G_{s} stabilizing X_{s}, where R_{s} is the Levi subgroup stabilizing both X_{s} and Y_{s}, and N_{s} is the unipotent radical. For every character $\chi : R_{\mathsf{s}} \to \mathbb{C}^{\times}$, view it as a character of P_{s} that is trivial on N_{s}. Write

$$I(\chi) := \mathrm{Ind}_{P_{\mathsf{s}}}^{G_{\mathsf{s}}} \chi \qquad \text{(normalized smooth induction)},$$

which is a Casselman-Wallach representation of G_{s} under the right translations. Note that the representations $I(\chi)$ and $I(\chi^{-1})$ are contragredients of each other with the G_{s}-invariant pairing

$$\langle\, , \rangle : I(\chi) \times I(\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\mathsf{s}}} f(g) \cdot f'(g)\, \mathrm{d}g.$$

Let ν_{χ} be a real number such that $|\chi|$ equals the composition of

$$R_{\mathsf{s}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(X_{\mathsf{s}}) \xrightarrow{|\det|^{\nu_{\chi}}} \mathbb{C}^{\times}. \tag{4.4}$$

Definition 4.6. *A classical signature s is split if $V_{\mathsf{s}} = X_{\mathsf{s}} \oplus Y_{\mathsf{s}}$.*

When s is split, P_{s} is called a Siegel parabolic subgroup of G_{s}.

Lemma 4.7. *Suppose that s is split and let $\chi : P_{\mathsf{s}} \to \mathbb{C}^{\times}$ be a character. Then there is a positive function Ψ on G_{s} with the following properties:*

- *Ψ is $(1-2|\nu_\chi| - \frac{|\mathsf{s}|}{2})$-bounded if $\star \in \{B, C, D, \widetilde{C}\}$, and $(2-2|\nu_\chi| - \frac{|\mathsf{s}|}{2})$-bounded if $\star \in \{C^*, D^*\}$;*
- *for all $f \in I(\chi)$, $f' \in I(\chi^{-1})$ and $g \in G_\mathsf{s}$,*

$$|\langle g.f, f'\rangle| \leqslant \Psi(g) \cdot |f|_{K_\mathsf{s}}|_\infty \cdot |f'|_{K_\mathsf{s}}|_\infty \qquad (|\ |_\infty \text{ stands for the suppernorm}). \tag{4.5}$$

Proof. Let f_0 denote the element in $I(|\chi|)$ such that $(f_0)|_{K_\mathsf{s}} = 1$, and likewise let f_0' denote the element in $I(|\chi^{-1}|)$ such that $(f_0')|_{K_\mathsf{s}} = 1$. Put

$$\Psi(g) := \langle g \cdot f_0, f_0'\rangle, \qquad g \in G_\mathsf{s}.$$

Then it is easy to see that (4.5) holds. Note that Ψ is an elementary spherical function, and the lemma then follows by using the well-known estimate of the elementary spherical functions ([86, Lemma 3.6.7]). □

4.3. **Matrix coefficient integrals against the oscillator representation.** We begin with the following lemma.

Lemma 4.8. *There exist continuous seminorms $|\cdot|_{\mathsf{s},\mathsf{s}'}$ and $|\cdot|^\vee_{\mathsf{s},\mathsf{s}'}$ on $\omega_{\mathsf{s},\mathsf{s}'}$ and $\omega^\vee_{\mathsf{s},\mathsf{s}'}$ respectively such that*

$$|\langle (g, g') \cdot u, v\rangle| \leqslant \Psi_\mathsf{s}^{|\mathsf{s}'|}(g) \cdot \Psi_{\mathsf{s}'}^{|\mathsf{s}|}(g') \cdot |u|_{\mathsf{s},\mathsf{s}'} \cdot |v|^\vee_{\mathsf{s},\mathsf{s}'}$$

for all $u \in \omega_{\mathsf{s},\mathsf{s}'}$, $v \in \omega^\vee_{\mathsf{s},\mathsf{s}'}$, and $(g, g') \in G_\mathsf{s} \times G_{\mathsf{s}'}$.

Proof. This follows from the proof of [50, Theorem 3.2]. □

Definition 4.9. *A Casselman-Wallach representation of $G_{\mathsf{s}'}$ is convergent for $\check{\Theta}^\mathsf{s}_{\mathsf{s}'}$ if it is ν-bounded for some $\nu > \nu_{\mathsf{s}'} - |\mathsf{s}|$.*

Example. Suppose that $\star' \neq \widetilde{C}$. Then the trivial representation of $G_{\mathsf{s}'}$ is convergent for $\check{\Theta}^\mathsf{s}_{\mathsf{s}'}$ if $|\mathsf{s}| > 2\nu_{\mathsf{s}'}$.

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is convergent for $\check{\Theta}^\mathsf{s}_{\mathsf{s}'}$. Consider the integrals

$$\begin{array}{rcl} (\pi' \times \omega_{\mathsf{s},\mathsf{s}'}) \times (\pi'^\vee \times \omega^\vee_{\mathsf{s},\mathsf{s}'}) & \rightarrow & \mathbb{C}, \\ ((u, v), (u', v')) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g \cdot u, u'\rangle \cdot \langle g \cdot v, v'\rangle \,\mathrm{d}g. \end{array} \tag{4.6}$$

Unless otherwise specified, all the measures on Lie groups occurring in this article are Haar measures.

Lemma 4.10. *The integrals in* (4.6) *are absolutely convergent and the map* (4.6) *is continuous and multi-linear.*

Proof. This is a direct consequence of Lemmas 4.3, 4.4 and 4.8. □

By Lemma 4.10, the integrals in (4.6) yield a continuous bilinear form

$$(\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'}) \times (\pi'^\vee \widehat{\otimes} \omega^\vee_{\mathsf{s},\mathsf{s}'}) \rightarrow \mathbb{C}. \tag{4.7}$$

Put

$$\bar{\Theta}^\mathsf{s}_{\mathsf{s}'}(\pi') := \frac{\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'}}{\text{the left kernel of (4.7)}}. \tag{4.8}$$

Proposition 4.11. *The representation $\bar{\Theta}^\mathsf{s}_{\mathsf{s}'}(\pi')$ of G_s is a quotient of $\check{\Theta}^\mathsf{s}_{\mathsf{s}'}(\pi')$, and is $(|\mathsf{s}'| - \nu_\mathsf{s})$-bounded.*

Proof. Note that the bilinear form (4.7) is $(G_{\mathsf{s}'} \times G_{\mathsf{s}'})$-invariant, as well as G_{s}-invariant. Thus $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ is a quotient of $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$, and is therefore a Casselman-Wallach representation. Its contragedient representation is identified with

$$(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi'))^{\vee} := \frac{\pi'^{\vee}\widehat{\otimes}\omega^{\vee}_{\mathsf{s},\mathsf{s}'}}{\text{the right kernel of (4.7)}}.$$

Lemmas 4.4 and 4.8 implies that there are continuous seminorms $|\cdot|_{\pi',\mathsf{s},\mathsf{s}'}$ and $|\cdot|_{\pi'^{\vee},\mathsf{s},\mathsf{s}'}$ on $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$ and $\pi'^{\vee}\widehat{\otimes}\omega^{\vee}_{\mathsf{s},\mathsf{s}'}$ respectively such that

$$|\langle g.u, v\rangle| \leqslant \Psi^{|\mathsf{s}'|}_{\mathsf{s}}(g)\cdot |u|_{\pi',\mathsf{s},\mathsf{s}'}\cdot |v|_{\pi'^{\vee},\mathsf{s},\mathsf{s}'}$$

for all $u \in \pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$, $v \in \pi'^{\vee}\widehat{\otimes}\omega^{\vee}_{\mathsf{s},\mathsf{s}'}$, and $g \in G_{\mathsf{s}}$. The proposition then easily follows in view of Lemma 4.3. □

4.4. **Unitarity.** For the notion of weakly containment of unitary representations, see [24] for example.

Lemma 4.12. *Suppose that $|\mathsf{s}| \geqslant \nu_{\mathsf{s}'}$. Then as a unitary representation of $G_{\mathsf{s}'}$, $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ is weakly contained in the regular representation.*

Proof. This has been known to experts (see [50, Theorem 3.2]). Lemmas 4.3, 4.4 and 4.8 implies that for a dense subspace of $\hat{\omega}_{\mathsf{s},\mathsf{s}'}|_{G_{\mathsf{s}'}}$, the diagonal matrix coefficients are almost square integrable. Thus the lemma follows form [24, Theorem 1]. □

However, if $\star' \in \{B, C^*, D^*\}$, $|\mathsf{s}| \neq \nu_{\mathsf{s}'}$ for the parity reason. For this reason, we introduce

$$\nu^{\circ}_{\mathsf{s}'} := \begin{cases} \nu_{\mathsf{s}'}, & \text{if } \star' \in \{C, D, \widetilde{C}\}; \\ \nu_{\mathsf{s}'} + 1, & \text{if } \star' \in \{B, C^*, D^*\}. \end{cases}$$

The following definition is a slight variation of definition 4.9.

Definition 4.13. *A Casselman-Wallach representation of $G_{\mathsf{s}'}$ is overconvergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ if it is ν-bounded for some $\nu > \nu^{\circ}_{\mathsf{s}'} - |\mathsf{s}|$.*

We will prove the following unitarity result in the rest of this subsection.

Theorem 4.14. *Assume that $|\mathsf{s}| \geqslant \nu^{\circ}_{\mathsf{s}'}$. Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is overconvergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$. If π' is unitarizable, then so is $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$.*

Recall the following positivity result of matrix coefficient integrals, which is a special case of [32, Theorem A. 5].

Lemma 4.15. *Let G be a real reductive group with a maximal compact subgroup K. Let π_1 and π_2 be two unitary representations of G such that π_2 is weakly contained in the regular representation. Let $u_1, u_2, \cdots, u_r$ ($r \in \mathbb{N}$) be vectors in π_1 such that for all $i, j = 1, 2, \cdots, r$, the integral*

$$\int_G \langle g\cdot u_i, u_j\rangle\, \Xi_G(g)\, \mathrm{d}g$$

is absolutely convergent, where Ξ_G is the bi-K-invariant Harish-Chandra's Ξ function on G. Let $v_1, v_2, \cdots, v_r$ be K-finite vectors in π_2. Put

$$u := \sum_{i=1}^{r} u_i \otimes v_i \in \pi_1 \otimes \pi_2.$$

Then the integral

$$\int_G \langle g\cdot u, u\rangle\, \mathrm{d}g$$

absolutely converges to a nonnegative real number.

Now we come to the proof of Theorem 4.14.

Proof of Theorem 4.14. Fix an invariant continuous Hermitian inner product on π', and write $\hat{\pi}'$ for the completion of π' with respect to this Hermitian inner product. The space $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$ is equipped with the inner product $\langle\,,\,\rangle$ that is the tensor product of the ones on π' and $\omega_{\mathsf{s},\mathsf{s}'}$. It suffices to show that

$$\int_{G_{\mathsf{s}'}} \langle g\cdot u, u\rangle \,\mathrm{d}g \geqslant 0$$

for all u in a dense subspace of $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$.

If $\nu^\circ_{\mathsf{s}'} < 0$, then $\star \in \{D, D^*\}$ and $|\mathsf{s}'| = 0$. The theorem is trivial in this case. Thus we assume that $\nu^\circ_{\mathsf{s}'} \geqslant 0$. We also assume that $\star = B$. The proof in the other cases is similar and is omitted.

Note that $\nu^\circ_{\mathsf{s}'} = \nu_{\mathsf{s}'} = |\mathsf{s}'|$ is even. Let $\mathsf{s}_1 = (B, p_1, q_1)$ and $\mathsf{s}_2 = (D, p_2, q_2)$ be two classical signatures such that

$$(p_1, q_1) + (p_2, q_2) = (p, q) \quad \text{and} \quad |\mathsf{s}_2| = \nu^\circ_{\mathsf{s}'}.$$

View $\hat{\omega}_{\mathsf{s}_2,\mathsf{s}'}$ as a unitary representation of the symplectic group $G_{(C,p',q')}$. Define π_2 to be its pull-back through the covering homomorphism $G_{\mathsf{s}'} \to G_{(C,p',q')}$. By Lemma 4.12, the representation π_2 of $G_{\mathsf{s}'}$ is weakly contained in the regular representation.

Put

$$\pi_1 := \hat{\pi}'\widehat{\otimes}_{\mathrm{h}}(\hat{\omega}_{\mathsf{s}_1,\mathsf{s}'}|_{G_{\mathsf{s}'}}) \qquad (\widehat{\otimes}_{\mathrm{h}} \text{ indicates the Hilbert space tensor product}).$$

Lemmas 4.4 and 4.8 imply that the integral

$$\int_{G_{\mathsf{s}'}} \langle g.u, v\rangle \cdot \Xi_{G_{\mathsf{s}'}}(g)\,\mathrm{d}g$$

is absolutely convergent for all $u, v \in \pi'\otimes\omega_{\mathsf{s}_1,\mathsf{s}'}$. The theorem then follows by Lemma 4.15. □

5. Double theta lifts and degenerate principal series

In this section, we relate double theta lifts with degenerate principal series representations.

Let

$$\dot{\mathsf{s}} = (\dot{\star}, \dot{p}, \dot{q}), \qquad \mathsf{s}' = (\star', p', q') \qquad \text{and} \qquad \mathsf{s}'' = (\star'', p'', q'')$$

be classical signatures such that

- $(q', p') + (p'', q'') = (\dot{p}, \dot{q})$;
- if $\dot{\star} \in \{B, D\}$, then $\star', \star'' \in \{B, D\}$;
- if $\dot{\star} \notin \{B, D\}$, then $\star' = \star'' = \dot{\star}$.

As in Section 4.1, put $\mathsf{s}'^- := (\star', q', p')$. We view $V_{\mathsf{s}'^-}$ and $V_{\mathsf{s}''}$ as subspaces of $V_{\dot{\mathsf{s}}}$ such that $(\langle\,,\,\rangle_{\dot{\mathsf{s}}}, J_{\dot{\mathsf{s}}}, L_{\dot{\mathsf{s}}})$ extends both $(\langle\,,\,\rangle_{\mathsf{s}'^-}, J_{\mathsf{s}'^-}, L_{\mathsf{s}'^-})$ and $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$, and $V_{\mathsf{s}'^-}$ and $V_{\mathsf{s}''}$ are perpendicular to each other under the form $\langle\,,\,\rangle_{\dot{\mathsf{s}}}$. Then we have an orthogonal decomposition

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''},$$

and both $G_{\mathsf{s}'^-}$ and $G_{\mathsf{s}''}$ are identified with subgroups of $G_{\dot{\mathsf{s}}}$.

Fix a linear isomorphism

$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^-} \tag{5.1}$$

as in (4.1), which induces an isomorphism

$$G_{\mathsf{s}'} \to G_{\mathsf{s}'^-}, \quad g \mapsto g^-. \tag{5.2}$$

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$. Write π'^{-} for the representation of $G_{\mathsf{s}'^{-}}$ that corresponds to π' under the isomorphism (5.2).

5.1. **Matrix coefficient integrals and double theta lifts.** We begin with the following lemma.

Lemma 5.1. *The function $(\Xi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}}$ on $G_{\mathsf{s}'}$ is $|\mathsf{s}''|$-bounded.*

Proof. This follows from the estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

Definition 5.2. *The Casselman-Wallach representation π'^{-} of $G_{\mathsf{s}'^{-}}$ is convergent for a Casselman-Wallach representation $\dot{\pi}$ of $G_{\dot{\mathsf{s}}}$ if there are real number ν' and $\dot{\nu}$ such that π' is ν'-bounded, $\dot{\pi}$ is $\dot{\nu}$-bounded, and*

$$\nu' + \dot{\nu} > -\,|\mathsf{s}''|\,.$$

Let $\dot{\pi}$ be a Casselman-Wallach representation of $G_{\dot{\mathsf{s}}}$ such that π'^{-} is convergent for $\dot{\pi}$. Consider the integrals

$$\begin{array}{rcl} (\pi'^{-} \times \dot{\pi}) \times ((\pi'^{-})^{\vee} \times \dot{\pi}^{\vee}) & \rightarrow & \mathbb{C}, \\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^{-} \cdot u, u' \rangle \cdot \langle g^{-} \cdot v, v' \rangle \, \mathrm{d}g. \end{array} \tag{5.3}$$

Lemma 5.3. *The integrals in* (5.3) *are absolutely convergent and the map* (5.3) *is continuous and multi-linear.*

Proof. Note that $(\Psi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}} = \Psi_{\mathsf{s}'}$. Thus the lemma follows from Lemmas 4.4 and 5.1. □

By Lemma 5.3, the integrals in (5.3) yield a continuous bilinear form

$$\langle\, , \,\rangle : (\pi'^{-} \widehat{\otimes} \dot{\pi}) \times ((\pi'^{-})^{\vee} \widehat{\otimes} \dot{\pi}^{\vee}) \rightarrow \mathbb{C}. \tag{5.4}$$

Put

$$\pi'^{-} * \dot{\pi} := \frac{\pi'^{-} \widehat{\otimes} \dot{\pi}}{\text{the left kernel of (5.4)}}. \tag{5.5}$$

This is a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

For every classical signature $\mathsf{s} = (\star, p, q)$, put

$$[\mathsf{s}] := \begin{cases} (C, p, q), & \text{if } \star = \tilde{C}; \\ \mathsf{s}, & \text{if } \star \neq \tilde{C}. \end{cases}$$

Proposition 5.4. *Suppose that $\mathsf{s} = (\star, p, q)$ is a classical signature such that both $\star'$ and $\star''$ equals the Howe dual of $\star$, and $2\nu_{\mathsf{s}} < |\dot{\mathsf{s}}|$. Assume that π' is ν'-bounded for some $\nu' > \nu_{\mathsf{s}'} - |\mathsf{s}|$. Then*

- *π' is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$;*
- *$\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ is convergent for $\check{\Theta}^{\mathsf{s}''}_{\mathsf{s}}$;*
- *the trivial representation 1 of $G_{\bar{\mathsf{s}}}$ is convergent for $\check{\Theta}^{\dot{\mathsf{s}}}_{\bar{\mathsf{s}}}$;*
- *π'^{-} is convergent for $\bar{\Theta}^{\dot{\mathsf{s}}}_{[\mathsf{s}]}(1)$;*
- *as representations of $G_{\mathsf{s}''}$, we have*

$$\bar{\Theta}^{\mathsf{s}''}_{\mathsf{s}}(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')) \cong \pi'^{-} * \bar{\Theta}^{\dot{\mathsf{s}}}_{[\mathsf{s}]}(1). \tag{5.6}$$

Proof. The first four claims in the proposition are obvious. We only need to prove the last one. Note that the integrals in

$$\begin{array}{rcl} (\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \omega_{\mathsf{s}'',\mathsf{s}}) \times ((\pi')^{\vee} \widehat{\otimes} \omega^{\vee}_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \omega^{\vee}_{\mathsf{s}'',\mathsf{s}}) & \rightarrow & \mathbb{C}, \\ (u,v) & \mapsto & \int_{G_{\mathsf{s}'} \times G_{\mathsf{s}}} \langle (g',g) \cdot u, v \rangle \, \mathrm{d}g' \, \mathrm{d}g \end{array} \tag{5.7}$$

are absolutely convergent and defines a continuous bilinear map. Also note that

$$\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}_2,\mathsf{s}} \cong \omega_{\dot{\mathsf{s}},\mathsf{s}}.$$

In view of Fubini's theorem, the lemma follows as both sides of (5.6) are isomorphic to the quotient of $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}}$ by the left kernel of the pairing (5.7).

□

5.2. **Matrix coefficient integrals against degenerate principal series.** We are particularly interested in the case when $\dot{\pi}$ is a degenerate principal series representation. Suppose that $\dot{\mathsf{s}}$ is split, and

$$p' \leqslant p'' \qquad \text{and} \qquad q' \leqslant q''.$$

Then there is a split classical signature $\mathsf{s}_0 = (\star_0, p_0, q_0)$ such that

- $\star_0 = \dot{\star}$;
- $(p', q') + (p_0, q_0) = (p'', q'')$.

We view $V_{\mathsf{s}'}$ and V_{s_0} as subspaces of $V_{\mathsf{s}''}$ such that $(\langle\, ,\, \rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$ extends both $(\langle\, ,\, \rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, L_{\mathsf{s}'})$ and $(\langle\, ,\, \rangle_{\mathsf{s}_0}, J_{\mathsf{s}_0}, L_{\mathsf{s}_0})$, and $V_{\mathsf{s}'}$ and V_{s_0} are perpendicular to each other under the form $\langle\, ,\, \rangle_{\mathsf{s}_2}$.

Put

$$V_{\mathsf{s}'}^{\triangle} := \{\iota_{\mathsf{s}'}(v) + v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\} \quad \text{and} \quad V_{\mathsf{s}'}^{\nabla} := \{\iota_{\mathsf{s}'}(v) - v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\}.$$

As before, we have that

$$V_{\mathsf{s}_0} = X_{\mathsf{s}_0} \oplus Y_{\mathsf{s}_0},$$

where X_{s_0} is a maximal J_{s_0}-stable totally isotropic subspace of V_{s_0}, and $Y_{\mathsf{s}_0} := L_{\mathsf{s}_0}(X_{\mathsf{s}_0})$. Suppose that

$$X_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0} \quad \text{and} \quad Y_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}.$$

In summary, we have decompositions

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}'} \oplus V_{\mathsf{s}_0} = (V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0}) \oplus (V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}).$$

As before, $X_{\dot{\mathsf{s}}}$ and X_{s_0} yield the Siegel parabolic subgroups

$$P_{\dot{\mathsf{s}}} = R_{\dot{\mathsf{s}}} \ltimes N_{\dot{\mathsf{s}}} \subset G_{\dot{\mathsf{s}}} \qquad \text{and} \qquad P_{\mathsf{s}_0} = R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}_0} \subset G_{\mathsf{s}_0}.$$

Write

$$P_{\mathsf{s}'',\mathsf{s}_0} = R_{\mathsf{s}'',\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$$

for the parabolic subgroup of $G_{\mathsf{s}''}$ stabilizing X_{s_0}, where $R_{\mathsf{s}'',\mathsf{s}_0}$ is the Levi subgroup stabilizing both X_{s_0} and Y_{s_0}, and $N_{\mathsf{s}'',\mathsf{s}_0}$ is the unipotent radical. We have an obvious homomorphism

$$G_{\mathsf{s}'} \times R_{\mathsf{s}_0} \to R_{\mathsf{s}'',\mathsf{s}_0},$$

which is a two fold covering map when $\dot{\star} = \widetilde{C}$, and an isomorphism in the other cases. For every $g \in G_{\mathsf{s}'}$, write $g^{\triangle} := g^- g$, which is an element of $R_{\dot{\mathsf{s}}}$.

Let $\dot{\chi} : R_{\dot{\mathsf{s}}} \to \mathbb{C}^\times$ be a character. It yields a degenerate principal series representation

$$I(\dot{\chi}) := \mathrm{Ind}_{P_{\dot{\mathsf{s}}}}^{G_{\dot{\mathsf{s}}}} \dot{\chi}$$

of $G_{\dot{\mathsf{s}}}$. Write

$$\chi_0 := \dot{\chi}|_{R_{\mathsf{s}_0}}, \tag{5.8}$$

and define a character

$$\chi' : G_{\mathsf{s}'} \to \mathbb{C}^\times, \quad g \mapsto \dot{\chi}(g^{\triangle}) \qquad \text{(this is a quadratic character)}. \tag{5.9}$$

Recall the representations π' of $G_{\mathsf{s}'}$ and π'^- of $G_{\mathsf{s}'^-}$. If $\dot{\mathsf{s}} = \widetilde{C}$, we assume that both π' and $\dot{\chi}$ are genuine. Then $(\pi' \otimes \chi') \otimes \chi_0$ descends to a Casselman-Wallach representation of $R_{\mathsf{s}'',\mathsf{s}_0}$. View it as a representation of $P_{\mathsf{s}'',\mathsf{s}_0}$ via the trivial action of $N_{\mathsf{s}'',\mathsf{s}_0}$, and form the representation $\mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi_1 \otimes \chi') \otimes \chi_0)$

Let $\nu_{\dot\chi} \in \mathbb{R}$ be as in (4.4). The rest of this subsection is devoted to a proof of the following theorem.

Theorem 5.5. *Assume that π' is ν' bounded for some*

$$\nu' > \begin{cases} 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 1, & \text{if } \dot\star \in \{B, C, D, \widetilde{C}\}; \\ 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 2, & \text{if } \dot\star \in \{C^*, D^*\}. \end{cases}$$

Then π'^- is convergent for $I_{\dot{\mathsf{s}}}(\dot\chi)$, and

$$\pi'^- * I(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Let the notation and assumptions be as in Theorem 5.5. Recall that the representations $I(\dot\chi)$ and $I(\dot\chi^{-1})$ are contragredients of each other with the $G_{\dot{\mathsf{s}}}$-invariant pairing

$$\langle\, ,\, \rangle : I(\dot\chi) \times I(\dot\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\dot{\mathsf{s}}}} f(g) \cdot f'(g)\, \mathrm{d}g,$$

The first assertion of Theorem 5.5 is a direct consequence of Lemma 4.7. As in (5.4), we have a continuous bilinear form

$$\begin{array}{rcl} (\pi'^- \widehat\otimes I(\dot\chi)) \times ((\pi'^-)^\vee \widehat\otimes I(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^- \cdot u, v \rangle\, \mathrm{d}g \end{array} \tag{5.10}$$

so that

$$\pi'^- * I(\dot\chi) := \frac{\pi'^- \widehat\otimes I(\dot\chi)}{\text{the left kernel of (5.10)}}. \tag{5.11}$$

Note that

$$G_{\dot{\mathsf{s}}}^\circ := P_{\dot{\mathsf{s}}} \cdot G_{\mathsf{s}''}$$

is open and dense in $G_{\dot{\mathsf{s}}}$, and its complement has measure zero in $G_{\dot{\mathsf{s}}}$. Moreover,

$$P_{\dot{\mathsf{s}}} \backslash G_{\dot{\mathsf{s}}}^\circ = (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}) \backslash G_{\mathsf{s}''}. \tag{5.12}$$

Form the normalized Schwartz induction

$$I^\circ(\dot\chi) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0.$$

The reader is referred to [21, Section 6.2] for the general notion of Schwartz inductions (in a slightly different unnormalized setting). Similarly, put

$$I^\circ(\dot\chi^{-1}) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0^{-1}.$$

Note that

(5.13) the modulus character of $P_{\dot{\mathsf{s}}}$ restricts to the modulus character of $R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$.

In view of (5.12) and (5.13), by extension by zero, $I^\circ(\dot\chi)$ is viewed as a closed subspace of $I(\dot\chi)$, and $I^\circ(\dot\chi^{-1})$ is viewed as a closed subspace of $I(\dot\chi^{-1})$. These two closed subspaces are $(G_{\mathsf{s}'} \times G_{\mathsf{s}''})$-stable.

Similar to (5.10), we have a continuous bilinear form

$$\begin{array}{rcl} (\pi'^- \widehat\otimes I^\circ(\dot\chi)) \times ((\pi'^-)^\vee \widehat\otimes I^\circ(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^- \cdot u, v \rangle\, \mathrm{d}g. \end{array} \tag{5.14}$$

Put

$$\pi'^- * I^\circ(\dot\chi) := \frac{\pi'^- \widehat\otimes I^\circ(\dot\chi)}{\text{the left kernel of (5.14)}}, \tag{5.15}$$

which is still a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

Lemma 5.6. *As representations of $G_{\mathsf{s}''}$,*
$$\pi'^{-} * I^\circ(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Proof. As Fréchet spaces, π'^{-} is obviously identified with π'. Note that the natural map $G_{\mathsf{s}'} \to (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0})\backslash G_{\mathsf{s}''}$ is proper and hence the following integrals are absolutely convergent and yield a $G_{\mathsf{s}''}$-equivariant continuous linear map:
$$\begin{array}{rcl} \xi : \pi'^{-}\widehat{\otimes} I^\circ(\dot\chi) & \to & \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0), \\ v \otimes f & \mapsto & \left(h \mapsto \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f(g^{-1}h)(g\cdot v)\,\mathrm{d}g\right). \end{array}$$
Moreover, this map is surjective (*cf.* [21, Section 6.2]). It is thus open by the open mapping theorem. Similarly, we have a open surjective $G_{\mathsf{s}''}$-equivariant continuous linear map
$$\begin{array}{rcl} \xi' : (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1}) & \to & \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi'^\vee \otimes \chi') \otimes \chi_0^{-1}), \\ v' \otimes f' & \mapsto & \left(h \mapsto \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f'(g^{-1}h)(g\cdot v')\,\mathrm{d}g\right). \end{array}$$
For all $v \otimes f \in \pi'^{-}\widehat{\otimes} I^\circ(\dot\chi)$ and $v' \otimes f' \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$, we have that
$$\begin{aligned} & \int_{G_{\mathsf{s}'}} \langle g^{-}\cdot(v\otimes f), v'\otimes f'\rangle\,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-}\cdot v, v'\rangle \cdot \langle g^{-}\cdot f, f'\rangle\,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-}\cdot v, v'\rangle \cdot \int_{K_{\mathsf{s}''}}\int_{G_{\mathsf{s}'}} (g^{-}\cdot f)(hx)\cdot f'(hx)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-}\cdot v, v'\rangle \cdot \int_{K_{\mathsf{s}''}}\int_{G_{\mathsf{s}'}} \chi'(g)\cdot f(g^{-1}hx)\cdot f'(hx)\,\mathrm{d}h\,\mathrm{d}x\,\mathrm{d}g \\ = & \int_{K_{\mathsf{s}''}}\int_{G_{\mathsf{s}'}}\int_{G_{\mathsf{s}'}} \langle (h^{-}(g^{-})^{-1})\cdot v, v'\rangle \cdot \chi'(hg^{-1})\cdot f(gx)\cdot f'(hx)\,\mathrm{d}g\,\mathrm{d}h\,\mathrm{d}x \\ = & \int_{K_{\mathsf{s}''}} \langle \xi(v\otimes f)(x), \xi'(v'\otimes f')(x)\rangle\,\mathrm{d}x \\ = & \langle \xi(v\otimes f), \xi'(v'\otimes f')\rangle. \end{aligned}$$
This implies the lemma. $\square$

Lemma 5.7. *Let $u \in \pi'^{-}\widehat{\otimes} I(\dot\chi)$. Assume that*
$$\int_{G_{\mathsf{s}'}} \langle g^{-}\cdot u, v\rangle\,\mathrm{d}g = 0 \tag{5.16}$$
for all $v \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$. Then (5.16) *also holds for all $v \in (\pi'^{-})^\vee\widehat{\otimes} I(\dot\chi^{-1})$.*

Proof. Take a sequence $(\eta_1, \eta_2, \eta_3, \cdots)$ of real valued smooth functions on $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$ such that

- for all $i \geqslant 1$, the support of η_i is contained in $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^\circ$;
- for all $i \geqslant 1$ and $x \in P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$, $0 \leqslant \eta_i(x) \leqslant \eta_{i+1}(x) \leqslant 1$;
- $\bigcup_{i=1}^\infty \eta_i^{-1}(1) = P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^\circ$.

Let $v \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$. Note that $\eta_i I(\dot\chi^{-1}) \subset I^\circ(\dot\chi^{-1})$. Thus $\eta_i v \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$.

Lemma 4.7 and Lebesgue's dominated convergence theorem imply that
$$\int_{G_{\mathsf{s}'}} \langle g^{-}\cdot u, v\rangle\,\mathrm{d}g = \lim_{i\to+\infty}\int_{G_{\mathsf{s}'}} \langle g^{-}\cdot u, \eta_i v\rangle\,\mathrm{d}g = 0.$$

This proves the lemma. □

Lemma 5.7 implies that we have a natural continuous linear map

$$\mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi'\otimes\chi')\otimes\chi_0)=\pi'^{-}*I^\circ(\dot\chi)\to\pi'^{-}*I(\dot\chi).$$

This map is clearly injective and $G_{\mathsf{s}''}$-equivariant. Similarly to Lemma 5.7, we know that the natural pairing

$$(\pi'^{-}*I(\dot\chi))\times((\pi'^{-})^\vee*I^\circ(\dot\chi^{-1}))\to\mathbb{C}$$

is well-defined and non-degenerate. Thus Theorem 5.5 follows by the following lemma.

Lemma 5.8. *Let G be a real reductive group. Let π be a Casselman-Wallach representation of G, and let $\tilde\pi$ be a smooth Fréchet representation of G with a G-equivariant injective continuous linear map*

$$\phi:\pi\to\tilde\pi.$$

Assume that there is a non-degenerate G-invariant continuous bilinear map

$$\langle\,,\,\rangle:\tilde\pi\times\pi^\vee\to\mathbb{C},$$

such that the composition of

$$\pi\times\pi^\vee\xrightarrow{(u,v)\mapsto(\phi(u),v)}\tilde\pi\times\pi^\vee\xrightarrow{\langle\,,\,\rangle}\mathbb{C}$$

is the natural pairing. Then ϕ is a topological isomorphism.

Proof. Let $u\in\tilde\pi$. Using the theorem of Dixmier-Malliavin [27, Theorem 3.3], we write

$$u=\sum_{i=1}^{s}\int_G\varphi_i(g)(g\cdot u_i)\,\mathrm{d}g\qquad(s\in\mathbb{N},\ u_i\in\tilde\pi),$$

where φ_i's are compactly supported smooth functions on G. As a continuous linear functional on $\pi^\vee$, we have that

$$\langle u,\,\cdot\,\rangle=\sum_{i=1}^{s}\int_G\varphi_i(g)\cdot\langle g\cdot u_i,\,\cdot\,\rangle\,\mathrm{d}g.$$

By [79, Lemma 3.5], the right-hand side functional equals $\langle u_0,\,\cdot\,\rangle$ for a unique $u_0\in\pi$. Thus $u=u_0$, and the lemma follows by the open mapping theorem. □

5.3. **Double theta lifts and parabolic induction.** In this subsection, we further assume that

$$\dot{\mathsf{s}}=(C,2k-1,2k-1),\ (D,2k-1,2k-1),\ (\widetilde{C},2k,2k),\ (C^*,2k,2k),\text{or}\ \ (D^*,2k,2k),$$

where $k\in\mathbb{N}^+$. We also assume that the character $\dot\chi:R_{\dot{\mathsf{s}}}\to\mathbb{C}^\times$ satisfies the following conditions:

$$(5.17)\quad\begin{cases}\dot\chi=1, & \text{if } \dot\star\in\{B,D\};\\ \dot\chi^2=1, & \text{if } \dot\star=C;\\ \dot\chi \text{ is genuine and } \dot\chi^4=1, & \text{if } \dot\star=\widetilde{C};\\ \dot\chi^2 \text{ equals the composition of } R_{\dot{\mathsf{s}}}\xrightarrow{\text{natural map}}\mathrm{GL}(Y_{\dot{\mathsf{s}}})\xrightarrow{\det}\mathbb{C}^\times, & \text{if } \dot\star=C^*;\\ \dot\chi^2 \text{ equals the composition of } R_{\dot{\mathsf{s}}}\xrightarrow{\text{natural map}}\mathrm{GL}(X_{\dot{\mathsf{s}}})\xrightarrow{\det}\mathbb{C}^\times, & \text{if } \dot\star=D^*.\end{cases}$$

If $\dot\star\notin\{C,\widetilde{C}\}$, the character $\dot\chi$ is uniquely determined by (5.17). If $\dot\star\in\{C,\widetilde{C}\}$, there are two characters satisfying (5.17), and we let $\dot\chi'$ denote the one other than $\dot\chi$.

The relationship between the degenerate principle series representation $I(\dot\chi)$ and Rallis quotients is summarized in the following lemma.

Lemma 5.9. *(a) If* $\dot{\mathsf{s}} = (C, 2k-1, 2k-1)$*, then*

$$I(\dot\chi) \oplus I(\dot\chi') \cong \bigoplus_{\mathsf{s}=(D,p,q),\, p+q=2k} \check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1).$$

(b) If $\dot{\mathsf{s}} = (\widetilde{C}, 2k, 2k)$*, then*

$$I(\dot\chi) \oplus I(\dot\chi') \cong \bigoplus_{\mathsf{s}=(B,p,q),\, p+q=2k+1} \check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1).$$

(c) If $\dot{\mathsf{s}} = (D, 2k-1, 2k-1)$*, then*

$$I(\dot\chi) \cong \check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) \oplus (\check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) \otimes \det), \qquad \text{where } \mathsf{s} = (C, k-1, k-1).$$

(d) If $\dot{\mathsf{s}} = (C^*, 2k, 2k)$*, then*

$$I(\dot\chi) \cong \check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1), \qquad \text{where } \mathsf{s} = (D^*, k, k).$$

(e) If $\dot{\mathsf{s}} = (D^*, 2k, 2k)$*, then there is an exact sequence of representations of* $G_{\dot{\mathsf{s}}}$*:*

$$0 \to \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) \to I(\dot\chi) \to \bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check\Theta_{\mathsf{s}_1}^{\dot{\mathsf{s}}}(1) \to 0.$$

(f) All the representations $\check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$*and* $\check\Theta_{\mathsf{s}_0}^{\dot{\mathsf{s}}}(1)$ *appearing in (a), (b), (c), (d), (e) are irreducible and unitarizable.*

Proof. See [47, Theorem 2.4], [48, Introduction], [49, Theorem 6.1] and [90, Sections 9 and 10]. □

Lemma 5.10. *For all* s *appearing in Lemma 5.9, the trivial representation* 1 *of* G_{s} *is overconvergent for* $\check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}$*, and*

$$\bar\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) = \check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1).$$

Proof. Note that the trivial representation 1 of G_{s} is 0-bounded, and $0 > \nu_{\mathsf{s}} + \nu_{\mathsf{s}}^\circ - |\dot{\mathsf{s}}|$. This implies the first assertion. Since $\check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$ is irreducible and $\bar\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$ is a quotient of $\check\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$, for the proof of the second assertion, it suffices to show that $\bar\Theta_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$ is nonzero.

Put

$$V_{\mathsf{s}}(\mathbb{R}) := \begin{cases} \text{the fixed point set of } J_{\mathsf{s}} \text{ in } V_{\mathsf{s}}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ V_{\mathsf{s}}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

As usual, realize $\hat\omega_{\dot{\mathsf{s}},\mathsf{s}}$ on the space of square integrable functions on $(V_{\mathsf{s}}(\mathbb{R}))^{\dot p}$ so that $\omega_{\dot{\mathsf{s}},\mathsf{s}}$ is identified with the space of the Schwartz functions, and G_{s} acts on it through the obvious transformation.

Take a positive valued Schwartz function ϕ on $(V_{\mathsf{s}}(\mathbb{R}))^{\dot p}$. Then

$$\langle g \cdot \phi, \phi \rangle = \int_{(V_{\mathsf{s}}(\mathbb{R}))^{\dot p}} \phi(g^{-1} \cdot x) \cdot \phi(x)\, \mathrm{d}x > 0, \quad \text{for all } g \in G_{\mathsf{s}}.$$

Thus

$$\int_{G_{\mathsf{s}}} \langle g \cdot \phi, \phi \rangle\, \mathrm{d}g \neq 0,$$

and the lemma follows. □

Lemma 5.11. *Assume that* π' *is* ν'*-bounded for some*

$$\nu' > \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \text{if } \dot{\mathsf{s}} = D^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \text{otherwise.} \end{cases}$$

Then for all s *appearing in Lemma 5.9 and all unitary character* α' *of* $G_{\mathsf{s}'}$*, the representation* $\pi' \otimes \alpha'$ *is convergent for* $\check\Theta_{\mathsf{s}'}^{\mathsf{s}}$*, and* $\bar\Theta_{\mathsf{s}'}^{\mathsf{s}}(\pi' \otimes \alpha')$ *is overconvergent for* $\check\Theta_{\mathsf{s}}^{\mathsf{s}''}$*.*

Proof. The first assertion follows by noting that

$$\nu_{\mathsf{s}'} - |\mathsf{s}| = \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \text{if } \dot{\mathsf{s}} = D^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \text{otherwise.} \end{cases}$$

The proof of second assertion is similar to the proof of the first assertion of Lemma 5.10. $\square$

Recall the character $\chi_0 := \dot{\chi}|_{R_{\mathsf{s}_0}}$ from (5.8). Similarly we define $\chi_0' := \dot{\chi}'|_{R_{\mathsf{s}_0}}$. Recall that π' is assumed to be genuine when $\star = \widetilde{C}$.

Theorem 5.12. *Assume that π' is ν'-bounded for some $\nu' > -\frac{|\mathsf{s}_0|}{2} - 1$.*
(a) If $\dot{\mathsf{s}} = (C, 2k-1, 2k-1)$, then

$$\bigoplus_{\mathsf{s}=(D,p,q),\, p+q=2k} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \oplus \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0').$$

(b) If $\dot{\mathsf{s}} = (\widetilde{C}, 2k, 2k)$, then

$$\bigoplus_{\mathsf{s}=(B,p,q),\, p+q=2k+1} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \oplus \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0').$$

(c) If $\dot{\mathsf{s}} = (D, 2k-1, 2k-1)$, then

$$\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \oplus \Big((\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi' \otimes \det))) \otimes \det\Big) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s} = (C, k-1, k-1)$ if $|\mathsf{s}'|$ is even, and $\mathsf{s} = (\widetilde{C}, k-1, k-1)$ if $|\mathsf{s}'|$ is odd.
(d) If $\dot{\mathsf{s}} = (C^*, 2k, 2k)$, then

$$\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s} = (D^*, k, k)$.
(e) If $\dot{\mathsf{s}} = (D^*, 2k, 2k)$, then there is an exact sequence

$$0 \to \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \to \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \to \mathcal{J} \to 0$$

of representations of $G_{\mathsf{s}''}$ such that $\mathcal{J}$ is a quotient of

$$\bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')).$$

Proof. Using Lemma 4.7, we know that π'^{-} is convergent for $I(\dot{\chi})$ (and convergent for $I(\dot{\chi}')$ when $\star \in \{C, \widetilde{C}\}$). Also note that the character χ' (see (5.9)) is trivial.

If we are in the situation (a), (b), (c) or (d), by using Theorem 5.5, Lemma 5.10 and Proposition 5.4, the theorem follows by applying the operation $\pi'^{-} * (\,\cdot\,)$ to the isomorphism in Lemma 5.9.

Now assume that we are in the situation (e). For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in part (e) of Lemma 5.9. Since π'^{-} is nuclear as a Fréchet space, the sequence

$$0 \to \pi'^{-} \widehat{\otimes} I_1 \to \pi'^{-} \widehat{\otimes} I_2 \to \pi'^{-} \widehat{\otimes} I_3 \to 0$$

is also topologically exact. Note that the natural map

$$\pi'^{-} * I_1 \to \pi'^{-} * I_2$$

is injective, and the natural map

$$\pi'^{-} \widehat{\otimes} I_3 \cong \frac{\pi'^{-} \widehat{\otimes} I_2}{\pi'^{-} \widehat{\otimes} I_1} \longrightarrow \frac{\pi'^{-} * I_2}{\pi'^{-} * I_1}$$

descends to a surjective map

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'-}} \to \frac{\pi'^{-} * I_2}{\pi'^{-} * I_1}.$$

Note that

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'-}} \cong \bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')).$$

Theorem 5.5 implies that

$$\pi'^{-} * I_2 \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0).$$

Lemma 5.10 and Proposition 5.4 imply that

$$\pi'^{-} * I_1 \cong \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')).$$

Therefore the theorem follows. □

6. Bounding the associated cycles

In this section, let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$. Let $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$, whose Zariski closure in $\mathfrak{g}_{\mathsf{s}}$ is denoted by $\overline{\mathcal{O}}$.

6.1. **Associated cycles.** In this subsection, we will recall the definition of associated cycles from [83].

Definition 6.1. *Let $H_{\mathbb{C}}$ be a complex linear algebraic group. Let $\mathcal{A}$ be a commutative $\mathbb{C}$-algebra carrying a locally algebraic linear action of $H_{\mathbb{C}}$ by algebra automorphisms. An $(\mathcal{A}, H_{\mathbb{C}})$-module is an $\mathcal{A}$-module σ equipped with a locally algebraic linear action of $H_{\mathbb{C}}$ such that the module structure map $\mathcal{A} \otimes \sigma \to \sigma$ is $H_{\mathbb{C}}$-equivariant. An $(\mathcal{A}, H_{\mathbb{C}})$-module is said to be finitely generated if it is so as an $\mathcal{A}$-module.*

For every affine complex algebraic variety Z, write $\mathbb{C}[Z]$ the algebra of regular functions on Z. Given a finitely generated $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module σ°, for each $K_{\mathsf{s},\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$, the set

$$\bigsqcup_{\mathbf{e} \in \mathscr{O}} \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]} \sigma^\circ$$

is naturally a $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$, where $\mathbb{C}_{\mathbf{e}}$ denotes the complex number field $\mathbb{C}$ viewing as a $\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]$-algebra via the evaluation map at $\mathbf{e}$. We define $\mathrm{AC}_{\mathscr{O}}(\sigma^\circ) \in \mathcal{K}_{\mathsf{s}}(\mathscr{O})$ to be the Grothendieck group element associated to this bundle. Define the associated cycle of σ° to be

$$\mathrm{AC}_{\mathcal{O}}(\sigma^\circ) := \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathrm{AC}_{\mathscr{O}}(\sigma^\circ) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

Write $I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}}$ for the radical ideal of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$ corresponding to the closed subvariety $\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}$. Every $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module is clearly an $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module. On the other hand, for each $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module σ,

$$\frac{(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i \cdot \sigma}{(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^{i+1} \cdot \sigma} \qquad (i \in \mathbb{N})$$

is naturally a $(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module. We say that σ is $\mathcal{O}$-bounded if

$$(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i \cdot \sigma = 0 \qquad \text{for some } i \in \mathbb{N}.$$

When σ is finitely generated, this is equivalent to saying that the support of σ is contained in $\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}$.

When σ is finitely generated and $\mathcal{O}$-bounded, we define its associated cycle to be

$$\mathrm{AC}_{\mathcal{O}}(\sigma) := \sum_{i\in\mathbb{N}} \mathrm{AC}_{\mathcal{O}}\left(\frac{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^i \cdot \sigma}{(I_{\overline{\mathcal{O}}\cap\mathfrak{p}_{\mathsf{s}}})^{i+1} \cdot \sigma}\right) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

The assignment $\mathrm{AC}_{\mathcal{O}}$ is additive in the following sense: the equality

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\sigma') + \mathrm{AC}_{\mathcal{O}}(\sigma'')$$

holds for every exact sequence $0 \to \sigma' \to \sigma \to \sigma'' \to 0$ of finitely generated $\mathcal{O}$-bounded $(\mathrm{S}(\mathfrak{p}_{\mathsf{s}}), K_{\mathsf{s},\mathbb{C}})$-modules.

Given a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module ρ of finite length, pick a filtration

$$\mathcal{F}: \quad \cdots \subset \rho_{-1} \subset \rho_0 \subset \rho_1 \subset \rho_2 \subset \cdots \tag{6.1}$$

of ρ that is good in the following sense:

- for each $i \in \mathbb{Z}$, ρ_i is a finite-dimensional K_{s}-stable subspace of ρ;
- $\mathfrak{g}_{\mathsf{s}} \cdot \rho_i \subset \rho_{i+1}$ for all $i \in \mathbb{Z}$, and $\mathfrak{g}_{\mathsf{s}} \cdot \rho_i = \rho_{i+1}$ when $i \in \mathbb{Z}$ is sufficiently large;
- $\bigcup_{i\in\mathbb{Z}} \rho_i = \rho$, and $\rho_i = 0$ for some $i \in \mathbb{Z}$.

Then the grading

$$\mathrm{Gr}(\rho) := \mathrm{Gr}(\rho, \mathcal{F}) := \bigoplus_{i\in\mathbb{Z}} \rho_i/\rho_{i+1}$$

is naturally a finitely generated $(\mathrm{S}(\mathfrak{p}_{\mathsf{s}}), K_{\mathsf{s},\mathbb{C}})$-module, where $\mathrm{S}(\mathfrak{p}_{\mathsf{s}})$ denotes the symmetric algebra. Recall that we have identify $\mathfrak{p}_{\mathsf{s}}$ with its dual space $\mathfrak{p}_{\mathsf{s}}^*$ by using the trace from. Hence $\mathrm{S}(\mathfrak{p}_{\mathsf{s}})$ is identified with $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$, and $\mathrm{Gr}(\rho)$ is a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module.

Recall that ρ is said to be $\mathcal{O}$-bounded if the associated variety of its annihilator ideal in $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})$ is contained in $\overline{\mathcal{O}}$.

Lemma 6.2. *Let ρ be a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module of finite length, and let $\mathcal{F}$ be a good filtration of ρ. Then ρ is $\mathcal{O}$-bounded if and only if $\mathrm{Gr}(\rho, \mathcal{F})$ is $\mathcal{O}$-bounded.*

Proof. This is a direct consequence of [83, Theorem 8.4]. ☐

When ρ is $\mathcal{O}$-bounded, it associated cycle is defined to be

$$\mathrm{AC}_{\mathcal{O}}(\rho) := \mathrm{AC}_{\mathcal{O}}(\mathrm{Gr}(\rho)) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O}).$$

This is independent of the good filtration (6.1). Moreover, taking the associated cycles is additive in the following sense: the equality

$$\mathrm{AC}_{\mathcal{O}}(\rho) = \mathrm{AC}_{\mathcal{O}}(\rho') + \mathrm{AC}_{\mathcal{O}}(\rho'')$$

holds for every exact sequence $0 \to \rho' \to \rho \to \rho'' \to 0$ of $\mathcal{O}$-bounded $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-modules of finite length.

6.2. **Algebraic theta lifts and commutative theta lifts.** We retain the notation of Section 3 and Section 4. Denote by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}}$ the $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$-submodule of $\omega_{\mathsf{s},\mathsf{s}'}$ generated by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$. This is an $(\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}} \times K_{\mathsf{s}'})$-module. For every $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module ρ', define its full theta lift (the algebraic theta lift) to be the $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho') := (\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}} \otimes \rho')_{\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'}} \qquad \text{(the coinvariant space)}.$$

It has finite length whenever ρ' has finite length ([40, Section 4]).

Recall from (3.4) the moment maps

$$\begin{array}{ccccc} \mathfrak{p}_{\mathsf{s}}^* = \mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}^* = \mathfrak{p}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array}$$

We view $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$ as an $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}] \otimes \mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]$-algebra by using the moment maps. The natural action of $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ on $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ yields a locally algebraic linear action of $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ on $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$. Thus $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$ is naturally a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}] \otimes \mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})$-module.

For every $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module σ', define its commutative theta lift to be

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma') := (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(the coinvariant space)},$$

which is naturally a $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module.

Lemma 6.3. *Suppose that σ' is a finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then the $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$ is also finitely generated.*

Proof. This follows from the structural analysis of the space of joint harmonics, as in [40, Section 4]. □

Lemma 6.4. *Suppose that ρ' is a $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then there exist a good filtration $\mathcal{F}'$ on ρ', a good filtration $\mathcal{F}$ on $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')$, and a surjective $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module homomorphism*

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\mathrm{Gr}(\rho', \mathcal{F}')) \to \mathrm{Gr}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho'), \mathcal{F}).$$

Proof. This follows from the discussions in [53, Section 3.2]. □

6.3. **Regular descent of a nilpotent orbit.** Recall from (3.6) the moment maps

$$\begin{array}{ccccc} \mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\!\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array}$$

Lemma 6.5. *The orbit $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$ if and only if*

$$\delta := |\mathsf{s}'| - |\nabla_{\mathrm{naive}}(\mathcal{O})| \geqslant 0.$$

When this is the case, the Young diagram of $\nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ is obtained from that of $\nabla_{\mathrm{naive}}(\mathcal{O})$ by adding δ boxes in the first column.

Proof. This is implied by [25, Theorem 3.6]. □

Definition 6.6. *The orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is regular for $\nabla_{\mathsf{s}'}^{\mathsf{s}}$ if either*

$$|\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|,$$

or

$$|\mathsf{s}'| > |\nabla_{\mathrm{naive}}(\mathcal{O})| \qquad \text{and} \qquad \mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O}).$$

In the rest of this section we assume that $\mathcal{O}$ is regular for $\nabla_{\mathsf{s}'}^{\mathsf{s}}$. Then $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Put $\mathcal{O}' := \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$, and let $\overline{\mathcal{O}'}$ denote the Zariski closure of $\mathcal{O}'$ in $\mathfrak{g}_{\mathsf{s}'}$.

Lemma 6.7. *As subsets of $\mathfrak{g}_{\mathsf{s}}$,*

$$\tilde{M}_{\mathsf{s}}(\tilde{M}_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}.$$

Proof. This is implied by [25, Theorems 5.2 and 5.6]. □

Lemma 6.8. *Suppose that σ' is an $\mathcal{O}'$-bounded $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then the $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$ is $\mathcal{O}$-bounded.*

Proof. Lemma 6.9 implies that

$$M_{\mathsf{s}}^{-1}(\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}) \supset M_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}).$$

Thus

$$(I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i \cdot \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \subset I_{\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}} \cdot \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \qquad \text{for some } i \in \mathbb{N},$$

and the lemma follows. Here $I_{\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}}$ denotes the radical ideal of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$ corresponding to the closed subvariety $\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}$. □

Put

$$\partial := \overline{\mathcal{O}} \backslash \mathcal{O} \qquad \text{and} \qquad \bar{\partial} := \mathfrak{g}_{\mathsf{s}} \backslash \partial.$$

Then $\bar{\partial}$ is an open subvariety of $\mathfrak{g}_{\mathsf{s}}$ and $\mathcal{O}$ is a closed subvariety of $\bar{\partial}$. Put

$$W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} := \tilde{M}_{\mathsf{s}}^{-1}(\bar{\partial}) \qquad \text{and} \qquad W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} := \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap \tilde{M}_{\mathsf{s}'}^{-1}(\mathcal{O}').$$

Then $W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$ is an open subvariety of $W_{\mathsf{s},\mathsf{s}'}$, and the following lemma implies that $W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$ is a closed subvariety of $W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$.

Lemma 6.9. *The set $W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$ is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit. Moreover, it is contained in $W_{\mathsf{s},\mathsf{s}'}^{\circ}$ and equals*

$$W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \cap \tilde{M}_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'}).$$

Proof. The first assertion is implied by [25, Theorem 3.6]. Recall from (3.7) that $W_{\mathsf{s},\mathsf{s}'}^{\circ} \cap \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O})$ is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit whose image under the moment map $\tilde{M}_{\mathsf{s}'}$ equals $\mathcal{O}'$. Thus

$$W_{\mathsf{s},\mathsf{s}'}^{\circ} \cap W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} = W_{\mathsf{s},\mathsf{s}'}^{\circ} \cap \tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}),$$

which is also a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit. Hence $W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \subset W_{\mathsf{s},\mathsf{s}'}^{\circ}$. The last assertion is a direct consequence of Lemma 6.9.

□

Write

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \qquad \text{and} \qquad \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}.$$

Then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$ is an open subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ and $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$ is a closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$. We have a decomposition

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} = \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}} \text{ that is contained in the image of } M_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'},$$

where

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'} := M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'},$$

which is a Zariski open and closed subset of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$. Lemmas 3.2 and 6.9 imply that $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathscr{O},\mathcal{O}'}$ is a single $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-orbit.

We defer the proof of the following proposition to Section 6.6.

Proposition 6.10. *The scheme theoretic fibre product*

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'})$$

is reduced, where $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$ is viewed as a $\mathfrak{p}_{\mathsf{s}'}$-scheme via the moment map $M_{\mathsf{s}'}$.

By Lemma 6.9 and Proposition 6.10, we know that

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} = \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}) = \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}), \tag{6.2}$$

which is a smooth closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}$.

For every element $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$, write $K_{\mathbf{e}}$ for the stabilizer of $\mathbf{e}$ in $K_{\mathsf{s},\mathbb{C}}$. Let $\mathbb{C}_{\mathbf{e}}$ be the field $\mathbb{C}$ viewing a $\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]$-algebra via the evaluation map at $\mathbf{e}$. Put

$$\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'} := M_{\mathsf{s}}^{-1}(\mathbf{e}) \cap M_{\mathsf{s}'}^{-1}(\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}) = M_{\mathsf{s}}^{-1}(\mathbf{e}) \cap M_{\mathsf{s}'}^{-1}(\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}),$$

which is a closed subvariety of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$. If $\mathbf{e}$ is not contained in the image of M_{s}, then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'} = \emptyset$. Otherwise, it is a single $(K_{\mathbf{e}} \times K_{\mathsf{s},\mathbb{C}})$-orbit, and Lemma 3.2 implies that it is also a single $K_{\mathsf{s},\mathbb{C}}$-orbit.

Lemma 6.11. *For every element* $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$*,*

$$\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}] = \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}].$$

Proof. As schemes, we have that

$$\begin{aligned} & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}) \\ = \ & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}} \times_{\mathfrak{p}_{\mathsf{s}'}} (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}) \\ = \ & \{\mathbf{e}\} \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \\ = \ & \{\mathbf{e}\} \times_{\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} ((\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}) \times_{\mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}) \\ = \ & \{\mathbf{e}\} \times_{\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \\ = \ & \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathbf{e},\mathcal{O}'}. \end{aligned}$$

The last equality holds because the morphism $M_{\mathsf{s}} : \mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'} \to \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ is $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-equivariant and hence smooth in the sense of algebraic geometry. This proves the lemma. □

6.4. **Commutative theta lifts and geometric theta lifts.** The purpose of this subsection is to prove the following

Proposition 6.12. *Suppose that* σ' *is a finitely generated* $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$*-module. Then*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')) = \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma'))$$

as elements of $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$*.*

By a quasi-coherent module over a scheme Z, we mean a quasi-coherent module over the structure sheaf of Z. We say that a quasi-coherent module $\mathcal{M}$ over a scheme Z descends to a closed subscheme Z_1 of Z if $\mathcal{M}$ is isomorphic to the push-forward of a quasi-coherent module over Z_1 via the closed embedding $Z_1 \to Z$. This is equivalent to saying that the ideal sheaf defining Z_1 annihilates $\mathcal{M}$.

When Z is an affine complex algebraic variety and σ is a $\mathbb{C}[Z]$-module, we write $\mathcal{M}_\sigma$ for the quasi-coherent module over Z corresponding to σ.

Lemma 6.13. *Let* σ *be a finitely generated* $\mathcal{O}$*-bounded* $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s}})$*-module. Assume that* $(\mathcal{M}_\sigma)|_{\bar{\partial} \cap \mathfrak{p}_{\mathsf{s}}}$ *descends to* $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$*. Then*

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma).$$

Proof. Let $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$. Note that for every ideal I of $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$,

$$(I \cdot \sigma)_{\mathbf{e}} = (I_{\mathbf{e}}) \cdot \sigma_{\mathbf{e}} \subset \sigma_{\mathbf{e}},$$

where a subscript $\mathbf{e}$ indicates the localization at $\mathbf{e}$. The assumption of the lemma implies that

$$((I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i)_{\mathbf{e}} \cdot \sigma_{\mathbf{e}} = 0 \qquad \text{for all } i \in \mathbb{N}^+.$$

Thus

$$((I_{\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}})^i \cdot \sigma)_{\mathbf{e}} = 0,$$

which implies the lemma. □

Lemma 6.14. *Suppose that Z is an affine complex algebraic variety with a transitive algebraic action of $K_{\mathsf{s},\mathbb{C}}$-action. Let σ be a finitely generated $(\mathbb{C}[Z], K_{\mathsf{s},\mathbb{C}})$-module. Let $z \in Z$ and write K_z for the stabilizer of z in $K_{\mathsf{s},\mathbb{C}}$. Then the natural map*

$$\sigma^{K_{\mathsf{s},\mathbb{C}}} \to (\mathbb{C}_z \otimes_{\mathbb{C}[Z]} \sigma)^{K_z}$$

is a linear isomorphism. Here $\mathbb{C}_z$ is the field $\mathbb{C}$ viewing as a $\mathbb{C}[Z]$-algebra via the evaluation map at z, and a superscript group indicates the space of invariant vectors under the group action.

Proof. Note that the set

$$\bigsqcup_{x \in Z} \mathbb{C}_x \otimes_{\mathbb{C}[Z]} \sigma$$

is naturally a $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over Z, and σ is identified with the space of algebraic sections of this bundle. Thus

$$\sigma \cong {}^{\mathrm{alg}}\,\mathrm{Ind}_{K_z}^{K_{\mathsf{s},\mathbb{C}}}(\mathbb{C}_x \otimes_{\mathbb{C}[Z]} \sigma) \qquad \text{(algebraically induced representation)}.$$

The lemma then follows by the algebraic version of the Frobenius reciprocity. □

Lemma 6.15. *Suppose that σ' is a finitely generated $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module, and write $\sigma := \check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$. Then the quasi-coherent module $(\mathcal{M}_\sigma)|_{\bar{\partial} \cap \mathfrak{p}_{\mathsf{s}}}$ descends to $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$.*

Proof. Write

$$\tilde{\sigma} := \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'} = (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]) \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} (\sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}),$$

to be viewed as a $\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}]$-module. Write $\tilde{\sigma}_0 := \sigma$, viewing as a $\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]$-module via the moment map M_{s}.

Proposition 6.10 and the equalities in (6.2) imply that $(\mathcal{M}_{\tilde{\sigma}})|_{\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\bar{\partial}}}$ descends to $\mathcal{X}_{\mathsf{s},\mathsf{s}'}^{\mathcal{O},\mathcal{O}'}$. This implies that $(\mathcal{M}_{\tilde{\sigma}_0})|_{\bar{\partial} \cap \mathfrak{p}_{\mathsf{s}}}$ descends to $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$. The lemma then follows since σ is a direct summand of $\tilde{\sigma}_0$. □

We are now ready to prove Proposition 6.12.

Proof of Proposition 6.12. Let σ' be a finitely generated $(\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module, and write $\sigma := \check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')$. Lemmas 6.15 and 6.13 imply that

$$\mathrm{AC}_{\mathcal{O}}(\sigma) = \mathrm{AC}_{\mathcal{O}}(\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma).$$

Suppose that $\mathbf{e} \in \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$. Then

$$\begin{aligned}
& \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma) \\
= \; & \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma \\
= \; & \mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}',\mathbb{C}}} \\
= \; & \Big((\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \mathbb{C}[\mathcal{X}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}]} \mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]) \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)_{K_{\mathsf{s}',\mathbb{C}}} \\
= \; & \Big(\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)_{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(by Lemma 6.11).}
\end{aligned}$$

If $\mathbf{e}$ is not in the image of M_{s}, then the above module is zero. Now we assume that $\mathbf{e}$ is in the image of M_{s} so that $\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ is a single $K_{\mathsf{s}',\mathbb{C}}$-orbit. Pick an arbitrary element $\phi \in \mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$, and write $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. Let

$$1 \to K_{\mathsf{s}_0,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi \xrightarrow{\text{the projection to the first factor}} (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1$$

be the exact sequence as in Lemma 3.2. Note that $K_{\mathsf{s}_0,\mathbb{C}}$ equals the stabilizer of ϕ in $K_{\mathsf{s}',\mathbb{C}}$.

Let $\mathbb{C}_\phi$ denote the field $\mathbb{C}$ viewing as a $\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}]$-algebra via the evaluation map at ϕ. Then we have that

$$\begin{aligned}
& \Big(\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)_{K_{\mathsf{s}',\mathbb{C}}} \\
= \; & \Big(\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)^{K_{\mathsf{s}',\mathbb{C}}} \qquad \text{(because } K_{\mathsf{s}',\mathbb{C}} \text{ is reductive)} \\
= \; & \Big(\mathbb{C}_\phi \otimes_{\mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}]} \mathbb{C}[\mathcal{X}^{\mathbf{e},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}] \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)^{K_{\mathsf{s}_0,\mathbb{C}}} \qquad \text{(by Lemma 6.14)} \\
= \; & \Big(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)^{K_{\mathsf{s}_0,\mathbb{C}}} \\
= \; & \Big(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)_{K_{\mathsf{s}_0,\mathbb{C}}} \qquad \text{(because } K_{\mathsf{s}_0,\mathbb{C}} \text{ is reductive).}
\end{aligned}$$

Putting all things together, we have an identification

$$\mathbb{C}_{\mathbf{e}} \otimes_{\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}]} (\mathbb{C}[\overline{\mathcal{O}} \cap \mathfrak{p}_{\mathsf{s}}] \otimes_{\mathbb{C}[\mathfrak{p}_{\mathsf{s}}]} \sigma) = \Big(\mathbb{C}_{\mathbf{e}'} \otimes_{\mathbb{C}[\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'}]} \sigma' \otimes \zeta_{\mathsf{s},\mathsf{s}'}\Big)_{K_{\mathsf{s}_0,\mathbb{C}}}.$$

It is routine to check that this identification respects the $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi$-actions, and thus the $K_{\mathbf{e}}$-actions. This proves the proposition. □

6.5. **Algebraic theta lifts and geometric theta lifts.** Proposition 6.12 has the following consequence.

Proposition 6.16. *Suppose that σ' is a finitely generated $\mathcal{O}'$-bounded $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-module. Then*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma')) \preceq \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\sigma'))$$

in $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$.

Proof. Suppose that $0 \to \sigma'_1 \to \sigma'_2 \to \sigma'_3 \to 0$ be an exact sequence of $\mathcal{O}'$-bounded finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}'}], K_{\mathsf{s}',\mathbb{C}})$-modules. Then

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_1) \to \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_2) \to \check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_3) \to 0$$

is an exact sequence of $\mathcal{O}$-bounded finitely generated $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-modules. Thus

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_2)) \preceq \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_1)) + \mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\sigma'_3)).$$

On the other hand, it is clear that

$$\vartheta_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma_2')) = \vartheta_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma_1')) + \vartheta_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma_3')).$$

The proposition then easily follows by Proposition 6.12. □

Finally, we are ready to prove the main result of this section. Recall that $\mathcal{O}$ is assumed to be regular for $\nabla_{\mathsf{s}'}^{\mathsf{s}}$.

Theorem 6.17. *Let ρ' be an $\mathcal{O}'$-bounded $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')$ is $\mathcal{O}$-bounded, and*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')) \preceq \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\rho')).$$

Proof. Let $\mathcal{F}'$ and $\mathcal{F}$ be good filtrations on ρ' and $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')$ respectively as in Lemma 6.4 so that there exists a surjective $(\mathbb{C}[\mathfrak{p}_{\mathsf{s}}], K_{\mathsf{s},\mathbb{C}})$-module homomorphism

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\mathrm{Gr}(\rho', \mathcal{F}')) \to \mathrm{Gr}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho'), \mathcal{F}). \tag{6.3}$$

The first assertion then follows from Lemmas 6.2 and 6.8.

Put $\sigma' := \mathrm{Gr}(\rho', \mathcal{F}')$. Then

$$\begin{array}{rll} & \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho')) & \\ = & \mathrm{AC}_{\mathcal{O}}(\mathrm{Gr}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho'), \mathcal{F})) & \\ \preceq & \mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\sigma')) & \text{(by (6.3))} \\ \preceq & \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\sigma')) & \text{(by Proposition 6.16)} \\ = & \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\rho')). & \end{array}$$

This proves the theorem. □

6.6. **Geometries of regular descent.** In this section, let (V, V') be a complex dual pair. We investigate the geometric properties of regular descent.

Let $\phi \in \tilde{M}_{\mathsf{s}'}^{-1}(\mathcal{O}) \cap W_{\mathsf{s},\mathsf{s}'}^{\circ}$, and $\mathbf{e}' = \tilde{M}_{\mathsf{s}'}(\phi)$. By the Jacobian criterion for regularity (see [52, Theorem 2.19]), the following sequence

$$0 \longrightarrow T_\phi \tilde{M}_{\mathsf{s},\mathsf{s}'}^{-1}(\mathbf{e}') \longrightarrow T_\phi W_{\mathsf{s},\mathsf{s}'} \xrightarrow{\mathrm{d}\tilde{M}_{\mathsf{s}'}} T_{\mathbf{e}'}\mathfrak{g}_{\mathsf{s}} \tag{6.4}$$

is exact if and only if ϕ is regular at $W_{\mathsf{s},\mathsf{s}'\mathbf{e}'}$.

We identify $T_\phi W_{\mathsf{s},\mathsf{s}'}$ and $T_{\mathbf{e}'}\mathfrak{g}_{\mathsf{s}'}$ with $W_{\mathsf{s},\mathsf{s}'}$ and $\mathfrak{g}_{\mathsf{s}'}$ respectively. Since $(G_{\mathsf{s}} \times (G_{\mathsf{s}'})_{\mathbf{e}'}) \cdot \phi$ is open in $\tilde{M}_{\mathsf{s}}^{-1}(\mathbf{e}')$, the map

$$\mathfrak{g}_{\mathsf{s}} \times (\mathfrak{g}_{\mathsf{s}'})_{\mathbf{e}'} \xrightarrow{(A,A')\mapsto A'\phi - \phi A} T_\phi \tilde{M}_{\mathsf{s}'}^{-1}(\mathbf{e}')$$

is surjective, where we identify $T_\phi \tilde{M}_{\mathsf{s}'}^{-1}(\mathbf{e}')$ as a subspace of $W_{\mathsf{s},\mathsf{s}'}$. Then the sequence (6.4) is exact if and only if the sequence (6.5) is exact.

$$\mathfrak{g} \oplus (\mathfrak{g}_{\mathsf{s}})_{\mathbf{e}'} \xrightarrow{(A,A')\mapsto A'\phi - \phi A} W \xrightarrow{b \mapsto \phi b^* + b\phi^*} \mathfrak{g}_{\mathsf{s}}. \tag{6.5}$$

Lemma 6.18. *The sequence* (6.5) *is exact and $W_{\mathbf{e}'}$ is regular at ϕ.*

Proof. Let $V_{\mathsf{s}_0'} := \phi V_{\mathsf{s}}$ and $V_{\mathsf{s}_1'}$ be the orthogonal complement of $V_{\mathsf{s}_0'}$ so that $V_{\mathsf{s}'} = V_{\mathsf{s}_0'} \oplus V_{\mathsf{s}_1'}$.

Suppose $|\mathsf{s}'| = \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O})$. Then every element $\phi \in W_{U,\mathbf{e}'} \subset \mathrm{Hom}(V, V')$ is a surjective morphism and $V_{\mathsf{s}_1'} = 0$. Therefore $\mathrm{d}M_{\mathsf{s}'} : T_\phi W_{\mathsf{s},\mathsf{s}'} \longrightarrow T_{\mathbf{e}'}\mathfrak{g}_{\mathsf{s}'}$ is surjective, which implies $M_{\mathsf{s}'}$ is smooth at ϕ (*cf.* [33, Proposition 10.4]). Pulling back $M_{\mathsf{s}'}$ via the inclusion $\mathbf{e}' \hookrightarrow \mathfrak{g}_{\mathsf{s}}$, we conclude that $W_{\bar{\partial},\mathbf{e}'}$ is smooth and so regular at ϕ. This also establishs the exactness of (6.5).

Now suppose $|\mathsf{s}'| > \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O})$ and $\mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O})$. We now establish the exactness of (6.5), which concludes the lemma.

Let $W_{\mathsf{s},\mathsf{s}'_i} := \mathrm{Hom}(V_{\mathsf{s}}, V_{\mathsf{s}'_i})$ and $\mathfrak{g}_{\mathsf{s}'_i} := \mathfrak{g}_{\mathsf{s}_i}$ for $i = 1, 2$. Let $*\colon \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}) \to \mathrm{Hom}(V_{\mathsf{s}'_1}, V_{\mathsf{s}'_0})$ denote the adjoint map. We have $W_{\mathsf{s},\mathsf{s}'} = W_{\mathsf{s},\mathsf{s}'_0} \oplus W_{\mathsf{s},\mathsf{s}'_1}$ and $\mathfrak{g}_{\mathsf{s}'} = \mathfrak{g}_{\mathsf{s}'_0} \oplus \mathfrak{g}_{\mathsf{s}'_1} \oplus \mathfrak{g}^{\boxslash}_{\mathsf{s}'}$, where

$$\mathfrak{g}^{\boxslash} := \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}) \right\}.$$

Now ϕ and $\mathbf{e}'$ are naturally identified with an element ϕ_0 in $W_{\mathsf{s},\mathsf{s}'_0}$ and an element $\mathbf{e}'_0$ in $\mathfrak{g}_{\mathsf{s}'_0}$ respectively. We have $(\mathfrak{g}_{\mathsf{s}'})_{\mathbf{e}'} = (\mathfrak{g}_{\mathsf{s}'_0})_{\mathbf{e}'_0} \oplus \mathfrak{g}_{\mathsf{s}'_1} \oplus \mathfrak{g}'^{\boxslash}_{\mathbf{e}'_0}$, where

$$\mathfrak{g}^{\boxslash}_{\mathsf{s}'\,\mathbf{e}'_0} = \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}) \text{ and } E\mathbf{e}'_0 = 0 \right\}.$$

Now maps in (6.5) have the following forms respectively:

$$\mathfrak{g}_{\mathsf{s}} \oplus (\mathfrak{g}_{\mathsf{s}'})_{\mathbf{e}'} \ni (A, A') = \left(A, (A'_0, A'_1, \left(\begin{smallmatrix} 0 & -E^* \\ E & 0 \end{smallmatrix}\right))\right) \mapsto A'\phi - \phi A = (A'_0\phi_0 - \phi_0 A, E\phi_0) \in W_{\mathsf{s},\mathsf{s}'}$$

and

$$W_{\mathsf{s},\mathsf{s}'} = \mathrm{Hom}(V_{\mathsf{s}}, V_{\mathsf{s}'_0}) \oplus \mathrm{Hom}(Vs, V_{\mathsf{s}'_1}) \ni (b_0, b_1) \mapsto \left(b_0^*\phi_0 + \phi_0^* b_0, \begin{pmatrix} b_0\phi_0^* + \phi_0 b_0^* & \phi_0 b_1^* \\ b_1\phi_0^* & 0 \end{pmatrix} \right) \in \mathfrak{g}_{\mathsf{s}} \oplus \mathfrak{g}_{\mathsf{s}'}.$$

Note that $G_{\mathsf{s}'_0} \cdot \mathbf{e}'_0 = \mathcal{O}'_0 := \nabla^{\mathsf{s}}_{\mathsf{s}'_0}(\mathcal{O})$, we have the exactness of the sequence

$$\mathfrak{g}_{\mathsf{s}} \oplus (\mathfrak{g}_{\mathsf{s}'_0})_{\mathbf{e}'_0} \longrightarrow W_{\mathsf{s},\mathsf{s}'_0} \longrightarrow \mathfrak{g}_{\mathsf{s}} \oplus \mathfrak{g}_{\mathsf{s}'_0}.$$

It left to show that the following sequence is exact:

$$\mathfrak{g}^{\boxslash}_{\mathsf{s}'\,\mathbf{e}'_0} \xrightarrow{\quad \eta_1 \quad} W_{\mathsf{s},\mathsf{s}'_1} \xrightarrow{\quad \eta_2 \quad} \mathrm{Hom}(V_{\mathsf{s}'_0}, V_{\mathsf{s}'_1}). \tag{6.6}$$

Here η_1 is given by $\left(\begin{smallmatrix} 0 & -E^* \\ E & 0 \end{smallmatrix}\right) \mapsto E\phi_0$ and η_2 is given by $b_2 \mapsto b_2\phi_0^*$. We have

$$\begin{aligned} \dim \mathrm{Im}(\eta_1) &= \dim \mathfrak{g}^{\boxslash}_{\mathsf{s}'\,\phi_0} \qquad (\phi_0 \text{ is surjective.}) \\ &= \dim V_{\mathsf{s}'_1} \cdot \dim(V_{\mathsf{s}'_0}/\,\mathrm{Im}(\mathbf{e}'_0)) \\ &= \dim V_{\mathsf{s}'_1} \cdot \dim \mathrm{Ker}(\mathbf{e}'_0) \\ &= \dim V_{\mathsf{s}'_1} \cdot \dim \mathbf{c}_2(\mathcal{O}) \end{aligned}$$

and

$$\begin{aligned} \dim \mathrm{Ker}(\eta_2) &= \dim V_{\mathsf{s}'_1} \cdot \dim(V_{\mathsf{s}}/\,\mathrm{Im}(\phi_0^*)) \\ &= \dim V_{\mathsf{s}'_1} \cdot (\dim V_{\mathsf{s}} - \dim V_{\mathsf{s}'_0}) \\ &= \dim V_{\mathsf{s}'_1} \cdot \mathbf{c}_1(\mathcal{O}) \\ &= \dim V_{\mathsf{s}'_1} \cdot \mathbf{c}_2(\mathcal{O}). \end{aligned}$$

Now the exactness of (6.6) and (6.5) follow by dimension counting. □

Now take $\phi \in \mathcal{X}_{\mathsf{s},\mathsf{s}'}$. Then the sequence (6.5) is compatible with the $(L_{\mathsf{s}}, L_{\mathsf{s}'})$-actions.

Lemma 6.19. *Suppose $\phi \in \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ and $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. Then following sequence is exact.*

$$\mathfrak{k}_{\mathsf{s}} \oplus (\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \longrightarrow \mathcal{X}_{\mathsf{s},\mathsf{s}'} \longrightarrow \mathfrak{p}_{\mathsf{s}'}. \tag{6.7}$$

In particular, $\mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ is regular at ϕ.

Proof. The exactness of the sequence is clear by taking eign-spaces of (6.6) under the $(L_{\mathsf{s}}, L_{\mathsf{s}'})$-action. Since $K_{\mathsf{s}}K_{\mathsf{s}'} \cdot \phi$ is open in $M_{\mathsf{s}'}^{-1}(\mathbf{e}')$, $\mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ is regular at ϕ by Jacobian criterion. □

We now recall the following lemma in EGA.

Lemma 6.20 ([31, Proposition 11.3.13]). *Suppose $f\colon X \to Y$ is a morphism between schemes, x is a point in X such that f is flat at x. Let $y = f(x)$.*
(a) If X is reduced at the point x, then Y is reduced at y.
(b) We assume f is of finite presentation at the porint x. If Y is reduced at x and x is reduced at the scheme theoretical fiber X_y, then X is reduced at x. □

Proof of Proposition 6.10. Consider the $K_{\mathsf{s}} \times K_{\mathsf{s}'}$-equivariant morphism

$$\mathcal{X}^{\bar{\partial},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}^{\bar{\partial}}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} (\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}) \longrightarrow \mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$$

Since $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$ is a finite union of $K_{\mathsf{s}'}$-orbits, the above morphism is flat by generic flatness [30, Théorème 6.9.1]. By Lemma 6.19, $\mathcal{X}^{\bar{\partial},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ is regular at every point ϕ since it is a open subscheme of $\mathcal{X}_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{p}_{\mathsf{s}'}} \mathbf{e}'$ and $\phi \in \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$. Clearly $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$ is reduced. We conclude that $\mathcal{X}^{\bar{\partial},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$ is reduced by Lemma 6.20. □

Using the same argument we also have the following lemma which we will not use in the paper.

Lemma 6.21. *Suppose $\mathcal{O}$ regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Then*

$$W_{\mathsf{s},\mathsf{s}'} \times_{\mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathsf{s}'}} \left((\mathfrak{g}_{\mathsf{s}} - \partial\mathcal{O}) \times \mathcal{O}'\right)$$

is a reduced scheme. □

Lemma 6.22. *Suppose $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$, $\mathscr{O}$ is a K_{s}-orbit in $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ and $\mathscr{O}' = \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) \subset \mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$. Suppose $\mathcal{E}' \in \mathrm{AOD}_{\mathsf{s}'}(\mathscr{O}')$. Then $\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$ is either zero or in $\mathrm{AOD}_{\mathsf{s}}(\mathscr{O})$.*

Proof. Note that $(\zeta_{\mathsf{s},\mathsf{s}'})^2 = (\bigwedge^{\mathrm{top}} \mathcal{X}_{\mathsf{s},\mathsf{s}'})^{-1}$. The lemma follows from Lemma 6.23 below. □

Lemma 6.23. *Suppose $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ and $\phi \in \mathcal{X}^{\mathcal{O},\mathcal{O}'}_{\mathsf{s},\mathsf{s}'}$. Let $\mathbf{e} = M_{\mathsf{s}}(\phi)$ and $\mathbf{e}' = M_{\mathsf{s}'}(\phi)$. Then*

$$\bigwedge^{\mathrm{top}} (\mathfrak{k}_{\mathsf{s}})_{\mathbf{e}} = \bigwedge^{\mathrm{top}} (\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \otimes \left(\bigwedge^{\mathrm{top}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}\right)^{-1} \tag{6.8}$$

as $(K_{\mathsf{s}} \times K_{\mathsf{s}'})_{\phi}$-module.

Proof. We retain the notation in the proof of Lemma 6.18 abd Lemma 6.19. Let $S_\phi := (K_{\mathsf{s}} \times K_{\mathsf{s}'})_\phi$, $\mathfrak{s}_\phi = \mathrm{Lie}(S_\phi)$, $S_{\phi_0} := (K_{\mathsf{s}} \times K_{\mathsf{s}'_0})_{\phi_0}$ and $\mathfrak{s}_{\phi_0} := \mathrm{Lie}(S_{\phi_0})$. Then $\mathfrak{s}_\phi = \mathfrak{s}_{\phi_0} \oplus \mathfrak{k}_{\mathsf{s}'_1}$ and $\mathfrak{s}_{\phi_0} \xrightarrow{\cong} (\mathfrak{k}_{\mathsf{s}})_{\mathbf{e}}$ as S_ϕ-module.

The sequence (6.7) leads to the following short exact sequence

$$0 \longrightarrow (\mathfrak{k}_{\mathsf{s}} \oplus (\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'})/\mathfrak{s}_\phi \longrightarrow \mathcal{X}_{\mathsf{s},\mathsf{s}'} \longrightarrow \mathfrak{p}_{\mathsf{s}'_0} \oplus \mathfrak{p}^{\boxslash}_{\mathsf{s}'} \longrightarrow 0,$$

where $\mathfrak{p}^{\boxslash}_{\mathsf{s}'} = \mathfrak{g}^{\boxslash}_{\mathsf{s}'} \cap \mathfrak{p}$.

Note that the S_ϕ-modules $\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathsf{s}}$, $\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathsf{s}'_1}$, $\bigwedge^{\mathrm{top}} \mathfrak{p}_{\mathsf{s}'_1}$ and $\bigwedge^{\mathrm{top}} \mathfrak{g}_{\mathsf{s}'}$ are trivial. Now

$$\begin{aligned}
&\bigwedge^{\mathrm{top}} (\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \otimes \left(\bigwedge^{\mathrm{top}} \mathcal{X}_{\mathsf{s},\mathsf{s}'}\right)^{-1} \\
&= \bigwedge^{\mathrm{top}} (\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'} \otimes \left(\bigwedge^{\mathrm{top}} \left((\mathfrak{k}_{\mathsf{s}} \oplus (\mathfrak{k}_{\mathsf{s}'})_{\mathbf{e}'})/\mathfrak{s}_\phi\right) \otimes \bigwedge^{\mathrm{top}} \mathfrak{p}_{\mathsf{s}'_0} \otimes \bigwedge^{\mathrm{top}} \mathfrak{p}^{\boxslash}_{\mathsf{s}'}\right)^{-1} \\
&= \bigwedge^{\mathrm{top}} (\mathfrak{s}_{\phi_0} \oplus \mathfrak{k}_{\mathsf{s}'_1}) \otimes \left(\bigwedge^{\mathrm{top}} (\mathfrak{k}_{\mathsf{s}} \oplus \mathfrak{k}_{\mathsf{s}'} \oplus \mathfrak{p}_{\mathsf{s}'_0} \oplus \mathfrak{p}^{\boxslash}_{\mathsf{s}'})\right)^{-1} \\
&= \bigwedge^{\mathrm{top}} \mathfrak{s}_{\phi_0} \otimes \left(\bigwedge^{\mathrm{top}} \mathfrak{g}_{\mathsf{s}'}\right)^{-1} \\
&\qquad (\text{by } \mathfrak{k}_{\mathsf{s}'} \oplus \mathfrak{p}_{\mathsf{s}'_0} \oplus \mathfrak{p}^{\boxslash}_{\mathsf{s}'} \oplus \mathfrak{p}_{\mathsf{s}'_1} = \mathfrak{g}_{\mathsf{s}'}\ ) \\
&= \bigwedge^{\mathrm{top}} (\mathfrak{k}_{\mathsf{s}})_{\mathbf{e}}.
\end{aligned}$$

This proves the lemma. □

7. Nilpotent orbits: combinatorics

In this section, we describes our combinatorical parameterization of nilpotent orbits and the local systems in details.

7.1. **Young diagrams, signed Young diagrams and admissible orbit data.** The set of $\mathrm{Nil}(\mathfrak{g})$ nilpotent $G_{\mathbb{C}}$-orbits in $\mathfrak{g}$ are parameterized by Young diagrams, see [23, Section 5.1]. The nilpotent obit $G_{\mathbb{C}} \cdot \mathbf{e}$ generated by a nilpotent element $\mathbf{e}$ in $\mathfrak{g}$ is attached to the Young diagram $\mathcal{O}$ such that such that

$$\mathbf{c}_i(\mathcal{O}) = \dim(\mathrm{Ker}(\mathbf{e}^{i+1})/\mathrm{Ker}(\mathbf{e}^i)), \quad \text{for all } i \in \mathbb{N}^+$$

By abuse of notaition, we identify $\mathcal{O}$ with the orbit $G_{\mathbb{C}} \cdot \mathbf{e}$.

Let $\mathfrak{sl}_2(\mathbb{C})$ be the complex Lie algebra consisting of 2×2 complex matrices of trace zero,

$$\mathring{X} := \begin{pmatrix} 1/2 & \sqrt{-1}/2 \\ \sqrt{-\mathbf{1}}/2 & -1/2 \end{pmatrix}, \quad \text{and} \quad \mathring{e} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

By the Jacobson-Morozov theorem, there is a Lie algebra homomorphism

$$\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g} \subset \mathfrak{gl}(V)$$

such that $\varphi(\mathring{X}) = \mathbf{e}$.

Let $\mathrm{St}_i = \mathrm{Sym}^{i-1}(\mathbb{C}^2)$ denote the realization of the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module φ_i on the $i-1$-symmetric power of the standard representation $\mathbb{C}^2$.

As an $\mathfrak{sl}_2(\mathbb{C})$-module via φ, we have

$$V = \bigoplus_{i \in \mathbb{N}^+} V_i \otimes_{\mathbb{C}} \mathrm{St}_i, \tag{7.1}$$

and

$${}^iV := \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathrm{St}_i, V)$$

is the multiplicity space and $\dim {}^iV = \mathbf{c}_i(\mathcal{O}) - \mathbf{c}_{i-1}(\mathcal{O})$. We view $\mathcal{O}$ as the function

$$\mathcal{O}\colon \mathbf{N}^+ \to \mathbb{N} \text{ such that } \mathcal{O}(i) = \dim {}^iV.$$

We equip St_i with a "standard" classical space structure $(\mathrm{St}_i, \langle\, ,\, \rangle_{\mathrm{St}_i}, J_{\mathrm{St}_i}, L_{\mathrm{St}_i})$.Here

- the classical signature of St_i is $(B, (i+1)/2, (i-1)/2)$ if i is odd and $(C, i/2, i/2)$ if i is even,
- J_{St_i} is the complex conjugation on $\mathrm{St}_i = \mathrm{Sym}^{i-1}(\mathbb{C}^2)$,
- $L_{\mathrm{St}_i} = (-1)^{\lfloor \frac{i-1}{2} \rfloor} \mathrm{Sym}^{i-1}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$,
- $\langle\, ,\, \rangle_{\mathrm{St}_i}$ is uniquely determined by J_{St_i} and L_{St_i} up to a positive scalar (We fix a form $\langle\, ,\, \rangle_{\mathrm{St}_i}$ for each i).

We define a equivalent relation on the set $\{ B, C, \widetilde{C}, D, C^*, D^* \}$ with the relation $B \sim D$ and $C \sim \widetilde{C}$.

Let $(V, \langle\, ,\, \rangle J, L)$ be a classical group with signature s. Let $\mathscr{O} \in \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$. By [75] (also see [83, Section 6]), there is a Lie-algebra homomorphism $\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g} \subset \mathfrak{gl}(V)$ such that $\varphi(\mathring{X}) \in \mathscr{O}$ and compatible with the classical space structure on V. More precisely, the multiplicity space ${}^iV$ have a unique classical space structure $({}^iV, \langle\, ,\, \rangle_{{}^iV}, J_i, L_i)$ with classical signature $\mathsf{s}_i = (\star_i, p_i, q_i)$ such that $\star_i \neq \widetilde{C}$ and restricted on the $\mathfrak{sl}_2(\mathbb{C})$-isotypic component ${}^iV \otimes \mathrm{St}_i$,

- $\langle\, ,\, \rangle$ is given by $\langle\, ,\, \rangle_{{}^iV} \otimes \langle\, ,\, \rangle_{\mathrm{St}_i}$,
- J is given by $J_i \otimes J_{\mathrm{St}_i}$,

- L is given by $L_i \otimes L_{\mathrm{St}_i}$,
- the symbol $\star_i$ satisfies the following condition

$$\begin{cases} \star_i \sim \star_{\mathsf{s}} & \text{if } i \text{ is odd} \\ \star_i \sim \star'_{\mathsf{s}} & \text{if } i \text{ is even} \end{cases} \tag{7.2}$$

Here $\star'_{\mathsf{s}}$ is the Howe dual of $\star_{\mathsf{s}}$, and $\sim$ is the equivalent relation on $\{ B, C, \widetilde{C}, C^*, D, D^* \}$ such that $B \sim D$ and $C \sim \widetilde{C}$. Then the classical signature s_i are independent of the choice of φ and uniquely determined the orbit $\mathscr{O}$.

Let

$$\overline{\mathsf{CS}} := \{ (\star, p, q) \mid \star \neq \widetilde{C} \text{ and } (\star, p, q) \text{ is a classical signature} \}.$$

We identify $\mathscr{O}$ with the function

$$\mathscr{O} \colon \mathbb{N}^+ \to \overline{\mathsf{CS}}, \qquad i \mapsto \mathsf{s}_i.$$

A important feature of a compatible φ map is that the Kostant-Sekiguchi correspondence is given by

$$G_{\mathsf{s}} \cdot \varphi(\mathring{e}) \leftrightarrow K_{\mathsf{s},\mathbb{C}} \cdot \varphi(\mathring{X}).$$

Let $\mathsf{m} \colon \overline{\mathsf{CS}} \to \mathbb{N}$ be the map given by $(\star, p, q) \mapsto p + q$. Now the complexification $G_{\mathbb{C}} \cdot \mathscr{O}$ of a rational nilpotent orbit is given by $\mathcal{O} := \mathsf{m} \circ \mathscr{O}$. Define

$$\mathsf{SYD}_\star(\mathcal{O}) := \left\{ \mathscr{O} \colon \mathbb{N}^+ \to \overline{\mathsf{CS}} \;\middle|\; \begin{array}{l} \star_{\mathscr{O}(i)} \text{ satisfies (7.2)}, \\ \mathsf{m} \circ \mathscr{O} = \mathcal{O} \end{array} \right\}$$

$$\mathsf{SYD}_\star := \bigsqcup_{\mathcal{O} \in \mathrm{Nil}_\star} \mathsf{SYD}_\star(\mathcal{O}).$$

Now $\mathsf{SYD}_\star$ is naturally identified with the set $\bigsqcup_{\mathsf{s}} \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ (the set of real nilpotent orbits) and $\bigsqcup_{\mathsf{s}} \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$. This is nothing but the signed Young diagram classification of rational nilpotent orbits, see [23].

For any $\mathscr{O} \in \mathsf{SYD}_\star$, let $\mathrm{Sign}(\mathscr{O}) = (p, q)$ if $\mathscr{F} \circ \mathscr{O}$ corresponds to a nilpotent orbit in $\mathrm{Nil}_{\mathsf{s}}(\mathfrak{p})$ with the classical signature $\mathsf{s} = (\star, p, q)$. When $\mathscr{O}(i) = (\mathsf{s}_i, p_i, q_i)$, the signature is computed by the following formula.

$$\begin{aligned} \mathrm{Sign}(\mathscr{O}) = & \sum_{i \in \mathbb{N}^+} \big(i \cdot p_{2i} + i \cdot q_{2i}, i \cdot p_{2i} + i \cdot q_{2i} \big) \\ & + \sum_{i \in \mathbb{N}^+} \big(i \cdot p_{2i-1} + (i-1) \cdot q_{2i-1}, (i-1) \cdot p_{2i-1} + i \cdot q_{2i-1} \big). \end{aligned}$$

We write

$$\mathsf{SYD}_{\mathsf{s}}(\mathcal{O}) = \{ \mathscr{O} \in \mathsf{SYD}_\star(\mathcal{O}) \mid \mathrm{Sign}(\mathscr{O}) = (p, q) \}.$$

which is identified with $\mathrm{Nil}_{\mathsf{s}}(\mathcal{O})$.

For a classical signature s, we define the complex group

$$K_{[\mathsf{s}],\mathbb{C}} := G_{\mathbb{C}}^{L_{\mathsf{s}}}.$$

Suppose $\mathscr{O} \in \mathrm{Nil}(\mathbf{p}_{\mathsf{s}})$, φ is compatible with $\mathscr{O}$ and $\mathbf{e} = \varphi(\mathring{X}) \in \mathscr{O}$. Let

$$K_{[\mathsf{s}],\mathbb{C}_{\mathbf{e}}} := \mathrm{Stab}_{K_{[\mathsf{s}],\mathbb{C}}}(\mathbf{e})$$

be the stabilizer of $\mathbf{e}$ in $K_{[\mathsf{s}],\mathbb{C}}$. Using the signed Young diagram classification, we have the following well known fact of the isotropic subgroup.

Lemma 7.1. *Then $K_{[\mathsf{s}],\mathbb{C}_{\mathbf{e}}} = R_{\mathbf{e}} \ltimes U_{\mathbf{e}}$, where $U_{\mathbf{e}}$ is the unipotent radical of $K_{[\mathsf{s}],\mathbb{C}_{\mathbf{e}}}$ and $R_{\mathbf{e}}$ is a reductive group canonically identified with*

$$\prod_{i \in \mathbb{N}^+} K_{[\mathsf{s}_{\mathscr{O}(i)}],\mathbb{C}}.$$

As a consequence, $R_{\mathbf{e}}$ is a products of complex general linear group and orthogonal groups

$$\prod_{\substack{\mathscr{O}(i)=(\star_i,p_i,q_i)\\ \star_i\in\{B,D\},p_i+q_i>0}} \mathrm{O}_{p_i}(\mathbb{C})\times\mathrm{O}_{q_i}(\mathbb{C}).$$

Fix a one-dimensional character χ of the connected component ${K_{\mathbf{e},\mathbb{C}}}^\circ$ of $K_{\mathbf{e},\mathbb{C}}$, let

$$\mathsf{LB}_\chi(\mathscr{O}) := \{\, \text{line bundle } \mathcal{E} \text{ on } \mathscr{O} \text{ such that } {K_{\mathbf{e},\mathbb{C}}}^\circ \text{ acts on the fiber } \mathcal{E}_x \text{ by } \chi \,\}.$$

By the structure of $R_{\mathbf{e}}$, a line bundle $\mathcal{E}\in\mathsf{LB}_\chi(\mathscr{O})$ is completely determined by the restruction of the $K_{\mathbf{e},\mathbb{C}}$-module $\mathcal{E}_{\mathbf{e}}$ on the the orthogonal factors of $R_{\mathbf{e}}$.

Let

$$\mathsf{MK} = \overline{\mathsf{CS}} \cup \{\, (\star,r,s) \mid \star\in\{B,D\}, r,s\in\mathbb{Z}\,\}.$$

We identify $\mathcal{E}\in\mathsf{LB}_\chi(\mathscr{O})$ with the map

$$\mathcal{E}\colon \mathbb{N}^+\to\mathsf{MK}$$

such that, for $\mathscr{O}(i)=(\star_i,p_iq_i)$,

- $\mathcal{E}(i) := \mathscr{O}(i)$ if $\star_i\notin\{B,D\}$;
- $\mathcal{E}(i) := (\star_i,(-1)^{\epsilon_i^+}p_i,(-1)^{\epsilon_i^-}q_i)$ if $\star_i\in\{B,D\}$, and the factor $K_{[\mathcal{E}(i)],\mathbb{C}}=\mathrm{O}_{p_i}(\mathbb{C})\times\mathrm{O}_{q_i}(\mathbb{C})$ acts by $\det^{\epsilon_i^+}\otimes\det^{\epsilon_i^-}$ on

$$\begin{cases}\mathcal{E}, & \text{when } \star=D,\\ \mathcal{E}\otimes\det^{-1}_{\sqrt{-1}}, & \text{when } \star=B.\end{cases}$$

.

Let $\mathscr{F}\colon\mathsf{MK}\to\overline{\mathsf{CS}}$ be the map given by $(\star,r,s)\mapsto(\star,|r|,|s|)$ and

$$\mathsf{MYD}(\mathscr{O}) := \left\{\, \mathcal{E}\colon\mathbb{N}^+\to\mathsf{MK} \;\middle|\; \mathscr{F}\circ\mathcal{E}=\mathscr{O} \,\right\}.$$

Thanks to the simple structure of $R_{\mathbf{e}}$, the above recipe allows us to identify $\mathsf{MYD}(\mathscr{O})$ with the set $\mathsf{LB}_\chi(\mathscr{O})$ when the later set is non-empty.

We wreite $\mathrm{Sign}(\mathcal{E}) := \mathrm{Sign}$

Let

$$\mathsf{MYD}_{\mathsf{s}}(\mathcal{O}) := \bigcup_{\mathscr{O}\in\mathrm{Nil}_{\mathsf{s}}(\mathcal{O})}\mathsf{MYD}(\mathscr{O}), \quad\text{and}\quad \mathsf{MYD}_\star(\mathcal{O}) := \bigcup_{\mathscr{O}\in\mathrm{Nil}_\star(\mathcal{O})}\mathsf{MYD}(\mathscr{O})$$

In particular, we identify $\mathrm{AOD}_{\mathsf{s}}(\mathcal{O})$ with a subset of $\mathsf{MYD}_\star(\mathcal{O})$, see Definition 1.9.

Set

$$\mathsf{MYD}_\star := \bigcup_{\mathcal{O}\in\mathrm{Nil}_\star}\mathsf{MYD}_\star(\mathcal{O}),$$

In the next section, we will define some operations on the free abelian groups $\mathbb{Z}[\mathsf{SYD}_\star]$ and $\mathbb{Z}[\mathsf{MYD}_\star]$.

Suppose $\mathcal{E}\in\mathsf{MYD}_\star$ such that $\mathcal{E}(i)=(\mathsf{s}_i,p_i,q_j)$. To ease the notation, we will use

$$(i_1^{(p_{i_1},q_{i_1})}i_2^{(p_{i_2},q_{i_2})}\cdots i_k^{(p_{i_k},q_{i_k})})_\star$$

to represent $\mathcal{E}$ where $i_1,\cdots,i_k$ are all the indices such that $(p_i,q_i)\neq(0,0)$. A Young diagram in $\mathsf{YD}_\star$ is represented similarly.

We adopt the usual convention to draw a singed Young diagram where the signature (p_i,q_i) equals to the total signature for the most right parts of all length i rows. We use "$\times$" and "$\diagup$" instants "+" and "−" to mark the rows when p_i or q_i are negative.

Example 7.2. *The expression* $(1^{(r,s)})_\star$ *represents the marked Young diagram in* $\mathsf{MYD}_\star$ *such that*

$$(1^{(r,s)})_\star(i) := \begin{cases} (\star, r, s) & \text{if } i = 1 \\ (\star_i, 0, 0) & \text{if } i > 0 \end{cases}$$

Here the symbols $\star_i$ *are uniquely determined by* (7.2).

The marked Young diagram $\mathcal{E} := (1^{(1,-1)}2^{(2,0)}3^{(0,-1)})_B$ *is drew as the following.*

$$\begin{array}{|c|c|c|} \hline / & \times & / \\ \hline - & + & \multicolumn{1}{c}{} \\ \cline{1-2} - & + & \multicolumn{1}{c}{} \\ \cline{1-2} + & \multicolumn{1}{c}{} & \\ \cline{1-1} / & \multicolumn{1}{c}{} & \\ \cline{1-1} \end{array}$$

The singed Young diagram $\mathscr{O} := \mathscr{F} \circ \mathcal{E}$ *is* $(1^{(1,1)}2^{(2,0)}3^{(0,1)})_B$ *and the complexification of* $\mathscr{O}$ *is* $(1^2\, 2^2\, 3^1)_B$.

7.2. **Operations on diagrams.** In this section, we define some operations on the signed Young diagrams and marked Young diagram.

Suppose $\star \in \{B, D\}$. We define a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on $\mathsf{MYD}_\star$. Let $(\epsilon^+, \epsilon^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathcal{E} \in \mathsf{MYD}_\star$ such that $(\star_i, p_i, q_i) := \mathcal{E}(i)$, we define $\mathcal{E} \otimes (\epsilon^+, \epsilon^-) \in \mathsf{MYD}_\star$ by

$$(\mathcal{E} \otimes (\epsilon^+, \epsilon^-))(i) := \begin{cases} (\star_i, (-1)^{\frac{\epsilon^+(i+1)+\epsilon^-(i-1)}{2}} p_i, (-1)^{\frac{\epsilon^+(i-1)+\epsilon^-(i+1)}{2}} q_i) & \text{if } i \text{ is odd,} \\ (\star_i, p_i, q_i) & \text{otherwise.} \end{cases}$$

For each $\mathcal{E} \in \mathrm{AOD}_\star$, its twist $\mathcal{E} \otimes (1^{-,+})^{\epsilon^+} \otimes (1^{+,1})^{\epsilon^-}$ is also in $\mathrm{AOD}_\star$. As elements in $\mathsf{MYD}_\star$, we have

$$\mathcal{E} \otimes (1^{-,+})^{\epsilon^+} \otimes (1^{+,-})^{\epsilon^-} = \mathcal{E} \otimes (\epsilon^+, \epsilon^-).$$

Suppose $\star \in \{C, \widetilde{C}, D^*\}$.

For $\mathcal{E} \in \mathsf{MYD}_\star$ such that $(\star_i, p_i, q_i) := \mathcal{E}(i)$, we define $\maltese\mathcal{E} \in \mathsf{MYD}_\star$ by

$$(\maltese\mathcal{E})(i) := \begin{cases} (\star_i, -p_i, -q_i) & \text{if } i \equiv 2 \pmod 4 \text{ and } \star_i \in \{B, D\}, \\ (\star_i, p_i, q_i) & \text{otherwise.} \end{cases}$$

Clearly, the a operation $\maltese$ defines an involution on $\mathsf{MYD}_\star$. For each $\mathcal{E} \in \mathsf{LB}_\chi(\mathscr{O})$, its twist $\mathcal{E} \otimes \det^{-1}_{\sqrt{-1}}$ is a line bundle in $\mathsf{LB}_{\chi'}(\mathscr{O})$ where χ' is a certain character. As elements in $\mathsf{MYD}_\star$, we have

$$\mathcal{E} \otimes \det_{\sqrt{-1}} = \maltese(\mathcal{E}).$$

We define the argumentation of a marked Young diagram. Suppose that $(p_0, q_0) \in \mathbb{Z} \times \mathbb{Z}$ and there is a $\star'_0 \sim \star'$ such that $\mathsf{s}_0 := (\star'_0, p_0, q_0) \in \overline{\mathsf{CS}}$. with $\star'_0 \sim \star'$. For each $\mathcal{E} \in \mathsf{MYD}_\star$, let $\mathcal{E} \cdot (p_0, q_0)$ be the marked Young diagram in $\mathsf{MYD}_{\star'}$ given by

$$(\mathcal{E} \cdot (p_0, q_0))(i) := \begin{cases} \mathsf{s}_0 & \text{if } i = 1, \\ \mathcal{E}(i-1) & \text{if } i > 1. \end{cases}$$

We will use the same notation to denote the natural extension of above operations to $\mathbb{Z}[\mathsf{MYD}_\star]$.

We define the partial order $\geq$ on $\mathbb{Z} \times \mathbb{Z}$ by

$$(a, b) \geq (c, d) \Leftrightarrow \begin{cases} a \geqslant c \geqslant 0 \\ b \geqslant d \geqslant 0 \end{cases} \quad \text{or} \quad \begin{cases} a \leqslant c \leqslant 0 \\ b \leqslant d \leqslant 0 \end{cases}.$$

For a $\mathcal{E} \in \mathsf{MYD}$ and $(p_0, q_0) \in \mathbb{Z} \times \mathbb{Z}$, we write

$$\mathcal{E} \sqsupseteq (p_0, q_0) \Leftrightarrow (p_1, q_1) \succeq (p_0, q_0) \text{ where } \mathcal{E}(1) = (\star_1, p_1, q_1).$$

Morever, we define an involution on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b) \mapsto (a, b)^- := (b, a)$.

Suppose $\mathcal{E} \sqsupseteq (p_0, q_0)$, we define the truncation of $\mathcal{E}$ by

$$\Lambda_{(p_0,q_0)}(\mathcal{E}) = \begin{cases} (\star_1, p_1 - p_0, q_1 - q_0) & \text{if } i = 1, \\ \mathcal{E}(i) & \text{if } i > 1. \end{cases}$$

We extend the truncation operation $\Lambda_{(p_0,q_0)}$ to $\mathbb{Z}[\mathsf{MYD}_\star]$ by setting $\Lambda_{p_0,q_0}(\mathcal{E}) := 0$ when $(p_0, q_0) \not\sqsubseteq \mathcal{E} \in \mathsf{MYD}_\star$. Note that $\Lambda_{(0,0)}$ is the identity map.

The augmentation and truncation of signed Young diagram are defined by the same formula.

7.3. **Theta lifts of local systems.** Let $\mathcal{O} \in \mathrm{Nil}_\star$ and $\mathcal{O}' \in \mathrm{Nil}_{\star'}$ such that $\mathcal{O}'$ is a regular descent of $\mathcal{O}'$. Let s and s' be two fixed classical signature, the theta lifting of marked Young diagrams is a homomorphism between

$$\check{\vartheta}^{\mathsf{s}}_{\mathsf{s}'} \colon \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}'}(\mathcal{O}')] \to \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}}(\mathcal{O})]$$

It suffice to define $\check{\vartheta}^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{E}')$ for each $\mathcal{E}' \in \mathsf{MYD}_{\mathsf{s}'}$ case by case. We set

$$t := \mathbf{c}_1(\mathcal{O}') - \mathbf{c}_2(\mathcal{O}).$$

Suppose $\mathsf{s}' = (\star', n, n)$ and $\mathsf{s} = (\star, p, q)$ such that $\star' \in \{ C, \widetilde{C}, D^* \}$ and $\star \in \{ D, B, C^* \}$. In this case, t is always an even integer. We define

$$\check{\vartheta}^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{E}') := \begin{cases} \left(\maltese^{\frac{p-q}{2}}(\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}'))\right) \cdot (p_0, q_0) & \text{if } (p_0, q_0) \succeq (0, 0) \text{ and } \star = D, \\ \left(\maltese^{\frac{p-q+1}{2}}(\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}'))\right) \cdot (p_0, q_0) & \text{if } (p_0, q_0) \succeq (0, 0) \text{ and } \star = B, \\ \left(\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}')\right) \cdot (p_0, q_0) & \text{if } (p_0, q_0) \succeq (0, 0) \text{ and } \star = C^*, \\ 0 & \text{otherwise} \end{cases} \tag{7.3}$$

where $(p_0, q_0) := (p, q) - (n, n) - {}^l\mathrm{Sign}(\mathcal{E}')^\sharp + (\frac{t}{2}, \frac{t}{2})$.

Suppose $\mathsf{s}' = (\star', p, q)$ and $\mathsf{s} = (\star, n, n)$ such that $\star' \in \{ D, B, C^* \}$ and $\star \in \{ C, \widetilde{C}, D^* \}$. We define

$$\check{\vartheta}^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{E}') := \begin{cases} \maltese^{\frac{p-q}{2}}\left(\sum_{j=0}^{t} \Lambda_{(j,t-j)}(\mathcal{E}') \cdot (n_0, n_0)\right) & \text{if } \star = C, \\ \maltese^{\frac{p-q-1}{2}}\left(\sum_{j=0}^{t} \Lambda_{(j,t-j)}(\mathcal{E}') \cdot (n_0, n_0)\right) & \text{if } \star = \widetilde{C}, \\ \sum_{j=0}^{\frac{t}{2}} \Lambda_{(2j,t-2j)}(\mathcal{E}') \cdot (n_0, n_0) & \text{if } \star = D^*, \end{cases} \tag{7.4}$$

where $2n_0 = \mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O})$.

7.4. **Induced orbits and double geometric lifts.** In this subsection, we provide various formulas for the induced orbits and double geometric theta lifts.

Suppose $\mathsf{s}' = (\star', p', q')$. In this sectoin, we let $\mathcal{O}' \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$. Fix $l \in \mathbb{N}$ such that $l \geqslant \mathbf{c}_2(\mathcal{O}')$.

It is elementary to check the following lemma on the orbit induction.

Lemma 7.3. *Let $\mathsf{s}'' = (\star', p' + l, q' + l)$ and $t = l - \mathbf{c}_1(\mathcal{O}')$. Let $\mathcal{O}'' := \mathrm{Ind}^{G_{\mathsf{s}'',\mathbb{C}}}_{G_{\mathsf{s}',\mathbb{C}}}(\mathcal{O}')$ and . $\mathscr{O}' \in \mathrm{Nil}_{\mathsf{s}'}(\mathcal{O}')$.*

(a) *Suppose* $\star' \in \{B, D\}$. *Then* $\mathcal{O}''$ *satisfies* $\mathbf{c}_i(\mathcal{O}'') = \mathbf{c}_{i-2}(\mathcal{O}')$ *for* $i \geqslant 4$ *and*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (l, l, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0 \text{ is even,} \\ (l+1, l-1, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0 \text{ is odd,} \\ (\mathbf{c}_1(\mathcal{O}'), l, l) & \text{if } t < 0. \end{cases}$$

We have

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \mathscr{O}' \cdot (\frac{t}{2}, \frac{t}{2}) \cdot (0,0) & \text{if } t \geqslant 0 \text{ is even,} \\ 2\left(\mathscr{O}' \cdot (\frac{t-1}{2}, \frac{t-1}{2}) \cdot (1,1)\right) & \text{if } t \geqslant 0 \text{ is odd,} \\ \sum_{p_0+q_0=-t} (\Lambda_{(p_0,q_0)} \mathscr{O}') \cdot (0,0) \cdot (p_0, q_0) & \text{if } t < 0. \end{cases}$$

(b) *Suppose* $\star' \in \{C, \widetilde{C}\}$. *Then* $\mathcal{O}''$ *satisfies* $\mathbf{c}_i(\mathcal{O}'') = \mathbf{c}_{i-2}(\mathcal{O}')$ *for* $i \geqslant 4$ *and*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (l, l, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0, \\ (\mathbf{c}_1(\mathcal{O}'), l, l) & \text{if } t < 0 \text{ is even,} \\ (\mathbf{c}_1(\mathcal{O}'), l+1, l-1) & \text{if } t < 0 \text{ is odd.} \end{cases}$$

We have

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \sum_{t=p_0+q_0} \mathscr{O}' \cdot (p_0, q_0) \cdot (0,0) & \text{if } t \geqslant 0, \\ (\Lambda_{(-\frac{t}{2}, -\frac{t}{2})} \mathscr{O}') \cdot (0,0) \cdot (-\frac{t}{2}, -\frac{t}{2}) & \text{if } t < 0 \text{ is even,} \\ \sum_{p_0+q_0=2} (\Lambda_{(-\frac{t-1}{2}, -\frac{t-1}{2})} \mathscr{O}') \cdot (p_0, q_0) \cdot (\frac{1-t}{2}, \frac{1-t}{2}) & \text{if } t < 0 \text{ is odd.} \end{cases}$$

(c) *Suppose* $\star' \in \{C^*, D^*\}$. *Note that* l *and* k *are even integers in this case. Then* $\mathcal{O}''$ *satisfies* $\mathbf{c}_i(\mathcal{O}'') = \mathbf{c}_{i-2}(\mathcal{O}')$ *for* $i \geqslant 4$ *and*

$$(\mathbf{c}_1(\mathcal{O}''), \mathbf{c}_2(\mathcal{O}''), \mathbf{c}_3(\mathcal{O}'')) = \begin{cases} (l, l, \mathbf{c}_1(\mathcal{O}')) & \text{if } t \geqslant 0, \\ (\mathbf{c}_1(\mathcal{O}'), l, l) & \text{if } t < 0. \end{cases}$$

When $\star' = D^*$, *we have*

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \sum_{p_0+q_0=\frac{t}{2}} \mathscr{O}' \cdot (2p_0, 2q_0) \cdot (0,0) & \text{if } t \geqslant 0, \\ \left(\Lambda_{(-\frac{t}{2}, -\frac{t}{2})} \mathscr{O}'\right) \cdot (0,0) \cdot (-\frac{t}{2}, -\frac{t}{2}) & \text{if } t < 0. \end{cases}$$

When $\star' = C^*$, *we have*

$$\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{O}') = \begin{cases} \mathscr{O}' \cdot (\frac{t}{2}, \frac{t}{2}) \cdot (0,0) & \text{if } t \geqslant 0, \\ \sum_{p_0+q_0=-\frac{t}{2}} \left(\Lambda_{(2p_0, 2q_0)} \mathscr{O}'\right) \cdot (0,0) \cdot (2p_0, 2q_0) & \text{if } t < 0. \end{cases}$$

□

Retain the notation in Lemma 7.3. Let

$$\mathscr{F} : \bigsqcup_{\mathsf{s}, \mathcal{O}} \mathcal{K}_{\mathsf{s}}(\mathcal{O}) \longrightarrow \bigsqcup_{\mathsf{s}, \mathcal{O}} \mathbb{Z}[\mathrm{Nil}_{\mathsf{s}}(\mathcal{O})]$$

be the map defined by $\mathcal{E} \mapsto \dim(\mathcal{E}_{\mathbf{e}''}) \mathscr{O}''$ where $\mathbf{e}'' \in \mathscr{O}'' \subset \mathcal{O}'' \cap \mathfrak{p}_{\mathsf{s}''}$ and $\mathcal{E} \in \mathcal{K}_{\mathsf{s}''}(\mathcal{O}'')$.

Lemma 7.4. *Let* $\mathcal{E}' \in \mathcal{K}_{\mathsf{s}'}(\mathcal{O})$.

(a) *Suppose* $\star' \in \{B, D\}$ *and* $2n = |\nabla_{\mathrm{naive}}(\mathcal{O}'')|$. *Let* $\mathsf{s} = (\star, n, n)$ *and*

$$\check{\vartheta}^2(\mathcal{E}') := \check{\vartheta}_{\mathsf{s}}^{\mathsf{s}''}(\check{\vartheta}_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{E}')) + \check{\vartheta}_{\mathsf{s}}^{\mathsf{s}''}(\check{\vartheta}_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{E}' \otimes \det)) \otimes \det.$$

Then

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = \begin{cases} 2\,\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')) & \text{if } t \geqslant 0 \text{ is even,} \\ \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')) & \text{if } t \geqslant -1 \text{ is odd.} \end{cases}$$

When $t < -1$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) \preceq \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality can be strict in general.*

(b) *Suppose* $\star' \in \{C, \widetilde{C}\}$. *Let* $d \in \mathbb{N}$ *and* $m = |\nabla_{\mathrm{naive}}(\mathcal{O}'')| + d$. *When* $t \geqslant 0$ *or* $(-t, d) \in (2\mathbb{N}) \times \{0\}$, *define*

$$\check{\vartheta}^2(\mathcal{E}') := \sum_{\substack{\mathsf{s}=(\star,p,q)\\ p+q=m}} \check{\vartheta}_{\mathsf{s}}^{\mathsf{s}''}(\check{\vartheta}_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{E}')).$$

When $t \geqslant 0$ *or* $(t, d) = (-2, 0)$,

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = (d+1)\,\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')).$$

When $t < -2$ *is even and* $d = 0$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) \preceq \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality can be strict in general*

(c) *Suppose* $\star' = C^*$ *and* $2n = |\nabla_{\mathrm{naive}}(\mathcal{O}'')|$. *Let* $\mathsf{s} = (\star, n, n)$ *and*

$$\check{\vartheta}^2(\mathcal{E}') := \check{\vartheta}_{\mathsf{s}}^{\mathsf{s}''}(\check{\vartheta}_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{E}')).$$

When $t \geqslant -2$,

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')).$$

When $t < -2$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) \preceq \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality is strict in general.*

(d) *Suppose* $\star' = D^*$. *Let* $d \in \mathbb{N}$ *and* $2m = |\nabla_{\mathrm{naive}}(\mathcal{O}'')| + 2d$. *When* $t \geqslant 0$ *or* $(-t, d) \in (2\mathbb{N}) \times \{0\}$, *define*

$$\check{\vartheta}^2(\mathcal{E}') := \sum_{\substack{\mathsf{s}=(\star,2p,2q)\\ 2p+2q=2m}} \check{\vartheta}_{\mathsf{s}}^{\mathsf{s}''}(\check{\vartheta}_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{E}')).$$

When $t \geqslant 0$,

$$\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = (d+1)\,\mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}')).$$

When $t < 0$ *and* $d = 0$, $\mathscr{F}(\check{\vartheta}^2(\mathcal{E}')) = \mathrm{Ind}_{G_{\mathsf{s}'}}^{G_{\mathsf{s}''}}(\mathscr{F}(\mathcal{E}'))$ *and the inequality can be strict in general.* □

8. Properties of painted bipartitions

8.1. **Vanishing propotition.** Because of the following proposition, we assume in the rest of this paper that $\check{\mathcal{O}}$ is quasi-distinghuished when $\star \in \{C^*, D^*\}$.

Proposition 8.1. *Suppose that* $\star \in \{C^*, D^*\}$. *If the set* $\mathrm{PBP}_\star(\check{\mathcal{O}})$ *is nonempty, then* $\check{\mathcal{O}}$ *is quasi-distinguished.*

Proof. Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. If $\star = C^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_i(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots.$$

This forces that $\check{\mathcal{O}}$ is quasi-distinguished.

If $\star = D^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_{i+1}(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots.$$

This also forces that $\check{\mathcal{O}}$ is quasi-distinguished. □

8.2. **Tails of painted bipartitions.** In the rest of this subsection, we assume that $\star \in \{B, D, C^*\}$. Let $(\imath, \jmath) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}}))$. Put

$$\star_{\mathbf{t}} := \begin{cases} D, & \text{if } \star \in \{B, D\}; \\ C^*, & \text{if } \star = C^*. \end{cases} \quad \text{and} \quad k := \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}})-\mathbf{r}_2(\check{\mathcal{O}})}{2} + 1 & \text{if } \star \in \{B, D\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})-\mathbf{r}_2(\check{\mathcal{O}})}{2} - 1 & \text{if } \star = C^*. \end{cases}$$

Here $k = \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, $\mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath)$, and $\mathbf{c}_1(\imath) - \mathbf{c}_1(\jmath)$ when $\star = B, C^*, D^*$, respectively.

Let $\check{\mathcal{O}}_{\mathbf{t}}$ be the following Young diagram that is determined by the pair $(\star, \check{\mathcal{O}})$.

- If $\star \in \{ B, D \}$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of two rows with lengths $2k - 1$ and 1.
- If $\star = C^*$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of one row with length $2k + 1$.

Note that in all these three cases $\check{\mathcal{O}}_{\mathbf{t}}$ has $\star_{\mathbf{t}}$-good parity and every element in $\mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ has the form

$$(8.1) \qquad \begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \varnothing \times D, \qquad \begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \varnothing \times D \qquad \text{or} \qquad \varnothing \times \begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times C^*,$$

respectively if $\star = B, D$ or C^*.

Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ be as before.

The case when $\star = B$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the first painted bipartition in (8.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{Q}(j, 1) \mid \mathbf{c}_1(\imath) + 1 \leqslant j \leqslant \mathbf{c}_1(\jmath) \}$$

with the set

$$\begin{cases} \{ c \}, & \text{if } \alpha = B^+, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ s \}, & \text{if } \alpha = B^-, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ \mathcal{Q}(\mathbf{c}_1(\imath), 1) \}, & \text{if } \mathbf{c}_1(\imath) > 0 \text{ and } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{r, d\}. \end{cases}$$

The case when $\star = D$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the second painted bipartition in (8.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{P}(j, 1) \mid \mathbf{c}_1(\jmath) + 2 \leqslant j \leqslant \mathbf{c}_1(\imath) \}$$

with the set

$$\begin{cases} \{ c \}, & \begin{array}{l} \text{if } \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}), \\ \quad (\mathcal{P}(\mathbf{c}_1(\jmath) + 1, 1), \mathcal{P}(\mathbf{c}_1(\jmath) + 1, 2)) = (r, c) \\ \quad \text{and } \mathcal{P}(l, 1) \in \{ r, d \}; \end{array} \\ \{ \mathcal{P}(\mathbf{c}_1(\jmath) + 1, 1) \}, & \text{otherwise.} \end{cases}$$

The case $\star = C^*$. When $k = 0$, we define the tail $\tau_{\mathbf{t}}$ of τ to be $\varnothing \times \varnothing \times C^*$. When $k > 0$, we define the tail $\tau_{\mathbf{t}}$ of τ to be the third painted bipartition in (8.1) such that

$$(x_1, x_2, \cdots, x_k) = (\mathcal{Q}(\mathbf{c}_1(\imath) + 1, 1), \mathcal{Q}(\mathbf{c}_1(\imath) + 2, 1), \cdots, \mathcal{Q}(\mathbf{c}_1(\jmath), 1)).$$

When $\star \in \{ B, D \}$, the symbol in the last box of the tail $\tau_{\mathbf{t}} \in \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ will be impotent for us. We write x_τ for it, namely

$$x_\tau := \mathcal{P}_{\tau_{\mathbf{t}}}(k, 1).$$

The following lemma is easy to check.

Lemma 8.2. *If $\star = B$, then*

$$x_\tau = s \Longleftrightarrow \begin{cases} \alpha = B^-; \\ \mathcal{Q}(\mathbf{c}_1(\jmath), 1) \in \{\bullet, s\}, \end{cases}$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{Q}(\mathbf{c}_1(\jmath), 1) = d.$$

If $\star = D$, then

$$x_\tau = s \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath), 1) = s,$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath), 1) = d.$$

□

8.3. **Some properties of the descent maps.** The key properties of the descent map when $\star \in \{ C, \widetilde{C}, D^* \}$ are summarized in the following proposition.

Proposition 8.3. *Suppose that $\star \in \{ C, \widetilde{C}, D^* \}$ and cosider the descent map*

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}'). \tag{8.2}$$

(a) If $\star = D^*$ or $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$, then the map (8.2) is bijective.
(b) If $\star \in \{C, \widetilde{C}\}$ and $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$, then the map (8.2) is injective and its image equals

$$\left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, x_{\tau'} \neq s \right\}.$$

Proof. We give the detailed proof of part (b) when $\star = \widetilde{C}$. The proofs in the other cases are similar and are left to the reader.

By the definition of descent map (see (2.1)), we have a well defined map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\}. \tag{8.3}$$

Suppose that τ' is an element in the codomain of the map (8.3). Similar to the proof of Lemma 2.1, there is a unique element in $\tau := \nabla^{-1}(\tau') \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ such that for all $i = 1, 2, \cdots, \mathbf{c}_1(\imath)$,

$$\mathcal{P}_\tau(i, 1) \in \{\bullet, s\},$$

and for all $(i, j) \in \mathrm{Box}(\tau')$,

$$\mathcal{P}_\tau(i, j+1) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}_{\tau'}(i, j) \in \{\bullet, s\}; \\ \mathcal{P}_{\tau'}(i, j), & \text{if } \mathcal{P}_{\tau'}(i, j) \notin \{\bullet, s\}, \end{cases}$$

and

$$\mathcal{Q}_\tau(i, j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}_{\tau'}(i, j) \in \{\bullet, s\}; \\ \mathcal{Q}_{\tau'}(i, j), & \text{if } \mathcal{Q}_{\tau'}(i, j) \notin \{\bullet, s\}. \end{cases}$$

Note that τ is in the domain of (8.3). It is then routine to check that the map

$$\nabla^{-1} : \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\} \to \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq c \right\}$$

and the map (8.3) are inverse to each other. Hence the map (8.3) is bijective.

Similarly, the map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) = c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^-, \mathcal{Q}_{\tau'}(\mathbf{c}_1(\imath), 1) \in \{r, d\} \right\}$$

is well-defined, and we show that it is bijective by explicitly constructing its inverse. In view of Lemma 8.2, this proves the proposition in the case we are considering.

□

As in the Introduction, $\mathcal{O}$ denotes the Barbasch-Vogan dual of $\check{\mathcal{O}}$. The following proposition is easy to verify:

Lemma 8.4. *Suppose $k \in \mathbb{N}^+$, $\star \in \{B, D, C^*\}$. Let $\check{\mathcal{O}}$ be the $\star$-good parity orbit such that $\mathbf{r}_3(\check{\mathcal{O}}) = 0$ and*

$$(\mathbf{r}_1(\check{\mathcal{O}}), \mathbf{r}_2(\check{\mathcal{O}})) = \begin{cases} (2k-2, 0) & \text{if } \star = B, \\ (2k-1, 1) & \text{if } \star = D, \\ (2k-1, 0) & \text{if } \star = C^*. \end{cases}$$

Then $\check{\mathcal{O}}$ is the regular nilpotent orbit and $\mathcal{O} = (1^{\mathbf{r}_1(\check{\mathcal{O}})+1})_\star$.

When $\star \in \{B, D\}$ the following map is bijective:

$$\begin{array}{ccc} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \longrightarrow & \{ (p, q, \varepsilon) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z} \mid \varepsilon = 1 \textit{ if } pq = 0, \textit{ and } p + q = |\mathcal{O}| \}, \\ \tau & \mapsto & (p_\tau, q_\tau, \varepsilon_\tau). \end{array}$$

When $\star = C^*$, the following map is bijective:

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \{ (p, q) \mid p, q \in 2\mathbb{N} \textit{ and } p + q = |\mathcal{O}| \}, \quad \tau \mapsto (p_\tau, q_\tau).$$

□

For every painted bipartition τ, write

$$\mathrm{Sign}(\tau) := (p_\tau, q_\tau).$$

When $\mathbf{r}_2(\check{\mathcal{O}}) > 0$, the double descent $\nabla^2(\tau) := \nabla(\nabla(\tau))$ is well-defined whenever $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$.

The key properties of the descent map when $\star \in \{D, B, C^*\}$ are summarized in the following two propositions.

Lemma 8.5. *Assume that $\star \in \{D, B, C^*\}$ and $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Write $\check{\mathcal{O}}'' := \check{\nabla}(\check{\mathcal{O}}')$ and consider the map*

$$\delta \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}), \qquad \tau \mapsto (\nabla^2(\tau), \tau_{\mathbf{t}}). \tag{8.4}$$

(a) Suppose that $\star = C^*$ or $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then the map (8.4) *is a bijective, and for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,*

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}), \mathbf{c}_2(\mathcal{O})) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{8.5}$$

(b) Suppose that $\star \in \{B, D\}$ and $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$. Then the map (8.4) *is injection and its image equals*

$$\left\{ (\tau'', \tau_0) \in \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}}) \;\middle|\; \begin{array}{l} \textit{either } x_{\tau''} = d, \textit{ or} \\ x_{\tau''} \in \{r, c\} \textit{ and } \mathcal{P}_{\tau_0}^{-1}(\{s, c\}) \neq \emptyset \end{array} \right\}. \tag{8.6}$$

Moreover, for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{8.7}$$

Proof. We give the detailed proof of part (b) when $\star = B$. The proofs in the other cases are similar and are left to the reader. Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ and $\tau'' = \nabla^2(\tau)$.

According to the descent algorithm, we have

$$\begin{aligned} \mathcal{P}(i,j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{P}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 1; \end{cases} \\ \mathcal{Q}(i,j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{Q}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 2; \end{cases} \end{aligned} \tag{8.8}$$

We label the boxes which are not considered in (8.8) as the following:

$$
\tau : \quad \begin{array}{|c|}\hline x_0 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline x_1 & x'' \\ \hline x_2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} x_k & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \alpha \quad \mapsto \quad \tau'' : \quad \varnothing \times \begin{array}{|c|}\hline y'' \\ \hline \end{array} \times \alpha'', \tag{8.9}
$$

where $k := \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, x_0, x_1 and y'' has coordinate $(\mathbf{c}_1(\imath), 1)$ in the corresponding painted Young diagram. By the descent algorithm, $x'' = y''$. Also note that $\mathbf{c}_2(\mathcal{O}) = 2\mathbf{c}_1(\imath)$.

We now discuss case by case according to the symbol x_1. When $x_1 = s$, we have $(x_0, \alpha'') = (c, B^-)$. Now

$$
\begin{aligned}
&\mathrm{Sign}(\tau) - \mathrm{Sign}(\tau'') \\
&= (2\mathbf{c}_1(\imath) - 2, 2\mathbf{c}_1(\imath) - 2) + (1,1) + (0,2) - (0,1) + \mathrm{Sign}(\alpha) + \mathrm{Sign}(x_2 \cdots x_k) \\
&= (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\tau_{\mathbf{t}}).
\end{aligned} \tag{8.10}
$$

Here Sign counts the signature of various symbols in the same way of (1.3). In second line of the above formula, the terms count the signatures of extra bullets, x_0, x_1, α'', α and $x_1 \cdots x_k$ respectively.

When $x_1 = \bullet$, we have $(x_0, \alpha'') = (\bullet, B^+)$. When $x_1 \in \{ r, d \}$, we have $x'' = y'' = d$ $(x_0, \alpha'') = (c, \alpha)$. In these two cases, the signature formula can be checked similarly. It also clear that the image of δ is in (8.6).

It is not hard to construct the inverse of δ from (8.6) to $\mathrm{PBP}_\star(\check{\mathcal{O}})$, and then the proposition follows. To illustrate the idea, we give the detail construction of $\delta^{-1}(\tau'', \tau_0)$ when $\mathcal{P}_{\tau_0}(k,1) = c$. Most of the entries in $\mathcal{P}$ and $\mathcal{Q}$ are defined by (8.8). We refer to (8.9) for the labeling of the rest of the entries, and set $\alpha := B^-$, $x'' := \mathcal{Q}_{\tau''}(\mathbf{c}_1(\imath), 1)$,

$$
(x_0, x_1) := \begin{cases} (\bullet, \bullet) & \text{if } \alpha'' = B^+, \\ (c, s) & \text{if } \alpha'' = B^-, \end{cases} \quad \text{and} \quad x_i := \mathcal{P}_{\tau_0}(i-1, 1) \quad \text{for } i = 2, 3, \cdots, k.
$$

[In fact, it suffice to check the proposition for the orbit $\check{\mathcal{O}}$ consists of three rows with lengths $2k$, 2, and 2.]

[Suppose $\star = C^*$. Then τ is obtained by attaching a column of $\mathbf{c}_1(\imath)$ bullets on the left of $\mathcal{P}$, and a column of $\mathbf{c}_1(\imath)$ bullets concatenated with $\tau_{\mathbf{t}}$ on the left of $\mathcal{Q}$. The bijectivity is clear. The signature formula follows from $2\mathbf{c}_1(\imath) = \mathbf{c}_2(\mathcal{O})$.

Suppose $\star = D$ and $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then τ is obtained by attaching a column of $\mathbf{c}_1(\jmath)$ bullets concatenated with $\tau_{\mathbf{t}}$ on the left of $\mathcal{P}$, and a column of $\mathbf{c}_1(\jmath)$ bullets on the left of $\mathcal{Q}$. The bijectivity is clear. The signature formula follows from $2\mathbf{c}_1(\jmath) = \mathbf{c}_2(\mathcal{O})$.

] □

Proposition 8.6. *Suppose that* $\star \in \{ D, B, C^* \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. *Then the map*

$$
\delta' \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}) \qquad \tau \mapsto (\nabla(\tau), \tau_{\mathbf{t}}) \tag{8.11}
$$

is injective. Moreover, (8.11) *is bijective unless* $\star \in \{ B, D \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$.

Proof. It follows from the injectivity results in Lemma 8.5 and Proposition 8.3. □

Combining the above propotition with Lemma 8.4, we get the following.

Corollary 8.7. *If* $\star \in \{ B, D, C^* \}$, *then the map*

$$
\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \rightarrow & \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array} \tag{8.12}
$$

is injective. $\square$

9. Proof of Theorem 3.9

to be added by Binyong.

10. Associated cycles and the exhaustion of special representations

By Theorem 3.9, we reduce the study of the associated cycles of into combinatorical problems. In this section, we will be solve these problems.

Recall that $\mathrm{AOD}_\star(\mathcal{O})$ can be parameterized by $\mathsf{MYD}_\star(\mathcal{O})$. We fist define the combinatorical companion of $\mathrm{AC}(\boldsymbol{\tau})$. Suppose $|\check{\mathcal{O}}| = 0$. For $\boldsymbol{\tau} \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, we define

$$\mathcal{L}_{\boldsymbol{\tau}} := \begin{cases} (1^{\mathrm{Sign}(\boldsymbol{\tau})})_\star \otimes (0,1) & \text{if } \star = B \\ (1^{(0,0)})_\star & \text{otherwise.} \end{cases}$$

Suppose $|\check{\mathcal{O}}| > 0$. For $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, let $\boldsymbol{\tau}' = \nabla(\boldsymbol{\tau})$ and we define

$$\mathcal{L}_{\boldsymbol{\tau}} := \begin{cases} \check{\vartheta}^{\mathsf{s}_{\boldsymbol{\tau}}}_{\mathsf{s}_{\boldsymbol{\tau}'}}(\mathcal{L}_{\boldsymbol{\tau}'} \otimes (\varepsilon_\wp, \varepsilon_\wp)) & \text{if } \star \in \{ C, \widetilde{C} \}, \\ \check{\vartheta}^{\mathsf{s}_{\boldsymbol{\tau}}}_{\mathsf{s}_{\boldsymbol{\tau}'}}(\mathcal{L}_{\boldsymbol{\tau}'}) & \text{if } \star = D^*, \\ \check{\vartheta}^{\mathsf{s}_{\boldsymbol{\tau}}}_{\mathsf{s}_{\boldsymbol{\tau}'}}(\mathcal{L}_{\boldsymbol{\tau}'}) \otimes (0, \varepsilon_\tau) & \text{if } \star \in \{ B, D \}, \\ \check{\vartheta}^{\mathsf{s}_{\boldsymbol{\tau}}}_{\mathsf{s}_{\boldsymbol{\tau}'}}(\mathcal{L}_{\boldsymbol{\tau}'}) & \text{if } \star = C^*. \end{cases}$$

See (7.3) and (7.4) for the definition of $\check{\vartheta}[\mathsf{s}'][\mathsf{s}]$.

For $\star \in \{ B, D \}$ and $\mathcal{E} \in \mathbb{Z}[\mathsf{MYD}_\star]$, we write

$$\mathcal{E}^+ := \Lambda_{(1,0)}(\mathcal{E}) \quad \text{and} \quad \mathcal{E}^- := \Lambda_{(0,1)}(\mathcal{E}).$$

We adopt the convention that $\mathcal{E} \cdot \varnothing = 0$ for $\mathcal{E} \in \mathbb{Z}[\mathsf{MYD}]$.

Using the signature formulas in Lemma 8.5, we have the following more explicit formulas on $\mathcal{L}_{\boldsymbol{\tau}}$.

Lemma 10.1. *Suppose $\check{\mathcal{O}}$ such that $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Let $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ and $\boldsymbol{\tau}' = \nabla(\boldsymbol{\tau})$.*

(a) Suppose $\star \in \{ C, \widetilde{C} \}$. Then

$$\mathcal{L}_{\boldsymbol{\tau}} = \begin{cases} \maltese^y\big((\mathcal{L}_{\boldsymbol{\tau}'} \otimes (\varepsilon_\wp, \varepsilon_\wp)) \cdot (n_0, n_0)\big) & \textit{if } \mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}) \\ \maltese^y\big((\mathcal{L}^+_{\boldsymbol{\tau}'} + \mathcal{L}^-_{\boldsymbol{\tau}'}) \cdot (0,0)\big) & \textit{if } \mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}) \end{cases} \tag{10.1}$$

where y is an integer determined by $\mathrm{Sign}(\boldsymbol{\tau}')$ *and $n_0 = (\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O}))/2$.*

(b) Suppose $\star \in \{ B, D \}$ and $\mathbf{r}_2(\mathcal{O}) > \mathbf{r}_3(\mathcal{O})$. Then

$$\mathcal{L}_{\boldsymbol{\tau}} = \big((\maltese^y \mathcal{L}_{\boldsymbol{\tau}'}) \cdot (p_{\tau_{\mathbf{t}}}, q_{\tau_{\mathbf{t}}})\big) \otimes (0, \varepsilon_\tau) \tag{10.2}$$

where y is an integer determined by $\mathrm{Sign}(\boldsymbol{\tau})$*. In particular,*

$$\mathcal{E}(1) = \mathcal{L}_{(\tau_{\mathbf{t}}, \varnothing)}(1) \tag{10.3}$$

for each $\mathcal{E} \in \mathsf{MYD}_\star$ has nonzero multiplicity in $\mathcal{L}_{\boldsymbol{\tau}}$.

(c) When $\mathbf{r}_2(\mathcal{O}) = \mathbf{r}_3(\mathcal{O})$,

$$\mathcal{L}_{\boldsymbol{\tau}} = \big((\maltese^t(\mathcal{L}^+_{\boldsymbol{\tau}''} \cdot (0,0)) \cdot \mathcal{T}^+_\tau + (\maltese^t(\mathcal{L}^-_{\boldsymbol{\tau}''} \cdot (0,0))) \cdot \mathcal{T}^-_\tau\big) \otimes (0, \varepsilon_\tau) \tag{10.4}$$

where $\tau'' = \nabla^2(\tau)$ *and*

$$t = \frac{|\mathrm{Sign}(\tau_{\mathbf{t}})|}{2},$$

$$\mathcal{T}_\tau^+ = \begin{cases} (p_{\tau_{\mathbf{t}}}, q_{\tau_{\mathbf{t}}} - 1) & \textit{when } q_{\tau_{\mathbf{t}}} - 1 \geqslant 0, \\ \varnothing & \textit{otherwise}, \end{cases}$$

$$\mathcal{T}_\tau^- = \begin{cases} (p_{\tau_{\mathbf{t}}} - 1, q_{\tau_{\mathbf{t}}}) & \textit{when } p_{\tau_{\mathbf{t}}} - 1 \geqslant 0, \\ \varnothing & \textit{otherwise}. \end{cases}$$

(d) *Suppose* $\star \in \{ D^* \}$. *Then*

$$\mathcal{L}_\tau = \mathcal{L}_{\tau'} \cdot (n_0, n_0),$$

where $n_0 = (\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O}))/2$.

(e) *Suppose* $\star \in \{ C^* \}$. *Then*

$$\mathcal{L}_\tau = \mathcal{L}_{\tau'} \cdot (p_{\tau_{\mathbf{t}}}, q_{\tau_{\mathbf{t}}}).$$

□

10.1. **Properties of $\mathcal{L}_\tau$.** In this section, we summarizes the properties of $\mathcal{L}_\tau$ in the following lemmas. Their proofs are given in the next three sections.

Lemma 10.2. *For each* $\tau \in \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}})$, $\mathcal{L}_\tau \neq 0$. *Moreover, the multiplicity of each* $\mathcal{E} \in \mathsf{MYD}_\star$ *in* $\mathcal{L}_\tau$ *is either* 0 *or* 1.

Lemma 10.3. *Suppose* $\star \in \{ C, \widetilde{C}, D^* \}$ *and* $\check{\mathcal{O}} \neq \varnothing$.

(a) *Let* $\tau_1 = (\tau_1, \wp_1)$ *and* $\tau_2 = (\tau_2, \wp_2)$ *be two elements in* $\mathrm{PBP}_{\star'}^{\mathrm{ext}}(\check{\mathcal{O}}')$ *such that* $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. *Then*

$$\mathrm{Sign}(\tau_1') = \mathrm{Sign}(\tau_2') \quad \textit{and} \quad \varepsilon_{\wp_1} = \varepsilon_{\wp_2},$$

where $\tau_i' := \nabla(\tau_i)$ *for* $i = 1, 2$.

(b) *When* $\check{\mathcal{O}}$ *is weakly-distinguished, the following map is injective.*

$$\tilde{\mathcal{L}} \colon \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}) \longrightarrow \mathbb{Z}[\mathsf{MYD}_\star(\mathcal{O})], \quad \tau \mapsto \mathcal{L}_\tau.$$

(c) *When* $\check{\mathcal{O}}$ *is quasi-distinguished, the image of the above map is* $\mathsf{MYD}_\star(\mathcal{O})$.

Lemma 10.4. *Suppose* $\star \in \{ B, D \}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$.

(a) *Let* $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}})$ *and* $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$ *such that*

$$\mathcal{L}_{\tau_1} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{\tau_2} \otimes (\epsilon_2, \epsilon_2).$$

Then

$$\epsilon_1 = \epsilon_2, \quad \varepsilon_{\tau_1} = \varepsilon_{\tau_2} \quad \textit{and} \quad \mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}.$$

(b) *When* $\check{\mathcal{O}}$ *is weakly-distinguished, the following map is injective.*

$$\mathcal{L} \colon \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}[\mathsf{MYD}_\star(\mathcal{O})], \quad (\tau, \epsilon) \mapsto \mathcal{L}_\tau \otimes (\epsilon, \epsilon).$$

(c) *When* $\check{\mathcal{O}}$ *is quasi-distinguished, the image of the above map is* $\mathsf{MYD}_\star(\mathcal{O})$.

Lemma 10.5. *Suppose* $\star = C^*$. *When* $\check{\mathcal{O}}$ *is quasi-distinguished, the image of the map*

$$\mathcal{L} : \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}) \longrightarrow \mathbb{Z}[\mathsf{MYD}_\star(\mathcal{O})], \quad \tau \mapsto \mathcal{L}_\tau$$

is $\mathsf{MYD}_\star(\mathcal{O})$.

Lemma 10.6. *Suppose* $\star \in \{ B, D \}$ *and* $\check{\mathcal{O}} \neq \varnothing$.

(a) *If* $x_\tau = s$, *then* $\mathcal{L}_\tau^+ = 0$ *and* $\mathcal{L}_\tau^- = 0$.

(b) *If* $x_\tau \in \{ r, c \}$, *then* $\mathcal{L}_\tau^+ \neq 0$ *and* $\mathcal{L}_\tau^- = 0$.

(c) *If* $x_\tau = d$, *then* $\mathcal{L}_\tau^+ \neq 0$ *and* $\mathcal{L}_\tau^- \neq 0$.

Lemma 10.7. *Suppose $\star \in \{B, D\}$, $\check{\mathcal{O}} \neq \varnothing$ and $\check{\mathcal{O}}$ is weakly-distinguished.*

(a) The map

$$\Upsilon_{\check{\mathcal{O}}}\colon \{\mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \text{ s.t. } \mathcal{L}^+_\tau \neq 0\} \to \mathbb{Z}[\mathsf{MYD}] \times \mathbb{Z}[\mathsf{MYD}], \quad \mathcal{L}_\tau \mapsto (\mathcal{L}^+_\tau, \mathcal{L}^-_\tau) \tag{10.5}$$

is injective.

(b) When $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$, the maps

$$\begin{array}{rcl}\Upsilon^+_{\check{\mathcal{O}}}\colon \{\mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \text{ s.t. } \mathcal{L}^+_\tau \neq 0\} & \to & \{\mathcal{L}^+_\tau\} \\ \mathcal{L}_\tau & \mapsto & \mathcal{L}^+_\tau \\ \Upsilon^-_{\check{\mathcal{O}}}\colon \{\mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \text{ s.t. } \mathcal{L}^+_\tau \neq 0\} & \to & \{\mathcal{L}^-_\tau\} \\ \mathcal{L}_\tau & \mapsto & \mathcal{L}^-_\tau \end{array}$$

are injective.

(c) When $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$, and $x_\tau = d$. Then there is $\mathcal{E} \in \mathsf{MYD}$ with non-zero coefficient in $\mathcal{L}^-_\tau$ such that $\mathcal{E} \sqsupseteq (1,0)$.

All the claims for $\star \in \{C^*, D^*\}$ in the above lemmas are straight forward to verify. We will focus on the non-quaternionic cases in the next two sections.

10.2. **The initial cases.** Suppose $\mathbf{c}_1(\mathcal{O}) = 0$. Then the group $G_\mathbb{C}$ is the trivial group, everything are clear.

We now assume $\mathbf{c}_1(\mathcal{O}) > 0$ and $\mathbf{c}_2(\mathcal{O}) = 0$.

Suppose $\star \in \{C, \widetilde{C}, D^*\}$. There there is $k \in \mathbb{N}^+$ such that $\mathcal{O} = (1^{2k})_\star$, and $\check{\mathcal{O}}$ is the regular nilpotent orbit in $\check{\mathfrak{g}}$. Now $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ has only one element, say τ_0, which maps to $\mathcal{L}_{\tau_0} = (1^{(k,k)})_\star$.

Suppose $\star \in \{B, D, C^*\}$. There is $k \in \mathbb{N}^+$ such that $\mathcal{O}$ and $\check{\mathcal{O}}$ are given by Lemma 8.4. When $\star \in \{B, D\}$,

$$\mathcal{L}_\tau = (1^{(p_\tau, (-1)^{\varepsilon_\tau} q_\tau)})_\star \quad \text{for all } \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}).$$

Thanks to Lemma 8.4, it is easy to see that

$$\begin{array}{rcl}\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \to & \mathsf{MYD}_\star(\mathcal{O}) \\ (\tau, \epsilon) & \mapsto & \mathcal{L}_\tau \otimes (\epsilon, \epsilon)\end{array}$$

is a bijection.

When $\star = C^*$, for $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$,

$$\mathcal{L}_\tau = (1^{(p_\tau, q_\tau)})_\star \quad \text{for all } \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}).$$

Now

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \to \mathsf{MYD}_\star(\mathcal{O}), \quad \tau \mapsto \mathcal{L}_\tau$$

is a bijection by Lemma 8.4.

In all the cases, the orbit $\check{\mathcal{O}}$ is quasi-distinghuished and all the lemmas in Section 10.1 are clear now.

10.3. **The general case for $\star \in \{C, \widetilde{C}\}$.** We now assume $\mathbf{c}_2(\mathcal{O}) > 0$ and retain the notation in Lemma 10.3.

For $\sum_{j=1}^b m_i \mathcal{E}_i \in \mathbb{Z}[\mathsf{MYD}_\star]$ with $m_j > 0$ and $\mathcal{E}_i \in \mathsf{MYD}_\star$, we define

$${}^d\mathrm{Sign}(\sum_{j=1}^b m_i \mathcal{E}_i) := \{\mathrm{Sign}(\mathscr{O}'_j) \mid j = 1, 2, \cdots, b\}.$$

where $\mathscr{O}'_j \in \mathsf{SYD}_{\star'}$ is given by $\mathscr{O}'_j(i) := \mathscr{F} \circ \mathcal{E}_j(i+1)$ for all $i \in \mathbb{N}^+$.

The case when $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$.

Proof of Lemma 10.2. Since $\mathcal{L}_{\tau'} \neq 0$, $\mathcal{L}_{\tau} \neq 0$ by (10.1). This proves Lemma 10.2.

Proof of Lemma 10.3 (a). Suppose $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (10.1), $\{\,\mathrm{Sign}(\tau_1')\,\} = {}^d\mathrm{Sign}(\mathcal{L}_{\tau_1}) = {}^d\mathrm{Sign}(\mathcal{L}_{\tau_2}) = \{\,\mathrm{Sign}(\tau_2')\,\}$. In particular, $\mathrm{Sign}(\tau_1') = \mathrm{Sign}(\tau_2')$ and the integer "y" in (10.1) are the same for τ_1 and τ_2. We get $\mathcal{L}_{\tau_1'} \otimes (\varepsilon_{\wp_1}) = \mathcal{L}_{\tau_2'} \otimes (\varepsilon_{\wp_2})$. So $\varepsilon_{\wp_1} = \varepsilon_{\wp_2}$ and $\mathcal{L}_{\tau_1'} = \mathcal{L}_{\tau_2'}$ by Lemma 10.4 (a).

When $\check{\mathcal{O}}$ is weakly-distinguished, $\check{\mathcal{O}}'$ is also weakly-distinguished and we get $\tau_1' = \tau_2'$ by Lemma 10.4 (b). Hence $\tau_1 = \tau_2$ by Proposition 8.3. This proves Lemma 10.3 (b).

When $\check{\mathcal{O}}$ is quasi-distinguished, $\check{\mathcal{O}}'$ is quasi-distinguished $\mathcal{L}_{\tau'} \in \mathsf{MYD}_{\star'}(\mathcal{O}')$ by Lemma 10.4 (c). Hence $\mathcal{L}_{\tau} \in \mathsf{MYD}_{\star}(\mathcal{O})$ by (10.1). This proves Lemma 10.3 (c).

Now we consider the rest case, where $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}) > 0$.

According to Proposition 8.3, $x_{\tau'} \neq s$. Thanks to Lemma 10.6 and (10.1), we have $\mathcal{L}^+_{\tau'} \neq 0$ and

$$
{}^d\mathrm{Sign}(\mathcal{L}_{\tau}) = \begin{cases} \{\,\mathrm{Sign}(\tau') - (1,0)\,\} & \text{if } x_{\tau'} \in \{\,r, c\,\}, \\ \{\,\mathrm{Sign}(\tau') - (1,0), \mathrm{Sign}(\tau') - (0,1)\,\} & \text{if } x_{\tau'} = d, \end{cases} \tag{10.6}
$$

In particular, $\mathcal{L}_{\tau} \neq 0$ and Lemma 10.2 is proven.

Suppose $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (10.6), $\mathrm{Sign}(\tau_1') = \mathrm{Sign}(\tau_2')$. Since we have $\varepsilon_{\wp_1} = \varepsilon_{\wp_2} = 0$, Lemma 10.3 (a) is proven.

Now (10.1) implies $\mathcal{L}^+_{\tau_1'} = \mathcal{L}^+_{\tau_2'}$ and $\mathcal{L}^-_{\tau_1'} = \mathcal{L}^-_{\tau_2'}$. By the injectivity of $\Upsilon_{\check{\mathcal{O}}'}$ in Lemma 10.7 (a), we get $\mathcal{L}_{\tau_1'} = \mathcal{L}_{\tau_2'}$. Suppose $\check{\mathcal{O}}$ is weakly-distinghuished, then $\check{\mathcal{O}}'$ is weakly-distinghuished. So $\tau_1' = \tau_2'$ by Lemma 10.4 (b). Now Proposition 8.3 implies $\tau_1 = \tau_2$, which proves Lemma 10.3 (b).

Note that $\check{\mathcal{O}}$ is not quasi-distinguished, Lemma 10.3 (c) is vacant.

10.4. **The general case for $\star \in \{\,B, D\,\}$.** To ease the following discussion, we adopt the following convension. Suppose $\mathcal{A} \in \mathbb{Z}[\mathsf{MYD}]$. We write $\mathcal{A} \rhd (p_0, q_0)$ if $\mathcal{E} \sqsupseteq (p_0, q_0)$ for all $\mathcal{E} \in \mathsf{MYD}$ with non-zero coefficient in $\mathcal{A}$, and write $\mathcal{A} \sqsupseteq (p_0, q_0)$ if there is an $\mathcal{E} \in \mathsf{MYD}$ with non-zero coefficient in $\mathcal{A}$ such that $\mathcal{E} \sqsupseteq (p_0, q_0)$.

We now assume $\mathbf{c}_2(\mathcal{O}) > 0$ and retain the notation in Lemma 10.4, 10.6 and 10.7.

We first consider the case when $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Lemma 10.2, i.e. $\mathcal{L}_{\tau} \neq 0$, follows from (10.2) directly.

Suppose $\mathcal{L}_{\tau_1} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{\tau_2} \otimes (\epsilon_2, \epsilon_2)$. By (10.3), $\mathcal{L}_{(\tau_1)_{\mathbf{t}}} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{(\tau_2)_{\mathbf{t}}} \otimes (\epsilon_2, \epsilon_2)$. By the result in the initial case, we get $\epsilon_1 = \epsilon_2$ and $\varepsilon_{\tau_1} = \varepsilon_{(\tau_1)_{\mathbf{t}}} = \varepsilon_{(\tau_2)_{\mathbf{t}}} = \varepsilon_{\tau_2}$. This proves Lemma 10.4 (a).

Now assume $\check{\mathcal{O}}$ is weakly-distinguished in addition. Then $\check{\mathcal{O}}'$ is also weakly-distinghuished. We get $\mathcal{L}_{\tau_1'} = \mathcal{L}_{\tau_2'}$ by (10.3) and so $\tau_1' = \tau_2'$. By Corollary 8.7, we conclude that $\tau_1 = \tau_2$. This proves Lemma 10.4 (b).

Suppose $\check{\mathcal{O}}$ is quasi-distinghuished. Then $\mathcal{L}_{\tau} \in \mathsf{MYD}_{\star}$ since $\mathcal{L}_{\tau'} \in \mathsf{MYD}_{\star'}$ by the quasi-distinguishness of $\check{\mathcal{O}}'$. This proves Lemma 10.4 (c).

Using (10.3), all the claims in Lemma 10.6 and 10.7 follows from that of $\check{\mathcal{O}}_{\mathbf{t}}$, see Section 8.2. (Note that we always have $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. So Lemma 10.6 (b) is not vacant.)

We now assume that $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$.

Let $\tau'' := (\tau'', \wp'') := \nabla^2(\tau)$. By Lemma 8.5, $x_{\tau''} \neq s$.

Proof of Lemma 10.2. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \geq (0,1)$, the first term in (10.4) dose not vanish since $\mathcal{L}^{+}_{\tau''} \neq 0$ by Lemma 10.6. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \ngeq (0,1)$, $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k,0)$ and $\mathcal{P}^{-1}_{\tau_{\mathbf{t}}}(\{s,c\}) = \varnothing$. Hence $x_{\tau''} = d$ by Lemma 8.5. Now $\mathcal{L}^{-}_{\tau''} \neq 0$ by Lemma 10.6 and the second term in (10.4) dose not vanish. This proves Lemma 10.2

Proof of Lemma 10.6. It is clear that $\mathcal{L}^{-}_{\tau} = 0$ when $x_\tau \in \{s,r,c\}$ because of the twist $\otimes(0,1)$ in Equation (10.4).

(i) Suppose $x_\tau = s$. We have $\mathrm{Sign}(\tau_{\mathbf{t}}) = (0,2k)$. where $2k = |\check{\mathcal{O}}_{\mathbf{t}}|$. Therefore, only the first term in (10.4) is non-zero and $\mathcal{E}(1) = (B,0,-(2k-1))$ for every $\mathcal{E} \in \mathsf{MYD}_\star$ with non-zero coefficient in $\mathcal{L}_\tau$. Hence $\mathcal{L}^{+}_{\tau} = 0$.

(ii) Suppose $x_\tau \in \{r,c\}$. If $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (1,1)$, the first term in (10.4) is non-zero and $\mathcal{L}_\tau \sqsupseteq (1,0)$. If $\mathrm{Sign}(\tau_{\mathbf{t}}) \nsucc (1,1)$, then $\mathcal{P}^{-1}_{\tau_{\mathbf{t}}}(\{s,c\}) = \varnothing$, $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k,0)$ and $x_{\tau''} = d$ by Lemma 8.5. Now the second term of (10.4) is non-zero and $\mathcal{T}^{-}_{\tau} \succ (1,0)$. In both cases, we conclude that $\mathcal{L}^{+}_{\tau} \neq 0$.

(iii) Suppose $x_\tau = d$. If $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (1,2)$, then the first term in (10.4) is non-zero and so $\mathcal{L}_\tau \sqsupseteq (1,1)$. Hence $\mathcal{L}^{+}_{\tau} \neq 0$ and $\mathcal{L}^{-}_{\tau} \neq 0$. If $\mathrm{Sign}(\tau_{\mathbf{t}}) \nsucc (1,2)$, then $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k-1,1)$ [1] and $x_{\tau''} = d$ by Lemma 8.5. So the both terms in (10.4) are non-zero, which leads to the conclusion.

Proof of Lemma 10.4 (a). The following proof only depends on Lemma 10.6. Suppose $\mathcal{L}' = \mathcal{L}_\tau \otimes (\epsilon,\epsilon)$. Let

$$\mathcal{L}'^{+} := \Lambda_{(1,0)}(\mathcal{L}'), \qquad \mathcal{L}'^{-} := \Lambda_{(0,1)}(\mathcal{L}'),$$
$$\widetilde{\mathcal{L}'}^{+} := \Lambda_{(1,0)}(\mathcal{L}' \otimes (1,1)), \text{ and} \qquad \widetilde{\mathcal{L}'}^{-} := \Lambda_{(0,1)}(\mathcal{L}' \otimes (1,1)).$$

Note that $\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O}) > 0$. So $\mathcal{L}^{+}_{\tau}$, $\mathcal{L}^{-}_{\tau}$, $\widetilde{\mathcal{L}'}^{+}$ and $\widetilde{\mathcal{L}'}^{-}$ can not all vanish. Using Lemma 10.6, we have the following criteria for $(\varepsilon_\tau, \epsilon)$:

$$(\varepsilon_\tau,\epsilon) = (0,0) \Leftrightarrow \begin{cases} \mathcal{L}'^{+} \neq 0 \\ \mathcal{L}'^{-} \neq 0 \end{cases};$$

$$(\varepsilon_\tau,\epsilon) = (0,1) \Leftrightarrow \begin{cases} \widetilde{\mathcal{L}'}^{+} \neq 0 \\ \widetilde{\mathcal{L}'}^{-} \neq 0 \end{cases};$$

$$(\varepsilon_\tau,\epsilon) = (1,0) \Leftrightarrow \begin{cases} \mathcal{L}'^{+} \neq 0 \\ \mathcal{L}'^{-} = 0 \end{cases} \text{ or } \begin{cases} \widetilde{\mathcal{L}'}^{+} = 0 \\ \widetilde{\mathcal{L}'}^{-} \neq 0 \end{cases};$$

$$(\varepsilon_\tau,\epsilon) = (1,1) \Leftrightarrow \begin{cases} \widetilde{\mathcal{L}'}^{+} \neq 0 \\ \widetilde{\mathcal{L}'}^{-} = 0 \end{cases} \text{ or } \begin{cases} \mathcal{L}'^{+} = 0 \\ \mathcal{L}'^{-} \neq 0 \end{cases}.$$

This implies Lemma 10.4 (a).

Proof of Lemma 10.4 (b). First not that $\mathbf{r}_1(\check{\mathcal{O}}'') = \mathbf{r}_3(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_5(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}'')$.

Suppose that $\mathcal{L}_{\tau_i} \sqsupseteq (0,1)$ or $\mathcal{L}_{\tau_i} \sqsupseteq (0,-1)$. The first term in (10.4) for both τ_1 and τ_2 must be non-zero and equal. This implies $\mathcal{L}^{+}_{\tau''_1} = \mathcal{L}^{+}_{\tau''_2}$. By Lemma 10.6 (b) and Lemma 10.4 (b), $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ and $\tau''_1 = \tau''_2$. Since $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by Lemma 8.5.

Now consider the case when $\mathcal{L}_{\tau_i} \not\sqsupseteq (0,1)$ and $\mathcal{L}_{\tau_i} \not\sqsupseteq (0,-1)$. In partcular, $x_\tau \neq d$.

By (10.4), we conclude that $\mathrm{Sign}((\tau_i)_{\mathbf{t}}) \in \{(2k,0),(2k-1,1)\}$.

We consider case by case according to the set $\{\mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i = 1,2\}$:

[1] $\tau_{\mathbf{t}} = d$ if it has length 1.

(i) Suppose $\{\,\mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i=1,2\,\} = \{\,(2k-1,1)\,\}$. We have $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$, and deduces $\tau''_1 = \tau''_2$.

(ii) Suppose $\{\,\mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i=1,2\,\} = \{\,(2k,0)\,\}$. We have $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$, and deduces $\tau''_1 = \tau''_2$ by Lemma 10.6 (b) and Lemma 10.4 (b).

(iii) Without loss of generality, we assume that $\mathrm{Sign}(\tau_{1\mathbf{t}}) = (2k-1,1)$ and $\mathrm{Sign}(\tau_{2\mathbf{t}}) = (2k,0)$. By (10.4), we get

$$\mathcal{L}^+_{\tau''_1}\cdot(0,0) = \maltese(\mathcal{L}^-_{\tau''_2}\cdot(0,0)),$$

which is contradict to Lemma 10.6 (c).

We finished the proof of Lemma 10.4 (b).

Lemma 10.4 (c) is vacant in the current case.

Proof of Lemma 10.7 (b). Note that the condition $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$ implies $2k = |\check{\mathcal{O}}_{\mathbf{t}}| \geqslant 4$.

We first prove the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$. Note that $\mathcal{L}_\tau \sqsupseteq (2,0)$ is equivalent to $\mathcal{L}^+_\tau \sqsupseteq (1,0)$. We clearly have the following bijection

$$\{\,\mathcal{L}_\tau \mid \tau\in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}_\tau \sqsupseteq (2,0)\,\} \to \{\,\mathcal{L}^+_\tau \mid \tau\in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^+_\tau \sqsupseteq (1,0)\,\},$$

since $\mathcal{L}_\tau) \lhd (1,0)$ for each element in the domain by (10.4).

Now we consider the set $\{\,\mathcal{L}_\tau \mid \tau\in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^+_\tau \neq 0, \mathcal{L}_\tau \not\sqsupseteq (2,0)\,\}$. Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in this set such that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$.

By (10.4), we have $\mathrm{Sign}(\tau_{i\mathbf{t}}) = (1,2k-1)$ and the first term of (10.4) non-vanish.

Note that $\mathcal{E}(1) = (B, 2k-2, (-1)^{\varepsilon_{\tau_i}}1)$ for each $\mathcal{E}\in\mathsf{MYD}$ having non-zero coefficient in the first term of (10.4). So $x_{\tau_1} = x_{\tau_2}$. Compairing the first term of (10.4) gives $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$. Note that $\check{\mathcal{O}}''$ is weakly-distinghuished and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$, we deduce that $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}''}$. Now we conclude that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by (10.4) and finished the proof of the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$.

The injectivity of $\Upsilon^-_{\check{\mathcal{O}}}$ is proved similarly.

Note that $\mathcal{L}_\tau \sqsupseteq (0,2)$ is equivalent to $\mathcal{L}^-_\tau) \sqsupseteq (0,1)$. We get the bijection

$$\{\,\mathcal{L}_\tau \mid \tau\in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}_\tau \sqsupseteq (0,2)\,\} \to \{\,\mathcal{L}^-_\tau \mid \tau\in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^-_\tau \sqsupseteq (0,1)\,\}.$$

Now we consider the set $\{\,\mathcal{L}_\tau \mid \tau\in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^-_\tau \neq 0, \mathcal{L}_\tau \not\sqsupseteq (0,2)\,\}$. Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in it such that $\mathcal{L}^-_{\tau_1} = \mathcal{L}^-_{\tau_2}$. By (10.4), we have $\mathrm{Sign}((\tau_i)_{\mathbf{t}}) \in \{\,(2k-1,1),(2k-2,2)\,\}$. We consider case by case according to the set $\{\,\mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i=1,2\,\}$:

(i) Suppose $\{\,\mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i=1,2\,\} = \{\,(2k-1,1)\,\}$. We have $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$, and so $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ by the injectivity of $\Upsilon^-_{\check{\mathcal{O}}''}$.

(ii) Suppose $\{\,\mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i=1,2\,\} = \{\,(2k-2,2)\,\}$. We get $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$, and so $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}''}$.

(iii) Without loss of generality, we assume that $\mathrm{Sign}(\tau_{1\mathbf{t}}) = (2k-1,1)$ and $\mathrm{Sign}(\tau_{2\mathbf{t}}) = (2k-2,2)$. But this will leads to $\maltese(\mathcal{L}^-_{\tau''_1}\cdot(0,0)) = \mathcal{L}^+_{\tau''_2}\cdot(0,0)$ which is contradict to Lemma 10.7 (c).

Now we conclude that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by (10.4) and finished the proof of the injectivity of $\Upsilon^-_{\check{\mathcal{O}}}$.

Proof of Lemma 10.7 (c). Note that we have $k\geqslant 2$ and $x_\tau = d$ under the stated condition. Suppose $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (1,2)$, the first term of (10.4) is non-vanish.

Otherwise, $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k-1,1) \geq (3,1)$. Since $\mathcal{L}^-_\tau \neq 0$, the second term of (10.4) must non-vanish.

In all the cases, $\mathcal{L}_\tau \sqsupseteq (1,1)$ and so $\mathcal{L}^-_\tau \sqsupseteq (1,0)$.

Proof of Lemma 10.7 (a). Suppose $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$. The injectivity followes from the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$.

It left to consider the case when $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$. Note that we have $k = 1$ and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$.

Suppose $\Upsilon_{\check{\mathcal{O}}}(\mathcal{L}_{\tau_1}) = \Upsilon_{\check{\mathcal{O}}}(\mathcal{L}_{\tau_2})$.

When $\mathcal{L}^-_{\tau_i} \neq 0$, we have $x_{\tau_1} = x_{\tau_2} = d$ and $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ by (10.4).

Suppose $\mathcal{L}^-_{\tau_i} = 0$. By the similar argument in the proof of Lemma 10.4 (b), we conclude that there are only two possibilities:

- $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ and $x_{\tau_1} = x_{\tau_2} = c$;
- $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$ and $x_{\tau_1} = x_{\tau_2} = r$.

In all the cases, we get $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}''}$ or $\Upsilon^-_{\check{\mathcal{O}}''}$. Hence $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by (10.4).

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