COUNTING UNIPOTENT REPRESENTATIONS OF REAL REUDCTIVE GROUPS

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Contents

1. Counting unipotent representations

In this section, let $G_\mathbb{C}$ be a connected complex reductive group and $\mathfrak{g}$ is its Lie algebra. Fix a antiholomorphic involution σ on $G_\mathbb{C}$ and a corresponding Cartan involution θ of $G_\mathbb{C}$. Let G be a finite central extension of a open subgroup os $G_\mathbb{C}^\sigma$ and

$$\mathrm{pr}\colon G \to G_\mathbb{C}^\sigma$$

be the canonical projection. Let $K = \mathrm{pr}^{-1}(G_\mathbb{C}^\sigma)$ and $W = W(G_\mathbb{C})$.

Let ${}^a\mathfrak{h}$ be the abstract Cartan subalgebra of $\mathfrak{g}$ and ${}^aX$ be the lattice of abstarct weight spaces. Let ${}^aR \subseteq {}^aX$, ${}^aR^+$ and ${}^aQ$ be the abstract root system, the set of positive roots and the root lattice. Let

$$\mathsf{C} = \left\{ \mu \in {}^a\mathfrak{h}^* \;\middle|\; \begin{array}{l} \text{either } \langle \mathrm{Re}(\mu), \check{\alpha}\rangle > 0 \text{ or} \\ \langle \mathrm{Re}(\mu), \check{\alpha}\rangle = 0 \text{ and } \sqrt{-1}\,\langle \mathrm{Im}(\mu), \check{\alpha}\rangle > 0 \end{array} \right\}$$

and $\overline{\mathsf{C}}$ be the closure of C in ${}^a\mathfrak{h}$.

We identify the set of infinitesimal characters with ${}^a\mathfrak{h}^*/W$.

1.1. **Coherent family.** For each finite dimensional $\mathfrak{g}$-module or $G_\mathbb{C}$-module F, let F^* be its contragredient representation and let $\Delta(F) \subseteq {}^aX$ denote the multi-set of weights in F.

Let $\Pi_{\Lambda_0}(G_\mathbb{C})$ be the set of irreducible finite dimensional representations of $G_\mathbb{C}$ with extermal weight in Λ_0 and $\mathcal{G}_\Lambda(G_\mathbb{C})$ be the subgroup generated by $\Pi_{\Lambda_0}(G_\mathbb{C})$. Let

$${}^aP := \{\, \mu \in {}^aX \mid \mu \text{ is a } {}^a\mathfrak{h}\text{-weight of an } F \in \Pi_{\mathrm{fin}}(G_\mathbb{C}) \,\}\,.$$

Via the highest weight theory, every W-orbit $W\cdot\mu$ in ${}^aP$ corresponds with the irreducible finite dimensional representation $F \in \Pi_{\mathrm{fin}}(G_\mathbb{C})$ with extermal weight μ.

Now the Grothendieck group $\mathcal{G}(G_\mathbb{C})$ of finite dimensional representation of $G_\mathbb{C}$ is identified with $\mathbb{Z}[{}^aP/W]$. In fact $\mathcal{G}(G_\mathbb{C})$ is a $\mathbb{Z}$-algebra under the tensor product and equiped with the involution $F \mapsto F^*$.

Fix a W-invariant sub-lattice $\Lambda_0 \subset {}^aX$ containing ${}^aQ$.

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Take a lattice $\Lambda = \lambda + \Lambda_0 \in \mathfrak{h}^*/\Lambda$ with $\lambda \in \overline{\mathsf{C}}$. Let

$$R(\lambda) := \{ \alpha \in {}^aR \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \} \quad \text{and} \quad W(\lambda) := \langle s_\alpha | \alpha \in R(\lambda) \rangle .$$

Definition 1.1. *Suppose $\mathcal{M}$ is an abelian group with $\mathcal{G}_\Lambda(G_\mathbb{C})$-action*

$$\mathcal{G}_\Lambda(G_\mathbb{C}) \times \mathcal{M} \ni (F, m) \mapsto F \otimes m.$$

In addition, we fix a subgroup $\mathcal{M}_{[\mu]}$ of $\mathcal{M}$ for each for each W-coset $[\mu] \in \Lambda/W$.

A function $f\colon \Lambda \to \mathcal{M}$ is called a coherent family based on Λ if it satisfies $f(\mu) \in \mathcal{M}_\mu$ and

$$F \otimes f(\mu) = \sum_{\nu \in \Delta(F)} f(\mu + \nu) \qquad \forall \mu \in \Lambda, F \in \Pi_{\Lambda_0}(G_\mathbb{C}).$$

Let $\mathrm{Coh}_\Lambda(\mathcal{M})$ be the abelian group of all coherent families based on Λ and value in $\mathcal{M}$.

In this paper, we will consider the following cases.

Example 1.2. *Suppose $\mathcal{M} = \mathbb{Q}$ and $F \otimes m = \dim(F) \cdot m$ for $F \in \Pi_{\Lambda_0}(G_\mathbb{C})$ and $m \in \mathcal{M}$. We let $\mathcal{M}_\mu = \mathcal{M}$ for every $\mu \in \Lambda$. When $\Lambda = \Lambda_0$, the set of W-harmonic polynomials on ${}^a\mathfrak{h}^*$ is natrually identified with $\mathrm{Coh}_\Lambda(\mathcal{M})$ via restriction (Vogan's result)*

Example 1.3. *Let $\mathcal{G}(\mathfrak{g}, K)$ be the Grothendieck group of finite length $(\mathfrak{g}, K)$-modules and $\mathcal{G}_\mu(\mathfrak{g}, K)$ be the subgroup of $\mathcal{G}(\mathfrak{g}, K)$ generated by the set of irreducible $(\mathfrak{g}, K)$-modules with infinitesimal character μ.*

Then $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$ is the group of coherent families of Harish-Chandra modules.

Example 1.4. *Fixing a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, let $\mathcal{G}(\mathfrak{g}, \mathfrak{b})$ be the Gorthendieck group of the category $\mathscr{O}^\mathfrak{b}$. The space $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, \mathfrak{b}))$ is defined similarly.*

Note that the lattice Λ is stable under the $W(\lambda)$ action. We can define $W(\lambda)$ action on $\mathrm{Coh}_\Lambda(\mathcal{M})$ by

$$w \cdot f(\mu) = f(w^{-1}\mu) \qquad \forall \mu \in \Lambda, w \in W(\lambda).$$

Translation principal assumption. Recall that we have fixed a set of simple roots of W_Λ. We make the following assumption for $\mathrm{Coh}_\Lambda(\mathcal{M})$.

- There is a basis $\{ \Theta_\gamma \mid \gamma \in \mathrm{Parm} \}$ of $\mathrm{Coh}_\Lambda(\mathcal{M})$ where Parm is a parameter set;
- For every $\mu \in \Lambda$, the evalutation at μ is surjective;
$$\mathrm{ev}_\mu\colon \mathrm{Coh}_\Lambda(\mathcal{M}) \to \mathcal{M}_{[\mu]} \qquad f \mapsto f(\mu)$$
is surjective;
- a subset $\tau(\gamma)$ of the simple roots of W_Λ is attached to each $\gamma \in \mathrm{Parm}$ such that $s \cdot \Theta_\gamma = -\Theta_\gamma$;
- $\Theta_\gamma(\mu) = 0$ if and only if $\tau(\gamma) \cap R_\mu \neq \varnothing$.
- $\{ \Theta_\gamma(\mu) \mid \tau(\gamma) \cap R_\mu = \varnothing \}$ form a basis of $\mathcal{M}_\gamma$.

The translation principle assumption implies

$$\mathrm{Ker}\,\mathrm{ev}_\mu = \mathrm{span}\,\{ \Theta_\gamma \mid \tau(\gamma) \cap R_\mu \neq \varnothing \} \tag{1.1}$$

Now we have the following counting lemma.

Lemma 1.5. *For each μ, we have*

$$\dim \mathcal{M}_{[}\mu] = \dim(\mathrm{Coh}_\Lambda(\mathcal{M}))_{W_\mu} = [\mathrm{Coh}_\Lambda(\mathcal{M}), 1_{W_\mu}].$$

Proof. Clearly

$$\mathrm{span}\,\{ \Theta_\gamma - w\Theta_\gamma \mid \gamma \in \mathrm{Parm}, w \in W_\Lambda \} \subseteq \ker \mathrm{ev}_\mu$$

since $w \cdot \Theta_\gamma(\mu) = \Theta_\gamma(w^{-1} \cdot \mu) = \Theta_\gamma(\mu)$ for $w \in W_\mu$. Combine this with (1.1), we conclude that

$$\mathrm{span}\,\{ \Theta_\gamma - w\Theta_\gamma \mid \gamma \in \mathrm{Parm}, w \in W_\Lambda \} = \ker \mathrm{ev}_\mu.$$

Therefore, ev_μ induces an isomorphism $(\mathrm{Coh}_\Lambda(\mathcal{M}))_{W_\mu} \to \mathcal{M}_{[\mu]}$. Now the dimension equality follows. $\square$

Example 1.6. *For the case of category $\mathscr{O}$. We can take* Parm $= W_\Lambda$. *Let* $\Theta_w(\mu) = L(w\mu)$ *for each* $w \in W_\Lambda$ *with* $\tau(w) = \{\, s_\alpha \mid \alpha \in \Delta^+, w\alpha \notin R^+ \,\}$.

Example 1.7. *In the Harish-Chandra module case,* Parm *consists of certain irreducible K-equavariant local systems on a K-orbit of the flag variety of G. $\tau(\gamma)$ is the τ-invariants of the parameter γ.*

2. Parameterize of Unipotent representations

We fix an abstract complex Cartan subgroup $\mathbf{H}_a$ and $\mathfrak{h}_a$ in $\mathbf{G}$ and a set of simple roots Π_a. Let $\mathcal{P}(\mathbf{G})$ be the set of all Langlands parameters of G-modules with character ρ (i.e. the infinitesimal character of the trivial representation). For $\gamma \in \mathcal{P}(\mathbf{G})$, let $\mathcal{L}(\gamma)$, $\mathcal{S}(\gamma)$ and Φ_γ be the corresponding Langlands quotient, standard module and coherent family such that $\Phi_\gamma(\rho) = \mathcal{L}(\gamma)$. Let $\mathcal{M}(\mathbf{G})$ be the span of $\mathcal{L}(\gamma)$. Let $\{\mathbb{B}\}$ be the set of all blocks. Then $\mathcal{P}(\mathbf{G}) = \bigsqcup_{\mathcal{B}} \mathcal{B}$. The Weyl group $W = W(G)$ acts on $\mathcal{M}(\mathbf{G})$ by coherent continuation. Let $\mathcal{M}_{\mathcal{B}}$ be the submodule of $\mathcal{M}(\mathbf{G})$ spand by $\gamma \in \mathcal{B}$, then

$$\mathcal{M}(\mathbf{G}) = \bigoplus_{\mathcal{B}} \mathcal{M}_{\mathcal{B}}$$

Let $\tau(\gamma) \subset \Pi_a$ be the τ-invariant of γ.

Let $\check{\mathcal{O}}$ be even orbit. $\lambda = \frac{1}{2}\check{h}$. Define

$$S(\lambda) = \{\, \alpha \in \Pi_a \mid \langle \alpha, \lambda \rangle = 0 \,\}.$$

Let $\mathcal{P}_\lambda(\mathbf{G})$ be the set of all Langlands parameters with infinitesimal character λ. Let $T_{\lambda,\rho}$ be the translation functor. Let

$$\mathcal{B}(S) = \{\, \gamma \in \mathcal{B} \mid S \cap \tau(\gamma) = \varnothing \,\}$$

and

$$\mathcal{P}(\mathbf{G}, S) = \bigsqcup_{\mathcal{B}} \mathcal{B}(S)$$

Then

$$\begin{aligned} \mathcal{P}(\mathbf{G}, S) &\longrightarrow \mathcal{P}_\lambda(\mathbf{G}) \\ \gamma &\longmapsto T_{\lambda,\rho}(\gamma) \end{aligned}$$

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Let

$$\mathcal{B}(S, \mathcal{O}) = \{\, \gamma \in \mathcal{B}(S) \mid \mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}} \,\}$$

Let

$$\begin{aligned} m_S(\sigma) &= [\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] \\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{M}_{\mathcal{B}}] \end{aligned}$$

Barbasch [10, Theorem 9.1] established the following theorem.

Theorem 2.1.

$$|\mathcal{B}(S, \mathcal{O})| = \sum_{\sigma} m_{\mathcal{B}}(\sigma) m_S(\sigma)$$

Here $\sigma \times \sigma$ running over the $W \times W$ appears in the double cell $\mathcal{C}(\mathcal{O})$.

Proof. We need to take the graded module of $\mathcal{M}(\mathbf{G})$ with respect to the $\overset{LR}{\leqslant}$. By abuse of notation, we identify the basis $\mathcal{P}(\mathbf{G})$ with its image in the graded module. Note that $S \cap \tau(\lambda) = \varnothing$ if and only if $W(S)$ acts on γ trivially by [70, Lemma 14.7]. On the other hand, by [70, Theorem 14.10, and page 58], $\mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}}$ only if γ generate a W-module in the double cell of $\mathcal{O}$. □

Now assume $S = S(\lambda)$. By [12, Cor 5.30 b) and c)], $[\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] = [\mathbf{1}|_{W(S)} : \sigma] \leqslant 1$.

3. Counting in type A

The results in this section are well known to the experts.

Let YD be the set of Young diagrams viewed as a finite multiset of positive integers. The set of nilpotent orbits in $\mathrm{GL}_n(\mathbb{C})$ is identified with Young diagram of n boxs.

Let $\check{G}_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$. Fix an orbit $\check{\mathcal{O}} \in \mathrm{Nil}(ckG)$, let $\check{\mathcal{O}}_e$ (resp. $\check{\mathcal{O}}_o$) be the partition conssists of all even (resp. odd) rows in $\check{\mathcal{O}}$.

Let S_n deonte the Weyl group of $\mathrm{GL}_n(\mathbb{C})$. Let $W_n := S_n \ltimes \{\pm 1\}^n$ denote the Weyl group of type B_n or C_n. Let sgn denote the sign representation of the Weyl group. The group W_n is naturally embeded in S_{2n}. For W_n, let ϵ denote the unique non-trivial character which is trivial on S_n. Note that ϵ is also the restriction of the sgn of S_{2n} on W_n.

3.1. **Special unipotent repesentations of $G = \mathrm{GL}_n(\mathbb{C})$.** By [12], the set of unipotent representations of $G = \mathrm{GL}_n(\mathbb{C})$ one-one corresponds to nilpotent orbits in $\mathrm{Nil}(\check{G}_{\mathbb{C}})$. Suppose $\check{\mathcal{O}}$ has rows

$$\mathbf{r}_1(\check{\mathcal{O}}) \geqslant \mathbf{r}_2(\check{\mathcal{O}}) \geqslant \cdots \geqslant \mathbf{r}_k(\check{\mathcal{O}}) > 0.$$

Then $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}})$ has columns $\mathbf{c}_i(\mathcal{O}) = \mathbf{r}_i(\check{\mathcal{O}})$ for all $i \in \mathbb{N}^+$. The map $\check{\mathcal{O}} \mapsto \mathcal{O}$ is a bijection.

We set $\mathsf{CP} = \mathsf{YD}$ be the set of Young diagrams. For $\tau \in \mathsf{CP}$ which has k columns, let 1_c be the trivial representations of $\mathrm{GL}_c(\mathbb{C})$.

$$\pi_\tau = 1_{\mathbf{c}_1(\tau)} \times 1_{\mathbf{c}_2(\tau)} \times \cdots \times 1_{\mathbf{c}_k(\tau)}.$$

The Vogan duality gives a duality between Harish-Chandra cells. In this case, Harish-Chandra cells is the double cell of Lusztig. Now we have a duality

$$\pi_\tau \leftrightarrow \pi_{\tau^t}.$$

Let $\tau' := \nabla(\tau)$ be the partition obtained by deleting the first column of τ. Let $\theta_{a,b}$ (resp. $\Theta_{a,b}$) be the theta lift (resp. big theta lift) from $\mathrm{GL}_a(\mathbb{C})$ to $\mathrm{GL}_b(\mathbb{C})$. Then we have

$$\pi_\tau = \theta_{|\tau'|,|\tau|}(\pi_\tau).$$

3.2. **Counting unipotent repesentations of $\mathrm{GL}_n(\mathbb{R})$.** Now let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G}_{\mathbb{C}})$. Recall the decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_e \cup \check{\mathcal{O}}_o$. Let $n_e = |\check{\mathcal{O}}_e|, n_o = |\check{\mathcal{O}}_o|$ and $\lambda_{\check{\mathcal{O}}} = \frac{1}{2}\check{h}$.

Then

$$W(\lambda_{\check{\mathcal{O}}}) \cong S_{|\check{\mathcal{O}}_e|} \times S_{|\check{\mathcal{O}}_o|}.$$

$$W_{\lambda_{\check{\mathcal{O}}}} = \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_e)} \times \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_o)}$$

By the formula of a-function, one can easily see that The cell in $W(\lambda_{\check{\mathcal{O}}})$ consists of the unique representatoin $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W[\lambda_{\check{\mathcal{O}}}]}(1)$. Now the W-cell is $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W} 1$ corresponds to the orbit $\mathcal{O} = \check{\mathcal{O}}^t$ under the Springer correspondence. [WLOG, we assume $\check{\mathcal{O}} = \check{\mathcal{O}}_o$.

Let $\sigma \in \widehat{S_n}$. We identify σ with a Young diagram. Let $c_i = \mathbf{c}_i(\sigma)$. Then $\sigma = J_{W'}^{S_n}\epsilon_{W'}$ where $W' = \prod S_{c_i}$ (see Carter's book). This impiles Lusztig's a-function takes value

$$a(\sigma) = \sum_i c_i(c_i - 1)/2$$

Compairing the above with the dimension formula of nilpotent Orbits [17, Collary 6.1.4], we get (for the formula, see Bai ZQ-Xie Xun's paper on GK dimension of $SU(p,q)$)

$$\tfrac{1}{2}\dim(\sigma) = \dim(L(\lambda)) = n(n-1)/2 - a(\sigma).$$

Here $\dim(\sigma)$ is the dimension of nilpotent orbit attached to the Young diagram of σ (it is the Springer correspondence, regular orbit maps to trivial representation, note that $a(\mathrm{triv}) = 0$), $L(\lambda)$ is any highest weight module in the cell of σ.

Return to our question, let $S' = \prod_i S_{\mathbf{c}_i(\check{\mathcal{O}})}$. We want to find the component σ_0 in $\mathrm{Ind}_{S'}^{S_n} 1$ whose $a(\sigma_0)$ is maximal, i.e. the Young diagram of σ_0 is minimal.

By the branaching rule, $\sigma \subset \mathrm{Ind}_{S'}^{S_n} 1$ is given by adding rows of lenght $\mathbf{c}_i(\check{\mathcal{O}})$ repeatly (Each time add at most one box in each column). Now it is clear that $\sigma_0 = \check{\mathcal{O}}^t$ is desired.

This agrees with the Barbasch-Vogan duality d_{BV} given by

$$\check{\mathcal{O}} \xrightarrow{\textit{Springer}} \check{\mathcal{O}} \xrightarrow{\otimes \mathrm{sgn}} \check{\mathcal{O}}^t \xrightarrow{\textit{Springer}} \check{\mathcal{O}}^t.$$

]

The $W_{[\lambda_{\check{\mathcal{O}}}]}$-module $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}$ is given by the following formula:

$$\begin{aligned} \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]} &\cong \mathcal{C}_{n_e} \otimes \mathcal{C}_{n_o} \quad \text{with} \\ \mathcal{C}_n &:= \bigoplus_{\substack{s,a,b \\ 2s+a+b=n}} \mathrm{Ind}_{W_s \times S_a \times S_b}^{S_n} \epsilon \otimes 1 \otimes 1. \end{aligned}$$

According to Vogan duality, we can obtain the above formula by tensoring sgn on the forumla of the unitary groups in [8, Section 4].

By branching rules of the symmetric groups, $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ can be parameterized by painted partition.

$$\mathrm{PP}_{\check{\mathcal{O}}} = \left\{\, \tau \colon \mathrm{Box}(\check{\mathcal{O}}^t) \to \{\, \bullet, c, d \,\} \;\middle|\; \begin{array}{l} \text{“}\bullet\text{” occures with even} \\ \text{mulitplicity in each column} \end{array} \right\}.$$

[The typical diagram of all columns with even length $2c$ are

$\bullet$	$\cdots$	$\bullet$	$\bullet$	$\cdots$	$\bullet$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bullet$	$\cdots$	$\bullet$	c	$\cdots$	c
$\bullet$	$\cdots$	$\bullet$	d	$\cdots$	d

The typical diagram of all columns with odd length $2c+1$ are

$\bullet$	$\cdots$	$\bullet$	$\bullet$	$\cdots$	$\bullet$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\bullet$	$\cdots$	$\bullet$	$\bullet$	$\cdots$	$\bullet$
c	$\cdots$	c	d	$\cdots$	d

]

Let $\mathrm{sgn}_n \colon \mathrm{GL}_n(\mathbb{R}) \to \{\, \pm 1 \,\}$ be the sign of determinant. Let 1_n be the trivial representation of $\mathrm{GL}_n(\mathbb{R})$. For $\tau \in \mathrm{PP}_{\check{\mathcal{O}}}$, we attache the representation

$$\pi_\tau := \bigtimes_j \underbrace{1_j \times \cdots \times 1_j}_{c_j\text{-terms}} \times \underbrace{\mathrm{sgn}_j \times \cdots \times \mathrm{sgn}_j}_{d_j\text{-terms}}. \tag{3.1}$$

Here

- j running over all column lengths in $\check{\mathcal{O}}^t$,
- d_j is the number of columns of length j ending with the symbol "d",
- c_j is the number of columns of length j ending with the symbol "$\bullet$" or "c", and
- "$\times$" denote the parabolic induction.

3.3. **Special Unipotent representations of** $G = \mathrm{GL}_m(\mathbb{H})$**.** Suppose that $\mathcal{O}$ is the complexificiation of a rational nilpotent $\mathrm{GL}_m(\mathbb{H})$-orbit. Then $\mathcal{O}$ has only even length columns. Therefore, $\mathrm{Unip}_{\check{\mathcal{O}}}(G) \neq \varnothing$ only if $\check{\mathcal{O}} = \check{\mathcal{O}}_e$.

In this case the coherent continuation representation is given by

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) = \mathrm{Ind}_{W_m}^{S_{2m}} \epsilon$$

and $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is a singleton. For each partition τ only having even columns, we define

$$\pi_\tau := \bigtimes_i 1_{\mathbf{c}_i(\tau)/2}.$$

3.4. **Counting special unipotent repesentations of** $\mathrm{U}(p,q)$**.** We call the parity of $|\check{\mathcal{O}}|$ the "good pairity". The other pairity is called the "bad parity". We write $\check{\mathcal{O}} = \check{\mathcal{O}}_g \cup \check{\mathcal{O}}_b$ where $\check{\mathcal{O}}_g$ and $\check{\mathcal{O}}_b$ consist of good pairity length rows and bad pairity rows respectively.

Let $(n_g, n_b) = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$. Now as the $S_{n_g} \times S_{n_b}$

$$\bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=n}} \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathrm{U}(p,q)) = \mathcal{C}_{gd} \otimes \mathcal{C}_{bd}$$

where

$$\mathcal{C}_{gd} = \bigoplus_{\substack{s,a,b\in\mathbb{N}\\ 2s+a+b=n_g}} \mathrm{Ind}_{W_s\times S_a\times S_b}^{S_{n_g}} 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn}$$

$$\mathcal{C}_{bd} = \begin{cases} \mathrm{Ind}_{W_{\frac{n_b}{2}}}^{S_{n_b}} 1 & \text{if } n_b \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

By the above formula, we have

Lemma 3.1. *(i) The set* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,q)) \neq \varnothing$ *if and only if* $p = q$ *and each row lenght in* $\check{\mathcal{O}}$ *has even multiplicity.*

(ii) Suppose $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \varnothing$*, let* $\check{\mathcal{O}}'$ *be the Young diagram such that* $\mathbf{r}_i(\check{\mathcal{O}}') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ *and* π' *be the unique special uinpotent representation in* $\mathrm{Unip}_{\check{\mathcal{O}}'}(\mathrm{GL}_p(\mathbb{C}))$*. Then the unique element in* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p))$ *is given by*

$$\pi := \mathrm{Ind}_P^{\mathrm{U}(p,p)} \pi'$$

where P *is a parabolic subgroup in* $\mathrm{U}(p,p)$ *with Levi factor equals to* $\mathrm{GL}_p(\mathbb{C})$*.*

(iii) In general, when $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \varnothing$*, we have a natural bijection*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}(n_1+p,n_2+p)) \\ \pi_0 & \mapsto & \mathrm{Ind}_P^{\mathrm{U}(n_1+p,n_2+p)} \pi' \otimes \pi_0 \end{array}$$

where P *is a parabolic subgroup with Levi factor* $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{U}(n_1,n_2)$*.*

The above lemma ensure us to reduce the problem to the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Now assume $\check{\mathcal{O}} = \check{\mathcal{O}}_g$ and so $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ corresponds to the blocks of the infinitesimal character of the trivial representation.

By [8, Theorem 4.2], Harish-Chandra cells in $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ are in one-one correspondence to real nilpotent orbits in $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t$.

[From the branching rule, the cell is parametered by painted partition

$$\mathrm{PP}(\mathrm{U}) := \left\{ \tau \in \mathrm{PP} \;\middle|\; \begin{array}{l} \mathrm{Im}(\tau) \subseteq \{ \bullet, s, r \} \\ \text{“}\bullet\text{” occures even times in each row} \end{array} \right\}.$$

The bijection $\mathrm{PP}(\mathrm{U}) \to \mathsf{SYD}, \tau \mapsto \mathscr{O}$ is given by the following recipe: The shape of $\mathscr{O}$ is the same as that of τ. $\mathscr{O}$ is the unique (upto row switching) signed Young diagram such that

$$\mathscr{O}(i, \mathbf{r}_i(\tau)) := \begin{cases} +, & \text{when } \tau(i, \mathbf{r}_i(\tau)) = r; \\ -, & \text{otherwise, i.e. } \tau(i, \mathbf{r}_i(\tau)) \in \{ \bullet, s \}. \end{cases}$$

Example 3.2.

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & r \\ \bullet & \bullet & & & \\ s & r & & & \\ s & & & & \\ r & & & & \end{array} \quad \mapsto \quad \begin{array}{ccccc} + & - & + & - & + \\ + & - & & & \\ - & + & & & \\ - & & & & \\ + & & & & \end{array}$$

]

Now the following lemma is clear.

Lemma 3.3. *When $\check{\mathcal{O}} = \check{\mathcal{O}}_g$, the associated varity of every special unipotent representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U})$ is irreducible. Moreover, the following map is a bijection.*

$$\begin{array}{ccc} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1, n_2)) & \longrightarrow & \{ \textit{rational forms of } \check{\mathcal{O}}^t \} \\ \pi_0 & \mapsto & \mathrm{AV}^{\mathrm{weak}}(\pi_0). \end{array}$$

□

Remark. When $\check{\mathcal{O}}_b \neq \varnothing$, the special unipotent representations can have reducible associated variety. Meanwhile, it is easy to see that the map $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) \ni \pi \mapsto \mathrm{AV}^{\mathrm{weak}}(\pi)$ is still injective.

We will show that every elements in $\mathrm{Unip}_{\check{\mathcal{O}}_g}$ can be constructed by iterated theta lifting. For each τ, let $\mathscr{O}$ be the corresponding real nilpotent orbit. Let $\mathrm{Sign}(\mathscr{O})$ be the signature of $\mathscr{O}$, $\nabla(\mathscr{O})$ be the signed Young diagram obtained by deleteing the first column of $\mathscr{O}$. Suppose $\mathscr{O}$ has k-columns. Inductively we have a sequence of unitary groups $\mathrm{U}(p_i, q_i)$ with $(p_i, q_i) = \mathrm{Sign}(\nabla^i(\mathscr{O}))$ for $i = 0, \cdots, k$. Then

$$\pi_\tau = \theta_{\mathrm{U}(p_1,q_1)}^{\mathrm{U}(p_0,q_0)} \theta_{\mathrm{U}(p_2,q_2)}^{\mathrm{U}(p_1,q_1)} \cdots \theta_{\mathrm{U}(p_k,q_k)}^{\mathrm{U}(p_{k-1},q_{k-1})}(1) \tag{3.2}$$

where 1 is the trivial representation of $\mathrm{U}(p_k, q_k)$.

Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Form the duality between cells of $\mathrm{U}(p,q)$ and $\mathrm{GL}(n, \mathbb{R})$. We have an ad-hoc (bijective) duality between unipotent representations:

$$\begin{array}{rccc} d_{\mathrm{BV}} \colon & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) & \to & \mathrm{Unip}_{\check{\mathcal{O}}^t}(\mathrm{GL}(\mathbb{R})) \\ & \pi_\tau & \mapsto & \pi_{d_{\mathrm{BV}}(\tau)} \end{array}$$

Here $\check{\mathcal{O}}^t = d_{\mathrm{BV}}(\check{\mathcal{O}})$ and $d_{\mathrm{BV}}(\tau)$ is the pained bipartition obtained by transposeing τ and replace s and r by c and d respectively. See (3.2) and (3.1) for the definition of special unipotent representations on the two sides.

4. Counting in type BC

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