COMBINATORICS FOR UNIPOTENT REPRESENTATIONS

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Contents

1. Relevent nilpotent orbits

Let

$$\mathcal{O} = (C_{2a}, C_{2a-1}, \cdots, C_1, C_0, C_{-1} = 0)$$

be a special orbit. The dual orbit

$$\check{\mathcal{O}} = [R_{2a} \geqslant R_{2a-1} \geqslant \cdots \geqslant R_1 \geqslant R_0].$$

The $\check{\mathcal{O}}$ is given by add a box on the longest row of $\mathcal{O}^T$ and then do B-collapse.

Definition 1.1. *Suppose* $\mathbf{G}$ *is type* C. *An special orbit* $\mathcal{O}$ *is called purely even if* $\check{\mathcal{O}}$ *only has odd rows. This is equivalent to the condition on columns of* $\mathcal{O}$ *such that (see Lemma 4.1)*

(a) $C_{2i} \equiv C_{2i-1} \pmod 2$*;*
(b) If C_{2i-1} *is even,* $C_{2i-1} > C_{2i-2}$*;*
(c) If C_{2i} *is odd,* $C_{2i} = C_{2i-1}$*.*

We call a purely even orbit $\mathcal{O}$ *is stablelly trivial when only even columns occures. This is equivalent to*

$$C_{2i} \equiv C_{2i-1} \equiv 0 \pmod 2 \text{ and } C_{2i-1} > C_{2i-2} \quad \forall i.$$

In this case all LS in the unipotent representations are irreducible. This is also equivalent to

$$R_{2i} > R_{2i-1} \quad \forall i$$

We call a purely even orbit $\mathcal{O}$ *is theta-admissible when each row columns appears at most* 2*-times. All unipotent representations of these orbits are expected to constructed by*

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theta correspondence. Moreover, they are parameterized by the their local systems. This is also equivalent to each row lenght appears at most 3-times in $\check{\mathcal{O}}$.

Suppose $\mathcal{O}$ is purely even. The dual orbit $\check{\mathcal{O}} = [R_{2a} \geqslant R_{2a-1} \geqslant \cdots \geqslant R_1 \geqslant R_0]$ is calculated in the following way:

$$[R_{2i}, R_{2i-1}] = \begin{cases} [C_0+1, 0] & i = 0 \\ [C_{2i}+1, C_{2i-1}-1] & i > 0, C_{2i} \text{ is even} \\ [C_{2i}, C_{2i-1}] & i > 0, C_{2i} \text{ is odd} \end{cases}$$

Definition 1.2. *Suppose* $\mathbf{G}$ *is type D. An special orbit*

$$\mathcal{O} = (C_{2a-1} \geqslant \cdots \geqslant C_0 \geqslant C_{-1} = 0) \quad C_{2i} \equiv C_{2i-1} \pmod 2, C_{2a-1} \equiv 0 \pmod 2$$

is called purley even if $\check{\mathcal{O}}$ only has odd rows. This is equivalent to the condition on columns of $\mathcal{O}$ such that (see Lemma 4.1)

(a) $C_{2i} \equiv C_{2i-1} \pmod 2$*;*
(b) If C_{2i-1} *is even,* $C_{2i-1} > C_{2i-2}$*;*
(c) If C_{2i} *is odd,* $C_{2i} = C_{2i-1}$*.*

Note that this is the same condition as the type C case.

Suppose $\mathcal{O}$ is purely even. The dual orbit $\check{\mathcal{O}} = [R_{2a-1} \geqslant \cdots \geqslant R_1 \geqslant R_0, R_{-1} = 0]$ is calculated in the following way:

$$[R_{2i}, R_{2i-1}] = \begin{cases} [C_0+1, 0] & i = 0 \\ [0, C_{2a-1}-1] & i = a \\ [C_{2i}+1, C_{2i-1}-1] & a > i > 0, C_{2i} \text{ is even} \\ [C_{2i}, C_{2i-1}] & a > i > 0, C_{2i} \text{ is odd} \end{cases}$$

2. Parameterize of Unipotent representations

We fix an abstract complex Cartan subgroup $\mathbf{H}_a$ and $\mathfrak{h}_a$ in $\mathbf{G}$ and a set of simple roots Π_a. Let $\mathcal{P}(\mathbf{G})$ be the set of all Langlands parameters of G-modules with character ρ (i.e. the infinitesimal character of the trivial representation). For $\gamma \in \mathcal{P}(\mathbf{G})$, let $\mathcal{L}(\gamma)$, $\mathcal{S}(\gamma)$ and Φ_γ be the corresponding Langlands quotient, standard module and coherent family such that $\Phi_\gamma(\rho) = \mathcal{L}(\gamma)$. Let $\mathcal{M}(\mathbf{G})$ be the span of $\mathcal{L}(\gamma)$. Let $\{\mathbb{B}\}$ be the set of all blocks. Then $\mathcal{P}(\mathbf{G}) = \bigsqcup_{\mathcal{B}} \mathcal{B}$. The Weyl group $W = W(G)$ acts on $\mathcal{M}(\mathbf{G})$ by coherent continuation. Let $\mathcal{M}_{\mathcal{B}}$ be the submodule of $\mathcal{M}(\mathbf{G})$ spand by $\gamma \in \mathcal{B}$, then

$$\mathcal{M}(\mathbf{G}) = \bigoplus_{\mathcal{B}} \mathcal{M}_{\mathcal{B}}$$

Let $\tau(\gamma) \subset \Pi_a$ be the τ-invariant of γ.

Let $\check{\mathcal{O}}$ be even orbit. $\lambda = \frac{1}{2}\check{h}$. Define

$$S(\lambda) = \{\, \alpha \in \Pi_a \mid \langle \alpha, \lambda \rangle = 0 \,\}.$$

Let $\mathcal{P}_\lambda(\mathbf{G})$ be the set of all Langlands parameters with infinitesimal character λ. Let $T_{\lambda,\rho}$ be the translation functor. Let

$$\mathcal{B}(S) = \{\, \gamma \in \mathcal{B} \mid S \cap \tau(\gamma) = \varnothing \,\}$$

and

$$\mathcal{P}(\mathbf{G}, S) = \bigsqcup_{\mathcal{B}} \mathcal{B}(S)$$

Then

$$\begin{aligned} \mathcal{P}(\mathbf{G}, S) &\longrightarrow \mathcal{P}_\lambda(\mathbf{G}) \\ \gamma &\longmapsto T_{\lambda,\rho}(\gamma) \end{aligned}$$

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Let

$$\mathcal{B}(S, \mathcal{O}) = \{\, \gamma \in \mathcal{B}(S) \mid \mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}} \,\}$$

Let

$$\begin{aligned} m_S(\sigma) &= [\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] \\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{M}_{\mathcal{B}}] \end{aligned}$$

Barbasch [10, Theorem 9.1] established the following theorem.

Theorem 2.1.

$$|\mathcal{B}(S, \mathcal{O})| = \sum_{\sigma} m_{\mathcal{B}}(\sigma) m_S(\sigma)$$

Here $\sigma \times \sigma$ running over the $W \times W$ appears in the double cell $\mathcal{C}(\mathcal{O})$.

Proof. We need to take the graded module of $\mathcal{M}(\mathbf{G})$ with respect to the $\overset{LR}{\leqslant}$. By abuse of notation, we identify the basis $\mathcal{P}(\mathbf{G})$ with its image in the graded module. Note that $S \cap \tau(\lambda) = \emptyset$ if and only if $W(S)$ acts on γ trivially by [70, Lemma 14.7]. On the other hand, by [70, Theorem 14.10, and page 58], $\mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}}$ only if γ generate a W-module in the double cell of $\mathcal{O}$. □

Now assume $S = S(\lambda)$. By [12, Cor 5.30 b) and c)], $[\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] = [\mathbf{1}|_{W(S)} : \sigma] \leqslant 1$.

3. Combinatorics of Weyl group representations

The irreducible representations of S_n are parameterized by Young diagrams. We use the notation that the trivial representation corresponds to a row and the sign representation corresponds to a column.

$$\mathit{triv} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline n & & b & o & x & e & s \\ \hline\end{array} \qquad \mathrm{sgn} \leftrightarrow \begin{array}{|c|}\hline n \\ \hline \\ \hline b \\ \hline o \\ \hline x \\ \hline e \\ \hline s \\ \hline\end{array}$$

[This notation coincide with the Springer correspondence, where a row of boxes represents the regular nilpotent orbit and the corresponding representation is the trivial representation. The trivial orbit yeilds the sign representation.]

Under the above paramterization, $\tau \mapsto \tau \otimes \mathrm{sgn}$ is described by the transpose of Young diagram. The branching rule is given by Littlewoods-Richardson. Let $\mathcal{K} = \bigoplus_n \mathrm{Groth}(S_n)$ be the graded ring of the Grothendieck groups of S_n. This yeilds well defined ring structure on the graded algebra

$$\mu\nu := \mathrm{Ind}_{S_{|\mu|} \times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu \otimes \nu.$$

Notation, we will use $[r_1, r_2, \cdots, r_k]$ or $(c_1, c_2, \cdots, c_k)$ denote the Young diagram, where $\{\, r_i \,\}$ denote the lengthes of its rows and $\{\, c_i \,\}$ denote the lengthes of its columns.

Let W_n denote the Weyl group of type BC: $W_n = S_n \ltimes \{\, \pm 1 \,\}^n$. Now $\mathrm{Irr}(W_n)$ is paramterized by a pair of Young diagram, we use the notation $\mu \times \nu$.

As the tensor algebra, $\bigoplus_n \mathrm{Groth}(W_n) \cong \mathcal{K} \otimes \mathcal{K}$ via $\mu \times \nu \mapsto \mu \otimes \nu$.

The branching rule is given by: $(\mu_1 \times \nu_1)(\mu_2 \times \nu_2) = (\mu_1\mu_2) \times (\nu_1\nu_2)$. The sgn repersentation of W_n is parameterized by $\varnothing \times (n)$. From the definitioin of the identification of bipartition with $\mathrm{Irr}(W_n)$, we also have

$$(\mu \times \nu) \otimes \mathrm{sgn} = \nu^t \times \mu^t.$$

We also will use the following branching formula:

$$\mathrm{Ind}_{S_n}^{W_n} \mathrm{triv} = \sum_{\substack{a,b,\\ a+b=n}} [a] \times [b],$$

and

$$\mathrm{Ind}_{S_n}^{W_n} \mathrm{sgn} = \sum_{\substack{a,b,\\ a+b=n}} (a) \times (b). \tag{3.1}$$

[Sketch of the proof of the first formula, the dimension of LHS is $n!2^n/n! = 2^n$. On the other hand, by Mackey theory, $[LHS : \mathrm{Ind}_{W_a\times W_b}^{W_n} \mathrm{triv} \otimes \chi] \geqslant 1$ since $S_n \cap W_a \times W_b = S_a \times S_b$. Therefore, $LHS \supset RHS$. On the other hand, dimension of RHS is $\sum_a n!/a!b! = 2^n$. This finished the proof. The proof of the second formula is similar. One also can obtain it by tensoring with sgn of the first formula.]

4. Combinatorics for type C

4.1. **The coherent continuation representation of type C.** The detail of the coherent continuation representation was discussed in [48, p221 (7)].

[In McGovern's paper, the coherent continuation representation is described as:

$$\sum_{t,s,a,b} \mathrm{Ind}_{W_t\times(W_s\ltimes W(A_1)^s)\times W_a\times W_b}^{W_{t+2s+a+b}} \mathrm{sgn} \otimes (\mathrm{triv} \otimes \mathrm{sgn}) \otimes \mathrm{triv} \otimes \mathrm{triv}$$

Now (4.1) was obtained by the following branching formula: [48, p220 (6)]

$$I_n := \mathrm{Ind}_{(W_s\ltimes W(A_1)^s)}^{W_{2s}} \mathrm{triv} \otimes \mathrm{sgn} = \sum \lambda \times \lambda$$

where λ running over all Young diagrams of size s. As McGovern claimed the proof of the above formula is similar to Barbasch's proof of [8, Lemma 4.1]:

$$\mathrm{Ind}_{W_n}^{S_{2n}} \mathrm{triv} = \sum \sigma \quad \text{where } \sigma \text{ has even rows only.}$$

Sketch of the proof (use branching rule and dimension counting): Note that $\dim I_n = \frac{(2p)!2^{2p}}{p!2^{2p}} = (2p)!/p! = \sum_\lambda \dim \lambda \times \lambda$ (For the last equality: $\dim \lambda \times \lambda = (2p)!(\dim \lambda)^2/(p!)^2$ where $\dim \lambda$ is the dimension of S_n representation determined by λ; But $\sum (\dim \lambda)^2 = p!$). On the other hand, $H := W_s \ltimes W(A_1)^s \cap W_s \times W_s = \Delta W_s \subset W_{2s}$. $\mathrm{triv} \otimes \mathrm{sgn}\,|_H =$ sgn of ΔW_s Therefore, $\lambda \times \lambda$ appears in I_n by Mackey formula. Now by dimension counting, we get the formula.]

The Cartan subgroups are parametrized by three integers $(a, 2s, b)$ where $a+2s+b = n$. For the coherent continuation representation we need four integers $(t, 2s, p, q)$, satisfying $t + 2s + p + q = n$. The corresponding representation is

$$\sum_{\substack{p,q,t,s,\\ \tau\in\widehat{S_s}}} \mathrm{Ind}_{S_t\times W_{2s}\times W_p\times W_q}^{W_n} [\mathrm{sgn} \otimes (\tau \times \tau) \otimes \mathrm{triv} \otimes \mathrm{triv}]. \tag{4.1}$$

The notation is set up to take the duality in [70] of types B and C into account. So we write r for the sign representation of S_t, and c and c' for the trivial representation of W_p, W_q.

Now representations in the coherent continuation are parameterized by the labeled diagram $\tau_L \times \tau_R$: dots are filled in both sides forming the same shape of Young diagram (coesponding to $\tau \times \tau$ in (4.1)); a column of totaly t "r"s then added next to dots on the both sides (by (3.1)), i.e. at most one r is added in each row of the existing dots diagram (corresponding to S_t in (4.1)); then add p "c"s and finally add q "c'"s on the left side of the diagram; each c or c' is added to at most one column. Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step.

Not in use right now. We label the rows of τ_L as $r_0, r_2, \ldots, r_{2m}$ and the rows of τ_R as $r_1, r_3, \ldots, r_{2m-1}$ to conform to the notation of the special symbol

4.2. **Barbasch-Vogan dual.** Let $\mathcal{O} = (C_{2a}, C_{2a-1}, \cdots, C_1, C_0, C_{-1} = 0)$ be a special orbit.

The dual orbit $\check{\mathcal{O}} = [R_{2a} \geqslant R_{2a-1} \geqslant \cdots \geqslant R_1 \geqslant R_0]$.

The $\check{\mathcal{O}}$ is given by add a box on the longest row of $\mathcal{O}^T$ and then do B-collapse.

Lemma 4.1. *$\check{\mathcal{O}}$ is not even if and only if there is i such that $C_{2i-1} = C_{2i-2}$ is an even number > 0. In particular, there is no special unipotent representations attached to $\mathcal{O}$.*

Proof. Note that R_{2a} must be an odd number. Let i be the largest number satisfies the condition above, then $R_{2i-1} = C_{2i-1}$ is an even number. Therefore $\check{\mathcal{O}}$ is not even.

If there is on such kind of columns, then all rows in $\check{\mathcal{O}}$ have odd lengths. ☐

[Note that $C_0, C_{-1} = 0$ is even, $(C_0, C_{-1}) = (C_0, 0) \leftrightarrow [C_0 + 1, 0] =: [R_0, 0]$.
Suppose that C_{2i}, C_{2i-1} are even and $C_{2i-1} > C_{2i-2}$, then $(C_{2i}, C_{2i-1}) \leftrightarrow [C_{2i} + 1, C_{2i-1} - 1] =: [R_{2i}, R_{2i-1}]$.
Suppose that $C_{2i} = C_{2i-1}$ are odd, then $(C_{2i}, C_{2i-1}) \leftrightarrow [C_{2i}, C_{2i-1}] =: [R_{2i}, R_{2i-1}]$.]

Definition 4.2. *Suppose* $\mathbf{G}$ *is type* C*. An special orbit* $\mathcal{O}$ *is called purley even if* $\check{\mathcal{O}}$ *only has odd rows.*

4.3. **Nilpotent orbit of type C and Springer correspondence.** Nilpotent orbits of type C are parameterized by partitions

$$\mathbf{p} := (C_{2a}, C_{2a-1}, \cdots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i} \equiv C_{2i-1} \pmod 2.$$

Spacial orbits are those orbits satisfing $C_{2i} = C_{2i-1}$ when C_{2i} is odd. [This is exactly the condition that the transpose of the Young diagram is also type C.]

The Springer correspondence restricted on the set of special orbit is given as the following: Write $C_0 = 2c_0$, $C_{2i} = 2c_{2i} + \epsilon_i$ and $C_{2i-1} = 2c_{2i-1} - \epsilon_i$. The (special) Weyl group representation associated to the special orbit $\mathcal{O}$ has columns

$$\tau_L \times \tau_R = (c_{2a-1}, \ldots, c_1) \times (c_{2a}, \ldots, c_0). \tag{4.2}$$

[The above formula could be deduced from Lusztig's interpration of Springer correspondence. Which is computed as the following: (We follow Lusztig's description of "generalized" Springer correspondence in [46, § 11, § 12], see also McGovern's description on p80 of his Left cells and domino tableaux paper. Warning: the algorithm of Lusztig and McGovern are different but essentially the same, we use McGovern's version.)

0. Suppose $\mathcal{O}^t$ is a Young diagram of type C has rows (later $\mathcal{O}^t$ will be the dual of a special orbit $\mathcal{O}$ of type C).

$$[C_{2a}, \cdots, C_0].$$

1. Form a string of numbers

$$C_0, C_1 + 1, \cdots, C_i + i, \cdots, C_{2a} + 2a.$$

Note that if $C_{2i} = C_{2i-1} = 2c_{2i}+1$ is odd, then $C_{2i}+2i = 2c_{2i}+1+2i = 2c_{2i-1}+(2i-1)$ and $C_{2i-1}+2i-1 = 2c_{2i}+2i$; if $C_{2i} = 2c_{2i}, C_{2i-1} = 2c_{2i-1}$ are both even, then $C_{2i}+2i = 2c_{2i}+2i$, $C_{2i-1}+2i-1 = 2c_{2i-1}+(2i-1)$.

Therefore, the even parts of the above sequence is

$$2c_0, \cdots, 2c_{2i}+2i, \cdots 2c_{2a}+2a,$$

the odd part of the above sequence is

$$2c_1+1, \cdots, 2c_{2i-1}+(2i-1), \cdots, 2c_{2a-1}+(2a-1).$$

2. Note that the odd parts is shorter than the even parts, we get the symbol

$$\begin{pmatrix} c_0 & & c_2+1 & & \cdots & & c_{2i}+i & & \cdots & & & & \cdots & & c_{2a}+a \\ & c_1 & & & & c_3+1 & & \cdots & & c_{2i+1}+i & & \cdots & & c_{2a-1}+a-1 & \end{pmatrix}$$

3. This symbol corresponds to the representation

$$[c_{2a}, \cdots, c_2, c_0] \times [c_{2a-1}, \cdots, c_3, c_1].$$

4. There is one flaw of the above calculation, we use columns instead of the row. Use duality to fix the problem which yeilds Barbasch's answer: Let $\pi_{\mathcal{O}}$ be the Special representation attached to the special orbit $\mathcal{O}$; then

$$\pi_{\mathcal{O}} = \pi_{\mathcal{O}^t} \otimes \mathrm{sgn} = (c_{2a-1}, \cdots, c_3, c_1) \times (c_{2a}, \cdots, c_2, c_0).$$

]

4.3.1. *Some left cells in type C.* Now recall some results about cells of Weyl group representation. A symbol (of defact 1) is a pair of integer sequence such that $0 < a_0 < \cdots < a_s$ and $b_0 < b_1 < \cdots < b_{s-1}$

$$\sigma := \begin{pmatrix} a_0 & & a_1 & & \cdots & & a_s \\ & b_0 & & b_1 & \cdots & b_{s-1} & \end{pmatrix}.$$

Now σ correspond to irreducible W_n representation with rows:

$$[a_s - s, \cdots, a_1 - 1, a_0] \times [b_{s-1} - (s-1), \cdots, b_1 - 1, b_0]$$

σ is called special if $a_0 \leqslant b_0 \leqslant a_1 \leqslant b_1 \cdots \leqslant b_{s-1} \leqslant a_S$. Define

$$\mathrm{rank}(\sigma) := \sum a_i + \sum b_i - (s(s+1)/2 + s(s-1)/2) = \sum a_i + \sum b_i - s^2.$$

On the set of symbols, one define the shift operation:

$$\begin{pmatrix} a_0 & & a_1 & & \cdots & & a_s \\ & b_0 & & b_1 & \cdots & b_{s-1} & \end{pmatrix} \mapsto \begin{pmatrix} 0 & & a_0+1 & & a_1+1 & & \cdots & & a_s+1 \\ & 0 & & b_0+1 & & b_1+1 & \cdots & b_{s-1}+1 & \end{pmatrix}$$

For a symbol σ, let $[\sigma]$ denote the equivalent class of symbols under the shift operation. Warning: Lusztig's definition of rank may be different in his different papers. [In general symbol is a pair of integers $\begin{pmatrix} A \\ B \end{pmatrix}$. $\mathrm{rank}(\begin{pmatrix} A \\ B \end{pmatrix}) := \sum a_i + \sum b_i - (\frac{|A|+|B|-1}{2})^2$ which equals to $\sum a_i + \sum b_i - s^2$]

Double cells of $\widehat{W}_n$ are parameterized by the special symbols class of rank n. The dobule cell $\mathcal{C}_\sigma$ of a given special symbol σ consists all representations corresponding to symbols with the same set of entries of the special symbol σ.

Now recall the definition of L. Fix a special symbol σ as above, let $T = \{ t_1, t_2, \cdots, t_p \}$ be the set of elements in σ only appears in the top row, and $B = \{ b_1, b_2, \cdots, b_{p-1} \}$ be the set of elements only appears in the bottom row. [(It is obvious that $|T| = |B| + 1$ by the definition of symbol since each number appears at most twice.)] $L := \{ (t_i, b_i) \mid i = 1, 2, \cdots, p-1 \} \subset \mathbb{F}_2^T \times \mathbb{F}_2^B$ (Here $|\mathbb{F}_2^S|$ is naturally identified with the power set of the set S). $R := \{ (t_{i+1}, b_i) \mid i = 1, 2, \cdots, p-1 \}$.

Left cells are more complicated, they are parameterized by supersmooth subspace of L, where L is a $\mathbb{F}_2$ vector space attached to the spacial symbol σ. There are two very important supersmooth subspaces L and 0.

The left cell $\mathcal{C}_L$ consists of all symbols obtained by transfering some t_i to the bottom row and corresponding b_i to the top row. The left cell $\mathcal{C}_0$ consists of all symbols obtained by transfering some t_{i+1} to the bottom row and corresponding b_i to the top row.

Another way to obtain $\mathcal{C}_L$: it consists of symbols obtained by interchanging some of the entries which only appears in the bottom row to their closest neighbors to the *left* which only appears in the top row.

Similarly $\mathcal{C}_0$ consists of symbols obtained by interchanging some of the entries which only appears in the bottom row to their closest neighbors to the *right* which only appears in the top row.

Note that a symbol is in certain $\mathcal{O}_L$ if only if its entry satisfies $a_{i-1} \leqslant b_i \leqslant a_{i+1}$ $\forall i$.

A symbol is in certain $\mathcal{O}_0$ if only if its entry satisfies $a_i \leqslant b_i \leqslant a_{i+2}$ $\forall i$.

This is equivalent to [48, p 219 (4)]. Note that a special symbol may have configuration

$$\begin{array}{cccccccccc} \cdots & & a_i & & \cdots & & a_{i+s} & & \cdots & \\ & \cdots & & b_i & & \cdots & & b_{i+s} & & \end{array}$$

This type of segment is irrelavent to the "interchange" operation.

Suppose we have a segement as the following:

$$\begin{array}{cccccccccccc} \cdots & & top_j & & a_{i+1} & & \cdots & & a_{i+s+1} & & & \cdots \\ & \cdots & & b_i & & & \cdots & b_{i+s} & & & bot_j & \end{array}$$

After the interchange, we will have the configuration:

$$\begin{array}{cccccccccc} \cdots & & a_{i+1} & & \cdots & & a_{i+s+1} & & bot_j & \cdots \\ & \cdots & & top_j & & b_i & & \cdots & & b_{i+s} \end{array}$$

By observing the above pictures, a moment though yields the claim.

View W_n as the Weyl group of type C, then $\mathcal{C}_L$ is called Lusztig and $\mathcal{C}_0$ is called Springer.

McGovern claims that: The set of all "Springer" symbols coincide with all symbols obtain by Springer correspondence from a nilpotent orbit of type C (with trivial local system).

I can not varify that. It seem that this dose not agree with Lusztig's results, see Carter's book p420 paragraph-3 on Springer correspondence. [This is McGovern's claim, haven't checked yet.]

Let $\tau := [\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_{r+1} \geqslant 0] \times [\nu_1 \geqslant \nu_2 \cdots \geqslant \nu_r] \in \widehat{W_n}$ then $\tau \in \mathcal{C}_L$ if and only if

$$\lambda_i + 1 \geqslant \mu_i \geqslant \lambda_{i+2} - 1. \tag{4.3}$$

[τ has symbol

$$\begin{pmatrix} \lambda_{r+1} & & \lambda_r + 1 & & \cdots & & \lambda_2 + r - 1 & & \lambda_1 + r \\ & \nu_r & & \nu_{r-1} + 1 & & \cdots & & \nu_1 + r - 1 & \end{pmatrix}$$

Now the condition on symbol reads

$$\lambda_{i+1} + r - i - 1 = \lambda_{i+2} + r + 1 - (i+2) \leqslant \mu_i + r - i \leqslant \lambda_i + r + 1 - i.$$

]

Similarly, $\tau \in \mathcal{C}_0$ if and only if

$$\lambda_{i-1} + 2 \geqslant \nu_i \geqslant \lambda_{i+1}.$$

From now on, let $\tau \in \widehat{W_n}$ be a irreducible representation occure in (4.1). Then τ are Lusztig. [It is to varity the (4.3). It follows from the dot-r-c diagram construction. $\nu_i \leqslant \lambda_i + 1$ is clear since we will add at most one "r" on the right young digram. The inequality $\lambda_{i+2} - 1 \leqslant \mu_i$ follows form the consideration of "exterme cases" for the row $i, i+1, i+2$:

$$\begin{array}{cccccc} \cdots & \bullet & r & c & \cdots & \bullet \\ \cdots & \bullet & * & c' & \times \cdots & \\ \cdots & * & * & & \cdots & \end{array}$$

where $*$ could be $\bullet$, r,c, or c' when ever it is proper. Anyway, for the "exterme case", $\lambda_{i+1} - 1 = \mu_i + 1 - 1 \leqslant \mu_i$.] By duality, $\tau \otimes \mathrm{sgn}$ are all Springer (in the sense of type C representation).

To be check: Now we reach the conclusion of Barabasch/McGovern that: The other representations in the primitive ideal cell are obtained by interchanging pairs coming from even columns,

$$(c_{2i}, c_{2i-1}) \longleftrightarrow (c_{2i-1} - 1, c_{2i} + 1). \tag{4.4}$$

4.4. **Reductions in type C.** Let $\mathcal{O} = (C_{2a}, C_{2a-1}, \cdots, C_1, C_0 \geqslant 0)$ be a special orbit.

Now we start to count the number of unipotent representations. In this section, we reduce the counting problem to the certain orbits with restricted shapes.

4.4.1. *Delete odd rows.*

Proposition 4.3. *Let $\mathcal{O}'$ be the special orbit obtained by deleteing all odd rows of $\mathcal{O}$. There is a bijection from the dot-c-r diagrams attached to $\mathcal{O}$ to that of $\mathcal{O}'$. In particular,* $\mathrm{unip}(\mathcal{O})$ *and* $\mathrm{unip}(\mathcal{O}')$ *has the same number of elements.*

Proof. Let $\tau_{\mathcal{O}} := \tau_L \times \tau_R$ be the special representation attached to $\mathcal{O}$. We refer to (4.2) for the shape of $\tau_L \times \tau_R$. All other representations are obtain from $\tau_{\mathcal{O}}$ by "switching columns" using recipy (4.4). We use $\tau' = \tau'_L \times \tau'_R$ denote an arbitary W-representation in the cell of $\tau_{\mathcal{O}}$.

Delete the longest odd rows: Suppose the longest rows of $\mathcal{O}$ have odd size. This is equivalent to $C_0 = 2c_0 > 0$. Note that the largest row occur an even number of times. Now all τ' has the shape that τ'_R has one more column (the column c_0) than τ'_L. Therefore, the column for c_0 must be filled with r's. The remainder of the rows corresponding to the column of size c_0 in both τ_L and τ_R must be filled with $\bullet$'s.

The multiplicity is not affected if the first c_0 rows are removed from both τ'_L and τ'_R. The computation of multiplicities is reduced to the case where the first C_0 rows removed $c_0 = 0$.

The situation is illustrated below, delete the boxed parts.

$$\begin{array}{l} \boxed{\begin{array}{ccc} \cdots & \bullet & \bullet \\ \cdots & \vdots & \vdots \\ \cdots & \bullet & \bullet \end{array}} \\ \cdots \; * \\ \cdots \; * \\ \cdots \; \vdots \end{array} \times \begin{array}{l} \boxed{\begin{array}{cccc} \cdots & \bullet & \bullet & r \\ \cdots & \vdots & \vdots & \vdots \\ \cdots & \bullet & \bullet & r \end{array}} \\ \cdots \; * \\ \cdots \; * \\ \cdots \; \vdots \end{array}$$

Delete the odd rows of length $2a - 1$ Now assume $C_0 = 0$ but $C_2 > C_1$. Note that $\mathcal{O}$ is a special orbit. So C_2 and C_1 must be even columns, and $c_2 > c_1$.

First consider $\tau_{\mathcal{O}} = \tau_L \times \tau_R$, it must be filled as the following: The first c_1 rows in the column for c_2 must be filled by $\bullet'$s or $r's$. The remaining rows in c_2 are filled with r's. The remainder of rows $c_1 + 1$ to c_2 in both τ_L and τ_R (columns $c_3, c_4 \dots$) must be filled with $\bullet'$s. Therefore, delete the boxed part do not affect the multiplicity, see below.

$$
\begin{array}{cccc}
 & c_1 & & c_2 \\
\cdots * & \bullet/r & \cdots * & \bullet/r \\
\cdots \vdots & \vdots & \cdots \vdots & \vdots \\
\cdots * & \bullet/r & \cdots * & \bullet/r \\
\boxed{\begin{array}{c}\cdots\bullet\\ \cdots\vdots\\ \cdots\bullet\end{array}} & & \boxed{\begin{array}{cc}\cdots\bullet & r\\ \cdots\vdots & \vdots\\ \cdots\bullet & r\end{array}} & \\
\cdots * & \times & \cdots * & \\
\cdots * & & \cdots * & \\
\cdots \vdots & & \cdots \vdots &
\end{array}
$$

For the other representations obtained from $\tau_{\mathcal{O}}$ without switching c_1 and c_2, the situation similar.

Now consider the representations switching $(c_1, c_2) \leftrightarrow (c_2 + 1, c_1 - 1)$. The shape of the dot-r-c diagram must be as the following. Now delete the boxed part will not affect the multiplicities.

$$
\begin{array}{cccc}
 & c_2+1 & & c_1-1 \\
\cdots * & \bullet/r & \cdots * & \bullet/r \\
\cdots \vdots & \vdots & \cdots \vdots & \vdots \\
\cdots * & \bullet/r & \cdots * & \bullet/r \\
\boxed{\begin{array}{cc}\cdots\bullet & r\\ \cdots\vdots & \cdots\\ \cdots\bullet & r\end{array}} & & \boxed{\begin{array}{c}\cdots\bullet\\ \cdots\vdots\\ \cdots\bullet\end{array}} & \\
\cdots \bullet & r/c & \cdots \bullet & \\
\cdots \bullet & r/c' & \cdots \bullet & \\
\cdots * & \times & \cdots * & \\
\cdots * & & \cdots * & \\
\cdots \vdots & & \cdots \vdots &
\end{array}
$$

They correspond to the next largest odd rows of $\mathcal{O}$ and do not affect the number of dot-r-c diagrams.

In summary, delete the odd row of length $2a - 1$ do not affect the number of dot-r-c diagrams.

The general case Argue exactly in the same way, we see that delete odd rows of any length in $\mathcal{O}$ will not affect the number of dot-r-c diagrams.

□

Theorem 4.4 (Type C)**.** *The multiplicities of a $\tau_L \times \tau_R$ coincide with the case when $C_0 = 0$ and $c_{2i} = c_{2i-1}$ when C_{2i}, C_{2i-1} are both even, and $c_{2i} = c_{2i-1} - 1$ in case $C_{2i}C_{2i-1}$ are both odd.*

4.5. **Some examples.** For each complex nilpotent orbit $\mathcal{O}$. Let $\mathrm{DRC}(\mathcal{O})$ be the set of dot-r-c diagrams attached to $\mathcal{O}$. Let $\mathrm{LS}(\mathcal{O})$ be the set of local systems obtained by theta correspondence.

Here is some examples about counting.

5. Parameterization of unipotent representations

5.1. **Type D.** Combinatorics of type D.

Type D. Label the columns $C_{2a-1} \geqslant \cdots \geqslant C_0 \geqslant 0$ satisfying $C_{2i} \equiv C_{2i-1} \pmod 2$, and $C_{2a-1} \equiv C_0 \equiv 0 \pmod 2$. As before, write $C_{2i} = 2c_{2i} + \epsilon_i$, $C_{2i-1} = 2c_{2i-1} - \epsilon_i$ and $C_0 = 2c_0$, $C_{2a-1} = 2c_{2a-1}$. Special nilpotent orbits satisfy $C_{2i} = C_{2i-1}$ whenever they are odd. We will reduce to the case $C_{2i} = C_{2i-1}$ which is the case of $\mathcal{O}$ having no even sized rows. The special Weyl group representation associated to $\mathcal{O}$ has columns

$$\tau_L \times \tau_R = (c_{2a-1}, \dots, c_1) \times (c_{2a-2}, \dots, c_0). \tag{5.1}$$

The other representations in the primitive ideal cell are obtained by interchanging pairs coming from even columns,

$$(c_{2i}, c_{2i-1}) \longleftrightarrow (c_{2i-1} + 1, c_{2i} - 1).$$

This is again as in [48]. The infinitesimal character $\lambda_{\mathcal{O}}$ is as in type C, (**??**).

Remark. Adding an even sized column of length at least $2c_{2a-1}$ gives a nilpotent orbit of type C, and implements the $\Theta-$correspondence.

In the case when all rows of odd size occur an even number of times, the corresponding unipotent representations can be parametrized by the local systems on the *real* orbits.

The unipotent representations of SO are parameterized by filling $\bullet$, r', r, c, c' in the relevent representations.

In the must of the cases, induce to the full orthogonal group will yields 2 different representations (Lets swich the left and right diagram to mark the difference). The only exception is that the diagram only contains "$\bullet$" in which case, the induce yields an irreducible O-moudle.

To compare with theta correspondence, we made the following choice. The theta lift from Sp always correspondes to the case where "r', r, c, c'" is filled on the left diagram.

Example 5.1. *We make the following choice of parameterization of* $\mathrm{O}(p,q)$ *representations. The advantage of the following choice is that, the occurance of "*c'*" is equalent to "spliting/doubling" of associate varieties.*

$$\begin{array}{l} s_0\left\{\begin{array}{c} r' \\ \vdots \\ r' \end{array}\right. \\ r_0\left\{\begin{array}{c} r \\ \vdots \\ r \\ r \end{array}\right. \end{array} \longleftrightarrow \mathbf{1}^{+,-}_{2r_0,2s_0} \qquad\qquad \begin{array}{l} s_0\left\{\begin{array}{c} r' \\ \vdots \\ r' \end{array}\right. \\ r_0\left\{\begin{array}{c} r \\ \vdots \\ r \end{array}\right. \\ \quad\ \ c \end{array} \longleftrightarrow \mathbf{1}^{+,-}_{2r_0+1,2s_0+1}$$

$$
\begin{array}{ll}
s_0 \left\{ \begin{array}{c} r' \\ \vdots \\ r' \end{array} \right. & \\
r_0 \left\{ \begin{array}{c} r \\ \vdots \\ r \end{array} \right. & \longleftrightarrow \mathbf{1}_{2r_0+2,2s_0+2} \\
\phantom{r_0 \{} \; c & \\
\phantom{r_0 \{} \; c' &
\end{array}
\qquad\qquad
\begin{array}{ll}
s_0 \left\{ \begin{array}{c} r' \\ \vdots \\ r' \end{array} \right. & \\
r_0 \left\{ \begin{array}{c} r \\ \vdots \\ r \\ r \end{array} \right. & \longleftrightarrow \mathbf{1}_{2r_0+1,2s_0+1} \\
\phantom{r_0 \{} \; c' &
\end{array}
$$

Now the lifting algorithm of diagrams is that

1. *replace r' by $\bullet$,*
2. *lifting case: add a colums of lenght c_{2a-1} to make a valid dot-r-c diagram.*
3. *lifting the determinant twist: do the usual lift, then move the left bottom corner "r" (if it exists) from the right diagram to the most left column of the left diagram. Otherwise, remove the left bottom corner "$\bullet$" and replace the most left column of the left diagram by $(\bullet \cdots \bullet\, cc')^T$.*
4. *generalized lifting case: add a colums of lenght $c_{2a-1} - 1$ and make a valid dot-r-c diagram. If there is no valid dot-r-c diagram, this representation dose not lift.*

$$
\begin{array}{ccccccc}
\pi & & \mathrm{DRC}(\pi) & & \mathrm{DRC}(\theta(\pi)) & & \mathcal{V}(\theta(\pi)) \\
 & & & & & & \begin{array}{cc} + & - \\ + & - \\ \vdots & \vdots \\ + & - \\ + & - \\ - & + \\ - & + \\ \vdots & \vdots \\ - & + \\ - & + \end{array} \\
\mathbf{1}^{+,-}_{2r_0,2s_0} & \longleftrightarrow & \begin{array}{c} r' \\ \vdots \\ r' \\ r \\ \vdots \\ r \end{array} & \longrightarrow & \begin{array}{ccc} \bullet & & \bullet \\ \vdots & & \vdots \\ \bullet & \times & \bullet \\ r & & r \\ \vdots & & \vdots \\ r & & r \end{array} & \longrightarrow &
\end{array}
$$

5.2. Lift drc diagram τ, corresponding to π from type C to type D

A type C special unipotent orbit attach a column of even length $C_{2a+1} = 2c_{2a+1}$.

Now the attach a column of length c_{2a+1} on the left of the τ_L. Allow all possible lables and change "$\bullet$" into "r'" to make a valid diagram, denote by $\tau' = \tau'_L \times \tau'_R$.

Count number of symbols to determine the signature of the orthogonal group. If "c'" is not presented in the new diagram, attach the diagram to the representation $\theta(\pi) \otimes \mathbf{1}^{+,-}$. If "$c'$" is presented, the new diagram is attached to $\theta(\pi)$.

By this choice, it is clear again that if no "$r/c/c'$" occure in the longest column of τ'_L then $\theta(\pi) \otimes \mathbf{1}^{+,-}$ will be killed by generalized lift. Moreover the appearance of c' implies a "splitting" of orbit in the generalized lift.

5.3. **The strategy of matching.** Let G and G' form a dual pair. Suppose a special nilpotent orbit $\mathcal{O}'$ is obtained by lift/generalized lift of $\mathcal{O}$ (i.e. by adding an even length column, or adding an odd length column and then delete a box on the previous column when is a symplectic group)

What we have to do is construct maps to form the following commutative diagram:

$$\begin{tikzcd} \mathrm{DRC}(\mathcal{O}) \arrow[r, dashed, "\eta"] \arrow[d, "\mathcal{L}"] & \mathrm{DRC}(\mathcal{O}') \arrow[d, "\mathcal{L}"] \\ \mathrm{LS}(\mathcal{O}) \arrow[r, dashed, "\vartheta"] & \mathrm{LS}(\mathcal{O}') \end{tikzcd}$$

If G is an orthogonal group, and $\mathcal{O}'$ is a normal lift of $\mathcal{O}$, the following additional diagram on the twisting of determinant also need to be establish: ($\mathcal{D}$ is the operation which change special shape diagram to the non-special shape)

$$\begin{tikzcd} \mathrm{DRC}(\mathcal{O}) \arrow[d, "\mathcal{L}"] \arrow[rrd, "\eta"] & & \\ \mathrm{LS}(\mathcal{O}) \arrow[r, "\vartheta"] \arrow[d, "\det\otimes"] & \mathrm{LS}(\mathcal{O}') & \mathrm{DRC}(\mathcal{O}') \arrow[l, "\mathcal{L}"'] \arrow[d, leftrightarrow, "\mathcal{D}"] \\ \mathrm{LS}(\mathcal{O}) \arrow[r, "\vartheta"] & \mathrm{LS}(\mathcal{O}') & \mathrm{DRC}(\mathcal{O}') \arrow[l, "\mathcal{L}"'] \end{tikzcd}$$

The map η and ϑ are the lifting map (running over all possible real forms of the real orthogonal group). The map $\mathcal{L}$ calculates the local system for each dot-r-c diagram.

The η maps is already defined in your notes, i.e. the algorithm of changing "r'" to "$\bullet$" (although I prefer a slightly different choice for the exceptional cases). From the definition of η, we could see all elements in $\mathrm{DRC}(\mathcal{O}')$ could be obtained from $\mathrm{DRC}(\mathcal{O})$ up to character twisting.

The key statement we must establish is that $\mathcal{L}$ is an injection.

6. Counting

In this section, we use $(c_1, c_2, ..., \star)$ denote the object obtained by adding columns $c_1, c_2, ...$ to $\star$ where $\star$ could be a Young diagram or a dot-r-c diagram.

Let $\mathrm{DRC}(\tau)$ denote all drc diagrams of shape to $\tau = \tau_L \times \tau_R$. Let $\nabla\tau = \nabla\tau_L \times \nabla\tau_R$ be the "descent" of representation where $\nabla\tau_L$ (resp. $\nabla\tau_R$) is obtained from τ_L (resp. τ_R) by reomoving the first column.

Clearly, ∇ induces a welldefined map $\nabla\colon \mathrm{DRC}(\tau) \longrightarrow \mathrm{DRC}(\nabla\tau)$. The map ∇ extends to a map

$$\nabla_{\mathcal{O}}\colon \mathrm{DRC}(\mathcal{O}) \longrightarrow \mathrm{DRC}(\nabla^2\mathcal{O}).$$

6.1. **Add even columns.** In this section we consider the process of adding two even length colums.

Example 6.1. *Suppose* $\mathcal{O} = (C_{2a}, C_{2a-1}, \cdots, C_1, C_0)$ *satisfies* C_{2a}, C_{2a-1} *are even. Attaching a even column* $(C_{2a} + 2d, C_{2a-1} + 2d)$ *yields a unipotent orbit*

$$\mathcal{O}'' = (C_{2a} + 2d, C_{2a-1} + 2d) + \mathcal{O}$$

We have

$$\#\mathrm{DRC}(\mathcal{O}'') = 8\#\mathrm{DRC}(\mathcal{O})$$

In particular, when $\mathcal{O}$ only has even length columns, we have

$$\#\mathrm{DRC}(\mathcal{O}) = \#\mathrm{LS}(\mathcal{O}).$$

This is easy by induction.

Example 6.2. *Assume n is odd. Consider $\mathcal{O} = (n, n)$ of $\mathrm{Sp}(2n)$. We have $\mathrm{DRC}(\mathcal{O}) = 2n + 1$. Check:*

$$\#\mathrm{LS}(\mathcal{O}) = 2n + 1.$$

In $\mathrm{LS}(\mathcal{O})$ there are n reducible local systems (the associated variety has two components). The rest $n + 1$ local systems are irreducible.

Suppose

$$\mathcal{O}' = (n + 2d - 1, n + 2d - 1, n, n)$$

Then

$$\#\mathrm{DRC}(\mathcal{O}) = 4(2d)n + 4(2d - 1)(n + 1).$$

Example 6.3. *Assume n is odd. Consider $\mathcal{O} = (C_{2a}, C_{2a}, \cdots)$ of $\mathrm{Sp}(2n)$ with C_{2a} even. Suppose*

$$\mathcal{O}' = (c_{2a} + n, c_{2a} + n) + \mathcal{O}$$

Then

$$\#\mathrm{DRC}(\mathcal{O}') = (2n + 1)\#\mathrm{DRC}(\mathcal{O}).$$

In $\mathrm{LS}(\mathcal{O}')$ there are n doubled local systems (the associated variety has two components). The rest $n + 1$ local systems are preserved.

Example 6.4. *Assume n is odd. Consider $\mathcal{O}_0 = (C_{2a}, C_{2a}, \cdots)$, C_{2a} $\mathcal{O} = (n, n)$ of $\mathrm{Sp}(2n)$. We have $\mathrm{DRC}(\mathcal{O}) = 2n + 1$. Check:*

$$\#\mathrm{LS}(\mathcal{O}) = 2n + 1.$$

In $\mathrm{LS}(\mathcal{O})$ there are n reducible local systems (the associated variety has two components). The rest $n + 1$ local systems are irreducible.

Suppose

$$\mathcal{O}' = (n + 2d - 1, n + 2d - 1, n, n)$$

Then

$$\#\mathrm{DRC}(\mathcal{O}) = 4(2d)n + 4(2d - 1)(n + 1).$$

7. Combinatorics for type B

In this section $\mathbf{G} = \mathrm{SO}(2n + 1, \mathbb{C})$.

7.1. Nilpotent orbit in type B.

Nilpotent orbits of $\mathbf{G}$ is parameterized Young diagrams with columns

$$\mathcal{O} = (C_{2a} \geqslant C_{2a-1} \geqslant \cdots C_1 \geqslant C_0 > 0)$$

satisfying

(a) $C_{2i+1} \equiv C_{2i} \pmod 2$ (This corresponds to the condition even rows occur even times.)

(b) $C_{2a} \equiv 1 \pmod 2$ (This corresponds to the total number of box is odd)

Note that $\mathcal{O}$ is special if and only if

$$C_{2i+1} = C_{2i} \text{ when } C_{2i} \text{ is even.}$$

This correspond to the condition that the transpose $\mathcal{O}^t$ is also type B.

Now let $\mathcal{O} = (C_{2a}, C_{2a-1}, \cdots, C_1, C_0)$ be a special nilpotent orbit of type B.

Then the Springer representation attached to $\mathcal{O}$ is given by (see[14, p 421])

[We follow Carter's notation. Note that $\mathcal{O}^t$ is also type B, with rows $[C_{2a}, C_{2a-1}, \cdots, C_1, C_0]$. Now $\lambda^* = [C_0, C_1+1, \cdots, C_{2a}+2a]$. Write $C_{2i} = 2c_{2i} + \epsilon_i$, $C_{2i+1} = 2c_{2i+1} - \epsilon_i$, $\epsilon_i = 0$ or 1.

Now the odd parts of λ^* are So

$$\begin{aligned}(2\xi_1^*+1, \cdots, 2\xi_{a+1}^*+1) &= (2c_0+1, \cdots, 2c_{2(a-1)}+1+2(a-1), 2c_{2a}+1+2a)\\ (2\eta_1^*, \cdots, 2\eta_a^*) &= (2c_1+1, \cdots, 2c_{2(a-1)}+1+2(a-1), 2c_{2a}+1+2a)\end{aligned}$$

(The argument: C_{2a} must be odd. Note that, the parity of $(C_{2i}+2i, C_{2i+1}+2i+1)$ are different. Moreover, $C_{2i} = C_{2i+1}$ when they are even. So sppose C_{2i} is even, $2\xi_{i+1}^*+1 = 2c_{2i+1}+2i+1 = 2c_{2i}+2i+1$, $2\eta_{i+1}^* = 2c_{2i}+2i = 2c_{2i+1}+2i$. Suppose C_{2i} is odd, $2\xi_{i+1}^*+1 = (2c_{2i}+1)+2i$ and $2\eta_{i+1}^* = 2c_{2i+1}-1+2i+1 = 2c_{2i+1}+2i$)

Therefore, $\xi_{i+1}^* = c_{2i}+i$, $\eta^*i+1 = c_{2i+i}+i$. Hence, $\xi_{i+1} = c_{2i}$, $\eta_{i+1} = c_{2i+1}$. Therefore, the Springer representation of $\mathcal{O}^t$ is

$$[c_0, c_2, \cdots, c_{2a}] \times [c_1, c_3, \cdots, c_{2a-1}].$$

Now tensor sgn yeilds that $\mathcal{O}$ corresponds to the special representation

$$\tau_L \times \tau_R = (c_{2a-1}, \cdots, c_3, c_1) \times (c_{2a}, \cdots, c_2, c_0).$$

]

$$\tau_L \times \tau_R = (c_{2a-1}, \ldots, c_1) \times (c_{2a}, c_{2a-2}, \ldots, c_0). \tag{7.1}$$

The other representations in the primitive ideal cell are given by interchanging

$$(c_{2i+1}, c_{2i}) \longleftrightarrow (c_{2i}, c_{2i+1}).$$

This corresponds to the Springer representation (associated to the trivial local system) for another nilpotent orbit.

Now assume $\mathcal{O} = (C_{2a}, C_{2a-1}, \cdots, C_0 > 0)$ satisfies

$$C_{2i} > C_{2i-1} \quad \forall i.$$

Note that C_{2a} is always odd. The Barbasch-Vogan duality map is given by (for special orbit) the type C-collapse of $\mathcal{O}^t$ then remove the last box.

Then $\check{\mathcal{O}} = (R_{2a}, \cdots, R_0)$ is given by

$$\begin{aligned} R_{2a} &= C_{2a} - 1\\ (R_{2i+1}, R_{2i}) &= \begin{cases} (C_{2i+1}, C_{2i}) & \text{if } C_{2i} \text{ is even}\\ (C_{2i+1}+1, C_{2i}-1) & \text{if } C_{2i} \text{ is odd}\end{cases}\end{aligned}$$

Note that the infinitesimal character of $\mathbf{G}$ is given by

$$\chi_{\check{\mathcal{O}}} = (\rho_{R_{2a}}, \cdots, \rho_{R_0})$$

where

$$\rho_R = (\frac{R-1}{2}, \frac{R-3}{2} \cdots, \frac{1}{2}) \in \tfrac{1}{2} + \mathbb{Z}^{R/2}.$$

Remark. Recall that a type B orbit $\mathcal{O}$ is called quasi-distinguish if it has columns

$$\mathcal{O} = (C_{2a}, \cdots, C_0)$$

where all C_k are odd integers and $C_{2i} > C_{2i-1}$. Now $\check{\mathcal{O}} = (R_{2a}, \cdots, R_0) = (C_{2a}-1, C_{2a-1}+1, C_{2a-2}-1, \cdots, C_1+1, C_0-1)$. Clearly, R_k are all even integers and $R_{2i+1} > R_{2i}$.

7.2. **dot-c-r diagram.** We fix the embedding $\iota\colon \mathrm{GL}(n,\mathbb{C}) \to \mathrm{SO}(2n+1,\mathbb{C})$ in the standard way (by fixing a polarization). The strong real form of $\mathbb{G}$ is given by the involution

$$x_t = \iota(\mathrm{diag}(\underbrace{1,\cdots,1}_{n-t},\underbrace{-1}_{t}))$$

Note that x_t corresponds to $\mathrm{SO}(2n-2t+1,2t)$. Therefore, the following counting procedure only counts $\mathrm{SO}(p,q)$ with q even (or equivalently only counts representations for $\mathrm{SO}(p,q)$ with $p>q$.

The Cartan subgroups are parametrized by four integers $(p,q,2s,t)$, satisfying $p+q+2s+t=n$. The corresponding representation is

$$\sum_{\tau\in\tilde{S}_s} Ind_{W_p\times W_q\times W_{2s}\times S_t}^{W_n}[\mathrm{sgn}\otimes\mathrm{sgn}\otimes(\tau\times\tau)\otimes\mathrm{triv}]. \tag{7.2}$$

The sum is over the $\tau\times\tau$ where τ is a partition of S_s. The representation $\tau\times\tau$ is labelled by dots, sign by r or r', and *triv* by c. Recall also the well known formula

$$Ind_{S_n}^{W_n}(\mathrm{triv}) = \sum_{a+b=n}(a)\times(b) \tag{7.3}$$

To induce we add r and r' at most one to each row to τ_R, and c at most one to each column to both τ_L and τ_R the total number being r.

8. Combinatorics for type M

In this section $\mathbf{G}=\mathrm{Sp}(2n,\mathbb{C})$ and $G=\mathrm{Mp}(2n,\mathbb{R})$. Note that the dual (real) group is $\check{G}=\mathrm{Mp}(2n,\mathbb{R})$ according to Renard-Trapa.

8.1. **Nilpotent orbit in type M.** Nilpotent orbits of $\mathbf{G}$ is parameterized Young diagrams with columns

$$\mathcal{O} = (C_{2a+1}\geqslant\cdots C_1\geqslant C_0>0)$$

satisfying

$C_{2i+1}\equiv C_{2i} \pmod 2$ (This corresponds to the condition odd rows occur even times.)

We call $\mathcal{O}$ *metaplectic special* if

$$C_{2i+1}=C_{2i} \text{ when } C_{2i} \text{ is even.}$$

Note that $\mathcal{O}$ is not a special orbit in general. The condition is equivalent to that adding a column of size $2n+1$, the orbit become special in type B (This operation implements the stable range theta lifting.)

Now let $\mathcal{O}=(C_{2a+1},\cdots,C_1,C_0)$ be a special nilpotent orbit of type M. We define

$$\check{\mathcal{O}} = (R_{2a+1},\cdots,R_1,R_0)$$

where

$$(R_{2i+1},R_{2i}) = \begin{cases}(C_{2i+1},C_{2i}) & \text{if } C_{2i} \text{ is even}\\ (C_{2i+1}+1,C_{2i}-1) & \text{if } C_{2i} \text{ is odd}\end{cases}$$

This definition matches the duality of type B.

Then the Springer representation attached to $\mathcal{O}$ is given by

$$\tau_L\times\tau_R = (c_{2a+1},c_{2a-1},\cdots,c_1)\times(c_{2a},c_{2a-2},\cdots,c_0)$$

where

$$C_{2i+1} = 2c_{2i+1}-\epsilon_i$$
$$C_{2i} = 2c_{2i}+\epsilon_i$$
$$\epsilon_i = 0 \text{ or } 1$$

The non-special representations are given by

$$(c_{2i+1}, c_{2i}) \leftrightarrow (c_{2i}, c_{2i+1})$$

9. Matching doc-r-c diagrams and local systems

Let $\mathbf{G} = \mathrm{Sp}, \mathrm{Mp}, \mathrm{O}$, and $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ is a complex nilpotent orbit.

Let $\mathrm{DRC}(\mathbf{G}, \mathcal{O})$ be the set of dot-r-c diagrams parameterizing unipotent representations of the real forms of $\mathbf{G}$.

The following set of orbit are essential to us:

Definition 9.1. *A nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ is called noticed if*

- *when G is type C, $\mathcal{O}$ is special and it has columns*

$$[C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0] \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k-1.$$

  Note that, by the definition of special orbit, we have $C_{2i} \equiv C_{2i-1} \pmod 2$, $C_{2i} = C_{2i-1}$ if C_{2i} is odd for $i = 1, \cdots k$.
- *when G is type M, $\mathcal{O}$ is metaplecitic special and it has columns*

$$[C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0] \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k-1.$$

  Note that, by the definition of metaplectic special, we have $C_{2i} \equiv C_{2i-1} \pmod 2$, $C_{2i} = C_{2i-1}$ if C_{2i} is even for $i = 1, \cdots k$.
- *when G is type D, $\mathcal{O}$ is special and it has columns*

$$[C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0] \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k.$$

  Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is noticed of type C, i.e. $C_{2i} \equiv C_{2i-1} \pmod 2$ and $C_{2i} = C_{2i-1}$ if C_{2i} is odd for $i = 1, \cdots k$.
- *when G is type B, $\mathcal{O}$ is special and it has columns*

$$[C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0] \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k.$$

  Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is noticed of type M, i.e. $C_{2i} \equiv C_{2i-1} \pmod 2$ and $C_{2i} = C_{2i-1}$ if C_{2i} is even for $i = 1, \cdots, k$.

Under the above notation, $\mathcal{O}$ is called essential if it is noticed and $C_{2i} = C_{2i-1}$ in addition. Let $\mathrm{Nil}^{\mathrm{n}}(\star)$ and $\mathrm{Nil}^{\mathrm{e}}(\star)$ be all the sets of noticed and essential orbits for type $\star = B,C,D,M$ respectively.

The essential descent of orbit $\mathcal{O} \in \mathrm{Nil}^{\mathrm{e}}(\star)$ is defined in the following way (under the above notation):

- *when $\star$ is B or D, $\mathrm{d}^{\mathrm{e}}(\mathcal{O}) := \nabla(\mathcal{O})$;*
- *when $\star$ is C or M,*

$$\mathrm{d}^{\mathrm{e}}(\mathcal{O}) := \begin{cases} \nabla(\mathcal{O}) & \text{if } C_{2k} \text{ is even and } \star = B, \\ \nabla(\mathcal{O}) & \text{if } C_{2k} \text{ is odd and } \star = M, \\ [C_{2k-1}+1, C_{2k-2}, C_{2k-3}, \cdots, C_0] & \text{otherwise.} \end{cases}$$

  Note that $\mathcal{O} \mapsto \mathrm{d}^{\mathrm{e}}(\mathcal{O})$ is a good generalized descent in the last case.

Let $\mathrm{LS}(\mathcal{O})$ denote the Grothendieck group of $\widetilde{\mathbf{K}}$-equivariant local systems on $\mathcal{O}$ for various real forms of $\mathbf{G}$. The following is our main theorem regards of the counting.

Theorem 9.2. *Suppose $\mathcal{O}$ is noticed . Then the map*

$$\mathrm{Ch}\colon \mathrm{Unip}(\mathcal{O}) \longrightarrow \mathrm{LS}(\mathcal{O})$$

is injective. Moreover, every element in $\mathrm{Unip}(\mathcal{O})$ can be obtained by iterated theta lifting.

By the counting argument above, it is suffice to establish the above theorem for essential orbits.

Note that type B dot-r-c diagram is extend by mark with A or B. Suppose $\mathbf{G}$ is of type B or D, we define a map

$$\begin{aligned} \mathrm{Sign}\colon\quad \mathrm{DRC^e}(B) \sqcup \mathrm{DRC^e}(D) &\longrightarrow \mathbb{N}\times\mathbb{N} \\ \tau &\longmapsto (p,q) \end{aligned}$$

Here (p,q) is the signature of the orthogonal group $\mathrm{O}(p,q)$ corresponding to the dot-r-c diagram τ, which is calculated by the following formula

$$p = \#\bullet + 2\#r + \#c + \#d + \#A$$
$$q = \#\bullet + 2\#s + \#c + \#d + \#B$$

For $\{\star,\star'\} = \{C,D\},\{B,M\}$, we define maps

$$\begin{aligned} \mathrm{d^e}\colon\quad \mathrm{DRC^e}(\star) \sqcup \mathrm{DRC^e}(\star') &\longrightarrow (\mathrm{DRC^e}(\star') \sqcup \mathrm{DRC^e}(\star)) \times \mathbb{Z}/2\mathbb{Z} \\ \tau &\longmapsto (\tau',\epsilon) \end{aligned}$$

such that

(a) τ is in $\mathrm{DRC^e}(\mathcal{O})$ where $\mathcal{O}$ is essential;
(b) $\tau' \in \mathrm{Nil^e}(\mathcal{O}')$ where $\mathcal{O}' := \mathrm{d}^e(\mathcal{O})$;
(c) $\epsilon = 0$ if $\mathcal{O} \mapsto \mathcal{O}'$.

Using d^e, we now define two maps $\mathcal{L}$ and π which are fitted into the following diagram.

$$\begin{tikzcd}
 & & & \mathrm{LS} \\
 & & \mathcal{L}_\tau & \\
 & \tau \arrow[r, mapsto] \arrow[ur, mapsto] & \pi_\tau \arrow[u, mapsto] & \\
 \mathrm{DRC^e} \arrow[rrr, "\pi"] \arrow[uuurrr, bend left, "\mathcal{L}"] & & & \mathrm{Unip} \arrow[uuu, "\mathrm{Ch}"']
\end{tikzcd}$$

Here

(a) τ is in $\mathrm{DRC^e}(\mathcal{O})$ where $\mathcal{O}$ is essential;
(b) π_τ is a unipotent representation of $\widetilde{G}$ attached to $\mathcal{O}$,
(c) $\widetilde{G}$ is $\mathrm{Mp}(2n,\mathbb{R})$ or $\mathrm{Sp}(2n,\mathbb{R})$ if τ is a diagram of type C or M;
(d) $\widetilde{G}$ is $\mathrm{O}(p,q)$ if τ is of type B or C and $(p,q) = \mathrm{Sign}(\tau)$;
(e) $\mathcal{L}_\tau = \mathrm{Ch}(\pi_\tau)$.

The definition of $\mathcal{L}$ and π are inductive with respect to the number of columns of $\mathcal{O}$:

i) Suppose $\mathcal{O} = [C_0 = 0]$ is type C or M. Then $\mathrm{DRC^e}(\mathcal{O}) = \{\tau_0\}$ is a singleton with $\tau_0 = (\varnothing,\varnothing)$. Define $\pi_\tau = \mathbf{1}$ the trivial representation of the trivial group $\mathrm{Sp}(0,\mathbb{R}) = \mathrm{Mp}(0,\mathbb{R}) = \mathbf{1}$. Here we assume the theta lift of $\mathbf{1}$ to $\mathrm{O}(p,q)$ for any signature p,q is the trivial representation $\mathbf{1}_{p,q}^{+,+}$ of $\mathrm{O}(p,q)$.
ii) We now assume $\tau \in \mathrm{DRC^e}(\mathcal{O})$ and $(\tau',\epsilon) = \mathrm{d}^e(\tau) \in \mathrm{DRC^e}(\mathcal{O}')$ where $\mathcal{O}' = \mathrm{d}^e(\mathcal{O})$.
   a) Suppose $\mathcal{O}$ is type C or M. Then $\widetilde{G} = \mathrm{Sp}(2n,\mathbb{R})$ or $\mathrm{Mp}(2n,\mathbb{R})$, and $\pi_{\tau'}$ is a unipotent representation of $\mathrm{O}(p,q)$ attached to τ' with $n = |\tau|$ and $(p,q) = \mathrm{Sign}(\tau')$.
   $$\pi_\tau := \bar{\Theta}_{\widetilde{G}',\widetilde{G}}(\pi_{\tau'} \otimes (\mathbf{1}_{p,q}^{-,-})^{\epsilon}).$$
   Here $\mathbf{1}_{p,q}^{-,-}$ is the determinant character of $\mathrm{O}(p,q)$.

b) Suppose $\mathcal{O}$ is type D or B. Then $\widetilde{G} = \mathrm{O}(p,q)$ and $\pi_{\tau'}$ is a unipotent representation of $\widetilde{G}' = \mathrm{Sp}(2n,\mathbb{R})$ or $\mathrm{Mp}(2n,\mathbb{R})$ attached to τ' with $n = |\tau'|$ and $(p,q) = \mathrm{Sign}(\tau)$.

$$\pi_\tau := \bar{\Theta}_{\widetilde{G}',\widetilde{G}}(\pi_{\tau'}) \otimes (\mathbf{1}_{p,q}^{+,-})^{\epsilon}.$$

Theorem 9.2 is a immediate consequence of the following

Proposition 9.3. *We have*

i) $\mathcal{L}_\tau$ *is non-zero for each* $\tau \in \mathrm{DRC}^{\mathrm{e}}$.

ii) For each essential orbit $\mathcal{O}$ *of type C or M, the following is an injection*

$$\begin{aligned} \mathrm{DRC}^{\mathrm{e}}(\mathcal{O}) &\longrightarrow \mathrm{LS}(\mathcal{O}) \\ \tau &\longmapsto \mathrm{Ch}(\pi_\tau) = \mathcal{L}_\tau \end{aligned}$$

iii) For each essential orbit $\mathcal{O}$ *of type D or B, the following is an injection*

$$\begin{aligned} \mathrm{DRC}^{\mathrm{e}}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow \mathrm{LS}(\mathcal{O}) \\ (\tau,\epsilon) &\longmapsto \mathrm{Ch}(\pi_\tau \otimes (\mathbf{1}_{p,q}^{-,-})^{\epsilon}) = \mathcal{L}_\tau \otimes (\mathbf{1}_{p,q}^{-,-})^{\epsilon} \end{aligned}$$

9.1. **The algorithm to define** d^e**.** From now on, we assume

$$\tau = ([\tau_{L,0}, \tau_{L,1}, \cdots], [\tau_{R,0}, \tau_{R,1}, \cdots])$$

and $(\tau', \epsilon) := \mathrm{d}^e(\tau)$

$$\tau' = ([\tau'_{L,0}, \tau'_{L,1}, \cdots], [\tau'_{R,0}, \tau'_{R,1}, \cdots])$$

The definition of sign ϵ

i) Suppose τ is type D or B, $\epsilon = 0$ if and only if $\#d(\tau) > \#d(\tau')$. This means:
   a) When τ is type D, $\mathrm{tail}(\tau_{L,0}) = d$.
   b) When τ is type B, $\mathrm{tail}(\tau_{R,0}) = d$ or $(\mathrm{tail}(\tau_{L,0}), \mathrm{tail}(\tau_{R,0}), \mathrm{tail}(\tau_{R,1})) = (c, r, d)$.

ii) Suppose τ is type C or M, $\epsilon = 0$ if and only if $|\tau_{L,0}| \geqslant |\tau_{R,0}|$.

Appendix A. The coherent continuation representation

We specialize the atlas setting to the classical group case. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle\gamma\rangle$.

A.1. **Type C.** References [3, Example 5.11]. The complex group is $\mathbf{G} = \mathrm{Sp}(2n,\mathbb{C})$. Use the standard coordiante of characters and cocharacters. Fix a Cartan subgroup $\mathbf{H}$. The coweight lattice $X_* = \mathbb{Z}^n$. The coroot lattice

$$P^\vee = \mathbb{Z}^n \cup (\mathbb{Z}^n + \tfrac{1}{2})^n = \{\, (a_1, \cdots, a_n) \mid a_i \text{ are all integers or are all half integers} \,\}$$

The outer involution $\gamma = 1$. The extended group is $\mathbf{G}^\Gamma = \mathbf{G} \times \Gamma$ $(\Gamma = \langle\gamma\rangle)$.
$Z(\mathbf{G}) = \{\, \pm 1 \,\}$. The strong real forms are parameterized by

$$((\tfrac{1}{2}[\mathbb{Z}^n \cup (\mathbb{Z}^n + \tfrac{1}{2})])/\mathbb{Z}^n)/W$$

The representative $\frac{1}{4}(1, \cdots, 1)$ corresponds to the split real form $G = \mathrm{Sp}(2n,\mathbb{R})$.
The representative

$$\frac{1}{2}(\underbrace{1, \cdots, 1}_{p}, \underbrace{0, \cdots, 0}_{q})$$

correspondes to $\mathrm{Sp}(p,q)$ Let

$$\begin{aligned}
\mathbf{N}^\Gamma &= \mathrm{Norm}_{\mathbf{G}^\Gamma}(\mathbf{H}) \\
\mathcal{I} &= \{\, \xi \in \mathbf{G}^\Gamma \backslash \mathbf{G} \mid \xi^2 \in Z(G) \,\} \\
\widetilde{\mathcal{X}} &= \mathcal{I} \cap \mathbf{N}^\Gamma \\
\mathcal{X} &= \widetilde{\mathcal{X}}/\mathbf{H} \quad \text{(The } \mathbf{H} \text{ action is by conjugation.)} \\
\mathcal{I}_W &= \{\, \tau \in W^\Gamma - W \mid \tau^2 = 1 \,\}
\end{aligned}$$

The set $\mathcal{I}/G$ parameterized the strong inner forms of $\mathbf{G}$ (corresponding to the outer automorphism γ). $\mathrm{Sp}(2n,\mathbb{R})$ corresponds to $\xi_0 = \mathrm{diag}(i,\cdots,i,-i,\cdots,-i)$. Let $z_0 = \xi_0^2 = -1$.

A.1.1. Note that we have γ act on $\mathbf{G}$ trivially. The set $\mathcal{I}_W$ is identified with Cartan involutions of H (for the inner class γ)[3, 9.13b]

$$\widetilde{\mathcal{X}} \longrightarrow \mathcal{X} \xrightarrow{\;p\;} \mathcal{I}_W \xleftrightarrow{1-1} \{\, \theta_{x,H} := \mathrm{Ad}_x \,\}$$

We may identify $\mathcal{I}_W$ with $\{\, \tau \in W \mid \tau^2 = 1 \,\}$. Then $\mathcal{I}_W/W$ is identified with the K-conjugacy class of θ-stable Cartan in $G = \mathrm{Sp}(2n,\mathbb{R})$.

Note $W = W_n = S_n \ltimes \{\, \pm 1 \,\}^n$.

Let $s_{i,j} = s_{e_i - e_j}$, $s_i = s_{2e_i}$ in the Weyl group W. Define $w_{i,j}$ and w_i (lift of s_i) also represent its lift in $\mathbf{N} = \mathrm{Norm}_{\mathbf{G}}(\mathbf{H})$.

$$w_{i,j} = \left(\begin{array}{cccc:cccc}
\ddots & & & & & & & \\
& 0 & 1 & & & & & \\
& -1 & 0 & & & & & \\
& & & \ddots & & & & \\
\hdashline
& & & & \ddots & & & \\
& & & & & 0 & -1 & \\
& & & & & 1 & 0 & \\
& & & & & & & \ddots
\end{array}\right)$$

$$w_i = \left(\begin{array}{ccc:ccc}
1 & & & & & \\
& 0 & & & 1 & \\
& & 1 & & & \\
\hdashline
& & & 1 & & \\
& -1 & & & 0 & \\
& & & & & 1
\end{array}\right)$$

Therefore, $\mathcal{I}_W$ has the following types of representatives: For each set of numbers p,t,r such that $2p+t+r=n$.

$$w_{p,t} = w_{1,2}w_{3,4}\cdots w_{2p-1,2p} \ltimes w_{2p+1}\cdots w_{2p+t}.$$

The θ-stable Cartan corresponding to $w_{p,t}$ is isomorphic to

$$H(\mathbb{R}) := H^{w_{p,t},\mathrm{conj}} = (\mathbb{C}^\times)^p \times (\mathbb{R}^\times)^t \times \mathbb{T}^r.$$

Now $H^\vee(\mathbb{R}) = (\mathbb{C}^\times)^p \times \mathbb{T}^t \times (\mathbb{R}^\times)^r$.

For $\tau \in \mathcal{I}_w$, $z \in Z(\mathbf{G})$, let

$$\begin{aligned}\mathcal{X}_0 &= \{\, x \in \mathcal{X} \mid x \text{ is } \mathbf{G}\text{-conjugate to } \xi_0 \,\} \\ &= \mathcal{X}(-1) = \{\, x \in \mathcal{X} \mid x^2 = -1 \,\} \\ \mathcal{X}_\tau(z) &= \{\, x \in \mathcal{X} \mid p(x) = \tau, x^2 = z \,\}\end{aligned}$$

Note that $\mathcal{X}_\tau(z)$ has a simply transitive action by $\mathbf{H}_{-\tau}/A_\tau$ [3, Prop 11.2]

$$|\mathcal{X}_\tau(z)| = 2^r$$

Since $\overline{p}\colon \mathcal{X}_0/W \to \mathcal{I}_W/W$ is bijective[3, Prop 12.12], we have

$$\mathcal{X}_0 = \bigcup_{\tau \in \mathcal{I}_W/W} W \cdot \mathcal{X}_\tau(z_0).$$

Let $W^\tau = \mathrm{Stab}_W(\tau)$. Let $h_k = \mathrm{diag}(1, \cdots, 1, \underbrace{i}_{k-th}, 1, \cdot, 1) \in (\mathbb{C}^\times)^n = \mathbf{H} \subset \mathbf{G}$, then $h_k^{-1} \in \mathbf{H}$ has the same shape with i replaced by $-i$. Now fix $\tau = w_{p,t}$. We can list elements in $\mathcal{X}_\tau(z_0)$ as the following: Fix $a + b = r$, let

$$x_{p,t,a,b} = w_{1,2}w_{3,4}\cdots, w_{2p-1,2p}w_{2p+1}\cdots w_{2p+t}h_{2p+t+1}\cdots h_{2p+t+a}h^{-1}_{2p+t+a+1}h^{-1}_{2p+t+a+b=n},$$

then

$$\mathcal{X}_\tau(z_0) = \bigcup_{a+b=r} W^\tau \cdot x_{p,t,a,b}$$

Fix the strong real form $x = x_{p,t,a,b}$. Then $\mathbf{K} = \mathbf{G}^\xi$ is the complexification of the maximal compact subgroup of G. Now the real Weyl group

$$\begin{aligned}W(\mathbf{G}(\mathbb{R}), \mathbf{H}(\mathbb{R})) &\cong W(K, H) = \mathrm{Stab}_W(x) \\ &\cong (W_p \ltimes W(A_1)^p) \times S_r \times W_t.\end{aligned}$$

Here $W(A_i)^p = \langle w_{1,2}, w_{3,4}, \cdots, w_{2p-1,2p}\rangle$ Here W_p is identified with $\triangle W_p \subset W_p \times W_p$ where $W_p \times W_p$ is the subgroup of W_{2p} via embedding

$$(a_1, a_3, \cdots, a_{2p-1}) \times (a_2, a_4, \cdots, a_{2p}) \mapsto (a_1, a_2, a_3, a_4, \cdots, a_{2p-1}, a_{2p}).$$

There is the list of real, imaginary, and complex root systems.

Real roots :

$$R^{\mathbb{R}} = \{\, \pm(e_{2i-1} + e_{2i}) \mid i = 1, \cdots, p \,\} \cup \{\, \pm(e_{2p+i} - e_{2p+j}), \pm(e_{2p+i} + e_{2p+j}) \mid i, j = 1, \cdots, t \,\}$$

It is type $(A_1)^p \times C_t$.

$$\check{\rho}^{\mathbb{R}} = \tfrac{1}{2}(\underbrace{1, -1, \cdots, 1, -1}_{2p\text{-terms}}, \underbrace{2t-1, 2t-3, \cdots, 1}_{t\text{-terms}}, 0, \cdots, 0)$$

Imaginary roots:

$$R^{i\mathbb{R}} = \{\, \pm(e_{2i-1} - e_{2i}) \mid i = 1, \cdots, p \,\} \cup \{\, \pm(e_{2p+t+i} - e_{2p+t+j}), \pm(e_{2p+t+i} + e_{2p+t+j}) \mid i, j = 1, \cdots, r \,\}$$

It is type $(A_1)^p \times C_r$

$$\check{\rho}^{i\mathbb{R}} = \tfrac{1}{2}(\underbrace{1, 1, \cdots, 1, 1}_{2p\text{-terms}}, 0, \cdots, 0, \underbrace{2r-1, 2r-3, \cdots, 1}_{r\text{-terms}})$$

Take $x_{p,t,a,b}$, The imaginary compact roots are:

$$\begin{aligned}R_{ic} = &\{\, e_{2p+t+i} - e_{2p+t+j} \mid 1 \leqslant i, j \leqslant a \,\} \cup \{\, e_{2p+t+i} - e_{2p+t+j} \mid a+1 \leqslant i, j \leqslant a+b = r \,\} \\ &\cup \{\, \pm(e_{2p+t+i} + e_{2p+t+j}) \mid 1 \leqslant i \leqslant a, a+1 \leqslant j \leqslant a+b = r \,\}\end{aligned}$$

Clearly R_{ic} is of type A_r. $W(R_{ic}) = S_r$ See [70, 3.13]

$$W_2(R_{ic}) := \{\, w \in W(R^{i\mathbb{R}}) \mid wR_{ic} = R_{ic} \,\} = W(A_1)^p \times W(R_{ic}).$$

Complex root system $R^{\mathbb{C}}$**:** Note that $R^{\mathbb{C}} := \{ \alpha \in R \mid \langle \alpha, \check{\rho}^{\mathbb{R}} \rangle = \langle \alpha, \check{\rho}^{i\mathbb{R}} \rangle = 0 \}$ Then
$R^{\mathbb{C}} = \{ \pm\alpha^C_{i,j} := \pm(e_{2i-1} - e_{2j-1}) \mid 1 \leqslant i < j \leqslant p \} \cup \{ \pm\tau\alpha^C_{i,j} = \mp(e_{2i} - e_{2j}) \mid 1 \leqslant i < j \leqslant p \}$.

Now $W(R^{\mathbb{C}})^\tau = \langle s_{\alpha_{i,i+1}} s_{\tau\alpha_{i,i+1}} | i = 1, \cdots, p \rangle$ is of type A_{p-1}.

Now the real Weyl group is of the form

$$W(\mathbf{K}, \mathbf{H}) = W(R^{\mathbb{C}})^\tau \ltimes (W(R^{\mathbb{R}}) \times W(\mathbf{G}^{\mathbf{A}}, \mathbf{H}))$$

Here $W_{ic} < W(\mathbf{G}^{\mathbf{A}}, \mathbf{H}) < W(R^{i\mathbb{R}})$, and

$$W(\mathbf{G}^{\mathbf{A}}, \mathbf{H}) = W_{ic} \ltimes A(H).$$

Note that $G = \mathrm{Sp}(2n, \mathbb{R})$ is large, $W(\mathbf{G}^{\mathbf{A}}, \mathbf{H}) = W_2(R_{ic})$.

In summary,

$$W(\mathbf{K}, \mathbf{H}) = \underbrace{\overbrace{S_p}^{(W^C)^\tau} \ltimes \overbrace{W(A_1)^p}^{\text{real}}}_{W_p} \ltimes \underbrace{\overbrace{W(A_1)^p}^{im}}_{A(H)} \times \overbrace{S_r}^{ic} \times \overbrace{W_t}^{real} \tag{A.1}$$

A.2. **Coherent continuation representation.** The full parameter space

$$\mathcal{Z} = \{ (x, y) \mid x \in \mathcal{X}, y \in \check{\mathcal{X}}, \theta_y = -\theta_x^T \}.$$

Since we consider the catogery of $G = \mathrm{Sp}(2n, \mathbb{R})$-modules at infinitesimal character $\rho(G)$, we choose $\check{z}_0 = 1$. Now the full parameter space of $\Pi(G, \rho(G))$ is given by

$$\begin{aligned} \mathcal{Z}_0 &= \{ (x, y) \mid x \in \mathcal{X}(-1), y \in \check{\mathcal{X}}(1), \theta_y = -\theta_x^T \} \\ &= \bigcup_{\tau \in \mathcal{I}_W / W} \{ (x, y) \mid x \in \mathcal{X}_\tau(-1), y \in \check{\mathcal{X}}(1), \theta_y = -\theta_x^T \} \end{aligned}$$

We now decompose $\mathcal{Z}_0$ into W-orbits. Note that $\mathcal{X}_0/W \to \mathcal{I}_W/W$ is bijective We choose $x_{p,t,r,0}$ as the representative of $\mathcal{X}_0/W \cong \mathcal{I}_W/W$. Note that $\mathrm{Stab}_W(x_{p,t,r,0})$ is calculated in (B.2).

Let

$$\check{\mathcal{X}}^x = \{ y \in \check{\mathcal{X}}(1) \mid \theta_y = -\theta_x^T \} = \check{\mathcal{X}}_{\check{\tau}}(1).$$

Here that $\check{\tau} \in \mathcal{I}_{\check{W}}$ is determined by θ_y. Therefore, $|\check{\mathcal{X}}^x| = 2^r$ (here there real tori $H(\mathbb{R})$ has t copies of $\mathbb{R}^\times$). Note that $W^{\check{\tau}} = (W^C)^\tau \ltimes (W(R^{i\mathbb{R}}) \times W(R^{\mathbb{R}})$, and except the factor $W(R^{\mathbb{R}})$ (the dual imaginary root group), other factors has trivial action on $\mathcal{X}_{\check{\tau}}(1)$. We could choose the following representatives of $\check{\mathcal{X}}^x/W(R^{\mathbb{R}})$ (it has $t+1$ members): for $a + b = t$, define

$$y_{p,a,b,r} := s_{1,2} \cdots s_{2p-1,2p} \check{h}_{2p+1} \cdots \check{h}_{2p+a} \check{s}_{2p+t+1} \cdots \check{s}_n.$$

Here $\check{h}_j = \mathrm{diag}(1, \cdots, 1, \underbrace{-1}_{j}, 1, \cdots, 1)$. Now

$$\mathrm{Stab}_{W(R^{\mathbb{R}})}(y_{p,a,b,r}) = W_a \times W_b.$$

Therefore,

$$\begin{aligned} \mathcal{Z}_0 &= \bigcup_{\tau \in \mathcal{I}_W / W} W \cdot \{ (x_{p,t,r,0}, y) \mid y \in \check{\mathcal{X}}_{\check{\tau}}(-1), \} \\ &= \bigcup_{p,a,b,r} W \cdot (x_{p,t,r,0}, y_{p,a,b,r}) \end{aligned}$$

Moreover, the stabilizer of $(x_{p,t,r,0}, y_{p,a,b,r})$ is

$$(W^C)^\tau \ltimes (W(\mathbf{G}^{\mathbf{A}}, \mathbf{H}) \times \mathrm{Stab}_{W(R^{\mathbb{R}})}(y_{p,a,b,r})) = W_p \ltimes \underbrace{W(A_1)^p \times S_r}_{W(\mathbf{G}^{\mathbf{A}},\mathbf{H})} \times W_a \times W_b$$

By taking filtration according to the dimension of real root system, the Cayley transform could be ignored. Then the coherent continuation representation is essentially given by the cross action. For the stabilizer, the $W(\mathbf{G}^{\mathbf{A}}, \mathbf{H})$ factor acts by -1 and other factors act trivially. This leads to the formula (4.1). Though, the notations are not matching.

Appendix B. Type D

Now let $\mathbf{G}_0 = \mathrm{SO}(2n, \mathbb{C})$. Let $\mathbf{G} = \mathrm{O}(2n, \mathbb{C})$. Fix a Cartan subgroup $\mathbf{H}$ and Borel $\mathbf{B}$. Let w_c be the element in $\mathrm{Norm}_{\mathbf{G}}(\mathbf{H})$ fixing $\mathbf{B}$. It is the element sending e_n to $-e_n$. (The upper triangler matrices form the Borel.)

$$\begin{pmatrix} I & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I \end{pmatrix}$$

Now $W := W(\mathbf{G})$ and $W' := W(\mathbf{G}_0)$ is the subgroup of W of index 2.

The cocharacter lattice $X_* = \mathbb{Z}^n$. The coweight lattice

$$P^\vee = \mathbb{Z}^n \cup (\mathbb{Z}^n + \tfrac{1}{2})^n = \{\, (a_1, \cdots, a_n) \mid a_i \text{ are all integers or are all half integers} \,\}$$

[Note that the root lattice of $\mathrm{SO}(2n, \mathbb{C})$ and $\mathrm{Sp}(2n, \mathbb{C})$ are the same $= \{\, (a_i) \mid \sum a_i \equiv 0 \pmod 2 \,\}$.
]

Note that $Z(\mathbf{G}) = Z(\mathbf{G}_0) = \{\, \pm 1 \,\}$

We consider the real forms $\mathrm{SO}(p, q)$. When p, q has the parity n, we take the outer involution $\gamma = 1$, together with $\mathrm{SO}^*(2n)$ form an inner class. When p, q has parity different with n, it relates to the outer involution $\gamma = \mathrm{diag}()$, these gives another inner class.

The set elements $\{\, \xi \mid \xi^2 \in Z(\mathbf{G}) \,\}$ in $\mathrm{SO}(2n, \mathbb{C})$ and $\mathrm{SO}(2n, \mathbb{C})w_c$ could be identified with the set of inolutions in $\mathrm{O}(2n, \mathbb{C})$.

When $\xi^2 = 1$, the $\mathbf{G}_0$-conjugacy class is the same as the $\mathbf{G}$-conjugacy class.

When $\xi^2 = -1$, there is only one $\mathbf{G}$-orbit which breaks into two $\mathbf{G}_0$-orbits. In fact, for each ξ under the suitable orthonormal basis, it has the form $\xi_{p,q} = \mathrm{diag}(\underbrace{1, \cdots, 1}_{p}, \underbrace{-1, \cdots, -1}_{q})$ $(p + q = 2n,\ \xi_{p,q}^2 = 1)$ or $\xi_0 = \mathrm{diag}(\underbrace{i, \cdots, i}_{n}, \underbrace{-i, \cdots, -i}_{n})$ under a "hyperbolic" basis. So there is an element in $\mathrm{O}(2n, \mathbb{C}) - \mathrm{SO}(2n, \mathbb{C})$ fixing ξ under the conjugation ation.

Suppose $\gamma = e$. The representative ξ_0 corresponds to the real form $G = \mathrm{SO}^*(2n)$. Clearly $z_0 = \xi_0^2 = -1$ in this case. The representative $\xi_{2s,2t}$ correspondes to $\mathrm{SO}(2s, 2t)$. Clearly, $z = \xi_{2s,2t}^2 = 1$ in this case.

Define

$$\mathcal{I}(\mathbf{G}) = \mathcal{I}(\mathbf{G}_0, e) \sqcup \mathcal{I}(\mathbf{G}_0, w_c) = \{\, \xi \in \mathbf{G} \mid \xi^2 \in Z(\mathbf{G}) \,\}$$

By the above discussion, we have

$$\mathcal{I}(\mathbf{G})/\mathbf{G} \xleftrightarrow{1-1} \{\, \mathrm{O}(p, q), \mathrm{O}^*(2n) \mid p + q = 2n \,\}$$

Define

$$\begin{aligned}
\widetilde{\mathcal{X}} &= \mathcal{I}(\mathbf{G}) \cap \mathrm{Norm}_{\mathbf{G}}(\mathbf{H}) = \mathcal{X}(\mathbf{G}) = \mathcal{X}(\mathbf{G}_0, e) \sqcup \mathcal{I}(\mathbf{G}_0, w_c) \\
&= \{ x \in \mathrm{Norm}_{\mathbf{G}}(\mathbf{H}) \mid x^2 \in Z(G) \} \\
\mathcal{X} &= \widetilde{\mathcal{X}}/\mathbf{H}
\end{aligned}$$

Define

$$\mathcal{I}_W = \{ w \in W \mid w^2 = 1 \} = \mathcal{I}_{W'}(\gamma = e) \sqcup \mathcal{I}_{W'}(\gamma = w_c)$$

Define $w_{i,j}$ and w_i be elements in $\mathbf{N} = \mathrm{Norm}_{\mathbf{G}}(\mathbf{H})$, let $s_{i,j}$ and s_i be their image in W:

$$w_{i,i+1} = \left(\begin{array}{cccc|cccc}
\ddots & & & & & & & \\
 & 0 & 1 & & & & & \\
 & 1 & 0 & & & & & \\
 & & & \ddots & & & & \\
\hline
 & & & & \ddots & & & \\
 & & & & & 0 & 1 & \\
 & & & & & 1 & 0 & \\
 & & & & & & & \ddots
\end{array}\right)$$

$$w_i = \left(\begin{array}{ccc|ccc}
1 & & & & & \\
 & 0 & & & 1 & \\
 & & 1 & & & \\
\hline
 & & & 1 & & \\
 & 1 & & & 0 & \\
 & & & & & 1
\end{array}\right)$$

Let

$$w'_{i,i+1} = \left(\begin{array}{cccc|cccc}
\ddots & & & & & & & \\
 & 0 & -1 & & & & & \\
 & -1 & 0 & & & & & \\
 & & & \ddots & & & & \\
\hline
 & & & & \ddots & & & \\
 & & & & & 0 & -1 & \\
 & & & & & -1 & 0 & \\
 & & & & & & & \ddots
\end{array}\right).$$

Then $w'_{i,i+1} = w_i w_{i,i+1} w_i^{-1}$.

Under W-action, $\mathcal{I}_W/W$ is parameterized by s, a, b such that $2s + a + b = n$. Representatives are given by:

$$w_{p,a,b} = w_{1,2} w_{3,4} \cdots w_{2s-1,2s} \ltimes w_{2s+1} \cdots w_{2s+a}.$$

Let $\tau = w_{s,a,b}$. Then

$$\begin{aligned}
W^{\tau} &= S_s \ltimes (\langle w_{2k-1} w_{2k} | k = 1, \cdots, s \rangle \times \langle w_{2k-1,2k} | k = 1, \cdots, s \rangle) \times W_a \times W_b \\
&\cong S_s \times (\mathbb{Z}_2^s \times \mathbb{Z}_2^s) \times W_a \times W_b.
\end{aligned}$$

Therefore, the twisted involution orbit $W \cdot \tau$ has size

$$\frac{2^n n!}{2^{2s} s! 2^a a! 2^b b!} = \frac{n!}{s! a! b!}$$

When $a = b = 0$, the W-orbit $W \cdot w_{p,a,b}$ splits into two W' orbits $W' \cdot w_{p,a,b}$ and $W' \cdot w'_{p,a,b}$ where

$$w'_{p,a,b} = w'_{1,2} w_{3,4} \cdots w_{2s-1,2s} \ltimes w_{2s+1} \cdots w_{2s+a}.$$

The θ-stable Cartan corresponding to $w_{s,a,b}$ is isomorphic to

$$H(\mathbb{R}) := H^{w_{s,a,b},\mathrm{conj}} = (\mathbb{C}^\times)^s \times (\mathbb{R}^\times)^a \times \mathbb{T}^b.$$

Now $H^\vee(\mathbb{R}) = (\mathbb{C}^\times)^s \times \mathbb{T}^a \times (\mathbb{R}^\times)^b$.

Consider the map

$$\tilde{\mathcal{X}} \longrightarrow \mathcal{X} := \tilde{\mathcal{X}}/\mathbf{H} \xrightarrow{\ p\ } \mathcal{I}_W$$

Let

$$h_i = (1, \ldots, 1, \underbrace{-1}_{i}, 1, \cdots, 1) \in (\mathbb{C}^\times)^n = \mathbf{H}$$

Now the W-orbit on $\mathcal{X}$ is given by the representatives

$$x_{s,a,p,q} = w_{1,2} w_{3,4} \cdots w_{2s-1,2s} w_{2s+1} \cdots w_{2s+a} h_{2s+a+1} \cdots h_{2s+a+p}.$$

Now

$$\mathcal{X}_\tau = \bigsqcup_{p+q=b} W^\tau \cdot x_{s,a,p,q}. \tag{B.1}$$

Moreover, only W_b factor of W^τ acts on $\mathcal{X}_\tau$ non-trivially.

Observe that $x = x_{s,a,p,q}$ gives the realform $\mathrm{O}(2s + a + 2p, 2s + a + 2q)$.

Here is the list of real, imaginary, and complex root systems.

Real roots :

$$R^{\mathbb{R}} = \{\, \pm(e_{2k-1} - e_{2k}) \mid k = 1, \cdots, s \,\} \cup \{\, e_{2s+k} \pm e_{2s+l} \mid k \neq l = 1, \cdots, a \,\}$$

It is type $(A_1)^s \times D_a$.

$$\check{\rho}^{\mathbb{R}} = \tfrac{1}{2}(\underbrace{1, -1, \cdots, 1, -1}_{2s\text{-terms}}, \underbrace{2a-2, 2a-4, \cdots, 0}_{a\text{-terms}}, 0, \cdots, 0)$$

Imaginary roots:

$$R^{i\mathbb{R}} = \{\, \pm(e_{2k-1} + e_{2k}) \mid k = 1, \cdots, s \,\} \cup \{\, e_{2s+a+k} \pm e_{2s+a+l} \mid k \neq l = 1, \cdots, b \,\}$$

It is type $(A_1)^s \times D_b$

$$\check{\rho}^{i\mathbb{R}} = \tfrac{1}{2}(\underbrace{1, 1, \cdots, 1, 1}_{2s\text{-terms}}, 0, \cdots, 0, \underbrace{2b-2, 2b-4, \cdots, 0}_{b\text{-terms}})$$

The imaginary compact roots are:

$$\begin{aligned} R_{ic} = &\{\, e_{2s+a+i} \pm e_{2s+a+j} \mid 1 \leqslant i, j \leqslant p \,\} \\ &\cup \{\, e_{2s+a+i} \pm e_{2s+a+j} \mid p+1 \leqslant i, j \leqslant p+q = b \,\} \end{aligned}$$

Clearly R_{ic} is of type $D_p \times D_q$. $W(R_{ic}) = W_p \times W_q$

Complex root system $R^{\mathbb{C}}$: Note that $R^{\mathbb{C}} := \{\, \alpha \in R \mid \langle \alpha, \check{\rho}^{\mathbb{R}} \rangle = \langle \alpha, \check{\rho}^{i\mathbb{R}} \rangle = 0 \,\}$ Then

$$\begin{aligned} R^{\mathbb{C}} = &\{\, \pm\alpha^C_{i,j} := \pm(e_{2i-1} - e_{2j-1}) \mid 1 \leqslant i < j \leqslant s \,\} \cup \{\, \pm\tau\alpha^C_{i,j} = (e_{2i} - e_{2j}) \mid 1 \leqslant i < j \leqslant s \,\} \\ &\cup \{\, \alpha_0 = \pm(e_{2s+a} - e_{2n}), \tau(\alpha_0) = \mp(e_{2s+a} + e_{2n}) \,\}. \end{aligned}$$

This term only appear when a, b are both non-zero

Now

$$W(R^{\mathbb{C}})^\tau = \begin{cases} \langle s_{\alpha_{i,i+1}} s_{\tau\alpha_{i,i+1}} | i = 1, \cdots, s \rangle & \text{type } A_{s-1} \text{ if } ab = 0 \\ \langle s_{\alpha_{i,i+1}} s_{\tau\alpha_{i,i+1}} | i = 1, \cdots, s \rangle \langle w_{cx} := w_{2s+a,n} w'_{2s+a,n} \rangle & \text{type } A_{s-1} \cdot A_1 \text{ if } ab \neq 0 \end{cases}$$

Here $w_{cx} = w_{2s+a,n}w'_{2s+a,n} = w_{2s+a}w_n$.

In summary,

$$\text{(B.2)}\quad W(\mathbf{K},\mathbf{H}) = W(\mathbf{G}(\mathbb{R}),\mathbf{H}(\mathbb{R})) = \underbrace{\overbrace{S_s}^{(W^C)^\tau} \ltimes \overbrace{W(A_1)^s}^{\text{real}}}_{W_s} \ltimes \underbrace{\overbrace{W(A_1)^s}^{im}}_{A(H)} \times \overbrace{W_p \times W_q}^{ic} \times \overbrace{W_a}^{real}$$

$$\begin{aligned} W(\mathbf{K}_0,\mathbf{H}) &= W(\mathbf{G}_0(\mathbb{R}),\mathbf{H}(\mathbb{R})) \\ \text{(B.3)}\qquad &= \underbrace{\overbrace{S_s \cdot P_2}^{(W^C)^\tau} \ltimes \overbrace{W(A_1)^s}^{\text{real}}}_{W_s} \ltimes \underbrace{\overbrace{W(A_1)^s}^{im}}_{A(H)} \times \overbrace{(W_p \times W_q)'}^{ic} \times \overbrace{W'_a}^{real} \\ &= S_s \ltimes (W(A_1)^s \times W(A_1)^s) \times (W_p \times W_q \times W_a)' \end{aligned}$$

Here $P_2 = \langle w_{2s+a,n}w'_{2s+a,n}\rangle$ if $ab \neq 0$, and trivial otherwise.

B.1. **Coherent continuation representation.** We consider the catgory of modules with infinitesimal character $\rho(\mathbf{G})$. Thefore we choose $\check{z}_0 = 1$. The set $\bigcup_{p+q=2n} \Pi(\mathrm{SO}(p,q),\rho(\mathbf{G}))$ is paramterized by

$$\mathcal{Z} = \{\, (x,y) \in \mathcal{X}(1) \times \check{\mathcal{X}}(1) \mid \theta_x = -\theta_y^T \,\}$$

Note that we allow $y \in \mathrm{O}(2n,\mathbb{C})$. In fact, if n is even x,y are both in SO or not (i.e. $\gamma = \check{\gamma} \in \mathrm{Out}(\mathrm{SO}(2n,\mathbb{C}))$). If n is odd, exactly one of x,y is in SO (i.e. , γ and $\check{\gamma}$ are different elemnts in $\mathrm{Out}(\mathrm{SO}(2n,\mathbb{C}))$).

We now decompose $\mathcal{Z}$ into W-orbits. Note that θ_x is determined by $\tau_{s,a,b}$ and $-\theta_x^T$ gives the element $\check{\tau}$ which is conjugate to $\tau_{s,b,a}$. By (B.1)

$$\begin{aligned} \{\, (x,y) \in \mathcal{Z} \mid x = x_{s,a,p,q} \,\} &= \{\, (x_{s,a,p,q}, y) \mid y \in \check{\mathcal{X}}(1)_{\check{\tau}} \,\} \\ &= \bigsqcup_{t+u=a, w\in W_a} (x_{s,a,p,q}, w\cdot y_{s,t,u,b}) \end{aligned}$$

with

$$y_{s,t,u,b} = w_{1,2}w_{3,4}\cdots w_{2s-1,2s}w_{2s+a+1}\cdots w_{2s+a+b}h_{2s+1}\cdots h_{2s+t}.$$

Clearly,

$$\mathrm{Stab}_W((x_{s,a,p,q}, y_{s,t,u,b})) = S_s \ltimes (W(A_1)^s \times W(A_1)^s) \times W_t \times W_u \times W_p \times W_q$$

Let sgn' denote the inflation of the sign character of S_n to W_n. Note that $\mathrm{sgn}' = (n)\times$ (Here (n) denote a column of lenght n)k. On the other hand, the sign charactor of W_n corresponds to bipartition $\times(n)$. Let sgn denote the sign charactor of $W(A_1)^s = (\mathbb{Z}/2\mathbb{Z})^s$.

We claim that restrict of the W_n-representation to W'_n yeilds the coherent continuation representation at the infintesimal character $\rho(G)$:

$$\mathbb{C}\mathcal{X} = \bigoplus_{2s+t+u+p+q=n} \mathrm{Ind}_{W_s\ltimes W(A_1)^s\times W_t\times W_u\times W_p\times W_q}^{W_n} \mathrm{triv}\otimes\mathrm{sgn}\otimes\mathrm{triv}\otimes\mathrm{triv}\otimes\mathrm{sgn}'\otimes\mathrm{sgn}'$$

Note that $W_s \ltimes W(A_1)^s \subset W'_n$. Note that $\mathrm{sgn}'|_{W'_a} = \mathrm{sgn}|_{W'_a}$.

$t = u = p = q = 0$**:** In this case, the generaters of $W(A_1)^s$ are reflections given by a (conjugate) of simple imaginary root. Therefore, they act by -1.

$p = q = 0$**:** $W'_n \cap (W_s \ltimes W(A_1)^s \times W_t \times W_u) = W_s \ltimes W(A_1)^s \times (W_t \times W_u)'$ Since the real roots acts by cross action, the stabilizer $(W_t \times W_u)' \subset W_a$

$r = u = 0$**:** $W_n' \cap (W_s \ltimes W(A_1)^s \times W_p \times W_q) = W_s \ltimes W(A_1)^s \times (W_p \times W_q)'$ Since the imaginary roots acts by -1*cross action, the stabilizer $(W_p \times W_q)' = (W_p' \times W_q')\langle w_{ic} \rangle$. Here $w_{ic} = w_{2s+p}w_{2s+p+q}$ is a product of two reflections of conjugations of simple imaginary root. Therefore w_{ic} acts trivially. So on $W_p \times W_q$ we must put either $\mathrm{sgn}' \otimes \mathrm{sgn}'$ or $\mathrm{sgn} \otimes \mathrm{sgn}$.

$p + q$ **and** $r + u$ **are non-zero:** In this case, $W_n' \cap (W_s \ltimes W(A_1)^s \times W_t \times W_u \times W_p \times W_q)$ contain the element $w_{cx} = w_{2s+a}w_{2s+a+b}$. Since w_{cx} is in the complex roots group, it must act trivially. This constrain forces us to put $\mathrm{sgn}' \otimes \mathrm{sgn}'$ on $W_p \times w_q$. (In this case $\mathrm{triv} \otimes \mathrm{triv} \otimes \mathrm{sgn}' \otimes \mathrm{sgn}'(w_{2s+a}w_{2s+a+b}) = 1$.)

Progress reports and questions

I have written a program to compute the combinatoric objects numerically. It can compute the following things:

(1) For each special nilpotent orbit of type B (odd special orthogonal group) and type C (symplectic group), it can generate all the "dot-c-r" diagrams.
(2) It can compute the theta lift of local systems on the nilpotent orbits for symplectic-even orthogonal groups dual pair and metaplectic-odd orthogonal groups dual pair.
(3) For each nilpotent orbit of the orthogonal, symplectic and metaplectic group, it can list all local systems obtained via iterated theta lift and twisting of characters.

Here are findings of the program. They agree with Barbasch's notes.

(1) Using stable range theta lift, we have an injective map from $\mathrm{Unip}(G)$ into $\mathrm{Unip}(G')$ where G is the smaller group, G' is split with split rank $\geqslant 2\operatorname{rank} G$. Therefore it is enough to understand the counting of unipotent representations for the symplectic groups (type C) and odd special orthogonal groups (type B).

    The countings for type B and C are similar due to Vogan duality. (In my program, the generation of dot-c-r diagram of type B is translated to the case of type C).
(2) **Symplectic group** Nilpotent orbits are parameterized by columns $\mathcal{O} := (C_{2a} \geqslant C_{2a-1} \cdots C_1 \geqslant C_0 \geqslant C_{-1} = 0)$ satisfying $C_{2i]} \equiv C_{2i-1} \pmod 2$. We only consider the special orbits which satisfy $C_{2i} = C_{2i-1}$ when C_{2i} is odd.
    (a) Reduction theorem Proposition 4.3 imples that all odd rows in $\mathcal{O}$ could be deleted. Example:

```
part0 = [16,14,13,13,6,4,1,1]
part01 = remove_odd_rows(part0)
print("deleting all odd rows of %s yields %s\n"%(part0, part01))
print_drc_diag_C(part0)
print_list_LS(part0, "C")
print_drc_diag_C(part01)
print_list_LS(part01, "C")
Output
deleting all odd rows of [16, 14, 13, 13, 6, 4, 1, 1] yields [12, 12, 11,
    11, 4, 4, 1, 1]
Type C partition: [16, 14, 13, 13, 6, 4, 1, 1]
Number of drc diagrams: 3520
Number of LS: 3520
Type C partition: [12, 12, 11, 11, 4, 4, 1, 1]
Number of drc diagrams: 3520 Type C
Number of LS: 3520
```

    (b) By the assumption of integral infinitesimal character, even length columns are only allowed to appear at most two times. Moreover, there is a reduction to the case that odd columns only appear twice. Warning: the reduction is not a one-one correspondence (See Barbasch's notes Theorem 2 on p9). Moreover, the number of local systems obtained by theta lifting is less than the number of unipotent representations, see example:

```
part0 = [3,3,3,3,1,1,1,1]
print_drc_diag_C(part0)
print_list_LS(part0,"C")
Output
Type C partition: [3, 3, 3, 3, 1, 1, 1, 1]
Number of drc diagrams: 49
Number of LS: 45
```

    I guess this phenomenon is due to that double theta lifting only relizes parabolic inductions(r-induction). (See Barbasch's notes p10)

(c) **Match the local systems.** The following is verified numerically (its' proof should be contained in Section 5 of Barbasch's notes, which I dose not fully understand.)

Theorem B.1. *Suppose $\mathcal{O}$ is a nilpotent orbit of type C such that even columns appear at most twice, and odd columns appear exactly twice. Then iterated theta lifting yields all unipotent representations attached to $\mathcal{O}$. Moreover, these representations are characterized by their local systems (note that these local systems could be reducible)*

An example for a "big orbit":

```
part0 = [21,21,18,16,12,12,11,11,6,4,1,1]
print_drc_diag_C(part0)
print_list_LS(part0, "C")
Output
Type C partition: [21, 21, 18, 16, 12, 12, 11, 11, 6, 4, 1, 1]
Number of drc diagrams: 286720
Number of LS: 286720
```

(3) **Odd orthogonal group** Nilpotent orbits are parameterized by columns

$$\mathcal{O} := (C_{2a} \geqslant C_{2a-1} \cdots C_1 \geqslant C_0 \geqslant 0)$$

satisfying $C_{2i+1} \equiv C_{2i} \pmod 2$ and C_{2a} is odd. We only consider the special orbits satisfying $C_{2i+1} = C_{2i}$ when C_{2i} is even. The dot-r-c diagram only counts unipotent representations of $\mathrm{SO}(2n+1-2k, 2k)$ with $k = 0, \cdots, n$ (all strong real forms). So

$$\# \bigcup_{p+q=2n+1} \mathrm{Unip}_{\mathcal{O}}(\mathrm{O}(p,q)) = 4\#\mathrm{DRC}(\mathcal{O})$$

(a) **Remove even rows** Following operations will not change the number of unipotent representations: (This is also in Barbasch's notes Section 3.2)
- If the column $C_{2i+1} = C_{2i} + 2r$ is odd, delete the $2r$ rows of length $2(a-i)$. (This also agrees with the theta correspondence)
- If the column $C_{2i+1} = C_{2i} + 2r$ is even, then delete $2r-2$ rows of length $2(a-i)$. When $C_{2i+1} = C_{2i} + 2$ is even, it can not be treated by theta lifting. Example:

```
part0 = [21,8,4,1,1]
part01 = [17,4,4,1,1] # deleting all even rows of part0
part02 = [19,6,4,1,1] # deleting 2(a-i)-2 even rows of part0
print_drc_diag_B(part0)
print_drc_diag_B(part01)
print_drc_diag_B(part02)
Output
Type B partition: [21, 8, 4, 1, 1]
The total number of drc diagrams: 896
Type B partition: [17, 4, 4, 1, 1]
The total number of drc diagrams: 376
Type B partition: [19, 6, 4, 1, 1]
The total number of drc diagrams: 896
```

(b) **Repeated columns** When a column occurs more than twice, theta lifting will not yields enough local systems.

(c) The case (a) and (b) seems already discussed in Section 3.2 of Barbasch notes (the reducition in metaplectic case).

(d) **Match the local systems.** The following is verified numerically.

Theorem B.2. *Suppose $\mathcal{O}$ is a nilpotent orbit of type B such that odd columns appear at most twice, and even columns appear exactly twice. Then iterated theta lifting yields all unipotent representations attached to $\mathcal{O}$. Moreover, these representations are characterized by their local systems.*

An example for a "big orbit":

```
partO = [21,16,16,12,12,11,11,4,4,1,1]
print_drc_diag_B(partO)
print_list_LS(partO, "B")
Output
Number of drc diagrams: 131456
Unipotent repn. of all forms: 525824
Number of LS: 525824
```

(e) **Metaplectic group to odd orthogonal groups** Let $\mathcal{O}'$ be a relevent nilpotent orbit of $\mathrm{Mp}(2m, \mathbb{C})$. By stable range theta lifting, the set of unipotent representations $\mathrm{Unip}_{\mathcal{O}'}(\mathrm{Mp}(2m, \mathbb{R}))$ is embedded to $\mathrm{Unip}_{\mathcal{O}}(\mathrm{SO}(2m + 3, 2m + 2))$. $\mathcal{O}$ is the orbit of $\mathrm{SO}(4m+5, \mathbb{C})$ obtained by attaching a column of length $2m+5$. I expect that the twisting of the strange character of $\mathrm{SO}(2m + 3, 2m + 2)$ is a free action on $\mathrm{Unip}_{\mathcal{O}}(\mathrm{SO}(2m + 3, 2m + 2))$. Then we will have:

$$2\#\mathrm{Unip}_{\mathcal{O}'}(\mathrm{Mp}(2m, \mathbb{R})) = \#\mathrm{Unip}_{\mathcal{O}}(\mathrm{SO}(2m + 3, 2m + 2))$$

Note that $\#\mathrm{Unip}_{\mathcal{O}}(\mathrm{SO}(2m + 3, 2m + 2))$ is counted by the dot-r-c diagram having the same number of r and r'.

(f) A question to Dan: In p2 of your notes 10_13_19, type B paragraph, shall we count the representations obtained from the special ones by interchanging c_{2i+1} and c_{2i}? According to my calculation, we should include these representations to make the numbers to match.

(4) **A remark:** In the formula of theta lifting of local systems, there is a twisting of character according to the signature of the orthogonal group. This twisting is crucial. Only the correct twisting leads to the matching of numbers of dot-c-r diagram and local systems.

(5) **TODO:**

(a) There are inductive structures of dot-r-c diagrams and local systems counting. I will try to prove Theorem B.1 and B.2 by reducing it to small rank cases. (This should be down by the same method in Barbasch's notes)

(b) Understand the relationship between our construction and Moeglin-Renard's work on Arthur packet. They have a notion of "good parity" which seems relevant.

(c) Understand the coherent continuation representations of type D and type M (the metaplectic group).

(d) Understand the cohomological and parabolic induction that appeared in the drc diagram reduction. To Dan: Are these inductions preserve unitarity?

(e) Deduce the Langlands parameter of unipotent representations from the drc diagram (It should not be too difficult, Adams already had an Atlas program to compute the L-parameters of all unipotent representations. But the program was not very efficient for classical groups.)

(f) Describe the L-parameter correspondence in our theta lifting construction. (Induction principle may be the guideline.)

(6) **The goal of our project.** Maybe we should write a series of papers on the subject. I propose only deal with the cases in Theorem B.1 and B.2 in our first paper. Then let us deal with the reductions in the second paper.

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