COUNTING SPECIAL UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS

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Contents

1. Introduction

Barbasch-Vogan [12, 16] developed a formula counting the number of special unipotent representations.

In this paper, we will calculate the number of special unipotent representations for real classical groups explicitly.

1.1. Counting theorem for the real classical groups of type BCD.

1.1. Counting theorem for the real classical groups of type BCD. In this section, we assume $\star \in \{ B, \widetilde{C}, C, C^*, D, D^* \}$.

Let

$$G_{\mathbb{C}} = \begin{cases} \mathrm{SO}(2n+1,\mathbb{C}) & \text{if } \star \in \{ B \}, \\ \mathrm{Sp}(2n,\mathbb{C}) & \text{if } \star \in \{ \widetilde{C}, C, C^* \}, \\ \mathrm{SO}(2n,\mathbb{C}) & \text{if } \star \in \{ D, D^* \}. \end{cases}$$

and the dual group of $G_{\mathbb{C}}$ is defined to be

$$\check{G}_{\mathbb{C}} = \begin{cases} \mathrm{Sp}(2n,\mathbb{C}) & \text{if } \star \in \{ B, \widetilde{C} \}, \\ \mathrm{SO}(2n+1,\mathbb{C}) & \text{if } \star \in \{ C, C^* \}, \\ \mathrm{SO}(2n,\mathbb{C}) & \text{if } \star \in \{ D, D^* \}. \end{cases}$$

Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G}_{\mathbb{C}})$. Let $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$ be the decomposition of $\check{\mathcal{O}}$ into good and bad parity parts.

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Let $\mathfrak{I}_\star(n)$ be the set of real groups given by

$$\mathfrak{I}_\star(n) := \begin{cases} \{\, \mathrm{SO}(2p+1, 2q) \mid p, q \in \mathbb{N} \text{ and } p+q = n \,\} & \text{if } \star = B \\ \{\, \mathrm{Sp}(2n, \mathbb{R}) \,\} & \text{if } \star = C \\ \{\, \mathrm{Mp}(2n, \mathbb{R}) \,\} & \text{if } \star = \widetilde{C} \\ \{\, \mathrm{Sp}(p, q) \mid p, q \in \mathbb{N} \text{ and } p+q = n \,\} & \text{if } \star = C^* \\ \{\, \mathrm{SO}(p, q) \mid p, q \in \mathbb{N} \text{ and } p+q = 2n \,\} & \text{if } \star = D \\ \{\, \mathrm{O}^*(2n) \,\} & \text{if } \star = D^* \end{cases}$$

and $\mathfrak{I}_\star = \bigcup_{n\in\mathbb{N}} \mathfrak{I}_\star(n)$.

Let $\mathrm{unip}_\star(\check{\mathcal{O}})$ be the set of special unipotent representations of groups in $\mathfrak{I}_\star$ attached to $\check{\mathcal{O}}$.

Theorem 1.1. *Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Then*

$$\left|\mathrm{unip}_\star(\check{\mathcal{O}})\right| \leq \left|\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})\right|.$$

Theorem 1.2. *Let $\check{\mathcal{O}}'_b$ be the partition such that $\check{\mathcal{O}}'_b \overset{r}{\sqcup} \check{\mathcal{O}}'_b = \check{\mathcal{O}}_b$ and*

$$\star' = \begin{cases} A^{\mathbb{R}} & \star \in \{\, B, \widetilde{C}, C, D \,\} \\ A^{\mathbb{H}} & \star \in \{\, C^*, D^* \,\} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \mathrm{Unip}_{\star'}(\check{\mathcal{O}}'_b) \times \mathrm{Unip}_\star(\check{\mathcal{O}}_g) & \longrightarrow & \mathrm{Unip}_\star(\check{\mathcal{O}}) \\ (\pi', \pi_0) & \mapsto & \pi' \rtimes \pi_0. \end{array}$$

2. Counting formula

Let G be a real reductive group in the Harish-Chandra class. Here G can be a non-linear group.

Fixing a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, we identify the set of infinitesimal character with $\mathfrak{h}^*/W$ where W is the Weyl group acting on $\mathfrak{h}$.

Let $\mathcal{I}_\mu$ be the maximal primitive ideal with infinitesimal character $W \cdot \mu$. Then there is a unique double cell $\mathcal{D}_\mu$ of $\mathrm{Irr}(W_{[\mu]})$ attached to $\mathcal{I}_\mu$ having associated variety $\mathcal{O}_\mu$.

Let

$$\Pi_{\mathcal{I}_\mu}(G) := \left\{\, \pi \in \mathrm{Irr}(G) \;\middle|\; \mathcal{I}_\mu \subseteq \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) \,\right\}.$$

Let $Q \subset \mathfrak{h}^*$ be the root lattice and write $[\mu] := \mu + Q$ for the coset $\mathfrak{h}^*/Q$ containing μ. Denote $\mathrm{Coh}_{[\mu]}(G)$ the space of coherent families based on $[\mu]$ which has a $W_{[\mu]}$-action.

For each $G_{\mathbb{C}}$-invariant closed subset S in the nilpotent cone of $\mathfrak{g}$, let

$$\mathrm{Coh}_{[\mu],\mathsf{S}}(G) := \left\{\, \Theta \in \mathrm{Coh}_{[\mu]} \;\middle|\; \mathrm{AV}_{\mathbb{C}}(\Theta(\mu')) \subset \mathsf{S} \text{ for } \mu' \in [\mu] \,\right\}.$$

The following theorem gives an upper bound of $|\Pi_{\mathcal{I}_{mu}}(G)|$.

Theorem 2.1 (Barbasch-Vogan). *Let $\Pi_{\mathcal{O},\mu}(G)$ denote the set of irreducible admissible G-module with complex associated variety $\overline{\mathcal{O}}$. Then*

$$\begin{aligned} \left|\Pi_{\mathcal{I}_\mu}(G)\right| &= \sum_{\sigma\in\mathcal{D}_\mu} [\sigma : \mathrm{Coh}_{[\mu]:\overline{\mathcal{O}}_\mu}(G)] \cdot [1_{W_\mu}, \sigma|_{W_\mu}] \\ &\leq \sum_{\sigma\in\mathcal{D}_\mu} [\sigma : \mathrm{Coh}_{[\mu]}(G)] \cdot [1_{W_\mu} : \sigma|_{W_\mu}]. \end{aligned}$$

Lemma 2.2 ([16, (5.26), Proposition 5.28])**.** *Let $\check{\mathcal{O}}$ be a nilpotent orbit in $\check{G}_{\mathbb{C}}$ and $\lambda_{\check{\mathcal{O}}}$ be the infinitesimal character attached to $\check{\mathcal{O}}$. Then the set*

$$^{L}\mathscr{C}(\check{\mathcal{O}}) := \left\{\, \sigma \in \mathcal{D}_{\lambda_{\check{\mathcal{O}}}} \;\middle|\; [1_{W_\mu} : \sigma] \neq 0 \,\right\}$$

is a left cell in $\mathcal{D}_{\lambda_{\check{\mathcal{O}}}}$ given by

$$(J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \operatorname{sgn}) \otimes \operatorname{sgn}.$$

Moreover, the multiplicity $[1_{W_\mu} : \sigma]$ is one when $\sigma \in {}^{L}\mathscr{C}(\check{\mathcal{O}})$.

Corollary 2.3. *Under the notation of Lemma 2.2, we have*

$$\left|\Pi_{\mathcal{I}_\mu}(G)\right| \leq \sum_{\sigma \in {}^{L}\mathscr{C}(\check{\mathcal{O}})} [\sigma : \mathrm{Coh}_{[\mu]}(G)]$$

2.1. **Coherent family.** Let Q be the $\mathfrak{h}$ root lattice of $\mathfrak{g}$. Let $\mathcal{G}$ be the Grothendieck group of finite dimensional $\mathfrak{g}$-modules occur in $S(\mathfrak{g})$. Note that $\mathcal{G}$ is also an algebra under tensor product. For each finite dimensional $\mathfrak{h}$-module F, let $\Delta(F)$ denote the multi-set of $\mathfrak{h}$-weights in F.

Via the highest weight theory, every W-orbit $W \cdot \mu$ in ${}^{a}Q$ corresponds with the irreducible finite dimensional representation F_μ with extremal weight μ.

For any $\lambda \in {}^{a}\mathfrak{h}^*$, we define the lattice

$$\Lambda := [\lambda] := \lambda + Q \subset \mathfrak{h}^*$$

and define

$$\begin{aligned} R_{[\lambda]} &:= \{\, \alpha \in R \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \,\}, \\ W_{[\lambda]} &:= \{\, w \in W \mid w \cdot \lambda - \lambda \in Q \,\} \\ R_\lambda &:= \{\, \alpha \in {}^{a}R \mid \langle \lambda, \check{\alpha} \rangle = 0 \,\}, \quad \text{and} \\ W_\lambda &:= \langle s_\alpha | \alpha \in R_\lambda \rangle = \langle w \in W | w \cdot \lambda = \lambda \rangle \subseteq W. \end{aligned} \tag{2.1}$$

It is known that $R_{[\lambda]}$ is a root system and

$$\begin{aligned} W_{[\lambda]} &= \mathrm{Stab}_W([\lambda]) = \langle s_\alpha | \alpha \in R_{[\lambda]} \rangle \subseteq W \quad \text{and} \\ W_\lambda &= \langle s_\alpha | \alpha \in R_\lambda \rangle \subseteq W_{[\lambda]}. \end{aligned}$$

In fact W_λ is a parabolic subgroup of $W_{[\lambda]}$ by Chevalley's theorem [96, Lemma 6.3.28].

When λ is regular, let

$$R^+_{[\lambda]} := \left\{\, \alpha \in R_{[\lambda]} \;\middle|\; \langle \check{\alpha}, \lambda \rangle > 0 \,\right\}$$

be the fixed positive root system.

In the following we define the notion of coherent family based on the lattice $[\lambda]$ in a quite general setting.

Definition 2.4. *Suppose that $\mathcal{M}$ is be an abelian group with $\mathcal{G}$-action:*

$$\mathcal{G} \times \mathcal{M} \ni (F, m) \mapsto F \otimes m.$$

In addition, a subgroup $\mathcal{M}_\mu$ of $\mathcal{M}$ is fixed for each $\mu \in [\lambda]$ such that $\mathcal{M}_\mu = \mathcal{M}_{w\cdot\mu}$ for any $\mu \in [\lambda]$ and $w \in W_{[\lambda]}$.

A function $f \colon \Lambda \to \mathcal{M}$ is called a coherent family based on Λ if it satisfies $f(\mu) \in \mathcal{M}_\mu$ and

$$F \otimes f(\mu) = \sum_{\nu \in \Delta(F)} f(\mu + \nu) \qquad \forall \mu \in \Lambda, F \in \mathcal{G}.$$

Let $\mathrm{Coh}_\Lambda(\mathcal{M})$ be the abelian group of all coherent families based on Λ and taking value in $\mathcal{M}$. We can define $W_{[\lambda]}$ action on $\mathrm{Coh}_[\lambda](\mathcal{M})$ by

$$w \cdot f(\mu) = f(w^{-1} \cdot \mu) \qquad \forall \mu \in \Lambda, w \in W_\Lambda.$$

In this paper, we will consider the following cases.

Example 2.5. *Suppose $\mathcal{M}$ is a field of characteristic zero and*

$$F \otimes m := \dim(F) \cdot m \quad \text{for all } F \in \mathcal{G} \text{ and } m \in \mathcal{M}.$$

We let $\mathcal{M}_\mu = \mathcal{M}$ for every $\mu \in \Lambda$. Then the set of W-harmonic polynomials on $\mathfrak{h}$ is naturally identified with $\mathrm{Coh}_{[\lambda]}(\mathcal{M})$ via the restriction on $[\lambda]$ by Vogan [95, Lemma 4.3].

[*Note that the polynomials are W-harmonic not necessary $W_{[\lambda]}$-harmonic. ($W_{[\lambda]}$-invariant differential operators are more than W-invariant differential operators.)*]

Example 2.6. *Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \subset \mathfrak{g}$, let $\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ be the Grothendieck group of the category $\mathcal{O}$ with coefficients in $\mathbb{C}$, i.e. the category of finitely generated $\mathcal{U}(\mathfrak{g})$-modules with semisimple $\mathfrak{h}$-action and locally finite $\mathfrak{n}$-action. For $\lambda \in \mathfrak{h}^*$, let $\mathcal{G}_{W\cdot\lambda}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ be the subgroup spanned by $\mathfrak{g}$-modules with infinitesimal character $W \cdot \lambda$. Here $\mathcal{G}$ acts on $\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ via the tensor product of $\mathfrak{g}$-modules.*

To ease the notation, we write

$$\mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) := \mathrm{Coh}_{[\lambda]}(\mathcal{G}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})).$$

For $\lambda \in \mathfrak{h}^*$, let $\rho := \sum_{\alpha \in \Delta(\mathfrak{n})} \alpha$,

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

be the Verma module with highest weight $\lambda - \rho$ and $L(\lambda)$ be the unique irreducible quotient of $M(\lambda)$.

Each $w \in W$ defines a coherent family

$$M_w(\mu) := M(w \cdot \mu) \quad \forall \mu \in [\lambda].$$

The map

$$\begin{array}{ccc} \mathbb{C}[W] & \longrightarrow & \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) \\ w & \mapsto & M_w \end{array}$$

is $W_{[\lambda]}$-equivariant isomorphism where $W_{[\lambda]}$ acts on $\mathbb{C}[W]$ by right translation.

One of the crucial property is that each irreducible module can be fitted into a coherent family. More precisely, the evaluation map descents to yields an isomorphism $\overline{\mathrm{ev}}_\mu$ in the following diagram.

$$\begin{tikzcd} \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) \arrow[r, "\Theta\mapsto\Theta(\mu)"] \arrow[d] & \mathcal{G}_{W\cdot\mu}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) \\ \left(\mathrm{Coh}_{[\lambda]}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})\right)_{W_\mu} \arrow[ur, hook, two heads, "\overline{\mathrm{ev}}_\mu"'] & \end{tikzcd} \tag{2.2}$$

[*The subjectivity is because of Verma modules form a basis of the category $\mathcal{O}$. The LHS of $\overline{\mathrm{ev}}_\mu$ has dimension $|W/W_\mu|$, the RHS has dimension $W \cdot \mu$. Now the isomorphism follows by dimension counting.*]

Example 2.7. *Suppose G is a reductive Lie group in the Harish-Chandra class. Let $\mathcal{G}(G)$ be the Grothendieck group of finite length admissible G-modules and $\mathcal{G}_\mu(G)$ be the subgroup of $\mathcal{G}(\mathfrak{g}, K)$ generated by the set of irreducible G-modules with infinitesimal character μ.*

By [96, 0.4.6], *we can naturally identify Q with the set of H^s-weights consisting the characters occurs in $S(\mathfrak{g})$ where H^s is a maximally split Cartan in G. Therefore, the set of irreducible G-submodules occur in $S(\mathfrak{g})$ is also naturally identified with $Q/W = \mathcal{G}$. We let $\mathcal{G}$ acts on $\mathcal{G}(G)$ by the tensor product of G-modules.*

[*Note that by the assumption that G is in the Harish-Chandra class, each irreducible $\mathfrak{g}$-submodule F embeds in $S(\mathfrak{g})$ is automatically globalized to a G-module. The point is that the globalization is independent of the embedding of F in $S(\mathfrak{g})$!*]

We write

$$\mathrm{Coh}_{[\lambda]}(G) := \mathrm{Coh}_{[\lambda]}(\mathcal{G}(G))$$

for the space of coherent family of G-modules.

Example 2.8. *Fix a $G_{\mathbb{C}}$-invariant closed subset S in the nilpotent cone of $\mathfrak{g}$. Let $\mathcal{G}_{\mathsf{S}}(G)$ be the Grothendieck group of finite length admissible G-modules whose complex associated varieties are contained in S. Define*

$$\mathcal{G}_{\mu,\mathsf{S}}(G) := \mathcal{G}_{\mu}(G) \cap \mathcal{G}_{\mathsf{S}}(G).$$

and write

$$\mathrm{Coh}_{[\lambda],\mathsf{S}}(G) := \mathrm{Coh}_{[\lambda]}(\mathcal{G}_{\mathsf{S}}(\mathfrak{g}, K)).$$

Note that

$$\mathrm{AV}_{\mathbb{C}}(\pi \otimes F) = \mathrm{AV}_{\mathbb{C}}(\pi).$$

for each finite length G-module π and finite dimensional G-module F. Therefore the $W_{[\lambda]}$-module

$$\mathrm{Coh}_{[\lambda],\mathsf{S}}(G) = (\mathrm{ev}_{\mu})^{-1}(\mathcal{G}_{\mu,\mathsf{S}}(G))$$

for any regular $\mu \in [\lambda]$.

Similarly, we define the space $\mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$ of coherent families in category $\mathcal{O}$ whose associated variety are contained in S. In particular,

$$\mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}) = (\mathrm{ev}_{\mu})^{-1}(\mathcal{G}_{\mu,\mathsf{S}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})).$$

A remarkable fact that the diagram Equation (2.2) still holds for $\mathrm{Coh}_{[\lambda],\mathsf{S}}(G)$. This is one of the first step towards the counting of the set

$$\mathrm{Irr}_{\mu,\mathsf{S}}(G) := \{\, \pi \in \mathrm{Irr}_{\mu}(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) \subset \mathsf{S} \,\}.$$

where

$$\mathrm{Irr}_{\mu}(G) := \{\, \pi \in \mathrm{Irr}(G) \mid \pi \text{ has infinitesimal character } \mu \,\}.$$

Lemma 2.9. *For each μ and closed $G_{\mathbb{C}}$-invariant subset S in the nilpotent cone of $\mathfrak{g}$, we have an isomorphism*

$$\overline{\mathrm{ev}}_{\mu}\colon \left(\mathrm{Coh}_{[\mu],\mathsf{S}}(G)\right)_{W_{\mu}} \longrightarrow \mathcal{G}_{\mu,\mathsf{S}}(G).$$

In particular,

$$|\mathrm{Irr}_{\mu,\mathsf{S}}(G)| = [1_{W_{\mu}} : \mathrm{Coh}_{[\mu],\mathsf{S}}(G)]$$

Proof. This is a consequence of the formal properties of the translation functor, especially the theory of τ-invariant.

The properties of the coherent family and translation principal: (We refers to [96, Section 7] for the proofs which also work in our (possibly non-linear) setting.)

(a) The evaluation map

$$\mathrm{ev}_{\mu}\colon \mathrm{Coh}_{[\mu]}(G) \to \mathcal{G}_{\mu}(G)$$

is surjective for any $\mu \in [\mu]$, see [96, Theorem 7.2.7].

From now on we fix a regular element $\lambda \in [\mu]$ such that μ is dominant with respect to $R^+_{[\lambda]}$

(b) The evaluation map ev_{λ} at λ is an isomorphism [96, Proposition 7.2.27].

For each $\pi \in \mathcal{G}_{\lambda}(G)$, let

$$\Theta_{\pi} := (\mathrm{ev}_{\lambda})^{-1}(\pi)$$

be the unique coherent family such that $\Theta_{\pi}(\lambda) = \pi$,

(c) If $\pi \in \mathrm{Irr}_\lambda(G)$, then $\Theta_\pi(\mu)$ is either zero or an irreducible G-module [96, Proposition 7.3.10, Corollary 7.3.23].

(d) For $\pi \in \mathrm{Irr}_\lambda(G)$,
$$\mathrm{AV}(\Theta_\pi(\mu)) = \mathrm{AV}(\pi)$$
whenever μ is dominant and $\Theta_\pi(\mu)$ non-zero. [This because $\pi = \psi_\mu^\lambda(\Theta_\pi(\mu))$ and $\Theta_\pi(\mu) = \psi_\lambda^\mu(\pi)$. Here is the translation functor from λ to μ see [96, Definition 4.5.7]. Translation dose not increase the associated variety.]

(e) If π and π' are in $\mathrm{Irr}_\lambda(G)$ such that $\Theta_\pi(\mu) = \Theta_{\pi'}(\mu)$ is non-zero, then $\pi = \pi'$.

For $\pi \in \mathrm{Irr}_\lambda(G)$, define the τ-invariant of π to be

$$\tau(\pi) := \left\{ \alpha \in R^+_{[\lambda]} \;\middle|\; \begin{array}{l} \alpha \text{ is simple and} \\ s_\alpha \cdot \Theta_\pi(\lambda) = -\Theta_\pi(\lambda) \end{array} \right\} \tag{2.3}$$

(f) $\Theta_\pi(\mu) = 0$ if and only if $\tau(\gamma) \cap R_\mu \neq \emptyset$ [96, Corollary 7.3.23 (c)].

Now we start to prove the lemma. By the translation principle,
$$\begin{aligned} &\{\, \Theta_\pi(\mu) \mid \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ s.t. } \Theta_\pi(\mu) \neq 0 \,\} \\ =&\{\, \Theta_\pi(\mu) \mid \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ s.t. } \tau(\pi) \cap R_\mu = \emptyset \,\}. \end{aligned}$$
forms a basis of $\mathcal{G}_{\mu,\mathsf{S}}(G)$. [The set consists of distinct (so linearly independent) irreducible G-modules by (c) and (e). They are spanning set by (a). For the support condition, see (d). The τ-invariant condition is by (f).] Hence
$$\begin{aligned} \ker \mathrm{ev}_\mu &= \mathrm{Span}\,\{\, \Theta_\pi \mid \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ s.t. } \tau(\pi) \cap R_\mu \neq \emptyset \,\} \\ &\subseteq \mathrm{Span}\,\left\{\, \tfrac{1}{2}(\Theta_\pi - s_\alpha \cdot \Theta_\pi) \;\middle|\; \pi \in \mathrm{Irr}_{\lambda,\mathsf{S}}(G) \text{ and } \alpha \in \tau(\pi) \cap R_\mu \,\right\} \\ &\qquad \text{(by the definition of } \tau(\pi) \text{ in (2.3).)} \\ &\subseteq \mathrm{Span}\,\left\{\, \Theta - w \cdot \Theta \;\middle|\; \Theta \in \mathrm{Coh}_{[\mu],\mathsf{S}}(G) \,\right\} \\ &\subseteq \ker \mathrm{ev}_\mu. \\ &\qquad \text{(by } w \cdot \Theta(\mu) = \Theta(w^{-1} \cdot \mu) = \Theta_\pi(\mu)) \end{aligned}$$
Since $\left(\mathrm{Coh}_{[\mu],\mathsf{S}}(G)\right)_{W_\mu} = \mathrm{Coh}_{[\mu],\mathsf{S}}(G) / \mathrm{Span}\,\{\, \Theta - w \cdot \Theta \mid \Theta \in \mathrm{Coh}_{[\mu],\mathsf{S}}(G) \,\}$, the lemma follows. □

3. Primitive ideals and Weyl group representations

3.1. Associated varieties of a primitive ideals and double cells in $W_{[\lambda]}$. In this section, we review the notion of double cells and its relation with the associated varieties of primitive ideals, see [11, 47]. We retain the notation in Example 2.6.

Let $\mathrm{Prim}_\lambda(\mathfrak{g})$ be the set of primitive ideals in $\mathcal{U}(\mathfrak{g})$ with infinitesimal character λ. Let $\lambda \in \mathfrak{h}^*$, each primitive ideal is the annihilator of a highest weight module by Duflo [28]. In other words, the following map is surjective
$$\begin{array}{ccc} W_{[\lambda]} & \longrightarrow & \mathrm{Prim}_\lambda(\mathfrak{g}) \\ w & \mapsto & I(w \cdot \lambda) := \mathrm{Ann}\, L(w \cdot \lambda). \end{array}$$

By the translation principal, we concentrate the discussion in the regular infinitesimal character case. From now on, we follows the convention in [11]. Let λ be a regular element in $\mathfrak{h}^*$ such that $R^+_{[\lambda]} \subset -\Delta(\mathfrak{n})$. [Here λ is regular anti-dominant ($\langle \lambda, \check{\alpha} \rangle \notin \mathbb{N}$ for each $\alpha \in \Delta(\mathfrak{n})$) with respect to the root system defining highest weight modules, but it is dominant with respect to $R^+_{[\lambda]}$.] For each $w \in W_{[\lambda]}$, define
$$a(w) := |\Delta(\mathfrak{n})| - \mathrm{GK\text{-}dim}(L(w\lambda)).$$
[Suppose λ is integral, then Under this definition, $a_{w_0} = |\Delta(\mathfrak{n})|$ and $a_e = 0$.] For

each w one can attach a polynomial $\widetilde{p}_w$ such that $\widetilde{p}_w(\mu) = \operatorname{rank}(\mathcal{U}(\mathfrak{g})/\operatorname{Ann}(L(w\mu)))$ when $\mu \in [\lambda]$ is dominant (i.e. $-\langle \mu, \check{\alpha}\rangle \notin \mathbb{N}^+$ for all $\alpha \in R^+_{[\lambda]}$). $\widetilde{p}_w$ is called the Goldie-rank polynomial attached to the primitive ideal $\operatorname{Ann}(L(w\lambda))$. Fix a dominant regular element δ in $\mathfrak{h}$ (i.e. $\langle \delta, \alpha\rangle > 0$ for each $\alpha \in \Delta(\mathfrak{n})$). Let

$$r_w = \sum_{y\in W_{[\lambda]}} a_{y,w}(y^{-1}\delta)^{a(w)} \in S(\mathfrak{h})$$

where $a_{y,w}$ is determined by the equation

$$L(w\lambda) = \sum_{y\in W_{[\lambda]}} a_{y,w} M(w\lambda)$$

in $\mathcal{G}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$. Then r_w is a positive multiple of $\widetilde{p}_w$ [45, Section 1.4].

A partial order $\underset{L}{\leq}$ can be define on $W_{[\lambda]}$ by the following condition [11, Proposition 2.9]

$$\begin{aligned} w_1 \underset{L}{\leq} w_2 &\Leftrightarrow I(w_1\lambda) \subseteq I(w_2\lambda) \\ &\Leftrightarrow L(w_2^{-1}\lambda) \text{ is a subquotient of } L(w_1^{-1}\lambda)\otimes S(\mathfrak{g}). \end{aligned} \tag{3.1}$$

We say $w_1 \underset{L}{\approx} w_2$ if and only if $w_1 \underset{L}{\leq} w_2 \underset{L}{\leq} w_1$. For $w \in W_{[\lambda]}$, we call

$$\mathscr{C}^L_w := \left\{\, w' \in W_{[\lambda]} \,\middle|\, w \underset{L}{\approx} w' \,\right\}$$

the left cell in $W_{[\lambda]}$ containing w.

In summary, we have a bijection

$$\begin{array}{ccc} W_{[\lambda]}/\underset{L}{\approx} & \longrightarrow & \operatorname{Prim}_\lambda(\mathfrak{g}) \\ \mathscr{C}^L_w & \mapsto & \operatorname{Ann} L(w\lambda). \end{array}$$

The partial order $\underset{R}{\leq}$ is defined by

$$w_1 \underset{R}{\leq} w_2 \Leftrightarrow w_1^{-1} \underset{L}{\leq} w_2^{-1}.$$

The partial order $\underset{LR}{\leq}$ is defined to be the minimal partial order containing $\underset{L}{\leq}$ and $\underset{R}{\leq}$. The relation $\underset{R}{\approx}$, $\underset{LR}{\approx}$, right cell $\mathscr{C}^R_w$ and double cells $\mathscr{C}^{LR}_w$ are defined similarly.

Since the Kazhdan-Lusztig conjecture has been proven (for the integral infinitesimal character case by [17, 19] and reduced to the integral infinitesimal character case by [90] (see [43, Section 13.13])), the definition of the order $\underset{L}{\leq}$ is the same as the partial order defined by Kazhdan-Lusztig [62] which only depends on the Coxeter group structure of $W_{[\lambda]}$, see [11, Corollary 2.3]. [Note that $x \underset{L}{<} y$ implies $a(x) < a(y)$ and $x \underset{LR}{\approx} y$ implies $a(x) = a(y)$.]

Note that the left cells are exactly the fibers of the map $w \mapsto \widetilde{p}_w$. Take a double cell $\mathscr{C}^{LR}_w$ in $W_{[\lambda]}$ and a set of representatives $\{\, w_1, w_2, \cdots, w_k \,\}$ of the left cells in $\mathscr{C}^{LR}_w$. Due to Barbasch-Vogan[10, 11] and Joseph[44–47], the following statements holds:

- the set of Goldie rank polynomials $\{\, \widetilde{p}_{w_i} \mid i = 1, 2, \cdots, k \,\}$ form a basis of a special representation σ_w of $W_{[\lambda]}$ realized in $S^{a(w)}(\mathfrak{h})$;
- the multiplicity of σ_w in $S^{a(w)}(\mathfrak{h})$ is one,
- $a(w)$ is the minimal degree m such that σ_w occurs in $S^m(\mathfrak{h})$ which is the fake degree and the generic degree of the special representation σ_w. [When $W_{[\lambda]} = W$, this is the definition of the fake degree. Otherwise, $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}^{W_\lambda}$ where $\mathfrak{h}_0$ is the span of coroots of $W_{[\lambda]}$. Then $S(\mathfrak{h}_0)$ is embeds in $S(\mathfrak{h})$.]

- the map $W_{[\lambda]}/\underset{LR}{\approx} \ni \mathscr{C}^{LR}_w \mapsto \sigma_w \in \mathrm{Irr}(W_{[\lambda]})$ yields a bijection between the set of double cells and the set $\mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]})$ of special representations of $W_{[\lambda]}$.
- Under the W action, the $W_{[\lambda]}$-module $\sigma_w \subset S^{a(w)}(\mathfrak{h})$ generates an irreducible W-module $\widetilde{\sigma}_w := j^W_{W_{[\lambda]}}\sigma_w$. The W-module $\widetilde{\sigma}_w$ corresponds to the a nilpotent orbit $\mathcal{O}_{\widetilde{\sigma}_w}$ with trivial local system. Now the complex associated variety
$$G^{ad}_{\mathbb{C}}\mathrm{AV}(L(w\lambda)) = \mathrm{AV}(\mathrm{Ann}(L(w\lambda))) = \overline{\mathcal{O}_{\widetilde{\sigma}_w}},$$
where $G^{ad}_{\mathbb{C}}$ is the adjoint group of $\mathfrak{g}$, see [47, Section 2.10].

[Note that the j-induction is not injective in general. For example, $j^{S_{a+b}}_{S_a\times S_b}\tau_a\otimes\tau_b = \tau_a \overset{c}{\sqcup} \tau_b$ where τ_a, τ_b are partitions.]

Following Joseph, we say two primitive ideals $I(w\lambda)$ and $I(w'\lambda)$ with $w, w' \in W_{[\lambda]}$are in the same *clan* if and only if $w \underset{LR}{\approx} w'$ or equivalently $\sigma_w = \sigma_{w'}$.

Obvious, the associated varieties of two primitive ideals in the same clan has the same associated variety. However the reverse dose not holds in the non-integral infinitesimal character case in general.

Coherent continuations. Now we recall the relationships between cells and the coherent continuation representations.

For each $w \in W$, we define a coherent family L_w by the condition $L_w(\lambda) = L(w\lambda)$.

For each $\mu \in \mathfrak{h}^*$, let $\mathcal{O}_{[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$ be the subcategory of the category $\mathcal{O}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$ consists of modules whose $\mathfrak{h}$-weights are contained in $[\mu]$,
$$\mathcal{G}_{W\cdot\lambda,[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) = \mathcal{G}_{[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) \cap \mathcal{G}_{W\cdot\lambda}(\mathfrak{g},\mathfrak{h},\mathfrak{n})$$
which is spanned by a block in the category $\mathcal{O}$ if $W\cdot\lambda \cap [\mu+\rho] \neq \emptyset$.

Consider the following subgroup of coherent continuation
$$\mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n};[\mu]) = \left\{\, \Theta \in \mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) \;\middle|\; \Theta(\lambda) \in \mathcal{G}_{W\cdot\lambda,[\mu]}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) \,\right\}.$$

We identify $\mathbb{C}[W_{[\lambda]}]$ with $\mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n},[\lambda-\rho])$ via $w \mapsto M_w$.

For each $w \in W_{[\lambda]}$, let
$$\overline{\mathscr{V}}^R_w := \mathrm{Span}\left\{\, L_{w'} \;\middle|\; w \underset{R}{\leq} w' \,\right\}$$
$$\mathscr{V}^R_w = \overline{\mathscr{V}}^R_w \Big/ \sum_{w \underset{L}{<} w'} \overline{\mathscr{V}}^R_{w'}$$

By (3.1), $\overline{\mathscr{V}}^R_w$ and $\mathscr{V}^R_w$ are $W_{[\lambda]}$-modules under right translation/coherent continuation action.

We define left $W_{[\lambda]}$-module $\overline{\mathscr{V}}^L_w$ and $\mathscr{V}^L_w$ using $\underset{L}{\leq}$ and $W_{[\lambda]}\times W_{[\lambda]}$-module $\overline{\mathscr{V}}^{LR}_w$ and $\mathscr{V}^{LR}_w$ using $\underset{LR}{\leq}$ similarly.

Suppose $\sigma_1, \sigma_2 \in \mathrm{Irr}(W_{[\lambda]})$. We define
$$\sigma_1 \underset{LR}{\leq} \sigma_2 \Leftrightarrow \exists w \in W_{[\lambda]} \text{ such that } \begin{cases} \sigma_1\otimes\sigma_1 \text{ occurs in } \mathscr{V}^{LR}_w \text{ and} \\ \sigma_2\otimes\sigma_2 \text{ occurs in } \overline{\mathscr{V}}^{LR}_w. \end{cases}$$

Now $\sigma_1 \underset{LR}{\approx} \sigma_2$ if and only if there exists a $w \in W_{[\lambda]}$ such that $\sigma_1\otimes\sigma_1$ and $\sigma_2\otimes\sigma_2$ both occur in $\mathscr{V}^{LR}_w$. Now $\underset{LR}{\leq}$ is a well defined partial order and $\underset{LR}{\approx}$ is an equivalent relation on $\mathrm{Irr}(W_{[\lambda]})$ respectively. We write ${}^{LR}\mathscr{C}_\sigma \subseteq \mathrm{Irr}(W_{[\lambda]})$ for the double cell containing σ. [A priori $\sigma_1 \underset{LR}{\approx} \sigma_2 \Leftrightarrow \sigma_1 \underset{LR}{\leq} \sigma_2 \underset{LR}{\leq} \sigma_1$.

But note that $\bigoplus_{w\in W_{[\lambda]}/\underset{LR}{\approx}} \mathscr{V}_w^{LR} \cong \mathbb{C}[W_{[\lambda]}]$ and $\sigma\otimes\sigma$ has multiplicity one in $\mathbb{C}[W_{[\lambda]}]$ which implies the claim.]

A left (resp. right cell) in $\mathrm{Irr}(W_{[\lambda]})$ is the multiset of the irreducible constituents in $\mathscr{V}_w^{L}$ (resp. left cell) for some $w \in W_{[\lambda]}$.

The equivalence of Barbasch-Vogan's definition and Lusztig's definition of cells in $\mathrm{Irr}(W_{[\lambda]})$ is a consequence of Kazadan-Lusztig conjecture, see [11, remarks after Corollary 2.16].

The structure of double and left cells are explicitly described in [63, Section 4]. In particular, σ_w is the unique special representation occurs in the double cell

$$^{LR}\mathscr{C}_w := \{\, \sigma \mid \sigma\otimes\sigma \text{ occurs in } \mathscr{V}_w^{LR} \,\} \subseteq \mathrm{Irr}(W_{[\lambda]}).$$

For this reason, we also write

$$^{LR}\mathscr{C}_\sigma := {}^{LR}\mathscr{C}_w$$

where $\sigma = \sigma_w$ is the unique special representation in $^{LR}\mathscr{C}_w$. The generic degree "a"-function is constant on the double cells and order preserving: for each $\sigma' \in {}^{LR}\mathscr{C}_\sigma$, the generic degree $a(\sigma') = a(\sigma)$; $a(\sigma') < a(\sigma'')$ if $\sigma' \underset{LR}{<} \sigma''$.

In summary, we have bijections

$$W_{[\lambda]}/\underset{LR}{\approx} \longleftrightarrow \mathrm{Irr}^{\mathrm{sp}}(W_{[\lambda]}) \longleftrightarrow \mathrm{Irr}(W_{[\lambda]})/\underset{LR}{\approx}.$$

We write $\mathscr{V}_\sigma^{LR}$ to be the unique double cell representation containing σ and $\overline{\mathscr{V}}_\sigma^{LR}$ to be the unique upper cone representation which is isomorphic to $\bigoplus_{\sigma\underset{LR}{\leq}\sigma'} \sigma'\otimes\sigma'$.

3.2. **Compare blocks.** Now we compare different blocks. Without of loss of generality, we assume λ is in the anti-dominant cone of $\Delta(\mathfrak{n})$, i.e $\langle \lambda, \check{\alpha}\rangle < 0$ for all $\alpha \in \Delta(\mathfrak{n})$.

Let $k = \left|W/W_{[\lambda]}\right|$ and

$$\{\, r_1, \cdots, r_k \,\} := \left\{\, r \;\middle|\; l(r) \text{ is minimal among elements in } rW_{[\lambda]} \,\right\}$$

be the set of distinguished representatives of the right cosets of $W_{[\lambda]}$ where $l(r)$ denote the length function with respect to the simple roots in $-\Delta(\mathfrak{n})$. In other words, r_i is the unique element in the coset $r_iW_{[\lambda]}$ such that $r_i\lambda$ is anti-dominant, i.e. $R^+_{[r_i\lambda]} \subseteq -\Delta(\mathfrak{n})$.

Now the map $L(w\lambda) \mapsto L(r_iw\lambda)$ with w running over $w \in W_{[\lambda]}$ induces an equivalence of category from $\mathcal{O}_{W\cdot\lambda,[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n})\cap$ to $\mathcal{O}_{W\cdot\lambda,[r_i\lambda]}$ by Soegel's theorem [43, 13.13]. In particular $w \mapsto r_iwr_i^{-1}$ induces isomorphism $W_{[\lambda]} \to W_{[r_i\lambda]}$ and preserves the cell structures. In other words, the following $W_{[\lambda]}$-module isomorphism

$$\begin{tikzcd}
 & \mathbb{C}[W] \arrow[dl, "w\mapsto M_w"'] \arrow[dr, "w\mapsto M_{r_iw}"] & \\
\mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n};[\lambda]) \arrow[rr] & & \mathrm{Coh}_{[\lambda]}(\mathfrak{g},\mathfrak{h},\mathfrak{n};[r_i\lambda])
\end{tikzcd}$$

maps cell representations to cell representations.

[Let $C = \{\, x \in \mathfrak{h} \mid \langle x, \alpha\rangle > 0 \ \forall \alpha \in \Delta(\mathfrak{n}) \,\}$ and $D_\lambda = \{\, x \in \mathfrak{h} \mid -\langle x, \beta\rangle > 0 \ \forall \beta \in R^+_{[\lambda]} \,\}$. C and D_λ are fundamental domains of $\mathfrak{h}$ under W and $W_{[\lambda]}$-actions. Clearly, $D_{w\lambda} = wD_\lambda$. The condition that $R^+_{[\mu]} \subset -\Delta(\mathfrak{n})$ is equivalent to $D_\mu \supset C$.

Now it is clear D_λ is the union of $r_i^{-1}C$ when r_i running over the preferred coset representatives of $W/W_{[\lambda]}$.]

Recall the definition of clan of the primitive ideals. By the comparing the definition of Goldie rank polynomials, we see that $I(r_iw)$ and $I(r_jw')$ in the same clan if and only if $w \underset{LR}{\approx} w'$. In other word, the clan is only depends on the $W_{[\lambda]}$-type of the double cell containing $L(w\lambda)$ where $w \in W$.

The above discussion yields the following.

Lemma 3.1. *Fix a $G_{\mathbb{C}}^{ad}$-invariant subset S in the nilpotent cone of $\mathfrak{g}$.*

Let

$$\begin{aligned} \mathsf{C}_{\mathsf{S}}^{sp} &:= \left\{ \sigma \in \operatorname{Irr}^{sp}(W_{[\lambda]}) \,\middle|\, \operatorname{Springer}(j_{W_{[\lambda]}}^{W}\sigma) \subseteq \mathsf{S} \right\}, \\ \mathsf{C}_{\mathsf{S}} &:= \left\{ \sigma' \in \operatorname{Irr}(W_{[\lambda]}) \,\middle|\, \exists \sigma \in \mathsf{C}_{\mathsf{S}}^{sp} \text{ such that} \sigma \underset{LR}{\leq} \sigma' \right\} \end{aligned} \tag{3.2}$$

Then, as an $W_{[\lambda]}$-module

$$\begin{aligned} \operatorname{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g},\mathfrak{h},\mathfrak{n}) &= \bigoplus_{i=1}^{k} \operatorname{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g},\mathfrak{h},\mathfrak{n};[r_i\lambda-\rho]) \\ &\cong \bigoplus_{i=1}^{k} \sum_{\sigma\in\mathsf{C}_{\mathsf{S}}^{sp}} \mathscr{V}_{\sigma}^{LR} \\ &\cong \bigoplus_{i=1}^{k} \bigoplus_{\sigma\in\mathsf{C}_{\mathsf{S}}} (\dim\sigma)\sigma \end{aligned}$$

As a baby case of the counting theorem for special unipotent representation of real reductive groups, we have the following counting theorem in the category $\mathcal{O}$.

We fix an regular element $\lambda \in [\mu]$ such that μ is dominant with respect to $R_{[\lambda]}^{+}$. Let $S_{[\lambda]}$ be the set of simple roots in $R_{[\lambda]}^{+}$. Let S_μ be the subset of simple roots in $R_{[\lambda]}^{+}$ orthogonal to μ. Observe that W_μ is always a parabolic subgroup attached to S_μ in $W_{[\lambda]} = W_{[\mu]}$. Let $\mathsf{D}_{\mathsf{S}_\mu}$ be the set of distinguished right coset representatives of $W_{[\lambda]}/W_\mu$. [$r \in \mathsf{D}_{\mathsf{S}_\mu}$ is the element with minimal lenght in rW_μ. Recall that $\tau(w) = \{\, \alpha \in S_{[\lambda]} \mid w\alpha \notin R_{[\lambda]}^{+} \,\}$ Note that $\tau(w) \cap R_\mu \neq \emptyset$ is equivalent to require that $w\mathsf{S}_\mu \subseteq R_{[\lambda]}^{+}$, i.e. w is a minimal length element. See for example, Carter, Simple groups of Lie type, Theorem 2.5.8.]

Theorem 3.2. *Let $\mathcal{O}$ be an nilpotent orbit in $\mathfrak{g}$ and $\mu \in \mathfrak{h}$. Let $\Pi_{W\cdot\mu,\mathcal{O}}$ be the set of irreducible highest weight modules π such that $\operatorname{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}}$. Let*

$$\begin{aligned} \mathcal{D}_{\mathcal{O},\mu}^{sp} &:= \left\{ \sigma \in \operatorname{Irr}^{sp}(W_{[\mu]}) \,\middle|\, \operatorname{Springer}(j_{W_{[\mu]}}^{W}\sigma) = \mathcal{O} \right\} \quad \textit{and} \\ \mathcal{D}_{\mathcal{O},\mu} &= \bigcup_{\sigma\in\mathcal{D}_{\mathcal{O},\mu}^{sp}} {}^{LR}\mathscr{C}_{\sigma}. \end{aligned} \tag{3.3}$$

Then

$$|\Pi_{W\cdot\mu,\mathcal{O}}| = \left|W/W_{[\mu]}\right| \cdot \sum_{\sigma\in\mathcal{D}_{\mathcal{O},\mu}} \left(\dim\sigma \cdot [1_{W_\mu} : \sigma]\right).$$

Let

$$\mathscr{C}_{\sigma,\mu}^{LR} = \mathscr{C}_{\sigma,\mu}^{LR} \cap \mathsf{D}_{\mathsf{S}_\mu}$$

and $\mathscr{C}_{\mathcal{O},\mu}^{LR} = \bigcup_{\sigma\in\mathcal{D}_{\mathcal{O}}^{sp}} \mathscr{C}_{\sigma,\mu}^{LR}$. Then

$$\Pi_{W\cdot\mu,\mathcal{O}} = \left\{\, L(r_i w) \,\middle|\, w \in \mathscr{C}_{\mathcal{O},\mu}^{LR} \textit{ and } i = 1,2,\cdots,k \,\right\}.$$

Here $\mathscr{V}_{\sigma}^{LR}$ is understood as a submodule of $\mathbb{C}[W_{[\lambda]}]$.

Fix $\mu \in \mathfrak{h}$ and let $\mathcal{I}_\mu$ be the maximal primitive ideal with infinitesimal character μ. Let $\Pi_{W\cdot\mu}$ be the set of irreducible highest weight modules π such that $\mathrm{Ann}(\pi) = \mathcal{I}_\mu$.

Let $a(\sigma)$ be the generic degree of a Weyl group representation σ.

In view of Theorem 2.1, we need the following lemma by Barbasch-Vogan.

Lemma 3.3 ([16, (5.26), Proposition 5.28])**.** *Let*

$$a_\mu = \max\{\, a(\sigma) \mid \sigma \in \mathrm{Irr}(W_{[\mu]}) \ \textit{and}\ [1_{W_\mu} : \sigma] \neq 0\,\}.$$

Let

$${}^{L}\mathscr{C}_\mu := \{\, \sigma \in \mathrm{Irr}(W_{[\mu]}) \mid a(\sigma) = a_\mu \ \textit{and}\ [1_{W_\mu} : \sigma] \neq 0\,\}.$$

Then ${}^{L}\mathscr{C}_\mu$ is a left cell of $W_{[\mu]}$ given by

$$ {}^{L}\mathscr{C}_\mu = (J_{W_\mu}^{W_{[\mu]}}\,\mathrm{sgn}) \otimes \mathrm{sgn} \tag{3.4}$$

which contains a unique special representation

$$\sigma_\mu = (j_{W_\mu}^{W_{[\mu]}}\,\mathrm{sgn}) \otimes \mathrm{sgn}\,.$$

Moreover, ${}^{L}\mathscr{C}_\mu$ is multiplicity free, which is equivalent to

$$[1_{W_\mu} : \sigma] = 1 \quad \textit{for each } \sigma \in {}^{L}\mathscr{C}_\mu.$$

Let

$$\mathcal{O}_\mu = \mathrm{Springer}(j_{W_{[\mu]}}^{W}\sigma_\mu). \tag{3.5}$$

Then

$${}^{L}\mathscr{C}_\mu = \left\{\, \sigma \in \mathsf{C}_{\overline{\mathcal{O}}_\mu} \;\middle|\; [1_{W_\mu} : \sigma] \neq 0 \,\right\}.$$

$\square$

[This is essentially contained in [16].

We adapt the notation in [16]: two special representations $\sigma \overset{LR}{\leq} \sigma'$ if and only if $\mathcal{O}_\sigma \supseteq \mathcal{O}_{\sigma'}$ where $\mathcal{O}_\sigma := \mathrm{Springer}(\sigma)$. The generic degree of σ is denoted by $a(\sigma)$. Note that the ordering of double cells/special representation is the same as the closure relation on special nilpotent orbits, see [16, Prop 3.23].

Note that induction maps left cone representation to a left cone representation [16, Prop 4.14 (a)]. Therefore $\mathrm{Ind}_{W_\mu}^{W_{[\mu]}}\mathrm{sgn}$ is a left cone representation. $J_{W_\mu}^{W_{[\mu]}}\mathrm{sgn}$ is a left cell (since J-induction preserves left cell [16, Prop 4.14 (b)]), it consists of the constituents in the induced representation with the minimal generic degree (by the definition of J-induction), it is also the set of constituents in $\mathrm{Ind}_{W_\mu}^{W_{[\mu]}}\mathrm{sgn}$ sit in the same $\underset{LR}{\approx}$ equivalence class (a unique double cell $\mathcal{D}$).

Recall that tensoring with sign (or rather twisting w_0) is an order reversing bijection of left cells in $W_{[\lambda_{\check{\mathcal{O}}}]}$ and induces a LR-order reversing bijection on $\mathrm{Irr}(W_{\lambda_{\check{\mathcal{O}}}})$, see [11, Prop. 2.25]. Therefore $\mathrm{Ind}_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} 1 = \left(\mathrm{Ind}_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}}\mathrm{sgn}\right) \otimes \mathrm{sgn}$ has a set of constituents which is maximal under the LR-order, in particular the generic degree takes maximal value on these representations.

Hence we get the conclusion.]

For each $\mu \in \mathfrak{h}^*$, let $\mathcal{I}_\mu$ be the maximal primitive ideal having infinitesimal character μ. Let $\Pi_{W\cdot\mu}$ be the set of all irreducible highest weight modules whose annihilator ideal are $\mathcal{I}_\mu$. Then

$$\Pi_{W\cdot\mu} = \Pi_{W\cdot\mu, \mathcal{O}_\mu}$$

where $\mathcal{O}_\mu$ is given by (3.5).

Combine the above theorem with Lemma 2.2, we have the following counting theorem.

Theorem 3.4. *Retain the notation in Lemma 3.3.*

$$|\Pi_{W\cdot\mu}| = \left|W/W_{[\mu]}\right| \cdot \dim {}^L\mathscr{C}_\mu.$$

Moreover,

$$\Pi_{W\cdot\mu} = \left\{ L(r_i w) \;\middle|\; w \in \mathscr{C}^{LR}_{\sigma_\mu} \cap \mathsf{D}_{\mathsf{S}_\mu} \text{ and } i = 1, 2, \cdots, k \right\}$$

□

3.3. **A variation.** In this section, let $(\mathfrak{g}, H, \mathfrak{n})$ be a triple that

- $\mathfrak{g}$ is a complex reductive Lie algebra,
- H is a Lie group such that $\mathfrak{h}_0 := \mathrm{Lie}(H)$ is a real form of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$,
- $\mathfrak{n}$ is a maximal nilpotent subalgebra stable under the $\mathfrak{h}$-action.

Let $\mathcal{O}'(\mathfrak{g}, H, \mathfrak{n})$ be the category of $(\mathfrak{g}, H)$-module such that $M \in \mathcal{O}'(\mathfrak{g}, H, \mathfrak{n})$ if and only if

- M is finitely generated as $\mathcal{U}(\mathfrak{g})$-module,
- $\mathfrak{n}$ acts on M locally nilpotently, and
- M decomposes in to a direct sum of finite dimension H-modules.

Let $\mathcal{G}(\mathfrak{g}, H, \mathfrak{n})$ be the Grothendieck group of $\mathcal{O}(\mathfrak{g}, H, \mathfrak{n})$.

We write H_0 for the connected component of H which is abelian. Since $H_0 = \exp(\mathfrak{h}_0)$ is central in H, for each $\phi \in \mathrm{Irr}(H)$ $\phi|_{H_0}$ is a multiple of character. Hence taking the derivative yields a well defined map

$$\mathrm{d}\colon \mathrm{Irr}(H) \longrightarrow \mathfrak{h}^*$$

sending ϕ to $\mathrm{d}\phi$.

Since d restricted on the lattice

$$\tilde{Q} := \{ \phi \in \mathrm{Irr}(H) \mid \phi \text{ occurs in } S(\mathfrak{g}) \}$$

is a bijection onto the root lattice Q [96, 0.4.6], we identify the root lattice Q with the $\tilde{Q}$ in $\mathrm{Irr}(H)$.

Now assume $\phi \in \mathrm{Irr}(H)$ and let

$$[\phi] := \left\{ \phi + \alpha \;\middle|\; \alpha \in \tilde{Q} \right\}$$

and $\mathcal{O}'(\mathfrak{g}, H, \mathfrak{n}; [\phi])$ be the subcategory of $\mathcal{O}'(\mathfrak{g}, H, \mathfrak{n})$ consists of modules whose H irreducible components are contained in $[\phi]$. Define $\mathrm{Coh}_{[\lambda]}(\mathfrak{g}, H, \mathfrak{n}; [\phi])$ to be the space of coherent families taking value in $\mathcal{O}'(\mathfrak{g}, H, \mathfrak{n}; [\phi])$ and $\mathrm{Coh}_{[\lambda]}(\mathfrak{g}, H, \mathfrak{n}; [\phi])$ to be its subspace whose complex associated variety is contained in S for a $\mathrm{Ad}(\mathfrak{g})$-invariant closed subset S in the nilpotent cone of $\mathfrak{g}$.

We have the following lemma.

Lemma 3.5. *Let $\phi \in \mathrm{Irr}(H)$ and fix a $\lambda \in \mathfrak{h}^*$ such that $[\mathrm{d}\phi + \rho] \cap W \cdot \lambda \neq \emptyset$. Then the forgetful functor*

$$\mathcal{F}\colon \mathcal{O}'(\mathfrak{g}, H, \mathfrak{n}) \longrightarrow \mathcal{O}'(\mathfrak{g}, \mathfrak{h}, \mathfrak{n})$$

induces a $W_{[\lambda]}$-module isomorphism

$$\mathcal{F}\colon \mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, H, \mathfrak{n}; [\phi]) \longrightarrow \mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, \mathfrak{h}, \mathfrak{n}; [\mathrm{d}\phi])$$

Proof. When S is the whole nilpotent cone, the isomorphism is given by identifying both sides with $\mathbb{C}[W_{[\lambda]}]$ via Verma modules such that

$$\widetilde{M_1}(\mathrm{d}\phi + \rho) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n}), H} \phi \mapsto (\dim \phi) \cdot M_1(\mathrm{d}\phi + \rho) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h}\oplus\mathfrak{n})} \mathrm{d}\phi.$$

For $\phi' \in \mathrm{Irr}(H)$, $M(\phi' + \rho)$ has a unique irreducible quotient $L(\phi')$ by the same argument of the same argument for the highest weight module. Now we have, $L(\phi' + \rho) = \dim(\phi') \cdot$

$L(\mathrm{d}\phi' + \rho)$. [These claims should be also much more clear from the D-module point of view. The middle extension functor only see the $\mathcal{D}_\lambda$-module structure and keeps the T-module structure automatically. Here T is the maximal compact subgroup of H.] Since the associated variety only depends on the $\mathcal{U}(\mathfrak{g})$-module structure, the rest part of the lemma follows. Check!! □

[Note that H_0 is abelian and $H = H_0$. By the assumption of Harish-Chandra class, $H = H_0 \times H/H_0$. Here H/H_0 is a finite group maybe non-abelian.]

The above lemma have the following immediate consequence.

Corollary 3.6. *Retain the notation in Lemma 3.1. Then, for $\sigma \in \mathrm{Irr}(W_{[\lambda]})$*

$$[\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, H, \mathfrak{n})] \neq 0 \Leftrightarrow \sigma \in \mathsf{C}_{\mathsf{S}}.$$

□

4. Harish-Chandra cells

In this section, let G be a real reductive group in the Harish-Chandra class. Here G could be a nonlinear group. We retain the notation in Example 2.7. We recall the argument before [68, Theorem 1].

We fix a Cartan involution θ on G and let $K = G^\theta$ be the maximal compact subgroup of G.

4.1. An embedding of coherent families of Harish-Chandra modules into that of category $\mathcal{O}$.

In this section, we recall a result in [21]. Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ be a Borel subalgebra in $\mathfrak{g}$ with the nilradical $\mathfrak{n}$ and $\mathfrak{h}$ a Cartan subalgebra in $\mathfrak{b}$. For a subalgebra $\mathfrak{u}$ of $\mathfrak{n}$ and $q \in \mathbb{N}$, Casian defined the localization functors $\gamma^q_{\mathfrak{u}}$ on the category of $\mathfrak{u}$-module. By [21, Proposition 4.8], $\gamma^q_{\mathfrak{u}}$ can be defined as the right derived functor of the functor $\gamma^0_{\mathfrak{u}}$ which sends a $\mathfrak{u}$-module M to

$$\gamma^0_{\mathfrak{u}}(M) := \left\{\, v \in M \;\middle|\; \mathfrak{u}^k v = 0 \text{ for some positive integer } k \,\right\}.$$

In particular, the $\mathfrak{u}$-action on $\gamma^q_{\mathfrak{u}}$ is locally nilpotent.

Suppose M is a $\mathfrak{g}$-module, we have (see [21, Proposition 4.14])

$$\operatorname{Ann} M \subseteq \operatorname{Ann}(\gamma^q_{\mathfrak{u}}(M)). \tag{4.1}$$

Moreover, $\gamma^q_{\mathfrak{u}}$ commutes with tensoring finite diemsional representations of $\mathfrak{g}$, i.e. for a finite dimensional $\mathfrak{g}$-module F there is a natural isomorphism (c.f. [21, Proposition 4.11])

$$F \otimes \gamma^q_{\mathfrak{u}}(M) \cong \gamma^q_{\mathfrak{u}}(F \otimes M). \tag{4.2}$$

If the $\mathfrak{g}$-module have finite dimensional $\mathfrak{n}$-cohomology, then $\gamma^q_{\mathfrak{n}}(M)$ is in the category $\mathcal{O}'$. See [21, Proposition 4.9].

Suppose M is a $(\mathfrak{g}, K)$-module. From the definition of $\gamma^q_{\mathfrak{u}}$, we can see that $\gamma^q_{\mathfrak{u}}(M)$ is naturally a $(\mathfrak{g}, K_L)$-module where K_L denote the normalizer of $\mathfrak{u}$ in K. Let $\mathfrak{l}$ be the normailzer of $\mathfrak{u}$ in $\mathfrak{g}$. Then there is the a spectrum sequence of $(\mathfrak{l}, K_L)$-module convergent to $H^{p+q}(\mathfrak{u}, M)$ (see [21, Proposition 4.4]):

$$H^q(\mathfrak{u}, \gamma^p_{\mathfrak{u}}(M)) \Rightarrow H^{p+q}(\mathfrak{u}, M). \tag{4.3}$$

For our application, we always assume M is a $(\mathfrak{g}, K)$-module, and take $\mathfrak{u} = \mathfrak{n}$.

Let H be a θ-stable Cartan subgroup of G, $T = H^\theta$ be the maximal compact subgroup of H and $\mathfrak{h} := \mathrm{Lie}(H)_{\mathbb{C}}$ is the corresponding Cartan subalgebra in $\mathfrak{g}$. We can view a finite dimensional H-module as a $(\mathfrak{h}, T)$-module and vice versa.

The localization functor $\gamma^q_{\mathfrak{n}}$ is compatible with coherent continuation.

Lemma 4.1. *Assume that each irreducible $(\mathfrak{g}, K)$-module have finite dimensional $\mathfrak{n}$-cohomology. For each $q \in \mathbb{N}$, we have the following map between $W_{[\lambda]}$-modules:*

$$\begin{array}{rcl} \gamma_{\mathfrak{n}}^q \colon \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, K) & \longrightarrow & \mathrm{Coh}_{[\lambda]}(\mathfrak{g}, H, \mathfrak{n}) \\ \Theta & \mapsto & \gamma_{\mathfrak{n}}^q \circ \Theta. \end{array}$$

Proof. This is a consequence of (4.1) and (4.2). [

$$\begin{aligned} F \otimes \gamma_{\mathfrak{n}}^q \Theta(\mu) &= \gamma_{\mathfrak{n}}^q (F \otimes \Theta(\mu)) \\ &= \gamma_{\mathfrak{n}}^q \Big(\sum_{\beta \in \Delta(F)} \Theta(\mu + \beta) \Big) \\ &= \sum_{\beta \in \Delta(F)} \gamma_{\mathfrak{n}}^q (\Theta(\mu + \beta)) \end{aligned}$$

] □

We fix a positive system of real roots $\Delta_{\mathbb{R}}^+$ and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ such that $\Delta_{\mathbb{R}}^+ \subset \Delta(\mathfrak{n})$. We let $e^\alpha \in \mathrm{Irr}(H)$ be the H-character on the α-root space.

For a finite length $(\mathfrak{g}, K)$-module M, we view its global character $\Theta_G(M)$ as a analytic function defined on the set G_{reg} of regular semisimple elements on G.

The following theorem is crucial.

Theorem 4.2 ([21, Theorem 3.1])**.** *Let M be a finite length $(\mathfrak{g}, K)$-module. The following statements hold.*

(i) The Lie algebra cohomology $H^q(\mathfrak{n}, M)$ is finite dimensional for each $q \in \mathbb{N}$. In particular, $\gamma_{\mathfrak{n}}^q(M)$ is in the category $\mathcal{O}'(\mathfrak{g}, H, \mathfrak{n})$ for eah $q \in \mathbb{N}$.

(ii) Let

$$H_{reg}^- := \left\{ h \in H \;\middle|\; \begin{array}{l} h \text{ is regular semisimple} \\ |e^\alpha(h)| < 1 \text{ for each real root } \alpha \in \Delta_{\mathbb{R}}^+ \end{array} \right\}$$

Then

$$\begin{aligned} \Theta_G(M)|_{H_{reg}^-} &= \frac{\sum_{q \in \mathbb{N}} (-1)^q \Theta_H \left(H^q(\mathfrak{n}, M) \right)}{\prod_{\alpha \in \Delta(\mathfrak{n})} (1 - e^\alpha)} \\ &= \frac{\sum_{p,q \in \mathbb{N}} (-1)^{p+q} \Theta_H \left(H^q(\mathfrak{n}, \gamma_{\mathfrak{n}}^p(M)) \right)}{\prod_{\alpha \in \Delta(\mathfrak{n})} (1 - e^\alpha)} \end{aligned} \tag{4.4}$$

□

Proof. The theorem is a recollection of Casian's results in loc. cit. The last equality in (4.4) follows from (4.3). The last expression in (4.4) is also a finite sum, since only finite may terms of $\gamma_{\mathfrak{n}}^p(M)$ are non-zero and they are in the category $\mathcal{O}'$. □

Retain the notation in Theorem 4.2, we write

$$\gamma_{\mathfrak{n}} := \sum_{q \in \mathbb{N}} (-1)^q \gamma_{\mathfrak{n}}^q \colon \mathcal{G}(\mathfrak{g}, K) \longrightarrow \mathcal{G}(\mathfrak{g}, H, \mathfrak{n})$$

Corollary 4.3 (c.f. [68])**.** *Let $H_1, H_2, \cdots, H_s$ form a set of representatives of the conjugacy class of θ-stable Cartan subgroup of G. Fix maximal nilpotent Lie subalgebra $\mathfrak{n}_i$ for each H_i as in Theorem 4.2. Then*

$$\gamma := \oplus_i \gamma_{\mathfrak{n}_i} : \;\; \mathcal{G}(\mathfrak{g}, K) \;\longrightarrow\; \bigoplus_{i=1}^s \mathcal{G}(\mathfrak{g}, H_i, \mathfrak{n}_i)$$

is an embedding of abelian groups.

Proof. Let $H_{i,\mathrm{reg}}$ be the set of regular semisimple elements in H_i. By the results of Harish-Chandra, taking the character of the elements induces an embedding of $\mathcal{G}(\mathfrak{g}, K)$ into the space of real analytic functions on $\bigsqcup_i H_{i,\mathrm{reg}}$. Since $H^-_{i,\mathrm{reg}}$ is open in $H_{i,\mathrm{reg}}$ and meets all the connected components of $H_{i,\mathrm{reg}}$, any real analytic function on $\bigsqcup_i H_{i,\mathrm{reg}}$ is determined by its restriction on $\bigsqcup_i H^-_{i,\mathrm{reg}}$. Now (4.4) implies that the global character of any $M \in \mathcal{G}(\mathfrak{g}, K)$ can be computed from the image $\gamma(M)$. $\square$

Corollary 4.4. *Retain the notation in Corollary 4.3. Let* S *be a* $G_{\mathbb{C}}^{ad}$*-invariant closed subset in the nilpotent cone of* $\mathfrak{g}$ *and* $\lambda \in \mathfrak{h}^*$*. Then* γ *induces an embedding of* $W_{[\lambda]}$*-module.*

$$\begin{array}{rccc} \gamma := \oplus_i \gamma_{\mathfrak{n}_i} : & \mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, K) & \longrightarrow & \bigoplus_{i=1}^{s} \mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, H_i, \mathfrak{n}_i) \\ & \Theta & \mapsto & \gamma \circ \Theta \end{array}$$

In particular, $[\sigma : \mathrm{Coh}_{[\lambda],\mathsf{S}}(\mathfrak{g}, K)] \neq 0$ *only if* $\sigma \in \mathsf{C}_{\mathsf{C}}$*.*

Proof. The maps is well-defined by (4.1). It is an embedding by Corollary 4.3. $\square$

Now we get the following theorem on the upper bound of small representations.

Theorem 4.5. *Let* $\mu \in \mathfrak{h}^*$*. Let* $\Pi_{W\cdot\mu}(G)$ *be the set of irreducible* $(\mathfrak{g}, K)$*-modules annihilated by the maximal primitive ideal* $\mathcal{I}_\mu$ *of infinitesimal character* μ*. Let* ${}^L\mathscr{C}_\mu$ *be the left cell of* $\mathrm{Irr}(W_{[\mu]})$ *defined in* (3.4) *and* $\mathcal{O}_\mu$ *be the nilpotent orbit defined by* (3.5)*. Then*

$$\begin{aligned} |\Pi_{W\cdot\mu}(G)| &= \sum_{\sigma \in {}^L\mathscr{C}_\mu} [\sigma : \mathrm{Coh}_{[\mu],\overline{\mathcal{O}}_\mu}(G)] \\ &\leq \sum_{\sigma \in {}^L\mathscr{C}_\mu} [\sigma : \mathrm{Coh}_{[\mu]}(G)] \end{aligned}$$

Proof. The equality follows from Lemma 2.9, Corollary 4.4 and Lemma 3.3. The inequality is also clear since $\mathrm{Coh}_{[\mu],\overline{\mathcal{O}}_\mu}(G)$ is a submodule of $\mathrm{Coh}_{[\mu]}(G)$. $\square$

Remark. Let λ be a regular element in $[\mu]$. Assume that for each Harish-Chandra cell representation $\mathscr{V}^{HC}$ at the infinitesimal character λ, the $W_{[\mu]}$-representations occur in $\mathscr{V}^{HC}$ are all contained in a unique double cell. Then one can prove that the inequality in the above theorem is sharp (c.f. [12]). However, we do not know such kind of claim holds in a general.

Another interesting consequence of Corollary 4.4 is that we can determine the complex associated variety attached to a Harish-Chandra cell by the character of the cell representation.

Lemma 4.6. *Let* $\mathscr{C}$ *be an Harish-Chandra cell at a regular infinitesimal character* λ *and* $\mathscr{V}$ *be the corresponding cell representation.* [*I guess the assumption of "regular" is unnecessary.*] *Then there is a unique special representation* $\sigma \in \mathrm{Irr}^{sp}(W_{[\lambda]})$ *such that* σ *has minimal generic degree in* $\mathscr{V}$ *and every irreducible modules* π *in* $\mathscr{C}$ *has complex associated variety*

$$\mathrm{AV}_{\mathbb{C}}(\pi) = \mathrm{Springer}(\mathrm{Ind}_{W_{[\lambda]}}^{W} \sigma).$$

Moreover, σ *is precisely the unique irreducible character with minimal fake degree occurring in* $\mathscr{V}$*.*

Proof. Let $\pi \in \mathscr{C}$ and the primitive ideal $\mathrm{Ann}(\pi)$ is attached to the special representation $\sigma \in \mathrm{Irr}(W_{[\lambda]})$. By Corollary 4.4, every irreducible character σ' occur in $\mathscr{V}$ satisfies $\sigma \underset{LR}{\leq} \sigma'$. On the other hand, σ do occurs in $\mathscr{V}$ by Barbasch-Vogan and King's result, see [21, Remark 3.2]. The rest part of the lemma is clear. $\square$

4.2. **Coherent continuation representation of Harish-Chandra modules.** Recall the definition of regular characters [96, Definition 6.6.1].

A regular character of G is a tuple a tuple $\gamma := (H, \Gamma, \bar{\gamma})$ such that

- H is a θ-stable Cartan subgroup of G,
- Γ is a continuous character of H,
- if α is imaginary, then $\langle \bar{\gamma}, \check{\alpha} \rangle$ is real and non-zero,
- the differential of Γ is
$$\bar{\gamma} + \rho_i - 2\rho_{ic}$$
where
$$\rho_i = \tfrac{1}{2} \sum_{\substack{\alpha \text{ imaginary} \\ \langle \bar{\gamma}, \check{\alpha} \rangle > 0}} \alpha \quad \text{and} \quad \rho_{ic} = \tfrac{1}{2} \sum_{\substack{\alpha \text{ compact imaginary} \\ \langle \bar{\gamma}, \check{\alpha} \rangle > 0}} \alpha.$$

We say γ is non-singular if $\bar{\gamma}$ is regular in $\mathfrak{h}^*$. The group K acts on the set of regular characters of G we let $[\gamma]$ to denote the K-conjugacy class of regular character containing γ.

Let $\mathcal{R}(G)$ denote the set of regular characters of G and $\mathcal{R}_\lambda(G)$ be the subset of $\mathcal{R}(G)$ consists of regular characters with infinitesimal character λ. Let $\mathcal{P}(G) := \mathcal{R}(G)/K$ and $\mathcal{P}_\lambda(G) := \mathcal{R}_\lambda(G)/K$ be the corresponding sets of K-conjugacy classes.

For each regular character γ, we attach a standard representation π_γ with infinitesimal character $\bar{\gamma}$ and let $\bar{\pi}_\gamma$ be the maximal completely reducible submodule of π_γ [96, 6.5.2,6.5.11,6.6.3]. Then $\bar{\pi}_\gamma$ and the image of π_γ in the Grothendieck group of $(\mathfrak{g}, K)$-module only depends on the K-conjugation class of γ. This justifies the consideration of $\mathcal{P}(G)$.

Now we assume that $\lambda \in \mathfrak{h}^*$ is a regular element. Then $\bar{\pi}_\gamma$ is the unique irreducible submodule in π_γ. The following map is bijective (Langlands classification)
$$\begin{array}{ccc} \mathcal{P}_\lambda(G) & \longrightarrow & \mathrm{Irr}_\lambda(G) \\ [\gamma] & \mapsto & \bar{\pi}_\gamma \end{array}$$
and $\{ \pi_\gamma \mid \gamma \in \mathcal{P}_\lambda(G) \}$ forms a basis of $\mathcal{G}_\lambda(G)$. As a consequence, we have an isomorphism of vector spaces
$$\begin{array}{ccc} \mathbb{C}[\mathcal{P}_\lambda(G)] & \longrightarrow & \mathrm{Coh}_{[\lambda]}(G) \\ [\gamma] & \mapsto & \Theta_\gamma \end{array}$$
where Θ_γ is the unique coherent family such that $\Theta_\gamma(\lambda) = \pi_\gamma$. In the following, we identify the two sides implicitly.

For any regular element $\lambda \in \mathfrak{h}^*$, let λ^a be the dominant element in the abstract Cartan $\mathfrak{h}^a$ corresponding to λ and
$$i_\lambda \colon W(\mathfrak{g}, \mathfrak{h}) \longrightarrow W^a$$
be the identification of $W(\mathfrak{g}, \mathfrak{h})$ with the abstract Weyl group W^a.

From now on we fix a regular element $\lambda \in \mathfrak{h}^*$.

Let $W^a_{[\lambda]} := i_\lambda(W_{[\lambda]})$ be the abstract integral Weyl group. We identify $W_{[\lambda]}$ with $W^a_{[\lambda]}$ implicitly.

In the following, we assume $\gamma = (H, \Gamma, \bar{\gamma}) \in \mathcal{R}_\lambda(G)$. On one hand, a element w in the real Weyl group $W(G, H)$ acts on γ by conjugation and gives a new regular character denoted by $w \cdot \gamma$. [Note that $W(G, H) = W(K_\mathbb{C}, H_\mathbb{C}) := N_{K_\mathbb{C}}(H_\mathbb{C})/K_\mathbb{C} \cap H_\mathbb{C}$ is a subgroup of $W(\mathfrak{g}, \mathfrak{h})$.]

On the other hand, $W_{[\bar{\gamma}]}$ acts on γ by cross action $(w, \gamma) \mapsto w \times gamma$ [96, 8.3.1]. On define the abstract Weyl group $W^a_{[\lambda]}$ acts on $\mathcal{R}_\lambda(G)$ by
$$w \times^a \gamma := \dot{w}^{-1} \times \gamma \text{ with } \dot{w} \in W_{[\lambda]}, w = i_{\bar{\gamma}}(\dot{w}),$$
see [97, Definition 4.2].

The cross action of $W^a_{[\lambda]}$ descents to an action on $\mathcal{P}_\lambda(G)$. reference? Let $W^a_{[\gamma]} \subset W^a_{[\lambda]}$ be the stabilizer of $[\gamma]$ under cross action. Then

$$W^a_{[\gamma]} := i_{\bar\gamma}(W_{[\gamma]}) \quad \text{with} \quad W_{[\gamma]} := \{\, w \in W(G,H) \mid w \times \gamma = w \cdot \gamma \,\}\,.$$

The θ-action on $\mathfrak{h}$ induces an action on $W(\mathfrak{g},\mathfrak{h})$. Then we have the following tower of groups

$$W_{[\gamma]} < W(G,H) < W(\mathfrak{g},\mathfrak{h})^\theta$$

and

$$W(\mathfrak{g},\mathfrak{h})^\theta = (W_{\mathbb{C}})^\theta \ltimes (W_{i\mathbb{R}} \times W_{\mathbb{R}})$$

where $W_{\mathbb{C}}$, $W_{i\mathbb{R}}$ and $W_{\mathbb{R}}$ are Weyl groups of complex, compact and real roots respectively (see [97, Proposition 3.12] and [3, (12.1)-(12.5)]).

Define a quadratic character on $W(\mathfrak{g},\mathfrak{h})^\theta$ by

$$\begin{array}{rccc} \mathrm{sgn}_{\mathfrak{h}}\colon W(\mathfrak{g},\mathfrak{h})^\theta = & (W_{\mathbb{C}})^\theta \ltimes (W_{i\mathbb{R}} \times W_{\mathbb{R}}) & \longrightarrow & \{\pm 1\} \\ & (w_{\mathbb{C}}, w_{i\mathbb{R}}, w_{\mathbb{R}}) & \mapsto & \mathrm{sgn}_{W_{i\mathbb{R}}}(w_{i\mathbb{R}}) \end{array}$$

where $\mathrm{sgn}_{W_{i\mathbb{R}}}$ denote the sign character of the imaginary Weyl group. Let $\mathfrak{h}^*_{i\mathbb{R}}$ be the span of imaginary roots, then

$$\mathrm{sgn}_{\mathfrak{h}}(w) = \det(w|_{\mathfrak{h}^*_{i\mathbb{R}}}).$$

Let $\mathrm{sgn}_{[\gamma]} := \mathrm{sgn}_{\mathfrak{h}}|_{W_{[\gamma]}}$ be the restriction of $\mathrm{sgn}_{\mathfrak{h}}$ on the cross stabilizer $W_{[\gamma]}$.

The following result is a unpublished result of Barbasch-Vogan based on results in [96, Chapter 8], a same argument works equally well with non-linear groups in the Harish-Chandra class.

Theorem 4.7 (c.f. [12, Proposition 2.4]). *Suppose* $\Pi_\lambda(G) = \bigsqcup_{i=1}^k W^a_{[\lambda]} \times^a [\gamma_i]$ *where* $\gamma_i = (H_i, \Gamma_i, \bar\gamma_i)$ *are representatives of the* $W^a_{[\lambda]}$*-orbits of* $\Pi_\lambda(G)$*. Then*

$$\mathrm{Coh}_{[\lambda]}(G) \cong \bigoplus_{i=1}^k \mathrm{Ind}_{W_{[\gamma_i]}}^{W^a_{[\lambda]}} \mathrm{sgn}_{[\gamma_i]}\,.$$

Sketch of the proof. To avoid the confusion, we use $t(w)$ to denote the coherent continuation action. The action of simple roots in R^+_λ on the basis Θ_γ is given calculated in [96, Chapter 8] and summarized in [97, Definition 14.4]. The calculation reduced to the representation theory of $\mathrm{SL}(2,\mathbb{R})$. For non-linear groups see [86, Definition 9.4] and note that the formula is exactly the same as that of linear groups for integral simple roots.

[Let $\mathcal{T}_{\lambda_1}^{\lambda_2}$ be the translation functor from infinitesimal character λ_1 to λ_2. Fix a By abstract non-sense, one have [96, Prop 7.2.22]

$$\Theta_\pi(\lambda) + s_\alpha \cdot \Theta_\pi(\lambda) = \mathcal{T}_{\lambda_0}^{\lambda} \mathcal{T}_{\lambda}^{\lambda_0}(\pi) =: \phi_\alpha\psi_\alpha(\pi)$$

where λ_0 is an element such that $\langle \lambda_0, \alpha\rangle = 0$ and $\langle \lambda_0, \beta\rangle > 0$ for all $\alpha \neq \beta \in R^+_{[\lambda]}$. By lifting to the covering group, we can assume that $\lambda - \lambda_0$ is a weight of a finite dimensional representation of G.

Now the computation reduces to compute the RHS $= \phi_\alpha\psi_\alpha(\pi)$ of the above equality. (Sometimes, we need [96, Prop 8.3.18] for the explicit computation of cross action.) This is computed case by case (let $t(w)$ denote the coherent continuation action):

- For compact imaginary roots, by Hecht-Schmid's "A proof of Blattner's conjecture", where RHS $= 0$.

$$t(s_\alpha)\gamma = -\gamma = -s_\alpha \times \gamma.$$

- For non-compact imaginary roots, reduce to $\mathrm{SL}(2,\mathbb{R})$ [96, 8.4.5,8.4.6].
$$t(s_\alpha)\gamma = -s_\alpha \times \gamma + R$$
where $R = c^\alpha(\gamma)$ or $\gamma_+^\alpha + \gamma_-^\alpha$ is a combination of regular characters on the Cartan subgroup H^α (which has higher $\mathbb{R}$-rank, in fact $\mathrm{rank}_{\mathbb{R}} H^\alpha = \mathrm{rank}_{\mathbb{R}} H + 1$).
- For real roots we can use [96, 8.3.19] which is a consquence of the (cohomological induction) construction of the standard module. The result says if w acts trivial on $\mathfrak{t} = \mathfrak{h}^\theta$, then $t(w^{-1})\gamma = w \times \gamma$. When α is a real root, then clearly s_α acts on $\mathfrak{t}$ trivially (since $-\alpha(x) = \theta(\alpha)(x) = \alpha(\theta(x)) = \alpha(x) = 0\ \forall x \in \mathfrak{t}$) and we have
$$t(s_\alpha)\gamma = s_\alpha \times \gamma.$$
- For complex root, use [96, 8.2.7] (whose proof relies on a long exact sequence in [96, 7.4.3(a)] which is also formal) we get: $\gamma + s_\alpha \times \gamma = \phi_\alpha \psi_\alpha(\gamma)$ (here $\gamma - n\alpha = s_\alpha \times \gamma$ by the definition of cross action). In particular, we have
$$t(s_\alpha)\gamma = s_\alpha \times \gamma.$$

]

Let
$$\mathcal{P}_{\lambda,r}(G) := \{\ [(H,\Gamma,\bar{\gamma})] \mid \text{real rank of } H \text{ is } r\ \}$$
and $\mathcal{P}_{\lambda,\geq r} = \bigsqcup_{l\geq r} \mathcal{P}_{\lambda,l}$. Define
$$\mathrm{Coh}_{[\lambda],\geq r}(G) := \mathrm{Span}\left\{\ \Theta_{[\gamma]}\ \middle|\ [\gamma] \in \mathcal{P}_{\lambda,\geq r}\ \right\}$$
From the explicit formula of the coherent continuation actions on the standard modules, we have that, for $[\gamma] \in \mathcal{P}_{\lambda,\geq r}(G)$,
$$t(s_\alpha)\ [\gamma] \equiv \begin{cases} -s_\alpha \times^a [\gamma] & \text{if } \alpha \text{ is imaginary,} \\ \ \ s_\alpha \times^a [\gamma] & \text{otherwise.} \end{cases}$$
Now it is elementary to deduce that, as $W^a_{[\lambda]}$-module,
$$\frac{\mathrm{Coh}_{[\lambda],\geq r}(G)}{\mathrm{Coh}_{[\lambda],\geq r+1}(G)} \cong \bigoplus_{W^a_{[\lambda]} \times^a [\gamma]} \mathrm{Ind}_{W_{[\gamma]}}^{W^a_{[\lambda]}} \mathrm{sgn}_{[\gamma]}$$
where the summation runs over the cross action orbits in $\mathcal{P}_{\lambda,r}(G)$. Since $W^a_{[\lambda]}$ is a finite group, we get the theorem by the complete reduciblity of $\mathrm{Coh}_{[\lambda]}(G)$. □

We remark that, when λ is integral, the set $\mathcal{P}_\lambda(G)$ can be enumerated using [3] (the algorithm is implemented in atlas) for linear groups. Under atlas' parameters, the cross action is also easy to calculate. For the metaplectic group, the problem was solved by Renard-Trapa [84, 85].

5. Counting in type A

The results in this section are well known to the experts.

Let YD be the set of Young diagrams viewed as a finite multiset of positive integers. The set of nilpotent orbits in $\mathrm{GL}_n(\mathbb{C})$ is identified with Young diagram of n boxes.

Let $\check{G}_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$. Fix an orbit $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G})$, let $\check{\mathcal{O}}_e$ (resp. $\check{\mathcal{O}}_o$) be the partition consists of all even (resp. odd) rows in $\check{\mathcal{O}}$.

Let S_n denote the Weyl group of $\mathrm{GL}_n(\mathbb{C})$. Let $W_n := S_n \ltimes \{\pm 1\}^n$ denote the Weyl group of type B_n or C_n. Let sgn denote the sign representation of the Weyl group. The group W_n is naturally embedded in S_{2n}. For W_n, let ϵ denote the unique non-trivial character which is trivial on S_n. Note that ϵ is also the restriction of the sgn of S_{2n} on W_n.

5.1. **Special unipotent representations of $G = \mathrm{GL}_n(\mathbb{C})$.** By [16], the set of unipotent representations of $G = \mathrm{GL}_n(\mathbb{C})$ one-one corresponds to nilpotent orbits in $\mathrm{Nil}(\check{G}_{\mathbb{C}})$. Suppose $\check{\mathcal{O}}$ has rows

$$\mathbf{r}_1(\check{\mathcal{O}}) \geq \mathbf{r}_2(\check{\mathcal{O}}) \geq \cdots \geq \mathbf{r}_k(\check{\mathcal{O}}) > 0.$$

Then $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}})$ has columns $\mathbf{c}_i(\mathcal{O}) = \mathbf{r}_i(\check{\mathcal{O}})$ for all $i \in \mathbb{N}^+$. The map $\check{\mathcal{O}} \mapsto \mathcal{O}$ is a bijection.

We set $\mathsf{CP} = \mathsf{YD}$ be the set of Young diagrams. For $\tau \in \mathsf{CP}$ which has k columns, let 1_c be the trivial representations of $\mathrm{GL}_c(\mathbb{C})$.

$$\pi_\tau = 1_{\mathbf{c}_1(\tau)} \times 1_{\mathbf{c}_2(\tau)} \times \cdots \times 1_{\mathbf{c}_k(\tau)}.$$

The Vogan duality gives a duality between Harish-Chandra cells. In this case, Harish-Chandra cells is the double cell of Lusztig. Now we have a duality

$$\pi_\tau \leftrightarrow \pi_{\tau^t}.$$

Let $\tau' := \nabla(\tau)$ be the partition obtained by deleting the first column of τ. Let $\theta_{a,b}$ (resp. $\Theta_{a,b}$) be the theta lift (resp. big theta lift) from $\mathrm{GL}_a(\mathbb{C})$ to $\mathrm{GL}_b(\mathbb{C})$. Then we have

$$\pi_\tau = \theta_{|\tau'|,|\tau|}(\pi_\tau).$$

5.2. **Counting unipotent representations of $\mathrm{GL}_n(\mathbb{R})$.** Now let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G}_{\mathbb{C}})$. Recall the decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_e \cup \check{\mathcal{O}}_o$. Let $n_e = \left|\check{\mathcal{O}}_e\right|, n_o = \left|\check{\mathcal{O}}_o\right|$ and $\lambda_{\check{\mathcal{O}}} = \frac{1}{2}\check{h}$.

Then

$$W_{\lambda_{\check{\mathcal{O}}}} \cong S_{|\check{\mathcal{O}}_e|} \times S_{|\check{\mathcal{O}}_o|}.$$

$$W_{\lambda_{\check{\mathcal{O}}}} = \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_e)} \times \prod_j S_{\mathbf{c}_j(\check{\mathcal{O}}_o)}$$

By the formula of a-function, one can easily see that The cell in $W(\lambda_{\check{\mathcal{O}}})$ consists of the unique representation $J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}}(1)$. Now the W-cell $(J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W} \mathrm{sgn}) \otimes \mathrm{sgn}$ consists a single representation

$$\tau_{\check{\mathcal{O}}} = \check{\mathcal{O}}_e^t \boxtimes \check{\mathcal{O}}_o^t.$$

The representation $j_{W_{\lambda_{\check{\mathcal{O}}}}}^{S_n} \tau_{\check{\mathcal{O}}}$ corresponds to the orbit $\mathcal{O} = \check{\mathcal{O}}^t$ under the Springer correspondence. [WLOG, we assume $\check{\mathcal{O}} = \check{\mathcal{O}}_o$.

Let $\sigma \in \widehat{S_n}$. We identify σ with a Young diagram. Let $c_i = \mathbf{c}_i(\sigma)$. Then $\sigma = J_{W'}^{S_n} \epsilon_{W'}$ where $W' = \prod S_{c_i}$ (see Carter's book). This implies Lusztig's a-function takes value

$$a(\sigma) = \sum_i c_i(c_i - 1)/2$$

Comparing the above with the dimension formula of nilpotent Orbits [24, Collary 6.1.4], we get (for the formula, see Bai ZQ-Xie Xun's paper on GK dimension of $SU(p,q)$)

$$\tfrac{1}{2}\dim(\sigma) = \dim(L(\lambda)) = n(n-1)/2 - a(\sigma).$$

Here $\dim(\sigma)$ is the dimension of nilpotent orbit attached to the Young diagram of σ (it is the Springer correspondence, regular orbit maps to trivial representation, note that $a(\mathrm{triv}) = 0$), $L(\lambda)$ is any highest weight module in the cell of σ.

Return to our question, let $S' = \prod_i S_{\mathbf{c}_i(\check{\mathcal{O}})}$. We want to find the component σ_0 in $\mathrm{Ind}_{S'}^{S_n} 1$ whose $a(\sigma_0)$ is maximal, i.e. the Young diagram of σ_0 is minimal.

By the branaching rule, $\sigma \subset \mathrm{Ind}_{S'}^{S_n} 1$ is given by adding rows of lenght $\mathbf{c}_i(\check{\mathcal{O}})$ repeatly (Each time add at most one box in each column). Now it is clear that $\sigma_0 = \check{\mathcal{O}}^t$ is desired.

This agrees with the Barbasch-Vogan duality d_{BV} given by

$$\check{\mathcal{O}} \xrightarrow{Springer} \check{\mathcal{O}} \xrightarrow{\otimes \mathrm{sgn}} \check{\mathcal{O}}^t \xrightarrow{Springer} \check{\mathcal{O}}^t.$$

]

The $W_{[\lambda_{\check{\mathcal{O}}}]}$-module $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}$ is given by the following formula:

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]} \cong \mathcal{C}_{n_e} \otimes \mathcal{C}_{n_o} \quad \text{with}$$
$$\mathcal{C}_n := \bigoplus_{\substack{s,a,b \\ 2s+a+b=n}} \mathrm{Ind}_{W_s \times S_a \times S_b}^{S_n} \epsilon \otimes 1 \otimes 1.$$

According to Vogan duality, we can obtain the above formula by tensoring sgn on the forumla of the unitary groups in [12, Section 4].

By branching rules of the symmetric groups, $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ can be parameterized by painted partition.

$$\mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}) = \left\{ \boldsymbol{\tau} := (\tau, \mathcal{P}) \;\middle|\; \begin{array}{l} \tau = \check{\mathcal{O}}^t \\ \mathrm{Im}(\mathcal{P}) \subset \{ \bullet, c, d \} \\ \#\{ i \mid \mathcal{P}(i,j) = \bullet \} \text{ is even} \end{array} \right\}. \tag{5.1}$$

For $\boldsymbol{\tau} := (\tau, \mathcal{P}) \in \mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}})$, we write $\mathcal{P}_{\boldsymbol{\tau}} := \mathcal{P}$.

[The typical diagram of all columns with even length $2c$ are

$$\begin{array}{|c|c|c|c|c|c|} \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bullet & \cdots & \bullet & c & \cdots & c \\ \hline \bullet & \cdots & \bullet & d & \cdots & d \\ \hline \end{array}$$

The typical diagram of all columns with odd length $2c+1$ are

$$\begin{array}{|c|c|c|c|c|c|} \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \hline c & \cdots & c & d & \cdots & d \\ \hline \end{array}$$

]

Let $\mathrm{sgn}_n \colon \mathrm{GL}_n(\mathbb{R}) \to \{ \pm 1 \}$ be the sign of determinant. Let 1_n be the trivial representation of $\mathrm{GL}_n(\mathbb{R})$. For $\boldsymbol{\tau} \in \mathrm{PP}\check{\mathcal{O}}$, we attache the representation

$$\pi_{\boldsymbol{\tau}} := \bigtimes_j \underbrace{1_j \times \cdots \times 1_j}_{c_j\text{-terms}} \times \underbrace{\mathrm{sgn}_j \times \cdots \times \mathrm{sgn}_j}_{d_j\text{-terms}}. \tag{5.2}$$

Here

- j running over all column lengths in $\check{\mathcal{O}}^t$,
- d_j is the number of columns of length j ending with the symbol "d",
- c_j is the number of columns of length j ending with the symbol "$\bullet$" or "c", and
- "$\times$" denote the parabolic induction.

5.3. **Special Unipotent representations of $G = \mathrm{GL}_m(\mathbb{H})$.** Suppose that $\mathcal{O}$ is the complexificiation of a rational nilpotent $\mathrm{GL}_m(\mathbb{H})$-orbit. Then $\mathcal{O}$ has only even length columns. Therefore, $\mathrm{Unip}_{\check{\mathcal{O}}}(G) \neq \emptyset$ only if $\check{\mathcal{O}} = \check{\mathcal{O}}_e$.

In this case the coherent continuation representation is given by

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) = \mathrm{Ind}_{W_m}^{S_{2m}} \epsilon$$

and $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is a singleton. For each partition τ only having even columns, we define

$$\pi_\tau := \bigtimes_i 1_{\mathbf{c}_i(\tau)/2}.$$

5.4. **Counting special unipotent repesentations of** $\mathrm{U}(p,q)$**.** We call the parity of $|\check{\mathcal{O}}|$ the "good pairity". The other pairity is called the "bad parity". We write $\check{\mathcal{O}} = \check{\mathcal{O}}_g \cup \check{\mathcal{O}}_b$ where $\check{\mathcal{O}}_g$ and $\check{\mathcal{O}}_b$ consist of good parity length rows and bad parity rows respectively.

Let $(n_g, n_b) = (|\check{\mathcal{O}}_g|, |\check{\mathcal{O}}_b|)$. Now as the $S_{n_g} \times S_{n_b}$

$$\bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=n}} \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathrm{U}(p,q)) = \mathcal{C}_g \otimes \mathcal{C}_b$$

where

$$\mathcal{C}_g = \bigoplus_{\substack{s,a,b\in\mathbb{N}\\ 2s+a+b=n_g}} \mathrm{Ind}_{W_s\times S_a\times S_b}^{S_{n_g}} 1\otimes \mathrm{sgn}\otimes\mathrm{sgn}$$

$$\mathcal{C}_b = \begin{cases} \mathrm{Ind}_{W_{\frac{n_b}{2}}}^{S_{n_b}} 1 & \text{if } n_b \text{ is even} \\ 0 & \text{otherwise.}\end{cases}$$

By the above formula, we have

Lemma 5.1. *(i) The set* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,q)) \neq \emptyset$ *if and only if* $p = q$ *and each row lenght in* $\check{\mathcal{O}}$ *has even multiplicity.*

(ii) Suppose $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$*, let* $\check{\mathcal{O}}'$ *be the Young diagram such that* $\mathbf{r}_i(\check{\mathcal{O}}') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ *and* π' *be the unique special uinpotent representation in* $\mathrm{Unip}_{\check{\mathcal{O}}'}(\mathrm{GL}_p(\mathbb{C}))$*. Then the unique element in* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p))$ *is given by*

$$\pi := \mathrm{Ind}_P^{\mathrm{U}(p,p)} \pi'$$

where P *is a parabolic subgroup in* $\mathrm{U}(p,p)$ *with Levi factor equals to* $\mathrm{GL}_p(\mathbb{C})$*.*

(iii) In general, when $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$*, we have a natural bijection*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}(n_1+p,n_2+p)) \\ \pi_0 & \mapsto & \mathrm{Ind}_P^{\mathrm{U}(n_1+p,n_2+p)} \pi'\otimes\pi_0 \end{array}$$

where P *is a parabolic subgroup with Levi factor* $\mathrm{GL}_p(\mathbb{C})\times\mathrm{U}(n_1,n_2)$*.*

The above lemma ensure us to reduce the problem to the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Now assume $\check{\mathcal{O}} = \check{\mathcal{O}}_g$ and so $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ corresponds to the blocks of the infinitesimal character of the trivial representation.

By [12, Theorem 4.2], Harish-Chandra cells in $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ are in one-one correspondence to real nilpotent orbits in $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t$.

[From the branching rule, the cell is parametered by painted partition

$$\mathrm{PP}(\mathrm{U}) := \left\{\, \tau\in\mathrm{PP} \;\middle|\; \begin{array}{l}\mathrm{Im}(\tau)\subseteq\{\bullet, s, r\}\\ \text{“}\bullet\text{” occures even times in each row}\end{array} \right\}.$$

The bijection $\mathrm{PP}(\mathrm{U}) \to \mathsf{SYD}, \tau\mapsto\mathscr{O}$ is given by the following recipe: The shape of $\mathscr{O}$ is the same as that of τ. $\mathscr{O}$ is the unique (upto row switching) signed Young diagram such that

$$\mathscr{O}(i,\mathbf{r}_i(\tau)) := \begin{cases} +, & \text{when } \tau(i,\mathbf{r}_i(\tau)) = r;\\ -, & \text{otherwise, i.e. } \tau(i,\mathbf{r}_i(\tau))\in\{\bullet, s\}.\end{cases}$$

Example 5.2.

$$
\begin{array}{|c|c|c|c|c|}
\hline
\bullet & \bullet & \bullet & \bullet & r \\
\hline
\bullet & \bullet \\
\cline{1-2}
s & r \\
\cline{1-2}
s \\
\cline{1-1}
r \\
\cline{1-1}
\end{array}
\quad \mapsto \quad
\begin{array}{|c|c|c|c|c|}
\hline
+ & - & + & - & + \\
\hline
+ & - \\
\cline{1-2}
- & + \\
\cline{1-2}
- \\
\cline{1-1}
+ \\
\cline{1-1}
\end{array}
$$

]

Now the following lemma is clear.

Lemma 5.3. *When $\check{\mathcal{O}} = \check{\mathcal{O}}_g$, the associated varity of every special unipotent representations in* $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U})$ *is irreducible. Moreover, the following map is a bijection.*

$$
\begin{array}{rcl}
\mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \{\, \textit{rational forms of } \check{\mathcal{O}}^t \,\} \\
\pi_0 & \mapsto & \mathrm{AV}^{\mathrm{weak}}(\pi_0).
\end{array}
$$

□

Remark. Note that the parabolic induction of an rational nilpotent orbit can be reducible. Therefore, when $\check{\mathcal{O}}_b \neq \emptyset$, the special unipotent representations can have reducible associated variety. Meanwhile, it is easy to see that the map $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) \ni \pi \mapsto \mathrm{AV}^{\mathrm{weak}}(\pi)$ is still injective.

We will show that every elements in $\mathrm{Unip}_{\check{\mathcal{O}}_g}$ can be constructed by iterated theta lifting. For each τ, let $\mathscr{O}$ be the corresponding real nilpotent orbit. Let $\mathrm{Sign}(\mathscr{O})$ be the signature of $\mathscr{O}$, $\nabla(\mathscr{O})$ be the signed Young diagram obtained by deleteing the first column of $\mathscr{O}$. Suppose $\mathscr{O}$ has k-columns. Inductively we have a sequence of unitary groups $\mathrm{U}(p_i,q_i)$ with $(p_i,q_i) = \mathrm{Sign}(\nabla^i(\mathscr{O}))$ for $i = 0,\cdots,k$. Then

$$
\pi_\tau = \theta_{\mathrm{U}(p_1,q_1)}^{\mathrm{U}(p_0,q_0)} \theta_{\mathrm{U}(p_2,q_2)}^{\mathrm{U}(p_1,q_1)} \cdots \theta_{\mathrm{U}(p_k,q_k)}^{\mathrm{U}(p_{k-1},q_{k-1})}(1) \tag{5.3}
$$

where 1 is the trivial representation of $\mathrm{U}(p_k,q_k)$.

Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Form the duality between cells of $\mathrm{U}(p,q)$ and $\mathrm{GL}(n,\mathbb{R})$. We have an ad-hoc (bijective) duality between unipotent representations:

$$
\begin{array}{rcl}
d_{\mathrm{BV}}\colon \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) & \rightarrow & \mathrm{Unip}_{\check{\mathcal{O}}^t}(\mathrm{GL}(\mathbb{R})) \\
\pi_\tau & \mapsto & \pi_{d_{\mathrm{BV}}(\tau)}
\end{array}
$$

Here $\check{\mathcal{O}}^t = d_{\mathrm{BV}}(\check{\mathcal{O}})$ and $d_{\mathrm{BV}}(\tau)$ is the pained bipartition obtained by transposeing τ and replace s and r by c and d respectively. See (5.3) and (5.2) for the definition of special unipotent representations on the two sides.

6. Counting in type BCD

In this section, we consider the case when $\check{\star} \in \{\, B, C, D \,\}$, i.e $\star \in \{\, B, \widetilde{C}, C, D, C^*, D^* \,\}$.

We identify $\mathfrak{h}^*$ with $\mathbb{Z}^n$ where $n = \mathrm{rank}(G_{\mathbb{C}})$ and let ρ be the half sum of all positive roots.

Recall that

$$
\text{good pairity} = \begin{cases} \text{odd} & \text{when } \star \in \{\, C, C^*, D, D^* \,\} \\ \text{even} & \text{when } \star \in \{\, B, \widetilde{C} \,\} \end{cases}
$$

Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathcal{G}})$ with decomposition $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$. Then $\check{\mathcal{G}}_{\lambda_{\check{\mathcal{O}}}} = \check{\mathcal{G}}_b \times \check{\mathcal{G}}_g$. Let n_b and n_g be the rank of $\check{\mathcal{G}}_b$ and $\check{\mathcal{G}}_g$ respectively. We have

$$
(n_b, n_g) = \begin{cases} (\frac{1}{2}\left|\check{\mathcal{O}}_b\right|, \frac{1}{2}(\left|\check{\mathcal{O}}_g\right| - 1)) & \text{when } \star \in \{\, C, C^* \,\} \\ (\frac{1}{2}\left|\check{\mathcal{O}}_b\right|, \frac{1}{2}\left|\check{\mathcal{O}}_g\right|) & \text{when } \star \in \{\, B, \widetilde{C}, D, D^* \,\} \end{cases}
$$

and integral Weyl group $W_{[\lambda_{\check{\mathcal{O}}}]}$is a product of two factors

$$W_{[\lambda_{\check{\mathcal{O}}}]} = W_b \times W_g$$

where

$$\begin{aligned} W_b &:= \begin{cases} \mathsf{W}_{n_b} & \text{when } \star \in \{ B, \widetilde{C} \} \\ \mathsf{W}'_{n_b} & \text{when } \star \in \{ C, C^*, D, D^* \} \end{cases} \\ W_g &:= \begin{cases} \mathsf{W}_{n_g} & \text{when } \star \in \{ B, C, C^* \} \\ \mathsf{W}'_{n_g} & \text{when } \star \in \{ \widetilde{C}, D, D^* \} \end{cases} \end{aligned} \tag{6.1}$$

When W_b or W_g is a Weyl group of type D_n, we always have the preferred embedding of S_n into W'_n given by the root system of $\check{\mathfrak{g}}$. The label I on the irreducible character of W_n is refer to this particular embedding.

More precisely, we identify bipartition with n parts with $\mathrm{Irr}(\mathsf{W}_n)$. To ease the notations, we let $(\tau_L, \tau_R)_I$ denote the unique irreducible character of W'_n given by

- the restriction of the irreducible character of W_n attached to (τ_L, τ_R) if $\tau_L \neq \tau_R$, and
- the character $\mathrm{Ind}_{\mathsf{S}_{\frac{n}{2}}}^{\mathsf{W}_n} \tau_L$ if $\tau_L = \tau_R$.

We remark that we always have

$$(\tau_L, \tau_R)_I = (\tau_R, \tau_L)_I$$

as W'_n-character.

6.1. **The left cell.** In this subsection, we described the Lusztig left cell attached to $\lambda_{\check{\mathcal{O}}}$ in each cases, where $\star \in \{ B, C, \widetilde{C}, C^*, D, D^* \}$.

To state the results, we made some definitions first. Define the irreducible W_b representation attached to $\check{\mathcal{O}}_b$ by the following formula

$$\tau_b := \begin{cases} \Big(\big(\frac{1}{2}(\mathbf{r}_2(\check{\mathcal{O}}_b)+1), \frac{1}{2}(\mathbf{r}_4(\check{\mathcal{O}}_b)+1), \cdots, \frac{1}{2}(\mathbf{r}_{2c}(\check{\mathcal{O}}_b)+1)\big), \\ \quad \big(\frac{1}{2}(\mathbf{r}_2(\check{\mathcal{O}}_b)-1), \frac{1}{2}(\mathbf{r}_4(\check{\mathcal{O}}_b)-1), \cdots, \frac{1}{2}(\mathbf{r}_{2c}(\check{\mathcal{O}}_b)-1)\big)\Big) & \text{if } \star \in \{ B, \widetilde{C} \}, \\ \Big(\big(\frac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \frac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \frac{1}{2}\mathbf{r}_{2c}(\check{\mathcal{O}}_b)\big), \\ \quad \big(\frac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}_b), \frac{1}{2}\mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \frac{1}{2}\mathbf{r}_{2c}(\check{\mathcal{O}}_b)\big)\Big)_I & \text{if } \star \in \{ C, C^*, D, D^* \}, \end{cases}$$

with $2c = \mathbf{c}_1(\check{\mathcal{O}}_b)$. Define partitions

- $\check{\mathcal{O}}'_b$ such that $\mathbf{r}_i(\check{\mathcal{O}}'_b) := \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ for all $i \in \mathbb{N}^+$,
- $\mathcal{O}'_b := (\check{\mathcal{O}}'_b)^t$, and
- $\check{\mathcal{O}}_b := \mathcal{O}'_b \overset{c}{\sqcup} \mathcal{O}'_b$.

Set

$$\mathfrak{P}_\star(\check{\mathcal{O}}_g) = \begin{cases} \{ (2i-1, 2i) \mid \mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) - \mathbf{r}_{2i}(\check{\mathcal{O}}_g) \geq 2, i \in \mathbb{N}^+ \} & \text{if } \star \in \left\{ C, \widetilde{C}, C^* \right\} \\ \{ (2i, 2i+1) \mid \mathbf{r}_{2i}(\check{\mathcal{O}}_g) - \mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) \geq 2, i \in \mathbb{N}^+ \} & \text{if } \star \in \{ B, D, D^* \}. \end{cases}$$

Let

$$\widetilde{A}(\check{\mathcal{O}}) := \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$$

be the power set of $\mathfrak{P}_\star(\check{\mathcal{O}}_g)$.

For each $\wp \in \widetilde{A}(\check{\mathcal{O}})$ we define an element $\tau_\wp$ in $\mathrm{Irr}(W_g)$. Here

$$\tau_\wp := \begin{cases} (\imath_\wp, \jmath_\wp) & \text{when } \star \in \{ B, C, C^* \} \\ (\imath_\wp, \jmath_\wp)_I & \text{when } \star \in \{ \widetilde{C}, D, D^* \} \end{cases}$$

and $(\imath_\wp, \jmath_\wp)$ are given by the following formulas:

- Suppose $\star \in \{ C, C^* \}$ and let $l = \min\{ i \mid \mathbf{r}_{2i}(\check{\mathcal{O}}_g) = 0 \}$. Then
$$(\mathbf{c}_l(\imath_\wp), \mathbf{c}_l(\jmath_\wp)) := (0, \tfrac{1}{2}(\mathbf{r}_{2l+1}(\check{\mathcal{O}}_g) - 1))$$
and, for all $1 \leq i < l$
$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_i(\jmath_\wp)) := \begin{cases} (\frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) - 1)) & \text{if } (2i-1, 2i) \notin \wp, \\ (\frac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) - 1)) & \text{otherwise.} \end{cases}$$
- Suppose $\star \in \{ D, D^* \}$ and let $l = \min\{ i \mid \mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) = 0 \}$. Then
$$\mathbf{c}_1(\imath_\wp) := \tfrac{1}{2}(\mathbf{r}_1(\check{\mathcal{O}}_g) + 1)$$
$$(\mathbf{c}_{l+1}(\imath_\wp), \mathbf{c}_l(\jmath_\wp)) := (0, \tfrac{1}{2}(\mathbf{r}_{2l}(\check{\mathcal{O}}_g) - 1))$$
and, for all $1 \leq i < l$
$$(\mathbf{c}_{i+1}(\imath_\wp), \mathbf{c}_i(\jmath_\wp)) := \begin{cases} \left(\frac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) - 1)\right) & \text{if } (2i, 2i+1) \notin \wp, \\ (\frac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}_g) + 1), \frac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) - 1)) & \text{otherwise.} \end{cases}$$
- Suppose $\star = B$. Then
$$\mathbf{c}_1(\jmath_\wp) := \tfrac{1}{2}\mathbf{r}_1(\check{\mathcal{O}}_g)$$
and for all $i \geq 1$
$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_{i+1}(\jmath_\wp)) := \begin{cases} (\frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g)) & \text{if } (2i, 2i+1) \notin \wp, \\ (\frac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g)) & \text{otherwise.} \end{cases}$$
- Suppose $\star = \widetilde{C}$. Then for all $i \geq 1$
$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_i(\jmath_\wp)) := \begin{cases} (\frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g)) & \text{if } (2i-1, 2i) \notin \wp, \\ (\frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g)) & \text{otherwise.} \end{cases}$$

For $\wp \subset \mathfrak{P}_\star(\check{\mathcal{O}}_g)$, let $\wp^c$ be the complement of $\wp$ in $\mathfrak{P}_\star(\check{\mathcal{O}}_g)$ and we have $\tau_\wp = \tau_{\wp^c}$ if $\star = \widetilde{C}$.

We define
$$\bar{A}(\check{\mathcal{O}}) = \begin{cases} \widetilde{A}(\check{\mathcal{O}})/\wp \sim \wp^c & \text{when } \star = \widetilde{C}, \\ \widetilde{A}(\check{\mathcal{O}}) & \text{otherwise.} \end{cases}$$

Here $\widetilde{A}(\check{\mathcal{O}})/\wp \sim \wp^c$ denote the quotient of $\widetilde{A}(\check{\mathcal{O}})$ by identifying $\wp$ with its complement $\wp^c$.

When $\star \neq \widetilde{C}$, $\bar{A}(\check{\mathcal{O}})$ is nothing but the Lusztig canonical quotient attached to $\check{\mathcal{O}}$. [This can be seem from the following lemma, c.f. [16, Proposition 5.28].]

Note that by definition, we have $\widetilde{A}(\check{\mathcal{O}}) = \widetilde{A}(\check{\mathcal{O}}_g)$ and $\bar{A}(\check{\mathcal{O}}) = \bar{A}(\check{\mathcal{O}}_g)$.

Recall that
$${}^L\mathscr{V}_{\check{\mathcal{O}}} := \left(J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W_{[\lambda_{\check{\mathcal{O}}}]}} \operatorname{sgn}\right) \otimes \operatorname{sgn}.$$
and ${}^L\mathscr{C}_{\check{\mathcal{O}}}$ is the multiset of irreducible components

Lemma 6.1 (c.f. Barbasch-Vogan[16, Proposition 5.28]). *In all the cases, ${}^L\mathscr{C}_{\check{\mathcal{O}}}$ is multiplicity free and we have the following bijections*
$$\begin{array}{ccccccc} \bar{A}(\check{\mathcal{O}}) & = & \bar{A}(\check{\mathcal{O}}_g) & \longrightarrow & {}^L\mathscr{C}(\check{\mathcal{O}}_g) & \longrightarrow & {}^L\mathscr{C}(\check{\mathcal{O}}) \\ & & \wp & \mapsto & \tau_\wp & \mapsto & \tau_b \otimes \tau_\wp. \end{array}$$

Moreover,
$$\tau_{\check{\mathcal{O}}} = \tau_b \otimes \tau_\emptyset$$

is the unique special representation in ${}^L\mathscr{C}_{\check{\mathcal{O}}}$ and

$$\text{Springer}(j_{W_{[\lambda_{\check{\mathcal{O}}}]}}^{W}(\tau_{\check{\mathcal{O}}})) = d_{\rm BV}(\check{\mathcal{O}}) = \mathcal{O}_b \overset{c}{\sqcup} d_{\rm BV}(\check{\mathcal{O}}_g). \tag{6.2}$$

[*The last equality could be checked using Sommer's formula on Springer correspondence directly: double columns* $(2c+1, 2c+1)$ *corresponds to* $B_{c=\alpha_{2i-1}} \times D_{c+1=\alpha_{2i}+1} = D_{c+1=\alpha_{2i-1}+1} \times C_{c=\alpha_{2i}}$ *factor in type* $B, \widetilde{C}$*. double columns* $(2c, 2c)$ *corresponds to factor* $D_{c=\beta_{2i-1}} \times C_{c=\beta_{2i}} = D_{c=\beta_{2i-1}} \times B_{c=\beta_{2i}}$ *in type* C, C^*, D, D^**.*] *Here* $d_{\rm BV}$ *is the metaplectic dual if* $\star = \widetilde{C}$ *and is the Barbasch-Vogan dual otherwise.*

Proof. For $\check{\mathcal{O}}_g$, the lemma is given by [16, Proposition 5.28]. For all the cases, the lemma follow from an induction on number of columns using Lusztig's formula of J-induction in [63, §4.4-4.6]. The equality (6.2) is due to Barbasch-Vogan for linear groups [16, Proposition A2].

[**Suppose $\star = C$.**

In this case, bad parity is even and each row length occur with even multiplicity. Suppose $\check{\mathcal{O}}_b = (C_1, C_1, C_2, C_2, \cdots, C_{k'}, C_{k'})$ with $c_1 = 2k$ and $k' = \mathbf{r}_1(\check{\mathcal{O}}_b)$.

$$W_{\lambda_{\check{\mathcal{O}}_b}} = S_{C_1} \times S_{C_2} \times \cdots S_{C_{k'}}.$$

The symbol of trivial representation of trivial group of type D is

$$\begin{pmatrix} 0, 1, \cdots, k-1 \\ 0, 1, \cdots, k-1 \end{pmatrix}.$$

Now it is easy to see that (use the similar computation as below)

$$J_{W_{\lambda_{\check{\mathcal{O}}_b}}}^{W_b} \text{sgn} = ((\tfrac{1}{2}C_1, \tfrac{1}{2}C_2, \cdots, \tfrac{1}{2}C_{k'}), (\tfrac{1}{2}C_1, \tfrac{1}{2}C_2, \cdots, \tfrac{1}{2}C_{k'})).$$

For the good parity part. Let $r_i' = \lfloor \frac{1}{2}(\mathbf{r}_i(\check{\mathcal{O}}_g) - \mathbf{r}_{i+1}(\check{\mathcal{O}}_g)) \rfloor$. Suppose $\check{\mathcal{O}}_g$ has $2l+1$ columns (superscripts denote the multiplicity)

$$\check{\mathcal{O}}_g = ((2l+1)^{2r_{2l+1}'+1}, 2l^{2r_{2l}'}, (2l-1)^{2r_{2l-1}'}, \cdots, 2^{2r_2'}, 1^{2r_1'})$$

and

$$W_{\lambda_{\check{\mathcal{O}}_g}} = W_l \times \underbrace{S_{2l+1} \times \cdots \times S_{2l+1}}_{2r_{2l+1}'\text{-terms}} \times \prod_{i<2l+1} \underbrace{S_i \times \cdots \times S_i}_{r_i'\text{-terms}}$$

The symbol of sign representation of W_l is

$$\begin{pmatrix} 0, 1, 2, \cdots, l \\ 1, 2, \cdots, l \end{pmatrix}.$$

The induction begins with the longest columns to the shorter columns

Induce to include all $2l+1$-length columns yields

$$\begin{pmatrix} r_{2l+1}' + 0, r_{2l+1}' + 1, r_{2l+1}' + 2, \cdots, r_{2l+1}' + l \\ r_{2l+1}' + 1, r_{2l+1}' + 2, \cdots, r_{2l+1}' + l \end{pmatrix}.$$

Now move the the shorter columns, we see that when even columns $(2i)^{2r_{2i}'}$ occurs, it adds $(i)^{r_{2i}'}$ columns on the both sides of the bipartition; when odd columns $(2i+1)^{r_{2i+1}'}$ occur, the bifurcation happens: one can

- attach columns $(i+1)^{r_{2i+1}'}$ on the left and columns $(i)^{r_{2i+1}'}$ on the right, which corresponds to $(2i+1, 2i+2) \neq \wp$, or
- attach columns $(i)^{r_{2i+1}'}$ on the left and columns $(i+1)^{r_{2i+1}'}$ on the right, which corresponds to $(2i+1, 2i+2) \in \wp$,

Therefore,

$$\begin{array}{rcc} J_{W_{\lambda_{\check{\mathcal{O}}_g}}}^{W_g} \operatorname{sgn} & \leftrightarrow & \mathbb{F}_2(\mathfrak{P}(\check{\mathcal{O}}_g)) \\ (\check{\tau}_L, \check{\tau}_R) =: \check{\tau}_\wp & \leftrightarrow & \wp \end{array}$$

where

$$\mathbf{r}_{l+1}(\check{\tau}_L) = r'_{2l+1} = \tfrac{1}{2}(\mathbf{r}_{2l+1}(\check{\mathcal{O}}_g) - 1)$$

and, if $(2i-1, 2i) \notin \wp$,

$$\mathbf{r}_i(\check{\tau}_L) = \sum_{l \geq 2i-1} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}) - 1)$$

$$\mathbf{r}_i(\check{\tau}_R) = 1 + \sum_{l \geq 2i} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) + 1)$$

if $(2i-1, 2i) \in \wp$,

$$\mathbf{r}_i(\check{\tau}_L) = \sum_{l \geq 2i} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) - 1)$$

$$\mathbf{r}_i(\check{\tau}_R) = 1 + \sum_{l \geq 2i-1} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}) + 1)$$

Since $\tau_\wp = \check{\tau}_\wp \otimes \operatorname{sgn}$, we get the claim.

We adopt the convention that

$$\mathsf{S}_{\mathcal{O}} := \prod_{i \in \mathbb{N}^+} \mathsf{S}_{\mathbf{c}_i(\mathcal{O})}$$

so that $j_{\mathsf{S}_{\mathcal{O}}}^{\mathsf{S}_{|\mathcal{O}|}} \operatorname{sgn} = \mathcal{O}$ for each partition $\mathcal{O}$.

Now consider the orbit under the Springer correspondence.

Let $\check{\mathcal{O}}'_b := [\mathbf{r}_2(\check{\mathcal{O}}_b), \mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \mathbf{r}_{2k}(\check{\mathcal{O}}_b)]$, $\mathcal{O}'_b := (\check{\mathcal{O}}'_b)^t$ and $\mathcal{O}_b := \mathcal{O}'_b \overset{c}{\sqcup} \mathcal{O}'_b$. Clearly, $\check{\mathcal{O}}_b = \check{\mathcal{O}}'_b \overset{r}{\sqcup} \check{\mathcal{O}}'_b$. Note that $\tau_b = j_{S_{\mathcal{O}'_b}}^{W'_b} \operatorname{sgn}$ (by the formula of fake degree see Lusztig or Carter's book). So, by induction by stage of j-induction, we have

$$\widetilde{\tau}_{\mathcal{O}} := j_{W'_b \times W_g}^{W_n} (\tau_b \otimes \tau_\emptyset) = j_{S_{\mathcal{O}'_b} \times W_g}^{W_n} \operatorname{sgn} \otimes \tau_\wp.$$

By Barbasch-Vogan, $\mathcal{O}_g := \operatorname{Springer}(\tau_\emptyset) = d_{BV}(\check{\mathcal{O}}_g)$, which is well know how to calculate. (In fact, one can deduce the result by our computation.)

Since the Springer correspondence commutes with parabolic induction, we get $\operatorname{Springer}(\widetilde{\tau}) = \operatorname{Ind}_{\mathrm{GL}_{\mathcal{O}'_b} \times \mathrm{Sp}(2g)}^{\mathrm{Sp}(2n)} 0 \times \mathcal{O}_g = \mathcal{O}_b \overset{c}{\sqcup} \mathcal{O}_g$.

Suppose $\star = D$.

The bad parity part is the same as that of the case when $\star = C$.

Now consider the good parity part.

$$\check{\mathcal{O}}_g = ((2l)^{2r'_{2l}+1}, (2l-1)^{2r'_{2l-1}}, (2l-2)^{2r'_{2l-2}}, \cdots, 2^{2r'_2}, 1^{2r'_1})$$

and

$$W_{\lambda_{\check{\mathcal{O}}_g}} = W'_l \times \underbrace{S_{2l} \times \cdots \times S_{2l}}_{2r'_{2l}\text{-terms}} \times \prod_{i<2l} \underbrace{S_i \times \cdots \times S_i}_{r'_i\text{-terms}}$$

The symbol of sign representation of W'_l is

$$\begin{pmatrix} 0, 1, \cdots, l-1 \\ 1, 2, \cdots, l \end{pmatrix}.$$

(Here we always made the choice of the top and bottom row to compatible with the type C case.)

Induce to include all $2l$-length columns yields

$$\begin{pmatrix} r'_{2l}+0, r'_{2l}+1, \cdots, r'_{2l}+l-1 \\ r'_{2l}+1, r'_{2l}+2, \cdots, r'_{2l}+l \end{pmatrix}.$$

Now move the the shorter columns. When odd columns $(2i+1)^{2r'_{2i+1}}$ occurs, it adds $(i)^{r'_{2i+1}}$ columns on the left and $(i+1)^{r'_{2i+1}}$ on the right. When even columns $(2i)^{r'_{2i}}$ occur, the bifurcation happens: one can

- attach columns $(i)^{r'_{2i}}$ on the left and columns $(i)^{r'_{2i}}$ on the right, which corresponds to $(2i, 2i+1) \neq \wp$, or
- attach columns $(i-1)^{r'_{2i}}$ on the left and columns $(i+1)^{r'_{2i}}$ on the right, which corresponds to $(2i, 2i+1) \in \wp$,

Therefore,

$$\begin{array}{ccc} \mathbb{F}_2(\mathfrak{P}(\check{\mathcal{O}}_g)) & \longrightarrow & J^{W_g}_{W_{\lambda_{\check{\mathcal{O}}_g}}} \operatorname{sgn} \\ \wp & \mapsto & (\check{\tau}_L, \check{\tau}_R) =: \check{\tau}_\wp \end{array}$$

where

$$\mathbf{r}_l(\check{\tau}_L) = r'_{2l} = \tfrac{1}{2}(\mathbf{r}_{2l}(\check{\mathcal{O}}_g) - 1)$$
$$\mathbf{r}_1(\check{\tau}_R) = 1 + \sum_i r'_i = \tfrac{1}{2}(\mathbf{r}_1(\check{\mathcal{O}}_g) + 1)$$

and, if $(2i, 2i+1) \notin \wp$,

$$\mathbf{r}_i(\check{\tau}_L) = \sum_{l \geq 2i} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) - 1)$$
$$\mathbf{r}_{i+1}(\check{\tau}_R) = 1 + \sum_{l \geq 2i+1} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}) + 1)$$

if $(2i, 2i+1) \in \wp$,

$$\mathbf{r}_i(\check{\tau}_L) = \sum_{l \geq 2i+1} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}) - 1)$$
$$\mathbf{r}_i(\check{\tau}_R) = 1 + \sum_{l \geq 2i} r'_l = \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) + 1)$$

Also note that $\check{\tau}_\wp = \check{\tau}_{\wp^c}$. The rest parts are the same as that of type C.

Suppose $\star = B$.

In this case, bad parity is odd and every odd row occurs with with even times.

We can write $r'_i := \lfloor \frac{1}{2}(\mathbf{r}_i(\check{\mathcal{O}}_b) - \mathbf{r}_{i-1}(\check{\mathcal{O}}_b)) \rfloor$

$$\check{\mathcal{O}}_b = ((2l)^{2r'_{2l}+1}, (2l-1)^{2r'_{2l-1}}, \cdots, 1^{2r'_1})$$

Then

$$W_{\lambda_{\check{\mathcal{O}}_b}} = W_l \times \underbrace{S_{2l} \times \cdots \times S_{2l}}_{2r'_{2l}\text{-terms}} \times \prod_{i<2l} \underbrace{S_i \times \cdots \times S_i}_{r'_i\text{-terms}}$$

(Note that in the product, $r'_i = 0$ if i is odd.) The computation of $\check{\sigma}_b = J^{W_b}_{W_{\lambda_{\check{\mathcal{O}}_b}}} \operatorname{sgn}$ is similar to that of the good parity for type C with no bifurcating, one deduce that J-induction and j-induction gives the same result.

$$\begin{aligned} \check{\sigma}_b &= \begin{pmatrix} 0, 1+r_l, 2+r_{l-1} \cdots, l+r_1 \\ 1+r_l, 2+r_{l-1}, \cdots, l+r_1 \end{pmatrix} \\ &= ([r_1, r_2, \cdots, r_l], [r_1+1, r_2+1, \cdots, r_l+1]) \end{aligned}$$

with $r_i = \frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_b) = \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_b)$. Now

$$\sigma_b = ((r_1+1, r_2+1, \cdots, r_l+1), (r_1, r_2, \cdots, r_l)) = j_{S_{\mathcal{O}'_b}}^{W_b} \operatorname{sgn}$$

where $\mathcal{O}'_b = (\mathbf{r}_2(\check{\mathcal{O}}_b), \mathbf{r}_4(\check{\mathcal{O}}_b), \cdots, \mathbf{r}_{2l}(\check{\mathcal{O}}_b))$. Under the Springer correspondence of type B, it corresponds to $\mathrm{Ind}_{\mathrm{GL}_b}^{\mathrm{SO}(2b+1)} \mathcal{O}'_b = \mathcal{O}'_b \overset{r}{\sqcup} \mathcal{O}'_b \overset{r}{\sqcup} (1)$.

Now we consider the good parity parts, where each row of $\check{\mathcal{O}}_g$ has even length.

Assume $r'_i := \frac{1}{2}(\mathbf{r}_i(\check{\mathcal{O}}_g) - \mathbf{r}_{i-1}(\check{\mathcal{O}}_g))$ and so

$$\check{\mathcal{O}}_g = ((2l+1)^{2r'_{2l+1}}, (2l)^{2r'_{2l}}, \cdots, 1^{2r'_1})$$

with $l = \min\{\, i \mid \mathbf{r}_{2i+2}(\check{\mathcal{O}}_g) = 0 \,\}$.

Then

$$W_{\lambda_{\check{\mathcal{O}}_g}} = \times \prod_{i \le 2l+1} \underbrace{S_i \times \cdots \times S_i}_{r'_i\text{-terms}}$$

Note that the trivial representation of the trivial group has symbol

$$\begin{pmatrix} 0, 1, 2, \cdots, l \\ 0, 1, \cdots, l-1 \end{pmatrix}.$$

Induce to include all $2l+1$-length columns yields

$$\begin{pmatrix} r'_{2l+1}+0, r'_{2l+1}+1, r'_{2l+1}+2, \cdots, r'_{2l+1}+l \\ r'_{2l+1}+0, r'_{2l+1}+1, \cdots, r'_{2l+1}+l-1 \end{pmatrix}.$$

Now move the the shorter columns. When odd columns $(2i+1)^{2r'_{2i+1}}$ occurs, it adds $(i+1)^{r'_{2i+1}}$ columns on the left and $(i)^{r'_{2i+1}}$ on the right. When even columns $(2i)^{r'_{2i}}$ occur, the bifurcation happens: one can

- attach columns $(i)^{r'_{2i}}$ on the left and columns $(i)^{r'_{2i}}$ on the right, which corresponds to $(2i, 2i+1) \neq \wp$, or
- attach columns $(i-1)^{r'_{2i}}$ on the left and columns $(i+1)^{r'_{2i}}$ on the right, which corresponds to $(2i, 2i+1) \in \wp$.

Therefore,

$$\begin{array}{ccc} \mathbb{F}_2(\mathfrak{P}(\check{\mathcal{O}}_g)) & \longrightarrow & J_{W_{\lambda_{\check{\mathcal{O}}_g}}}^{W_g} \operatorname{sgn} \\ \wp & \mapsto & (\check{\tau}_L, \check{\tau}_R) =: \check{\tau}_\wp \end{array}$$

where

$$\mathbf{r}_1(\check{\tau}_L) = \sum_i r'_i = \tfrac{1}{2}\mathbf{r}_1(\check{\mathcal{O}}_g)$$

and, if $(2i, 2i+1) \notin \wp$,

$$\mathbf{r}_{i+1}(\check{\tau}_L) = \sum_{l \ge 2i+1} r'_l = \tfrac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}})$$

$$\mathbf{r}_i(\check{\tau}_R) = \sum_{l \ge 2i} r'_l = \tfrac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}})$$

if $(2i, 2i+1) \in \wp$,

$$\mathbf{r}_{i+1}(\check{\tau}_L) = \sum_{l \ge 2i} r'_l = \tfrac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}})$$

$$\mathbf{r}_i(\check{\tau}_R) = \sum_{l \ge 2i+1} r'_l = \tfrac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}})$$

Some remarks on the BV-dual. The calculation of $\mathcal{O}_g$ from $\tau_\emptyset$ can be reduced to the case of quasi-distinguished orbits (other case are deduced from this by parabolic induction, corresponds to attach two even columns for the balanced pairs). Compare Sommer's description of Springer correspondence with ours, we deduce that

$$\mathcal{O}_g = (\mathbf{r}_1(\check{\mathcal{O}}_1)+1, \mathbf{r}_2(\check{\mathcal{O}}_2)-1, \mathbf{r}_3(\check{\mathcal{O}}_3)+1, \cdots, \mathbf{r}_{2l}(\check{\mathcal{O}}_{2l})-1, \mathbf{r}_{2l+1}(\check{\mathcal{O}}_{2l+1})+1)$$

The rest parts are similar to that of type D.]

We sketch the proof for the case when $\star = \widetilde{C}$.

For a partition $\mathcal{O}$, we set

$$\mathsf{S}_{\mathcal{O}} := \prod_{i\in\mathbb{N}^+} \mathsf{S}_{\mathbf{c}_i(\mathcal{O})}$$

so that $j_{\mathsf{S}_{\mathcal{O}}}^{\mathsf{S}_{|\mathcal{O}|}} \mathrm{sgn} = \mathcal{O}$.

We first consider the good parity (even) part.

Now we consider the good parity parts, where each row of $\check{\mathcal{O}}_g$ has even length.

We set $r'_i := \frac{1}{2}(\mathbf{r}_i(\check{\mathcal{O}}_g) - \mathbf{r}_{i-1}(\check{\mathcal{O}}_g))$, $l = \min\{\, i \mid \mathbf{r}_{2i+1}(\check{\mathcal{O}}_g) = 0\,\}$, and write

$$\check{\mathcal{O}}_g = ((2l)^{2r'_{2l}}, (2l-1)^{2r'_{2l-1}}, \cdots, 1^{2r'_1})$$

where $i^{r'}$ denotes r'-copies of length i columns. The Weyl group of good parity is W'_{n_g} with $n_g = \frac{1}{2}\left|\check{\mathcal{O}}_g\right|$. For $1 \le k \le l$, let

$$\vec{S}_i = \underbrace{\mathsf{S}_i \times \cdots \times \mathsf{S}_i}_{r'_i\text{-terms}} \quad \text{and} \quad n_k = \sum_{i=k}^{l} i\cdot r'_i.$$

Then $W_{\lambda_{\check{\mathcal{O}}_g}} = \prod_{i=1}^{l} \vec{S}_i$ and

$$\begin{aligned} {}^L\check{\mathscr{V}}_{\check{\mathcal{O}}_g} &:= J_{W_{\lambda_{\check{\mathcal{O}}_g}}}^{W_g} \mathrm{sgn} \\ &= J_{\vec{S}_1\times \mathsf{W}'_{n_2}}^{W_{n_1}} \Big(\mathrm{sgn} \otimes J_{\vec{S}_2\times\mathsf{W}'_{n_3}}^{\mathsf{W}'_{n_2}} \Big(\mathrm{sgn}\otimes \cdots \big(J_{\vec{S}_l}\,\mathrm{sgn}\big)\cdots\Big)\Big) \end{aligned}$$

Applying [63, (4.6.2)] inductively, we see that the operation $J_{\vec{S}_i\times\mathsf{W}'_{n_{i+1}}}^{\mathsf{W}'_{n_i}}(\mathrm{sgn}\otimes\_\_)$ doubles (resp. keeps) the number of irreducible components if i is odd (resp. even).

Suppose $\mathfrak{P}_\star(\check{\mathcal{O}}_g) = \emptyset$. We define

$$\tilde{A}'(\check{\mathcal{O}}) := \mathbb{F}_2[\emptyset]$$

to be the trivial group. Then ${}^L\check{\mathscr{V}}_{\check{\mathcal{O}}_g}$ is irreducible and marked by label I. Hence, ${}^L\mathscr{C}(\check{\mathcal{O}}_g)$, ${}^L\mathscr{C}(\check{\mathcal{O}})$ and $\tilde{A}'(\check{\mathcal{O}})$ are naturally identified.

Now assume $\mathfrak{P}_\star(\check{\mathcal{O}}_g) \neq \emptyset$. Then the two parts of the bipartition of an irreducible component are different. Let $i_0 := \min\left\{\, i \,\middle|\, (2i-1, 2i) \in \mathfrak{P}_\star(\check{\mathcal{O}}_g) \,\right\}$. Then we have a bijection

$$\tilde{A}'(\check{\mathcal{O}}) := \{\, \wp \in \mathbb{F}_2[\mathfrak{P}_\star(\check{\mathcal{O}}_g)] \mid (2i_0-1, 2i_0) \notin \wp \,\} \longrightarrow {}^L\check{\mathscr{V}}_{\check{\mathcal{O}}_g}$$

to record the bifurcation when attaching odd length columns. Here we send $\wp$ to $\check{\tau}_\wp := (\check{\tau}_L, \check{\tau}_R)$ such that

$$(\mathbf{r}_i(\check{\tau}_L), \mathbf{r}_i(\check{\tau}_R)) := \begin{cases} (\frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g)) & \text{if } (2i-1,2i)\notin \wp \\ (\frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g)) & \text{if } (2i-1,2i)\in \wp \end{cases}$$

We obtain the structure of ${}^L\mathscr{C}_{\check{\mathcal{O}}_g}$ by tensoring the sign representation.

[Note that the trivial representation of the trivial group is represented by the symbol

$$\begin{pmatrix} 0,1,2,\cdots,l-1 \\ 0,1,2,\cdots,l-1 \end{pmatrix}_I.$$

Now move the the shorter columns. When even columns $(2i)^{2r'_{2i}}$ occurs, it adds $(i)^{r'_{2i}}$ columns on the left and $(i)^{r'_{2i}}$ on the right. When odd columns $(2i-1)^{r'_{2i-1}}$ occur, the bifurcation happens: one can

- attach columns $(i-1)^{r'_{2i-1}}$ on the left and columns $(i)^{r'_{2i-1}}$ on the right, which corresponds to $(2i-1,2i)\neq\wp$, or
- attach columns $(i)^{r'_{2i-1}}$ on the left and columns $(i-1)^{r'_{2i-1}}$ on the right, which corresponds to $(2i-1,2i)\in\wp$.

Note that when we first encounter the longest odd column, we make the choice that the size of left part is larger than that of the right part. Now If $(2i-1,2i)\notin\wp$,

$$\mathbf{r}_i(\check\tau_L)=\sum_{l\geq 2i} r'_l=\tfrac12\mathbf{r}_{2i}(\check{\mathcal{O}}_g)$$
$$\mathbf{r}_i(\check\tau_R)=\sum_{l\geq 2i-1} r'_l=\tfrac12\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g)$$

if $(2i-1,2i)\in\wp$,

$$\mathbf{r}_i(\check\tau_L)=\sum_{l\geq 2i-1} r'_l=\tfrac12\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g)$$
$$\mathbf{r}_i(\check\tau_R)=\sum_{l\geq 2i} r'_l=\tfrac12\mathbf{r}_{2i}(\check{\mathcal{O}}_g)$$

]

Now we consider the bad parity (odd) part. Suppose $\check{\mathcal{O}}_b$ is nonempty such that

$$\check{\mathcal{O}}_b=(2c_0,2c_1,2c_1,2c_2,2c_2,\cdots,2c_k,2c_k)$$

where $2k+1=\mathbf{r}_1(\check{\mathcal{O}}_b)$ and $2c_i=\mathbf{c}_{2i+1}(\check{\mathcal{O}}_b)$. Now

$$W_{\lambda_{\check{\mathcal{O}}_b}}=\mathsf{W}_{c_0}\times\mathsf{S}_{2c_1}\times\mathsf{S}_{2c_2}\times\cdots\times\mathsf{S}_{2c_k}$$

and

$$\begin{aligned}\check\tau_b &:= J_{W_{\lambda_{\check{\mathcal{O}}_b}}}^{W_b}\operatorname{sgn}=((c_1,c_2,\cdots,c_k),(c_0,c_1,\cdots,c_l))\\ &=\big([\tfrac12(\mathbf{r}_2(\check{\mathcal{O}})-1),\tfrac12(\mathbf{r}_4(\check{\mathcal{O}})-1),\cdots,\tfrac12(\mathbf{r}_{2c_0}(\check{\mathcal{O}})-1)],\\ &\qquad[\tfrac12(\mathbf{r}_2(\check{\mathcal{O}})+1),\tfrac12(\mathbf{r}_4(\check{\mathcal{O}})+1),\cdots,\tfrac12(\mathbf{r}_{2c_0}(\check{\mathcal{O}})+1)]\big)\end{aligned}$$

is irreducible by [63, (4.5.4)]. Tensoring with sign yields the formula of τ_b. Moreover, by the fake degree formula (see [20, p 376]), we have

$$\tau_b=\check\tau_b\otimes\operatorname{sgn}=j_{\mathsf{S}_{\mathcal{O}'_b}}^{\mathsf{W}_{n_b}}\operatorname{sgn}.$$

where $\mathcal{O}'_b:=(\check{\mathcal{O}}'_b)^t:=(\mathbf{r}_2(\check{\mathcal{O}}),\mathbf{r}_4(\check{\mathcal{O}}),\cdots,\mathbf{r}_{2c_0}(\check{\mathcal{O}}))$.

Now we sketch the proof of (6.2).

Recall the definition of the metaplectic Barbasch-Vogan dual in [8]. The duality map commutes with parabolic induction: Suppose $\check{\mathfrak{l}}\subset\check{\mathfrak{g}}$ is a parabolic subalgebra of $\check{\mathfrak{g}}$ and $\mathfrak{l}$ is the corresponding parabolic subalgebra in $\mathfrak{g}$, then

$$d_{\mathrm{BV}}(\check{\mathcal{O}})=\operatorname{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(d_{\mathrm{BV}}(\check{\mathcal{O}}_{\check{\mathfrak{l}}}))\tag{6.3}$$

for each nilpotent orbit $\check{\mathcal{O}}$ in $\check{\mathfrak{g}}$ such that $\check{\mathcal{O}}_{\check{\mathfrak{l}}}:=\check{\mathcal{O}}\cap\check{\mathfrak{l}}\neq\emptyset$. This is clear by reducing to the type B case, see [8, Proposition 3.8]. By removing pairs of rows with the same lengths in $\check{\mathcal{O}}$, we reduced to check the equality in the case when $\check{\mathcal{O}}_b=\emptyset$ and $\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g)>\mathbf{r}_{2i}(\check{\mathcal{O}}_g)$

for all i such that $i \leq \mathbf{c}_1(\check{\mathcal{O}}_g)$. In this case, both sides of (6.2) can be easily computed directly, which equals to

$$(\mathbf{r}_1(\check{\mathcal{O}})-1, \mathbf{r}_2(\check{\mathcal{O}})+1, \cdots, \mathbf{r}_{2c-1}(\check{\mathcal{O}})-1, \mathbf{r}_{2c}(\check{\mathcal{O}})+1)$$

with $c = \min \{ i \mid \mathbf{r}_{2i+1}(\check{\mathcal{O}}) = 0 \}$.

[Compare Sommer's description of Springer correspondence we deduce that the RHS is

$$\mathcal{O}_g = (\mathbf{r}_1(\check{\mathcal{O}}_1)-1, \mathbf{r}_2(\check{\mathcal{O}}_2)+1, \mathbf{r}_3(\check{\mathcal{O}}_3)+1, \cdots, \mathbf{r}_{2l-1}(\check{\mathcal{O}}_{2l-1})-1, \mathbf{r}_{2l}(\check{\mathcal{O}}_{2l})+1)$$

The LHS is calculated by $((((\check{\mathcal{O}}^t)_D)^+)^-)_C$. We write $R_i = \mathbf{r}_i(\check{\mathcal{O}}) = 2r_i$. Now under our assumption, $R_{2i-1} > R_{2i}$, we have

$$\begin{aligned}
((((\check{\mathcal{O}}^t)_D)^+)^-)_C &= ((((R_1, R_2, \cdots, R_{2l-1}, R_{2l})_D)^+)^-)_C \\
&= ((R_1-1, R_2, \cdots, R_{2l-1}, R_{2l}, 1))_C \\
&= (R_1-1, R_2+1, \cdots, R_{2l-1}-1, R_{2l}+1)
\end{aligned}$$

So the proof is done.

]

At last, one can see that $d_{\mathrm{BV}}(\check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g) = \check{\mathcal{O}}_b^t \overset{c}{\sqcup} d_{\mathrm{BV}}(\check{\mathcal{O}}_g)$ using (6.3). $\square$

Since the representation theory of W_n is more elementary than that of W'_n, we prefer to induces every thing to W_n. We record the following technical lemma for the later use.

Lemma 6.2. *Suppose $\star = \widetilde{C}$. For any $\wp \in \tilde{A}(\check{\mathcal{O}})$, let $\widetilde{\tau}_\wp = (\imath_\wp, \jmath_\wp)$. Then*

$$\mathrm{Ind}_{\mathsf{W}_{n_b}\times\mathsf{W}'_{n_g}}^{\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}} \tau_b \boxtimes \tau_\wp = \begin{cases} \tau_b \boxtimes \widetilde{\tau}_\emptyset & \text{if } \tilde{A}(\check{\mathcal{O}}) = \emptyset, \\ \tau_b \boxtimes \widetilde{\tau}_\wp \oplus \tau_b \boxtimes \widetilde{\tau}_{\wp^c} & \text{otherwise.} \end{cases}$$

Let

$${}^L\widetilde{\mathscr{V}}_{\check{\mathcal{O}}} := \mathrm{Ind}_{W_b\times W_g}^{\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}} {}^L\mathscr{V}_{\check{\mathcal{O}}}$$

and ${}^L\widetilde{\mathscr{C}}_{\check{\mathcal{O}}}$ be the set of irreducible components of the $\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}$-module ${}^L\widetilde{\mathscr{V}}_{\check{\mathcal{O}}}$. Then we have a bijection

$$\begin{array}{ccccccc} \tilde{A}(\check{\mathcal{O}}) & = & \tilde{A}(\check{\mathcal{O}}_g) & \longrightarrow & {}^L\widetilde{\mathscr{C}}(\check{\mathcal{O}}_g) & \longrightarrow & {}^L\widetilde{\mathscr{C}}(\check{\mathcal{O}}) \\ & & \wp & \mapsto & \widetilde{\tau}_\wp & \mapsto & \tau_b \otimes \widetilde{\tau}_\wp. \end{array}$$

Suppose $\star \in \{ D, D^* \}$. For any $\wp \in \tilde{A}(\check{\mathcal{O}})$, let $\widetilde{\tau}_\wp = (\imath_\wp, \jmath_\wp)$ and $\widetilde{\tau}_b = \mathrm{Ind}_{\mathsf{W}'_{n_b}}^{\mathsf{W}_{n_b}} \tau_b$. Then

$$\mathrm{Ind}_{\mathsf{W}'_{n_b}\times\mathsf{W}'_{n_g}}^{\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}} \tau_b \boxtimes \tau_\wp = \begin{cases} \widetilde{\tau}_b & \text{if } n_g = 0 \\ \widetilde{\tau}_b \boxtimes \widetilde{\tau}_\wp \oplus \widetilde{\tau}_b \boxtimes \widetilde{\tau}_\wp^s & \text{otherwise.} \end{cases}$$

where $\widetilde{\tau}_\wp^s := \widetilde{\tau}_\wp \otimes \breve{\mathrm{sgn}} \neq \widetilde{\tau}_\wp$.

Proof. The lemma follows immediately from the above lemma on the explicit descriptions of the left cells. $\square$

6.2. **Coherent continuation representations.** Before we start to describe the coherent continuation representations we first recall some subgroups of the Weyl group and the related branching rule.

In the following $a, b, c, d, n, p, q, r, s, t \in \mathbb{N}$. We view W_{2t} and S_{2t} as the reflection group acts on $\mathbb{C}^t$ as usual. Let $\mathsf{H}_t := \mathsf{W}_t \ltimes \{ \pm 1 \}^t$ be the subgroup in W_{2t} such that

- the first factor W_t sits in S_{2t} commuting with the involution

$$(12)(34)\cdots((2t-1)(2t)).$$

- The element $(1,\cdots,1,\underbrace{-1}_{i\text{-th term}},1,\cdots,1)\in\{\pm1\}^t$ acts on $\mathbb{C}^{2t}$ by

$$(x_1,x_2,\cdots,x_{2t})\mapsto(x_1,\cdots,x_{2i-2},-x_{2i},-x_{2i-1},x_{2i+1},\cdots,x_{2t}).$$

Note that H_t is also a subgroup of W'_{2t}. Define the quadratic character

$$\widetilde{\mathrm{sgn}}:=q\otimes\mathrm{sgn}\colon\begin{array}{ccc}\mathsf{H}_t=\mathsf{W}_t\ltimes\{\pm1\}^t & \longrightarrow & \{\pm1\}\\ (g,(a_1,a_2,\cdots,a_t)) & \mapsto & a_1a_2\cdots a_t.\end{array}$$

The most important formulas are

$$\mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}_{2t}}\widetilde{\mathrm{sgn}}=\sum_{\sigma\in\mathrm{Irr}(\mathsf{S}_t)}(\sigma,\sigma)\quad\text{and}\quad\mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}'_{2t}}\widetilde{\mathrm{sgn}}=\sum_{\sigma\in\mathrm{Irr}(\mathsf{S}_t)}(\sigma,\sigma)_I,\tag{6.4}$$

see [68, p220 (6)].

We denote $\overline{\mathrm{sgn}}$ the inflation of the sign representation of S_n to W_n. Then

$$\mathrm{Ind}_{\mathsf{S}_n}^{\mathsf{W}_n}\mathrm{sgn}=\bigoplus_{a+b=n}\mathrm{Ind}_{W_a\times W_b}^{W_n}\overline{\mathrm{sgn}}\boxtimes\mathrm{sgn}=\bigoplus_{a+b=n}((a,),(b,)).$$

Let $\breve{\mathrm{sgn}}:=\mathrm{sgn}\otimes\overline{\mathrm{sgn}}$ which is the quadratic character of $\mathsf{W}_n=\mathsf{S}_n\ltimes\{\pm1\}^n$ given by

$$(s,(x_1,x_2,\cdots,x_n))\mapsto x_1x_2\cdots x_n.$$

[In fact, $\breve{\mathrm{sgn}}$ is the unique non-trivial quadratic character of $\mathsf{W}_n=\mathsf{S}_n\ltimes\{\pm1\}^n$ which is trivial on S_n.] Then

$$\mathrm{Ind}_{\mathsf{S}_n}^{\mathsf{W}_n}1=\bigoplus_{a+b=n}\mathrm{Ind}_{W_a\times W_b}^{W_n}1\boxtimes\breve{\mathrm{sgn}}=\bigoplus_{a+b=n}([a,],[b,]).$$

[Now (6.4) was obtained by the following branching formula: [68, p220 (6)]

$$I_n:=\mathrm{Ind}_{(W_s\ltimes W(A_1)^s)}^{W_{2s}}\mathrm{triv}\otimes\mathrm{sgn}=\sum\lambda\times\lambda$$

where λ running over all Young diagrams of size s. As McGovern claimed the proof of the above formula is similar to Barbasch's proof of [12, Lemma 4.1]:

$$\mathrm{Ind}_{W_n}^{S_{2n}}\mathrm{triv}=\sum\sigma\quad\text{where }\sigma\text{ has even rows only.}$$

Sketch of the proof (use branching rule and dimension counting): Note that $\dim I_n=\frac{(2p)!2^{2p}}{p!2^{2p}}=(2p)!/p!=\sum_\lambda\dim\lambda\times\lambda$ (For the last equality: $\dim\lambda\times\lambda=(2p)!(\dim\lambda)^2/(p!)^2$ where $\dim\lambda$ is the dimension of S_n representation determined by λ; But $\sum(\dim\lambda)^2=p!$). On the other hand, $H:=W_s\ltimes W(A_1)^s\cap W_s\times W_s=\Delta W_s\subset W_{2s}$. $\mathrm{triv}\otimes\mathrm{sgn}|_H=\mathrm{sgn}$ of ΔW_s Therefore, $\lambda\times\lambda$ appears in I_n by Mackey formula. Now by dimension counting, we get the formula.]

We now define various Weyl representations case by case. They will be used to state the formula of coherent continuation representations. In the following σ is running over all irreducible representations of S_t.

- Suppose $\star=B$, $p+q=2m+1$ is odd. Define

$$\mathcal{C}_b^n:=\bigoplus_{2t+c+d=n}\mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_c\times\mathsf{W}_d}^{\mathsf{W}_n}\widetilde{\mathrm{sgn}}\boxtimes1\boxtimes1$$

$$\mathcal{C}_g^{p,q}:=\bigoplus_{\substack{0\le p-(2t+a+2r)\le1\\0\le q-(2t+a+2s)\le1}}\mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a\times\mathsf{W}_s\times\mathsf{W}_r}^{\mathsf{W}_n}\widetilde{\mathrm{sgn}}\boxtimes1\boxtimes\mathrm{sgn}\boxtimes\mathrm{sgn}$$

[Here is a point which could cases confusion: Although the real Weyl group is $\mathsf{H}_t\times\mathsf{W}_a\times\mathsf{W}_s\times\mathsf{W}_r$, the cross stabilizer is much smaller $=\mathsf{H}_t\times\mathsf{S}_a\times\mathsf{W}_s\times\mathsf{W}_r$! This is dual to the fact that, for the split Cartan and real root e_i, $\mathrm{sgn}\circ e_i\colon H\to\mathbb{R}^\times\to$

$\{\pm 1\}$ is non-trivial! The good infinitesimal character takes half-integer values. So s_{e_i} never cross stabilizing a regular character at these infinitesimal characters]

- Suppose $\star = C^*$. Define

$$\mathcal{C}_b^n := \begin{cases} \mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}_n} \widetilde{\mathrm{sgn}} & \text{if } n = 2t \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{C}_g^{2p,2q} := \bigoplus_{(t+s,t+r)=(p,q)} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_s\times\mathsf{W}_t}^{\mathsf{W}_{p+q}} \widetilde{\mathrm{sgn}} \otimes \mathrm{sgn} \otimes \mathrm{sgn}$$

- Suppose $\star = C$. Define

$$\mathcal{C}_b^n := \bigoplus_{2t+a=n} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}_n} \widetilde{\mathrm{sgn}} \otimes 1$$

$$\mathcal{C}_g^{n,n} := \bigoplus_{2t+a+c+d=n} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a\times\mathsf{W}_c\times\mathsf{W}_d}^{W_n} \widetilde{\mathrm{sgn}} \otimes \mathrm{sgn} \otimes 1 \otimes 1$$

- Suppose $\star = \widetilde{C}$. Define

$$\mathcal{C}_b^n := \bigoplus_{2t+c+d=n} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_c\times\mathsf{W}_d}^{\mathsf{W}_n} \widetilde{\mathrm{sgn}} \boxtimes 1 \boxtimes 1$$

$$\mathcal{C}_g^{n,n} := \bigoplus_{2t+a+a'=n} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a\times\mathsf{S}_{a'}}^{\mathsf{W}_n} \widetilde{\mathrm{sgn}} \otimes \mathrm{sgn} \otimes 1$$

- Suppose $\star = D$ and $p+q = 2m$ is even. Define

$$\mathcal{C}_b^n := \bigoplus_{2t+a=n} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}_n} \widetilde{\mathrm{sgn}} \otimes 1$$

$$\mathcal{C}_g^{p,q} := \bigoplus_{\substack{2t+c+d+2r=p \\ 2t+c+d+2s=q}} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{W}_s\times\mathsf{W}_r\times\mathsf{W}'_c\times\mathsf{W}_d}^{\mathsf{W}_{(p+q)/2}} \widetilde{\mathrm{sgn}} \otimes \overline{\mathrm{sgn}} \otimes \overline{\mathrm{sgn}} \otimes 1 \otimes 1$$

- Suppose $\star = D^*$. Define

$$\mathcal{C}_b^n := \begin{cases} \mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}'_n} \widetilde{\mathrm{sgn}} & \text{if } n = 2t \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{C}_g^{n,n} := \bigoplus_{2t+a=n} \mathrm{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}'_n} \widetilde{\mathrm{sgn}} \otimes \mathrm{sgn}$$

Now assume $\mathrm{rank}_{\mathbb{C}} G_{\mathbb{C}} = n$. We identify $\mathfrak{h}^*$ with $\mathbb{C}^n$. Let Q be the root lattice in $\mathfrak{h}^*$ which is

$$Q = \begin{cases} \mathbb{Z}^n & \text{if } \star = B \\ \{ (a_i) \in \mathbb{Z}^n \mid \sum_{i=1}^n a_i \text{ is even} \} & \text{if } \star \in \{ C, \widetilde{C}, C^*, D, D^* \} \end{cases}$$

For $n_b, n_g \in \mathbb{N}$ such that $n_b + n_g = n$, we consider the lattice

$$\Lambda_{n_b,n_g} = \begin{cases} (\underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_b\text{-terms}}, \underbrace{0, \cdots, 0}_{n_g\text{-terms}}) + Q & \text{when } \star \in \{ C, C^*, D, D^* \} \\ (\underbrace{0, \cdots, 0}_{n_b\text{-terms}}, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_g\text{-terms}}) + Q & \text{when } \star \in \{ B, \widetilde{C} \}. \end{cases}$$

Clearly,

$$W_{\Lambda_{n_b,n_g}} = W_b \times W_g$$

with W_b and W_g defined by (6.1).

Here $\mathsf{PBP}_*(\_)$ denote the set of painted bi-partitions running over all relevant shapes.

We define

$$\mathrm{Sign}(G) := \begin{cases} (p,q) & \text{if } G = \mathrm{SO}(p,q) \\ (n,n) & \text{if } G = \mathrm{Sp}(2n,\mathbb{R}) \text{ or } \mathrm{Mp}(2n,\mathbb{R}) \\ (2p,2q) & \text{if } G = \mathrm{Sp}(p,q) \\ (n,n) & \text{if } G = \mathrm{O}^*(2n) \end{cases}$$

Proposition 6.3. *As $W_{\Lambda_{n_b,n_g}} := W_b \times W_g$-module, the coherent continuation representation $\mathrm{Coh}_{\Lambda_{n_b,n_g}}(G)$ is isomorphic to the restriction to $W_{\Lambda_{n_b,n_g}}$ of*

$$\begin{cases} \mathcal{C}_b^{n_b} \boxtimes \mathcal{C}_g^{p,q} & \text{if } \star \in \{ B, C, \widetilde{C}, C^*, D \} \\ \mathrm{Ind}_{\mathsf{W}'_{n_b}\times \mathsf{W}'_{n_g}}^{W''} (\mathcal{C}_b^{n_b} \boxtimes \mathcal{C}_g^{p,q}) & \text{if } \star\ D^* \end{cases}$$

with $(p,q) = \mathrm{Sign}(G) - (n_b, n_b)$ and

$$W'' := \left(\mathsf{W}_{n_b} \times \mathsf{W}_{n_g}\right) \cap \mathsf{W}'_{n_g+n_b} \text{when } \star = D^*.$$

Remark. When $\star \in \{ C^*, D^* \}$ and n_b is odd, $\mathrm{Coh}_{\Lambda_{n_b,n_g}}(G) = 0$ by the proposition.

Sketch of the proof. When G is linear, this can be calculated using [3], see also [68, Applications]. When G is the metaplectic group, this is a direct consequence of [84, 85]. We give a sketch of the argument. $\square$

We define

$$G'_n := \begin{cases} \mathrm{GL}(n,\mathbb{R}) & \text{when } \star \in \{ B, C, \widetilde{C}, D \}, \\ \mathrm{GL}(\frac{1}{2}n,\mathbb{H}) & \text{when } \star \in \{ C^*, D^* \}. \end{cases}$$

Proposition 6.4. *In all the cases,*

$$\mathsf{PBP}_{\star,b}(\check{\mathcal{O}}_b) = \mathrm{PP}_{G'}(\check{\mathcal{O}}'_b) = \mathrm{Unip}_{G'}(\check{\mathcal{O}}'_b).$$

Proof. Suppose $\star \in \{ C^*, D^* \}$. Then

$$\left|\mathsf{PBP}_{\star,b}(\check{\mathcal{O}}_b)\right| = \left|\mathrm{PP}_{G'}(\check{\mathcal{O}}'_b)\right| = \left|\mathrm{Unip}_{\check{\mathcal{O}}'_b}(G'_{n_b})\right| = 1.$$

Suppose $\star \in \left\{ B, C, \widetilde{C}, D \right\}$, $\tau_b = (\tau_{L,b}, \tau_{R,b})$ and $\tau'_b = \mathcal{O}'_b$ It is easy to see that we have a bijection:

$$\begin{array}{ccc} \mathsf{PBP}_{\star,b}(\check{\mathcal{O}}_b) & \longrightarrow & \mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b) \\ (\boldsymbol{\tau}_L, \boldsymbol{\tau}_R) & \mapsto & \boldsymbol{\tau}' \end{array}$$

where $\boldsymbol{\tau}'$ is defined by the condition that

$$\boldsymbol{\tau}_L(\mathbf{c}_j(\tau_{L,b}), j) = d \Longleftrightarrow \boldsymbol{\tau}'(\mathbf{c}_j(\tau'_b), j) = d.$$

[Now the claim follows for the fact that the bottom rows in $\boldsymbol{\tau}_L$ can be filled by $\bullet/c$ or d and

$$\mathbf{c}_i(\tau_{L,b}) = \mathbf{c}_j(\tau_{L,b}) \Leftrightarrow \mathbf{c}_i(\tau'_b) = \mathbf{c}_j(\tau'_b) \quad \forall i, j \in \mathbb{N}^+.$$

]

$\square$

Proposition 6.5. *In all the cases, we have*

$$\sum_{\wp \in \tilde{A}'(\check{\mathcal{O}})} [\tau_b \otimes \tau_\wp : \mathrm{Coh}_{\Lambda_{n_b,n_g}}(G)] = \left|\mathsf{PBP}_{\star,b}(\check{\mathcal{O}}_b)\right| \cdot \left|\mathsf{PBP}_{G_g}(\check{\mathcal{O}}_g)\right|.$$

Proof. Suppose $\star = C^*$. For the bad parity, $n_b = 2n_b'$ must be even.

$$\begin{aligned}\mathcal{C}_b^{n_b} &= \operatorname{Res}_{\mathsf{W}_{n_b}}^{\mathsf{W}'_{n_b}} \operatorname{Ind}_{\mathsf{H}_{n_b'}}^{\mathsf{W}_{n_b}} \widetilde{\operatorname{sgn}} \\ &= \bigoplus_{\sigma\in\operatorname{Irr}(\mathsf{S}_{n_b'})} \left((\sigma,\sigma)_I \oplus (\sigma,\sigma)_{II}\right).\end{aligned}$$

For the good parity,

$$\mathcal{C}_g^{2p,2q} = \bigoplus_{(t+s,t+r)=(p,q)} \bigoplus_{\sigma} \operatorname{Ind}_{\mathsf{W}_{2t}\times\mathsf{W}_s\times\mathsf{W}_r}^{\mathsf{W}_{p+q}} (\sigma,\sigma)\otimes\operatorname{sgn}\otimes\operatorname{sgn}$$

Now the branching rule implies the irreducible components of $\mathcal{C}_g^{2p,2q}$ are given by the dot-diagram attaching two columns on the right, which we mark them by s and r respectively.

Suppose $\star = C$

$$\begin{aligned}&[\tau_b \boxtimes \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}'_{n_b}\times\mathsf{W}_{n_g}} \\ =&[\operatorname{Ind}_{\mathsf{W}'_{n_b}\times\mathsf{W}_{n_g}}^{\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}} \tau_b \boxtimes \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}} \\ =&[\widetilde{\tau}_b \boxtimes \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}} \\ =&\#\mathsf{PBP}_\star(\check{\mathcal{O}}_b)\cdot\#\mathsf{PBP}_\star(\check{\mathcal{O}}_g;\wp)\end{aligned}$$

Suppose $\star = D^*$. Let

$$\widetilde{\tau}_b = \operatorname{Ind}_{\mathsf{W}'_{n_b}}^{\mathsf{W}_{n_b}} \tau_b \quad \text{and} \quad \widetilde{\tau}_\wp := (\imath_\wp, \jmath_\wp).$$

Note that $\widetilde{\tau}_\wp|_{W_g} = \tau_\wp$ form the explicit description of $\tau_\wp$ and $\mathbf{c}_1(\imath_\wp) \geq \mathbf{c}_1(\jmath_{sP})$.

Suppose $n_g = 0$ first and let $t = n_b/2$, then $\tau_b = (\tau_b', \tau_b')_I$ and

$$\begin{aligned}[\tau_b : \mathcal{C}_b]_{W_b} =&[\tau_b : \operatorname{Ind}_{\mathsf{H}_t}^{\mathsf{W}'_{n_b}} \widetilde{\operatorname{sgn}}] \\ =&[(\tau_b',\tau_b')_I : \bigoplus_\sigma (\sigma,\sigma)_I] \\ =& 1.\end{aligned}$$

Suppose $n_b = 0$ then

$$\begin{aligned}[\tau_\wp : \mathcal{C}_g]_{\mathsf{W}'_{n_g}} =&[\widetilde{\tau}_\wp|_{W_g} : \sum_{2t+a=n_g} \operatorname{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}'_{n_g}} \widetilde{\operatorname{sgn}}\otimes\operatorname{sgn}]_{\mathsf{W}'_{n_g}} \\ =&[\widetilde{\tau}_\wp : \sum_{2t+a=n_g} \operatorname{Ind}_{\mathsf{H}_t\times\mathsf{S}_a}^{\mathsf{W}'_{n_g}} \widetilde{\operatorname{sgn}}\otimes\operatorname{sgn}]_{\mathsf{W}_{n_g}} \\ =&\mathsf{PBP}_\star(\check{\mathcal{O}}_g;\wp)\end{aligned}$$

Now consider the general case, we assume n_b and n_g are both non-zero. We claim that $\operatorname{Ind}_{W_b\times W_g}^{W''} \tau_b\boxtimes\tau_\wp = (\widetilde{\tau}_b\otimes\widetilde{\tau}_\wp)|_{W''}$: first, the dimension of the two sides are equal ($W_b\times W_g$ has index 2 in W''); second,

$$\begin{aligned}&[\operatorname{Ind}_{W_b\times W_g}^{W''} \tau_b\boxtimes\tau_\wp : (\widetilde{\tau}_b\otimes\widetilde{\tau}_\wp)|_{W''}] \\ =&[\operatorname{Ind}_{\mathsf{W}'_{n_b}\times\mathsf{W}'_{n_g}}^{\mathsf{W}_{n_b}\times\mathsf{W}_{n_g}} \tau_b\boxtimes\tau_\wp : \widetilde{\tau}_b\otimes\widetilde{\tau}_\wp] \\ =&[\widetilde{\tau}_b\boxtimes\widetilde{\tau}_\wp \oplus \widetilde{\tau}_b\boxtimes(\widetilde{\tau}_\wp\otimes\breve{\operatorname{sgn}}) : \widetilde{\tau}_b\otimes\widetilde{\tau}_\wp] = 1\end{aligned}$$

where $\widetilde{\tau}_\wp\otimes\breve{\operatorname{sgn}}$ has the bipartition obtained by switching the left and right side of $\widetilde{\tau}_\wp$.

$$\begin{aligned}
&[\tau_b \boxtimes \tau_\wp : \mathrm{Ind}_{\mathsf{W}'_{n_b}\times \mathsf{W}'_{n_g}}^{W''} \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}'_{n_b}\times \mathsf{W}'_{n_g}} \\
=&[\mathrm{Ind}_{W_b\times W_g}^{W''} \tau_b \boxtimes \widetilde{\tau}_\wp : \mathrm{Ind}_{W_b\times W_g}^{W''} \mathcal{C}_b \boxtimes \mathcal{C}_g]_{W''} \\
=&[(\widetilde{\tau}_b \boxtimes \widetilde{\tau}_\wp)|_{W''} : \mathrm{Ind}_{W_b\times W_g}^{W''} \mathcal{C}_b \boxtimes \mathcal{C}_g]_{W''} \\
=&[\widetilde{\tau}_b \boxtimes \widetilde{\tau}_\wp : \mathrm{Ind}_{W_b\times W_g}^{\mathsf{W}_{n_b}\times \mathsf{W}_{n_g}} \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_g}\times \mathsf{W}_{n_b}} \\
=&\#\mathsf{PBP}_\star(\check{\mathcal{O}}_b)\cdot \#\mathsf{PBP}_\star(\check{\mathcal{O}}_g;\wp)
\end{aligned}$$

Suppose $\star = D$. Suppose that $n_g \neq 0$. Since $\mathbf{c}_1(\imath_\wp) > \mathbf{c}_1(\jmath_\wp)$, we have $\widetilde{\tau}_\wp^s := \widetilde{\tau}_\wp \otimes \widetilde{\mathrm{sgn}} \not\cong \widetilde{\tau}_\wp$.

$$\begin{aligned}
&[\tau_b \boxtimes \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}'_{n_b}\times \mathsf{W}'_{n_g}} \\
=&[\mathrm{Ind}_{\mathsf{W}'_{n_b}\times \mathsf{W}'_{n_g}}^{\mathsf{W}_{n_b}\times \mathsf{W}_{n_g}} \tau_b \boxtimes \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_b}\times \mathsf{W}_{n_g}} \\
=&[\widetilde{\tau}_b \boxtimes \widetilde{\tau}_\wp \oplus \widetilde{\tau}_b \boxtimes \widetilde{\tau}_\wp^s : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{W''} \\
=&[\widetilde{\tau}_b \boxtimes \widetilde{\tau}_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_g}\times \mathsf{W}_{n_b}} \\
=&\#\mathsf{PBP}_\star(\check{\mathcal{O}}_b)\cdot \#\mathsf{PBP}_\star(\check{\mathcal{O}}_g;\wp)
\end{aligned}$$

The terms involving $\widetilde{\tau}_\wp^s$ vanish since every irreducible component (σ_L, σ_R) in $\mathcal{C}_g$ satisfies $\sigma_L \supseteq \sigma_R$ but $\mathbf{c}_1(\imath_\wp) > \mathbf{c}_1(\jmath_\wp)$.

Suppose that $n_g = 0$.

$$\begin{aligned}
&[\tau_b : \mathcal{C}_b^{n_b}]_{\mathsf{W}'_{n_b}} \\
=&[\mathrm{Ind}_{\mathsf{W}'_{n_b}}^{\mathsf{W}_{n_b}} \tau_b : \bigoplus_{2t+a=n} \mathrm{Ind}_{\mathsf{H}_t\times \mathsf{S}_a}^{\mathsf{W}_n} \widetilde{\mathrm{sgn}} \otimes 1] \\
=&[\widetilde{\tau}_b : \bigoplus_{2t+a=n} \mathrm{Ind}_{\mathsf{H}_t\times \mathsf{S}_a}^{\mathsf{W}_n} \widetilde{\mathrm{sgn}} \otimes 1]
\end{aligned}$$

In any cases, the counting formula holds. [There is place to confuse: Why there shouldn't be double the size of special unipotent representations?

In fact, $\mathrm{AC}_{\mathbb{C}}(\pi)$ can only be the fixed type, say $\mathcal{O}_I$! Note that we fixed an infinitesimal character which has half-integral values. This choice implicitly force us to fix real Siegel parabolic when we do induction from GL! The non-trivial outer automorphism, say c, will permute the infinitesimal character to the another one and we then will have $\mathrm{AC}_{\mathbb{C}}({}^c\pi)i = \mathcal{O}_{II}$.

Using Barbasch's formula of wavefront, we see that the induction $\pi_I := \mathrm{Ind}_{\mathrm{GL}}^{\mathrm{SO}} \pi'$ must be irreducible, where π' is a unipotent repn. of GL. This will also implies $\mathrm{Ind}_{\mathrm{GL}}^{\mathrm{O}} \pi'$ is irreducible and restricted to two SO-modules, π_I and π_{II}.

]

Suppose $\star = \widetilde{C}$. Then

$$\begin{aligned}
&[\tau_b \boxtimes \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_b}\times \mathsf{W}_{n_g}} \\
=&[\tau_b \boxtimes \mathrm{Ind}_{\mathsf{W}'_{n_g}}^{\mathsf{W}_{n_g}} \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_b}\times \mathsf{W}_{n_g}} \\
=&\sum_{\wp\in \tilde{A}(\check{\mathcal{O}})} [\tau_b \boxtimes \widetilde{\tau}_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_b}\times \mathsf{W}_{n_g}} \\
=&\#\mathsf{PBP}_\star(\check{\mathcal{O}}_b)\cdot \#\mathsf{PBP}_\star(\check{\mathcal{O}}_g)
\end{aligned}$$

Suppose $\star = B^*$. Then

$$\begin{aligned}&\sum_{\wp\in \tilde{A}'(\check{\mathcal{O}})} [\tau_b \boxtimes \tau_\wp : \mathcal{C}_b \boxtimes \mathcal{C}_g]_{\mathsf{W}_{n_b}\times \mathsf{W}'_{n_g}}\\ =&\#\mathsf{PBP}_\star(\check{\mathcal{O}}_b)\cdot \#\mathsf{PBP}_\star(\check{\mathcal{O}}_g;\wp)\end{aligned}$$

□

6.3. **Reduction to the good parity case.** In this section, G is a classical group or metaplectic group.

Lemma 6.6. *Suppose $n_g = 0$. Then we have a bijection*

$$\begin{array}{rccc}\mathfrak{I}_b : & \mathrm{Unip}_{G'}(\check{\mathcal{O}}'_b) & \longrightarrow & \mathrm{Unip}_G(\check{\mathcal{O}})\\ & \pi' & \mapsto & \pi := \mathrm{Ind}_P^G \pi'\end{array}$$

where P is the standard parabolic subgroup of G whose Levi subgroup is isomorphic to G'. Let $\mathscr{O}'$ be the real nilpotent orbit in G' such that $\mathscr{O}'_{\mathbb{C}} = \mathcal{O}$. and $\mathscr{O}$ be the real induction of $\mathscr{O}'$ to G. Then

$$\mathrm{AC}(\pi) = \mathscr{O}$$

is multiplicity one.

Proof. Note that $\mathrm{WF}(\pi) = \mathscr{O}'$. It follows from Barabasch's formula on wavefront cycle that the associated variety of $\pi := \mathrm{Ind}_P^G \pi'$ is $\mathscr{O}$ with multiplicity one. This immediately implies that π is irreducible. In fact, if π is reducible, π must contain a irreducible sub-quotient with infinitesimal character $\lambda_{\check{\mathcal{O}}}$ and GK-dimension $< \frac{1}{2}\dim_{\mathbb{C}}\mathcal{O}$, this is contradict to Lemma 3.3. [First, it is not obvious to me that the wavefront of $\mathrm{Ind}_P^G \pi$ must be contained in $\mathrm{Ind}\,\mathrm{WF}(\pi)$. But it is clear that the leading term must be $\sum_{\mathscr{O} \text{ open in } \mathrm{WF}(\pi)} \mathrm{Ind}\,\mathscr{O}$. So the boundaries has less GK-dimension.

Suppose π_0 is the sub-quotient with less GK-dimension. On the other hand, the maximal primitive ideal $\mathcal{I}_{\check{\mathcal{O}}}$ with infinitesimal character must contains $\mathrm{Ann}\,\pi_0$. In other words, $\mathrm{AV}_{\mathbb{C}}(\pi_0) \subseteq \overline{\mathcal{O}}$ which implies GK-dimension of $\pi_0 \geq \frac{1}{2}\dim_{\mathbb{C}}\mathcal{O}$, a contradiction.]
Note that representations in $\mathrm{Unip}_{G'}(\check{\mathcal{O}}'_b)$ have distinct cuspidal data/Langlands parameter. This implies that $\mathrm{Ind}_P^G \pi'$ has distinct cuspidal data/Langlands when π' varies. Recall that $\star' \in \{A^{\mathbb{R}}, A^{\mathbb{C}}, A^{\mathbb{H}}\}$ depends on G. Therefore, $\mathfrak{I}_b$ is injective. The bijection follows from the counting inequality below:

$$\left|\mathrm{PP}_{\star'}(\check{\mathcal{O}}'_b)\right| = \left|\mathrm{Unip}_{G'}\right|(\check{\mathcal{O}}'_b) \leq \left|\mathrm{Unip}_G(\check{\mathcal{O}})\right| \leq \left|\mathsf{PBP}_{\star,b}(\check{\mathcal{O}}_b)\right| = \left|\mathrm{PP}_{\star'}(\check{\mathcal{O}}'_b)\right|.$$

□

Case by case, we set

$$(G_b, G_g) = \begin{cases}(\mathrm{SO}(n_b, n_b+1), \mathrm{SO}(p,q)) & \text{when } \star = B\\ (\mathrm{Sp}(2n_b,\mathbb{R}), \mathrm{Sp}(2n_g,\mathbb{R})) & \text{when } \star = C\\ (\mathrm{Sp}(n_b,n_b), \mathrm{Sp}(p,q)) & \text{when } \star = C^*\\ (\mathrm{Mp}(2n_b,\mathbb{R}), \mathrm{Mp}(2n_g,\mathbb{R})) & \text{when } \star = \widetilde{C}\\ (\mathrm{O}^*(n_b), \mathrm{O}^*(n_g)) & \text{when } \star = D^*\\ (\mathrm{SO}(n_b,n_b), \mathrm{SO}(p,q)) & \text{when } \star = D\end{cases}$$

Let G' be the Levi of the Siegel parabolic in G_b.

Proposition 6.7. *There is a bijection*

$$\begin{array}{rccc}\mathfrak{I}\colon & \mathrm{Unip}_{G'}(\check{\mathcal{O}}'_b)\times \mathrm{Unip}_{G_g}(\check{\mathcal{O}}_g) & \longrightarrow & \mathrm{Unip}_G(\check{\mathcal{O}})\\ & (\pi',\pi_0) & \mapsto & \pi'\rtimes\pi_0.\end{array} \tag{6.5}$$

First we translate the problem to regular infinitesimal character using the following lemma: The method was already carefully explained in [67] and see [33, Section 3] for a comprehensive account of the problem.

Lemma 6.8 (c. f. [33, Lemma 3.3]). *Suppose that*

(i) G_1 and G_2 are two real reductive groups in the Harish-Chandra class

(ii) there is an isomorphism

$$f\colon (\mathfrak{h}_1^a)^* \to (\mathfrak{h}_1^a)^*$$

of the dual of the abstract Cartans of G_1 and G_2;

(iii) $\lambda_1 \in (\mathfrak{h}_1^a)^*$ and $\lambda_2 = f(\lambda_1)$ are regular dominant elements;

(iv) f induces a bijection between $R^+_{[\lambda_1]}$ and $R^+_{[\lambda_2]}$, so that we can identify $W_{[\lambda_1]}$ with $W_{[\lambda_2]}$ via f;

Let Coh_1 be a $W_{[\lambda_1]}$ submodule of $\mathrm{Coh}_{[\lambda_1]}(G_1)$ such that $\mathrm{ev}_{\lambda_1}(\mathrm{Coh}_1)$ is spanned by irreducible G_1-modules. Suppose

$$\varphi\colon \mathrm{Coh}_1 \to \mathrm{Coh}_{[\lambda_2]}(G_2) \tag{6.6}$$

is an injection between $W_{[\lambda_1]} = W_{[\lambda_2]}$-modules such that

- *$\varphi(\Theta)(\lambda_2)$ is irreducible if $\Theta \in \mathrm{Coh}_1$ and $\Theta(\lambda_1)$ is irreducible.*

Then for any $\mu_1 \in [\lambda_1]$, the evaluation at μ_1 induces an injection

$$\varphi_{\mu_1}\colon \mathrm{ev}_{\mu_1}(\mathrm{Coh}_1) \longrightarrow \mathrm{ev}_{f(\mu_1)}(\mathrm{Coh}_{[\lambda_2]}(G_2)).$$

such that $\varphi_{\mu_1}(\pi)$ is irreducible if π is irreducible.

Proof. The injectivity of f_{μ_1} is clear from Lemma 2.9 and the injectivity of (6.6). [Note that $W_{[\lambda_1]}$ is a finite group!]

We now prove the second claim. It easy to reduce to the case when μ_1 is $R^+_{[\lambda_1]}$ dominant, c.f. [33, Lemma 3.3]. Let $\Theta \in \mathrm{Coh}_1$ such that $\Theta(\mu_1) = \pi$ and $\Theta(\lambda_1)$ is irreducible (the existence of Θ is an abstract property of the coherent continuation). By our assumption $\varphi(\Theta)(\lambda_1)$ is irreducible. Therefore $\varphi_{\mu_1}(\pi) := \varphi(\Theta)(f(\mu_1))$ must be irreducible since it is non zero by the first claim. □

In our case, we use the Kazhdan-Lusztig-Vogan theory to obtain the injection (6.6) and then reduce the problem to bad and good parts separately. Note that the method does not really relies on the Vogan duality and nor require the whole coherent continuation module are isomorphism.

In additional to (i)-(iv) in Lemma 6.8, we made the following assumptions (c. f. [33, §3E]):

(v) there is an injection

$$\varphi\colon B \to \mathcal{P}_{\lambda_2}(G_2)$$

where $B \subset \mathcal{P}_{\lambda_1}(G_1)$ is a union of blocks of conjugate classes of regular characters with infinitesimal character λ_1.

(vi) for $\gamma_1 \in B$ and $\gamma_2 = \varphi(\gamma_1) \in \mathcal{P}_{\lambda_2}(G_2)$ the following conditions are satisfied

(a) $f \circ \theta_{\gamma_1} = \theta_{\gamma_2} \circ f$ where θ_{γ_i} are the Cartan involution induced on the corresponding abstract root systems. [This condition implies that, the notion of compact/complex/real and $\alpha \in R^+(\gamma), \theta(\alpha) \notin R^+(\gamma)$ are preserved by f. In particular, the integral length function l^I (defined up to a shifting) of G_1 and G_2 can be uniformly identified.]

(b) For simple roots in $R^+_{\lambda_i}$, the notions of noncompact/real type I/II are preserved by f: α_1 is noncompact type I if and only if $\alpha_2 := f(\alpha_1)$ is noncompact type I, etc;

(c) The cross actions are compatible:

$$f(w \times [\gamma_1]) = f(w) \times [f(\gamma_1)] \qquad \forall \gamma_1 \in B_1, w \in W_{[\lambda_1]}.$$

(d) The Cayley transforms are compatible:

$$f(c^{\alpha_1}(\gamma_1)) = c^{f(\alpha_1)}(f(\gamma_1)) \qquad \forall \gamma_1 \in B_1, \alpha_1 \text{is noncompact imaginary}$$

and

$$f(c_{\alpha_1}(\gamma_1)) = c_{f(\alpha_1)}(f(\gamma_1))$$

for all $\gamma_1 \in B_1, \alpha_1$ is real and satisfies parity condition.

Under the above assumptions, let $\mathcal{G}_B$ be the span of $\{\, \bar{\pi}_\gamma \mid \gamma \in B \,\}$ in the Grothendieck group, and $\mathrm{Coh}_1 := {\mathrm{ev}_\lambda}^{-1}(\mathcal{G}(B))$. Then φ is given by sending Θ_{γ_1} to $\Theta_{f(\gamma_1)}$. In fact, the above map on coherent continuation representations can be lifted to maps between Hecke-modules and the validity of Kazhdan-Lusztig-Vogan conjecture implies the preservation of irreducibility of φ at λ_1.

We set $G_1 := G_b \times G_g$ and $G_2 := G$. There is a natural map

$$f\colon G_1 = G_b \times G_g \longrightarrow G = G_2.$$

The map is given by [33, §3G] when $\star = B$ and given by [85, §5] when $\star = \widetilde{C}$. In the other cases, they are natural embeddings. The maps between abstract Cartans $f\colon (\mathfrak{h}_1^a)^* \to (\mathfrak{h}_1^a)^*$ are given by putting the coordinate of G_b before G_g.

Let $\lambda \in (\mathfrak{h}_1^a)^*$ be a regular dominant element in $[\lambda_{\check{\mathcal{O}}}]$. Then $\lambda_1 := f^{-1}(\lambda)$ is regular dominant for G_1. Let $B = \mathcal{P}_{\lambda_1}(G_1)$ in all the case. The map φ is given by the natural maps between real Cartans. This is clear when f is an embedding. See [33, §3] when G is a special orthogonal group and [85, §5] when G is a metaplectic group.

Proof of Proposition 6.7: the injectivity of $\mathfrak{I}$. Let $\mu_1 = f^{-1}(\lambda_{\check{\mathcal{O}}})$. Since coherent continuation is compatible with induction [96, Proposition 7.4.1], we have the following commutative diagram

$$\begin{tikzcd}
 & \mathrm{Coh}_{[\lambda_b']}(G_b') \otimes \mathrm{Coh}_{[\lambda_g]}(G_g) \arrow[dl, "\mathrm{Ind}_{P_b}^{G_b} \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{Ind}_P^G"] & \\
\mathrm{Coh}_{[\lambda_b]}(G_b) \otimes \mathrm{Coh}_{[\lambda_g]}(G_g) \arrow[rr, "\varphi"] \arrow[d, "\mathrm{ev}_{\mu_1}"'] & & \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) \arrow[d, "\mathrm{ev}_{\check{\mathcal{O}}}"] \\
\mathcal{G}_{\mu_1}(G_b \times G_g) \arrow[rr, "\varphi_{\mu_1}"] & & \mathcal{G}_{\lambda_{\check{\mathcal{O}}}}
\end{tikzcd}$$

Here P_b is the parabolic subgroup of G_b whose Levi subgroup is isomorphic to G_b' and P is the parabolic of G whose Levi subgroup is isomorphic to $G_b' \times G_g$. The horizontal line φ is an injection by the above setting, which implies φ_{μ_1} is injective by Lemma 6.8. Note that φ_{μ_1} sends $\mathrm{Ind}_{P_b}(\pi') \otimes \pi_0$ to $\mathrm{Ind}_P^G(\pi' \otimes \pi_0)$. [Let $\Theta' \otimes \Theta_0 \in \mathrm{Coh}_{[\lambda_b']}(G_b') \otimes \mathrm{Coh}_{[\lambda_g]}(G_g)$ be coherent family such that $\Theta' \otimes \Theta_0(\lambda_{\check{\mathcal{O}}}) = \pi' \otimes \pi_0$. Now $\mathrm{Ind}_{P_b}(\pi') \otimes \pi_0 = (\mathrm{Ind}_{P_b} \otimes \mathrm{id})(\Theta' \otimes \Theta_0)(\mu_1)$ and $\mathrm{Ind}_P(\pi' \otimes \pi_0) = \varphi(\mathrm{Ind}_{P_b} \otimes \mathrm{id}(\Theta' \otimes \Theta_0))(\lambda_{\check{\mathcal{O}}}) = (\mathrm{Ind}_P^G)(\Theta' \otimes \Theta_0)(\lambda_{\check{\mathcal{O}}})$] Now the proposition follows from Lemma 6.6. $\square$

Lemma 6.9. *We have*

$$\left|\mathrm{Unip}_G(\check{\mathcal{O}})\right| = \left|\mathrm{Unip}_{G_b}(\check{\mathcal{O}}_b)\right| \cdot \left|\mathrm{Unip}_{G_g}(\check{\mathcal{O}}_g)\right|$$

Proof. It suffice to assume n_b and n_g are both non-zero Fix a regular dominant infinitesimal character $\lambda \in [\lambda_{\check{\mathcal{O}}}]$ and write $(\lambda_b, \lambda_g) := f^{-1}(\lambda)$. It requires a some more precise information about the blocks/cell of G. Suppose $\star \neq D^*$. Then

$$\varphi\colon \mathcal{P}_{\lambda_b}(G_b) \times \mathcal{P}_{\lambda_g}(G_g) \longrightarrow \mathcal{P}_{\lambda}(G)$$

is a bijection. In particular each Harish-Chandra in $\mathcal{P}_\lambda(G)$ is a product of cells in $\mathcal{P}_{\lambda_b}(G_b)$ and cells in $\mathcal{P}_{\lambda_g}(G_g)$. By Lemma 4.6, this implies φ restricted to an isomorphism

$$\mathrm{Coh}_{[\lambda_b],\overline{\mathcal{O}}_b}(G_g) \otimes \mathrm{Coh}_{[\lambda_g],\overline{\mathcal{O}}_g}(G_b) \xrightarrow{\varphi} \mathrm{Coh}_{[\lambda],\overline{\mathcal{O}}}(G).$$

g and the lemma follows.

Now consider the case where $\star = D^*$. In this case, $\mathcal{P}_\lambda(G)$ has 2 blocks. Meanwhile $\mathcal{P}_{\lambda_b}(G_b)$ and $\mathcal{P}_{\lambda_g}(G_g)$ have only 1 block. Let $B_1 := (\mathcal{P}_{\lambda_b}(G_b) \times \mathcal{P}_{\lambda_g}(G_g))$ and B_2 be the other block. For any W'_n-module τ, let τ^s denote the twist of τ by any non-trivial element in $\mathsf{W}_n/\mathsf{W}'_n$.

Then

$$\mathrm{Coh}_{B_1} \cong \mathcal{C}_b \otimes \mathcal{C}_g = \mathrm{Coh}_{[\lambda_b],\overline{\mathcal{O}}_b}(G_g) \otimes \mathrm{Coh}_{[\lambda_g],\overline{\mathcal{O}}_g}(G_b),$$

$\mathrm{Coh}_{B_2} \cong \mathcal{C}_b^s \otimes \mathcal{C}_g^s$ and

$$\mathrm{Coh}_{[\lambda]}(G) = \mathrm{Coh}_{B_1} \oplus \mathrm{Coh}_{B_2}.$$

Clearly $\mathrm{Coh}_{[\lambda],\overline{\mathcal{O}}}$ is compatible with the above decomposition and we have

$$\mathrm{Coh}_{[\lambda_b],\overline{\mathcal{O}}_b}(G_g) \otimes \mathrm{Coh}_{[\lambda_g],\overline{\mathcal{O}}_g}(G_b) \xrightarrow{\varphi} \mathrm{Coh}_{[\lambda],\overline{\mathcal{O}}}(G) \cap \mathrm{Coh}_{B_1}.$$

Observe that $[\tau_b : \mathcal{C}_b^s] = 0$. Therefore

$$\begin{aligned}
\left|\mathrm{Unip}_G(\check{\mathcal{O}})\right| &= \sum_{\tau_b \boxtimes \tau_g \in {}^L\mathscr{C}_{\check{\mathcal{O}}}} [\tau_b \otimes \tau_g : \mathrm{Coh}_{[\lambda],\overline{\mathcal{O}}}(G) \cap \mathrm{Coh}_{B_1}] \\
&\quad + \sum_{\tau_b \boxtimes \tau_g \in {}^L\mathscr{C}_{\check{\mathcal{O}}}} [\tau_b \otimes \tau_g : \mathrm{Coh}_{[\lambda],\overline{\mathcal{O}}}(G) \cap \mathrm{Coh}_{B_2}] \\
&= \sum_{\tau_b \boxtimes \tau_g \in {}^L\mathscr{C}_{\check{\mathcal{O}}}} [\tau_b \otimes \tau_g : \mathrm{Coh}_{[\lambda_b],\overline{\mathcal{O}}_b}(G_g) \otimes \mathrm{Coh}_{[\lambda_g],\overline{\mathcal{O}}_g}(G_b)] \\
&= \left|\mathrm{Unip}_{G_b}(\check{\mathcal{O}}_b)\right| \cdot \left|\mathrm{Unip}_{G_g}(\check{\mathcal{O}}_g)\right|
\end{aligned}$$

□

Proof of Proposition 6.7: the bijectivity of $\mathfrak{I}$. Now the bijectivity of $\mathfrak{I}$ follows by the counting in Lemma 6.6 and Lemma 6.9. □

7. Some case by case computation.

The below will be merged with the definition of painted bi-partition and the computation of coherent continuation representations in various cases.

7.1. Counting special unipotent representation of $G = \mathrm{Sp}(p,q)$. .

The dual group of $G_{\mathbb{C}} = \mathrm{Sp}(2n,\mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{SO}(2n+1,\mathbb{C})$.

Let

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_b) = \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath,\jmath) = \tau_b \\ \mathrm{Im}\mathcal{P} \subset \{\bullet\}, \mathrm{Im}\mathcal{Q} \subset \{\bullet\} \end{array} \right\}$$

Proposition 7.1. *Suppose* $\check{\mathcal{O}} \in \mathrm{Nil}(\mathrm{SO}(2n+1,\mathbb{C}))$ *with decomposition* $\check{\mathcal{O}} = \check{\mathcal{O}}_b \overset{r}{\sqcup} \check{\mathcal{O}}_g$.

(i) *The size of* $\mathrm{Unip}_\star(\check{\mathcal{O}})$ *is counted by* $\mathsf{PBP}_\star(\check{\mathcal{O}}_g)$.

(ii) The set $\mathsf{PBP}_\star(\check{\mathcal{O}}_g)$ *is non-empty only if*

$$\mathbf{r}_{2i-1}(\check{\mathcal{O}}) > \mathbf{r}_{2i}(\check{\mathcal{O}})$$

for each $i \in \mathbb{N}^+$ *with* $\mathbf{r}_{2i}(\check{\mathcal{O}}) > 0$.

(iii) We have

$$\left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g)\right| = |\mathrm{Nil}_{C^*}(\mathcal{O}_g)|$$

with $\mathcal{O}_g := d_{\mathrm{BV}}(\check{\mathcal{O}}_g)$.

(iv) When $\mathrm{Unip}_{C^*}(\check{\mathcal{O}}) \neq \emptyset$, *there is a bijection*

$$\begin{array}{ccccc} \mathrm{Nil}_{C^*}(\mathcal{O}_g) & \longrightarrow & \mathrm{Unip}_{C^*}(\check{\mathcal{O}}_g) & \longrightarrow & \mathrm{Unip}_{C^*}(\check{\mathcal{O}}) \\ \mathscr{O}_g & \mapsto & \pi_{\mathscr{O}_g} & \mapsto & \pi' \rtimes \pi_{\mathscr{O}_g}. \end{array}$$

Here π' *is the unique special unipotent representation of* $\mathrm{GL}_p(\mathbb{H})$ *with associated variety* $\mathcal{O}_b$, $\pi_{\mathscr{O}_g}$ *is the unique special unipotent representation of* $\mathrm{Sp}(p,q)$ *with associated variety* $\mathscr{O}_g$ *such that* $\mathrm{Sign}(\mathscr{O}_g) = (2p, 2q)$, *and* $\pi' \rtimes \pi_{\mathscr{O}_g}$ *denote the parabolic induction.*

Proof. (i) Suppose $(2i-1, 2i) \in \wp \subset \mathfrak{P}(\check{\mathcal{O}}_g)$. Let $(\imath, \jmath) := \tau_\wp$. Then $\mathbf{c}_i(\imath) > \mathbf{c}_i(\jmath)$ which implies that $\mathsf{PBP}_{\star,\wp}(\check{\mathcal{O}}_g) = \emptyset$.

(ii) The first claim is similar to part (i), see [9, Proposition 10.1].

(iii) This is [68, Theorem 6], also see [9].

(iv) It follows from **??** and the Vogan duality. See appendix.

□

[Let $W = W(G_\mathbb{C})$ where $G_\mathbb{C}$ is natrualy embeded in $\mathrm{GL}(n,\mathbb{C})$. Let $s_{\varepsilon_i i, \varepsilon_j j}$ be the permutation matrix of index i, j, where the (i,j)-th entry is ε_i, the (j,i)-th entry is ε_j and the other place is the identity matrix. Let $w^\epsilon_{i,\pm j}$ be the element such that

- it is in $G_\mathbb{C}$ and entries in $\{0\} \cup \mu_4$
- it lifts the element $e_i \leftrightarrow \pm e_j$ in the Weyl group.
- $(w^\epsilon_{i,\pm j})^2 = \epsilon 1 \in \{\pm 1\}$.

Let

$$h^+_{\pm i} = \mathrm{diag}(1, \cdots, 1, \pm 1, 1, \cdots, 1)$$

where ± 1 is the i-th place. Let

$$h^-_{\pm i} = \mathrm{diag}(1, \cdots, 1, \pm\sqrt{-1}, 1, \cdots, 1)$$

where $\pm\sqrt{-1}$ is the i-th place. Let $e_{\pm i} = s_{\pm i, \pm(n-i+1)}$.

Let $\breve{w}^\pm_{i,\pm j}$ be the lift of $e_i \leftrightarrow \pm e_j$ such that

- it is in $G_\mathbb{C}$ and entries in $\{0\} \cup \mu_4$
- it lifts the element $e_i \leftrightarrow \pm e_j$ in the Weyl group.
- $(w^\epsilon_{i,\pm j})^2|_{[i,j]} = \epsilon 1 \in \{\pm 1\}$ here "$|_{[i,j]}$" means restricts on the $e_i, e_j, -e_i, -e_j$-weights space.

Let

$$\begin{aligned} x_{b,s,r} &= w^+_{1,2} \cdots w^+_{2b-1,2b}\, h^+_{2b+1} \cdots h^+_{2b+s} \\ y^+_{b,s,r} &= \breve{w}^+_{1,-2} \cdots \breve{w}^+_{2b-1,-2b}\, e_{2b+1} \cdots e_{2b+s+r} \\ y^-_{b,s,r} &= \breve{w}^-_{1,-2} \cdots \breve{w}^-_{2b-1,-2b} \end{aligned}$$

We compute the parameter space

$$\mathcal{Z}_g = \bigcup_{2b+s+r=m} W_m \cdot (x^+_{b,s,r}, y^+_{b,s,r})$$

$$\mathcal{Z}_b = \bigcup_{2b=m} W_m \cdot (x^+_{b,0,0}, y^-_{b,0,0})$$

]

7.2. **Counting special unipotent representation of $G = \mathrm{Sp}(2n,\mathbb{R})$.** In this section, we consider the case where $\star = C$.

Recall (7.2) for the definition of the lattice Λ_{n_b,n_g}.

Lemma 7.2. *We have the following formula on the coherent continuation representations based on Λ_{n_b,n_g}:*

$$\mathrm{Coh}_{\Lambda_{n_b,n_g}}(\mathrm{Sp}(2n,\mathbb{R})) = \mathcal{C}_g \otimes \mathcal{C}_b$$

with

$$\mathcal{C}_g = \bigoplus_{2t+a+c+d=n_g} \mathrm{Ind}_{H_t\times S_a\times W_c\times W_d}^{W_{n_g}} 1\otimes \mathrm{sgn}\otimes 1\otimes 1$$

$$\mathcal{C}_b = \mathrm{Res}_{W_{n_b}}^{W'_{n_b}}\left(\bigoplus_{2t+a=n_b} \mathrm{Ind}_{H_t\times S_a}^{W_{n_b}} \widetilde{\mathrm{sgn}}\otimes 1\right)$$

□

[Consider $\mathcal{C}_b$. The real Cartan must be $\mathbb{C}^{\times t}\times \mathbb{R}^{\times t}$ In real Weyl group is $H_t\times W_a$ generated by The H_t action is known. W_a is generated by the reflections of $e_i\pm e_j$ $n-2t<i\neq j\leq n$. We consider the regular character $\gamma = *\otimes \underbrace{||^{\frac12}\otimes\cdots\otimes ||^{\frac12}}_{a\text{-terms}}$. The cross action of $s_{e_{n-1}+e_n}$ is given by

$$\begin{aligned} s_{e_{n-1}+e_n}\times\gamma &= *\otimes \underbrace{||^{\frac12}\otimes\cdots\otimes ||^{\frac12}\otimes \mathrm{sgn}\,||^{-\frac12}\otimes \mathrm{sgn}\,||^{-\frac12}}_{a\text{-terms}}\\ &\neq s_{e_{n-1}+e_n}\cdot\gamma\\ &= *\otimes \underbrace{||^{\frac12}\otimes\cdots\otimes ||^{\frac12}\otimes ||^{-\frac12}\otimes ||^{-\frac12}}_{a\text{-terms}}. \end{aligned}$$

Now we see that the cross stabilizer should be $H_t\times S_a$.]

Now $[\tau_b : \mathcal{C}_b]$ is counted by the size of the following set

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_b) := \left\{ (\imath,\jmath,\mathcal{P},\mathcal{Q}) \;\middle|\; \begin{array}{l}(\imath,\jmath)=\tau_b\\ \mathrm{Im}\mathcal{P}\subset\{\bullet,c\},\mathrm{Im}\mathcal{Q}\subset\{\bullet,d\}\end{array}\right\}$$

and for each $\wp\subset\mathfrak{P}(\check{\mathcal{O}}_g)$, the multiplicity $[\tau_\wp:\mathcal{C}_g]$ is counted by the size of

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g,\wp) := \left\{ (\imath,\jmath,\mathcal{P},\mathcal{Q}) \;\middle|\; \begin{array}{l}(\imath,\jmath)=\tau_\wp\\ \mathrm{Im}\mathcal{P}\subset\{\bullet,r,c,d\},\mathrm{Im}\mathcal{Q}\subset\{\bullet,s\}\end{array}\right\}$$

and for each $\wp$

Recall the definition of $\mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b)$ in (5.1).

We define

$$\mathsf{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}_g) := \mathsf{PBP}_\star(\check{\mathcal{O}}_g)\times \overline{\mathsf{A}}(\check{\mathcal{O}}_g).$$

Now we consider the good parity part. We defer the proof of the following lemma in the Appendix C

Lemma 7.3. *For each $\wp\in\subset\mathfrak{P}(\check{\mathcal{O}}_g)$, we have*

$$\left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g,\wp)\right| = \left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g,\emptyset)\right|.$$

In particular, we have

$$\mathrm{Unip}_\star(\check{\mathcal{O}}_g) = \left|\mathsf{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}_g)\right|.$$

7.3. **Type $\widetilde{C}$.** The left cell.

Lemma 7.4. *Suppose $\star = \widetilde{C}$, $\check{\mathcal{O}}_b$ has $2k$ rows. Here each row in $\check{\mathcal{O}}_b$ has odd length and each row in $\check{\mathcal{O}}_g$ has even length, and $W_{[\lambda_{\check{\mathcal{O}}}]} = W_b \times W'_g$ where $b = \frac{1}{2}\left|\check{\mathcal{O}}_b\right|$ and $g = \frac{1}{2}\left|\check{\mathcal{O}}_g\right|$. Let*

$$\tau_b = \Big((\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)+1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)+1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)+1}{2}),$$
$$(\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)-1}{2}, \frac{\mathbf{r}_4(\check{\mathcal{O}}_b)-1}{2}, \cdots, \frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)-1}{2}) \Big) \in \mathrm{Irr}(W_b).$$

Set

$$\mathfrak{P}(\check{\mathcal{O}}_g) = \{\, (2i-1, 2i) \mid \mathbf{r}_{2i-1}(\check{\mathcal{O}}_g) > \mathbf{r}_{2i}(\check{\mathcal{O}}_g), \text{ and } i \in \mathbb{N}^+ \,\}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}}) = \mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

For $\wp \in \overline{\mathsf{A}}(\check{\mathcal{O}})$, let

$$\tau_\wp := (\imath, \jmath) \in \mathrm{Irr}(W'_g)$$

such that for all $i \geq 1$

$$(\mathbf{c}_i(\imath), \mathbf{c}_i(\jmath)) := \begin{cases} (\frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g)) & \text{if } (2i-1, 2i) \notin \wp, \\ (\frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i-1}(\check{\mathcal{O}}_g)) & \text{otherwise.} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \overline{\mathsf{A}}(\check{\mathcal{O}}) & \longrightarrow & {}^{L}\mathscr{C}(\check{\mathcal{O}}) \\ \wp & \mapsto & \tau_b \otimes \tau_\wp, \end{array}$$

such that $\tau_b \otimes \tau_\wp$ is the special representation in ${}^{L}\mathscr{C}(\check{\mathcal{O}})$.

We remark that if $\mathfrak{P}(\check{\mathcal{O}}_g) = \emptyset$ the the representation $\tau_\emptyset$ has label I.

[The bad parity part is the same as the case when $\star = B$.

For the good parity part, note that the trivial representation of the trivial group has symbol

$$\begin{pmatrix} 0, 1, \cdots, r \\ 0, 1, \cdots, r \end{pmatrix}.$$

Here we assume $\check{\mathcal{O}}_g$ has at most $2r$ rows.

Now the bifurcation happens for the odd length column.]

The coherent continuation representation.

Fix n_b, n_g such that $n_b + n_g = n$. Let

$$\Lambda_{n_b,n_g} = (\underbrace{0, \cdots, 0}_{n_b\text{-terms}}, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_g\text{-terms}}) + Q. \tag{7.1}$$

We use Renard-Trapa's result.

Lemma 7.5. *We have the following formula on the coherent continuation representations based on Λ_{n_b,n_g}:*

$$\bigoplus_{p+q=n} \mathrm{Coh}_{\Lambda_{n_b,n_g}}(\mathrm{Mp}(2n, \mathbb{R})) \cong \mathcal{C}_b \otimes \mathcal{C}_g.$$

with

$$\mathcal{C}_g = \mathrm{Res}^{W'_{n_g}}_{W_{n_g}} \left(\bigoplus_{\substack{t,a,a'\in\mathbb{N}\\ 2t+a+a'=n_g}} \mathrm{Ind}^{W_{n_b}}_{H_t\times S_a\times S_{a'}} \widetilde{\mathrm{sgn}}\otimes 1\otimes \mathrm{sgn} \right)$$

$$\mathcal{C}_b = \bigoplus_{\substack{t,c,d\in\mathbb{N}\\ 2t+c+d=n_b}} \mathrm{Ind}^{W_{n_b}}_{H_t\times W_c\times W_d} \widetilde{\mathrm{sgn}}\otimes 1\otimes 1$$

Proof. This is contained in [84, 85]. By [85, Theorem 5.2] and [85, Corollary 4.5 (2)], $\mathrm{Coh}_{\Lambda_{n_b,n_g}}(\mathrm{Mp}(2n,\mathbb{R}))$ is dual to $\mathcal{C}_g\otimes\check{\mathcal{C}}_b$ where $\mathcal{C}_g$ is the coherent continuation representation based on the lattice Λ_{0,n_g} and

$$\begin{aligned}\check{\mathcal{C}}_{2n_b,\mathbb{R}} &\cong \bigoplus_{\substack{p,q\in\mathbb{N}\\ p+q=n_b}} \mathrm{Coh}_{\Lambda_{n_b,0}}(\mathrm{Sp}(p,q))\\ &= \bigoplus_{\substack{t,c,d\in\mathbb{N}\\ 2t+c+d=n_b}} \mathrm{Ind}^{W_{n_b}}_{H_t\times W_c\times W_d} \widetilde{\mathrm{sgn}}\otimes \mathrm{sgn}\otimes \mathrm{sgn}\end{aligned}$$

The main result in [84] implies that

$$\mathcal{C}_g = \mathrm{Res}^{W'_{n_g}}_{W_{n_g}} \left(\bigoplus_{\substack{t,a,a'\in\mathbb{N}\\ 2t+a+a'=n_g}} \mathrm{Ind}^{W_{n_b}}_{H_t\times S_a\times S_{a'}} \widetilde{\mathrm{sgn}}\otimes 1\otimes \mathrm{sgn} \right)$$

is self-dual.

Tensor with the sign representation yields the lemma. □

7.4. **Type** B. The dual group of $G_{\mathbb{C}}=\mathrm{SO}(2n+1,\mathbb{C})$ is $\check{G}_{\mathbb{C}}=\mathrm{Sp}(2n,\mathbb{C})$. The odd is the bad parity, and even is the good parity.

Fix n_b, n_g such that $n_b+n_g=n$. Let

$$\Lambda_{n_b,n_g} = (\underbrace{\tfrac{1}{2},\cdots,\tfrac{1}{2}}_{n_b\text{-terms}},\underbrace{0,\cdots,0}_{n_g\text{-terms}})+Q. \tag{7.2}$$

Suppose $\check{\mathcal{O}}\in\mathrm{Nil}(\mathrm{SO}(2n+1,\mathbb{C}))$ with decomposition $\check{\mathcal{O}}=\check{\mathcal{O}}_b\overset{r}{\sqcup}\check{\mathcal{O}}_g$.

Let $\check{\mathcal{O}}'_b$ be the Young diagram such that $\mathbf{r}_i(\check{\mathcal{O}}'_b)=\mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ and $\mathcal{O}'_b$ be the transpose of $\check{\mathcal{O}}'_b$.

7.5. **The left cell.**

Lemma 7.6. *Suppose* $\star=B$, $\check{\mathcal{O}}_b$ *has* $2k$ *rows. Here each row in* $\check{\mathcal{O}}_b$ *has odd length and each row in* $\check{\mathcal{O}}_g$ *has even length, and* $W_{[\lambda_{\check{\mathcal{O}}}]}=W_b\times W_g$ *where* $b=\frac{1}{2}\left|\check{\mathcal{O}}_b\right|$ *and* $g=\frac{1}{2}\left|\check{\mathcal{O}}_g\right|$. *Let*

$$\begin{aligned}\tau_b = \Bigg(&(\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)+1}{2},\frac{\mathbf{r}_4(\check{\mathcal{O}}_b)+1}{2},\cdots,\frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)+1}{2}),\\ &(\frac{\mathbf{r}_2(\check{\mathcal{O}}_b)-1}{2},\frac{\mathbf{r}_4(\check{\mathcal{O}}_b)-1}{2},\cdots,\frac{\mathbf{r}_{2k}(\check{\mathcal{O}}_b)-1}{2})\Bigg)\in\mathrm{Irr}(W_b).\end{aligned}$$

Set

$$\mathfrak{P}(\check{\mathcal{O}}_g)=\{\,(2i,2i+1)\mid \mathbf{r}_{2i+1}(\check{\mathcal{O}}_g)>\mathbf{r}_{2i}(\check{\mathcal{O}}_g),\ \text{and } i\in\mathbb{N}^+\,\}$$

and $\overline{\mathsf{A}}(\check{\mathcal{O}})=\mathbb{F}_2[\mathfrak{P}(\check{\mathcal{O}}_g)]$.

For $\wp \in \overline{\mathsf{A}}(\check{\mathcal{O}})$, let

$$\tau_\wp := (\imath, \jmath) \in \mathrm{Irr}(W_b)$$

such that

$$\mathbf{c}_1(\jmath) := \tfrac{1}{2}\mathbf{r}_1(\check{\mathcal{O}}_g)$$

and for all $i \geq 1$

$$(\mathbf{c}_i(\imath), \mathbf{c}_{i+1}(\jmath)) := \begin{cases} (\frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g)) & \text{if } (2i, 2i+1) \notin \wp, \\ (\frac{1}{2}\mathbf{r}_{2i+1}(\check{\mathcal{O}}_g), \frac{1}{2}\mathbf{r}_{2i}(\check{\mathcal{O}}_g)) & \text{otherwise.} \end{cases}$$

Then we have the following bijection

$$\begin{array}{ccc} \overline{\mathsf{A}}(\check{\mathcal{O}}) & \longrightarrow & {}^L\mathscr{C}(\check{\mathcal{O}}) \\ \wp & \mapsto & \tau_b \otimes \tau_\wp, \end{array}$$

such that $\tau_b \otimes \tau_\wp$ is the special representation in ${}^L\mathscr{C}(\check{\mathcal{O}})$.

7.6. **Counting special unipotent representation of $G = \mathrm{SO}(2p+1, 2q)$ with $p+q = n$.** .

The dual group of $G_{\mathbb{C}} = \mathrm{SO}(2n, \mathbb{C})$ is $\check{G}_{\mathbb{C}} = \mathrm{SO}(2n+1, \mathbb{C})$.

Let

$$\Lambda_{n_1,n_2} = (\underbrace{0, \cdots, 0}_{n_1\text{-terms}}, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{n_2\text{-terms}},)$$

We set $(2n_g, 2n_b) = (\left|\check{\mathcal{O}}_g\right|, \left|\check{\mathcal{O}}_b\right|)$.

Lemma 7.7. *We have the following formula on the coherent continuation representations based on Λ_{n_1,n_2}:*

$$\bigoplus_{p+q=n} \mathrm{Coh}_{\Lambda_{n_1,n_2}}(\mathrm{SO}(2p+1, 2q)) \cong \mathcal{C}_b \otimes \mathcal{C}_g.$$

with

$$\mathcal{C}_g = \bigoplus_{\substack{t,s,r\in\mathbb{N} \\ 2t+s+r=n_g}} \mathrm{Ind}_{H_t \times S_a \times W_s \times W_r}^{W_{n_g}} \widetilde{\mathrm{sgn}} \otimes 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn}$$

$$\mathcal{C}_b = \bigoplus_{\substack{t,c,d\in\mathbb{N} \\ 2t+c+d=n_b}} \mathrm{Ind}_{H_t \times W_c \times W_d}^{W_{n_b}} \widetilde{\mathrm{sgn}} \otimes 1 \otimes 1$$

From now on, we set $(2n_1, 2n_2) := (2g, 2b) := (\left|\check{\mathcal{O}}_g\right|, \left|\check{\mathcal{O}}_b\right|)$.

Clearly $[\tau_b : \mathcal{C}_b]$ is counted by the size of the following set

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_b) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath, \jmath) = \tau_b \\ \mathrm{Im}\mathcal{P} \subset \{\bullet, c, d\}, \mathrm{Im}\mathcal{Q} \subset \{\bullet\} \end{array} \right\}$$

and for each $\wp \subset \mathfrak{P}(\check{\mathcal{O}}_g)$, the multiplicity $[\tau_\wp : \mathcal{C}_g]$ is counted by the size of

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \wp) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath, \jmath) = \tau_\wp \\ \mathrm{Im}\mathcal{P} \subset \{\bullet, c\}, \mathrm{Im}\mathcal{Q} \subset \{\bullet, s, r, d\} \end{array} \right\}$$

and for each $\wp$.

Recall the definition of $\mathrm{PP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_b)$ in (5.1).

Lemma 7.8. *For each $\wp \in\subset \mathfrak{P}(\check{\mathcal{O}}_g)$, we have*

$$\left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \wp)\right| = \left|\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \emptyset)\right|.$$

In particular, we have

$$\mathrm{Unip}_\star(\check{\mathcal{O}}_g) = \left|\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}_g)\right|.$$

Here $\mathrm{Unip}_\star(\check{\mathcal{O}}_g)$ *denote the set of all unipotent representations attached to the inner class* $\mathrm{SO}(2p+1,2q)$.

7.7. **Type D.** Clearly $[\tau_b : \mathcal{C}_b]$ is counted by the size of the following set

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_b) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath,\jmath) = \tau_b \\ \mathrm{Im}\mathcal{P} \subset \{ \bullet, c, d \}, \mathrm{Im}\mathcal{Q} \subset \{ \bullet \} \end{array} \right\}$$

and for each $\wp \subset \mathfrak{P}(\check{\mathcal{O}}_g)$, the multiplicity $[\tau_\wp : \mathcal{C}_g]$ is counted by the size of

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \wp) := \left\{ (\imath, \jmath, \mathcal{P}, \mathcal{Q}) \;\middle|\; \begin{array}{l} (\imath,\jmath) = \tau_\wp \\ \mathrm{Im}\mathcal{P} \subset \{ \bullet, s, r, c, d \}, \mathrm{Im}\mathcal{Q} \subset \{ \bullet \} \end{array} \right\}$$

and for each $\wp$.

Appendix A. Combinatorics of Weyl group representations in the classical types

A.1. **The $\jmath$-induction.** If μ and ν are two partitions representing two symmetric groups representations. Then

$$\jmath_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu \boxtimes \nu = \mu \cup \nu,$$

where $\mu \cup \nu$ is the partition such that

$$\{ \mathbf{c}_i(\mu\cup\nu) \mid i \in \mathbb{N}^+ \} = \{ \mathbf{c}_i(\mu) \mid i \in \mathbb{N}^+ \} \cup \{ \mathbf{c}_i(\nu) \mid i \in \mathbb{N}^+ \}$$

as multisets.

[Use the inductive by stage of $\jmath$-induction

$$\begin{aligned} &\jmath_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu \boxtimes \nu \\ &= \jmath_{S_{|\mu|}\times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \jmath_{\prod_i S_{\mathbf{c}_i(\mu)} \times \prod_j S_{\mathbf{c}_j(\nu)}}^{S_{|\mu|}\times S_{|\nu|}} \mathrm{sgn} \\ &= \jmath_{\prod_i S_{\mathbf{c}_i(\mu\cup\nu)}}^{S_{|\mu|+|\nu|}} \mathrm{sgn} \\ &= \mu \cup \nu. \end{aligned}$$

]

We have

$$\jmath_{S_n}^{W_n} \mathrm{sgn} = \begin{cases} \left({\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k}, {\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k}\right) & \text{if } n = 2k \text{ is even,} \\ \left({\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k+1}, {\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k}\right) & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

[The symbol of trivial of trivial group is

$$\binom{0,1,\cdots,k}{0,\cdots,k-1}.$$

Apply the formula [63, 4.5.4], When n is even, the symbol of the induce is

$$\binom{0,2,\cdots,k+1}{1,\cdots,k}.$$

corresponds to $\left({\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k}, {\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k}\right)$.

When n is even, the symbol of the induce is

$$\binom{1,2,\cdots,k+1}{1,\cdots,k}.$$

corresponds to $\left({\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k+1}, {\begin{array}{|c|}\hline \ \\ \hline \vdots \\ \hline \vdots \\ \hline \ \\ \hline\end{array}}_{k}\right)$.]

If $\tau = (\tau_L, \tau_R)$ and $\sigma = (\sigma_L, \sigma_R)$ be two bipartition. Then

$$\jmath_{W_{|\tau|}\times W_{|\sigma|}}^{W_{|\tau|+|\sigma|}} \tau \boxtimes \sigma = (\tau_L \cup \sigma_L, \tau_R \cup \sigma_R)$$

Appendix B. Metaplectic Barbasch-Vogan duality and Weyl group representations

In this section, we exam primitive ideals for $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ at half integral infinitesimal character and explain the metaplectic Barbasch-Vogan dual in terms of the cells of W'_n.

Let $\check{\mathcal{O}}$ be a metaplectic "good" nilpotent orbit in $\mathfrak{sp}(2n, \mathbb{C})$, i.e. $\mathbf{r}_i(\check{\mathcal{O}})$ is even for each $i \in \mathbb{N}^+$.

Then $W_{[\lambda_{\check{\mathcal{O}}}]} = W'_n$ and the left cell is given by

$$(J_{W_{\check{\mathcal{O}}}}^{W'_n} \operatorname{sgn}) \otimes \operatorname{sgn} \qquad \text{with} \qquad W_{\check{\mathcal{O}}} = \prod_{i \in \mathbb{N}^+} S_{\mathbf{c}_{2i}(\check{\mathcal{O}})}.$$

Here $W_{\check{\mathcal{O}}}$ is an subgroup of S_n and the embedding of S_n in W'_n is fixed. When n is even,we the symbol of $J_{S_n}^{W'_n} \operatorname{sgn}$ is degenerate and we label it by "I".

Let

$$\mathrm{PP}\tilde{C}(\check{\mathcal{O}}) = \{ (2i-1, 2i+2) \mid i \in \mathbb{N}^+, \mathbf{r}_{2i-1}(\check{\mathcal{O}}) \neq \mathbf{r}_{2i}(\check{\mathcal{O}}) \}.$$

For each $\wp \subset \mathrm{PP}\tilde{C}(\check{\mathcal{O}})$, let $\tau_\wp := (\imath_\wp, \jmath_\wp)$ be the bipartition given by

$$(\mathbf{r}_i(\imath_\wp), \mathbf{r}_i(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}), & \text{if } (2i-1, 2i) \notin \wp, \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}), & \text{otherwise.} \end{cases}$$

If $\tau_\wp$ is degenerate, it represent the element in $\widehat{W'_n}$ with label I. Let $\wp^c$ be the complement of $\wp$ in $\mathrm{PP}\tilde{C}(\check{\mathcal{O}})$. Then $\tau_\wp$ and $\tau_{\wp^c}$ represent the same irreducible W'_n-module.

Using the induction formula in [?L, (4.6.6) (4.6.7)], we have the following lemma.

Lemma B.1. *The representation*

$${}^L\mathscr{C}'(\check{\mathcal{O}}) := J_{W_{\check{\mathcal{O}}}}^{W'_n} \operatorname{sgn}$$

is multiplicity free. The map

$$\mathbb{Z}[\mathrm{PP}\tilde{C}(\check{\mathcal{O}})]/\sim \longrightarrow \{ \tau \in \widehat{W'_n} \mid \tau \subset {}^L\mathscr{C}'(\check{\mathcal{O}}) \} \qquad \wp \mapsto \tau_\wp$$

is a bijection where $\sim$ is the equivalent relation identifying $\wp$ with $\wp^c$. Moreover, the special representation in ${}^L\mathscr{C}'(\check{\mathcal{O}})$ is $\tau_\emptyset$. ☐

In the following, we write

$$\tau_{\check{\mathcal{O}}} := \tau_\emptyset$$

for the unique special representation in ${}^L\mathscr{C}'(\check{\mathcal{O}})$.

Let $\tau_{\check{\mathcal{O}}} = (\imath, \jmath)$ such that $\mathbf{r}_i(\imath) \leq \mathbf{r}_i(\jmath)$ for all $i \in \mathbb{N}^+$. Then the unique special representation in $\operatorname{Ind}_{W_{\check{\mathcal{O}}}}^{W'_n} 1$ where the a function take maximal value is given by the bipartition

$$\tau_{\mathcal{O}} := (\jmath^t, \imath^t).$$

Let $\sigma'(\mathcal{O})$ denote the $\tau_{\mathcal{O}}$-isotypic component of W'_n-harmonic polynomials in $S(\mathfrak{h})$. Comparing the fake degree formulas of type C and D (see [20, Proposition 11.4.3, 11.4.4]), we conclude that the W_n-representation

$$\sigma(\mathcal{O}) := \mathbb{C}[W_n] \cdot \sigma'(\mathcal{O}) \tag{B.1}$$

is irreducible whose type is also given by the bipartition $\tau_{\mathcal{O}}$. The type of $\sigma(\mathcal{O})$ is $j_{W'}^{W} \sigma'(\mathcal{O})$.

Lemma B.2. *Suppose σ' is a W'_n-special representation.Let $\check{\mathcal{O}}'$ be the partition of type D corresponds to the W'_n-special representation $\sigma' \otimes \operatorname{sgn}_D$. Then $\sigma := j_{W'_n}^{W_n} \sigma'$ corresponds to the partition $\check{\mathcal{O}}'^t$ under the Springer correspondence of type C.*

Proof. First note that $\check{\mathcal{O}}'$ is a specail nilpotent orbit of type D_n, therefore $\check{\mathcal{O}}'^t$ is a partition of type C_n (see [24, Proposition 6.3.7]).

By the explicitly formula of the Lusztig-Spaltenstein duality, we known that the D-collapsing $(\check{\mathcal{O}}'^t)_D$ of $\check{\mathcal{O}}'^t$ corresponds to the special representation σ' of type D.

The Springer correspondence algorithm for classical groups can be naturally extends to all partitions. Sommers showed that two partitions are mapped to the same Weyl group representations if and only if they has the same D-collapsing [91, Lemma 9]. Note that the algorithms computing the Springer correspondence for type C and D are essentially the same (see [20, Section 13.3] or [91, Section 7]). By the injectivity of the Springer correspondence, we conclude that the type C partition $\check{\mathcal{O}}'^t$ must corresponds to σ. □

The following lemma is a direct consequence of Lemma B.2.

Lemma B.3. *Suppose $\check{\mathcal{O}}$ has good parity of type $\tilde{C}$. Under the Springer correspondence of type C, the representation $\sigma(\check{\mathcal{O}})$ (see* (B.1)*) corresponds to the nilpotent orbit $\mathcal{O}$ defined by*

$$(\mathbf{c}_{2i-1}(\mathcal{O}), \mathbf{c}_{2i}(\mathcal{O})) = \begin{cases} (\mathbf{r}_{2i-1}(\check{\mathcal{O}}), \mathbf{r}_{2i}(\check{\mathcal{O}})), & \text{if } \mathbf{r}_{2i-1}(\check{\mathcal{O}}), \mathbf{r}_{2i}(\check{\mathcal{O}}) \\ (\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1, \mathbf{r}_{2i}(\check{\mathcal{O}})+1), & \text{otherwise} \end{cases}$$

for all $i \in \mathbb{N}^+$.

Proof. Apply the algorithm of Springer correspondence of type D to the representation $\sigma'(\check{\mathcal{O}})$ gives the above formula. □

[A technical point, the representation of W_n' is given by symbol $\binom{\xi}{\mu}$ or $\binom{\mu}{\xi}$. However, using the Springer correspondence formula, only one of the arrangement can give a valid type D partition.

It is a interesting fact that a type D orbit $\mathcal{O}$ is special if and only if $\mathcal{O}^t$ is of type C.
]

Lemma B.4. *Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_b \cup \check{\mathcal{O}}_g$. Then $\mathcal{O} = \mathcal{O}_b \cup \mathcal{O}_g$ where $\mathcal{O}_b = \check{\mathcal{O}}_b^t$ and $\mathcal{O}_g = \tilde{d}_{BV}(\check{\mathcal{O}}_g)$.*

[We take the convention that $2\mathcal{O} = [2r_i]$ if $\mathcal{O} = [r_i]$. We also write $[r_i] \cup [r_j] = [r_i, r_j]$. $\dagger\mathcal{O} = [r_i + 1]$.

We suppose

$$\check{\mathcal{O}}_b = [2r_1+1, 2r_1+1, \cdots, 2r_k+1, 2r_k+1] = (2c_0, 2c_1, 2c_1, \cdots, 2c_l, 2c_l)$$

where $l = r_1$.

Now

$$\begin{aligned} W_{\check{\mathcal{O}}_b} &= W_{c_0} \times S_{2c_1} \times S_{2c_2} \times \cdots \times S_{2c_l} \\ \check{\sigma}_b &:= j_{W_{\check{\mathcal{O}}_b}}^{W_b} \operatorname{sgn} = ((c_1, c_2, \cdots, c_k), (c_0, c_1, \cdots, c_l)) \\ &= ([r_1, r_2, \cdots, r_k], [r_1+1, r_2+1, \cdots, r_k+1]) \end{aligned}$$

Therefore

$$\sigma_b = \check{\sigma}_b \otimes \operatorname{sgn} = ((r_1+1, r_2+1, \cdots, r_k+1), (r_1, r_2, \cdots, r_k))$$

which corresponds to the orbit

$$\mathcal{O}_b = (2r_1+1, 2r_1+1, 2r_2+1, 2r_2+1, \cdots, 2r_k+1, 2r_k+1) = \check{\mathcal{O}}_b^t.$$

This implies

$$\sigma_b = j_{W_{L_b}}^{W_b} \operatorname{sgn}, \quad \text{where } W_{L,b} = \prod_{i=1}^{k} S_{2r_i+1}.$$

(Note that $\mathcal{O}'_b = (2r_1+1, 2r_2+1, \cdots, 2r_k+1)$ which corresponds to $j_{W_{L_b}}^{S_b}$ sgn and $\mathrm{ind}_L^G \mathcal{O}'_b = \mathcal{O}_b$.)

Now we deduce that

$$\begin{aligned}\sigma &:= j_{W_b\times W_g}^{W_n} \sigma_b \otimes \sigma_g \\ &= j_{W_{L_b}\times W_g}^{W_n} \operatorname{sgn} \otimes \sigma_g\end{aligned}$$

where $W_{L_b} \times W_g$ is a parabolic subgroup of W_n corresponds to the Levi factor L of type

$$A_{2r_1+1} \times A_{2r_2+1} \times \cdots \times A_{2r_k+1} \times W_g.$$

Therefore

$$\mathcal{O} = \mathrm{ind}_L^G \operatorname{triv} \times \mathcal{O}_g = \mathcal{O}_b \cup \mathcal{O}_g.$$

We claim that the map $\tilde{d}\colon \check{\mathcal{O}} \mapsto \mathcal{O}$ defined here coincide with our Metaplectic BV duality paper.

Note that $\tilde{d}_{\mathrm{BV}}$ is compatible with parabolic induction. Suppose $\check{\mathfrak{l}}$ is a Levi subgroup of $\check{\mathfrak{g}}$ and $\check{\mathcal{O}}_{\check{\mathfrak{l}}} := \check{\mathcal{O}} \cap \check{\mathfrak{l}} \neq \emptyset$. Then

$$\tilde{d}_{\mathrm{BV}}(\check{\mathcal{O}}) = \mathrm{ind}_{\mathfrak{l}}^{\mathfrak{g}} \tilde{d}_{\mathrm{BV}}(\check{\mathcal{O}}_{\check{\mathfrak{l}}}).$$

(Since $\tilde{d}_{\mathrm{BV}}$ commute with the descent map, the claim follows from the prosperity of d_{BV})

The map $\tilde{d}$ also compatible with parabolic induction. It suffice to consider the case where $\check{\mathfrak{l}}$ is a maximal parabolic of type $A_l \times C_n$ and the orbit is trivial on the A_l factor. Suppose l is the bad parity, the claim is clear by our computation for the bad parity case.

Now suppose $l = 2m$ has good parity. If there is a i such that $\mathbf{r}_{2i+1}(\check{\mathcal{O}}) = \mathbf{r}_{2i+2}(\check{\mathcal{O}}) = 2m$. Then $\mathbf{c}_{2i+1}(\mathcal{O}) = \mathbf{c}_{2i+2}(\mathcal{O}) = 2m$ and the claim follows.

Otherwise, we can assume

$$R_{2i-1} := \mathbf{r}_{2i-1}(\check{\mathcal{O}}) > \mathbf{r}_{2i}(\check{\mathcal{O}}) = \mathbf{r}_{2i+1}(\check{\mathcal{O}}) > \mathbf{r}_{2i+2}(\check{\mathcal{O}}) =: R_{2i+2}.$$

and then

$$\mathcal{O} = (\cdots, R_{2i-1} - 1, 2m+1, 2m-1, R_{2i+2}, \cdots)$$

One check again that $\mathcal{O} = \mathrm{ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{O}_{\mathfrak{l}}$. (Note that the induction operation is add two length $2m$ columns and then apply C-collapsing.)

Now it suffice to check that $\tilde{d}(\check{\mathcal{O}}) = \tilde{d}_{\mathrm{BV}}(\check{\mathcal{O}})$ for every orbits $\check{\mathcal{O}}$ whose rows are multiplicity free) (i.e. $\check{\mathcal{O}}$ is distinguished). This is clear by the explicit formula for the both sides.]

Appendix C. Matching specail shape and non-special shape painted bipartitions

C.1. Proof of ...

Proof. Let i be the minimal integer such that $(2i-1, 2i) \in \wp$. We define $\wp' = \wp - \{\,(2i-1, 2i)\,\}$. We will establish a bijection

$$\mathsf{PBP}_\star(\check{\mathcal{O}}_g, \wp') \longrightarrow \mathsf{PBP}_\star(\check{\mathcal{O}}_g, \wp).$$

□

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