The weak Arthur packets of real classical groups

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(joint with Dan Barbasch, Binyong Sun and Chengbo Zhu)

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(BIRS-IASM, Arthur packets)

Jordan-Chevalley decomposition

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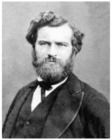

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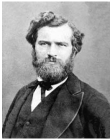

- Classification of conjugation classes in G:

$$G/\sim = \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \{\, su \mid u \in G_s \,\} / \sim \xleftrightarrow{bij.} \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \mathrm{unip}(G_s)$$

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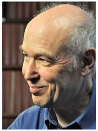

$$\mathrm{Irr}(G) = \bigsqcup_{s \in \check{G}_{\text{s.s.}}/\sim} \mathcal{E}(G, s).$$

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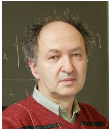

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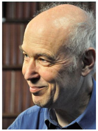

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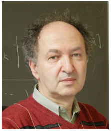

Lusztig's map to the unipotent packet.

$$\mathcal{E}(G, s) \xrightarrow[\mathcal{L}_s]{bij.} \mathcal{E}(\check{G}_s, 1)$$

Representations of Real Lie groups

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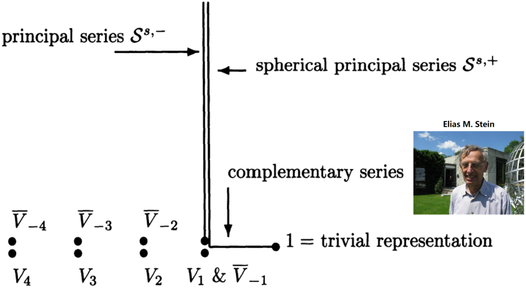

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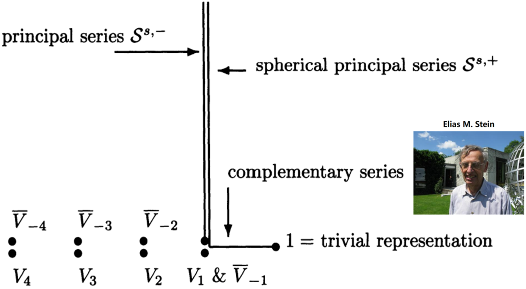

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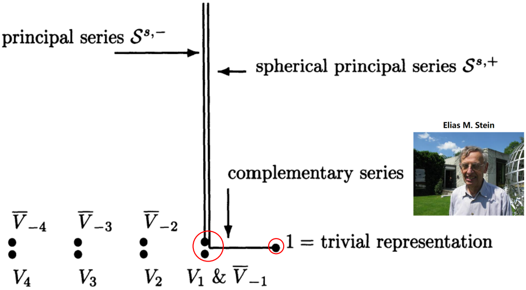

- *Open problem:* Structure of the unitary dual!

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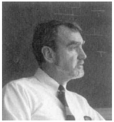

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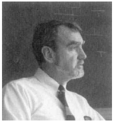

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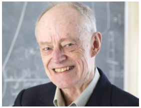
-  *Conjecture:* $\exists$ unipotent Arthur packets

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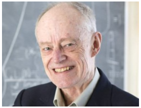
-  *Conjecture:* $\exists$ unipotent Arthur packets

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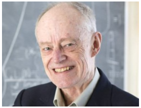
-  *Conjecture:* $\exists$ unipotent Arthur packets
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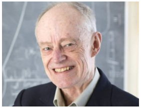
-  *Conjecture:* $\exists$ unipotent Arthur packets

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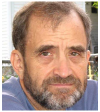
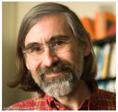
  "Sur Les paquets d'Arthur des groupes classiques réels" (2020 JEMS)

Barbasch-Vogan’s definition of special unip. repn.

- Barbasch  Vogan  $\mathrm{Unip}(G) :=$ repn. with smallest “size”

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  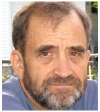 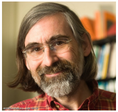 $\mathrm{Unip}(G) :=$ repn. with smallest "size"
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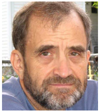
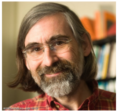
- Barbasch  Vogan  $\mathrm{Unip}(G) :=$ repn. with smallest "size"
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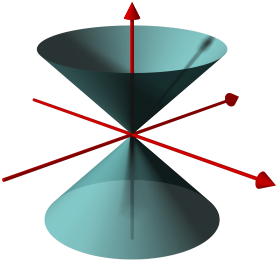

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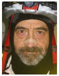
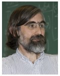
- $\mathrm{Nil}^{\mathrm{special}}(\mathbf{G}) \ni \mathcal{O} :=$ the  Spaltenstein    dual of $\check{\mathcal{O}}$.

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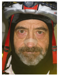
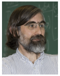
- $\mathrm{Nil}^{\mathrm{special}}(\mathbf{G}) \ni \mathcal{O} :=$ the  Spaltenstein    dual of $\check{\mathcal{O}}$.

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- *Barbasch-Vogan* 1985: Classification for complex groups.
- *Barbasch* 1989: Proved the conj. for complex classical groups.

Special unip. repn. of simply conn. classical groups

Theorem (Barbasch-M.-Sun-Zhu)

Suppose G is a simply connected real classical group, i.e. one of the following groups

$$\mathrm{SU}(p,q), \mathrm{Spin}(p,q), \mathrm{Spin}(2n,\mathbb{H}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{Mp}(2n,\mathbb{R}), \mathrm{Sp}(p,q)$$

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All *special unipotent repn.* of G are *unitarizable*.

Answer for *exceptional groups*

J. Adams, S. Miller, M. van Leeuven, and D. A. Vogan

atlas  + other method.

Counting G-module with a specific ann. variety

- Fix inf. char. $\mu \in \mathfrak{h}^*/W$

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Lemma:

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Lemma: If $\mathfrak{g}$ has no E_8 factor, then

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Applications

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- genuine special unipotent repn. of $\mathrm{Spin}(p,q)$ are some obvious irreducibly parabolicaly induced module.

Nilpotent orbits with "good/bad parity"

- Bad parity (must occur with even multiplicity in $\check{\mathcal{O}}$):

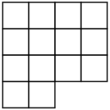
$$\begin{cases} \text{even number}, & \text{when } \check{G} \text{ is type } B \text{ or } D \\ \text{odd number}, & \text{when } \check{G} \text{ is type } C \end{cases}$$

- $\check{\mathcal{O}}$ has *"good parity"* if $\check{\mathcal{O}}$ only contains

$$\begin{cases} \text{odd rows}, & \text{when } \check{G} \text{ is type } B \text{ or } D \\ \text{even rows}, & \text{when } \check{G} \text{ is type } C \end{cases}$$

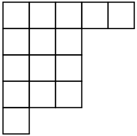
- $\lambda_{\check{\mathcal{O}}}$ is integral.

$\mathcal{O}$



$\mathrm{Sp}(14,\mathbb{C})$

$\check{\mathcal{O}}$



$\mathrm{SO}(15,\mathbb{C})$

Reduction to the "good parity"

- Consider $G = \mathrm{Sp}(2n, \mathbb{R})$.
- $\check{\mathcal{O}}$ decompose into two parts $\check{\mathcal{O}}_g$ (good parity) and $\check{\mathcal{O}}_b$ (bad parity).
- Assume $\check{\mathcal{O}}_b = \{r_1, r_1, \cdots, r_k, r_k\}$.

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Theorem Let $\check{\mathcal{O}}'_b = \{\, r_1, \cdots, r_k \,\} \in \mathrm{Nil}_{\mathrm{GL}}$.

$$\begin{array}{ccc} \mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}) \times \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{Sp}) & \xrightarrow{bij.} & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}) \\ (\pi', \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}_{|\check{\mathcal{O}}'_b|} \times \mathrm{Sp}(2n_0, \mathbb{R}) \ltimes U}^{\mathrm{Sp}(2n, \mathbb{R})} \pi' \otimes \pi_0 \end{array}$$

$$\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}) = \left\{ \mathrm{Ind} \bigotimes_{j=1}^{k} \mathrm{sgn}_{\mathrm{GL}(r_j, \mathbb{R})}^{\epsilon_j} \;\middle|\; \epsilon_j \in \mathbb{Z}/2\mathbb{Z} \right\}$$

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- Use *theta correspondence* to study $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
- We assume $\check{\mathcal{O}}$ has good parity from now on.

Counting unipotent representations I

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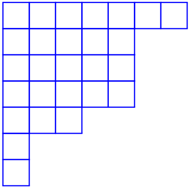
- $[\tau : \mathscr{G}_{\rho_G}(G)]$ is counted by painted bi-partitions $\mathrm{PBP}(\check{\mathcal{O}})$.

Example of PBP

$$\check{\mathcal{O}} = [7,5,5,5,3,1,1] =$$



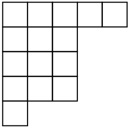
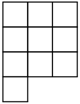
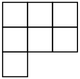
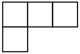
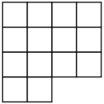
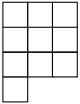
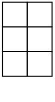
$\mathrm{PBP}_{\check{\mathcal{O}}}(\mathrm{Sp}(2n,\mathbb{R}))$

$$\begin{array}{|c|c|c|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & c & \\ \cline{1-2} r & d & \\ \cline{1-2} d & & \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \bullet & \bullet & \bullet \\ \hline \bullet & s & \\ \cline{1-2} \end{array}$$

Associated character

$$\mapsto \begin{array}{|c|c|c|c|c|c|}\hline * & = & * & = & * & = \\ \hline * & = & * & = & * & = \\ \hline = & * & = & * & & \\ \cline{1-4} * & = & * & = & & \\ \cline{1-4} * & = & * & = & & \\ \cline{1-4} - & + & & & & \\ \cline{1-2} \end{array} \cup \begin{array}{|c|c|c|c|c|c|}\hline * & = & * & = & * & = \\ \hline * & = & * & = & * & = \\ \hline = & * & = & * & & \\ \cline{1-4} = & * & = & * & & \\ \cline{1-4} * & = & * & = & & \\ \cline{1-4} + & - & & & & \\ \cline{1-2} \end{array}$$

Inductive structure of nilpotent orbits

$\widehat{\mathbf{G}}_i$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$
$\check{\mathcal{O}}_i$				
$\mathcal{O}_i$				
$\mathbf{G}_i$	$\mathrm{Sp}(14,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{Sp}(6,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

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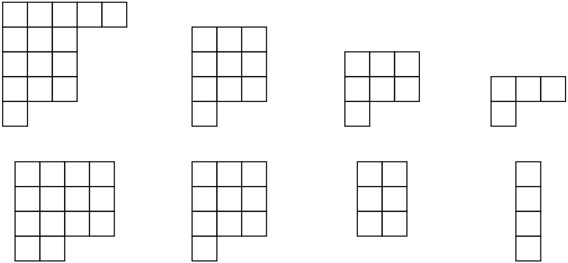
$\check{\mathcal{O}}_i$



$\mathcal{O}_i$

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Kraft

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resolution of singularities of nilpotent orbit closures.

Example of descent sequences

$\mathbf{G}_i^\vee$	$SO(15,\mathbb{C})$	$O(10,\mathbb{C})$	$SO(7,\mathbb{C})$	$O(4,\mathbb{C})$
$\check{\mathcal{O}}_i$	$\ydiagram{5,3,3,3,1}$	$\ydiagram{3,3,3,1}$	$\ydiagram{3,3,1}$	$\ydiagram{3,1}$
$\mathcal{O}_i$	$\begin{ytableau} - & + & - & + \\ + & - & + & - \\ + & - & + & - \\ + & - \end{ytableau}$	$\begin{ytableau} + & - & + \\ - & + & - \\ - & + & - \\ - \end{ytableau}$	$\begin{ytableau} - & + \\ + & - \\ + & - \end{ytableau}$	$\begin{ytableau} + \\ + \\ - \\ - \end{ytableau}$
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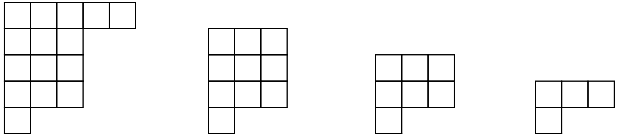
$\check{\mathcal{O}}_i$				
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Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

- $\chi = \bigotimes_{j=0}^{2a} \chi_j$, a 1-dim repn. of $\prod_{j=0}^{2a} G_j$.
- $\chi_j \in \{\mathbf{1}, \mathrm{sgn}^{+,-}, \mathrm{sgn}^{-,+}, \det\}$
- Define a smooth repn. of $G = G_{2a}$ (the symplectic group).

$$\pi_\chi := (\omega_{G_{2a},G_{2a-1}} \widehat{\otimes} \omega_{G_{2a-1},G_{2a-2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1,G_0} \otimes \chi)_{G_{2a-1} \times G_{2a-2} \times \cdots \times G_0}$$

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$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}.$$

Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

	$\mathrm{Sp}(6, \mathbb{R})$		
$\mathrm{O}(4,0)$		$\theta(\mathrm{sgn}^{+,-})$	
$\mathrm{O}(3,1)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(2,2)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(1,3)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(0,4)$			$\theta(\mathrm{sgn}^{-,+})$

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\mathrm{LS}(\check{\mathcal{O}}') \arrow[u, "\vartheta^{\mathrm{geo}}"] & \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}}') \arrow[l, two heads, "\mathrm{AC}"'] \arrow[r, leftrightarrow] & \bigcup \mathrm{Unip}_{\check{\mathcal{O}}'}(G') \arrow[u, "\theta"']
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- The injectivity of theta lifting is crucial!

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- θ-lifting $\approx \sqrt{\text{parabolic induction}}$
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- θ-lifting $\approx \sqrt{\text{parabolic induction}}$
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- Sharp formula of Asso. Cycles: lower bound=upper bound
  - *Algebra+Analysis:* $\rightsquigarrow$ lower bound.
  - *Algebraic Geometry:* double fibration of moment maps
    $\rightsquigarrow$ upper bound
- Exhaustion: lower bound=upper bound (Combinatorics)
  - Character theory (Kazhdan-Lusztig-Vogan theory)
    $\rightsquigarrow$ upper bound by counting "tableaux".
  - Asso. Cycle+Injectivity of θ $\rightsquigarrow$ lower bound.

Preprints (Barbasch, M. , Sun and Zhu)

- *Definition for metaplectic groups*

  On the notion of metaplectic Barbasch-Vogan duality
  https://arxiv.org/abs/2010.16089

- *Construction and unitarity using θ-lifting*

  Special unipotent representations: orthogonal and symplectic groups
  https://arxiv.org/abs/1712.05552

- *Counting unipotent representations*
  https://arxiv.org/abs/2205.05266

Thank you for your attention!