Special unipotent representations of real classical groups and theta correspondence

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Classical groups and special unipotent representations

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D_n	$\mathrm{O}(p, 2n-p)$	$\mathrm{O}(2n, \mathbb{C})$	$\mathrm{O}(2n, \mathbb{C})$	D_n
C_n	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$	B_n
B_n	$\mathrm{O}(p, 2n+1-p)$	$\mathrm{O}(2n+1, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$	C_n
$\tilde{C}_n$	$\mathrm{Mp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$	C_n
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All *specail unipotent representations* of G are *unitarizable*.

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- Lemma:

$$\#\{\, \pi \in \mathrm{Irr}_\mu(\mathfrak{g}, K)(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \,\}$$
$$= \sum_{\substack{\tau \in \mathcal{D} \\ \mathcal{D} \rightsquigarrow \mathcal{O}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathcal{K}_\lambda(\mathfrak{g}, K)]$$

Nilpotent orbits with "good/bad parity"

- Bad parity (must occurs with even multiplicity in $\check{\mathcal{O}}$):

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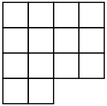
- Example of good parity:

$\mathcal{O}$



$\mathrm{Sp}(14, \mathbb{C})$

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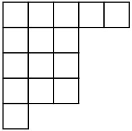

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- Use *theta correspondence* to construct $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.

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- Use *theta correspondence* to construct $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
- We assume $\check{\mathcal{O}}$ has good parity from now on.

Descent of nilpotent orbits: $\mathbf{G} = \mathrm{Sp}(2n, \mathbb{C})$

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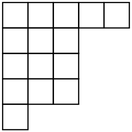
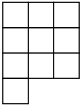
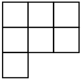
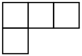
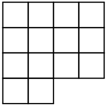
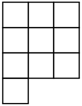
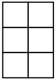
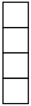
Example of descent sequences

$\mathbf{G}_i^\vee$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$
$\mathcal{O}_i^\vee$				
$\mathcal{O}_i$				
$\mathbf{G}_i$	$\mathrm{Sp}(14,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{Sp}(6,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

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Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

$$\begin{array}{cccc}
 & & \mathrm{Sp}(6,\mathbb{R}) & \\
\mathrm{O}(4,0) & & \theta(\mathrm{sgn}^{+,-}) & \\
\mathrm{O}(3,1) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(2,2) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(1,3) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(0,4) & & & \theta(\mathrm{sgn}^{-,+})
\end{array}$$

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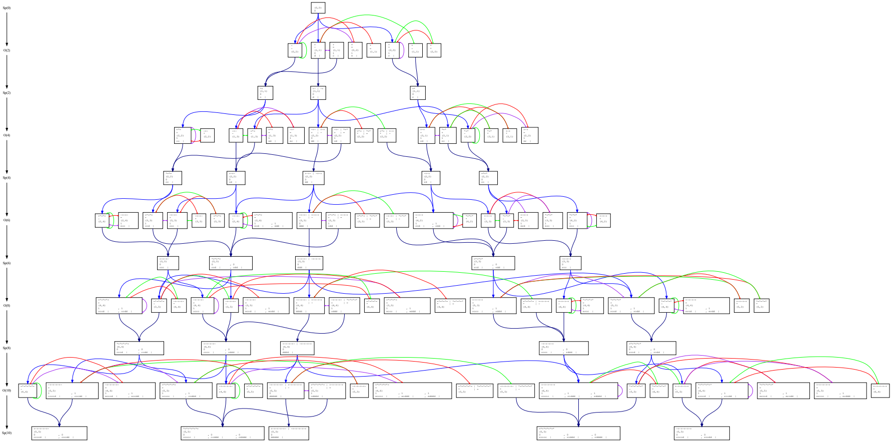
- The injectivity of theta lifting is crucial!


Sp(0)
O(2)
Sp(2)
O(4)
Sp(4)
O(6)
Sp(6)
O(8)
Sp(8)
O(10)
Sp(10)

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- Question: How to describe $\pi_{\psi,\eta}$ explicitly?

Thank you for your attention!