

COMBINATORICS FOR UNIPOTENT REPRESENTATIONS

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1. STATEMENT OF THE MAIN RESULT

1.1. Nilpotent Orbits. Let $\mathbf{G} = \mathrm{Sp}, \mathrm{O}$, and $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ is a complex nilpotent orbit. “Type M” means that $\tilde{\mathbf{G}}$ is a metaplectic group. its dual group $\check{\mathbf{G}}$ is a symplectic group with the same rank, and its duality means the metaplectic dual. Let $\mathrm{DRC}(\mathcal{O})$ be the set of dot-r-c diagrams parameterizing unipotent representations of the real forms of \mathbf{G} .

We say an orbit $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathbf{G}})$ is purely even (good parity in Moeglin-Renard’s notation) if

all row lengths of $\check{\mathcal{O}}$ are even when $\check{\mathbf{G}}$ is a symplectic group.
all row lengths of $\check{\mathcal{O}}$ are odd when $\check{\mathbf{G}}$ is an orthogonal group.

Definition 1.1. Now we describe the nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ who is the Barbasch-Vogan dual of a purely even orbit $\check{\mathcal{O}}$. If G is

Type C Then \mathcal{O} is special and it has columns

$(C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0)$ such that $C_{2i+1} > C_{2i}$ if C_{2i} is even for $i = 0, \dots, k-1$.

Note that, by the definition of special orbit, we have $C_{2i} \equiv C_{2i-1} \pmod{2}$, $C_{2i} = C_{2i-1}$ if C_{2i} is odd for $i = 1, \dots, k$.

Type M Then \mathcal{O} is metaplectic special and it has columns

$(C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0)$ such that $C_{2i+1} > C_{2i}$ if C_{2i} is odd for $i = 0, \dots, k-1$.

Note that, by the definition of metaplectic special, we have $C_{2i} \equiv C_{2i-1} \pmod{2}$, $C_{2i} = C_{2i-1}$ if C_{2i} is even for $i = 1, \dots, k$.

Type D and \mathcal{O} is special and it has columns

$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0)$ such that $C_{2i+1} > C_{2i}$ if C_{2i} is even for $i = 0, \dots, k$.

Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is purely even of type C and C_{2k+1} is always even.

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Type B and \mathcal{O} is special and it has columns

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is odd for } i = 0, \dots, k.$$

Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is purely even of type M and C_{2k+1} is always odd.

Let $\text{Nil}^{\text{dpe}}(\star)$ the set of duals of purely even orbits for type $\star = B, C, D, M$ respectively.

1.2. Descent of orbits in the purely even case. For $\{\star, \star'\} = \{C, D\}, \{B, M\}$, we define the notion of descent of orbits. The descent map is defined in the following way (under the above notation):

$$\overline{\nabla}: \quad \text{Nil}^{\text{dpe}}(\star) \longrightarrow \text{Nil}^{\text{dpe}}(\star')$$

$$\mathcal{O} \longmapsto \mathcal{O}'$$

such that

- when \star is B or D, $\mathcal{O}' := \nabla(\mathcal{O})$;
- when \star is C or M,

$$\mathcal{O}' := \begin{cases} \nabla(\mathcal{O}) & \text{if } C_{2k} \text{ is even and } \star = B \\ & \text{or } C_{2k} \text{ is odd and } \star = M; \\ (C_{2k-1} + 1, C_{2k-2}, C_{2k-3}, \dots, C_0) & \text{otherwise.} \end{cases}$$

In the second case, $C_{2k} = C_{2k-1}$ and $\mathcal{O} \mapsto \overline{\nabla}(\mathcal{O})$ is a good generalized descent.

The following set of orbit are essential to us:

Definition 1.2. A nilpotent orbit $\mathcal{O} \in \text{Nil}(\mathbf{G})$ is called noticed if G is

Type D and \mathcal{O} is special and it has columns

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \dots, k.$$

Type B and \mathcal{O} is special and it has columns

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \dots, k.$$

Type C and $\overline{\nabla}(\mathcal{O})$ is noticed of type D.

Type M and $\overline{\nabla}(\mathcal{O})$ is notice of type B.

Furthermore, We say a noticed orbit \mathcal{O} of type D or B is +noticed if $C_{2k+1} - C_{2k} \geq 2$.

The following inductive definition of noticed orbit is easy to verify.

Lemma 1.3. An orbit \mathcal{O} of type D (resp. type B) is noticed if and only if

- i) $\overline{\nabla}^2(\mathcal{O})$ is noticed, when C_{2k} is even (resp. odd), or
- ii) $\overline{\nabla}^2(\mathcal{O})$ is +noticed, when C_{2k} is odd (resp. even).

1.3. Relevant Weyl group representations and dot-r-c diagrams. Retain the notation in Definition 1.2. Now we describe all relevant representations of the Weyl group appear in the counting algorithm.

[We use “s” and “d” to denote “r” and “c” in Dan’s notes respectively.]

Type D We define c_i as the following

$$c_{2i} := \lfloor \frac{C_{2i}}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1} + 1}{2} \rfloor \quad \forall i = 0, 1, \dots, k,$$

$$c_{2k+1} := \frac{C_{2k+1}}{2}$$

The (special) Weyl group representation associated to the special orbit \mathcal{O} has columns

$$\tau = \tau_L \times \tau_R = (c_{2k+1}, c_{2k-1}, \dots, c_1) \times (c_{2k}, \dots, c_0).$$

The other representations is obtained from the special one by interchanging following pairs when $c_{2i-1} \neq c_{2i} + 1$ and $i = 1, \dots, k$ (these are pairs coming from even length columns)

$$(c_{2i-1}, c_{2i}) \longleftrightarrow (c_{2i} + 1, c_{2i-1} - 1).$$

[Note that we always have $c_{2i-1} - 1 \geq c_{2i-2}$ by our assumptions of the shape of the orbit.]

The dot-r-c diagrams of type D: For each representation $\tau = \tau_L \times \tau_R$, dots are filled in both sides forming the same shape of Young diagram, a column of “s” is added next to dots on the left side diagram, a column of “r” is added next to dots on the left side diagram, add a row of “c”s and finally add a row of “d”s on the left side of the diagram; Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step. [At most one r/s is added in each row of the existing dots diagram and at most one c/d is added to each column.]

In summary, s/r/c/d are all added on the left diagram. Fixing the representation τ , the left diagram τ_L determin τ_R completely.

Type C Set $C_{2k+1} = 0$. The Weyl group representations are obtained by the same formula of Type D.

The dot-r-c diagrams of type C are obtained in almost the same way as that of type D, except that s are added on the right side of the diagram.

Type B We define c_i as the following

$$c_{2i} := \lfloor \frac{C_{2i} + 1}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1}}{2} \rfloor \quad \forall i = 1, 2, \dots, k,$$

$$c_{2k+1} := \frac{C_{2k+1} - 1}{2}$$

Note that C_{2k+1} is always odd and $C_0 = 0$. The (special) Weyl group representation associated to the special orbit \mathcal{O} has columns

$$\tau = \tau_L \times \tau_R = (c_{2k}, \dots, c_2) \times (c_{2k+1}, c_{2k-1}, \dots, c_1).$$

The other representations is obtained from the special one by interchanging following pairs when $c_{2i-1} \neq c_{2i} + 1$ and $i = 1, \dots, k$ (these are pairs coming from odd length columns)

$$(c_{2i}, c_{2i-1}) \longleftrightarrow (c_{2i-1}, c_{2i}).$$

[Note that we always have $c_{2i-1} \geq c_{2i-2}$ by our assumptions of the shape of the orbit.]

The dot-r-c diagrams of type B: For each representation $\tau = \tau_L \times \tau_R$, dots are filled in both sides forming the same shape of Young diagram, a column of “s” is added next to dots on the right side of the diagram, a column of “r” is added next to dots on the right side diagram, add a row of “d”s on the right side of the diagram, and finally add a row of “c”s on the left side of the diagram. Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step. [At most one r/s is added in each row of the existing dots diagram and at most one c/d is added to each column.]

In summary, s/r/d are added on the right and c is added on the left.

Each dot-r-c diagram of type B is extended into two diagrams by adding an extra label “a” or “b”. This extension helps us to determine the signature of special orthogonal groups.

Fixing the representation τ , the right diagram τ_R determin τ_L completely.

Type M Set $C_{2k+1} = 0$. The Weyl group representations are obtained by the same formula of Type B.

The dot-r-c diagrams of type M are obtained in almost the same way as the type B except that s are added on the left side of the diagram and we do not add labels a/b.

We let $\text{DRC}(\mathcal{O})$ denote the set of dot-r-c diagrams defined above. Note that for type B, there is an additional mark a/b attached to the diagram.

In addition, for $\star = B, C, D, M$, let

$$\text{DRC}(\star) := \bigsqcup_{\mathcal{O} \in \text{Nil}^{\text{dpe}}(\star)} \text{DRC}(\mathcal{O}).$$

1.4. Main theorem. Let $\text{LS}(\mathcal{O})$ denote the Grothendieck group of $\tilde{\mathbf{K}}$ -equivariant local systems on \mathcal{O} for various real forms of \mathbf{G} . The following is our main theorem on the counting of unipotent representations.

Let $\tilde{\mathcal{O}}$ be a purely even orbit and \mathcal{O} be its dual orbit.

Theorem 1.4. *We can define an injective map using convergence range theta lifting:*

$$\begin{aligned} \pi : \text{DRC}(\mathcal{O}) &\longrightarrow \text{Unip}(\mathcal{O}) \\ \tau &\longmapsto \pi_\tau \end{aligned}$$

- i) When \mathcal{O} is of type C or type M, π is a bijection.
- ii) When \mathcal{O} is of type D or type B, the following map is a bijection

$$\begin{aligned} \text{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow \text{Unip}(\mathcal{O}) \\ (\tau, \varepsilon) &\longmapsto \pi_\tau \otimes (\det)^\varepsilon \end{aligned}$$

- iii) All unipotent representations attached to \mathcal{O} are unitarizable.
- iv) Suppose \mathcal{O} is noticed. The associated character map is an injective map:

$$\text{Ch}_{\mathcal{O}} : \text{Unip}(\mathcal{O}) \hookrightarrow \text{LS}(\mathcal{O})$$

- v) Suppose \mathcal{O} is quasi-distinguished. The image of associated character map is exactly the set of admissible orbit data. In other words, we have a bijection

$$\text{Ch}_{\mathcal{O}} : \text{Unip}(\mathcal{O}) \hookrightarrow \text{LS}^{\text{aod}}(\mathcal{O})$$

According to Moeglin-Renard’s work, Arthur’s unipotent Arthur packet (defined by endoscopic transfer) is constructed by iterated theta lifting. The theorem implies that ABV’s unipotent Arthur packet coincide with Arthur’s.

Two unipotent representation π_1 and π_2 are called in the same LS packet if $\mathcal{L}_{\pi_1} = \mathcal{L}_{\pi_2}$. For a local system $\mathcal{L} \in \text{LS}(\mathcal{O})$, let

$$[\mathcal{L}] = \{ \pi \mid \text{Ch}_{\mathcal{O}}(\pi) = \mathcal{L} \}$$

denote the set of all unipotent representations attached to \mathcal{L} . We call $[\mathcal{L}]$ a LS packet.

Remark. 1. We use convergence range theta lift to construct π inductively with respect to the number of columns in \mathcal{O} .

2. When \mathcal{O} is not noticed, a LS packet may not be a singleton and we use the injectivity of theta lifting for a fixed dual pair in an essential way.

3. The size of LS packet $[\mathcal{L}]$ could be arbitrary large when the size of \mathcal{O} groups. For example, \mathcal{O} is the regular orbit of $\mathrm{Sp}(2n, \mathbb{C})$, there is only 5 possible local systems when $n \geq 3$. But there are $2n + 1$ unipotent representations.
4. One will see easily from the proof that the associated character $\mathrm{Ch}_{\mathcal{O}}(\pi)$ for every $\pi \in \mathrm{Unip}(\mathcal{O})$ is always multiplicity free. Even when $\mathrm{Ch}_{\mathcal{O}}(\pi)$ is not irreducible, the character of the local system attached the same row length in different irreducible components are always the same.

2. MATCHING DOC-R-C DIAGRAMS, LOCAL SYSTEMS, AND UNIPOTENT REPRESENTATIONS

2.1. Descent maps of dot-r-c diagram in the algorithm. Suppose \mathbf{G} is of type B or D, we define a map

$$\begin{aligned} \mathrm{Sign}: \quad \mathrm{DRC}(B) \sqcup \mathrm{DRC}(D) &\longrightarrow \mathbb{N} \times \mathbb{N} \\ \tau &\longmapsto (p, q) \end{aligned}$$

Here (p, q) is the signature of the orthogonal group $\mathrm{O}(p, q)$ corresponding to the dot-r-c diagram τ , which is calculated by the following formula

$$\begin{aligned} p &= \# \bullet + 2\#r + \#c + \#d + \#A \\ q &= \# \bullet + 2\#s + \#c + \#d + \#B \end{aligned}$$

Suppose \mathcal{O} is the trivial orbit of $\mathrm{Sp}(2c_0, \mathbb{R})$ ($c_0 \geq 0$) or $\mathrm{Mp}(0, \mathbb{R})$ ($c_0 = 0$). Let

$$(2.1) \quad \tau_{\mathcal{O}} = \begin{array}{c} \emptyset \\ \times \\ s \end{array} \begin{array}{c} s \\ \vdots \\ s \end{array}$$

such that there are c_0 -entries marked by “ s ” in the right diagram. Then $\mathrm{DRC}(\mathcal{O}) = \{\tau_{\mathcal{O}}\}$ is a singleton. Define $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$ the trivial representation of $\mathrm{Sp}(2c_0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R}) = \mathbf{1}$. That by convention, $\mathrm{Sp}(0, \mathbb{R}) = \mathrm{Mp}(0, \mathbb{R})$ is the trivial group. Let

$$\mathrm{DRC}_0(C) = \{\tau_{\mathcal{O}} \mid \mathcal{O} = (2c_0), c_0 = 0, 1, 2, \dots\} \text{ and } \mathrm{DRC}_0(M) = \{\tau_{(0)} = \emptyset \times \emptyset\}.$$

The algorithm of matching dot-r-c diagrams is encoded in the descent map $\overline{\nabla}$ of dot-r-c diagrams defined below. For $(\star, \star') = (D, C), (B, M)$, we define maps

$$\begin{aligned} \overline{\nabla}: \quad \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) &\longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star)) \times \mathbb{Z}/2\mathbb{Z} \\ \tau &\longmapsto (\tau', \epsilon). \end{aligned}$$

Here

- i) τ is in $\mathrm{DRC}(\mathcal{O})$ where $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(\star) \sqcup \mathrm{Nil}^{\mathrm{dpe}}(\star')$;
- ii) $\tau' \in \mathrm{DRC}(\mathcal{O}')$ where $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$;
- iii) $i\epsilon = 0$ when $\mathcal{O} \mapsto \mathcal{O}'$ is a generalized descent from type C or M to type B or D.

To ease the notations, we define

$$\overline{\nabla}_1: \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) \longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star))$$

such that $\overline{\nabla}_1(\tau)$ is the first component of $\overline{\nabla}(\tau)$.

We define \mathcal{L} and π inductively with respect to the number of columns of \mathcal{O} .

- i) Our reduction terminates when \mathcal{O} is the trivial orbit of $\mathrm{Sp}(2c_0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R})$. In this case, $\mathrm{DRC}(\mathcal{O}) = \{\tau_{\mathcal{O}}\}$ which is attached to the trivial representation. We remark that the theta lift of the trivial representation $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$ of $\mathrm{Sp}(0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R})$ to $\mathrm{O}(p, q)$ for any signature p, q is the trivial representation $\mathbf{1}_{p,q}^{+,+}$ of $\mathrm{O}(p, q)$.¹
- ii) Suppose $\mathcal{O} \in \mathrm{Nil}(\star) \sqcup \mathrm{Nil}(\star')$ is a nontrivial orbit. Let $\tau \in \mathrm{DRC}(\mathcal{O})$ and

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \in \mathrm{DRC}(\mathcal{O}') \times \mathbb{Z}/2\mathbb{Z} \text{ where } \mathcal{O}' = \overline{\nabla}(\mathcal{O}).$$

- a) Suppose $\tau \in \mathrm{DRC}(C)$ or $\mathrm{DRC}(M)$. Now $\tilde{G} = \mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Mp}(2n, \mathbb{R})$ with $n = |\tau|$. By induction, τ' is attached to a unipotent representation $\pi_{\tau'}$ of $\tilde{G}' = \mathrm{O}(p, q)$ where $(p, q) = \mathrm{Sign}(\tau')$. We define

$$\pi_{\tau} := \bar{\Theta}_{\tilde{G}', \tilde{G}}(\pi_{\tau'} \otimes (\det)^{\epsilon}).$$

Here \det is the determinant character of $\mathrm{O}(p, q)$.

- b) Suppose $\tau \in \mathrm{DRC}(D)$ or $\mathrm{DRC}(B)$. Now $\tilde{G} = \mathrm{O}(p, q)$ with $(p, q) = \mathrm{Sign}(\tau)$. By induction, τ' is attached to a unipotent representation of \tilde{G}' where $\tilde{G}' = \mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Mp}(2n, \mathbb{R})$ and $n = |\tau'|$. We define

$$\pi_{\tau} := \bar{\Theta}_{\tilde{G}', \tilde{G}}(\pi_{\tau'} \otimes (\mathbf{1}_{p,q}^{+,-})^{\epsilon}).$$

We let

$$\begin{aligned} \mathcal{L}_{\tau} &:= \mathrm{Ch}_{\mathcal{O}}(\pi_{\tau}), \text{ and} \\ {}^{\ell}\mathrm{LS}(\mathcal{O}) &:= \{ \mathcal{L}_{\tau} \mid \tau \in \mathrm{DRC}(\mathcal{O}) \} \end{aligned}$$

to be the set of local systems obtained from our algorithm.

We summarize our definitions in the following diagram with $\star = \mathcal{O}/C/D/B/M$.

$$\begin{array}{ccccc} \mathrm{DRC}(\star) & \xrightarrow{\pi} & \mathrm{Unip}(\star) & & \\ \downarrow \mathcal{L} & \begin{array}{ccc} \tau & \xrightarrow{\quad} & \pi_{\tau} \\ \downarrow & & \downarrow \\ \mathcal{L}_{\tau} & \xleftarrow{\quad} & \mathcal{L}_{\tau} \end{array} & \downarrow \mathrm{Ch} & \\ \mathcal{K}_{\mathrm{M}}(\star) & \xrightarrow{\quad} & \mathrm{LS}(\star) & & \end{array}$$

Here $\mathcal{K}_{\mathrm{M}}(\star)$ is a combinatorially defined monoid parameterizing the local systems on the relevant nilpotent orbits, see.

3. THE DEFINITION OF $\overline{\nabla}$

From now on, we assume

$$\tau = (\tau_{L,0}, \tau_{L,1}, \dots) \times (\tau_{R,0}, \tau_{R,1}, \dots).$$

For any τ let τ^{\bullet} be the subdiagram consisting of \bullet and s entries only.

¹When $c_0 = 0$, this is an assignment by convention.

3.1. dot-s switching algorithm. We say a bipartition $\tau = \tau_L \times \tau_R$ is interlaced with larger left part if τ has columns $(\tau_{L,0} \geq \tau_{L,1}, \dots) \times (\tau_{R,0}, \tau_{R,1}, \dots)$ such that

$$\tau_{L,0} \geq \tau_{R,0} \geq \tau_{L,1} \geq \tau_{R,1} \geq \tau_{L,2} \geq \dots$$

$\text{bi}\mathcal{P}_L$ denote the set of all interlaced bipartitions with larger left parts. Similarly, define $\text{bi}\mathcal{P}_R$ be the set of bipartitions with larger right parts:

$$\text{bi}\mathcal{P}_R = \{ \tau = (\tau_{L,0} \geq \tau_{L,1}, \dots) \times (\tau_{R,0}, \tau_{R,1}, \dots) \mid \tau_{L,0} \geq \tau_{R,0} \geq \tau_{L,1} \geq \tau_{R,1} \geq \tau_{L,2} \geq \dots \}$$

Let DS_L (resp. DS_R) be the set of filled diagrams with only “s” entries on the left (resp. right) diagram and the rest are all “•”.

Note that for each $\tau \in \text{DS}_L$, its shape τ is in $\text{bi}\mathcal{P}_L$. On the other hand, for each $\tau \in \text{bi}\mathcal{P}_L$, we can fill “•” in all entries on the right diagram and corresponding entries on the left, and then fill “s” in the rest entries on the left. This procedure yields a valid dot-s diagram. In summary, $\text{bi}\mathcal{P}_L$ and DS_L are naturally bijective to each other.

Let $\bar{\nabla}_L: \text{bi}\mathcal{P}_L \rightarrow \text{bi}\mathcal{P}_R$ (resp. $\bar{\nabla}_R: \text{bi}\mathcal{P}_R \rightarrow \text{bi}\mathcal{P}_L$) be the operation of deleting the longest column on the left (resp. on the right). Then the operation gives a well defined maps between DS_L and DS_R making the following diagrams commute (the vertical maps are the natural identification discussed above):

$$\begin{array}{ccc} \text{bi}\mathcal{P}_L & \xrightarrow{\bar{\nabla}_L} & \text{bi}\mathcal{P}_R \\ \parallel & & \parallel \\ \text{DS}_L & \xrightarrow{\bar{\nabla}_L} & \text{DS}_R \end{array} \quad \begin{array}{ccc} \text{bi}\mathcal{P}_R & \xrightarrow{\bar{\nabla}_R} & \text{bi}\mathcal{P}_L \\ \parallel & & \parallel \\ \text{DS}_R & \xrightarrow{\bar{\nabla}_R} & \text{DS}_L \end{array}$$

By abuse of notation, we also call the dashed arrow above $\bar{\nabla}_L$ and $\bar{\nabla}_R$.

3.2. Bijection between “special” and “non-special” diagrams for type C. We define a bijection between “special” diagram τ^s and “non-special” diagrams τ .

Definition 3.1. Let $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0) \in \text{Nil}^{\text{dpe}}(C)$ with $k \geq 0$. Let

$$\tau = (\tau_{2k-1}, \dots, \tau_1, \tau_{-1} = 0) \times (\tau_{2k}, \dots, \tau_0).$$

be a Weyl group representation attached to $\mathcal{O} \in \text{Nil}^{\text{dpe}}(C)$. We say τ has

- special shape if $\tau_{2k-1} \leq \tau_{2k} + 1$;
- non-special shape if $\tau_{2k-1} > \tau_{2k} + 1$.

Clearly, this gives a partition of the set of Weyl group representations attached to \mathcal{O} .

Suppose $k \geq 1$, $C_{2k} = 2c_{2k}$ and $C_{2k-1} = 2c_{2k-1}$. The special shape representation τ^s

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k-1}, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \dots, \tau_0)$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k} + 1, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \dots, \tau_0)$$

In the pairing $\tau^s \leftrightarrow \tau^{ns}$, τ_j is unchanged for $j \leq 2k - 2$. We call τ^s the special shape and τ^{ns} the corresponding non-special shape

Definition 3.2. Suppose C_{2k} is even. We retain the notation in Definition 3.1 where the shapes τ^s and τ^{ns} correspond with each other. We define a bijection between $\text{DRC}(\tau^s)$ and $\text{DRC}(\tau^{ns})$:

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \longleftrightarrow & \text{DRC}(\tau^{ns}) \\ \tau^s & \longleftrightarrow & \tau^{ns} \end{array}$$

Without loss of generality, we can assume that $\tau^s \in \text{DRC}(\tau^s)$ and $\tau^{ns} \in \text{DRC}(\tau^{ns})$ have the following shapes:

$$(3.1) \quad \tau^s : \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ x_0 * \cdots & v_0 \cdots & & \\ x_1 x_2 \cdots & x_3 & & \\ & \begin{array}{c} s \\ \vdots \\ s \end{array} & & \end{array} \longleftrightarrow \tau^{ns} : \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ y_0 * \cdots & w_0 \cdots & & \\ \begin{array}{c} r \\ \vdots \\ r \end{array} y_2 \cdots & & & \\ & \begin{array}{c} y_1 \\ y_3 \end{array} & & \end{array}$$

(Here the grey parts have length $(C_{2k} - C_{2k-1})/2$ (could be zero length), the row contains x_0 and y_0 may be empty, and $*/\cdots$ are arbitrary entries. We remark that the value of v_0 and w_0 are completely determined by x_0 and y_0 . In particular, v_0 and w_0 are non-empty if and only if $x_0/y_0 \neq \emptyset$, i.e. $C_{2k-1} \geq 4$.)

The correspondence between (x_0, x_1, x_2, x_3) , with (y_0, y_1, y_2, y_3) is given by Table 1. The rest part of the diagrams are unchanged.

We define

$$\text{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \text{DRC}(\tau^s) \quad \text{and} \quad \text{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \text{DRC}(\tau^{ns})$$

where τ^s (resp. τ^{ns}) runs over all special (resp. non-special) shape representations attached to \mathcal{O} . Clearly $\text{DRC}(\mathcal{O}) = \text{DRC}^s(\mathcal{O}) \sqcup \text{DRC}^{ns}(\mathcal{O})$.

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

3.3. Special and non-special shapes of type D. Now we define the notion of special and non-special shape of type D.

Definition 3.3. Let

$$\tau = (\tau_{2k+1}, \tau_{2k-1}, \dots, \tau_1) \times (\tau_{2k}, \dots, \tau_0)$$

be a Weyl group representation attached to $\mathcal{O} \in \text{Nil}^{\text{dpe}}(D)$. We say τ has

- special shape if $\tau_{2k-1} \leq \tau_{2k} + 1$;
- non-special shape if $\tau_{2k-1} > \tau_{2k} + 1$.

Suppose $\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \dots, C_0 \geq 0)$ with $k \geq 1$.² A representation τ^s attached to \mathcal{O} has the shape

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \dots, \tau_0)$$

is paired with

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k+1}, c_{2k} + 1, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \dots, \tau_0).$$

In the pairing $\tau^s \leftrightarrow \tau^{ns}$, τ_j is unchanged for $j \leq 2k - 2$.

3.4. Definition of $\bar{\nabla}$ of Type D and C. The induction starts with type C: For the trivial orbit \mathcal{O} of $\tilde{G} = \text{Sp}(2c_0, \mathbb{R})$, $\text{DRC}(\mathcal{O}) = \{\tau_{\mathcal{O}}\}$ and $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$ is the trivial representation, see (2.1).

²Note that C_{2k} and C_{2k-1} are non-zero even integers.

3.4.1. The definition of ϵ .

- (1). When $\tau \in \text{DRC}(D)$, ϵ is determined by the “basal disk” of τ (see (6.3) for the definition of x_τ):

$$\epsilon_\tau := \begin{cases} 0, & \text{if } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

- (2). When $\tau \in \text{DRC}(C)$, the ϵ is determined by the lengths of $\tau_{L,0}$ and $\tau_{R,0}$:

$$\epsilon_\tau := \begin{cases} 0, & \text{if } \tau \text{ has special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| \leq 1; \\ 1, & \text{if } \tau \text{ has non-special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| > 1. \end{cases}$$

3.4.2. Initial cases. Let \mathcal{O} be a nilpotent orbit of type D with at most 2 columns.

- (3). Suppose $\mathcal{O} = (2c_1, 2c_0)$. Then $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$ is the trivial orbit of $\text{Sp}(2c_0, \mathbb{R})$. $\text{DRC}(\mathcal{O})$ consists of diagrams of shape $\tau_L \times \tau_R = (c_1,) \times (c_0,)$. The set $\text{DRC}(\mathcal{O}')$ is a singleton $\{\tau_{\mathcal{O}'}\}$, and every element $\tau \in \text{DRC}(\mathcal{O})$ maps to $\tau_{\mathcal{O}'}$ (see (2.1)):

$$\text{DRC}(\mathcal{O}) \ni \tau : \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \mapsto \tau_{\mathcal{O}'} : \begin{array}{c} \emptyset \\ \vdots \\ s \end{array} \times \begin{array}{c} s \\ \vdots \\ s \end{array}$$

	τ'	τ^s	τ^{ns}	
	$\begin{array}{ccc} x'_0 & * & * \\ x'_1 & x'_2 & \\ & \times & \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ x_1 & x_2 & x_3 \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} y_0 & * & * * \\ \begin{array}{c} r \\ \vdots \\ r \end{array} & y_2 & \\ & \times & \\ y_1 & & \\ y_3 & & \end{array}$	
$\begin{array}{l} x'_1 = s \\ \text{then} \\ z_0 = \emptyset / s \\ x_0 = \emptyset / \bullet \\ x_1 = \bullet \\ x_2 = x'_2 \end{array}$	$\begin{array}{ccc} x'_0 & * & * \\ s & x_2 & \\ & \times & \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ \bullet & x_2 & \bullet \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ \begin{array}{c} r \\ \vdots \\ r \end{array} & x_2 & \\ & \times & \\ c & & \\ d & & \end{array}$	$x_2 \neq r$
		$\begin{array}{ccc} x_0 & * & * * \\ \bullet & r & \bullet \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ \begin{array}{c} r \\ \vdots \\ r \end{array} & c & \\ & \times & \\ r & & \\ d & & \end{array}$	$x_2 = r$
$\begin{array}{l} x'_1 \neq s \\ \text{then} \\ x_1 = x'_1 \\ x_2 = x'_2 \end{array}$	$\begin{array}{ccc} x'_0 & * & * \\ x'_1 & x_2 & \\ & \times & \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ x_1 & x_2 & s \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ \begin{array}{c} r \\ \vdots \\ r \end{array} & x_2 & \\ & \times & \\ r & & \\ x_1 & & \end{array}$	$\begin{array}{l} x'_0 \neq c \\ \text{then} \\ x'_0 = \emptyset / s / r \\ x_0 = \emptyset / \bullet / r \end{array}$
		$\begin{array}{ccc} c & * & * * \\ d & x_2 & s \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} r & * & * * \\ \begin{array}{c} r \\ \vdots \\ r \end{array} & x_2 & \\ & \times & \\ c & & \\ d & & \end{array}$	$\begin{array}{l} x'_0 = c \\ \text{then} \\ x_0 = x'_0 = c \\ x_1 = x'_1 = d \\ x_2 = x'_2 = \emptyset / d \end{array}$

TABLE 1. “special-non-special” switch

We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \text{ and } x_\tau := x_n$$

We call $\mathbf{p}_\tau = \mathbf{x}_\tau$ the “peduncle” part of τ and x_τ the “basal disk” of τ . Note that \mathbf{x}_τ consists of entries marked by $s/r/c/d$.

3.4.3. *The descent from C to D.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

- (4). Suppose $|\tau_{L,0}| - |\tau_{R,0}| < 2$, i.e. τ has special shape. Keep r, c, d unchanged, delete $\tau_{R,0}$ and fill the remaining part with “•” and “s” by $\bar{\nabla}_R(\tau^\bullet)$ using the dot-s switching algorithm.
- (5). Suppose $|\tau_{L,0}| - |\tau_{R,0}| \geq 2$. We define

$$\tau' = \bar{\nabla}_1(\tau) := \bar{\nabla}_1(\tau^s)$$

where τ^s is the special diagram corresponding to τ defined in Definition 3.2.

Lemma 3.4. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$ with $k \geq 1$ and C_{2k} even. Let $\mathcal{O}' := \bar{\nabla}(\mathcal{O}) = (C_{2k-1}, \dots, C_0)$. Then*

$$\begin{array}{ccc} \text{DRC}^s(\mathcal{O}) & \xrightarrow{\bar{\nabla}_1} & \text{DRC}(\mathcal{O}') \\ \text{DRC}^{ns}(\mathcal{O}) & \xrightarrow{\bar{\nabla}_1} & \text{DRC}(\mathcal{O}') \end{array}$$

are a bijections.

Proof. The claim for $\text{DRC}^s(\mathcal{O})$ is clear by the definition of descent. The claim for $\text{DRC}^{ns}(\mathcal{O})$ reduces to that of $\text{DRC}^s(\mathcal{O})$ using Definition 3.2. \square

Lemma 3.5. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$ with $k \geq 1$ and C_{2k} odd. Let $\mathcal{O}' := \bar{\nabla}(\mathcal{O}) = (C_{2k-1} + 1, \dots, C_0)$. Then the following map is a bijection*

$$\text{DRC}(\mathcal{O}) \xrightarrow{\bar{\nabla}_1} \{ \tau' \in \text{DRC}(\mathcal{O}') \mid x_\tau \neq s \}.$$

Proof. This is clear by the descent algorithm. \square

3.4.4. *Descent from D to C.* Now we consider the general case of the descent from type D to type C . So we assume \mathcal{O} has at least 3 columns. First note that the shape of $\tau' = \bar{\nabla}(\tau)$ is the shape of τ deleting the longest column on the left.

The definition splits in cases below. In all these cases we define

$$(3.2) \quad \mathbf{x}_\tau := x_1 \cdots x_n \text{ and } x_\tau := x_n$$

which is marked by $s/r/c/d$. For the part marked by $*/\dots$, $\bar{\nabla}$ keeps r, c, d and maps the rest part consisting of • and s by dot-s switching algorithm.

- (6). When $C_{2k} = C_{2k-1}$ is odd, we could assume τ and τ' have the following forms with $n = (C_{2k+1} - C_{2k} + 1)/2$.

$$\tau : \begin{array}{ccccccc} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & w_0 & * & \dots & \dots \\ x_1 & y_1 & \dots & \times & & & \\ \vdots & & & & & & \\ & & & & & & x_n \end{array} \mapsto \tau' : \begin{array}{ccccccc} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & & w_0 & * & \dots & \dots \\ z_1 & \dots & \times & & & & \end{array}$$

Suppose $y_1 = c$ and $(x_1, \dots, x_n) = (r, \dots, r)$ or (r, \dots, r, d) , then let $z_1 = r$; otherwise, set $z_1 := y_1$. We define

$$\mathbf{p}_\tau := \begin{matrix} x_1 & y_1 \\ \vdots & \\ x_n & \end{matrix}.$$

- (7). When C_{2k} is even and τ has special shape, the descent is given by the following diagram

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots \\ \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots \\ \bullet \\ \vdots \\ \bullet \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \ \dots \end{array} \times \begin{array}{c} \dots\dots\dots \\ s \\ \vdots \\ s \end{array}$$

- (8). When C_{2k} is even and τ has non-special shape, then

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots\dots\dots \\ s \ r \ \dots\dots\dots \\ \vdots \\ s \ r \\ x_0 \ y_1 \\ x_1 \ y_2 \\ \vdots \\ x_n \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \ \dots\dots\dots \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots\dots\dots \\ r \ \vdots \ \dots\dots\dots \\ r \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \ \dots\dots\dots \end{array}$$

where y_1, y_2 are non empty. In most of the case, we just delete the longest column on the left and set $(z_1, z_2) = (y_1, y_2)$. The exceptional cases are listed below:

- When $x_0 = r$, we have $(y_1, y_2) = (c, d)$. We let

$$\begin{array}{c} x_0 \ y_1 \quad r \ c \quad z_1 \quad r \\ x_1 \ y_2 = x_1 \ d \mapsto z_2 := r \\ \vdots \quad \vdots \\ x_n \quad x_n \end{array}$$

- When $x_0 = c$, we have $n = 1$ and $(x_0, x_1) = (y_1, y_2) = (c, d)$. We let

$$\begin{array}{c} x_0 \ y_1 = c \ c \mapsto z_1 := r \\ x_1 \ y_2 = d \ d \quad z_2 := c \end{array}$$

- We remark that x_0 never equals to d .

3.4.5. *A key property in the descent case.* We retain the notation in Definition 3.3, where

$$\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \dots, C_0) \text{ such that } k \geq 1.$$

Let $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$. The following lemma is the key property satisfied by our definition

Lemma 3.6. *Let τ^s and τ^{ns} are two representations attached to \mathcal{O} as in Definition 3.3. Then*

- i) *For every $\tau \in \text{DRC}(\tau^s) \sqcup \text{DRC}(\tau^{ns})$, the shape of $\overline{\nabla}^2(\tau)$ is*

$$\tau'' = (c_{2k-1}, \tau_{2k-3}, \tau_1) \times (\tau_{2k-2}, \dots, \tau_0)$$

- ii) *Let $\mathcal{O}_1 = (2(c_{2k+1} - c_{2k}),) \in \text{Nil}(D)$ be the trivial orbit of $\text{O}(2(c_{2k-1} - c_{2k}), \mathbb{C})$. We define $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$ by $\delta(\tau) = (\overline{\nabla}^2(\tau), \mathbf{x}_\tau)$. The following maps are bijections*

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \xrightarrow{\delta} & \text{DRC}(\tau'') \times \text{DRC}(\mathcal{O}_1) \xleftarrow{\delta} \text{DRC}(\tau^{ns}) \\ \tau^s & \longmapsto & (\overline{\nabla}^2(\tau^s), \mathbf{x}_{\tau^s}) \\ & & (\overline{\nabla}^2(\tau^{ns}), \mathbf{x}_{\tau^{ns}}) \longleftarrow \tau^{ns} \end{array}$$

In particular, we obtain an one-one correspondence $\tau^s \leftrightarrow \tau^{ns}$ such that $\delta(\tau^s) = \delta(\tau^{ns})$.

iii) Suppose τ^s and τ^{ns} correspond as the above such that $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$. Then

$$(3.3) \quad \text{Sign}(\tau^s) = \text{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \text{Sign}(\tau'') + \text{Sign}(\mathbf{x}).$$

Proof. Suppose τ^s has special shape. the the behavior of τ^s under the descent map $\bar{\nabla}$ is illustrated as the following:

$$\tau^s : \begin{array}{c} \begin{array}{|c|} \hline \dots \\ * \dots \\ \bullet \\ \vdots \\ \bullet \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ * \dots \\ \bullet \\ \vdots \\ \bullet \\ \hline \end{array} \mapsto \tau'^s \times \begin{array}{|c|} \hline \dots \\ * \dots \\ s \\ \vdots \\ s \\ \hline \end{array} \mapsto \tau'' : \begin{array}{|c|} \hline \dots \\ * \dots \\ \vdots \\ * \dots \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ * \dots \\ \vdots \\ * \dots \\ \hline \end{array} \\ x_1 \\ \vdots \\ x_n \end{array}$$

Note that τ^s is obtained from τ'' by attaching the grey part consisting totally $2c_{2k}$ dots and \mathbf{x} . Now the claims for τ^s is clear.

Now consider the descent of a non-special diagram τ^{ns} :

$$\tau^{ns} : \begin{array}{c} \begin{array}{|c|} \hline \dots \\ * y_0 * \dots \\ s \ r \ y_2 \dots \\ \vdots \\ s \ r \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ u_0 \dots \\ \hline \end{array} \mapsto \tau'^{ns} : \begin{array}{c} \begin{array}{|c|} \hline \dots \\ y'_0 * \dots \\ r \ y'_2 \dots \\ \vdots \\ r \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ w_0 \dots \\ \hline \end{array} \mapsto \tau'' : \begin{array}{|c|} \hline \dots \\ x'_0 * \dots \\ x'_1 x'_2 \dots \\ \hline \end{array} \\ x_0 \ y_1 \\ x_1 \ y_3 \\ \vdots \\ x_n \end{array}$$

The bijection follows from the observation that 1. τ^{ns} is obtained by attaching the grey part x_0, \dots, x_n and y_0, y_1, y_2, y_3 to the $*/\dots$ part of τ'' ; 2. the value of (y_0, y_1, y_2, y_3) is completely determined by (x'_0, x'_1, x'_2) and $\mathbf{x} = x_1 \cdots x_n$.

We leave it to the reader for checking case by case that the grey part of τ^{ns} has signature $(C_{2k} - 2, C_{2k} - 2)$ and

$$\text{Sign}(x_0 y_0 y_1 y_2 y_3) - \text{Sign}(x'_0 x'_1 x'_2) = (2, 2).$$

Therefore (6.4) follows.

[First assume that $C_{2k} = 2$. Note that we can not/(or now?) assume $k = 1$, consider the orbit $\mathcal{O}'' = (2, 1, 1, 1, 1)$.

- (i) $x'_1 = s$, now $y'_0 = \emptyset/\bullet$ when $x'_0 = \emptyset/s$
 - (a) Suppose $x'_2 \neq r$. Then $(y'_2, y'_1, y'_3) = (x'_2, r, d)$, $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, x'_2, c, d)$. Therefore, the sign difference is $\text{Sign}(s, c, d) - \text{Sign}(s) = (2, 2)$.
 - (b) Suppose $x'_2 = r$. Then $(y'_2, y'_1, y'_3) = (c, r, d)$. $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, c, r, d)$. Therefore, the sign difference is $\text{Sign}(s, c, r, d) - \text{Sign}(s, r) = (2, 2)$.
- (ii) $x'_1 \neq s$.
 - (a) Suppose $x'_0 \neq c$. Then $x'_0 = \emptyset/s/r$, $x'_1 = r/c/d$, $(y'_0, y'_2, y'_1, y'_3) = (\emptyset/\bullet/r, x'_2, r, x'_1)$.
 - 1). $x'_1 = r$. Then $(x_0, y_0, y_2, y_1, y_3) = (r, x'_0, x'_2, c, d)$. Therefore, the sign difference is $\text{Sign}(r, c, d) - \text{Sign}(r) = (2, 2)$.
 - 2). $x'_1 = c$. Then $x_1 = d$. $(x_0, y_0, y_2, y_1, y_3) = (c, x'_0, x'_2, c, d)$. Therefore, the sign difference is $\text{Sign}(c, c, d) - \text{Sign}(c) = (2, 2)$.
 - 3). $x'_1 = d$. Then $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, y'_2, y'_1, y'_3) = (s, x'_0, x'_2, r, d)$. Therefore, the sign difference is $\text{Sign}(s, r, d) - \text{Sign}(d) = (2, 2)$.
 - (b) $(x_0, y_0, y_2, y_1, y_3) = (s, r, x'_2, c, d)$. Therefore, the sign difference is $\text{Sign}(s, r, x'_2, c, d) - \text{Sign}(c, d, x'_2) = (2, 2)$.

- (b) Suppose $x'_0 = c$. Then $x'_1 = d$, $(y'_0, y'_2, y'_1, y'_3) = (r, x'_2, c, d)$. $(x_0, y_0, y_2, y_1, y_3) = (s, r, x'_2, c, d)$. Therefore, the sign difference is $\text{Sign}(s, r, x'_2, c, d) - \text{Sign}(c, d, x'_2) = (2, 2)$.

]

□

3.4.6. *A key proposition in the generalized descent case.* We now assume that $k \geq 1$ and C_{2k} is odd, i.e. $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a generalized descent.

Without loss of generality, we can assume the dot-r-c diagrams have the following shapes with $n = (C_{2k+1} - C_{2k} + 1)/2$:

$$(3.4) \quad \tau : \begin{array}{c} \begin{array}{|c|c|c|c|} \hline \dots & \dots & \dots & \dots \\ \hline * & * & * & * \\ \hline \end{array} & \begin{array}{|c|c|} \hline \dots & \dots \\ \hline * & * \\ \hline \end{array} & \xrightarrow{\bar{\nabla}_1} & \tau' : \begin{array}{c} \begin{array}{|c|c|c|} \hline \dots & \dots & \dots \\ \hline * & * & * \\ \hline \end{array} & \begin{array}{|c|c|} \hline \dots & \dots \\ \hline * & * \\ \hline \end{array} & \xrightarrow{\bar{\nabla}_1} & \tau'' : \begin{array}{c} \begin{array}{|c|c|c|} \hline \dots & \dots & \dots \\ \hline * & * & * \\ \hline \end{array} & \begin{array}{|c|c|} \hline \dots & \dots \\ \hline * & * \\ \hline \end{array} \end{array}$$

The diagram τ is obtained from the diagram of τ by adding \bullet at the grey parts, changing $x_{\tau''}$ to x_0 and attaching $x_1 \cdots x_n$.

We define

$$\mathbf{p}_\tau := \begin{array}{c} x_1 \ x_0 \\ \vdots \\ x_n \end{array}$$

and call \mathbf{p}_τ the peduncle part of τ . Let $\mathcal{O}_1 = (C_{2k+1} - C_{2k} + 1,) \in \text{Nil}^{\text{dpe}}(D)$. We define $\mathbf{u}_\tau \in \text{DRC}(\mathcal{O}_1)$ by the following formula:

$$(3.5) \quad \mathbf{u}_\tau := u_1 \cdots u_n = \begin{cases} r \cdots rc, & \text{when } \mathbf{p}_\tau = \begin{array}{c} r \\ \vdots \\ r \end{array} \begin{array}{c} c \\ \\ \end{array} \\ r \cdots cd, & \text{when } \mathbf{p}_\tau = \begin{array}{c} r \\ \vdots \\ d \end{array} \begin{array}{c} c \\ \\ \end{array} \\ x_1 \cdots, x_n & \text{otherwise.} \end{cases}$$

Let $\tilde{\mathcal{O}} = (C_{2k+1} - C_{2k} + 1, 1, 1,) \in \text{Nil}^{\text{dpe}}(D)$ and $\tilde{\mathcal{O}}'' = \bar{\nabla}^2(\tilde{\mathcal{O}}) = (2) \in \text{Nil}^{\text{dpe}}(D)$. Then $\mathbf{p}_\tau \in \text{DRC}(\tilde{\mathcal{O}})$ and $x_\tau \in \text{Nil}^{\text{dpe}}(D)$. There is some restriction on the possibilities of \mathbf{u}_τ .

Lemma 3.7. *Let*

$$\tau = \begin{array}{c} x_1 \ x_0 \\ \vdots \\ x_n \end{array} \in \text{DRC}(\tilde{\mathcal{O}}) \quad \text{and} \quad \tau'' := \bar{\nabla}_1^2(\tau).$$

Then we have the following properties which is easy to verify.

- i) If $\tau'' = r$, then $s \in \mathbf{u}_\tau$ or $c \in \mathbf{u}_\tau$. In particular, $\text{Sign}(\mathbf{u}_\tau) > (0, 1)$. If $\text{Sign}(\mathbf{u}_\tau) \nprec (0, 2)$, we have

$$\tau = \begin{array}{c} r \ c \\ \vdots \\ r \end{array} \quad \text{and} \quad \mathbf{u}_\tau = \begin{array}{c} r \\ \vdots \\ c \end{array}.$$

- ii) If $\tau'' = c$, then $s \in \mathbf{u}_\tau$ or $c \in \mathbf{u}_\tau$. In particular, $\text{Sign}(\mathbf{u}_\tau) \geq (0, 1)$.

- iii) If $\tau'' = d$, there is no restriction on \mathbf{u}_τ .

- iv) We have $x_n = d$ if and only $d \in \mathbf{u}_\tau$.

- v) If $x_n = d$ and $\tau'' = r/c$, we have $n \geq 2$ and $\text{Sign}(\mathbf{u}_\tau) \geq (0, 2)$.

- vi) If $\mathbf{u}_\tau = r \cdots r$, we have $\tau'' = d$.

Lemma 3.8. *The map $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$ given by $\tau \mapsto (\bar{\nabla}_1^2(\tau), \mathbf{u}_\tau)$ is injective. Moreover,*

$$\text{Sign}(\tau) = \text{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \text{Sign}(\mathbf{u}_\tau).$$

The map $\tilde{\delta}: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ given by $\tau \mapsto (\bar{\nabla}_1^2(\tau), \text{Sign}(\tau), \epsilon_\tau)$ is injective.

Proof. By our algorithm,

$$(x_{\tau''}, \mathbf{u}_{\tau}) = (x_{\nabla_1^2(\mathbf{p}_{\tau})}, \mathbf{u}_{\mathbf{p}_{\tau}}).$$

The injectivity of δ and the signature formula follows directly from the definition of \mathbf{u}_{τ} . Now the injectivity of $\tilde{\delta}$ follows from the injectivity of $\mathbf{u}_{\tau} \mapsto (\text{Sign}(\mathbf{u}_{\tau}), \epsilon)$ by Lemma 5.2. \square

4. CONVENTION ON THE LOCAL SYSTEM

An irreducible local system is represented by a *marked Young diagram*. A general local system is understood as the union of irreducible ones. By our computation later, it would be clear that the local systems appeared in our cases are always *multiplicity one*.

4.1. Commutative free monoid. Given a set \mathbf{A} , the commutative free monoid $\mathbf{C}(\mathbf{A})$ on \mathbf{A} consists all finite multisets with elements drawn from \mathbf{A} . It is equipped with the binary operation, say \bullet such that

$$\{a_1, \dots, a_{k_1}\} \bullet \{b_1, \dots, b_{k_2}\} := \{a_1, \dots, a_{k_1}, b_1, \dots, b_{k_2}\}.$$

In the following, we always identify \mathbf{A} with the subset of $\mathbf{C}(\mathbf{A})$ consists of singletons.

Suppose \mathbf{M} is a commutative monoid, a map $f: \mathbf{A} \rightarrow \mathbf{M}$ is uniquely extends to a morphism $\mathbf{C}(\mathbf{A}) \rightarrow \mathbf{M}$ which is still called f by abuse of notations.

In particular a map $f: \mathbf{A} \rightarrow \mathbf{B}$ between sets uniquely extends to an morphism $f: \mathbf{C}(\mathbf{A}) \rightarrow \mathbf{C}(\mathbf{B})$ by

$$f(\{a_1, \dots, a_k\}) = \{f(a_1), \dots, f(a_k)\}.$$

4.2. Marked Young diagram. We now explain the combinatorial objects parameterizing local systems. Basically, we use $+/-$ (resp. $*/=$) to denote the associated character on the corresponding factor of the isotropic group. Let $\star = B, C, D, M$.

- **Young diagram** Let \mathbf{YD} be the commutative free monoid $\mathbf{C}(\mathbb{N}^+)$ on the set of positive integers $\mathbb{Z}_{\geq 1}$. An element $\mathcal{O} = \{R_1, \dots, R_k\}$ is identified with the Young diagram whose set of row lengths is \mathcal{O} . Therefore, we identify \mathbf{YD} with the set of Young diagrams. Let

$$|\cdot| : \mathbf{YD} \longrightarrow \mathbb{N}$$

be the map sending $\{R_1, \dots, R_k\}$ to $\sum_{i=1}^k R_i$.

Let $\mathbf{YD}(\star)$ to denote the Young diagrams corresponding to partitions of type \star . The set $\mathbf{YD}(\star)$ parameterizes the set of nilpotent orbits of type \star . Note that $\mathbf{YD}(C) = \mathbf{YD}(M)$.

- **Signed Young diagram** Let \mathbf{S} be the set of non-empty finite length strings of alternating signs of the form $+ - + \dots$ or $- + - \dots$. Let $\mathbf{SYD} := \mathbf{C}(\mathbf{S})$ be the monoid on \mathbf{S} .

The monoid \mathbf{SYD} is identified with the set of all signed Young diagrams: $\mathcal{O} = \{\mathbf{r}_1, \dots, \mathbf{r}_k\} \in \mathbf{SYD}$ is identified with the signed Young diagram whose rows are filled by \mathbf{r}_j . The map $\mathbf{S} \rightarrow \mathbb{Z}_{\geq 1}$ sending a string to its length extends to the morphism

$$\begin{aligned} \mathcal{Y}: \mathbf{SYD} &\longrightarrow \mathbf{YD} \\ \{\mathbf{r}_1, \dots, \mathbf{r}_k\} &\longmapsto \{|\mathbf{r}_1|, \dots, |\mathbf{r}_k|\} \end{aligned}$$

where $|\mathbf{r}_j|$ denote the length of the string \mathbf{r}_j . By abused of notation, we also denote $|\cdot| \circ \mathcal{Y}$ by $|\cdot|$.

We define $\text{SYD}(\star)$ to be a subset of the preimage of $\text{YD}(\star)$ under the above map: When $\star = C, M$ (resp. $\star = D, B$),

$$\text{SYD}(\star) = \left\{ \mathcal{O} \in \text{SYD} \left| \begin{array}{l} \mathcal{Y}(\mathcal{O}) \in \text{YD}(\star) \\ \text{For each positive odd number (resp.} \\ \text{even number) } r, \text{ the strings } + - \cdots \\ \text{and } - + \cdots \text{ of the length } r \text{ ap-} \\ \text{pears in } \mathcal{O} \text{ with the same multiplic-} \\ \text{ity (could be 0).} \end{array} \right. \right\}$$

The set $\text{SYD}(\star)$ parameterizes the real nilpotent orbits of type \star and \mathbf{K} -orbits in \mathfrak{p} .

We define two signature maps $\mathbf{S} \rightarrow \mathbb{N} \times \mathbb{N}$ as the following. For $\mathbf{r} \in \mathbf{S}$, define

$$\text{Sign}(\mathbf{r}) := (\# + (\mathbf{r}), \# - (\mathbf{r})) \quad \text{and}$$

$${}^l\text{Sign}(\mathbf{r}) := \begin{cases} (1, 0) & \text{if } \mathbf{r} = + \cdots \\ (0, 1) & \text{if } \mathbf{r} = - \cdots \end{cases}$$

The above defined signature maps naturally extends to signature maps $\text{Sign}: \text{SYD} \rightarrow \mathbb{N} \times \mathbb{N}$ and $\text{Sign}: \text{SYD} \rightarrow \mathbb{N} \times \mathbb{N}$.

- **Marked Young diagram** Let \mathbf{M} be the set of finite length strings of alternating marks of the form $+ - + \cdots, - + - \cdots, * = * \cdots$, or $= * = \cdots$.

Let $\text{MYD} := \mathbf{C}(\mathbf{M})$ which is identified with the set of Young diagrams marked with alternating marks $+/-$, $*/=$. Define $\mathcal{F}: \mathbf{M} \rightarrow \mathbf{S}$ by the formula

$$\mathcal{F}(\mathbf{r}) := \begin{cases} + - + \cdots & \text{if } \mathbf{r} = + - + \cdots \text{ or } * = * \cdots \\ - + - \cdots & \text{if } \mathbf{r} = - + - \cdots \text{ or } = * = \cdots \end{cases} \quad \forall \mathbf{r} \in \mathbf{M}$$

such that $\mathcal{F}(\mathbf{r})$ and \mathbf{r} have the same length. \mathcal{F} extends to the map $\mathcal{F}: \text{MYD} \rightarrow \text{SYD}$.

By abused of notation, we also denote $| \circ \mathcal{Y}$ by $| \cdot |$.

For $\mathcal{L} \in \text{MYD}$, we define

$$\text{Sign}(\mathcal{L}) = \text{Sign}(\mathcal{F}(\mathcal{L})) \quad \text{and} \quad {}^l\text{Sign}(\mathcal{L}) = {}^l\text{Sign}(\mathcal{F}(\mathcal{L})).$$

We define $\text{MYD}(\star)$ to be a subset of the preimage of the above map:

$$\text{MYD}(\star) = \left\{ \mathcal{L} \in \text{MYD}(\star) \left| \begin{array}{l} \mathcal{F}(\mathcal{L}) \in \text{SYD}(\star); \\ \mathbf{r}_1 = \mathbf{r}_2 \text{ if } \mathbf{r}_1, \mathbf{r}_2 \in \mathcal{L} \text{ and } \mathcal{F}(\mathbf{r}_1) = \mathcal{F}(\mathbf{r}_2); \\ \mathbf{r} = + - + \cdots \text{ or } - + - \cdots \text{ if the length} \\ \text{of } \mathbf{r} \text{ is odd (resp. even) when } \star \in \{C, M\} \\ \text{(resp. } \star \in \{B, D\}). \end{array} \right. \right\}$$

We will use $\text{MYD}(\star)$ to parameterize irreducible local systems attached to the nilpotent orbits of type \star .

For monoid YD , SYD or MYD , the identity element is the empty set \emptyset , and the monoid binary operator is denoted by “ \cdot ”. The \emptyset is always in $\text{MYD}(\star)$ by convention. According to the definition of $\text{MYD}(\star)$, it could happen that $\mathcal{L}_1 \cdot \mathcal{L}_2 \notin \text{MYD}(\star)$ for $\mathcal{L}_1, \mathcal{L}_2 \in \text{MYD}(\star)$.

We define an involution

$$(4.1) \quad \overline{}: \mathbf{M} \rightarrow \mathbf{M}$$

on \mathbf{M} by switching the symbols $+$ with $*$ and $-$ with $=$. For example $\overline{+ - +} = * = *$ and $\overline{=*} = - +$.

We define the map

$$(4.2) \quad \dagger: \mathbf{M} \rightarrow \mathbf{M}$$

by extending the length of the strings to the left by one. For example $\dagger(+ - +) = - + - +$ and $\dagger(=*) = *=*$.

For $\mathcal{L}, \mathcal{C} \in \text{MYD}$, the expression $\mathcal{L} \succ \mathcal{C}$ means that there is $\mathcal{B} \in \text{MYD}$ such that the factorization $\mathcal{L} = \mathcal{B} \cdot \mathcal{C}$ holds.

In the following, we will represent an element $\mathcal{L} \in \text{MYD}$ by the Young diagram whose rows are filled with the strings in \mathcal{L} . The terminology “ l -row” means a row of length l . If we do not care whether the mark is $*$ or $+$ (resp. $=$ or $=$), we will use $*$ (resp. \div) to mark the corresponding position.

Example 4.1. *The following two diagrams $\dagger_{p,q}$ and $\dagger_{p,q}$ are in $\text{MYD}(B) \cup \text{MYD}(D)$.*

$$\dagger_{p,q} := \begin{array}{c} \dagger \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array}$$

represents the local system on the trivial orbit of $O(p, q)$ which has the trivial character on $O(p)$ and \det on $O(q)$. Similarly,

$$\dagger_{p,q} := \begin{array}{c} \dagger \\ \vdots \\ + \\ - \\ \vdots \\ - \end{array}$$

represents the trivial local system on the trivial orbit.

4.2.1. Local systems. We define \mathcal{K}_M to be the free commutative monoid generated by MYD . The binary operation in \mathcal{K}_M is denoted by “ $+$ ”.

The operation “ \cdot ” naturally extends to \mathcal{K}_M :

$$\left(\sum_i m_i \mathcal{L}_i \right) \cdot \left(\sum_j m_j \mathcal{L}'_j \right) := \sum_{i,j} m_i m_j (\mathcal{L}_i \cdot \mathcal{L}'_j).$$

where $\mathcal{L}_i, \mathcal{L}'_j \in \text{MYD}$, $m_i, m_j \in \mathbb{Z}_{\geq 0}$.

Now $(\mathcal{K}_M, +, \cdot)$ is in fact a commutative semiring. We let 0 be the additive identity element in \mathcal{K}_M and \emptyset is the multiplicative identity. Note that $\mathcal{L} \cdot 0 = 0$ for any $\mathcal{L} \in \mathcal{K}_M$ by definition.

For $\mathcal{L}, \mathcal{D} \in \mathcal{K}_M$, $\mathcal{L} \succ \mathcal{D}$ means that there exists $\mathcal{B} \in \mathcal{K}_M$ such that $\mathcal{L} = \mathcal{B} \cdot \mathcal{D}$. In addition, we write $\mathcal{L} \supset \mathcal{D}$ if there is $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{K}_M$ such that $\mathcal{L}_1 \succ \mathcal{D}$ and $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}$.

Let $\mathcal{K}_M(\star)$ be the sub-monoid of \mathcal{K}_M generated by $\text{MYD}(\star)$. Note that $\mathcal{K}_M(\star)$ is not a sub semi-ring of \mathcal{K}_M .

For $\mathcal{L} = \sum_{i=1}^k \mathcal{L}_i \in \mathcal{K}_M$ with $\mathcal{L}_i \in \text{MYD}$, we let

$$\text{Sign}(\mathcal{L}) = \{ \text{Sign}(\mathcal{L}_i) \mid i = 1, \dots, k \} \text{ and } {}^l\text{Sign}(\mathcal{L}) = \{ {}^l\text{Sign}(\mathcal{L}_i) \mid i = 1, \dots, k \}$$

4.3. Operations on $\mathcal{K}_M(\star)$. The operation \dagger defined in (4.2) extends to an operation

$$\dagger: \mathcal{K}_M \longrightarrow \mathcal{K}_M$$

on \mathcal{K}_M which adding a column on the left of the original marked Young diagram.

4.3.1. Character twists on $\mathcal{K}_M(D)$ and $\mathcal{K}_M(B)$. Recall that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ parameterizes the characters of a indefinite orthogonal group $O(p, q)$:

$$\chi = (\chi^+, \chi^-) \longleftrightarrow (\mathbf{1}^{-,+})^{\chi^+} \otimes (\mathbf{1}^{+,-})^{\chi^-}.$$

We define the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on \mathbf{M} by:

$$\mathbf{r} \otimes \chi := \begin{cases} \bar{\mathbf{r}} & \text{if } p + q \equiv 1 \text{ and } \chi^+ p + \chi^- q \equiv 1 \pmod{2} \\ \mathbf{r} & \text{otherwise} \end{cases}$$

where $\mathbf{r} \in \mathbf{M}$, $\chi = (\chi^+, \chi^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $(p, q) = \text{Sign}(\mathbf{r})$.

The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action extends to actions on $\mathcal{K}_{\mathbf{M}}(D)$ and $\mathcal{K}_{\mathbf{M}}(B)$.

4.3.2. *Twists on $\mathcal{K}_{\mathbf{M}}(C)$ and $\mathcal{K}_{\mathbf{M}}(M)$.* We define an involution $\mathbf{\Psi}_C: \mathbf{M} \rightarrow \mathbf{M}$ on \mathbf{M} by

$$\mathbf{\Psi}(\mathbf{r}) := \begin{cases} \bar{\mathbf{r}} & \text{if } |\mathbf{r}| \equiv 2 \pmod{4} \\ \mathbf{r} & \text{otherwise.} \end{cases}$$

The involution $\mathbf{\Psi}$ extends to an involution on $\mathcal{K}_{\mathbf{M}}$. We will only apply $\mathbf{\Psi}$ on $\mathcal{L} \in \mathcal{K}_{\mathbf{M}}(C)$ or $\mathcal{K}_{\mathbf{M}}(M)$.

4.3.3. *Truncations.* We define the Truncation maps

$$\mathbf{T}^+: \mathcal{K}_{\mathbf{M}} \longrightarrow \mathcal{K}_{\mathbf{M}}$$

such that for $\mathcal{L} \in \text{MYD}$

$$\mathbf{T}^+(\mathcal{L}) = \begin{cases} \mathcal{L}_1 & \text{if there is } \mathcal{L}_1 \in \text{MYD} \text{ such that } \mathcal{L}_1 \cdot + = \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathbf{T}^-: \mathcal{K}_{\mathbf{M}} \longrightarrow \mathcal{K}_{\mathbf{M}}$ similarly such that $\mathcal{L} = \mathbf{T}^-(\mathcal{L}) \cdot -$ if $\mathcal{L} > -$ and $\mathbf{T}^-(\mathcal{L}) = 0$ otherwise for $\mathcal{L} \in \text{MYD}$.

To symplify the notation, we write $\mathcal{L}^\pm := \mathbf{T}^\pm(\mathcal{L})$ for $\mathcal{L} \in \mathcal{K}_{\mathbf{M}}$.

4.4. Theta lifts of local systems.

4.4.1. *Lift from type C to D.* For each signature $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p + q$ is even, we define a map

$$\vartheta_{CD, (p, q)}: \mathcal{K}_{\mathbf{M}}(C) \rightarrow \mathcal{K}_{\mathbf{M}}(D)$$

as the following: Let $\mathcal{L} \in \text{MYD}(C)$. Let $(p_1, q_1) = {}^l\text{Sign}(\mathcal{L})$ and

$$(p_0, q_0) = (p, q) - \text{Sign}(\mathcal{L}) - (q_1, p_1).$$

We define

$$\vartheta_{CD, (p, q)}(\mathcal{L}) = \begin{cases} (\dagger \mathbf{\Psi}^{\frac{p-q}{2}}(\mathcal{L})) \cdot \dagger_{p_0, q_0} & \text{if } p_0 \geq 0 \text{ and } q_0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

4.4.2. *Lift from type D to type C.* For each non-zero integer $n \in \mathbb{N}$, we define a map

$$\vartheta_{DC, n}: \mathcal{K}_{\mathbf{M}}(D) \rightarrow \mathcal{K}_{\mathbf{M}}(C)$$

as the following: Let $\mathcal{L} \in \text{MYD}(D)$. Let $(p_1, q_1) = {}^l\text{Sign}(\mathcal{L})$ and $(p, q) = \text{Sign}(\mathcal{L})$. Let

$$n_0 = n - (p + q)/2 - (p_1 + q_1)/2$$

We define

$$\vartheta_{DC, n}(\mathcal{L}) = \begin{cases} \mathbf{\Psi}^{\frac{p-q}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geq 0} \\ \mathbf{\Psi}^{\frac{p-q}{2}}(\dagger \mathcal{L}^+ + \dagger \mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that $\mathbf{\Psi}^{\frac{p-q}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) = (\mathbf{\Psi}^{\frac{p-q}{2}}(\dagger \mathcal{L})) \cdot \dagger_{n_0, n_0}$.

[We take a splitting $\text{O}(p, q) \times \text{Sp}(n, \mathbb{R})$ in to the big metaplectic group Mp such that $\text{O}(p, \mathbb{C}) \times \text{O}(q, \mathbb{C})$ acts on the Fock model linearly. The maximal compact $K_{\text{Sp}(n, \mathbb{R})} = \text{U}(2n)$ acts on the Fock model by the character $\zeta = \det^{(p-q)/2}$. For the component of the rational

nilpotent orbit $\mathrm{Sp}(2n, \mathbb{R})$, the factor of the component group is Sp if the corresponding row has odd length. Otherwise the component group is $\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})$ where (p_{2k}, q_{2k}) is the signature corresponding to the $2k$ -rows. $\zeta|_{\mathrm{O}(p_{2k})} = \det_{\mathrm{O}(p_{2k})}^{(p-q)k/2}$ which is nontrivial if and only if $p - q \equiv 2 \pmod{4}$ and $2k \equiv 2 \pmod{4}$. This gives the formula above.]

In the proof later, we don't care above the marks for rows with length longer than 1 in the most of the case. So we will omit $\boxtimes_{(p,q)}$. When there is no confusion, we will simply write ϑ for $\vartheta_{CD,(p,q)}$ and $\vartheta_{CD,(p,q)}$.

4.4.3. *Lift from type M to B.* For each signature $(p, q) \in \mathbb{N} \times \mathbb{N}$ such that $p + q$ is odd, we define a map

$$\vartheta_{MB,(p,q)}: \mathcal{K}_M(M) \rightarrow \mathcal{K}_M(B)$$

as the following: Let $\mathcal{L} \in \mathrm{MYD}(M)$. Let $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and

$$(p_0, q_0) = (p, q) - \mathrm{Sign}(\mathcal{L}) - (q_1, p_1).$$

We define

$$\vartheta_{MB,(p,q)}(\mathcal{L}) = \begin{cases} (\dagger \boxtimes^{\frac{p-q+1}{2}}(\mathcal{L})) \cdot \dagger_{p_0, q_0} & \text{if } p_0 \geq 0 \text{ and } q_0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

4.4.4. *Lift from type B to type M.* For each non-zero integer $n \in \mathbb{N}$, we define a map

$$\vartheta_{BM,n}: \mathcal{K}_M(B) \rightarrow \mathcal{K}_M(M)$$

as the following: Let $\mathcal{L} \in \mathrm{MYD}(D)$. Let $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and $(p, q) = \mathrm{Sign}(\mathcal{L})$. Let

$$n_0 = n - (p + q)/2 - (p_1 + q_1)/2$$

We define

$$\vartheta_{BM,n}(\mathcal{L}) = \begin{cases} \boxtimes^{\frac{p-q-1}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geq 0} \\ \boxtimes^{\frac{p-q-1}{2}}(\dagger \mathcal{L}^+ + \dagger \mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that $\boxtimes^{\frac{p-q-1}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) = (\boxtimes^{\frac{p-q-1}{2}}(\dagger \mathcal{L})) \cdot \dagger_{n_0, n_0}$.

[For odd orthogonal-metaplectic group case, we still take the splitting such that $\mathrm{O}(p, q)$ acts linearly.

The maximal compact $\tilde{K}_{\mathrm{Sp}(n, \mathbb{R})} = \widetilde{\mathrm{U}(2n)}$ acts on the Fock model by the character $\zeta = \det^{(p-q)/2}$.

Then $\det^{(p-q)/2}|_{\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})} = \det_{\mathrm{O}(p_{2k})}^{\frac{(p-q)k}{2}} \boxtimes \det_{\mathrm{O}(q_{2k})}^{\frac{(p-q)k}{2}}$. Note that $p - q$ is an odd number. When k is even the character on $\widetilde{\mathrm{O}(p_{2k})/\mathrm{O}(q_{2k})}$ are $\mathbf{1}$ or \det . When k is odd the character would be $\det^{\frac{1}{2}}$ or $\det^{\frac{1}{2}+1}$.

We assume the default character on $2k$ -rows is $\det^{\frac{k}{2}}$, and we use $+/-$ to mark the row. Otherwise, we use $*/=$ to mark the rows.]

4.5. Examples and remarks.

4.5.1. Example.

Example 4.2. Let $\mathcal{T} := \dagger \dagger (\dagger_{2,1}) + \dagger \dagger (\dagger_{1,2})$ and $\mathcal{P} = \dagger_{1,3}$. Then

$$\mathcal{T} \cdot \mathcal{P} = \begin{array}{cc} \begin{array}{c} * \div * \\ * \div * \\ \div * \div \end{array} & \begin{array}{c} * \div * \\ \div * \div \\ \div * \div \end{array} \\ + & \cup + \\ = & = \\ = & = \\ = & = \end{array}.$$

4.5.2. *Signs of the local systems obtained by iterated lifting.* Suppose $\mathcal{L} \in \mathcal{K}_M$ is obtained by iterated theta lifting and character twisting in Section 4.3.1. Then the set ${}^l\text{Sign}(\mathcal{L})$ has restricted possibilities. First, $\text{Sign}(\mathcal{L})$ is always a singleton.

When \mathcal{O} is of type B/D, ${}^l\text{Sign}(\mathcal{L})$ is a singleton $\{(p_1, q_1)\}$ where

$$(p_1, q_1) = \text{Sign}(\mathcal{L}) - (n_0, n_0) \quad \text{and} \quad 2n_0 = |\overline{\nabla}(\mathcal{O})|.$$

When \mathcal{O} is of type C/M, ${}^l\text{Sign}(\mathcal{L}) = \{(p_1, q_1)\}$ is a singleton in the most of the case. The ${}^l\text{Sign}(\mathcal{L})$ may have two elements if \mathcal{L} is obtained by good generalized lifting, and it has the form ${}^l\text{Sign}(\mathcal{L}) = \{(p_1 + 1, q_1), (p_1, q_1 + 1)\}$.

5. PROOF OF MAIN THEOREM IN THE TYPE C/D CASE

We do induction on the number of columns of the nilpotent orbits.

5.1. **Notation.** In this section, \mathcal{O} is always an nilpotent orbit of type D in $\text{Nil}^{\text{dpe}}(D)$. We always let

$$\begin{aligned} \mathcal{O} &= (C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0), \\ \mathcal{O}' &= \overline{\nabla}(\mathcal{O}) = (C_{2k}, C_{2k-1}, \dots, C_1, C_0). \end{aligned}$$

When $k \geq 1$, we let

$$\mathcal{O}'' = \overline{\nabla}^2(\mathcal{O}) = \begin{cases} (C_{2k-1}, \dots, C_1, C_0) & \text{when } C_{2k} \text{ is even,} \\ (C_{2k-1} + 1, \dots, C_1, C_0) & \text{when } C_{2k} \text{ is odd.} \end{cases}$$

For a $\tau \in \text{DRC}(\mathcal{O})$, we let

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \quad \text{and} \quad (\tau'', \epsilon') = \overline{\nabla}(\tau') \quad (\text{when } k \geq 1).$$

5.1.1. *Main proposition.* The aim of this section is to prove the following main proposition holds for $\mathcal{O} \in \text{Nil}^{\text{dpe}}(D)$.

Proposition 5.1. *We have:*

- i) \mathcal{L}_τ is non-zero. In particular, $\pi_\tau \neq 0$.
- ii) Suppose $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$, then $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.
- iii) Suppose $\tau_1 \neq \tau_2 \in \text{DRC}(\mathcal{O})$, then $\pi_{\tau_1} \neq \pi_{\tau_2}$.
- iv) The local system \mathcal{L}_τ is disjoint with their determinant twist:

$$(5.1) \quad \{\mathcal{L}_\tau \mid \tau \in \text{DRC}(\mathcal{O})\} \cap \{\mathcal{L}_\tau \otimes \det \mid \tau \in \text{DRC}(\mathcal{O})\} = \emptyset.$$

v) We have

- (i) $x_\tau = s$, then $\mathcal{L}_\tau^+ = \mathcal{L}_\tau^- = 0$.
- (ii) $x_\tau = r/c$, then $\mathcal{L}_\tau^+ \neq 0$ and $\mathcal{L}_\tau^- = 0$.
- (iii) $x_\tau = d$, then $\mathcal{L}_\tau^+ \neq 0$ and $\mathcal{L}_\tau^- \neq 0$.

vi) If $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a usual descent, we have a simpler factorization:

$$(5.2) \quad \mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_{x_\tau}.$$

- vii) When \mathcal{O}' is noticed, the map $\mathcal{L}: \text{DRC}(\mathcal{O}') \rightarrow {}^\ell\text{LS}(\mathcal{O}')$ is a bijection.
- viii) When \mathcal{O} is noticed, the map $\mathcal{L}: \text{DRC}(\mathcal{O}) \rightarrow {}^\ell\text{LS}(\mathcal{O})$ is a bijection.
- ix) When \mathcal{O} is +noticed, the maps

$$\begin{aligned} \Upsilon_{\mathcal{O}}^+ : \{\mathcal{L}_\tau \mid \mathcal{L}_\tau^+ \neq 0\} &\longrightarrow \{\mathcal{L}_\tau^+\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^+ & \text{and} \\ \Upsilon_{\mathcal{O}}^- : \{\mathcal{L}_\tau \mid \mathcal{L}_\tau^- \neq 0\} &\longrightarrow \{\mathcal{L}_\tau^-\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^- \end{aligned}$$

are injective.
 x) When \mathcal{O} is noticed, the map

$$(5.3) \quad \begin{array}{ccc} \Upsilon_{\mathcal{O}}: \{ \mathcal{L}_{\tau} \mid \mathcal{L}_{\tau}^{+} \neq 0 \} & \longrightarrow & \{ (\mathcal{L}_{\tau}^{+}, \mathcal{L}_{\tau}^{-}) \mid \mathcal{L}_{\tau}^{+} \neq 0 \} \\ \mathcal{L}_{\tau} & \longmapsto & (\mathcal{L}_{\tau}^{+}, \mathcal{L}_{\tau}^{-}) \end{array}$$

is injective.³

Note that, $\text{DRC}(\mathcal{O})$ counts the unipotent representations of special orthogonal groups a priori. But Proposition 5.1 (iv)), implies that the representation π_{τ} and $\pi_{\tau} \otimes \det$ are two different representations of the orthogonal groups. Therefore $\text{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z}$ counts the set of unipotent representations of orthogonal groups.

The main theorem would be a consequence of Proposition 5.1.

5.1.2. *How to think about the factorization of local systems.* When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is the usual descent, the local system \mathcal{L}_{τ} could be compared to a water hydra, see in Figure 1.

The tentacles part $\mathcal{T}_{\tau} = {}^{\dagger}\mathcal{L}_{\tau'}$ could be reducible. The peduncle part $\mathcal{P}_{\tau} := \mathcal{L}_{\mathbf{x}_{\tau}}$ and $\mathcal{L}_{\tau} = \mathcal{T}_{\tau} \cdot \mathcal{P}_{\tau}$.

When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a generalized descent, the situation is more complicated as the local system \mathcal{L}_{τ} may consists of two hydras, see (5.8).

5.2. The initial case: When $k = 0$, $\mathcal{O} = (C_1 = 2c_1, C_0 = 2c_0)$ has at most two columns. Now $\pi_{\tau'}$ is the trivial representation of $\text{Sp}(2c_0, \mathbb{R})$. The lift $\pi_{\tau'} \mapsto \bar{\Theta}(\pi_{\tau'})$ is in fact a stable range theta lift. For $\tau \in \text{DRC}(\mathcal{O})$, let $(p_1, q_1) = \text{Sign}(\mathbf{x}_{\tau})$ and x_{τ} be the foot of τ . It is easy to see that (using associated character formula)

$$\mathcal{L}_{\pi_{\tau}} = \mathcal{T}_{\tau} \cdot \mathcal{P}_{\tau}, \text{ where } \mathcal{T}_{\tau} := {}^{\dagger}\mathcal{L}_{c_0, c_0}, \text{ and } \mathcal{P}_{\tau} := \begin{cases} {}^{\dagger}\mathcal{L}_{p_1, q_1} & x_{\tau} \neq d, \\ {}^{\dagger}\mathcal{L}_{p_1, q_1} & x_{\tau} = d. \end{cases}$$

Note that $\mathcal{L}_{\pi_{\tau}}$ is irreducible.

³In fact, we only need to know whether \mathcal{L}_{τ}^{-} is non-zero or not.

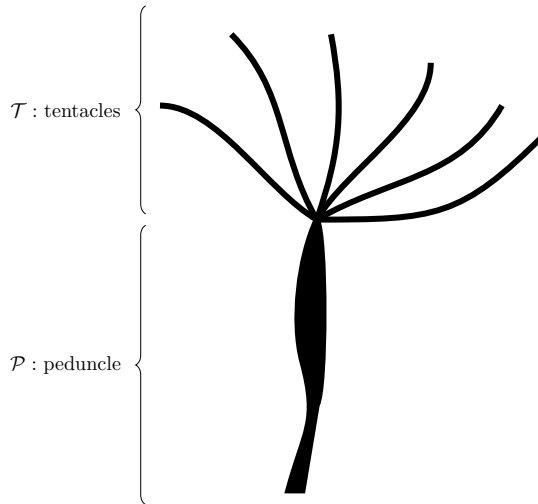


FIGURE 1. Freshwater hydra

Let $\mathcal{O}_1 = (2(c_1 - c_0)) \in \text{Nil}^{\text{dpe}}(D)$. Using the above formula, one can check that the following maps are bijections

$$\begin{array}{ccc} \text{DRC}(\mathcal{O}_1) & \longleftarrow & \text{DRC}(\mathcal{O}) \longrightarrow {}^\ell\text{LS}(\mathcal{O}) \\ \mathbf{x}_\tau & \longleftarrow & \tau \longrightarrow \mathcal{L}_\tau. \end{array}$$

To see that $\text{DRC}(\mathcal{O}) \ni \tau \mapsto \mathcal{L}_\tau$ is an injection, one check the following lemma holds:

Lemma 5.2. *The map $\text{DRC}(\mathcal{O}_1) \longrightarrow \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ given by $\tau \mapsto (\text{Sign}(\tau), \epsilon_\tau)$ is injective (see Section 6.2.1 for the definition of ϵ_τ). Moreover, $\epsilon_\tau = 0$ only if $\text{Sign}(\tau) \geq (0, 1)$.*

Moreover, $\mathcal{L}_\tau \otimes \det \notin {}^\ell\text{LS}(\mathcal{O})$ for any $\tau \in \text{DRC}(\mathcal{O})$. In fact, if $\text{Sign}(\mathbf{x}_\tau) \geq (1, 0)$, \mathcal{L}_τ on the 1-rows with +-signs is trivial and $\mathcal{L}_\tau \otimes \det$ has non-trivial restriction. When $\text{Sign}(\mathbf{x}_\tau) \not\geq (1, 0)$, $\mathbf{x}_\tau = s \cdots s$, all 1-rows of \mathcal{L}_τ are marked by “=” and all 1-rows of $\mathcal{L}_\tau \otimes \det$ are marked by “-”.

Therefore our main Theorem 1.4 and main proposition Proposition 5.1 holds for \mathcal{O} .

5.3. The descent case. Now we assume $k \geq 1$ and C_{2k} is even. In this case $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a usual descent.

5.3.1. Unipotent representations attached to \mathcal{O}' . Therefore, $\text{LS}(\mathcal{O}'') \longrightarrow \text{LS}(\mathcal{O}')$ given by $\mathcal{L} \mapsto \vartheta(\mathcal{L})$ is an injection between abelian groups.

For $\mathcal{L}'' \in {}^\ell\text{LS}(\mathcal{O}'') \sqcup {}^\ell\text{LS}(\mathcal{O}'') \otimes \det$, let $(p_1, q_1) = {}^l\text{Sign}(\mathcal{L})$ and $(p_2, q_2) = (C_{2k} - q_1, C_{2k} - p_1)$. Then

$$\vartheta(\mathcal{L}'') = \dagger \mathcal{L}'' \cdot \dagger_{p_2, q_2}.$$

By (5.1), we have a injection

$$(5.4) \quad \begin{array}{ccc} {}^\ell\text{LS}(\mathcal{O}'') \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & {}^\ell\text{LS}(\mathcal{O}') \\ (\mathcal{L}'', \epsilon') & \longmapsto & \vartheta(\mathcal{L}'' \otimes \det^{\epsilon'}). \end{array}$$

In particular, we have $\mathcal{L}_{\tau'} \neq 0$ and $\pi_{\tau'} \neq 0$ for every $\tau' \in \text{DRC}(\mathcal{O}')$.

5.3.2. Unipotent representations attached to \mathcal{O} . Now suppose $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (5.4), $\epsilon'_1 = \epsilon'_2$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. In particular, τ'_1 and τ'_2 have the same signature. By Lemma 3.4 and the induction hypothesis, $\tau'_1 \neq \tau'_2$ and so $\pi_{\tau'_1} \otimes \det^{\epsilon'_1} \neq \pi_{\tau'_2} \otimes \det^{\epsilon'_2}$. Now the injectivity of theta lifting yields

$$\pi_{\tau'_2} = \bar{\Theta}(\pi_{\tau'_1} \otimes \det^{\epsilon'_1}) \neq \bar{\Theta}(\pi_{\tau'_2} \otimes \det^{\epsilon'_2}) = \pi_{\tau'_2}$$

Suppose \mathcal{O}' is noticed. Then $\mathcal{O}'' = \bar{\nabla}(\mathcal{O}')$ is noticed by definition and $(\tau'', \epsilon') \mapsto \text{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is an injection into $\text{LS}(\mathcal{O}'')$. Now (5.4) implies $\tau' \mapsto \mathcal{L}_{\tau'}$ is also an injection.

Suppose \mathcal{O}' is quasi-distinguished. Then $\mathcal{O}'' = \bar{\nabla}(\mathcal{O}')$ is quasi-distinguished by definition and $(\tau'', \epsilon') \mapsto \text{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is a bijection with $\text{LS}^{\text{aod}}(\mathcal{O}'')$. Since the component group of each K-nilpotent orbit in \mathcal{O}' is naturally isomorphic to the component group of its descent. So we deduce that $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection onto $\text{LS}^{\text{aod}}(\mathcal{O}')$.

This proves the main proposition for \mathcal{O}' .

Now let $\tau \in \text{DRC}(\mathcal{O})$. By (6.4), we have

$$(5.5) \quad \mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau \neq 0, \text{ where } \mathcal{T}_\tau := \dagger \mathcal{L}_{\tau'}, \text{ and } \mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}.$$

In particular, $\pi_\tau \neq 0$ and we could determine \mathbf{x}_τ from the marks on the 1-rows of \mathcal{L}_τ .

Now suppose $\tau_1 \neq \tau_2$ and $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (5.5), the result in Section 5.2 and Lemma 3.6 (ii), we have $\mathbf{x}_{\tau_1} = \mathbf{x}_{\tau_2}$, $\epsilon_1 = \epsilon_2$ and $\tau'_1 \neq \tau'_2$. By (5.5), we have $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. Applying the main

proposition of \mathcal{O}' , we have $\epsilon'_1 = \epsilon'_2$ and $\tau'_1 \neq \tau'_2$. Now by the injectivity of theta lifing, we conclude that

$$\pi_{\tau_2} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (1^{+, -})^{\epsilon_1} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (1^{+, -})^{\epsilon_2} = \pi_{\tau'_2}$$

5.3.3. Note that \mathcal{L}_τ is obtained from $\mathcal{L}_{\tau'}$ by attaching the peduncle determined by \mathbf{x}_τ . The claims about noticed and quasi-distinguished orbit are easy to verify. We leave them to the reader.

5.4. **The general descent case.** We assume $k \geq 1$ and all the properties are satisfied by $\tau'' \in \text{DRC}(\mathcal{O}'')$. We retain the notation in Section 3.4.6.

5.4.1. *Local systems.* We now describe the local systems $\mathcal{L}_{\tau'}$ and \mathcal{L}_τ more precisely using our formula of the associated character and the induction hypothesis. Note that these local systems are always non-zero which implies that $\pi_{\tau'}$ and π_τ are unipotent representations attached to \mathcal{O}' and \mathcal{O} respectively.

First note that in τ' and τ'' , $x_{\tau''} \neq s$ by the definition of dot-r-c diagram.⁴ By the induction hypothesis, $\mathcal{L}_{\tau''}^+ \neq 0$.

Let $(p_1'', q_1'') := {}^l\text{Sign}(\mathcal{L}_{\tau''})$. Then

$$(5.6) \quad {}^l\text{Sign}(\mathcal{L}_{\tau''}^+) = (p_0 - 1, q_0), \text{ and } {}^l\text{Sign}(\mathcal{L}_{\tau''}^-) = (p_0, q_0 - 1) \text{ if } \mathcal{L}_{\tau''}^- \neq \emptyset.$$

$$(5.7) \quad \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''}) = \boxtimes^{\frac{|\text{Sign}(\tau'')|}{2}} (\dagger \mathcal{L}_{\tau''}^+ + \dagger \mathcal{L}_{\tau''}^-) \neq 0$$

where ${}^l\text{Sign}(\dagger \mathcal{L}_{\tau''}^+) = (q_1'', p_1'' - 1)$ and ${}^l\text{Sign}(\dagger \mathcal{L}_{\tau''}^-) = (q_1'' - 1, p_1'')$ (if $\dagger \mathcal{L}_{\tau''}^- \neq 0$).

Let $(p_1, q_1) := {}^l\text{Sign}(\mathcal{L}_\tau)$ and $(e, f) := (p_1 - p_1'' + 1, q_1 - q_1'' + 1)$. Using Lemma 3.8 one can show that

$$(e, f) = \text{Sign}(\mathbf{u}_\tau).$$

[It suffice to consider the most left three columns of the peduncle part: this part has signature ${}^l\text{Sign}(\mathcal{P}_\tau) + (1, 1) = \text{Sign}(\mathbf{u}_\tau x_{\tau''}) = \text{Sign}(\mathbf{u}_\tau) + {}^l\text{Sign}(\mathcal{D}_{\tau''})$. Therefore,

$$\text{Sign}(\mathbf{u}_\tau) = {}^l\text{Sign}(\mathcal{P}_\tau) - {}^l\text{Sign}(\mathcal{D}_{\tau''}) + (1, 1) = {}^l\text{Sign}(\mathcal{L}_\tau) - {}^l\text{Sign}(\mathcal{L}_{\tau''}) + (1, 1).$$

]

Now

$$(5.8) \quad \mathcal{L}_\tau = \begin{cases} (1^{+, -} \otimes \dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^+) \cdot \mathcal{P}_\tau^+ + (1^{+, -} \otimes \dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^-) \cdot \mathcal{P}_\tau^- & \text{if } x_\tau \neq d \\ (\dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^+) \cdot \mathcal{P}_\tau^+ + (\dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^-) \cdot \mathcal{P}_\tau^- & \text{if } x_\tau = d \end{cases}$$

where

$$t = \frac{|\text{Sign}(\tau) - \text{Sign}(\tau'')|}{2} = \frac{|\text{Sign}(\mathbf{u}_\tau)|}{2}$$

$$\mathcal{P}_\tau^+ = \begin{cases} \dagger_{e, f-1} & \text{when } f \geq 1 \text{ and } x_\tau \neq d \\ \dagger_{e, f-1} & \text{when } f \geq 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{P}_\tau^- = \begin{cases} \dagger_{e-1, f} & \text{when } e \geq 1 \text{ and } x_\tau \neq d \\ \dagger_{e-1, f} & \text{when } e \geq 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 5.3. *The local system $\mathcal{L}_\tau \neq 0$.*

⁴Suppose $\tau'' \in \text{DRC}(\mathcal{O}'')$ has the shape in (3.4) and $x_{\tau''} = s$. By the induction hypothesis, $\mathcal{L}_{\tau''}^+ = \mathcal{L}_{\tau''}^- = 0$ and so the theta lift of $\pi_{\tau''}$ vanishes.

Proof. Since $x_{\tau''} \neq s$, $\mathcal{L}_{\tau''}^+ \neq 0$. When $\text{Sign}(\mathbf{u}_{\tau}) \geq (0, 1)$, the first term in (5.8) dose not vanish. Now suppose $\text{Sign}(\mathbf{u}_{\tau}) \not\geq (0, 1)$. We have $x_{\tau''} = d$ by Lemma 3.7 and $\text{Sign}(\mathbf{u}_{\tau}) > (1, 0)$. Therefore, $\mathcal{L}_{\tau''}^- \neq 0$ by the induction hypothesis and the second term in (5.8) dose not vanish. Hence $\mathcal{L}_{\tau} \neq 0$ in all cases. \square

5.4.2. *Unipotent representations attached to \mathcal{O}' .* In this section, we let $\tau' \in \text{DRC}(\mathcal{O}')$ and $\tau'' = \bar{\nabla}_1(\tau') \in \text{DRC}(\mathcal{O}'')$. First note that $x_{\tau''} \neq s$ and so $\mathcal{L}_{\tau''}^+$ always non-zero.

Recall (5.7). We claim that we can recover $\mathcal{L}_{\tau''}^+$ and $\mathcal{L}_{\tau''}^-$ from $\mathcal{L}_{\tau'}$:

Lemma 5.4. *The map $\Omega_{\mathcal{O}'}: \mathcal{L}_{\tau'} \mapsto (\mathcal{L}_{\tau''}^+, \mathcal{L}_{\tau''}^-)$ is a well defined map.*

Proof. When ${}^l\text{Sign}(\mathcal{L}_{\tau'})$ has two elements, $x_{\tau''} = d$ and ${}^l\text{Sign}(\mathcal{L}_{\tau'}) = \{(q_1'', p_1'' - 1), (q_1'' - 1, p_1'')\}$. In (5.7), $\dagger\mathcal{L}_{\tau''}^+$ (resp. $\dagger\mathcal{L}_{\tau''}^-$) consists of components whose first column has signature $(q_1'', p_1'' - 1)$ (resp. $(q_1'' - 1, p_1'')$). When ${}^l\text{Sign}(\mathcal{L}_{\tau'})$ has only one elements, $x_{\tau} = r/c$, ${}^l\text{Sign}(\mathcal{L}_{\tau'}) = \{(q_1'', p_1'' - 1)\}$, $\mathcal{L}_{\tau'} = \dagger\mathcal{L}_{\tau''}^+$ and $\mathcal{L}_{\tau''}^- = 0$

In any case, we get

$$(5.9) \quad \text{Sign}(\tau'') = (n_0, n_0) + (p_1'', q_1'') \quad \text{where } 2n_0 = |\nabla(\mathcal{O}'')|.$$

Using the signature $\text{Sign}(\tau'')$, we could recover the twisting characters in the theta lifting of local system. Therefore $\mathcal{L}_{\tau''}^+$ and $\mathcal{L}_{\tau''}^-$ can be recovered from $\mathcal{L}_{\tau'}$. \square

Lemma 5.5. *Suppose $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$. Then $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.*

Proof. It suffice to consider the case when $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (5.9) in the proof of Lemma 5.4, $\text{Sign}(\tau'_1) = \text{Sign}(\tau'_2)$. On the other hand, $\epsilon_1 = \epsilon_2 = 0$ and $\tau'_1 \neq \tau'_2$ by Lemma 3.5. So

$$\pi_{\tau'_1} = \bar{\Theta}(\pi_{\tau'_1}) \neq \bar{\Theta}(\pi_{\tau'_2}) = \pi_{\tau'_2}$$

by the injectivity of theta lift. \square

5.4.3. *Unipotent representations attached to \mathcal{O} .*

Lemma 5.6. *Suppose $\tau_1 \neq \tau_2 \in \text{DRC}(\mathcal{O})$. Then $\pi_{\tau_1} \neq \pi_{\tau_2}$.*

Proof. It suffice to consider the case that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. Clearly, $\text{Sign}(\tau_1) = \text{Sign}(\tau_2)$. On the other hand, the twisting $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ since it is determined by the local system: $\epsilon_{\tau_i} = 0$ if and only if $\mathcal{L}_{\tau_i} \supset -$.

We conclude that $\tau'_1 \neq \tau'_2$ and $\tau'_1 \neq \tau'_2$ by Lemma 3.8 and Lemma 3.5. By Lemma 5.5 and the injectivity of theta lift,

$$\pi_{\tau_1} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (\mathbf{1}^{+, -})^{\epsilon_{\tau_1}} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (\mathbf{1}^{+, -})^{\epsilon_{\tau_2}} = \pi_{\tau_2}$$

\square

5.4.4. *Noticed orbits.* In this section, we prove the claims about noticed and +noticed orbits.

Lemma 5.7. *Suppose \mathcal{O}' is noticed. Then*

(i) *The map $\{\mathcal{L}_{\tau''} \mid x_{\tau''} \neq s\} \longrightarrow \{\mathcal{L}_{\tau'}\} = {}^{\ell}\text{LS}(\mathcal{O}')$ given by $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''})$ is a bijection.*

(ii) *The map $\text{DRC}(\mathcal{O}') \rightarrow {}^{\ell}\text{LS}(\mathcal{O}')$ given by $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection.*

Proof. (i) Note that \mathcal{O}'' is noticed by definition. Hence $\Upsilon_{\mathcal{O}''}$ is invertible. Recall Lemma 5.4, we see that

$$(\Upsilon_{\mathcal{O}''})^{-1} \circ \Omega_{\mathcal{O}'}: \mathcal{L}_{\tau'} \mapsto (\mathcal{L}_{\tau''}^+, \mathcal{L}_{\tau''}^-) \mapsto \mathcal{L}_{\tau''}$$

gives the inverse of the map in the claim.

(ii) Suppose $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$. By Lemma 3.5, $\tau''_1 \neq \tau''_2$. Hence $\mathcal{L}_{\tau'_1} \neq \mathcal{L}_{\tau'_2}$ by the induction hypothesis. Therefore, $\mathcal{L}_{\tau'_1} \neq \mathcal{L}_{\tau'_2}$ by (i) of the claim. \square

Lemma 5.8. *Suppose \mathcal{O} is noticed, $\mathcal{L}: \text{DRC}(\mathcal{O}) \mapsto {}^\ell\text{LS}(\mathcal{O})$ given by $\tau \mapsto \mathcal{L}_\tau$ is bijective.*

Proof. We prove the claim by contradiction. We assume $\tau_1 \neq \tau_2$ such that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$.

Recall (5.8): there is an pair of non-negative integers $(e_i, f_i) = \text{Sign}(\mathbf{u}_{\tau_i})$ such that

$$\mathcal{L}_{\tau_i} = \dagger \dagger \mathcal{L}_{\tau''_i}^+ \cdot \mathcal{P}_{\tau_i}^+ + \dagger \dagger \mathcal{L}_{\tau''_i}^- \cdot \mathcal{P}_{\tau_i}^-.$$

(i) Suppose that \mathcal{L}_{τ_i} contains a 1-row marked by $-$ or $=$. Then we have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$, which can be read from the mark $-/ =$. Moreover, the term $\dagger \dagger \mathcal{L}_{\tau''_i}^+ \cdot \mathcal{P}_{\tau_i}^+ \neq 0$ and we can recover it from \mathcal{L}_{τ_i} . So $\mathcal{L}_{\tau''_1}^+ = \mathcal{L}_{\tau''_2}^+$. Since \mathcal{O}'' is +noticed, $\tau''_1 = \tau''_2$ by (5.10) and the induction hypothesis. Note that $\text{Sign}(\tau_1) = \text{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by Lemma 3.8 which is contradict to our assumption.

(ii) Now we assume that \mathcal{L}_{τ_i} does not contains a 1-row marked by $-/ =$. By (5.8), we conclude that $x_{\tau''} \neq d$ and $\mathbf{u}_\tau = r \cdots r$ or $r \cdots rc$. Therefore \mathbf{p}_{τ_i} must be one of the following form by Lemma 3.7:

$$\mathbf{p}_1 = \begin{array}{c} r \ c \\ \vdots \\ r \\ r \end{array}, \quad \mathbf{p}_2 = \begin{array}{c} r \ c \\ \vdots \\ r \\ c \end{array}, \quad \text{and} \quad \mathbf{p}_3 = \begin{array}{c} r \ d \\ \vdots \\ r \\ r \end{array}$$

We consider case by case according to the set $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\}$:

- (a) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1\}$ or $\{\mathbf{p}_2\}$. In these cases, $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by Lemma 3.8. On the other hand, $\mathcal{L}_{\tau''_1}^+ = \mathcal{L}_{\tau''_2}^+$ by (5.8). Since \mathcal{O}'' is +noticed, we get $\tau''_1 = \tau''_2$ a contradiction.
- (b) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_3\}$. We still have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by Lemma 3.8. On the other hand, $\mathcal{L}_{\tau''_1}^- = \mathcal{L}_{\tau''_2}^-$ by (5.8). This again contradict to the injectivity of $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau''}^-$.
- (c) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1, \mathbf{p}_2\}$. By the argument above, we have $\mathcal{L}_{\tau''_1}^+ = \mathcal{L}_{\tau''_2}^+$. Now $\tau''_1 \neq \tau''_2$ since $\{x_{\tau''_1}, x_{\tau''_2}\} = \{r, c\}$. This contradict to that \mathcal{O}'' is +noticed again.
- (d) Suppose $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ or $\mathbf{p}_2, \mathbf{p}_{\tau_2} = \mathbf{p}_3$. Then

$$\mathcal{L}_{\tau_1} = \dagger \boxtimes^{\frac{|\mathbf{u}_{\tau_1}|}{2}} \dagger \mathcal{L}_{\tau''_1}^+ \cdot \mathcal{P}_{\tau_1}^+ \quad \text{and} \quad \mathcal{L}_{\tau_2} = \dagger \boxtimes^{\frac{|\mathbf{u}_{\tau_2}|}{2}} \dagger \mathcal{L}_{\tau''_2}^- \cdot \mathcal{P}_{\tau_2}^-.$$

This implies $|\text{Sign}(\tau''_1)| \equiv |\text{Sign}(\tau''_2)| + 2 \pmod{4}$ and $\boxtimes \dagger \mathcal{L}_{\tau''_1}^+ = \dagger \mathcal{L}_{\tau''_2}^-$.

Claim 5.9. *We have $\mathcal{L}_{\tau''_2}^- \supset +$.*

Proof. If \mathcal{O}'' is obtained by the usual descent, we have $\mathcal{L}_\tau \geq \begin{smallmatrix} - \\ + \end{smallmatrix}$ and we are done.

Now assume \mathcal{O}'' is obtained by generalized descent and \mathcal{O}'' is +noticed, i.e. $\mathbf{u}_{\tau''}$ has at least length 2. Suppose $\text{Sign} \mathbf{u}_{\tau''} > (1, 2)$. Applying (5.8) to τ'' , we see that the first term is non-vanishing and $\mathcal{L}_{\tau''} \supseteq \dagger_{(1,1)}$.

Otherwise, $\text{Sign}(\mathbf{u}_{\tau''}) = (2n_0 - 1, 1) \geq (3, 1)$ where n_0 is the length of $\mathbf{u}_{\tau''}$. By (5.8), the second term is non-vanishing and $\mathcal{L}_{\tau''} > \dagger_{(2n_0-1,1)} > +$. \square

The claim leads to a contradiction: Thanks to the character twist in the theta lifting formula of the local system, we see that the the associated character restricted on the 2-row $- +$ of $\boxtimes \dagger \mathcal{L}_{\tau''_1}^+$ and $\dagger \mathcal{L}_{\tau''_2}^-$ must be are different, a contradiction.

We finished the proof of the lemma. \square

5.4.5. The maps $\Upsilon_{\mathcal{O}}^+$, $\Upsilon_{\mathcal{O}}^-$ and $\Upsilon_{\mathcal{O}}$.

Lemma 5.10. *Suppose \mathcal{O} is +noticed. The map $\Upsilon_{\mathcal{O}}^+ : \{\mathcal{L}_{\tau} \mid \mathcal{L}_{\tau}^+ \neq 0\} \rightarrow \{\mathcal{L}_{\tau}^+ \neq 0\}$ is injective.*

Proof. We have two cases:

(i) \mathcal{L}_{τ} contain an irreducible component $> \begin{smallmatrix} + \\ + \end{smallmatrix}$. This is equivalent to \mathcal{L}_{τ}^+ has an irreducible component $> \begin{smallmatrix} + \\ + \end{smallmatrix}$. Now all irreducible components of $\mathcal{L}_{\tau} > \begin{smallmatrix} + \\ + \end{smallmatrix}$ and $\mathcal{L}_{\tau} \mapsto \mathcal{L}_{\tau}^+$ will not kill any irreducible components. Hence we could recover \mathcal{L}_{τ} from \mathcal{L}_{τ}^+ .

(ii) Suppose that $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ do not contain a component $> \begin{smallmatrix} + \\ + \end{smallmatrix}$.

By Lemma 5.8 $\tau_1 \neq \tau_2$.⁵ We now show that $\mathcal{L}_{\tau_1}^+ = \mathcal{L}_{\tau_2}^+ \neq 0$ leads to a contradiction. Clearly, $\text{Sign}(\tau_1) = \text{Sign}(\tau_2)$. By (5.8) and checking the properties in Lemma 3.7, we have

$$\mathbf{p}_{\tau_i} = \begin{matrix} s & x_{\tau''} \\ \vdots & \\ s & \end{matrix} \quad \text{where } x_{\tau''} = c/d, x_{\tau_i} = c/d.$$

On the other hand, when \mathbf{p}_{τ_i} have the above form, the first term of (5.8) is non-vanishing. So we conclude that $\mathcal{L}_{\tau_1}^+ = \mathcal{L}_{\tau_2}^+$ implies

$$(5.10) \quad \mathcal{L}_{\tau_1''}^+ = \mathcal{L}_{\tau_2''}^+.$$

Moreover, we see that every components of \mathcal{L}_{τ_i} has a 1-row of mark “- / =” and we can determine ϵ_{τ_i} by the mark - / = of the 1-row appeared in $\mathcal{L}_{\tau_i}^+$. In particular, we have $\epsilon_1 = \epsilon_2$. Now Lemma 3.8 implies $\tau_1'' \neq \tau_2''$. This is contradict to (5.10) and our induction hypothesis since \mathcal{O}'' is also +noticed. □

Lemma 5.11. *Suppose \mathcal{O} is noticed. Then the map $\Upsilon_{\mathcal{O}} : \mathcal{L}_{\tau} \mapsto (\mathcal{L}_{\tau}^+, \mathcal{L}_{\tau}^-)$ is injective.*

Proof. It suffice to consider the case where \mathcal{O} is noticed but not +noticed, i.e. $C_{2k+1} = C_{2k} + 1$. In this case, $n = 1$. The peduncle \mathbf{p}_{τ} have four possible cases:

$$\begin{matrix} r & c, & c & c, & c & d, & \text{or} & d & d. \end{matrix}$$

By (5.8), $\mathcal{L}_{\tau}^+ = \dagger \dagger \mathcal{L}_{\tau''}^+$.

Now suppose $\tau_1 \neq \tau_2$ such that $\mathcal{L}_{\tau_1}^+ = \mathcal{L}_{\tau_2}^+$. Since \mathcal{O}'' is +noticed, we have $\tau_1'' = \tau_2''$ by the induction hypothesis (see Lemma 5.10). By Lemma 3.8, this only happens when $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{ \begin{smallmatrix} c & d \\ d & d \end{smallmatrix} \}$. Since one of $\mathcal{L}_{\tau_i}^-$ is zero and the other is non-zero, we conclude that $\Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_1}) \neq \Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_2})$. □

Lemma 5.12. *Suppose \mathcal{O} is +noticed. The map $\Upsilon_{\mathcal{O}}^- : \{\mathcal{L}_{\tau} \mid \mathcal{L}_{\tau}^- \neq 0\} \rightarrow \{\mathcal{L}_{\tau}^- \neq 0\}$ is injective.*

Proof. The proof is similar to that of Lemma 5.10. We have two cases:

(i) \mathcal{L}_{τ} contain an irreducible component $> \begin{smallmatrix} - \\ - \end{smallmatrix}$. This is equivalent to \mathcal{L}_{τ}^- has an irreducible component $> \begin{smallmatrix} - \\ - \end{smallmatrix}$ and $\mathcal{L}_{\tau} \mapsto \mathcal{L}_{\tau}^-$ will not kill any irreducible components. Hence we could recover \mathcal{L}_{τ} from \mathcal{L}_{τ}^- .

(ii) Now we make a weaker assumption that $\tau_1 \neq \tau_2 \in \text{DRC}(\mathcal{O})$ such that \mathcal{L}_{τ_1} and \mathcal{L}_{τ_2} do not contain a component $> \begin{smallmatrix} - \\ - \end{smallmatrix}$.

⁵ $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ clearly implies $\tau_1 \neq \tau_2$.

By (5.8) and the properties in Lemma 3.7, we see that \mathbf{p}_{τ_i} must be one of the following

$$\mathbf{p}_1 = \begin{array}{c} r \\ \vdots \\ r \\ d \end{array} \quad \text{or} \quad \mathbf{p}_2 = \begin{array}{c} r \\ \vdots \\ r \\ d \end{array}.$$

Without loss of generality, we have the following possibilities:

- (a) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_1$. We have $\mathcal{L}_{\tau_1''}^+ = \mathcal{L}_{\tau_2''}^+$ and obtain the contradiction.
- (b) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_2$. We have $\mathcal{L}_{\tau_1''}^- = \mathcal{L}_{\tau_2''}^-$ and obtain the contradiction.
- (c) $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ and $\mathbf{p}_{\tau_2} = \mathbf{p}_2$. Now we apply the same argument in the case (d) of the proof of Lemma 5.8. We get $\boxtimes \dagger \mathcal{L}_{\tau_1''}^+ = \dagger \mathcal{L}_{\tau_2''}^-$, $\mathcal{L}_{\tau_2''}^- \supset +$ and a contradiction by looking at the associated character on the 2-row $- +$.

This finished the proof. \square

6. TYPE B/M CASE

6.1. “Special” and “non-special” diagrams.

6.1.1. type M.

Definition 6.1. Let $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0) \in \text{Nil}^{\text{dpe}}(M)$ with $k \geq 1$. Let

$$\tau = (\tau_{2k}, \dots, \tau_2) \times (\tau_{2k-1}, \dots, \tau_1).$$

be a Weyl group representation attached to $\mathcal{O} \in \text{Nil}^{\text{dpe}}(M)$. We say τ has

- special shape if $\tau_{2k} \geq \tau_{2k-1}$;
- non-special shape if $\tau_{2k} < \tau_{2k-1}$.

Clearly, this gives a partition of the set of Weyl group representations attached to \mathcal{O} .

Suppose $k \geq 1$, $C_{2k} = 2c_{2k} - 1$ and $C_{2k-1} = 2c_{2k-1} + 1$. The special shape representation τ^s

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k-1}, \tau_{2k-3}, \dots, \tau_1).$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k-1}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k}, \tau_{2k-3}, \dots, \tau_1)$$

In the pairing $\tau^s \leftrightarrow \tau^{ns}$, τ_j is unchanged for $j \leq 2k - 2$. We call τ^s the special shape and τ^{ns} the corresponding non-special shape

Definition 6.2. Suppose C_{2k} is odd. We retain the notation in Definition 6.1 where the shapes τ^s and τ^{ns} correspond with each other. We define a bijection between $\text{DRC}(\tau^s)$ and $\text{DRC}(\tau^{ns})$:

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \longleftrightarrow & \text{DRC}(\tau^{ns}) \\ \tau^s & \longleftrightarrow & \tau^{ns} \end{array}$$

Without loss of generality, we can assume that $\tau^s \in \text{DRC}(\tau^s)$ and $\tau^{ns} \in \text{DRC}(\tau^{ns})$ have the following shapes:

$$(6.1) \quad \tau^s : \begin{array}{c} \dots\dots\dots \\ y_1 \dots\dots x_0 \dots\dots \\ \begin{array}{|c|} \hline s \\ \hline \vdots \\ \hline s \\ \hline \end{array} \\ y_2 \end{array} \times \longleftrightarrow \tau^{ns} : \begin{array}{c} \dots\dots\dots \\ y_0 \dots\dots x_1 \dots\dots \\ \begin{array}{|c|} \hline r \\ \hline \vdots \\ \hline r \\ \hline \end{array} \\ x_2 \end{array}$$

Here the grey parts have length $(C_{2k} - C_{2k-1})/2$ (could be zero length). The entries x_0, x_1, y_0 and y_1 are either all empty or all non-empty. The correspondence between (x_0, x_1, x_2) , with (y_0, y_1, y_2) is given by switching r and s , d and c . i.e.

$$\begin{cases} y_i = s \Leftrightarrow x_i = r \\ y_i = c \Leftrightarrow x_i = d \end{cases} \quad \text{for } i = 0, 1, 2.$$

We define

$$\text{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \text{DRC}(\tau^s) \quad \text{and} \quad \text{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \text{DRC}(\tau^{ns})$$

where τ^s (resp. τ^{ns}) runs over all special (resp. non-special) shape representations attached to \mathcal{O} . Clearly $\text{DRC}(\mathcal{O}) = \text{DRC}^s(\mathcal{O}) \sqcup \text{DRC}^{ns}(\mathcal{O})$.

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

6.1.2. *Special and non-special shapes of type B.* Now we define the notion of special and non-special shape of type B.

Definition 6.3. *Let*

$$\tau = (\tau_{2k}, \dots, \tau_2) \times (\tau_{2k+1}, \tau_{2k-1}, \dots, \tau_1)$$

be a Weyl group representation attached to $\mathcal{O} \in \text{Nil}^{\text{dpe}}(B)$. We say τ has

- special shape if $\tau_{2k} \geq \tau_{2k-1}$;
- non-special shape if $\tau_{2k} < \tau_{2k-1}$.

Suppose $k \geq 1$ and

$$(6.2) \quad \mathcal{O} = (C_{2k+1} = 2c_{2k+1} + 1, C_{2k} = 2c_{2k} - 1, C_{2k-1} = 2c_{2k-1} + 1, \dots, C_1 > 0, C_0 = 0).$$

A representation τ^s attached to \mathcal{O} has the shape

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \dots, \tau_1)$$

is paired with

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k-1}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k+1}, c_{2k}, \tau_{2k-3}, \dots, \tau_1).$$

In the pairing $\tau^s \leftrightarrow \tau^{ns}$ described as the above, τ_j are unchanged for $j \leq 2k - 2$.

6.2. Definition of $\overline{\nabla}$ for type B and M.

6.2.1. The definition of ϵ .

- (1). When $\tau \in \text{DRC}(B)$, ϵ is determined by the “basal disk” of τ (see (6.3) for the definition of x_τ):

$$\epsilon_\tau := \begin{cases} 0, & \text{if } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

- (2). When $\tau \in \text{DRC}(M)$, the ϵ is determined by the lengths of $\tau_{L,0}$ and $\tau_{R,0}$:

$$\epsilon_\tau := \begin{cases} 0, & \text{if } |\tau_{L,0}| - |\tau_{R,0}| \geq 0; \\ 1, & \text{if } |\tau_{L,0}| - |\tau_{R,0}| < 0. \end{cases}$$

6.2.2. *Initial cases.*

- (3). Suppose $\mathcal{O} = (2c_1 + 1)$ has only one column. Then $\mathcal{O}'_0 := \overline{\nabla}(\mathcal{O})$ is the trivial orbit of $\text{Mp}(0, \mathbb{R})$. $\text{DRC}(\mathcal{O})$ consists of diagrams of shape $\tau_L \times \tau_R = \emptyset \times (c_1,)$. The set $\text{DRC}(\mathcal{O}'_0)$ is a singleton $\{\tau'_0 = \emptyset \times \emptyset\}$, and every element $\tau \in \text{DRC}(\mathcal{O})$ maps to τ_0 (see (2.1)):

$$\text{DRC}(\mathcal{O}) \ni \tau := \begin{array}{c} \emptyset \quad x_1 \\ \times \quad \vdots \\ \quad x_n \\ m_\tau \end{array} \mapsto \tau'_0 := \emptyset \times \emptyset$$

Here $m_\tau = a$ or b is the mark of τ . We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := x_1 \cdots x_n m_\tau.$$

We call $\mathbf{p}_\tau = \mathbf{x}_\tau$ the “peduncle” part of τ .

6.2.3. *The descent from M to B.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

- (4). Suppose $|\tau_{L,0}| \geq |\tau_{R,0}|$, i.e. τ has special shape. Keep r, d unchanged, delete $\tau_{L,0}$ and fill the remaining part with “•” and “s” by $\overline{\nabla}_R(\tau^\bullet)$ using the dot-s switching algorithm. The mark $m_{\tau'}$ of $\tau' := \overline{\nabla}_1(\tau)$ is given by the following formula:

$$m_{\tau'} := \begin{cases} a & \text{if } \tau_{L,0} \text{ ends with } s \text{ or } \bullet, \\ b & \text{if } \tau_{L,0} \text{ ends with } c. \end{cases}$$

- (5). Suppose $|\tau_{L,0}| < |\tau_{R,0}|$, i.e. τ has non-special shape. We define

$$\overline{\nabla}_1(\tau) := \overline{\nabla}_1(\tau^s)$$

where τ^s is the special diagram corresponding to τ defined in Definition 6.2.

Lemma 6.4. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$ with $k \geq 1$ and C_{2k} is odd. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1}, \dots, C_0)$. Then*

$$\begin{array}{ccc} \text{DRC}^s(\mathcal{O}) & \xrightarrow{\overline{\nabla}_1} & \text{DRC}(\mathcal{O}') \\ \text{DRC}^{ns}(\mathcal{O}) & \xrightarrow{\overline{\nabla}_1} & \text{DRC}(\mathcal{O}') \end{array}$$

are a bijections.

Proof. The claim for $\text{DRC}^s(\mathcal{O})$ is clear by the definition of descent. The claim for $\text{DRC}^{ns}(\mathcal{O})$ reduces to that of $\text{DRC}^s(\mathcal{O})$ using Definition 6.2. \square

Lemma 6.5. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$ with $k \geq 1$ and C_{2k} even. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1} + 1, \dots, C_0)$. Then the following map is a bijection*

$$\text{DRC}(\mathcal{O}) \xrightarrow{\overline{\nabla}_1} \left\{ \tau' \in \text{DRC}(\mathcal{O}') \left| \begin{array}{l} m_{\tau'} \neq b \text{ and} \\ \tau'_{R,0} \text{ dose not ends with } s. \end{array} \right. \right\}.$$

Proof. This is clear by the descent algorithm. \square

6.2.4. *Descent from B to M.* Now we define the general case of the descent from type B to type M. We assume that $k \geq 1$ i.e. \mathcal{O} has at least 3 columns. First note that the shape of $\tau' = \overline{\nabla}(\tau)$ is the shape of τ deleting the most left column on the right diagram.

The definition splits in cases below. In all these cases we define

$$(6.3) \quad \tilde{\mathbf{x}}_\tau := x_1 \cdots x_n m_\tau \text{ and } \tilde{x}_\tau := x_n$$

which is marked by $s/r/c/d$. For the part marked by $*/\cdots$, $\overline{\nabla}$ keeps r, c, d and maps the rest part consisting of \bullet and s by dot-s switching algorithm.

- (6). When C_{2k} is odd and τ has special shape, the descent is given by the following diagram

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ \bullet \\ \vdots \\ \bullet \\ y_0 \end{array} \times \begin{array}{c} \dots\dots\dots \\ * * \dots\dots \\ \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ s \\ \vdots \\ s \\ y'_0 \end{array} \times \dots\dots\dots$$

Here the grey columns has length $(C_{2k} - C_{2k-1})/2$ and $n = (C_{2k+1} - C_{2k})/2$. The “*/...” part of τ' is given by keeping the corresponding entries marked by $r/d/c$ in τ unchange and filling s/\bullet accordingly in the rest of the entries. The entry y'_0 of τ' is given by the following formula:

$$y'_0 := \begin{cases} s & \text{if } x_1 = \bullet \text{ or } (x_1, m_\tau) = (r/d, a) \\ c & \text{if } x_1 = s \text{ or } (x_1, m_\tau) = (r/d, b). \end{cases}$$

- (7). When C_{2k} is odd and τ has non-special shape, then

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ s \\ \vdots \\ s \\ x_1 \end{array} \times \begin{array}{c} \dots\dots\dots \\ * * \dots\dots \\ s \ r \\ \vdots \\ s \ r \\ x_0 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ r \\ \vdots \\ r \\ x'_0 \end{array} \times \dots\dots\dots$$

Here the grey columns has length $(C_{2k} - C_{2k-1})/2$ and $n = (C_{2k+1} - C_{2k})/2$. The “*/...” part of τ' is given by keeping the corresponding entries marked by $r/d/c$ in τ unchange and filling s/\bullet accordingly in the rest of the entries. The entry x'_0 of τ' is given by the following formula:

$$x'_0 := \begin{cases} r & \text{if } (x_1, m_\tau) = (r/d, a), \\ d & \text{if } (x_1, m_\tau) = (r/d, b), \\ x_0 & \text{if } x_1 = s. \end{cases}$$

Note that

- (8). When $C_{2k} = C_{2k-1}$ is even, we could assume τ and τ' have the following forms with $n = (C_{2k+1} - C_{2k} + 1)/2$.

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ y_0 \dots\dots \end{array} \times \begin{array}{c} \dots\dots\dots \\ * * \dots\dots \\ x_1 \ x_0 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ y'_0 \dots\dots \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ x'_0 \end{array}$$

In the most of the case, τ' is obtained from τ by deleting the first column of τ_R , keeping $c/r/d$ and filling s/\bullet accordingly. The exceptional cases are when $x_1 = r/d$. In the exceptional case we always have $x_0 = d$. We define

$$x'_0 := d, \\ y'_0 := \begin{cases} s & \text{if } m_\tau = a, \\ c & \text{if } m_\tau = b. \end{cases}$$

Note that this definition is similar to that of the special shape diagrams in the descent case. We define the “peduncle” of τ to be

$$\mathbf{p}_\tau = \begin{matrix} x_1 & x_0 \\ \vdots & \\ x_n & \\ m_\tau & \end{matrix}.$$

In all the cases, let $\mathcal{O}_1 = (2n,)$ is the trivial nilpotent of type D. For each $\tau \in \text{DRC}(\mathcal{O})$, we define $\mathbf{x}_\tau \in \text{DRC}(\mathcal{O}_1)$ by require the following identity of multisets:

$$\{\text{entry of } \mathbf{x}_\tau\} = \begin{cases} \{c, x_2, \dots, x_n\} & \text{if } (x_1, m_\tau) = (\bullet/s, a), \\ \{s, x_2, \dots, x_n\} & \text{if } (x_1, m_\tau) = (\bullet/s, b), \\ \{x_1, x_2, \dots, x_n\} & \text{otherwise } (x_1 = r/d). \end{cases}$$

In the above definition we include the initial case where \mathcal{O} is a trivial orbit of type B. We define $\mathbf{x}_\tau = \emptyset$ when \mathcal{O} is the trivial orbit of $\text{O}(1, \mathbb{C})$.

We define the “basal disk” of τ as the following:

$$x_\tau := \begin{cases} \mathbf{x}_\tau & \text{if } C_{2k+1} = C_{2k} + 1. \\ c & \text{if } x_n m_\tau = sa \text{ or } \bullet a \\ x_n & \text{otherwise.} \end{cases}$$

6.2.5. *A key proposition for descent case.* Now we assume \mathcal{O} is given by (6.2) Let $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$.

The following lemma is the key property satisfied by our definition

Lemma 6.6. *Suppose $k \geq 1$ and τ^s and τ^{ns} are two representations attached to \mathcal{O} as in Definition 6.3. Then*

i) *For every $\tau \in \text{DRC}(\tau^s) \sqcup \text{DRC}(\tau^{ns})$, the shape of $\overline{\nabla}^2(\tau)$ is*

$$\tau'' = (\tau_{2k-2}, \dots, \tau_2) \times (c_{2k-1}, \tau_{2k-3}, \tau_1)$$

ii) *We define $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$ by $\delta(\tau) = (\overline{\nabla}^2(\tau), \mathbf{x}_\tau)$. The following maps are bijective*

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \xrightarrow{\delta} & \text{DRC}(\tau'') \times \text{DRC}(\mathcal{O}_1) \xleftarrow{\delta} \text{DRC}(\tau^{ns}) \\ \tau^s & \longmapsto & (\overline{\nabla}^2(\tau^s), \mathbf{x}_{\tau^s}) \\ & & (\overline{\nabla}^2(\tau^{ns}), \mathbf{x}_{\tau^{ns}}) \longleftarrow \tau^{ns} \end{array}$$

In particular, we obtain an one-one correspondence $\tau^s \leftrightarrow \tau^{ns}$ such that $\delta(\tau^s) = \delta(\tau^{ns})$.

iii) *Suppose τ^s and τ^{ns} correspond as the above such that $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$. Then*

$$(6.4) \quad \text{Sign}(\tau^s) = \text{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \text{Sign}(\tau'') + \text{Sign}(\mathbf{x}).$$

Note that $\text{Sign}(\mathbf{x}) \in \mathbb{N} \times \mathbb{N}$.

Proof. By our assumption, we have $c_{2k+1} \geq c_{2k} > c_{2k-1}$ and

$$\tau^s = (c_{2k}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k+1}, c_{2k-1}).$$

The behavior of τ^s under the descent map $\overline{\nabla}$ is illustrated as the following:

$$(6.5) \quad \tau^s : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ \bullet \\ \vdots \\ \bullet \end{array} \times \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau'^s : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ s \\ \vdots \\ s \\ x'_0 \end{array} \times \cdots \mapsto \tau'' : \cdots \times \cdots \times \cdots \times \cdots \times \cdots \times m_{\tau''}$$

The grey part consists of totally $2(c_{2k} - 1) = C_{2k} - 1$ dots. Hence the total signature of the grey part is $(C_{2k} - 1, C_{2k} - 1)$. Note that τ^s is obtained from the “ $* \cdots$ ” part of τ'' by attaching the grey part and x_0, \dots, x_n, m_τ .

The descent of a non-special diagram τ^{ns} :

$$(6.6) \quad \tau^{ns} : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \end{array} \times \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ s \text{ } r \\ \vdots \\ s \text{ } r \\ x_1 \text{ } x_0 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau'^{ns} : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \end{array} \times \begin{array}{c} \text{.....} \\ * \text{ } \cdots \\ r \\ \vdots \\ r \\ x'_0 \end{array} \mapsto \tau'' : \cdots \times \cdots \times \cdots \times \cdots \times \cdots \times m_{\tau''}$$

Here the grey parts in τ consists of $C_{2k} - 1$ marks of \bullet , s , or r and s and r occur with the same multiplicity. Hence the total signature of the grey part is $(C_{2k} - 1, C_{2k} - 1)$.

To prove the lemma, it suffice to verify the following claims which we leave to the reader:

- In both the special shape and non-special shape cases, the following map is bijective:

$$x_0 x_1 \cdots x_n m_{\tau^{s/ns}} \mapsto (\mathbf{x}_{\tau^{s/ns}}, m_{\tau''}).$$

- The following equation of signatures holds

$$\text{Sign}(x_0 x_1 \cdots x_n m_\tau) - \text{Sign}(m_{\tau''}) = \text{Sign}(\mathbf{x}_\tau).$$

In the verification, one can use the fact that $m_\tau = m_{\tau''}$ when $x_1 = r/d$.

[Suppose $x_1 = r/d$. Then $x_0 = c/d$ and

$$\text{Sign}(x_0 \cdots x_n m_\tau) - \text{Sign}(m_{\tau''}) = (1, 1) + \text{Sign}(x_1 \cdots x_n).$$

Now we consider the generic cases, i.e. $x_1 = \bullet/s$: In this case, we claim that $\text{Sign}(x_0 x_1) - \text{Sign}(m_{\tau''}) = (1, 2)$. Now

$$\begin{aligned} & \text{Sign}(x_0 x_1 \cdots x_n m_\tau) - \text{Sign}(m_{\tau''}) \\ &= (1, 1) + \text{Sign}(x_2 \cdots x_n m_\tau) + (0, 1) \\ &= \begin{cases} (1, 1) + \text{Sign}(x_2 \cdots x_n) + \text{Sign}(c) & \text{if } m_\tau = a \\ (1, 1) + \text{Sign}(x_2 \cdots x_n) + \text{Sign}(s) & \text{if } m_\tau = b \end{cases} \end{aligned}$$

First consider the special shape diagram $\tau = \tau^s$.

- (i) Suppose $m_{\tau''} = a$. Then $x_0 \times x_1 = \bullet \times \bullet$.
- (ii) Suppose $m_{\tau''} = b$. Then $x_0 \times x_1 = c \times s$.

Now consider the non-special shape diagram $\tau = \tau^{ns}$.

- (i) Suppose $m_{\tau''} = a$. Then $\emptyset \times x_1 x_0 = \emptyset \times sr$.
- (ii) Suppose $m_{\tau''} = b$. Then $\emptyset \times x_1 x_0 = \emptyset \times sd$.

The matching between τ^s and τ^{ns} is also clear by the above listing of cases.]

□

6.2.6. *A key proposition for the generalized descent case.* Now assume C_{2k} is even. By our assumption, we have $c_{2k+1} \geq c_{2k} = c_{2k-1}$.

$$(6.7) \quad \tau : \begin{array}{c} \text{.....} \\ y_0 \text{.....} \end{array} \times \begin{array}{c} \text{.....} \\ x_1 x_0 \cdots \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \text{.....} \\ y'_0 \text{.....} \end{array} \times \begin{array}{c} \text{.....} \\ x'_0 \cdots \end{array} \mapsto \tau'' : \begin{array}{c} \text{.....} \\ \text{.....} \\ x'' \cdots \\ m_{\tau''} \end{array}$$

Here the gray parts in τ are two columns of \bullet of length $C_{2k}/2 - 1$.

Lemma 6.7. *In (6.7), $x'' = x_0$. The map $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$ given by $\tau \mapsto (\bar{\nabla}_1^2(\tau), \mathbf{x}_\tau)$ is injective. Moreover,*

$$\text{Sign}(\tau) = \text{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \text{Sign}(\mathbf{x}_\tau).$$

The map $\tilde{\delta}: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ given by $\tau \mapsto (\bar{\nabla}_1^2(\tau), \text{Sign}(\tau), \epsilon_\tau)$ is injective.

Proof. The claim $x'' = x_0$, the injectivity of δ and the signature formula follows directly from our algorithm.

Now the injectivity of $\tilde{\delta}$ follows from $\mathbf{x}_\tau \mapsto (\text{Sign}(\mathbf{x}_\tau), \epsilon)$ is injective by Lemma 5.2. [We can fully recover τ from \mathbf{x}_τ and τ'' .

Suppose \mathbf{x}_τ dose not contains s or c . Then $x_1 = r/d$, $m_\tau = m_{\tau''}$, $y_0 = c$. Hence the claim holds.

$$\text{Sign}(\tau) = \text{Sign}(\tau'') + (C_{2k} - 2, C_{2k} - 2) + \text{Sign}(c) + \text{Sign}(\mathbf{x}_\tau)$$

Now assume \mathbf{x}_τ contain at least one s or c . Now

$$m_\tau = \begin{cases} a & \text{if } \mathbf{x}_\tau \text{ contains } c \\ b & \text{otherwise} \end{cases}$$

Therefore, $\text{Sign}(x_2 \cdots x_n m_\tau) = \text{Sign}(\mathbf{x}_\tau) - (0, 1)$.

Clearly, we can recover $x_2 \cdots x_n$ from \mathbf{x}_τ by deleting the c (if it exists) or a s . On the other hand, (y_0, x_1) is completely determined by $m_{\tau''}$:

$$(y_0, x_1) = \begin{cases} (\bullet, \bullet) & \text{if } m_{\tau''} = a \\ (c, s) & \text{if } m_{\tau''} = b \end{cases}$$

Hence $\text{Sign}(y_0 x_1) = \text{Sign}(m_{\tau''}) + (1, 2)$.

Now $\text{Sign}(y_0 x_0 x_1 \cdots x_n m_\tau) = \text{Sign}(\mathbf{x}_\tau) + \text{Sign}(x'' m_{\tau''}) + (1, 2) - (0, 1)$. This yields the signature identity.] \square

6.3. Proof of the type BM case. We will reduce the proof to the type CD case.

In this section, $k \geq 1$ and $\mathcal{O} = (C_{2k+1}, C_{2k}, \cdots, C_1, C_0 = 0) \in \text{Nil}^{\text{dpe}}(B)$. $\mathcal{O}' := \bar{\nabla}(\mathcal{O})$ and $\mathcal{O}'' := \bar{\nabla}(\mathcal{O}')$.

Take $\tau \in \text{DRC}(\mathcal{O})$, we marks the entries in τ as in (6.5) to (6.7).

6.3.1. *Usual decent case.* Thanks to Lemma 6.6, the proof in the descent case is exactly the same as that in Section 5.3.

6.3.2. *Reduction to type CD case in the general descent case.* Suppose C_{2k} is even. We define $\tilde{\mathcal{O}} = (C_{2k+1} - C_{2k} + 1, 1, 1) \in \text{Nil}^{\text{dpe}}(D)$ and $\tilde{\mathcal{O}}'' := \bar{\nabla}_1(\tilde{\mathcal{O}}) = (2)$.

The key proposition of reduction.

Proposition 6.8. *We have the following properties:*

- (a) When $\tau'' = \bar{\nabla}_1^2(\tau)$ for certain $\tau \in \text{DRC}(\mathcal{O})$, we have $x_{\tau''} \neq s$;
- (b) For each τ'' such that $x_{\tau''} \neq s$, we have a bijection

$$\begin{aligned} \bar{\delta}: \{ \tau \in \text{DRC}(\mathcal{O}) \mid \bar{\nabla}_1^2(\tau) = \tau'' \} &\longrightarrow \{ (x_{\tilde{\tau}}, \mathbf{u}_{\tilde{\tau}}) \mid \tilde{\tau} \in \text{DRC}(\tilde{\mathcal{O}}) \text{ s.t. } \bar{\nabla}_1^2(\tilde{\tau}) = x_{\tau''} \} \\ \tau &\longmapsto (x_{\tau''}, \mathbf{x}_{\tau}) \end{aligned}$$

where $\mathbf{u}_{\tilde{\tau}}$ is defined in (3.5).

The situation could be summarized in the following commutative diagram:

$$\begin{array}{ccccccc} & & \tau & \xrightarrow{\delta} & (\tau'', \mathbf{x}_{\tau}) & \longmapsto & \tau'' \\ & & \downarrow & & \downarrow & & \downarrow \\ x_{\tau} & \longleftarrow & \mathbf{p}_{\tau} & \longrightarrow & (x_{\tau''}, \mathbf{x}_{\tau}) & \longrightarrow & x_{\tau''} \\ & \uparrow & \downarrow & & \parallel & & \parallel \\ x_{\tilde{\tau}} & \longleftarrow & \tilde{\tau} & \xrightarrow{\delta} & (\tilde{\tau}'', \mathbf{u}_{\tilde{\tau}}) & \longrightarrow & \tilde{\tau}'' \end{array}$$

We remark that x_{τ} and $x_{\tilde{\tau}}$ are equal in the most of the case, especially when $n = 1$. But there are exceptional cases.

Proof. This follows from a case by case verification according to our algorithm. We list all possible cases below:

When $n = 1$, see Table 2.

Now assume $n \geq 1$. We let $\mathbf{x} = x_1 x_2 \cdots x_n$ and define

$$\mathbf{x}' = \begin{cases} \mathbf{x} \text{ deleting "c"} & \text{if } c \in \mathbf{x} \\ \mathbf{x} \text{ deleting "s"} & \text{if } c \notin \mathbf{x}, s \in \mathbf{x} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now all the cases are listed in Table 3. □

Thanks to $\bar{\delta}$ defined in the above lemma, we can use (5.8). The rest of the proof is similar to that in Section 5.4. We leave the details to the reader.

REFERENCES

- [1] J. Adams, *Discrete spectrum of the reductive dual pair $(O(p, q), Sp(2m))$* , Invent. Math. **74** (1983), no. 3, 449–475.
- [2] J. Adams, B. Barbasch, and D. A. Vogan, *The Langlands classification and irreducible characters for real reductive groups*, Progress in Math., vol. 104, Birkhauser, 1991.
- [3] Jeffrey Adams and Fokko du Cloux, *Algorithms for representation theory of real reductive groups*, Journal of the Institute of Mathematics of Jussieu **8** (2009), no. 2, 209–259, DOI 10.1017/S1474748008000352.
- [4] J. Arthur, *On some problems suggested by the trace formula*, Lie group representations, II (College Park, Md.), Lecture Notes in Math. 1041 (1984), 1–49.
- [5] ———, *Unipotent automorphic representations: conjectures*, Orbites unipotentes et représentations, II, Astérisque **171–172** (1989), 13–71.
- [6] L. Auslander and B. Kostant, *Polarizations and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

- [7] D. Barbasch, *Unipotent representations for real reductive groups*, Proceedings of ICM (1990), Kyoto (2000), 769–777.
- [8] Dan Barbasch and David Vogan, *Weyl Group Representations and Nilpotent Orbits*, Representation Theory of Reductive Groups: Proceedings of the University of Utah Conference 1982, 1983, pp. 21–33.
- [9] D. Barbasch, *Orbital integrals of nilpotent orbits*, The mathematical legacy of Harish-Chandra, Proc. Sympos. Pure Math. **68** (2000), 97–110.
- [10] ———, *The unitary spherical spectrum for split classical groups*, J. Inst. Math. Jussieu **9** (2010), 265–356.
- [11] ———, *Unipotent representations and the dual pair correspondence*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 47–85.
- [12] D. Barbasch and D. A. Vogan, *Unipotent representations of complex semisimple groups*, Annals of Math. **121** (1985), no. 1, 41–110.
- [13] R. Brylinski, *Dixmier algebras for classical complex nilpotent orbits via Kraft-Procesi models. I*, The orbit method in geometry and physics (Marseille, 2000). Progress in Math. **213** (2003), 49–67.
- [14] Roger W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993.
- [15] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G* , Canad. J. Math. **41** (1989), 385–438.
- [16] F. Du Cloux, *Sur les représentations différentiables des groupes de Lie algébriques*, Ann. Sci. École Norm. Sup. **24** (1991), no. 3, 257–318.
- [17] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, Van Nostrand Reinhold Co., 1993.
- [18] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda, *Nilpotent orbits and complex dual pairs*, J. Algebra **190** (1997), no. 2, 518 – 539.
- [19] ———, *Dual pairs and Kostant-Sekiguchi correspondence. II. Classification of nilpotent elements*, Central European J. Math. **3** (2005), 430–474.
- [20] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) **102** (1978), 307–330.
- [21] M. Duflo, *Théorie de Mackey pour les groupes de Lie algébriques*, Acta Math. **149** (1982), no. 3-4, 153–213.
- [22] R. Gomez and C.-B. Zhu, *Local theta lifting of generalized Whittaker models associated to nilpotent orbits*, Geom. Funct. Anal. **24** (2014), no. 3, 796–853.
- [23] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).
- [24] ———, *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966).
- [25] M. Harris, J.-S. Li, and B. Sun, *Theta correspondences for close unitary groups*, Arithmetic Geometry and Automorphic Forms, Adv. Lect. Math. (ALM) **19** (2011), 265–307.
- [26] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag, 1983.
- [27] H. He, *Unipotent representations and quantum induction*, arXiv:math/0210372 (2002).
- [28] J.-S. Huang and J.-S. Li, *Unipotent representations attached to spherical nilpotent orbits*, Amer. J. Math. **121** (1999), no. 3, 497–517.
- [29] J.-S. Huang and C.-B. Zhu, *On certain small representations of indefinite orthogonal groups*, Represent. Theory **1** (1997), 190–206.
- [30] R. Howe, *θ -series and invariant theory*, Automorphic Forms, Representations and L -functions, Proc. Sympos. Pure Math, vol. 33, 1979, pp. 275–285.
- [31] ———, *On a notion of rank for unitary representations of the classical groups*, Harmonic analysis and group representations, Liguori, Naples (1982), 223–331.
- [32] ———, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.
- [33] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Piatetski-Shapiro, I. et al. (eds.), The Schur lectures (1992). Ramat-Gan: Bar-Ilan University, Isr. Math. Conf. Proc. 8, (1995), 1–182.
- [34] D. Jiang, B. Liu, and G. Savin, *Raising nilpotent orbits in wave-front sets*, Represent. Theory **20** (2016), 419–450.
- [35] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.

- [36] B. Kostant, *Quantization and unitary representations*, Lectures in Modern Analysis and Applications III, Lecture Notes in Math. **170** (1970), 87–208.
- [37] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.
- [38] S. S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.
- [39] S. S. Kudla, *Some extensions of the Siegel-Weil formula*, In: Gan W., Kudla S., Tschinkel Y. (eds) Eisenstein Series and Applications. Progress in Mathematics, vol 258. Birkhäuser Boston (2008), 205–237.
- [40] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence II*, Israel J. Math. **100** (1997), 29–59.
- [41] ———, *Degenerate principal series of metaplectic groups and Howe correspondence*, D. Prasad et al. (eds.), Automorphic Representations and L-Functions, Tata Institute of Fundamental Research, India, (2013), 379–408.
- [42] J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), no. 2, 237–255.
- [43] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2006.
- [44] H. Y. Loke and J. Ma, *Invariants and K -spectrums of local theta lifts*, Compositio Math. **151** (2015), 179–206.
- [45] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, j. London Math. Soc. **19** (1979), 41–52.
- [46] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272, DOI 10.1007/BF01388564. MR732546
- [47] G. W. Mackey, *Unitary representations of group extensions*, Acta Math. **99** (1958), 265–311.
- [48] W. M. McGovern, *Cells of Harish-Chandra modules for real classical groups*, Amer. J. of Math. **120** (1998), 211–228.
- [49] C. Mœglin, *Front d’onde des représentations des groupes classiques p -adiques*, Amer. J. Math. **118** (1996), 1313–1346.
- [50] ———, *Paquets d’Arthur Spéciaux Unipotents aux Places Archimédiennes et Correspondance de Howe*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 469–502.
- [51] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p -adique*, Lecture Notes in Mathematics, vol. 1291, Springer, 1987.
- [52] K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*, Astérisque **273** (2001), 1–163.
- [53] K. Nishiyama, H. Ochiai, and C.-B. Zhu, *Theta lifting of nilpotent orbits for symmetric pairs*, Trans. Amer. Math. Soc. **358** (2006), 2713–2734.
- [54] K. Nishiyama and C.-B. Zhu, *Theta lifting of unitary lowest weight modules and their associated cycles*, Duke Math. J. **125** (2004), 415–465.
- [55] T. Ohta, *The closures of nilpotent orbits in the classical symmetric pairs and their singularities*, Tohoku Math. J. **43** (1991), no. 2, 161–211.
- [56] ———, *Induction of nilpotent orbits for real reductive groups and associated varieties of standard representations*, Hiroshima Math. J. **29** (1999), no. 2, 347–360.
- [57] ———, *Nilpotent orbits of \mathbb{Z}_4 -graded Lie algebra and geometry of moment maps associated to the dual pair $(U(p, q), U(r, s))$* , Publ. RIMS **41** (2005), no. 3, 723–756.
- [58] A. Paul and P. Trapa, *Some small unipotent representations of indefinite orthogonal groups and the theta correspondence*, University of Aarhus Publ. Series **48** (2007), 103–125.
- [59] V. L. Popov and E. B. Vinberg, *Invariant Theory*, Algebraic Geometry IV: Linear Algebraic Groups, Invariant Theory, Encyclopedia of Mathematical Sciences, vol. 55, Springer, 1994.
- [60] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloq. Math. **70** (1996), 93–102.
- [61] ———, *Characters, dual pairs, and unitary representations*, Duke Math. J. **69** (1993), no. 3, 547–592.
- [62] S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.
- [63] S. Sahi, *Explicit Hilbert spaces for certain unipotent representations*, Invent. Math. **110** (1992), no. 2, 409–418.
- [64] J. Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138.
- [65] W. Schmid and K. Vilonen, *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Annals of Math. **151** (2000), no. 3, 1071–1118.

- [66] E. Sommers, *Lusztig's canonical quotient and generalized duality*, J. Algebra **243** (2001), no. 2, 790–812.
- [67] T. A. Springer and R. Steinberg, *Seminar on algebraic groups and related finite groups; Conjugate classes*, Lecture Notes in Math., vol. 131, Springer, 1970.
- [68] B. Sun and C.-B. Zhu, *A general form of Gelfand-Kazhdan criterion*, Manuscripta Math. **136** (2011), 185–197.
- [69] P. Trapa, *Special unipotent representations and the Howe correspondence*, University of Aarhus Publication Series **47** (2004), 210–230.
- [70] D. A. Vogan, *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. **49** (1982), no. 4, 943–1073. MR683010
- [71] ———, *Unitary representations of reductive Lie groups*, Ann. of Math. Stud., vol. 118, Princeton University Press, 1987.
- [72] ———, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups, Proc. Conf., Brunswick/ME (USA) 1989, Prog. Math. **101** (1991), 315–388.
- [73] ———, *The method of coadjoint orbits for real reductive groups*, Representation theory of Lie groups (Park City, UT, 1998). IAS/Park City Math. Ser. **8** (2000), 179–238.
- [74] ———, *Unitary representations of reductive Lie groups*, Mathematics towards the Third Millennium (Rome, 1999). Accademia Nazionale dei Lincei, (2000), 147–167.
- [75] N. R. Wallach, *Real reductive groups I*, Academic Press Inc., 1988.
- [76] ———, *Real reductive groups II*, Academic Press Inc., 1992.
- [77] H. Weyl, *The classical groups: their invariants and representations*, Princeton University Press, 1947.
- [78] S. Yamana, *Degenerate principal series representations for quaternionic unitary groups*, Israel J. Math. **185** (2011), 77–124.

x_τ	$\tilde{x}_{\tau''}$ $m_{\tau''}$	\mathbf{p}_τ	$(x_{\tau''}, \mathbf{x}_\tau)$	$\tilde{\tau}_L$
r	r a	$\bullet \begin{smallmatrix} r \\ b \end{smallmatrix}$	(r, s)	$s \ r$
		$\bullet \begin{smallmatrix} r \\ a \end{smallmatrix}$	(r, c)	$r \ c$
r	r b	$s \begin{smallmatrix} r \\ b \end{smallmatrix}$	(r, s)	$s \ r$
		$s \begin{smallmatrix} r \\ a \end{smallmatrix}$	(r, c)	$r \ c$
c	s a	$\bullet \begin{smallmatrix} s \\ b \end{smallmatrix}$	(c, s)	$s \ c$
		$\bullet \begin{smallmatrix} s \\ a \end{smallmatrix}$	(c, c)	$c \ c$
d	d a	$\bullet \begin{smallmatrix} d \\ b \end{smallmatrix}$	(d, s)	$s \ d$
		$\bullet \begin{smallmatrix} d \\ a \end{smallmatrix}$	(d, c)	$c \ d$
		$r \begin{smallmatrix} d \\ a \end{smallmatrix}$	(d, r)	$r \ d$
		$d \begin{smallmatrix} d \\ a \end{smallmatrix}$	(d, d)	$d \ d$
	d b	$s \begin{smallmatrix} d \\ b \end{smallmatrix}$	(d, s)	$s \ d$
		$s \begin{smallmatrix} d \\ a \end{smallmatrix}$	(d, c)	$c \ d$
		$r \begin{smallmatrix} d \\ b \end{smallmatrix}$	(d, r)	$r \ d$
		$d \begin{smallmatrix} d \\ b \end{smallmatrix}$	(d, d)	$d \ d$

TABLE 2. Reduction when $n = 1$

x_τ	$\tilde{x}_{\tau''}$ $m_{\tau''}$	\mathbf{p}_τ	$x_{\tau''}, \mathbf{x}_\tau$	$\tilde{\tau}_L$	conditions	
r	r a	$\begin{smallmatrix} \bullet & r \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	r, \mathbf{x}	$\mathbf{x} \ r$	$x_1 \neq r$	$c \in \mathbf{X}$
		$\begin{smallmatrix} r & c \\ \mathbf{x}' & \end{smallmatrix}$		$x_1 = r$		
	$\begin{smallmatrix} \bullet & r \\ \mathbf{x}' & \\ b & \end{smallmatrix}$	r, \mathbf{x}	$\mathbf{x} \ r$	$c \notin \mathbf{X}, s \in \mathbf{X}$		
r	r b	$\begin{smallmatrix} s & r \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	r, \mathbf{x}	$\mathbf{x} \ r$	$x_1 \neq r$	$c \in \mathbf{X}$
		$\begin{smallmatrix} r & c \\ \mathbf{x}' & \end{smallmatrix}$		$x_1 = r$		
	$\begin{smallmatrix} s & r \\ \mathbf{x}' & \\ b & \end{smallmatrix}$	r, \mathbf{x}	$\mathbf{x} \ r$	$c \notin \mathbf{X}, s \in \mathbf{X}$		
c	s a	$\begin{smallmatrix} \bullet & s \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	c, \mathbf{x}	$\mathbf{x} \ c$	$c \in \mathbf{X}$	$s \in \mathbf{X}$ is automatic
		$\begin{smallmatrix} \bullet & s \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$c \notin \mathbf{X}$	
d	d a	$\begin{smallmatrix} \bullet & d \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	d, \mathbf{x}	$\mathbf{x} \ d$	$c \in \mathbf{X}$	
		$\begin{smallmatrix} \bullet & d \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$x_1 = s$	$c \notin \mathbf{X}$
		$\begin{smallmatrix} \mathbf{x} & d \\ a & \end{smallmatrix}$			$x_1 \neq s$	$c \notin \mathbf{X}$
d	d b	$\begin{smallmatrix} s & d \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	d, \mathbf{x}	$\mathbf{x} \ d$	$c \in \mathbf{X}$	
		$\begin{smallmatrix} s & d \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$x_1 = s$	$c \notin \mathbf{X}$
		$\begin{smallmatrix} \mathbf{x} & d \\ b & \end{smallmatrix}$			$x_1 \neq s$	$c \notin \mathbf{X}$

TABLE 3. Reduction when $n \geq 1$

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