## 1. The descents of painted bipartitions

As before, let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity. Put

$$l := l_{\star,\check{\mathcal{O}}} := \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2}; & \text{if } \star \in \{B, \widetilde{C}\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})-1}{2}, & \text{if } \star \in \{C, C^*\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \star \in \{D, D^*\}. \end{cases} \tag{1.1}$$

This is the length of the leading column of every element of $\mathrm{PBP}_\star(\check{\mathcal{O}})$.

In various context, we use $\varnothing$ to denote the empty set, the empty Young diagram or the painted Young diagram whose underlying Young diagram is empty. For every Young diagram $\imath$, its descent, which is denoted by $\nabla(\jmath)$, is defined to be the Young diagram obtained from $\jmath$ by removing the first column. By convention, $\nabla(\varnothing) = \varnothing$.

In the rest of this section, we assume that $\check{\mathcal{O}} \neq \varnothing$, and write $\check{\mathcal{O}}'$ for its dual descent. Write $\star'$ for the Howe dual of $\star$ so that $\check{\mathcal{O}}'$ has $\star'$-good parity. Put

$$l' := l_{\star',\check{\mathcal{O}}'}$$

1.1. **Naive descents of painted bipartitions.** In this subsection, let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ be a painted bipartition such that $\star_\tau = \star$. Write $\star'$ for the Howe dual of $\star$ and put

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } \mathcal{P}_\tau(l_{\star,\check{\mathcal{O}}}, 1), 1) \neq c; \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } \mathcal{P}_\tau(l_{\star,\check{\mathcal{O}}}, 1), 1) = c; \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{1.2}$$

[

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ does not occur in the leading column of } \tau; \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ occurs in the leading column of } \tau; \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{1.3}$$

]

**Lemma 1.1.** *If $\star \in \{B, C, C^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\imath, \nabla(\jmath))$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,
$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}(i,j), & \text{if } \mathcal{P}(i,j) \notin \{\bullet, s\}; \end{cases}$$
- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,
$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j+1), & \text{if } \mathcal{Q}(i,j+1) \notin \{\bullet, s\}. \end{cases}$$

*Proof.* First assume that the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$. Then the image of $\mathcal{P}$ is in fact contained in $\{\bullet\}$, and $(\imath, \jmath)$ is right interlaced in the sense that

$$\mathbf{c}_1(\jmath) \geqslant \mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\jmath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\jmath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots.$$

Hence $(\imath', \jmath') := (\imath, \nabla(\jmath))$ is left interlaced in the sense that

$$\mathbf{c}_1(\imath') \geqslant \mathbf{c}_1(\jmath') \geqslant \mathbf{c}_2(\imath') \geqslant \mathbf{c}_2(\jmath') \geqslant \mathbf{c}_3(\imath') \geqslant \mathbf{c}_3(\jmath') \geqslant \cdots.$$

Then it is clear that there is unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ such that images of $\mathcal{P}'$ and $\mathcal{Q}'$ are both contained in $\{\bullet, s\}$. This proves the lemma in the special case when the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$.

The proof of the lemma in the general case is easily reduced to this special case. $\square$

**Lemma 1.2.** *If $\star \in \{\widetilde{C}, D, D^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\nabla(\imath), \jmath)$;
- *for all $(i, j) \in \mathrm{Box}(\imath')$,*

$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i, j+1) \in \{\bullet, s\}; \\ \mathcal{P}(i, j+1), & \text{if } \mathcal{P}(i, j+1) \notin \{\bullet, s\}; \end{cases}$$

- *for all $(i, j) \in \mathrm{Box}(\jmath')$,*

$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i, j) \in \{\bullet, s\}; \\ \mathcal{Q}(i, j), & \text{if } \mathcal{Q}(i, j) \notin \{\bullet, s\}. \end{cases}$$

*Proof.* The proof is similar to that of Lemma 1.1. $\square$

**Definition 1.3.** *In the notation of Lemma 1.1 and 1.2, we call $\tau'$ the naive descent of $\tau$, to be denoted by $\nabla_{\mathrm{naive}}(\tau)$.*

*Example.* If

$$\tau = \begin{array}{cccc} \bullet & \bullet & \bullet & c \\ \bullet & s & c & \\ s & & & \\ c & & & \end{array} \times \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & r & d \\ d & d & \end{array} \times \widetilde{C},$$

then

$$\nabla_{\mathrm{naive}}(\tau) = \begin{array}{ccc} \bullet & \bullet & c \\ \bullet & c & \end{array} \times \begin{array}{ccc} \bullet & \bullet & s \\ \bullet & r & d \\ d & d & \end{array} \times B^-.$$

1.2. **Descents of painted bipartitions.** Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ and write

$$\tau'_{\mathrm{naive}} = (\imath', \mathcal{P}'_{\mathrm{naive}}) \times (\jmath', \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'$$

for the naive descent of $\tau$. This is clearly an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$.

The following two lemmas are easily verified and we omit the proofs. We will give an example for each of them.

**Lemma 1.4.** *Suppose that*

$$\begin{cases} \alpha = B^+; \\ \mathbf{r}_2(\check{\mathcal{O}}) > 0; \\ \mathcal{Q}(l, 1) \in \{r, d\}. \end{cases}$$

*Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

*such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ and for all $(i, j) \in \mathrm{Box}(\imath')$,*

$$\mathcal{P}'(i,j) = \begin{cases} s, & \text{if } (i, j) = (l', 1); \\ \mathcal{P}'_{\mathrm{naive}}(i, j), & \text{otherwise.} \end{cases}$$

*Example.* If

$$\tau = \begin{array}{|c|c|}\hline \bullet & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline \bullet & r \\ \hline r & d \\ \hline\end{array} \times B^+,$$

then

$$\tau'_{\text{naive}} = \begin{array}{|c|c|}\hline s & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline\end{array} \times \widetilde{C} \qquad \text{and} \qquad \tau' = \begin{array}{|c|c|}\hline s & c \\ \hline s & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline\end{array} \times \widetilde{C}.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $4, 4, 4, 2$, and $l' = 2$.

**Lemma 1.5.** *Suppose that*

$$\begin{cases} \alpha = D; \\ \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0; \\ \mathcal{P}(l'+1, 1) = r; \\ \mathcal{P}(l'+1, 2) = c; \\ \mathcal{P}(l, 1) \in \{ r, d \}. \end{cases}$$

*Then there is a unique element in* $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

*such that* $\mathcal{Q}' = \mathcal{Q}'_{\text{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i, j) = \begin{cases} r, & \text{if } (i, j) = (l'+1, 1); \\ \mathcal{P}'_{\text{naive}}(i, j), & \text{otherwise.} \end{cases}$$

*Example.* If

$$\tau = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & s \\ \hline \bullet & s \\ \hline r & c \\ \hline\end{array} \times \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \end{array} \times D,$$

then

$$\tau'_{\text{naive}} = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline c \\ \hline\end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \end{array} \times C, \qquad \text{and} \qquad \tau' = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline r \\ \hline\end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{2}{c}{} \end{array} \times C.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $7, 7, 7, 3$, and $l' = 3$.

**Definition 1.6.** *We define the descent of* $\tau$ *to be*

$$\nabla(\tau) := \begin{cases} \tau', & \text{if the condition of Lemma 1.4 or 1.5 holds;} \\ \nabla_{\text{naive}}(\tau), & \text{otherwise,} \end{cases}$$

*which is an element of* $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$. *Here* $\tau'$ *is as in Lemmas 1.4 and 1.5.*

In conclusion, we have defined the descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

the Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, New York 14853
*Email address*: dmb14@cornell.edu

School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China
*Email address*: hoxide@sjtu.edu.cn

Institute for Advanced Study in Mathematics, Zhejiang University, Hangzhou, 310058, China
*Email address*: sunbinyong@zju.edu.cn

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
*Email address*: matzhucb@nus.edu.sg