1. Special unipotent representations of $\mathrm{U}(p,q)$

In this section, let $\star \in \{ A^*, \widetilde{A}^* \}$ and

$$G := \begin{cases} \mathrm{U}(p,q) & \text{when } \star = A^* \\ \widetilde{\mathrm{U}}(p,q) & \text{when } \star = \widetilde{A}^* \end{cases}$$

where $\widetilde{\mathrm{U}}(p,q)$ is the determinant square double cover of $\mathrm{U}(p,q)$.

For each nilpotent orbit $\check{\mathcal{O}} \in \mathrm{Nil}(\check{G})$, let

$$\mathrm{Unip}_G(\check{\mathcal{O}}) := \begin{cases} \{ \pi \in \mathrm{Irr}(G) \mid \mathrm{Ann}\,\pi = I_{\check{\mathcal{O}}} \} & \text{if } \star = A^*, \\ \{ \pi \in \mathrm{Irr}(G) \mid \mathrm{Ann}\,\pi = I_{\check{\mathcal{O}}} \text{ and } \pi \text{ is geninue} \} & \text{if } \star = A^*. \end{cases}$$

In this section we count the number of elements in $\mathrm{Unip}_G(\check{\mathcal{O}})$.

We define

$$\text{“good parity ”} = \begin{cases} \text{the parity of } \left|\check{\mathcal{O}}\right| & \text{if } \star = A^*, \\ \text{the parity of } \left|\check{\mathcal{O}}\right| + 1 & \text{if } \star = \widetilde{A}^*, \end{cases}$$

and the other parity is called the bad parity.

We make the following decomposition:

$$\check{\mathcal{O}} = \check{\mathcal{O}}_g \overset{r}{\sqcup} \check{\mathcal{O}}_b$$

where every row in $\check{\mathcal{O}}_g$ has the good parity and every row in $\check{\mathcal{O}}_b$ has the bad parity.

Let $(n_{\mathrm{g}}, n_{\mathrm{b}}) = (\left|\check{\mathcal{O}}_g\right|, \left|\check{\mathcal{O}}_b\right|)$. Clearly

$$W_{[\lambda_{\check{\mathcal{O}}}]} = \mathsf{S}_{n_{\mathrm{g}}} \times \mathsf{S}_{n_{\mathrm{b}}}$$
$$\tau_{\lambda_{\check{\mathcal{O}}}} = \check{\mathcal{O}}_g^t \boxtimes \check{\mathcal{O}}_b^t$$

We define $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}$ to be the coherent continuations of genuine representations if $\star = \widetilde{A}^*$.

Lemma 1.1. *Let*

$$\mathcal{C}_{p,q} = \bigoplus_{\substack{t,s,r \in \mathbb{N} \\ t+r=p, t+s=q}} \mathrm{Ind}_{\mathsf{W}_t \times \mathsf{S}_s \times \mathsf{S}_r}^{\mathsf{S}_{n_g}} 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn}$$

$$\mathcal{C}_b = \begin{cases} \mathrm{Ind}_{\mathsf{W}_{\frac{n_{\mathrm{b}}}{2}}}^{\mathsf{S}_{n_{\mathrm{b}}}} 1 & \text{if } n_{\mathrm{b}} \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Then, as the $W_{[\lambda_{\check{\mathcal{O}}}]} = \mathsf{S}_{n_g} \times \mathsf{S}_{n_b}$ module,

$$\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G) = \begin{cases} \mathcal{C}_{p-\frac{1}{2}n_{\mathrm{b}}, q-\frac{1}{2}n_{\mathrm{b}}} \otimes \mathcal{C}_b & \text{if } n_{\mathrm{b}} \text{ is even and } p,q \geq \frac{1}{2}n_{\mathrm{b}} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For non-linear group, tensoring the genuine $\det^{1/2}$-character yields a $W_{[\lambda]}$-module isomorphism $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}+(\frac{1}{2},\cdots,\frac{1}{2})]}(\mathrm{U}(p,q)) \longrightarrow \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(G)$. So the problem reduces to the linear group case, which follows from a routine calculation. □

Note that the structure of Harish-Chandra cell is determined by [?Bo]. As a consequence, we have the following sharp counting theorem.

Theorem 1.2. *Retain the above setting. Then [label=()]*
$\mathrm{Unip}_G(\check{\mathcal{O}}) \neq \emptyset$ *only if* $p, q \geq \frac{1}{2}n_{\mathrm{b}}$ *and* $p+q$ *is even when* $G = \widetilde{U}(p,q)$.
In that case Suppose $\min(p,q) < \mathfrak{n}$ Unip

(i) The set $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,q)) \neq \emptyset$ *if and only if* $p = q$ *and each row lenght in* $\check{\mathcal{O}}$ *has even multiplicity.*

(ii) Suppose $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$*, let* $\check{\mathcal{O}}'$ *be the Young diagram such that* $\mathbf{r}_i(\check{\mathcal{O}}') = \mathbf{r}_{2i}(\check{\mathcal{O}}_b)$ *and* π' *be the unique special uinpotent representation in* $\mathrm{Unip}_{\check{\mathcal{O}}'}(\mathrm{GL}_p(\mathbb{C}))$*. Then the unique element in* $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p))$ *is given by*

$$\pi := \mathrm{Ind}_P^{\mathrm{U}(p,p)} \pi'$$

where P *is a parabolic subgroup in* $\mathrm{U}(p,p)$ *with Levi factor equals to* $\mathrm{GL}_p(\mathbb{C})$*.*

(iii) In general, when $\mathrm{Unip}_{\check{\mathcal{O}}_b}(\mathrm{U}(p,p)) \neq \emptyset$*, we have a natural bijection*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}(n_1+p,n_2+p)) \\ \pi_0 & \mapsto & \mathrm{Ind}_P^{\mathrm{U}(n_1+p,n_2+p)} \pi' \otimes \pi_0 \end{array}$$

where P *is a parabolic subgroup with Levi factor* $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{U}(n_1,n_2)$*.*

The above lemma ensure us to reduce the problem to the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Now assume $\check{\mathcal{O}} = \check{\mathcal{O}}_g$ and so $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ corresponds to the blocks of the infinitesimal character of the trivial representation.

By [?BV.W, Theorem 4.2], Harish-Chandra cells in $\mathrm{Coh}_{[\check{\mathcal{O}}]}$ are in one-one correspondence to real nilpotent orbits in $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t$.

[From the branching rule, the cell is parametered by painted partition

$$\mathrm{PP}(\mathrm{U}) := \left\{\, \tau \in \mathrm{PP} \;\middle|\; \begin{array}{l} \mathrm{Im}(\tau) \subseteq \{\, \bullet, s, r \,\} \\ \text{“}\bullet\text{” occures even times in each row} \end{array} \right\}.$$

The bijection $\mathrm{PP}(\mathrm{U}) \to \mathsf{SYD}, \tau \mapsto \mathscr{O}$ is given by the following recipe: The shape of $\mathscr{O}$ is the same as that of τ. $\mathscr{O}$ is the unique (upto row switching) signed Young diagram such that

$$\mathscr{O}(i, \mathbf{r}_i(\tau)) := \begin{cases} +, & \text{when } \tau(i, \mathbf{r}_i(\tau)) = r; \\ -, & \text{otherwise, i.e. } \tau(i, \mathbf{r}_i(\tau)) \in \{\, \bullet, s \,\}. \end{cases}$$

Example 1.3.

$$\begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet & r \\ \hline \bullet & \bullet \\ \cline{1-2} s & r \\ \cline{1-2} s \\ \cline{1-1} r \\ \cline{1-1} \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|c|} \hline + & - & + & - & + \\ \hline + & - \\ \cline{1-2} - & + \\ \cline{1-2} - \\ \cline{1-1} + \\ \cline{1-1} \end{array}$$

]

Now the following lemma is clear.

Lemma 1.4. *When* $\check{\mathcal{O}} = \check{\mathcal{O}}_g$*, the associated varity of every special unipotent representations in* $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U})$ *is irreducible. Moreover, the following map is a bijection.*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{U}(n_1,n_2)) & \longrightarrow & \{\, \textit{rational forms of } \check{\mathcal{O}}^t \,\} \\ \pi_0 & \mapsto & \mathrm{AV}^{\mathrm{weak}}(\pi_0). \end{array}$$

□

Remark 1.5. Note that the parabolic induction of an rational nilpotent orbit can be reducible. Therefore, when $\check{\mathcal{O}}_b \neq \emptyset$, the special unipotent representations can have reducible associated variety. Meanwhile, it is easy to see that the map $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) \ni \pi \mapsto \mathrm{AV}^{\mathrm{weak}}(\pi)$ is still injective.

We will show that every elements in $\mathrm{Unip}_{\check{\mathcal{O}}_g}$ can be constructed by iterated theta lifting. For each τ, let $\mathscr{O}$ be the corresponding real nilpotent orbit. Let $\mathrm{Sign}(\mathscr{O})$ be the signature of $\mathscr{O}$, $\nabla(\mathscr{O})$ be the signed Young diagram obtained by deleteing the first column of $\mathscr{O}$.

Suppose $\mathscr{O}$ has k-columns. Inductively we have a sequence of unitary groups $\mathrm{U}(p_i, q_i)$ with $(p_i, q_i) = \mathrm{Sign}(\nabla^i(\mathscr{O}))$ for $i = 0, \cdots, k$. Then

$$\pi_\tau = \theta_{\mathrm{U}(p_1,q_1)}^{\mathrm{U}(p_0,q_0)} \theta_{\mathrm{U}(p_2,q_2)}^{\mathrm{U}(p_1,q_1)} \cdots \theta_{\mathrm{U}(p_k,q_k)}^{\mathrm{U}(p_{k-1},q_{k-1})}(1) \tag{1.1}$$

where 1 is the trivial representation of $\mathrm{U}(p_k, q_k)$.

Suppose $\check{\mathcal{O}} = \check{\mathcal{O}}_g$. Form the duality between cells of $\mathrm{U}(p, q)$ and $\mathrm{GL}(n, \mathbb{R})$. We have an ad-hoc (bijective) duality between unipotent representations:

$$\begin{array}{rcl} d_{\mathrm{BV}}\colon \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{U}) & \to & \mathrm{Unip}_{\check{\mathcal{O}}^t}(\mathrm{GL}(\mathbb{R})) \\ \pi_{\boldsymbol{\tau}} & \mapsto & \pi_{d_{\mathrm{BV}}(\boldsymbol{\tau})} \end{array}$$

Here $\check{\mathcal{O}}^t = d_{\mathrm{BV}}(\check{\mathcal{O}})$ and $d_{\mathrm{BV}}(\boldsymbol{\tau})$ is the pained bipartition obtained by transposeing $\boldsymbol{\tau}$ and replace s and r by c and d respectively. See (**??**) and (**??**) for the definition of special unipotent representations on the two sides.