The weak Arthur packets of real classical groups

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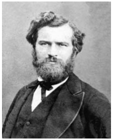
(BIRS-IASM, Arthur packets)

Jordan-Chevalley decomposition

- For each $g \in G := \mathrm{GL}_n(\mathbb{C})$, $\exists!$ pair (s, u) such that
$$g = su = us,$$
- s is semisimple, and u is unipotent ($u - 1$ is nilpotent.)

Camille Jordan



Claude Chevalley

- Classification of conjugation classes in G:

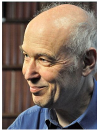
$$G/\sim = \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \{ su \mid u \in G_s \} / \sim \xleftrightarrow{bij.} \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \mathrm{unip}(G_s)$$

Finite group of Lie type: $G := \mathbf{G}(\mathbb{F}_q)$

- Deligne-Lusztig and Lusztig (1970s–1980s)
- Jordan decomposition:

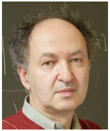
  Define dual group $\check{\mathrm{G}}$. E.g. $\mathrm{Sp}_{2n} = G \leftrightsquigarrow \check{\mathrm{G}} = \mathrm{SO}_{2n+1}$

Deligne



$$\mathrm{Irr}(G) = \bigsqcup_{s \in \check{\mathrm{G}}_{\mathrm{s.s.}}/\sim} \mathcal{E}(G, s).$$

Lusztig



Lusztig's map to the unipotent packet.

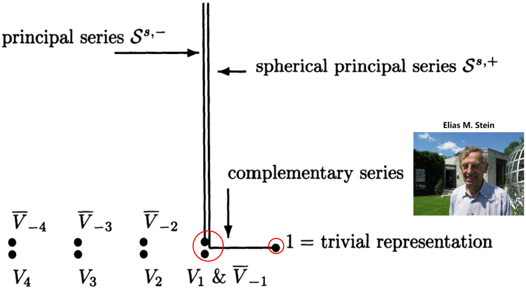
$$\mathcal{E}(G, s) \xrightarrow[\mathcal{L}_s]{bij.} \mathcal{E}(\check{\mathrm{G}}_s, 1)$$

Representations of Real Lie groups

- $G \in \{\, \mathrm{GL}_n(\mathbb{R}), \mathrm{U}(p,q), \mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{Mp}(2n,\mathbb{R}), \cdots \,\}$

$$\{\text{ discrete series }\} \subset \{\text{ tempered }\} \subset \{\text{ unitary }\} \subset \mathrm{Irr}(G)$$

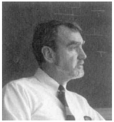
- Unitary dual of $\mathrm{SL}_2(\mathbb{R})$:



- Open problem: Structure of the unitary dual!

Langlands and Arthur

- $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\,\mathbb{C}^\times$ such that $jzj^{-1} = \bar{z}, j^2 = -1 \in \mathbb{C}^\times$.
- Langlands dual group ${}^LG = \check{G} \rtimes \langle j \rangle$.
  Eg. $\mathrm{Sp}_{2n}(\mathbb{R}) = G \longleftrightarrow {}^LG = \mathrm{SO}_{2n+1}(\mathbb{C}) \times \langle j \rangle$
- Langlands



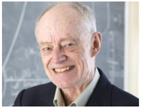
$$\mathrm{Irr}(G) \xrightarrow{\text{finite to one}} \{\, \phi\colon W_{\mathbb{R}} \to {}^LG \,\} / \check{G}.$$

- Arthur $\quad \mathrm{Irr}_{\mathrm{temp}}(G) \subset \mathrm{Irr}_A(G) \subset \mathrm{Irr}_{\mathrm{unit}}(G)$

$$\mathrm{Irr}_A(G) \xrightarrow{\text{finite to finite}} \{\, \psi\colon W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG \,\} / \check{G}.$$

Arthur's unipotent representations

- $\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$
- Unipotent parameter: $\psi|_{\mathbb{C}^\times}$ = trivial

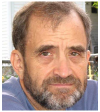
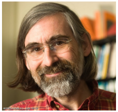
  $\Leftrightarrow$ nilpotent orbit of a real from of $\check{G}$.
-  Conjecture: $\exists$ unipotent Arthur packets

  "On some problems suggested by the trace formula" 1980's

  Mœglin Renard
-   Reduction to unipotent Arthur packet

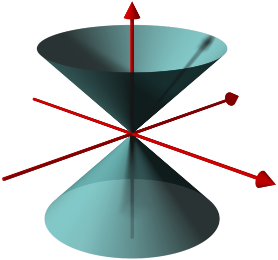
  "Sur Les paquets d'Arthur des groupes classiques réels" (2020 JEMS)

Barbasch-Vogan's definition of special unip. repn.

- Barbasch  Vogan  $\mathrm{Unip}(G) :=$ repn. with smallest "size"
- Wavefront cycle/Associated cycle:

$$AC : \mathrm{Irr}(G) \longrightarrow \mathcal{K}(\text{equiv. sheaves on nil-cone})$$

- E.g.: $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{R}^3$.



Barbasch-Vogan's definition of special unip. repn. II

G: a real reductive group.

- $\check{\mathcal{O}}$: a nilpotent orbit in $\check{\mathrm{G}} \rightsquigarrow$ inf. char. $\chi_{\check{\mathcal{O}}}$.

  $\rightsquigarrow$ the maximal primitive ideal $\mathcal{I}_{\check{\mathcal{O}}} \subset \mathcal{U}(\mathfrak{g})$.

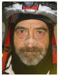
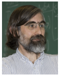
Duflo

- Definition (Barbasch-Vogan):

  An irr. adm. G-module is called special unipotent if

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) = \mathcal{I}_{\check{\mathcal{O}}}.$$

$\iff \pi$ has inf. char. $\chi_{\check{\mathcal{O}}}$ and $\mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}}$

- $\mathrm{Nil}^{\mathrm{special}}(\mathbf{G}) \ni \mathcal{O} :=$ the  Spaltenstein    dual of $\check{\mathcal{O}}$.

- $\mathrm{Unip}_{\check{\mathcal{O}}}(G) := \{$ special unipotent repn. attached to $\check{\mathcal{O}}\}$.

Conjecture/Open problems

- Major open problem: Classify the unitary dual:

$$\mathrm{Irr}_{\mathrm{unit}}(G) = \{\text{ irr. unitary repn. of } G\}.$$

- Philosophy: $\mathsf{Unip}_{\check{\mathcal{O}}}(G)$ = the building blocks of the unitary dual.

  Special unip. repn. are not enough in general

  E.g., Complex Spin group, by Wong-Zhang.

  Losev, Mason-Brown and Matvieievskyi for the generalization.

- Conjecture (1980s): $\mathsf{Unip}_{\check{\mathcal{O}}}(G)$ consists of unitary repn.

  Question 1: Size of $\mathsf{Unip}_{\check{\mathcal{O}}}(G)$?

  Question 2: Construction of unipotent repn.

  Question 3: Character formula?

- Barbasch-Vogan 1985: Classification for complex groups.
- Barbasch 1989: Proved the conj. for complex classical groups.

Special unip. repn. of simply conn. classical groups

Theorem (Barbasch-M.-Sun-Zhu)

Suppose G is a simply connected real classical group, i.e. one of the following groups

$$\mathrm{SU}(p,q), \mathrm{Spin}(p,q), \mathrm{Spin}(2n,\mathbb{H}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{Mp}(2n,\mathbb{R}), \mathrm{Sp}(p,q)$$

Arthur-Barbasch-Vogan's unitarity conj. for special unipotent repn. holds:

All special unipotent repn. of G are unitarizable.

Answer for exceptional groups

J. Adams, S. Miller, M. van Leeuven, and D. A. Vogan

atlas  + other method.

Counting G-module with a specific ann. variety

- Fix inf. char. $\mu \in \mathfrak{h}^*/W$
- integral Weyl group

$$W(\mu) := \{\, w \in W \mid \langle \mu - w\mu, \check{\alpha} \rangle \in \mathbb{Z},\ \forall \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \,\}$$

double cell $\mathcal{D} \subset \mathrm{Irr}(W(\mu)) \rightsquigarrow$ the special repn. τ_0

$\rightsquigarrow$ truncated ind. $J_{W(\mu)}^{W} \tau_0 \xrightarrow{\text{Springer corr.}} \mathcal{O}$.

$$W_\mu = \{\, w \in W \mid w\mu = \mu \,\}.$$

- $\mathscr{G}_\mu(G)$: the coherent countinuation repn. of $W(\mu)$.

Lemma: If $\mathfrak{g}$ has no E_8 factor, then

$$\begin{aligned} &\#\{\, \pi \in \mathrm{Irr}_\mu(\mathfrak{g}, K)(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \,\} \\ &= \sum_{\substack{\mathcal{D} \rightsquigarrow \mathcal{O} \\ \tau \in \mathcal{D}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathscr{G}_\mu(G)] \end{aligned}$$

Applications

- $\mathsf{Unip}_{\check{\mathcal{O}}}(\mathrm{SL}_n(\mathbb{H}))$ = restricition of $\mathsf{Unip}_{\check{\mathcal{O}}}(\mathrm{GL}_n(\mathbb{H}))$.
- $\mathsf{Unip}_{\check{\mathcal{O}}}(\mathrm{SL}_n(\mathbb{R}))$ = components in the rest. of $\mathsf{Unip}_{\check{\mathcal{O}}}(\mathrm{GL}_n(\mathbb{R}))$.
- $\mathsf{Unip}_{\check{\mathcal{O}}}(\mathsf{SU}_(p,q))$ = rest. of $\mathsf{Unip}_{\check{\mathcal{O}}}$(det double cover of $\mathsf{U}(p,q)$).
- genuine special unipotent repn. of $\mathsf{Spin}(p,q)$ are some obvious irreducibly parabolicaly induced module.

Nilpotent orbits with "good/bad parity"

- Bad parity (must occur with even multiplicity in $\check{\mathcal{O}}$):

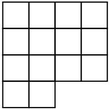
$$\begin{cases} \text{even number,} & \text{when } \check{\mathrm{G}} \text{ is type } B \text{ or } D \\ \text{odd number,} & \text{when } \check{\mathrm{G}} \text{ is type } C \end{cases}$$

- $\check{\mathcal{O}}$ has "good parity" if $\check{\mathcal{O}}$ only contains

$$\begin{cases} \text{odd rows,} & \text{when } \check{\mathrm{G}} \text{ is type } B \text{ or } D \\ \text{even rows,} & \text{when } \check{\mathrm{G}} \text{ is type } C \end{cases}$$

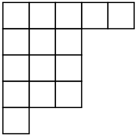
$\check{\mathcal{O}}$ good parity $\rightsquigarrow \chi_{\check{\mathcal{O}}}$ is integral.

$\mathcal{O}$



$\mathrm{Sp}(14, \mathbb{C})$

$\check{\mathcal{O}}$



$\mathrm{SO}(15, \mathbb{C})$

Reduction to the "good parity"

- Consider $G = \mathrm{Sp}(2n, \mathbb{R})$.
- $\check{\mathcal{O}}$ decompose into two parts $\check{\mathcal{O}}_g$ (good parity) and $\check{\mathcal{O}}_b$ (bad parity).
- Assume $\check{\mathcal{O}}_b = \{r_1, r_1, \cdots, r_k, r_k\}$.

Theorem Let $\check{\mathcal{O}}_b' = \{\, r_1, \cdots, r_k \,\} \in \mathrm{Nil}_{\mathrm{GL}}$.

$$\begin{array}{ccc} \mathrm{Unip}_{\check{\mathcal{O}}_b'}(\mathrm{GL}) \times \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{Sp}) & \xrightarrow{bij.} & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}) \\ (\pi', \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}_{|\check{\mathcal{O}}_b'|} \times \mathrm{Sp}(2n_0, \mathbb{R}) \ltimes U}^{\mathrm{Sp}(2n, \mathbb{R})} \pi' \otimes \pi_0 \end{array}$$

$$\mathrm{Unip}_{\check{\mathcal{O}}_b'}(\mathrm{GL}) = \left\{ \mathrm{Ind} \bigotimes_{j=1}^{k} \mathrm{sgn}_{\mathrm{GL}(r_j, \mathbb{R})}^{\epsilon_j} \;\middle|\; \epsilon_j \in \mathbb{Z}/2\mathbb{Z} \right\}$$

- Use theta correspondence to study $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
- We assume $\check{\mathcal{O}}$ has good parity from now on.

Counting unipotent representations I

- Example: $G = \mathrm{Sp}(2n, \mathbb{R})$ and $\chi_{\check{\mathcal{O}}} \in \rho_G +$ weight lattice.

$$W(\chi_{\check{\mathcal{O}}}) = S_n \ltimes \{\pm 1\}^n,$$
$$\mathscr{G}_{\rho_G}(G) = \sum_{\substack{b,c,d,r,\\ \sigma \in \widehat{S_r}}} \mathrm{Ind}_{S_r \times W_{2b} \times W_c \times W_d}^{W_n} \mathrm{sgn} \otimes (\sigma \times \sigma) \otimes \mathbf{1} \otimes \mathbf{1}.$$
$$\updownarrow$$
$$\text{max. torus } \mathbb{C}^b \times \mathrm{U}(1)^r \times \mathbb{R}^{c+d}$$

$\check{\mathcal{O}} \rightsquigarrow$ special repn. $\tau = \mathrm{Springer}(d_{BV}(\check{\mathcal{O}}))$

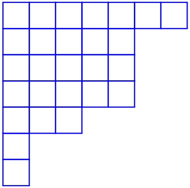
Lusztig left cell $\mathcal{C} = \{\tau_1, \cdots, \tau_{2^l}\}$ containing τ

- $[\tau_i : 1_{W_{\lambda_{\check{\mathcal{O}}}}}] = 1$ and $[\tau_i : \mathscr{G}_{\rho_G}(G)] = [\tau : \mathscr{G}_{\rho_G}(G)]$

$$\#\mathrm{Unip}_{\check{\mathcal{O}}}(G) = 2^l \cdot [\tau : \mathscr{G}_{\rho_G}(G)]$$

- $[\tau : \mathscr{G}_{\rho_G}(G)]$ is counted by painted bi-partitions $\mathrm{PBP}(\check{\mathcal{O}})$.

Example of PBP

$\check{\mathcal{O}} = [7,5,5,5,3,1,1] =$ 

$\mathrm{PBP}_{\check{\mathcal{O}}}(\mathrm{Sp}(2n,\mathbb{R}))$

$\bullet$	$\bullet$	$\bullet$
$\bullet$	c	
r	d	
d		

$\times$

$\bullet$	$\bullet$	$\bullet$
$\bullet$	s	

$\mapsto$

Associated character

$*$	$=$	$*$	$=$	$*$	$=$
$*$	$=$	$*$	$=$	$*$	$=$
$=$	$*$	$=$	$*$		
$*$	$=$	$*$	$=$		
$*$	$=$	$*$	$=$		
$-$	$+$				

$\cup$

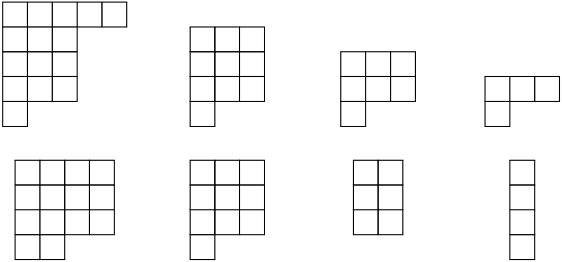
$*$	$=$	$*$	$=$	$*$	$=$
$*$	$=$	$*$	$=$	$*$	$=$
$=$	$*$	$=$	$*$		
$=$	$*$	$=$	$*$		
$*$	$=$	$*$	$=$		
$+$	$-$				

Inductive structure of nilpotent orbits

$\widehat{\mathbf{G}}_i$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

$\check{\mathcal{O}}_i$

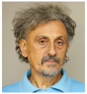
$\mathcal{O}_i$



$\mathbf{G}_i$	$\mathrm{Sp}(14,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{Sp}(6,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

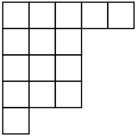
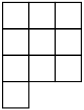
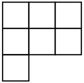
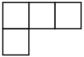
Kraft

Procesi



resolution of singularities of nilpotent orbit closures.

Example of descent sequences

$\mathbf{G}_i^\vee$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$
$\check{\mathcal{O}}_i$				
$\mathcal{O}_i$	$-+-+$ $+-+-$ $+-+-$ $+-$	$+-+$ $-+-$ $-+-$ $-$	$-+$ $+-$ $+-$	$+$ $+$ $-$ $-$
G_i	$\mathrm{Sp}(14,\mathbb{R})$	$\mathrm{O}(4,6)$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathrm{O}(2,2)$

Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

- $\chi = \bigotimes_{j=0}^{a} \chi_j$, a character of $\prod_{j=0}^{a} G_j$.
- $\chi_j \in \{\mathbf{1}, \mathrm{sgn}^{+,-}, \mathrm{sgn}^{-,+}, \det\}$ when G_j is an orthogonal group.
- Define a smooth repn. of $G = G_a$

$$\pi_\chi := (\omega_{G_a, G_{a-1}} \widehat{\otimes} \omega_{G_{a-1}, G_{a-2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1, G_0} \otimes \chi)_{G_{a-1} \times G_{a-2} \times \cdots \times G_0}$$

Theorem (Barbasch-M.-Sun-Zhu)

Suppose $\check{\mathcal{O}}$ is an orbit with good parity. Then

- either $\pi_\chi = 0$ or
- $\pi_\chi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$ and unitarizable.
- Moreover,

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}.$$

Example: Coincidences of theta liftings

Lift to $G = \mathrm{Sp}(6,\mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4,\mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

	$\mathrm{Sp}(6,\mathbb{R})$		
$\mathrm{O}(4,0)$		$\theta(\mathrm{sgn}^{+,-})$	
$\mathrm{O}(3,1)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(2,2)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(1,3)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(0,4)$			$\theta(\mathrm{sgn}^{-,+})$

Matching unipotent representations with PBP

- $\mathrm{PBP}(\check{\mathcal{O}})$ is complicate.
- $\mathrm{LS}(\check{\mathcal{O}}) = \{\, \mathrm{AC}(\pi_\chi)\,\}$ is also complicate.
- Proof of Exhaustion

  Define descent of painted bi-part., compatible with the theta!

$$\begin{tikzcd}
\mathrm{LS}(\check{\mathcal{O}}) & \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}}) \arrow[l, two heads, "\mathrm{AC}"'] \arrow[d, "\nabla"'] \arrow[r, leftrightarrow] & \bigcup \mathrm{Unip}_{\check{\mathcal{O}}}(G) \\
\mathrm{LS}(\check{\mathcal{O}}') \arrow[u, "\vartheta^{\mathrm{geo}}"] & \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}}') \arrow[l, two heads, "\mathrm{AC}"'] \arrow[r, leftrightarrow] & \bigcup \mathrm{Unip}_{\check{\mathcal{O}}'}(G') \arrow[u, "\theta"']
\end{tikzcd}$$

For $\tau \in \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})$, define

$$\pi_\tau := \Theta(\pi_{\nabla(\tau)} \otimes \chi'_\tau) \otimes \chi_\tau$$

Lifting of Associated characters I

- Example $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p,q))$

$$\begin{tikzcd}
 & M_{p,n} \oplus M_{q,n} \arrow[dl, "\varphi'"'] \arrow[dr, "\varphi"] & \\
M_{p,q} \cong \mathfrak{p}' & w = (A,B) \arrow[dl, mapsto] \arrow[dr, mapsto] & \mathfrak{p} \cong \mathrm{Sym}_n \oplus \mathrm{Sym}_n \\
X' = AB^T & & X = (A^T A, B^T B)
\end{tikzcd}$$

- $\overline{\mathcal{O}} \cap \mathfrak{p} \supset \varphi(\varphi'^{-1}(\mathfrak{p}' \cap \mathcal{O}'))$ where $\mathcal{O}$ is a cplx. nil. **G**-orbit.
- Upper bound of associated cycle: we can define

$$\vartheta^{\mathrm{geo}} \colon \mathcal{K}_{\mathcal{O}'}(G') \longrightarrow \mathcal{K}_{\mathcal{O}}(G)$$

such that

$$\mathrm{AC}(\Theta(\pi')) \preceq \vartheta^{\mathrm{geo}}(\mathrm{AC}(\pi')),$$

for every π' with $\mathrm{AV}(\pi') \subset \overline{\mathcal{O}'}$

Lifting of Associated Characters II

- Recall $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$
- For $\mathscr{L}' \in \mathcal{K}_{\mathcal{O}'}(G')$, $\mathscr{L} = \vartheta(\mathscr{L}') \in \mathcal{K}_{\mathcal{O}}(G)$,

$$\mathscr{L}_X = \vartheta_w(\mathscr{L}_{X'}) := \det{}^{(p-q)/2}|_{K_X} \otimes (\mathscr{L}'_{X'})^{K'_{2,X'}} \circ \alpha_w,$$

  $\alpha_w \colon K_X \longrightarrow K'_{1,X'}$: a homomorphism between isotropic subgroups.
- The twisting is crucial.

  $\Rightarrow$ admissible orbit data $\rightsquigarrow$ admissible orbit data.
- Support of $\vartheta(\mathscr{L}')$ could be reducible.

Key ideas in the proof

- double θ-lift $\approx$ parabolic induction

  θ-lift $\approx \sqrt{\text{parabolic induction}}$

  use the structure of degenerate principle series.
- Estimation of matrix coeff. integrals $\rightsquigarrow$ unitarity
- Sharp formula of Asso. Char.: lower bound=upper bound
  - double θ-lift $\rightsquigarrow$ lower bound.
  - double fibration of moment maps $\rightsquigarrow$ upper bound
- Exhaustion: lower bound=upper bound (Combinatorics)
  - Character theory (Kazhdan-Lusztig-Vogan theory) $\rightsquigarrow$ upper bound by counting painted bipartitions.
  - Asso. Char.+Injectivity of θ $\rightsquigarrow$ lower bound.

Relevant papers

- Definition for metaplectic groups

  On the notion of metaplectic Barbasch-Vogan duality
  https://arxiv.org/abs/2010.16089
- Counting and reduction to good parity
  https://arxiv.org/abs/2205.05266
- Construction and unitarity using θ-lifting
  https://arxiv.org/abs/1712.05552

Thank you for your attention!