SPECIAL UNIPOTENT REPRESENTATIONS : ORTHOGONAL AND SYMPLECTIC GROUPS

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Contents

1. Introduction and the main results

1.1. **Unitary representations and the orbit method.** A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [39, 40]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [6, 39]. For more general Lie groups, work of Mackey and Duflo [24, 51] suggest that one should focus attention on reductive Lie groups. As expounded by Vogan in his writings (see for example [76, 78, 79]), the problem finally is to

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quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We mention two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [4, 5], Adams, Barbasch and Vogan established some important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [2]. The problem of finding (integral) special unipotent representations for complex semisimple groups (as well as their distribution characters) was solved earlier by Barbasch and Vogan [14] and the unitarity of these representations was established by Barbasch for complex classical groups [7]. In a similar vein, Barbasch outlined his proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [8]. The second major development is Vogan's theory of associated varieties [77] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a representation attached to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [35], the description of discrete spectrum by Adams [1] and the preservation of unitarity in stable range theta lifting by Li [46]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [12, 15, 30, 32, 33, 63, 66, 68, 74]. Particularly worth mentioning were the work of Przebinda [66] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [30] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [25, 48, 57, 59], which among other things demonstrate the tight link between the orbit method and Howe's theory.

In the present article we will demonstrate that the orbit method and Howe's theory in fact have perfect synergy when it comes to unipotent representations. (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [12, 30, 54, 74].) We will restrict our attention to a real classical group G of orthogonal or symplectic type and we will construct all unipotent representations of $\widetilde{G}$ attached to $\mathcal{O}$ (in the sense of Barbasch and Vogan) via the method of theta lifting. Here $\widetilde{G}$ is either G, or the metaplectic cover of G if G is a real symplectic group, and $\mathcal{O}$ is a member of our preferred set of complex nilpotent orbits, large enough to include all those which are rigid special. Here we wish to emphasize that there have been extensive investigations of unipotent representations for real reductive groups by Vogan and his collaborators (see e.g. [2, 76, 77]), only for unitary groups complete results are known (*cf.* [13, 74]), in which case all such representations may also be described in terms of cohomological induction.

1.2. **Special unipotent representations of classical groups of type B, C or D.** In this article, we are aimed to understand special unipotent representations of classical groups of type B, C or D. As the cases of complex orthogonal groups and complex symplectic groups are well-understood (see [14] and [12]), we will focus on the following groups:

$$\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p,q),\ \mathrm{O}^*(2n), \tag{1.1}$$

where $p, q, n \in \mathbb{N} := \{0, 1, 2, \cdots\}$. Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the real metaplectic group, namely the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n = 0$.

Let G be one the groups in (1.1). As usual, we view it as a real form of $G_{\mathbb{C}}$, or a double cover of a real form of $G_{\mathbb{C}}$ in the metaplecitic case, where

$$G_{\mathbb{C}} := \begin{cases} \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \text{ or } \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2p+2q}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Respectively write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ for the Lie algebras of G and $G_{\mathbb{C}}$. Then $\mathfrak{g}$ is viewed as a complexification of $\mathfrak{g}_{\mathbb{R}}$.

Let $r_{\mathfrak{g}}$ denote the rank of $\mathfrak{g}$. Let $W_{r_{\mathfrak{g}}}$ denote the subgroup of $\mathrm{GL}_{r_{\mathfrak{g}}}(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. Then as usual, Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} = (\mathrm{S}(\mathbb{C}^{r_{\mathfrak{g}}}))^{W_{r_{\mathfrak{g}}}}. \tag{1.2}$$

Here and henceforth, "U" indicates the universal enveloping algebra of a Lie algebra, a superscript group indicate the space of invariant vectors under the group action, and "S" indicates the symmetric algebra. Unless $G_{\mathbb{C}}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. By (1.2), we have the following parameterization of characters of $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$:

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}, \mathbb{C}) = W_{r_{\mathfrak{g}}} \backslash (\mathbb{C}^{r_{\mathfrak{g}}})^* = W_{r_{\mathfrak{g}}} \backslash \mathbb{C}^{r_{\mathfrak{g}}}$$

Here "$\mathrm{Hom}_{\mathrm{alg}}$" indicates the set of $\mathbb{C}$-algebra homomorphisms, and a superscript " $*$ " over a vector space indicates the dual space.

We define the Langlands dual of G to be the complex group

$$\check{G} := \begin{cases} \mathrm{Sp}_{p+q-1}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is odd}; \\ \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is even}; \\ \mathrm{O}_{2n+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{O}_{2p+2q+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$.

Denote by $\mathrm{Nil}(\check{\mathfrak{g}})$ the set of nilpotent $\check{G}$-orbits in $\check{\mathfrak{g}}$, namely the set of all orbits of nilpotent matrices under the adjoint action of $\check{G}$ in $\check{\mathfrak{g}}$. When no confusion is possible, we will not distinguish a nilpotent orbit in $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ with its corresponding Young diagram. In particular, the zero orbit is also regarded as the Young with one nonempty column at most.

Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. It determines a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$ as in what follows. For every integer $a \geqslant 0$, write

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd}; \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even}; \end{cases}$$

By convention, $\rho(1)$ and $\rho(0)$ are the empty sequence. Write $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ ($s \geqslant 0$) for the row lengths of $\check{\mathcal{O}}$. Define

$$\chi(\check{\mathcal{O}}) := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0), \tag{1.3}$$

to be viewed as a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_\mathbb{C}} \to \mathbb{C}$. Here the number of 0's is

$$\left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor.$$

Recall the following well-known result of Dixmier ([16, Section 3]): for every algebraic character χ of $\mathrm{Z}(\mathfrak{g})$, there exists a unique maximal ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of χ. As an easy consequence, we know that there is a unique maximal $G_\mathbb{C}$-stable ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.

Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation ([18,81]) if its Harish-Chandra module has finite length. When $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a metaplectic group, write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then a representation of G is said to be genuine if ε_G acts on it through the scalar multiplication by -1. Following Barbasch and Vogan ([2, 14]), we make the following definition.

Definition 1.1. *Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. An irreducible Casselman-Wallach representation π of G is attached to $\check{\mathcal{O}}$ if*

- *$I_{\check{\mathcal{O}}}$ annihilates π; and*
- *V is genuine if G is a metaplectic group.*

Write $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ for the set of all isomorphism classes of irreducible Casselman-Wallach representations of G that are attached to $\check{\mathcal{O}}$. We say that an irreducible Casselman-Wallach representation of G is special unipotent if it is attached to $\check{\mathcal{O}}$, for some $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. We will construct all the special unipotent representations, and show that they are all unitarizable as predicted by the Arthur-Barbasch-Vogan conjecture [2, Introduction].

1.3. **Combinatorics : bipartitions.** For every Young diagram $\mathcal{O}$, write

$$\mathbf{r}_1(\mathcal{O}) \geqslant \mathbf{r}_2(\mathcal{O}) \geqslant \mathbf{r}_3(\mathcal{O}) \geqslant \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\mathcal{O}) \geqslant \mathbf{c}_2(\mathcal{O}) \geqslant \mathbf{c}_3(\mathcal{O}) \geqslant \cdots$$

for its column lengths. Denote by $|\mathcal{O}| := \sum_{i=1}^{\infty} \mathbf{r}_i(\mathcal{O})$ the total size of $\mathcal{O}$.

We introduce six symbols B, C, D, $\widetilde{C}$, C^* and D^* to indicate the types of the groups that we are considering as in (1.1), namely odd real orthogonal groups, real symplectic groups, even real orthogonal groups, real metaplectic groups, quaternionic symplectic groups and quaternionic orthogonal groups, respectively. Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and suppose that G has type $\star$.

Following [55, Definition 4.1], We say that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity if

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \check{G} \text{ is a complex symplectic group; and} \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \check{G} \text{ is a complex orthogonal group.} \end{cases}$$

The study of the special unipotent representations in general case reduced to the case when $\check{\mathcal{O}}$ has $\star$-good parity. See Appendix A. Now we suppose that $\check{\mathcal{O}}$ has $\star$-good parity. Equivalently but in a completely combinatorial setting, we consider $\check{\mathcal{O}}$ as a Young diagram that has $\star$-good parity in the following sense:

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \star \in \{B, \widetilde{C}\}; \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \star \in \{C, D, C^*, D^*\}; \text{ and} \\ \text{the total size } \left|\check{\mathcal{O}}\right| \text{ is odd if and only if } \star \in \{C, C^*\}. \end{cases}$$

Definition 1.2. *A $\star$-pair is a pair $(i, i+1)$ of successive positive integers such that*
$$\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$
A $\star$-pair $(i, i+1)$ is said to be

- *empty in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*
- *tailed in $\check{\mathcal{O}}$, if $\star \in \{C, D, C^*, D^*\}$ and $\mathbf{r}_i(\check{\mathcal{O}}) > \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *distinguished in $\check{\mathcal{O}}$, if it is not empty, balanced or tailed in $\check{\mathcal{O}}$.*

Let $\mathrm{DP}_\star(\check{\mathcal{O}})$ denote the set of all $\star$-pairs that are distinghuished in $\check{\mathcal{O}}$. For every subset $\wp \subset \mathrm{DP}_\star(\check{\mathcal{O}})$, we define a pair
$$(\imath_\wp, \jmath_\wp) := (\imath_\star(\check{\mathcal{O}}, \wp), \jmath_\star(\check{\mathcal{O}}, \wp))$$
of Young diagrams as in what follows.

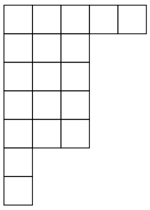
If $\star = B$, then
$$\mathbf{c}_1(\jmath_\wp) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$
and for all $i \geqslant 1$,
$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_{i+1}(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}), & \text{if } (2i, 2i+1) \in \wp; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}), & \text{otherwise.} \end{cases}$$
If $\star = \widetilde{C}$, then for all $i \geqslant 1$,
$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_i(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}), & \text{if } (2i-1, 2i) \in \wp; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}), & \text{otherwise.} \end{cases}$$
If $\star = D$ or D^*, then
$$\mathbf{c}_1(\imath_\wp) = \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0; \\ 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0, \end{cases}$$
and for all $i \geqslant 1$,
$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_{i+1}(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{if } (2i, 2i+1) \in \wp; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ (0, 0), & \text{if } (2i, 2i+1) \text{ is empty in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$
If $\star = C$ or C^*, then for all $i \geqslant 1$,
$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_i(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})+1}{2}), & \text{if } (2i-1, 2i) \in \wp; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (0, 0), & \text{if } (2i-1, 2i) \text{ is empty in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$
In all cases, it is known that (see [17, Section13.2])
$$(\imath_\star(\check{\mathcal{O}}, \wp), \jmath_\star(\check{\mathcal{O}}, \wp)) = (\imath_\star(\check{\mathcal{O}}', \wp'), \jmath_\star(\check{\mathcal{O}}', \wp')) \quad \text{if and only if} \quad \begin{cases} \check{\mathcal{O}} = \check{\mathcal{O}}'; \\ \wp = \wp', \end{cases} \tag{1.4}$$
where $\check{\mathcal{O}}'$ is another Young diagram that has $\star$-good parity, and $\wp' \subset \mathrm{DP}_\star(\check{\mathcal{O}}')$. Put
$$\mathrm{BP}_\star(\check{\mathcal{O}}) := \{(\imath_\wp, \jmath_\wp) \mid \wp \subset \mathrm{DP}_\star(\check{\mathcal{O}})\}$$

and

$$\mathrm{BP}_\star := \bigcup_{\check{\mathcal{O}} \text{ is Young diagram that has } \star\text{-good parity}} \mathrm{BP}_\star(\check{\mathcal{O}}).$$

The latter is a disjoint union by (1.4), and its elements are called $\star$-bipartitions. Recall that the set $\mathrm{BP}_\star(\check{\mathcal{O}})$ parametrizes certain Weyl group representations ([17, Section 11.4]).

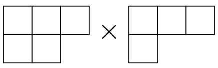
Example. Suppose that $\star = C$, and $\check{\mathcal{O}}$ is the following Young diagram which has $\star$-good parity.

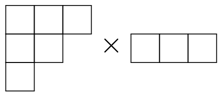


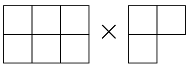
Then

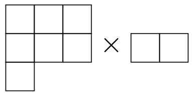
$$\mathrm{DP}_\star(\check{\mathcal{O}}) = \{(1,2),(5,6)\}.$$

and $(\imath_\wp, \jmath_\wp)$ has the following form.

$\wp = \varnothing$: 

$\wp = \{(1,2)\}$: 

$\wp = \{(5,6)\}$: 

$\wp = \{(1,2),(5,6)\}$: 

Here and henceforth, when no confusion is possible, we write $\alpha \times \beta$ for a pair (α, β). We will also write $\alpha \times \beta \times \gamma$ for a triple (α, β, γ).

1.4. **Combinatorics : painted bipartitions.** Let $\imath$ be a Young diagram. Write $\mathrm{Box}(\imath)$ for the set of all boxes in $\imath$, which has $|\imath|$ elements in total. We say that a Young diagram $\imath'$ is contained in $\imath$ (and write $\imath' \subset \imath$) if

$$\mathbf{r}_i(\imath') \leqslant \mathbf{r}_i(\imath) \qquad \text{for all } i = 1, 2, 3, \cdots .$$

When this is the case, $\mathrm{Box}(\imath')$ is viewed as a subset of $\mathrm{Box}(\imath)$ concentrating on the upper-left corner. We say that a subset of $\mathrm{Box}(\imath)$ is a Young subdiagram if it equals $\mathrm{Box}(\imath')$ for a Young diagram $\imath' \subset \imath$. In this case, we call $\imath'$ the Young diagram corresponding to this Young subdiagram.

We introduce five symbols $\bullet$, s, r, c and d.

Definition 1.3. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is a Young subdiagram of* $\mathrm{Box}(\imath)$ *when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$ when $S = \{s\}$ or $\{r\}$;*
- *every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$ when $S = \{c\}$ or $\{d\}$.*

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram and a painting on it.

Example. Suppose that $\tau = \ydiagram{2,1}$, then there are $25 + 12 + 6 + 2 = 45$ paintings on τ in total as listed below.

$\begin{ytableau} \bullet & \alpha \\ \beta \end{ytableau}$ $\quad \alpha, \beta \in \{\bullet, s, r, c, d\}$ $\qquad$ $\begin{ytableau} s & \alpha \\ \beta \end{ytableau}$ $\quad \alpha \in \{r, c, d\}, \beta \in \{s, r, c, d\}$

$\begin{ytableau} r & \alpha \\ \beta \end{ytableau}$ $\quad \alpha \in \{c, d\}, \beta \in \{r, c, d\}$ $\qquad$ $\begin{ytableau} c & \alpha \\ d \end{ytableau}$ $\quad \alpha \in \{c, d\}$

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.4. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- *$(\imath, \jmath) \in \mathrm{BP}_\alpha$ if $\alpha \notin \{B^+, B^-\}$, and $(\imath, \jmath) \in \mathrm{BP}_B$ if $\alpha \in \{B^+, B^-\}$;*
- *$\mathcal{P}^{-1}(\bullet)$ and $\mathcal{Q}^{-1}(\bullet)$ corresponding to the same Young diagram;*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \textit{if } \alpha = B^+ \textit{ or } B^-; \\ \{\bullet, r, c, d\}, & \textit{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \textit{if } \alpha = D; \\ \{\bullet, s, c\}, & \textit{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \textit{if } \alpha = C^*; \\ \{\bullet, s\}, & \textit{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \textit{if } \alpha = B^+ \textit{ or } B^-; \\ \{\bullet, s\}, & \textit{if } \alpha = C; \\ \{\bullet\}, & \textit{if } \alpha = D; \\ \{\bullet, r, d\}, & \textit{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \textit{if } \alpha = C^*; \\ \{\bullet, r\}, & \textit{if } \alpha = D^*. \end{cases}$$

With the notation of Definition 1.4, we define in what follows some objects attached to the painted bipartition τ:

$$\star_\tau, \ \check{\mathcal{O}}_\tau, \ \wp_\tau, \ |\tau|, \ \varepsilon_\tau, \ p_\tau, \ q_\tau, \ G_\tau \quad \dim \tau.$$

$\star_\tau$: This is the symbol given by

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

$\check{\mathcal{O}}_\tau$ and $\wp_\tau$: This is the unique pair such that

- $\check{\mathcal{O}}_\tau$ is a Young diagram that has $\star_\tau$-good parity;
- $\wp_\tau \subset \mathrm{DP}_{\star_\tau}(\check{\mathcal{O}}_\tau)$; and
- $(\imath, \jmath) = (\imath_{\star_\tau}(\check{\mathcal{O}}_\tau, \wp_\tau), \jmath_{\star_\tau}(\check{\mathcal{O}}_\tau, \wp_\tau))$.

$|\tau|$: This is the natural number

$$|\tau| := |\imath| + |\jmath|.$$

Note that

$$|\tau| := \begin{cases} \frac{|\check{\mathcal{O}}_\tau| - 1}{2}, & \text{if } \star_\tau = C \text{ or } C^*; \\ \frac{|\check{\mathcal{O}}_\tau|}{2}, & \text{otherwise.} \end{cases}$$

ε_τ: This is an element of $\mathbb{Z}/2\mathbb{Z}$. It equals 0 in the following cases:

(a) $\star_\tau = B$, $|\tau| > 0$ and the symbol d occurs in the first column of the painted Young diagram $(\jmath, \mathcal{Q})$;
(b) $\star_\tau = D$, $|\tau| > 0$ and the symbol d occurs in the first column of the painted Young diagram $(\imath, \mathcal{P})$;
(c) $\star_\tau \in \{C, \widetilde{C}\}$ and $(1,2) \notin \wp_\tau$.

In all other case, $\varepsilon_\tau = 1$.

p_τ and q_τ: This is a pair of natural numbers given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := \# \bullet + 2\# r + \# c + \# d + \# B^+; \\ q_\tau := \# \bullet + 2\# s + \# c + \# d + \# B^-. \end{cases}$$

G_τ: This is a classical group given by

$$G_\tau := \begin{cases} \mathrm{O}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

$\dim \tau$: This is the dimension of the standard representation of the complexification of G_τ, equvalently,

$$\dim \tau := \begin{cases} 2\,|\tau| + 1, & \text{if } \star_\tau = B; \\ 2\,|\tau|\,, & \text{otherwise.} \end{cases}$$

Example. Suppose that

$$\tau = \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet & \multicolumn{1}{|c}{} \\ \cline{1-1} c & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & r & d \\ \hline \bullet & \multicolumn{3}{|c}{} \\ \cline{1-1} d & \multicolumn{3}{|c}{} \\ \cline{1-1} \end{array} \times B^+.$$

Then

$$\star_\tau = B, \qquad \check{\mathcal{O}}_\tau = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \multicolumn{4}{|c}{} \\ \cline{1-2} \ & \ & \multicolumn{4}{|c}{} \\ \cline{1-2} \ & \ & \multicolumn{4}{|c}{} \\ \cline{1-2} \ & \ & \multicolumn{4}{|c}{} \\ \cline{1-2} \end{array}, \qquad \wp_\tau = \{(6,7)\}, \quad |\tau| = 4 + 6 = 10, \qquad \varepsilon_\tau = 0,$$

and

$$p_\tau = 6{+}2{+}1{+}2{+}1 = 12, \qquad q_\tau = 6{+}0{+}1{+}2{+}0 = 9, \qquad G_\tau = \mathrm{O}(12, 9), \qquad \dim \tau = 21.$$

Write PBP for the set of all painted bipartitions. Then

$$\mathrm{PBP} = \bigsqcup_{(\star, \check{\mathcal{O}})} \mathrm{PBP}_\star(\check{\mathcal{O}}),$$

where $\star$ runs over the set $\{B, \mathrm{C}, D, \widetilde{C}, C^*, D^*\}$, $\check{\mathcal{O}}$ runs over all Young diagrams that has $\star$-good parity, and

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) := \{\tau \in \mathrm{PBP} \mid \star_\tau = \star,\ \check{\mathcal{O}}_\tau = \check{\mathcal{O}}\}.$$

1.5. **Descents of painted bipartitions and theta lifts.** Let $\star$ and $\check{\mathcal{O}}$ be as before. For every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, in what follows we will construct a representation π_τ of G_τ by theta lift. For the starting case when $\check{\mathcal{O}}$ is the empty Young diagram, define

$$\pi_\tau := \begin{cases} \text{the one dimensional genuine representation,} & \text{if } \star = \widetilde{C} \text{ so that } G_\tau = \widetilde{\mathrm{Sp}}_0(\mathbb{R}); \\ \text{the one dimensional trivial representation,} & \text{if } \star \neq \widetilde{C}. \end{cases}$$

Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or } \ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or } \ D^*.$$

We call $\star'$ the Howe dual of $\star$. Now we assume that $\check{\mathcal{O}}$ is nonempty, and define its decent to be the Young diagram

$$\check{\mathcal{O}}' := \text{the one obtained from } \check{\mathcal{O}} \text{ by removing the first row.}$$

Note that $\check{\mathcal{O}}'$ has $\star'$-good parity. In section 2.2, we will define the descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

Let $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. Put $\tau' := \nabla(\tau)$. Let $(W_{\tau,\tau'}, \langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\tau'})$ be a real symplectic space of dimension $\dim\tau \cdot \dim\tau'$. As usual, there are continuous homomorphisms $G_\tau \to \mathrm{Sp}(W_{\tau,\tau'})$ and $G_{\tau'} \to \mathrm{Sp}(W_{\tau,\tau'})$ whose images form a reductive dual pair in $\mathrm{Sp}(W_{\tau,\tau'})$. We form the semidirect prduct

$$J_{\tau,\tau'} := (G_\tau \times G_{\tau'}) \ltimes \mathrm{H}(W_{\tau,\tau'}),$$

where

$$\mathrm{H}(W_{\tau,\tau'}) := W_{\tau,\tau'} \times \mathbb{R}$$

is the Heisenberg group with group multiplication

$$(w,t)(w',t') := (w + w', t + t' + \langle w, w'\rangle_{\tau,\tau'}), \quad w, w' \in W_{\tau,\tau'},\ t, t' \in \mathbb{R}.$$

Let $\omega_{\tau,\tau'}$ be a suitably normalized smooth oscillator representation of $J_{\tau,\tau'}$ such that every $t \in \mathbb{R} \subset J_{\tau,\tau'}$ acts on it through the scalar multiplication by $e^{2\pi\sqrt{-1}\,t}$ (the letter π often denotes a representation, but here it stands for the circumference ratio). See Section 12.2.1 for details.

For every Casselman-Wallach representation π' of $G_{\tau'}$, write

$$\check{\Theta}_{\tau'}^{\tau}(\pi') := (\omega_{\tau,\tau'} \widehat{\otimes} \pi')_{G_{\tau'}} \qquad \text{(the Hausdorff coinvariant space)},$$

where $\widehat{\otimes}$ indicates the complete projective tensor product. This is a Casselman-Wallach representation of G_τ. Now we define the representation π_τ of G_τ by induction on $\mathbf{c}_1(\check{\mathcal{O}})$:

$$\pi_\tau := \begin{cases} \check{\Theta}_{\tau'}^{\tau}(\pi_{\tau'}) \otimes (1_{p_\tau,q_\tau}^{+,-})^{\varepsilon_\tau}, & \text{if } \star_\tau = B \text{ or } D; \\ \check{\Theta}_{\tau'}^{\tau}(\pi_{\tau'} \otimes \det^{\varepsilon_\tau}), & \text{if } \star_\tau = C \text{ or } \widetilde{C}; \\ \check{\Theta}_{\tau'}^{\tau}(\pi_{\tau'}), & \text{if } \star_\tau = C^* \text{ or } D^*. \end{cases}$$

Here $1_{p_\tau,q_\tau}^{+,-}$ denotes the character of $\mathrm{O}(p_\tau, q_\tau)$ whose restriction to $\mathrm{O}(p_\tau) \times \mathrm{O}(q_\tau)$ equals $1 \otimes \det$ (1 stands for the trivial character).

Put

$$\mathrm{Unip}_\star(\check{\mathcal{O}}) := \begin{cases} \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|+1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = B; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{|\check{\mathcal{O}}|-1}(\mathbb{R})), & \text{if } \star = C; \\ \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = D; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{Sp}}_{|\check{\mathcal{O}}|}(\mathbb{R})), & \text{if } \star = \widetilde{C}; \\ \bigsqcup_{p,q\in\mathbb{N}, 2p+2q=|\check{\mathcal{O}}|-1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q)), & \text{if } \star = C^*; \\ \mathrm{Unip}_{\check{\mathcal{O}}}\, \mathrm{O}^*(|\check{\mathcal{O}}|), & \text{if } \star = D^*. \end{cases}$$

We are now ready to formulate our first main theorem.

Theorem 1.5. *Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.*
(a) For every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, the representation π_τ of G_τ is irreducible and attached to $\check{\mathcal{O}}$.
(b) If $\star \in \{B, D\}$ and $|\check{\mathcal{O}}| > 0$, then the map

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \rightarrow & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ (\tau, \epsilon) & \mapsto & \pi_\tau \otimes \det^\epsilon \end{array}$$

is bijective.
(c) In all other cases, the map

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \rightarrow & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ \tau & \mapsto & \pi_\tau \end{array}$$

is bijective.

By the above theorem (and the reduction in Appendix A), we have explicitly constructed all special unipotent representations of the classical groups in (1.1). By this construction and the method of matrix coefficient integrals, we are able to prove the unitarity of these representations.

Theorem 1.6. *All special unipotent representations of the classical groups in* (1.1) *are unitarizable.*

1.6. **Associated cycles.** Let $G, G_\mathbb{C}, \mathfrak{g}_\mathbb{R}, \mathfrak{g}, \check{\mathfrak{g}}$ and $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ be as in Section 1.2. Fix a maximal compact subgroup K of G whose Lie algebra is denoted by $\mathfrak{k}_\mathbb{R}$. Then we have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the complexification of $\mathfrak{k}_\mathbb{R}$, and $\mathfrak{g}$ is equipped with the trace form. By taking the dual spaces, we have a decomposition

$$\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*,$$

Write $K_\mathbb{C}$ for the complexification of the complex group K. It is a complex algebraic group with an obvious algebraic action on $\mathfrak{p}^*$.

As before, suppose that G has type $\star$ and $\check{\mathcal{O}}$ has $\star$-good parity. Write $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}^*)$ for the Barbarsch-Vogan dual of $\check{\mathcal{O}}$ so that its Zariski closure in $\mathfrak{g}^*$ equals the associated variety of the ideal $I_{\check{\mathcal{O}}} \subset \mathrm{U}(\mathfrak{g})$. The algebraic variety $\mathcal{O} \cap \mathfrak{p}^*$ is a finite union of $K_\mathbb{C}$-orbits. Given such an orbit $\mathscr{O}$, write $\mathcal{K}_{\mathscr{O}}(K_\mathbb{C})$ for the Grothendieck group of the category of $K_\mathbb{C}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{\mathcal{O}}(K_\mathbb{C}) := \bigoplus_{\mathscr{O} \text{ is a } K_\mathbb{C}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathcal{K}_{\mathscr{O}}(K_\mathbb{C}).$$

We say that a Casselman-Wallach representation of G is $\mathcal{O}$-bounded if the associated variety of its annihilator ideal is contained in the Zariski closure of $\mathcal{O}$. Note that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are $\mathcal{O}$-bounded. Write $\mathcal{K}(G)_{\mathcal{O}-\text{bounded}}$ for the Grothendieck group of the category of all such representations. From [77, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{AC}_{\mathcal{O}}\colon \mathcal{K}(G)_{\mathcal{O}-\text{bounded}} \longrightarrow \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}}).$$

We call $\mathrm{AC}_{\mathcal{O}}(\pi)$ the associated cycle of π, where π is an $\mathcal{O}$-bounded Casselman-Wallach representation of G. This is a very important invariant attached to π.

Following Vogan [77, Section 8], we make the following definition.

Definition 1.7. *Let $\mathscr{O}$ be a $K_{\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}^*$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $K_{\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_X$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_X$;*
- *if $\star = \widetilde{C}$, then ε_G acts on $\mathcal{E}$ by the scalar multiplication by -1.*

Here $X \in \mathscr{O}$, $\mathcal{E}_X$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_X$ denotes the Lie algebra of the stabilizer of X in $K_{\mathbb{C}}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_X$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$.

Note that in the situation of the classical groups we consider here, all admissible orbit data are line bundles. Denote by $\mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$. Put

$$\mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}}) \subset \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}}).$$

Recall that a nilpotent orbit in $\check{\mathfrak{g}}$ is said to be distinguished if it is has no intersection with every proper Levi subalgebra of $\check{\mathfrak{g}}$. Combinatorially, this is equivalent to saying that no pair of rows of the Young diagram has equal nonzero length. Note that all distinguished nilpotent orbits in $\check{\mathfrak{g}}$ has $\star$-good parity.

Definition 1.8. *(a) The orbit $\check{\mathcal{O}}$ (which has $\star$-good parity) is said to be quasi-distinguished if there is no $\star$-pair that is balanced in $\check{\mathcal{O}}$.*
(b) If $\star \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be be pseudo-distinguished if there is no positive even integer i such that $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = \mathbf{r}_{i+2}(\check{\mathcal{O}}) = \mathbf{r}_{i+3}(\check{\mathcal{O}}) > 0$.
(c) If $\star \in \{C, \widetilde{C}, C^*\}$ so that its Howe dual $\star' \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be be pseudo-distinguished if either it is the empty Young diagram or it is nonempty and its descent $\check{\mathcal{O}}'$ (which is a Young diagram that has $\star'$-good parity) is pseudo-distinguished.

We will calculate the associated character of π_τ for every bipartition τ. The calculation is a key ingredient in the proof of Theorem 1.5. Especially, we have the following theorem concerning the associated characters of the special unipotent representations.

Theorem 1.9. *Let G be a group in* (1.1) *that has type $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For every $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the associated cycle $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$ is a nonzero sum of pairwise distinct elements of $\mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}})$.
(b) If $\check{\mathcal{O}}$ is pseudo-distinguished, then the map

$$\mathrm{AC}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$$

is injective.
(c) If $\check{\mathcal{O}}$ is quasi-distinguished, then the map $\mathrm{AC}_{\check{\mathcal{O}}}$ induces a bijection

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}}).$$

Remark. Suppose that $\star \in \{C^*, D^*\}$ so that G is quaternionic. Then there is precisely one admissible orbit datum over $\mathscr{O}$ for each $K_{\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}^*$. Thus

$$\mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}}) = K_{\mathbb{C}} \backslash (\mathcal{O} \cap \mathfrak{p}^*).$$

If $\check{\mathcal{O}}$ is not quasi-distinghuished, then $\mathcal{O} \cap \mathfrak{p}^*$ is empty (see [20, Theorems 9.3.4 and 9.3.5]), and hence $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is also empty.

2. The descents of painted bipartitions

Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and let $\check{\mathcal{O}}$ be a Young diagrams that has $\star$-good parity. A painted bipartition $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ is said to be pre-special if

$$\begin{cases} (1,2) \notin \wp_\tau, & \text{when } \star \in \{C, \widetilde{C}, C^*\}; \\ (2,3) \notin \wp_\tau, & \text{when } \star \in \{B, D, D^*\}. \end{cases}$$

Write

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) = \mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}}) \sqcup \mathrm{PBP}^{\mathrm{ns}}_\star(\check{\mathcal{O}}),$$

where $\mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}})$ is the set of pre-special elements in $\mathrm{PBP}_\star(\check{\mathcal{O}})$, and $\mathrm{PBP}^{\mathrm{ns}}_\star(\check{\mathcal{O}})$ is the complementary set of $\mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}})$. We remark that $\mathrm{PBP}^{\mathrm{ns}}_\star(\check{\mathcal{O}})$ is empty in the following cases:

- $\star \in \{ C^*, D^* \}$;
- $\star \in \{ C, \widetilde{C} \}$ and $(1,2) \notin \mathrm{DP}(\check{\mathcal{O}})$;
- $\star \in \{ B, D \}$ and $(2,3) \notin \mathrm{DP}(\check{\mathcal{O}})$.

2.1. **Switching painted bipartitions.** In this subsection, we assume that $\star \in \{C, \widetilde{C}\}$ and $(1,2) \in \mathrm{DP}_\star(\check{\mathcal{O}})$. We will define a bijective map

$$\mathrm{PBP}^{\mathrm{ns}}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}}), \quad \tau \mapsto$$

in what follows.

Let $\wp \in \mathrm{DP}_\star(\check{\mathcal{O}})$. Put

$$\mathrm{PBP}_\star(\check{\mathcal{O}}, \wp) = \{\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \mid \wp_\tau = \wp\}.$$

We say that a painted bipartition τ is pre-special if $\wp_\tau$ has no intersection with the set $\{(1,2), (2,3)\}$. Write

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) = \mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}}) \sqcup \mathrm{PBP}^{\mathrm{ns}}_\star(\check{\mathcal{O}}),$$

where $\mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}})$ is the set of pre-special elements in $\mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}})$, and $\mathrm{PBP}^{\mathrm{ns}}_\star(\check{\mathcal{O}})$ is the complementary set.

For every painted bipartition $\tau = (\imath, \mathcal{P}_{\mathrm{l}}) \times (\jmath, \mathcal{P}_{\mathrm{r}}) \times \alpha$, we call the pained Young diagram $(\imath, \mathcal{P}_{\mathrm{l}})$ the left painting of τ, and call $(\jmath, \mathcal{P}_{\mathrm{r}})$ the right painting of τ.

2.2. **Descending painted bipartitions.** Assume that $\left|\check{\mathcal{O}}\right| > 0$ and let $\check{\mathcal{O}}'$ be its descent as defined in the Introduction. Let $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. In what follows we will define case by case a painted bipartition $\tau' := \nabla(\tau) \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$, to be called the descent of τ.

Lemma 2.1. *Suppose that τ*

$\star = C$. In this case,

is nonempty Put

$$\mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}}) := \{\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \mid \ \tau \text{ is pre-special}\}.$$

and the map (2.1) has been defined. In the rest of this subsection we assume that $\mathrm{PBP}_\star(\check{\mathcal{O}}) \neq \mathrm{PBP}^{\mathrm{ps}}_\star(\check{\mathcal{O}})$. Then either

$$\star \in \{B, D\} \quad \text{and} \quad (2,3) \in \mathrm{DP}_\star(\check{\mathcal{O}})$$

or
$$\star \in \{C, \widetilde{C}\} \quad \text{and} \quad (1,2) \in \mathrm{DP}_\star(\check{\mathcal{O}}).$$
We define the map (2.1) case by case.

The goal of this subsection is to define a map
$$\mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_\star^{\mathrm{ps}}(\check{\mathcal{O}}), \quad \tau \mapsto \tau^{\mathrm{ps}}. \tag{2.1}$$
Firstly, for every $\tau \in \mathrm{PBP}_\star^{\mathrm{ps}}(\check{\mathcal{O}})$, we define $\tau^{\mathrm{ps}} := \tau$. In particular, the map (2.1) has been defined when $\mathrm{PBP}_\star(\check{\mathcal{O}}) = \mathrm{PBP}_\star^{\mathrm{ps}}(\check{\mathcal{O}})$.
$\star = B$:
$\star = C$:
$\star = C$: keep the left painted Young diagram unchanged.
$\star = C$: removing the first column of the left painted Young diagram.

Lemma 2.2. *Suppose that $\star \in \{C^*, D^*\}$. Then $\wp_\tau = \emptyset$ for all $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$.*

By Lemma 2.2, $\mathrm{PBP}_\star(\check{\mathcal{O}}) = \mathrm{PBP}_\star^{\mathrm{ps}}(\check{\mathcal{O}})$ when $\star \in \{C^*, D^*\}$.

3. The main results

3.1. **Infinitesimal characters.** Let $\epsilon = \pm 1$. Let V be an ϵ-symmetric complex bilinear space, namely a finite dimensional complex vector space equipped with a non-degenerate ϵ-symmetric bilinear form $\langle\, ,\, \rangle_V : V \times V \to \mathbb{C}$. Write G_V for the isometry group of V, and write $\mathfrak{g}_V$ for the Lie algebra of G_V.

We identify $\mathfrak{g}_V^*$ with $\mathfrak{g}_V$ via the trace form. Let $\mathrm{Nil}(\mathfrak{g}_V^*)$ $(= \mathrm{Nil}(\mathfrak{g}_V))$ denote the set of nilpotent G_V-orbits in $\mathfrak{g}_V^*$ $(= \mathfrak{g}_V)$. The following result is due to Dixmier.

We now proceed to our setup in order to state the main technical result of this article.

Let $\mathbf{G}$ be either a complex orthogonal group or a complex symplectic group. Put $\epsilon = 1$ or -1, when $\mathbf{G}$ is respectively orthogonal or symplectic. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$ and let $\mathcal{O}$ be a nilpotent $\mathbf{G}$-orbit in $\mathfrak{g}^*$. Unless otherwise specified, a superscript " $*$ " indicates the dual space. By identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via the trace form, $\mathcal{O}$ is also viewed as a nilpotent $\mathbf{G}$-orbit in $\mathfrak{g}$. Denote by $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ the set of nilpotent $\mathbf{G}$-orbits in $\mathfrak{g}$.

In the usual way, a nilpotent $\mathbf{G}$-orbit $\mathcal{O}$ in $\mathfrak{g}$ is parameterized by its Young diagram. In this article, we will label a Young diagram $\mathbf{d}$ by its column partition $[c_0, c_1, \cdots, c_k]$, where
$$c_0 \geqslant c_1 \geqslant \cdots \geqslant c_k > 0$$
represent lengths of the columns of $\mathbf{d}$. If $\mathbf{G}$ is a trivial group, then the only nilpotent $\mathbf{G}$-orbit is labeled by the empty sequence $\emptyset$, and we set $k = -1$ in this case.

Fix a parity $\mathbb{p} \in \mathbb{Z}/2\mathbb{Z}$ throughout the paper. Let $\mathcal{P}_{\mathbf{G}}$ be the set of partitions/Young diagrams parameterizing the set $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$. Set $\epsilon_l = \epsilon(-1)^l$ for $l \geqslant 0$ and consider the following subset of $\mathcal{P}_{\mathbf{G}}$:
$$\mathcal{P}_{\mathbf{G}}^{\mathbb{p}} := \left\{ \mathbf{d} = [c_0, c_1, \cdots, c_k] \in \mathcal{P}_{\mathbf{G}} \;\middle|\; \begin{array}{l} \bullet \text{ all } c_i\text{'s have parity } \mathbb{p}; \\ \bullet\ c_l \geqslant c_{l+1} + 2, \text{ if } 0 \leqslant l \leqslant k-1 \text{ and} \\ \quad \epsilon_l = 1. \end{array} \right\}. \tag{3.1}$$

Let $\mathrm{Nil}_{\mathbf{G}}^{\mathbb{p}}(\mathfrak{g})$ be the subset of $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ corresponding to partitions in $\mathcal{P}_{\mathbf{G}}^{\mathbb{p}}$. Write $\mathbf{V}$ for the standard module of $\mathbf{G}$. Note that if the group $\mathbf{G}$ is orthogonal and nontrivial, then there is no partition in $\mathcal{P}_{\mathbf{G}}$ satisfying the first condition in (3.1) unless $\mathbb{p}$ equals the parity of $\dim \mathbf{V}$.

Remarks. 1. If $\mathbb{p}$ is the parity of $\dim \mathbf{V}$, then $\mathrm{Nil}_{\mathbf{G}}^{\mathbb{p}}(\mathfrak{g})$ is precisely the set of stably trivial special nilpotent orbits in $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$. If $\mathbf{G}$ is a symplectic group, then all nilpotent orbits in $\mathrm{Nil}_{\mathbf{G}}^1(\mathfrak{g})$ are metaplectic-special, as defined in [38]. See also [53].

2. All rigid special nilpotent orbits in $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ are stably trivial. The reader is referred to [12, Section 2] for facts on stably trivial special nilpotent orbits.
3. Every orbit $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$ is irreducible as an algebraic variety [72, IV 2.27].

Let m be the rank of $\mathbf{G}$. We identify a Cartan subalgebra of $\mathfrak{g}$ with $\mathbb{C}^m$, using the standard coordinates. Let a superscript group indicate the space of invariant vectors under the group action. Then Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{\mathbf{G}} = (\mathrm{S}(\mathbb{C}^m))^{W_m},$$

where "U" indicates the universal enveloping algebra, "S" indicates the symmetric algebra, and W_m denotes the subgroup of $\mathrm{GL}_m(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. We may thus parameterize algebraic characters of $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ by $(\mathbb{C}^m)^*/W_m \cong \mathbb{C}^m/W_m$. Unless $\mathbf{G}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$.

For any positive integer r, denote

$$\rho_r^{\epsilon} = \begin{cases} (\frac{r}{2}-1, \frac{r}{2}-2, \cdots, \frac{r}{2}-\lfloor\frac{r}{2}\rfloor) & \in (\frac{1}{2}\mathbb{Z})^{\lfloor\frac{r}{2}\rfloor}, \quad \text{if } \epsilon = 1; \\ (\frac{r}{2}, \frac{r}{2}-1, \cdots, \frac{r}{2}-\lfloor\frac{r-1}{2}\rfloor) & \in (\frac{1}{2}\mathbb{Z})^{\lfloor\frac{r+1}{2}\rfloor}, \quad \text{if } \epsilon = -1. \end{cases}$$

Definition 3.1. *For a nilpotent orbit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ given by the partition $\mathbf{d} = [c_0, c_1, \cdots, c_k]$, define the following element $\lambda_{\mathcal{O}}$ of $\mathbb{C}^m$:*

$$\lambda_{\mathcal{O}} := (\rho_{c_0}^{\epsilon_0}, \rho_{c_1}^{\epsilon_1}, \cdots, \rho_{c_k}^{\epsilon_k}) \in (\tfrac{1}{2}\mathbb{Z})^m.$$

By abuse of notation, $\lambda_{\mathcal{O}}$ also represents its image in $\mathbb{C}^m/W_m$, as well as the corresponding algebraic character of $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$.

Remark. If $\mathbb{p}$ is the parity of $\dim \mathbf{V}$, then for every $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$, $\lambda_{\mathcal{O}}$ equals the algebraic character of $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ determined by $\frac{1}{2}{}^L h$ as in [14, Equation (1.5)] (*cf.* [74, Proposition 1.12] and [71, Section 7]).

We will work in the category of Casselman-Wallach representations [18, 81]. Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation if its Harish-Chandra module has finite length.

Let G be a real form of $\mathbf{G}$, namely, G is the fixed point group of an anti-holomorphic involutive automorphism of $\mathbf{G}$. Thus G is a real orthogonal group, a real symplectic group, a quaternionic orthogonal group, or a quaternionic symplectic group. When G is a real symplectic group, write

$$1 \longrightarrow \{1, \varepsilon_G\} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

for the metaplectic cover of G. It does not split unless G is trivial. We say that a Casselman-Wallach representation π of $\widetilde{G}$ is $\mathbb{p}$-genuine if ε_G acts on π through the scalar multiplication by $(-1)^{\mathbb{p}}$. In all other cases, we simply set $\widetilde{G} := G$, and all Casselman-Wallach representations of $\widetilde{G}$ are defined to be $\mathbb{p}$-genuine. When no confusion is possible, the notion of "$\mathbb{p}$-genuine" will be used in similar settings without further explanation.

For a Casselman-Wallach representation π of $\widetilde{G}$, denote $\mathrm{Ann}(\pi)$ its annihilator ideal in $\mathrm{U}(\mathfrak{g})$. Let $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$. Following Barbasch and Vogan [2, 14], we make the following definition.

Definition 3.2. *An irreducible Casselman-Wallach representation π of $\widetilde{G}$ is said to be $\mathcal{O}$-unipotent if*

- *the associated variety of $\mathrm{Ann}(\pi)$ is contained in the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$,*
- *the action of $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ on π is given by the character $\lambda_{\mathcal{O}}$, and*
- *π is $\mathbb{p}$-genuine.*

Remarks. 1. The first two conditions of Definition 3.2 may also be given in terms of a certain primitive ideal $I_{\mathcal{O}}$ of $\mathrm{U}(\mathfrak{g})$. We adopt a more direct approach.
2. We comment on the third condition of Definition 3.2 by the example of $G = \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. Take $\mathcal{O}$ the principal nilpotent orbit, then up to isomorphisms, there are six irreducible Casselman-Wallach representations of $\widetilde{G}$ satisfying the first two conditions. Four of these representations also satisfy the third condition and are thus $\mathcal{O}$-unipotent in our definition. They are the irreducible components of the two oscillator representations. The other two descend to the two irreducible principal series representations of G with infinitesimal character $\lambda_{\mathcal{O}}$. One of them is unitarizable and the other is not. They are not considered as unipotent representations.

Let $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ denote the set of isomorphism classes of $\mathcal{O}$-unipotent irreducible Casselman-Wallach representations of $\widetilde{G}$. Write $\mathcal{K}_{\mathcal{O}}(\widetilde{G})$ for the Grothedieck group of the category of Casselman-Wallach representations π of $\widetilde{G}$ such that the associated variety of $\mathrm{Ann}(\pi)$ is contained in $\overline{\mathcal{O}}$. We view $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ as a subset of $\mathcal{K}_{\mathcal{O}}(\widetilde{G})$.

Fix a maximal compact subgroup K of G. Write $\mathbf{K}$ for the Zariski closure of K in $\mathbf{G}$, which is a universal complexification of K. Write $\widetilde{K} \to K$ for the one or two fold covering map induced by the covering map $\widetilde{G} \to G$. Denote by $\widetilde{\mathbf{K}}$ the universal complexification of $\widetilde{K}$.

Write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of $\mathbf{K}$, and $\mathfrak{p}$ is its orthogonal complement in $\mathfrak{g}$ with respect to the trace form. Under the adjoint action of $\mathbf{K}$, the complex variety $\mathcal{O} \cap \mathfrak{p}$ is a union of finitely many orbits, each of dimension $\frac{\dim_{\mathbb{C}} \mathcal{O}}{2}$. For any $\mathbf{K}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}$, let $\mathcal{K}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ denote the Grothedieck group of the category of $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}}) := \bigoplus_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}_{\mathscr{O}}(\widetilde{\mathbf{K}}).$$

Following Vogan [77, Section 8], we make the following definition.

Definition 3.3. *Let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_X$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_X$;*
- *$\mathcal{E}$ is $\mathbb{p}$-genuine.*

Here $X \in \mathscr{O}$, $\mathcal{E}_X$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_X$ denotes the Lie algebra of the stabilizer of X in $\mathbf{K}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_X$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$.

Note that in the situation of classical groups we consider here, all admissible orbit data are line bundles. Denote by $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}})$. Put

$$\mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) := \bigsqcup_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) \subset \mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

According to Vogan [77, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{Ch}_{\mathcal{O}} : \mathcal{K}_{\mathcal{O}}(\widetilde{G}) \to \mathcal{K}_{\mathcal{O}}(\widetilde{\mathbf{K}}). \tag{3.2}$$

The main result of this article is the following theorem.

Theorem 3.4. *Let $\mathcal{O} \in \mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{g})$. The homomorphism* (3.2) *restricts to a bijective map*

$$\mathrm{Ch}_{\mathcal{O}} : \Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G}) \longrightarrow \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}}). \tag{3.3}$$

Moreover, every representation in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ is unitarizable.

Remarks. 1. When G is a quaternionic orthogonal group or an quaternionic symplectic group, the first assertion of Theorem 3.4 is proved in Theorems 6 and 10 of [52]. See also [74, Theorem 3.1].
2. The bijectivity in Equation (3.3) implies in particular that the associated variety of a representation in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ is the closure of a single nilpotent $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$.

Key to the proof of Theorem 3.4 is the existence of unitarizable unipotent representations with the prescribed nilpotent admissible orbit data. More precisely we shall first prove the following result (see Theorem 14.1):

- for every $\mathcal{E} \in \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$, there exists a unitarizable $\mathcal{O}$-unipotent representation π of $\widetilde{G}$ such that $\mathrm{Ch}_{\mathcal{O}}(\pi) = \mathcal{E}$.

Our construction of the $\mathcal{O}$-unipotent representation π for a given $\mathcal{E} \in \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ will be guided by a geometric process which we call descent. We describe it briefly. Write $\mathbf{G} = \mathbf{G}_{\mathbf{V}}$, where $\mathbf{V}$ is the standard module of $\mathbf{G}$ as before, which is an ϵ-symmetric bilinear space. From the pair $(\mathbf{V}, \mathcal{O})$, the decent process yields another pair $(\mathbf{V}', \mathcal{O}')$, where $\mathbf{V}'$ is an ϵ'-symmetric bilinear space, and $\mathcal{O}'$ is a nilpotent $\mathbf{G}'$-orbit in $\mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}'}(\mathfrak{g}')$. Here $\epsilon\epsilon' = -1$, and $\mathbf{G}' = \mathbf{G}_{\mathbf{V}'}$. The real form G of $\mathbf{G}$ is specified by an additional structure on $\mathbf{V}$, which we call $(\epsilon, \dot\epsilon)$-space, where $\dot\epsilon = \pm 1$. Assume that $\mathscr{O}$ is a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$. In a similar way, the decent process yields from the pair $(\mathbf{V}, \mathscr{O})$ another pair $(\mathbf{V}', \mathscr{O}')$, where $\mathbf{V}'$ is an $(\epsilon', \dot\epsilon')$-space ($\dot\epsilon\dot\epsilon' = -1$), giving rise to a real form G' of $\mathbf{G}'$.

The geometric process inverse to the descent will be called lifting. Apart from lifting a nilpotent orbit, one can in fact lift an equivariant algebraic vector bundle on the nilpotent orbit which has the additional property of sending an admissible orbit datum over $\mathscr{O}'$ to an admissible orbit datum over $\mathscr{O}$. Moreover, by twisting with characters of the component group of $\mathbf{K}$ if necessary, this lifting map of admissible orbit data is surjective (see Lemma 7.21). A key observation here is that the geometric processes outlined are completely governed by a double fiberation of moment maps mentioned earlier. (Some of the basic facts about this double fiberation first appeared in a paper of Kraft and Procesi on the geometry of conjugate classes in classical groups [41].) This summarises the geometric preparations regarding our nilpotent orbits and concludes our discussion on the contents of Section 7.

To construct the $\mathcal{O}$-unipotent representation π, we will apply theta lifting for the dual pair (G', G), starting from an $\mathcal{O}'$-unipotent representation of $\widetilde{G'}$ with an associated character $\mathcal{E}' \in \mathcal{K}^{\mathrm{aod}}_{\mathcal{O}'}(\widetilde{\mathbf{K}}')$. Our model of theta lifting is provided by matrix coefficient integrals against the oscillator representation in the so-called convergent range. The critical task before us is to show that the lifted representation will not vanish and furthermore the associated character $\mathcal{E}$ of the lifted representation is simply the lift of $\mathcal{E}'$. As a first step, we show an upper bound of $\mathcal{E}$ by the lift of $\mathcal{E}'$ by some largely geometric considerations. To achieve the equality, we adapt the idea of He mentioned earlier, namely by applying another theta lifting to an appropriate group (of the same type as G') and then relating the double theta lift, via a variant of the well-known doubling method (*cf.* [56, 67]), to a certain degenerate principal series representation of an appropriately large classical group (whose structure is explicitly described in [44, 45, 83]). The end result is that the lifted representation must have the anticipated associated character, by comparing associated cycles within the said degenerate principal series representation. To summarize the contents, we take care of the analytic preparations, namely growth of matrix coefficients, their integrals against oscillator representations as well as degenerate principal series in Section 12, and we combine geometry and analysis to determine the associated characters of theta lifts in the convergent range (in a general setting) in Section 13.

In Section 14, we complete the construction of unipotent representations by an iterative procedure. For the exhaustion part of the main theorem, we use in a crucial way the result of Barbasch [8, 11] counting unipotent representations for a real symplectic group and a split real odd orthogonal group. Putting together our construction of unipotent representations with the prescribed nilpotent admissible orbit data, the counting result of Barbasch for the specific cases just mentioned, as well as a general technique of embedding unitary representations (via stable range theta lifting), our main theorem follows for all the cases considered in this article.

Finally the unitarizability of all $\mathcal{O}$-unipotent representations is a consequence of the exhaustion and the construction by our concrete models. Specifically we achieve this by showing the nonnegativity of a certain matrix coefficient integral, using a general technique of Harris, Li and Sun [28]. Note that the unitarizability of the constructed representations also follows from the earlier work of He on theta lifting in the so-called strongly semistable range ([30, Chapter 5] or [31]).

Notation. We adopt the following convention throughout the article. Boldface letters denote objects defined over $\mathbb{C}$ such as a complex vector space or a complex Lie group. If an object with " $'$" (e.g. $\mathbf{V}'$) is defined, all related notions will be implicitly defined by adding a prime (e.g. $\mathbf{G}'$). Epsilons, e.g. $\epsilon, \dot{\epsilon}, \epsilon'$, always mean a sign taking values in $\{\pm 1\}$. A superscript " ^{t}" indicates the transpose and "tr" the trace, of a matrix. A superscript " $^{\vee}$" indicates the contragredient representation in the appropriate contexts. Unless otherwise specified, "Ind" and "ind" indicate the normalized smooth and Schwartz induction[1], respectively. "$\langle\,,\,\rangle$" denotes natural parings or Hermitian inner products in the appropriate contexts. All measures appearing in integrals on real reductive groups are the Haar measures. The largest integer less than or equal to a real number a is denoted by $\lfloor a \rfloor$. Let "id" indicate the identity map in various contexts. Let $\mathbf{i}$ denote a fixed $\sqrt{-1}$.

4. Statement of the main result(new)

4.1. **Nilpotent Orbits.** Let $\mathbf{G} = \mathrm{Sp}, \mathrm{O}$, and $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ is a complex nilpotent orbit. "Type M" means that $\widetilde{G}$ is a metaplectic group. its dual group $\check{\mathbf{G}}$ is a symplectic group with the same rank, and its duality means the metaplectic dual. Let $\mathrm{DRC}(\mathcal{O})$ be the set of dot-r-c diagrams parameterizing unipotent representations of the real forms of $\mathbf{G}$.

We say an orbit $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathbf{G}})$ is purely even (good parity in Moeglin-Renard's notation) if

all row lengths of $\check{\mathcal{O}}$ are even when $\check{\mathbf{G}}$ is a symplectic group.
all row lengths of $\check{\mathcal{O}}$ are odd when $\check{\mathbf{G}}$ is an orthogonal group.

Definition 4.1. *Now we describe the nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ who is the Barbasch-Vogan dual of a purely even orbit $\check{\mathcal{O}}$. If G is*

Type C *Then $\mathcal{O}$ is special and it has columns*

$$(C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is even for } i = 0, \cdots, k-1.$$

Note that, by the definition of special orbit, we have $C_{2i} \equiv C_{2i-1} \pmod 2$, $C_{2i} = C_{2i-1}$ if C_{2i} is odd for $i = 1, \cdots k$.

[1]For Schwartz induction, see [19].

Type M *Then $\mathcal{O}$ is metaplecitic special and it has columns*

$$(C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is odd for } i = 0, \cdots, k-1.$$

Note that, by the definition of metaplectic special, we have $C_{2i} \equiv C_{2i-1} \pmod 2$, $C_{2i} = C_{2i-1}$ if C_{2i} is even for $i = 1, \cdots k$.

Type D *and $\mathcal{O}$ is special and it has columns*

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is even for } i = 0, \cdots, k.$$

Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is purely even of type C and C_{2k+1} is always even.

Type B *and $\mathcal{O}$ is special and it has columns*

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is odd for } i = 0, \cdots, k.$$

Note that, the condition is equivalent to $\nabla(\mathcal{O})$ is purely even of type M and C_{2k+1} is always odd.

Let $\mathrm{Nil}^{\mathrm{dpe}}(\star)$ the set of duals of purely even orbits for type $\star =$ B,C,D,M respectively.

4.2. **Descent of orbits in the purely even case.** For $\{\star, \star'\} = \{C, D\}, \{B, M\}$, we define the notion of descent of orbits. The descent map is defined in the following way (under the above notation):

$$\begin{array}{rccc} \overline{\nabla}: & \mathrm{Nil}^{\mathrm{dpe}}(\star) & \longrightarrow & \mathrm{Nil}^{\mathrm{dpe}}(\star') \\ & \mathcal{O} & \longmapsto & \mathcal{O}' \end{array}$$

such that

- when $\star$ is B or D, $\mathcal{O}' := \nabla(\mathcal{O})$;
- when $\star$ is C or M,

$$\mathcal{O}' := \begin{cases} \nabla(\mathcal{O}) & \text{if } C_{2k} \text{ is even and } \star = B \\ & \text{or } C_{2k} \text{ is odd and } \star = M; \\ (C_{2k-1}+1, C_{2k-2}, C_{2k-3}, \cdots, C_0) & \text{otherwise.} \end{cases}$$

In the second case, $C_{2k} = C_{2k-1}$ and $\mathcal{O} \mapsto \overline{\nabla}(\mathcal{O})$ is a good generalized descent.

The following set of orbit are essential to us:

Definition 4.2. *A nilpotent orbit $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$ is called* noticed *if G is*

Type D *and $\mathcal{O}$ is special and it has columns*

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k.$$

Type B *and $\mathcal{O}$ is special and it has columns*

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \cdots, k.$$

Type C *and $\overline{\nabla}(\mathcal{O})$ is noticed of type D.*

Type M *and $\overline{\nabla}(\mathcal{O})$ is notice of type B.*

Furthermore, We say a noticed orbit $\mathcal{O}$ of type D or B is +noticed *if $C_{2k+1} - C_{2k} \geqslant 2$.*

The following inductive definition of noticed orbit is easy to verify.

Lemma 4.3. *An orbit $\mathcal{O}$ of type D (resp. type B) is noticed if and only if*

i) $\overline{\nabla}^2(\mathcal{O})$ is noticed, when C_{2k} is even (resp. odd), or
ii) $\overline{\nabla}^2(\mathcal{O})$ is +noticed, when C_{2k} is odd (resp. even).

4.3. **Relavent Weyl group representations and dot-r-c diagrams.** Retain the notation in Definition 4.2. Now we describe all relevent representations of the Weyl group appear in the counting algorithm.

Type D We define c_i as the following

$$c_{2i} := \lfloor \frac{C_{2i}}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1}+1}{2} \rfloor \quad \forall i = 0, 1, \cdots, k,$$
$$c_{2k+1} := \frac{C_{2k+1}}{2}$$

The (special) Weyl group representation associated to the special orbit $\mathcal{O}$ has columns

$$\tau = \tau_L \times \tau_R = (c_{2k+1}, c_{2k-1}, \ldots, c_1) \times (c_{2k}, \ldots, c_0).$$

The other representations is obtained from the special one by interchanging following pairs when $c_{2i-1} \neq c_{2i} + 1$ and $i = 1, \cdots, k$ (these are pairs coming from even length columns)

$$(c_{2i-1}, c_{2i}) \longleftrightarrow (c_{2i} + 1, c_{2i-1} - 1).$$

The dot-r-c diagrams of type D: For each representation $\tau = \tau_L \times \tau_R$, dots are filled in both sides forming the same shape of Young diagram, a column of "s" is added next to dots on the left side diagram, a column of "r" is added next to dots on the left side diagram, add a row of "c"s and finally add a row of "d"s on the left side of the diagram; Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step.

In summary, $s/r/c/d$ are all added on the left diagram. Fixing the representation τ, the left diagram τ_L determin τ_R completely.

Type C Set $C_{2k+1} = 0$. The Weyl group representations are obtained by the same formula of Type D.

The dot-r-c diagrams of type C are obtained in almost the same way as that of type D, except that s are added on the right side of the diagram.

Type B We define c_i as the following

$$c_{2i} := \lfloor \frac{C_{2i}+1}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1}}{2} \rfloor \quad \forall i = 1, 2, \cdots, k,$$
$$c_{2k+1} := \frac{C_{2k+1}-1}{2}$$

Note that C_{2k+1} is always odd and $C_0 = 0$. The (special) Weyl group representation associated to the special orbit $\mathcal{O}$ has columns

$$\tau = \tau_L \times \tau_R = (c_{2k}, \ldots, c_2) \times (c_{2k+1}, c_{2k-1}, \ldots, c_1).$$

The other representations is obtained from the special one by interchanging following pairs when $c_{2i-1} \neq c_{2i} + 1$ and $i = 1, \cdots, k$ (these are pairs coming from odd length columns)

$$(c_{2i}, c_{2i-1}) \longleftrightarrow (c_{2i-1}, c_{2i}).$$

The dot-r-c diagrams of type B: For each representation $\tau = \tau_L \times \tau_R$, dots are filled in both sides forming the same shape of Young diagram, a column of "s" is added next to dots on the right side of the diagram, a column of "r" is added next to dots on the right side diagram, add a row of "d"s on the right side of the diagram, and finally add a row of "c"s on the left side of the diagram. Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step.

In summary, $s/r/d$ are added on the right and c is added on the left.

Each dot-r-c diagram of type B is extended into two diagrams by adding an extra label "a" or "b". This extension helps us to determine the signature of special orthogonal groups.

Fixing the representation τ, the right diagram τ_R determin τ_L completely.

Type M Set $C_{2k+1} = 0$. The Weyl group representations are obtained by the same formula of Type B.

The dot-r-c diagrams of type M are obtained in almost the same way as the type B except that s are added on the left side of the diagram and we do not add labels a/b.

We let $\mathrm{DRC}(\mathcal{O})$ denote the set of dot-r-c diagrams defined above. Note that for type B, there is an additional mark a/b attached to the diagram.

In addition, for $\star = B, C, D, M$, let

$$\mathrm{DRC}(\star) := \bigsqcup_{\mathcal{O}\in \mathrm{Nil}^{\mathrm{dpe}}(\star)} \mathrm{DRC}(\mathcal{O}).$$

4.4. **Main theorem.** Let $\mathrm{LS}(\mathcal{O})$ denote the Grothendieck group of $\widetilde{\mathbf{K}}$-equivariant local systems on $\mathcal{O}$ for various real forms of $\mathbf{G}$. The following is our main theorem on the counting of unipotent representations.

Let $\check{\mathcal{O}}$ be a purely even orbit and $\mathcal{O}$ be its dual orbit.

Theorem 4.4. *We can define an injective map using convergence range theta lifting:*

$$\begin{aligned} \pi :\ \mathrm{DRC}(\mathcal{O}) &\longrightarrow \mathrm{Unip}(\mathcal{O}) \\ \tau &\longmapsto \pi_\tau \end{aligned}$$

i) When $\mathcal{O}$ is of type C or type M, π is a bijection.

ii) When $\mathcal{O}$ is of type D or type B, the following map is a bijection

$$\begin{aligned} \mathrm{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow \mathrm{Unip}(\mathcal{O}) \\ (\tau, \varepsilon) &\longmapsto \pi_\tau \otimes (\det)^\varepsilon \end{aligned}$$

iii) All unipotent representations attached to $\mathcal{O}$ are unitarizable.

iv) Suppose $\mathcal{O}$ is noticed. The associated character map is an injective map:

$$\mathrm{Ch}_{\mathcal{O}}:\quad \mathrm{Unip}(\mathcal{O}) \hookrightarrow \mathrm{LS}(\mathcal{O})$$

v) Suppose $\mathcal{O}$ is quasi-distinguished. The image of associated character map is exactly the set of admissible orbit data. In other words, we have a bijection

$$\mathrm{Ch}_{\mathcal{O}}:\quad \mathrm{Unip}(\mathcal{O}) \hookrightarrow\!\!\!\!\rightarrow \mathrm{LS}^{\mathrm{aod}}(\mathcal{O})$$

According to Moeglin-Renard's work, Arthur's unipotent Arthur packet (defined by endoscopic transfer) is constructed by iterated theta lifting. The theorem implies that ABV's unipotent Arthur packet coincide with Arthur's.

Two unipotent representation π_1 and π_2 are called in the same LS packet if $\mathcal{L}_{\pi_1} = \mathcal{L}_{\pi_2}$. For a local system $\mathcal{L} \in \mathrm{LS}(\mathcal{O})$, let

$$[\mathcal{L}] = \{\, \pi \mid \mathrm{Ch}_{\mathcal{O}}(\pi) = \mathcal{L} \,\}$$

denote the set of all unipotent representations attached to $\mathcal{L}$. We call $[\mathcal{L}]$ a LS packet.

Remark. 1. We use convergence range theta lift to construct π inductively with respect to the number of columns in $\mathcal{O}$.
2. When $\mathcal{O}$ is not noticed, a LS packet may not be a singleton and we use the injectivity of theta lifting for a fixed dual pair in an essential way.
3. The size of LS packet $[\mathcal{L}]$ could be arbitrary large when the size of $\mathcal{O}$ groups. For example, $\mathcal{O}$ is the regular orbit of $\mathrm{Sp}(2n, \mathbb{C})$, there is only 5 possible local systems when $n \geqslant 3$. But there are $2n+1$ unipotent representations.
4. One will see easily from the proof that the associated character $\mathrm{Ch}_{\mathcal{O}}(\pi)$ for every $\pi \in \mathrm{Unip}(\mathcal{O})$ is always multiplicity free. Even when $\mathrm{Ch}_{\mathcal{O}}(\pi)$ is not irreducible, the character of the local system attached the same row length in different irreducible components are always the same.

5. Matching doc-r-c diagrams, local systems, and unipotent representations

5.1. **Descent maps of dot-r-c diagram in the algorithm.** Suppose $\mathbf{G}$ is of type B or D, we define a map

$$\begin{aligned} \mathrm{Sign}\colon\quad \mathrm{DRC}(B) \sqcup \mathrm{DRC}(D) &\longrightarrow \mathbb{N} \times \mathbb{N} \\ \tau &\longmapsto (p, q) \end{aligned}$$

Here (p, q) is the signature of the orthogonal group $\mathrm{O}(p, q)$ corresponding to the dot-r-c diagram τ, which is calculated by the following formula

$$\begin{aligned} p &= \# \bullet + 2\#r + \#c + \#d + \#A \\ q &= \# \bullet + 2\#s + \#c + \#d + \#B \end{aligned}$$

Suppose $\mathcal{O}$ is the trivial orbit of $\mathrm{Sp}(2c_0, \mathbb{R})$ ($c_0 \geqslant 0$) or $\mathrm{Mp}(0, \mathbb{R})$ ($c_0 = 0$). Let

$$\tau_{\mathcal{O}} = \boxed{\varnothing} \times \begin{array}{|c|} \hline s \\ \hline \vdots \\ \hline s \\ \hline \end{array} \tag{5.1}$$

such that there are c_0-entries marked by "s" in the right diagram. Then $\mathrm{DRC}(\mathcal{O}) = \{ \tau_{\mathcal{O}} \}$ is a singleton. Define $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$ the trivial representation of $\mathrm{Sp}(2c_0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R}) = \mathbf{1}$. That by convention, $\mathrm{Sp}(0, \mathbb{R}) = \mathrm{Mp}(0, \mathbb{R})$ is the trivial group. Let

$$\mathrm{DRC}_0(C) = \{ \tau_{\mathcal{O}} \mid \mathcal{O} = (2c_0), c_0 = 0, 1, 2, \cdots \} \text{ and } \mathrm{DRC}_0(M) = \{ \tau_{(0)} = \varnothing \times \varnothing \}.$$

The algorithm of matching dot-r-c diagrams is encoded in the descent map $\overline{\nabla}$ of dot-r-c diagrams defined below. For $(\star, \star') = (D, C), (B, M)$, we define maps

$$\begin{aligned} \overline{\nabla}\colon\quad \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) &\longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star)) \times \mathbb{Z}/2\mathbb{Z} \\ \tau &\longmapsto (\tau', \epsilon). \end{aligned}$$

Here

i) τ is in $\mathrm{DRC}(\mathcal{O})$ where $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(\star) \sqcup \mathrm{Nil}^{\mathrm{dpe}}(\star')$;
ii) $\tau' \in \mathrm{DRC}(\mathcal{O}')$ where $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$;
iii) i$\epsilon = 0$ when $\mathcal{O} \mapsto \mathcal{O}'$ is a generalized descent from type C or M to type B or D.

To ease the notations, we define

$$\overline{\nabla}_1\colon \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) \longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star))$$

such that $\overline{\nabla}_1(\tau)$ is the first component of $\overline{\nabla}(\tau)$.

We define $\mathcal{L}$ and π inductively with respect to the number of columns of $\mathcal{O}$.

i) Our reduction terminates when $\mathcal{O}$ is the trivial orbit of $\mathrm{Sp}(2c_0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R})$. In this case, $\mathrm{DRC}(\mathcal{O}) = \{\, \tau_{\mathcal{O}} \,\}$ which is attached to the trivial representation. We remark that the theta lift of the trivial representation $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$ of $\mathrm{Sp}(0, \mathbb{R})$ or $\mathrm{Mp}(0, \mathbb{R})$ to $\mathrm{O}(p, q)$ for any signature p, q is the trivial representation $\mathbf{1}^{+,+}_{p,q}$ of $\mathrm{O}(p, q)$.[2]

ii) Suppose $\mathcal{O} \in \mathrm{Nil}(\star) \sqcup \mathrm{Nil}(\star')$ is a nontrivial orbit. Let $\tau \in \mathrm{DRC}(\mathcal{O})$ and

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \in \mathrm{DRC}(\mathcal{O}') \times \mathbb{Z}/2\mathbb{Z} \text{ where } \mathcal{O}' = \overline{\nabla}(\mathcal{O}).$$

a) Suppose $\tau \in \mathrm{DRC}(C)$ or $\mathrm{DRC}(M)$. Now $\widetilde{G} = \mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Mp}(2n, \mathbb{R})$ with $n = |\tau|$. By induction, τ' is attached to a unipotent representation $\pi_{\tau'}$ of $\widetilde{G}' = \mathrm{O}(p, q)$ where $(p, q) = \mathrm{Sign}(\tau')$. We define

$$\pi_\tau := \bar{\Theta}_{\widetilde{G}', \widetilde{G}}(\pi_{\tau'} \otimes (\det)^{\epsilon}).$$

Here det is the determinant character of $\mathrm{O}(p, q)$.

b) Suppose $\tau \in \mathrm{DRC}(D)$ or $\mathrm{DRC}(B)$. Now $\widetilde{G} = \mathrm{O}(p, q)$ with $(p, q) = \mathrm{Sign}(\tau)$. By induction, τ' is attached to a unipotent representation of $\widetilde{G}'$ where $\widetilde{G}' = \mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Mp}(2n, \mathbb{R})$ and $n = |\tau'|$. We define

$$\pi_\tau := \bar{\Theta}_{\widetilde{G}', \widetilde{G}}(\pi_{\tau'}) \otimes (\mathbf{1}^{+,-}_{p,q})^{\epsilon}.$$

We let

$$\mathscr{L}_\tau := \mathrm{Ch}_{\mathcal{O}}(\pi_\tau), \text{ and}$$
$${}^{\ell}\mathrm{LS}(\mathcal{O}) := \{\, \mathcal{L}_\tau \mid \tau \in \mathrm{DRC}(\mathcal{O}) \,\}$$

to be the set of local systems obtained from our algorithm.

We summarize our definitions in the following diagram with $\star = \mathcal{O}/C/D/B/M$.

$$\begin{tikzcd}
\mathrm{DRC}(\star) \arrow[rrr, "\pi"] \arrow[dddd, "\mathcal{L}"'] & & & \mathrm{Unip}(\star) \arrow[dddd, "\mathrm{Ch}"] \\
& \tau \arrow[r, mapsto] \arrow[dd, mapsto] & \pi_\tau \arrow[dd, mapsto] & \\
& & & \\
& \mathcal{L}_\tau \arrow[r, leftrightarrow] & \mathscr{L}_\tau & \\
\mathcal{K}_{\mathsf{M}}(\star) \arrow[rrr, hook, two heads] & & & \mathrm{LS}(\star)
\end{tikzcd}$$

Here $\mathcal{K}_{\mathsf{M}}(\star)$ is a combinatiorially defined monoid parameterizing the local systems on the relevent nilpotent orbits, see.

6. The definition of $\overline{\nabla}$

From now on, we assume

$$\tau = (\tau_{L,0}, \tau_{L,1}, \cdots) \times (\tau_{R,0}, \tau_{R,1}, \cdots).$$

For any τ let $\tau^{\bullet}$ be the subdiagram consisting of $\bullet$ and s entries only.

[2]When $c_0 = 0$, this is an assignment by convention.

6.1. **dot-s switching algorithm.** We say a bipartition $\tau = \tau_L \times \tau_R$ is interlaced with lager left part if τ has columns $(\tau_{L,0} \geqslant \tau_{L,1}, \cdots,) \times (\tau_{R,0}, \tau_{R,1}, \cdots,)$ such that

$$\tau_{L,0} \geqslant \tau_{R,0} \geqslant \tau_{L,1} \geqslant \tau_{R,1} \geqslant \tau_{L,2} \geqslant \cdots .$$

$\mathrm{bi}\mathcal{P}_{\mathrm{L}}$ denote the set of all interlaced bipartitions with larger left parts. Similarly, define $\mathrm{bi}\mathcal{P}_{\mathrm{R}}$ be the set of bipartitoins with larger right parts:

$$\mathrm{bi}\mathcal{P}_{\mathrm{R}} = \{\, \tau = (\tau_{L,0} \geqslant \tau_{L,1}, \cdots,) \times (\tau_{R,0}, \tau_{R,1}, \cdots,) \mid \tau_{L,0} \geqslant \tau_{R,0} \geqslant \tau_{L,1} \geqslant \tau_{R,1} \geqslant \tau_{L,2} \geqslant \cdots \}$$

Let DS_{L} (resp. DS_{R}) be the set of of filled diagrams with only "s" entries on the left (resp. right) diagram and the rest are all "$\bullet$".

Note that for each $\boldsymbol{\tau} \in \mathrm{DS}_{\mathrm{L}}$, its shape τ is in $\mathrm{bi}\mathcal{P}_{\mathrm{L}}$. On the other hand, for each $\tau \in \mathrm{bi}\mathcal{P}_{\mathrm{L}}$, we can fill "$\bullet$" in all entries on the right diagram and corresponding entries on the left, and than fill "s" in the rest entries on the left. This procedure yields a valid dot-s diagram. In summary, $\mathrm{bi}\mathcal{P}_{\mathrm{L}}$ and DS_{L} are naturally bijective to each other.

Let $\overline{\nabla}_{\mathrm{L}} \colon \mathrm{bi}\mathcal{P}_{\mathrm{L}} \to \mathrm{bi}\mathcal{P}_{\mathrm{R}}$ (resp. $\overline{\nabla}_{\mathrm{R}} \colon \mathrm{bi}\mathcal{P}_{\mathrm{R}} \to \mathrm{bi}\mathcal{P}_{\mathrm{L}}$) be the operation of deleting the longest column on the left (resp. on the right). Then the operation gives a well defined maps between DS_{L} and DS_{R} making the following diagrams commute (the vertical maps are the natural identification discussed above):

$$\begin{tikzcd}
\mathrm{bi}\mathcal{P}_{\mathrm{L}} \arrow[r, "\overline{\nabla}_{\mathrm{L}}"] \arrow[d, equal] & \mathrm{bi}\mathcal{P}_{\mathrm{R}} \arrow[d, equal] \\
\mathrm{DS}_{\mathrm{L}} \arrow[r, dashed, "\overline{\nabla}_{\mathrm{L}}"] & \mathrm{DS}_{\mathrm{R}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{bi}\mathcal{P}_{\mathrm{R}} \arrow[r, "\overline{\nabla}_{\mathrm{R}}"] \arrow[d, equal] & \mathrm{bi}\mathcal{P}_{\mathrm{L}} \arrow[d, equal] \\
\mathrm{DS}_{\mathrm{R}} \arrow[r, dashed, "\overline{\nabla}_{\mathrm{R}}"] & \mathrm{DS}_{\mathrm{L}}
\end{tikzcd}$$

By abuse of notation, we also call the dashed arrow above $\overline{\nabla}_{\mathrm{L}}$ and $\overline{\nabla}_{\mathrm{R}}$.

6.2. Definition of $\overline{\nabla}$ of Type D and C.

6.2.1. *Bijection between "special" and "non-special" diagrams for type C.* We define a bijection between "special" diagram $\boldsymbol{\tau}^s$ and "non-special" diagrams $\boldsymbol{\tau}$.

Definition 6.1. *Let* $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_1, C_0, C_{-1} = 0) \in \mathrm{Nil}^{\mathrm{dpe}}(C)$ *with* $k \geqslant 0$*. Let*

$$\tau = (\tau_{2k-1}, \cdots, \tau_1, \tau_{-1} = 0) \times (\tau_{2k}, \cdots, \tau_0).$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(C)$*. We say* τ *has*

- special shape *if* $\tau_{2k-1} \leqslant \tau_{2k} + 1$*;*
- non-special shape *if* $\tau_{2k-1} > \tau_{2k} + 1$.

Clearly, this gives a partition of the set of Weyl group representations attached to $\mathcal{O}$*.*

Suppose $k \geqslant 1$, $C_{2k} = 2c_{2k}$ *and* $C_{2k-1} = 2c_{2k-1}$*. The special shape representation* τ^s

$$\tau^s = \tau^s_L \times \tau^s_R = (c_{2k-1}, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \cdots, \tau_0)$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau^{ns}_L \times \tau^{ns}_R = (c_{2k} + 1, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \cdots, \tau_0)$$

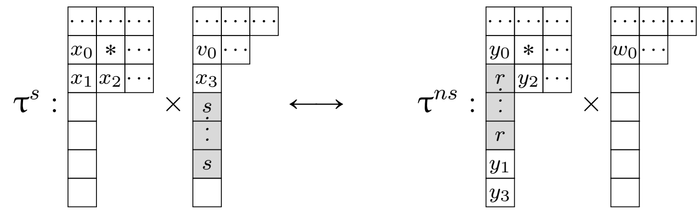
In the paring $\tau^s \leftrightarrow \tau^{ns}$, τ_j *is unchanged for* $j \leqslant 2k-2$*. We call* τ^s *the* special shape *and* τ^{ns} *the corresponding* non-special shape

Definition 6.2. *Suppose* C_{2k} *is even. We retain the notation in Definition 6.1 where the shapes* τ^s *and* τ^{ns} *correspond with each other. We define a bijection between* $\mathrm{DRC}(\tau^s)$ *and* $\mathrm{DRC}(\tau^{ns})$*:*

$$\mathrm{DRC}(\tau^s) \longleftrightarrow \mathrm{DRC}(\tau^{ns})$$
$$\boldsymbol{\tau}^s \longleftrightarrow \boldsymbol{\tau}^{ns}$$

Without loss of generality, we can assume that $\boldsymbol{\tau}^s \in \mathrm{DRC}(\tau^s)$ *and* $\boldsymbol{\tau}^{ns} \in \mathrm{DRC}(\tau^{ns})$ *have the following shapes:*

(6.1)



(Here the grey parts have length $(C_{2k} - C_{2k-1})/2$ *(could be zero length), the row contains* x_0 *and* y_0 *may be empty, and* $*/\cdots$ *are arbitrary entries. We remark that the value of* v_0 *and* w_0 *are completely determined by* x_0 *and* y_0*. In particular,* v_0 *and* w_0 *are non-empty if and only if* $x_0/y_0 \neq \emptyset$*, i.e.* $C_{2k-1} \geqslant 4$*.)*

The correspondence between (x_0, x_1, x_2, x_3)*, with* (y_0, y_1, y_2, y_3) *is given by Table 1. The rest part of the diagrams are unchanged.*

We define

$$\mathrm{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \mathrm{DRC}(\tau^s) \quad \text{and} \quad \mathrm{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \mathrm{DRC}(\tau^{ns})$$

where τ^s *(resp.* τ^{ns}*) runs over all specail (resp. non-special) shape representations attached to* $\mathcal{O}$*. Clearly* $\mathrm{DRC}(\mathcal{O}) = \mathrm{DRC}^s(\mathcal{O}) \sqcup \mathrm{DRC}^{ns}(\mathcal{O})$*.*

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

6.2.2. *Special and non-special shapes of type D.* Now we define the notion of special and non-special shape of type D.

Definition 6.3. *Let*

$$\tau = (\tau_{2k+1}, \tau_{2k-1}, \cdots, \tau_1) \times (\tau_{2k}, \cdots, \tau_0)$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(D)$*. We say* τ *has*

- special shape *if* $\tau_{2k-1} \leqslant \tau_{2k} + 1$*;*
- non-special shape *if* $\tau_{2k-1} > \tau_{2k} + 1$.

Suppose $\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \cdots, C_0 \geqslant 0)$ *with* $k \geqslant 1$.[3] *A representation* τ^s *attached to* $\mathcal{O}$ *has the shape*

$$\tau^s = \tau^s_L \times \tau^s_R = (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \cdots, \tau_0)$$

is paired with

$$\tau^{ns} = \tau^{ns}_L \times \tau^{ns}_R = (c_{2k+1}, c_{2k} + 1, \tau_{2k-3}, \cdots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \cdots, \tau_0).$$

In the paring $\tau^s \leftrightarrow \tau^{ns}$*,* τ_j *is unchanged for* $j \leqslant 2k - 2$*.*

6.2.3. *Convention.* The induction starts with type C: For the trivial orbit $\mathcal{O}$ of $\widetilde{G} = \mathrm{Sp}(2c_0, \mathbb{R})$, $\mathrm{DRC}(\mathcal{O}) = \{\, \boldsymbol{\tau}_{\mathcal{O}} \,\}$ and $\pi_{\boldsymbol{\tau}_{\mathcal{O}}} = \mathbf{1}$ is the trivial representation, see (5.1).

[3]Note that C_{2k} and C_{2k-1} are non-zero even integers.

6.2.4. *The defintion of* ϵ.

(1). When $\tau \in \mathrm{DRC}(D)$, ϵ is determined by the "basal disk" of τ (see (6.2) for the definition of x_τ):

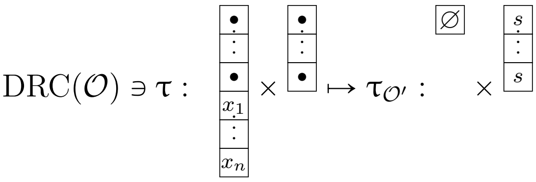
$$\epsilon_\tau := \begin{cases} 0, & \text{if } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

(2). When $\tau \in \mathrm{DRC}(C)$, the ϵ is determined by the lengths of $\tau_{L,0}$ and $\tau_{R,0}$:

$$\epsilon_\tau := \begin{cases} 0, & \text{if } \tau \text{ has special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| \leqslant 1; \\ 1, & \text{if } \tau \text{ has non-special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| > 1. \end{cases}$$

6.2.5. *Initial cases.* Let $\mathcal{O}$ be a nilpotent orbit of type D with at most 2 columns.

(3). Suppose $\mathcal{O} = (2c_1, 2c_0)$. Then $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$ is the trivial orbit of $\mathrm{Sp}(2c_0, \mathbb{R})$. $\mathrm{DRC}(\mathcal{O})$ consists of diagrams of shape $\tau_L \times \tau_R = (c_1,) \times (c_0,)$. The set $\mathrm{DRC}(\mathcal{O}')$ is a singleton $\{\tau_{\mathcal{O}'}\}$, and every element $\tau \in \mathrm{DRC}(\mathcal{O})$ maps to $\tau_{\mathcal{O}'}$ (see (5.1)):



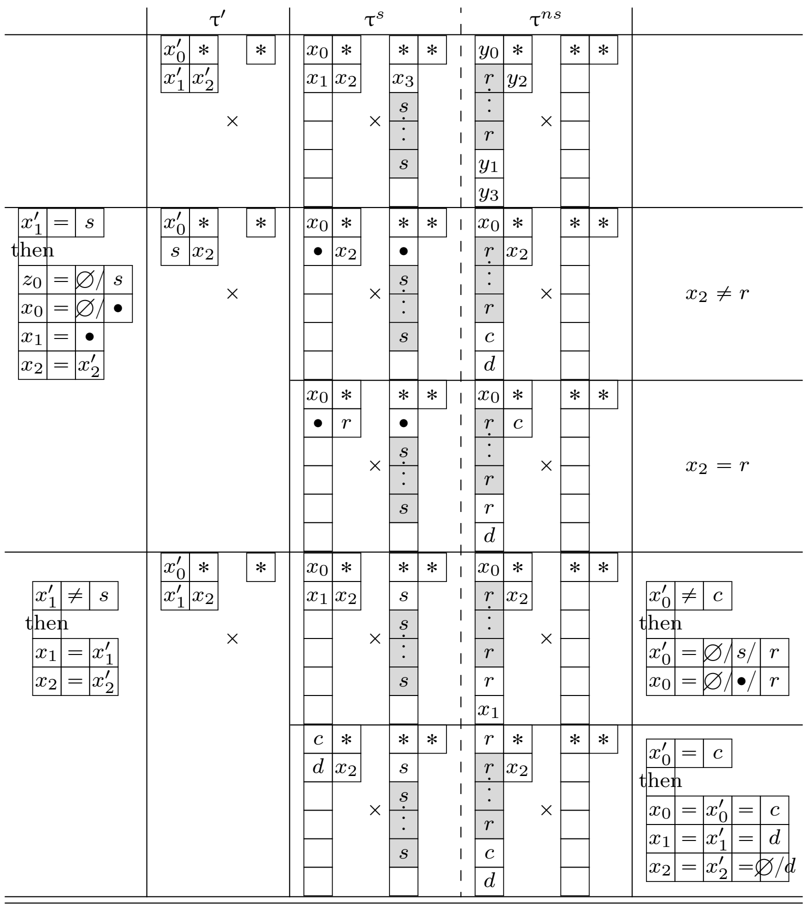


TABLE 1. "special-non-special" switch

We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := \begin{array}{|c|}\hline x_1 \\ \hline \vdots \\ \hline x_n \\ \hline \end{array} \text{ and } x_\tau := x_n$$

We call $\mathbf{p}_\tau = \mathbf{x}_\tau$ the "peduncle" part of τ and x_τ the "basal disk" of τ. Note that $\mathbf{x}_\tau$ consists of entries marked by $s/r/c/d$.

6.2.6. *The descent from C to D.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

(4). Suppose $|\tau_{L,0}| - |\tau_{R,0}| < 2$, i.e. τ has special shape. Keep r, c, d unchanged, delete $\tau_{R,0}$ and fill the remaining part with "$\bullet$" and "s" by $\overline{\nabla}_{\mathrm{R}}(\tau^\bullet)$ using the dot-s switching algorithm.

(5). Suppose $|\tau_{L,0}| - |\tau_{R,0}| \geqslant 2$. We define

$$\tau' = \overline{\nabla}_1(\tau) := \overline{\nabla}_1(\tau^s)$$

where τ^s is the special diagram corresponding to τ defined in Definition 6.2.

Lemma 6.4. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} even. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1}, \cdots, C_0)$. Then*

$$\mathrm{DRC}^s(\mathcal{O}) \xrightarrow{\quad\overline{\nabla}_1\quad} \mathrm{DRC}(\mathcal{O}')$$

$$\mathrm{DRC}^{ns}(\mathcal{O}) \xrightarrow{\quad\overline{\nabla}_1\quad} \mathrm{DRC}(\mathcal{O}')$$

are a bijections.

Proof. The claim for $\mathrm{DRC}^s(\mathcal{O})$ is clear by the definition of descent. The claim for $\mathrm{DRC}^{ns}(\mathcal{O})$ reduces to that of $\mathrm{DRC}^s(\mathcal{O})$ using Definition 6.2. $\square$

Lemma 6.5. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} odd. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1} + 1, \cdots, C_0)$. Then the following map is a bijection*

$$\mathrm{DRC}(\mathcal{O}) \xrightarrow{\quad\overline{\nabla}_1\quad} \{\, \tau' \in \mathrm{DRC}(\mathcal{O}') \mid x_\tau \neq s \,\}.$$

Proof. This is clear by the descent algorithm. $\square$

6.2.7. *Descent from D to C.* Now we consider the general case of the descent from type D to type C. So we assume $\mathcal{O}$ has at least 3 columns. First note that the shape of $\tau' = \overline{\nabla}(\tau)$ is the shape of τ deleting the longest column on the left.

The definition splits in cases below. In all these cases we define

$$\mathbf{x}_\tau := x_1 \cdots x_n \text{ and } x_\tau := x_n \tag{6.2}$$

which is marked by $s/r/c/d$. For the part marked by $*/\cdots$, $\overline{\nabla}$ keeps r, c, d and maps the rest part consisting of $\bullet$ and s by dot-s switching algorithm.

(6). When $C_{2k} = C_{2k-1}$ is odd, we could assume τ and τ' hase the following forms with $n = (C_{2k+1} - C_{2k} + 1)/2$.

$$\tau : \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & \cdots \\ \hline x_1 & y_1 & \cdots \\ \hline \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{x_n} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline w_0 & * & \cdots \\ \hline \end{array} \mapsto \tau' : \begin{array}{|c|c|}\hline \cdots & \cdots \\ \hline * & \cdots \\ \hline z_1 & \cdots \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline w_0 & * & \cdots \\ \hline \end{array}$$

Suppose $y_1 = c$ and $(x_1, \cdots, x_n) = (r, \cdots, r)$ or $(r, \cdots, r, d)$, then let $z_1 = r$; otherwise, set $z_1 := y_1$. We define

$$\mathbf{p}_\tau := \begin{array}{|c|c|}\hline x_1 & y_1 \\ \hline \multicolumn{1}{|c|}{\vdots} \\ \cline{1-1} \multicolumn{1}{|c|}{x_n} \\ \cline{1-1} \end{array}.$$

(7). When C_{2k} is even and τ has special shape, the descent is given by the following diagram

$$\tau: \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & \cdots \\ \hline \bullet & \multicolumn{2}{c}{} \\ \vdots & \multicolumn{2}{c}{} \\ \bullet & \multicolumn{2}{c}{} \\ \cline{1-1} x_1 & \multicolumn{2}{c}{} \\ \cline{1-1} \vdots & \multicolumn{2}{c}{} \\ \cline{1-1} x_n & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & \cdots \\ \hline \bullet & \multicolumn{2}{c}{} \\ \vdots & \multicolumn{2}{c}{} \\ \bullet & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \mapsto \tau': \begin{array}{|c|c|}\hline \cdots & \cdots \\ \hline * & \cdots \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & \cdots \\ \hline s & \multicolumn{2}{c}{} \\ \vdots & \multicolumn{2}{c}{} \\ s & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

(8). When C_{2k} is even and τ has non-special shape, then

$$\tau: \begin{array}{|c|c|c|c|}\hline \cdots & \cdots & \cdots & \cdots \\ \hline * & * & \cdots & \cdots \\ \hline s & r & \cdots & \cdots \\ \hline \vdots & \vdots & \multicolumn{2}{c}{} \\ s & r & \multicolumn{2}{c}{} \\ \cline{1-2} x_0 & y_1 & \multicolumn{2}{c}{} \\ \cline{1-2} x_1 & y_2 & \multicolumn{2}{c}{} \\ \cline{1-2} \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} x_n & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & \cdots & \cdots \\ \hline \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \mapsto \tau': \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & \cdots \\ \hline r & * & \cdots \\ \hline \vdots & \multicolumn{2}{c}{} \\ r & \multicolumn{2}{c}{} \\ \cline{1-1} z_1 & \multicolumn{2}{c}{} \\ \cline{1-1} z_2 & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & \cdots & \cdots \\ \hline \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \; & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

where y_1, y_2 are non empty. In most of the case, we just delete the longest column on the left and set $(z_1, z_2) = (y_1, y_2)$. The exceptional cases are listed below:

- When $x_0 = r$, we have $(y_1, y_2) = (c, d)$. We let

$$\begin{array}{|c|c|}\hline x_0 & y_1 \\ \hline x_1 & y_2 \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} x_n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline r & c \\ \hline x_1 & d \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} x_n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \mapsto \begin{array}{|c|}\hline z_1 \\ \hline z_2 \\ \hline \end{array} := \begin{array}{|c|}\hline r \\ \hline r \\ \hline \end{array}$$

- When $x_0 = c$, we have $n = 1$ and $(x_0, x_1) = (y_1, y_2) = (c, d)$. We let

$$\begin{array}{|c|c|}\hline x_0 & y_1 \\ \hline x_1 & y_2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline c & c \\ \hline d & d \\ \hline \end{array} \mapsto \begin{array}{|c|}\hline z_1 \\ \hline z_2 \\ \hline \end{array} := \begin{array}{|c|}\hline r \\ \hline c \\ \hline \end{array}.$$

- We remark that x_0 never equals to d.

6.2.8. *A key property in the descent case.* We retain the notation in Definition 6.3, where

$$\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \cdots, C_0) \text{ such that } k \geqslant 1.$$

Let $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$. The following lemma is the key property satisfied by our definition

Lemma 6.6. *Let τ^s and τ^{ns} are two representations attached to $\mathcal{O}$ as in Definition 6.3. Then*

i) For every $\tau \in \mathrm{DRC}(\tau^s) \sqcup \mathrm{DRC}(\tau^{ns})$, the shape of $\overline{\nabla}^2(\tau)$ is

$$\tau'' = (c_{2k-1}, \tau_{2k-3}, \tau_1) \times (\tau_{2k-2}, \cdots, \tau_0)$$

ii) Let $\mathcal{O}_1 = (2(c_{2k+1} - c_{2k}),) \in \mathrm{Nil}(D)$ be the trivial orbit of $\mathrm{O}(2(c_{2k-1} - c_{2k}), \mathbb{C})$. We define $\delta\colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ by $\delta(\tau) = (\overline{\nabla}^2(\tau), \mathbf{x}_\tau)$. The following maps are bijections

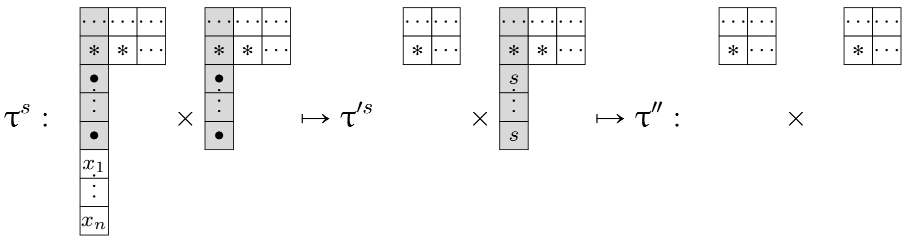
$$\begin{array}{ccccc} \mathrm{DRC}(\tau^s) & \xrightarrow{\delta} & \mathrm{DRC}(\tau'') \times \mathrm{DRC}(\mathcal{O}_1) & \xleftarrow{\delta} & \mathrm{DRC}(\tau^{ns}) \\ \tau^s & \longmapsto & (\overline{\nabla}^2(\tau^s), \mathbf{x}_{\tau^s}) & & \\ & & (\overline{\nabla}^2(\tau^{ns}), \mathbf{x}_{\tau^{ns}}) & \longleftarrow & \tau^{ns} \end{array}$$

In particular, we obtain an one-one correspondence $\tau^s \leftrightarrow \tau^{ns}$ such that $\delta(\tau^s) = \delta(\tau^{ns})$.

iii) Suppose τ^s *and* τ^{ns} *correspond as the above such that* $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$. *Then*

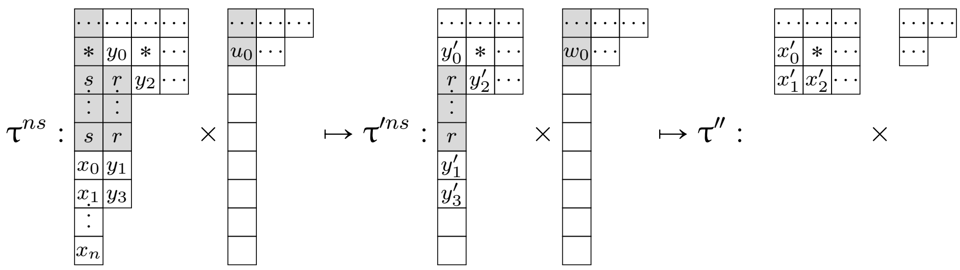
$$\operatorname{Sign}(\tau^s) = \operatorname{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \operatorname{Sign}(\tau'') + \operatorname{Sign}(\mathbf{x}). \tag{6.3}$$

Proof. Suppose τ^s has special shape. the the behavior of τ^s under the descent map $\overline{\nabla}$ is illostrated as the following:



Note that τ^s is obtained from τ'' by attaching the grey part consisting totally $2c_{2k}$ dots and $\mathbf{x}$. Now the claims for τ^s is clear.

Now consider the descent of a non-special diagram τ^{ns}:



The bijection follows from the observation that 1. τ^{ns} is obtained by attaching the grey part $x_0, \cdots, x_n$ and y_0, y_1, y_2, y_3 to the $*/\cdots$ part of τ''; 2. the value of (y_0, y_1, y_2, y_3) is completely determined by (x'_0, x'_1, x'_2) and $\mathbf{x} = x_1 \cdots x_n$.

We leave it to the reader for checking case by case that the grey part of τ^{ns} has signature $(C_{2k} - 2, C_{2k} - 2)$ and

$$\operatorname{Sign}(x_0 y_0 y_1 y_2 y_3) - \operatorname{Sign}(x'_0 x'_1 x'_2) = (2, 2).$$

Therefore (6.3) follows.

[First assume that $C_{2k} = 2$. Note that we can not/(or now?) assume $k = 1$, consider the orbit $\mathcal{O}'' = (2, 1, 1, 1, 1)$.

(i) $x'_1 = s$, now $y'_0 = \varnothing/\bullet$ when $x'_0 = \varnothing/s$
   (a) Suppose $x'_2 \neq r$. Then $(y'_2, y'_1, y'_3) = (x'_2, r, d)$, $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, x'_2, c, d)$. Therefore, the sign difference is $\operatorname{Sign}(s, c, d) - \operatorname{Sign}(s) = (2, 2)$.
   (b) Suppose $x'_2 = r$. Then $(y'_2, y'_1, y'_3) = (c, r, d)$. $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, c, r, d)$. Therefore, the sign difference is $\operatorname{Sign}(s, c, r, d) - \operatorname{Sign}(s, r) = (2, 2)$.

(ii) $x'_1 \neq s$.
   (a) Suppose $x'_0 \neq c$. Then $x'_0 = \varnothing/s/r$, $x'_1 = r/c/d$, $(y'_0, y'_2, y'_1, y'_3) = (\varnothing/\bullet/r, x'_2, r, x'_1)$.
      1). $x'_1 = r$. Then $(x_0, y_0, y_2, y_1, y_3) = (r, x'_0, x'_2, c, d)$. Therefore, the sign difference is $\operatorname{Sign}(r, c, d) - \operatorname{Sign}(r) = (2, 2)$.
      2). $x'_1 = c$. Then $x_1 = d$. $(x_0, y_0, y_2, y_1, y_3) = (c, x'_0, x'_2, c, d)$. Therefore, the sign difference is $\operatorname{Sign}(c, c, d) - \operatorname{Sign}(c) = (2, 2)$.
      3). $x'_1 = d$. Then $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, y'_2, y'_1, y'_3) = (s, x'_0, x'_2, r, d)$. Therefore, the sign difference is $\operatorname{Sign}(s, r, d) - \operatorname{Sign}(d) = (2, 2)$.

   $(x_0, y_0, y_2, y_1, y_3) = (s, r, x'_2, c, d)$. Therefore, the sign difference is $\operatorname{Sign}(s, r, x'_2, c, d) - \operatorname{Sign}(c, d, x'_2) = (2, 2)$.

(b) Suppose $x'_0 = c$. Then $x'_1 = d$, $(y'_0, y'_2, y'_1, y'_3) = (r, x'_2, c, d)$. $(x_0, y_0, y_2, y_1, y_3) = (s, r, x'_2, c, d)$. Therefore, the sign difference is $\mathrm{Sign}(s, r, x'_2, c, d) - \mathrm{Sign}(c, d, x'_2) = (2, 2)$.

] ◻

6.2.9. *A key proposition in the generalized descent case.* We now assume that $k \geqslant 1$ and C_{2k} is odd, i.e. $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a generalized descent.

Without loss of generality, we can assume the dot-r-c diagrams have the following shapes with $n = (C_{2k+1} - C_{2k} + 1)/2$:

$$\tau : \begin{array}{|c|c|c|c|}\hline \cdots & \cdots & \cdots & \cdots \\ \hline * & * & * & * \\ \hline x_1 & x_0 & \cdots & \cdots \\ \hline \vdots \\ \cline{1-1} x_n \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & \cdots \\ \hline \end{array} \xmapsto{\overline{\nabla}_1} \tau' : \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & * \\ \hline x_{\tau''} & \cdots & \cdots \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & \cdots \\ \hline \end{array} \xmapsto{\overline{\nabla}_1} \tau'' : \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline * & * & * \\ \hline x_{\tau''} & \cdots & \cdots \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \cdots & \cdots \\ \hline * & \cdots \\ \hline \end{array} \tag{6.4}$$

The diagram τ is obtained from the diagram of τ by adding $\bullet$ at the grey parts, changing $x_{\tau''}$ to x_0 and attaching $x_1 \cdots x_n$.

We define

$$\mathbf{p}_\tau := \begin{array}{|c|c|}\hline x_1 & x_0 \\ \hline \vdots \\ \cline{1-1} x_n \\ \cline{1-1} \end{array}$$

and call $\mathbf{p}_\tau$ the peduncle part of τ. Let $\mathcal{O}_1 = (C_{2k+1} - C_{2k} + 1,) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. We define $\mathbf{u}_\tau \in \mathrm{DRC}(\mathcal{O}_1)$ by the following formula:

$$\mathbf{u}_\tau := u_1 \cdots u_n = \begin{cases} r \cdots rc, & \text{when } \mathbf{p}_\tau = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots \\ \cline{1-1} r \\ \cline{1-1} \end{array} \\ r \cdots cd, & \text{when } \mathbf{p}_\tau = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots \\ \cline{1-1} d \\ \cline{1-1} \end{array} \\ x_1 \cdots , x_n & \text{otherwise.} \end{cases} \tag{6.5}$$

Let $\tilde{\mathcal{O}} = (C_{2k+1} - C_{2k} + 1, 1, 1,) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$ and $\tilde{\mathcal{O}}'' = \overline{\nabla}^2(\tilde{\mathcal{O}}) = (2) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. Then $\mathbf{p}_\tau \in \mathrm{DRC}(\tilde{\mathcal{O}})$ and $x_\tau \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. There is some restriction on the possibilities of $\mathbf{u}_\tau$.

Lemma 6.7. *Let*

$$\tau = \begin{array}{|c|c|}\hline x_1 & x_0 \\ \hline \vdots \\ \cline{1-1} x_n \\ \cline{1-1} \end{array} \in \mathrm{DRC}(\tilde{\mathcal{O}}) \quad \textit{and} \quad \tau'' := \overline{\nabla}_1^2(\tau).$$

Then we have the following properties which is easy to verify.

i) If $\tau'' = r$, then $s \in \mathbf{u}_\tau$ or $c \in \mathbf{u}_\tau$. In particular, $\mathrm{Sign}(\mathbf{u}_\tau) \succ (0, 1)$. If $\mathrm{Sign}(\mathbf{u}_\tau) \nsucc (0, 2)$, we have

$$\tau = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots \\ \cline{1-1} r \\ \cline{1-1} \end{array} \quad \textit{and} \quad \mathbf{u}_\tau = \begin{array}{|c|}\hline r \\ \hline \vdots \\ \hline c \\ \hline \end{array}.$$

ii) If $\tau'' = c$, then $s \in \mathbf{u}_\tau$ or $c \in \mathbf{u}_\tau$. In particular, $\mathrm{Sign}(\mathbf{u}_\tau) \succeq (0, 1)$.
iii) If $\tau'' = d$, there is no restriction on $\mathbf{u}_\tau$.
iv) We have $x_n = d$ if and only $d \in \mathbf{u}_\tau$.
v) If $x_n = d$ and $\tau'' = r/c$, we have $n \geqslant 2$ and $\mathrm{Sign}(\mathbf{u}_\tau) \succeq (0, 2)$.
vi) If $\mathbf{u}_\tau = r \cdots r$ or $r \cdots rd$, we have $\tau'' = d$.

Lemma 6.8. *The map $\delta\colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ given by $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathbf{u}_\tau)$ is injective. Moreover,*

$$\mathrm{Sign}(\tau) = \mathrm{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \mathrm{Sign}(\mathbf{u}_\tau).$$

The map $\tilde{\delta}\colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ given by $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathrm{Sign}(\tau), \epsilon_\tau)$ is injective.

Proof. By our algorithm,

$$(x_{\tau''}, \mathbf{u}_\tau) = (x_{\overline{\nabla}_1^2(\mathbf{p}_\tau)}, \mathbf{u}_{\mathbf{p}_\tau}).$$

The the injectivity of δ and the siginature formula follows directly from the definition of $\mathbf{u}_\tau$. Now the injectivity of $\tilde{\delta}$ follows from the injectivity of $\mathbf{u}_\tau \mapsto (\mathrm{Sign}(\mathbf{u}_\tau), \boldsymbol{\epsilon})$ by Lemma 9.2. □

7. Classical groups and their nilpotent orbits

7.1. **Complex orthogonal and symplectic groups.** Let $\epsilon \in \{\pm 1\}$. Let $\mathbf{V}$ be a finite dimensional complex vector space equipped with an ϵ-symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle_{\mathbf{V}}$, to be called an ϵ-symmetric bilinear space. Denote its isometry group by

$$\mathbf{G} := \mathbf{G}_{\mathbf{V}} := \{ g \in \mathrm{End}_{\mathbb{C}}(\mathbf{V}) \mid \langle g \cdot v_1, g \cdot v_2 \rangle_{\mathbf{V}} = \langle v_1, v_2 \rangle_{\mathbf{V}}, \quad \text{for all } v_1, v_2 \in \mathbf{V} \}.$$

The Lie algebra of $\mathbf{G}$ is given by

$$\mathfrak{g} := \mathfrak{g}_{\mathbf{V}} := \{ X \in \mathrm{End}_{\mathbb{C}}(\mathbf{V}) \mid \langle X \cdot v_1, v_2 \rangle_{\mathbf{V}} + \langle v_1, X \cdot v_2 \rangle_{\mathbf{V}} = 0, \quad \text{for all } v_1, v_2 \in \mathbf{V} \}.$$

Note that $\mathbf{G}_{\mathbf{V}}$ is a trivial group if and only if $\mathbf{V} = \{0\}$.

7.2. **Real form, Cartan involution, and $(\epsilon, \dot{\epsilon})$-space.** We recall some facts about real structures and Cartan involutions on the complex classical group $\mathbf{G}$. See [60, Section 1.2-1.3]. Let $\dot{\epsilon} \in \{\pm 1\}$.[4]

Definition 7.1. *An $\dot{\epsilon}$-real form of $\mathbf{V}$ is a conjugate linear automorphism J of $\mathbf{V}$ such that*

$$J^2 = \epsilon\dot{\epsilon} \quad \text{and} \quad \langle Jv_1, Jv_2 \rangle_{\mathbf{V}} = \overline{\langle v_1, v_2 \rangle_{\mathbf{V}}}, \quad \text{for all } v_1, v_2 \in \mathbf{V}.$$

For J as in Definition 7.1, write $\mathbf{G}^J$ for the centralizer of J in $\mathbf{G}$ (we will use similar notation without further explanation). It is a real form of $\mathbf{G}$ as in Table 2.

ϵ \ $\dot{\epsilon}$	1	−1
1	real orthogonal group	quaternionic orthogonal group
−1	quaternionic symplectic group	real symplectic group

Table 2. The real classical group $\mathbf{G}^J$

Remarks. 1. The real form $\mathbf{G}^J$ may also be described as the isometry group of a $\dot{\epsilon}$-Hermitian space over a division algebra D, where D is $\mathbb{R}$ if $\epsilon\dot{\epsilon} = 1$ and the quaternion algebra $\mathbb{H}$ if $\epsilon\dot{\epsilon} = -1$.
2. Every real form of $\mathbf{G}$ is of the form $\mathbf{G}^J$ for some J.
3. The conjugate linear map J is an equivalent formulation of Adams-Barbasch-Vogan's notion of strong real forms [2, Definition 2.13].

Definition 7.2. *An $\dot{\epsilon}$-Cartan form of $\mathbf{V}$ is a linear automorphism L of $\mathbf{V}$ such that*

$$L^2 = \dot{\epsilon} \quad \text{and} \quad \langle Lv_1, Lv_2 \rangle_{\mathbf{V}} = \langle v_1, v_2 \rangle_{\mathbf{V}}, \quad \text{for all } v_1, v_2 \in \mathbf{V}.$$

Given an $\dot{\epsilon}$-Cartan form L of $\mathbf{V}$, conjugation by L yields an involution of the algebraic group $\mathbf{G}$. Thus we have a symmetric subgroup $\mathbf{G}^L$ of $\mathbf{G}$.

We omit the proof of the following lemma (*cf.* [60, Section 1.3]).

Lemma 7.3. *For every $\dot{\epsilon}$-real form J of $\mathbf{V}$, up to conjugation by $\mathbf{G}^J$, there exists a unique $\dot{\epsilon}$-Cartan form L of $\mathbf{V}$ such that*

[4] In [60], $\dot{\epsilon}$ is denoted by ω.

(a) $LJ = JL$;
(b) the Hermitian form $(u,v) \mapsto \langle Lu, Jv\rangle_{\mathbf{V}}$ *on* $\mathbf{V}$ *is positive definite.*

Similarly, for every $\dot\epsilon$*-Cartan form* L *of* $\mathbf{V}$*, up to conjugation by* $\mathbf{G}^L$*, there exists a unique* $\dot\epsilon$*-real form* J *of* $\mathbf{V}$ *such that the above two conditions hold.*

For L as in Lemma 7.3, conjugation by L induces a Cartan involution on $\mathbf{G}^J$.

Definition 7.4. *An* ϵ*-symmetric bilinear space* $\mathbf{V}$ *with a pair* (J, L) *as in Lemma 7.3 is called an* $(\epsilon, \dot\epsilon)$-space.

In view of Lemma 7.3 and when no confusion is possible, we also name the pairs $(\mathbf{V}, L)$ and $(\mathbf{V}, J)$ as $(\epsilon, \dot\epsilon)$-spaces. We define the notion of a non-degenerate $(\epsilon, \dot\epsilon)$-subspace and isomorphisms of $(\epsilon, \dot\epsilon)$-spaces in an obvious way.

Definition 7.5. *The signature of an* $(\epsilon, \dot\epsilon)$*-space* $(\mathbf{V}, J, L)$ *is defined by the following recipe:*

$$\mathrm{Sign}(\mathbf{V}) := \mathrm{Sign}(\mathbf{V}, L) := (n^+, n^-),$$

where n^+ *and* n^- *are respectively the dimensions of* $+1$ *and* -1 *eigenspaces of* L *if* $\dot\epsilon = 1$*, and the dimensions of* $+\mathbf{i}$ *and* $-\mathbf{i}$ *eigenspaces of* L *if* $\dot\epsilon = -1$*. For every* L*-stable subquotient* $\mathbf{E}$ *of* $\mathbf{V}$*, the signature* $\mathrm{Sign}(\mathbf{E})$ *is defined analogously.*

For a fixed pair $(\epsilon, \dot\epsilon)$, the isomorphism class of an $(\epsilon, \dot\epsilon)$-space $(\mathbf{V}, J, L)$ is determined by $\mathrm{Sign}(\mathbf{V})$. In view of Lemma 7.3, we define the signatures of the pairs $(\mathbf{V}, L)$ and $(\mathbf{V}, J)$ as that of $(\mathbf{V}, J, L)$. The value of $\mathrm{Sign}(\mathbf{V})$ is listed in Table 3 for the real classical group $\mathbf{G}^J$.

$\mathbf{G}^J$	$\mathrm{Sign}(\mathbf{V})$
$\mathrm{O}(p,q)$	(p,q)
$\mathrm{Sp}(2n,\mathbb{R})$	(n,n)
$\mathrm{O}^*(2n)$	(n,n)
$\mathrm{Sp}(p,q)$	$(2p,2q)$

TABLE 3. Signature of $(\epsilon, \dot\epsilon)$-spaces

7.3. **Metaplectic cover.** Let (J, L) be as in Lemma 7.3. Put $G := \mathbf{G}^J$ and $\mathbf{K} := \mathbf{G}^L$. Then $K := G \cap \mathbf{K}$ is a maximal compact subgroup of both G and $\mathbf{K}$. As a slight modification of G, we define

$$\widetilde{G} := \begin{cases} \text{the metaplectic double cover of } G, & \text{if } G \text{ is a real symplectic group;} \\ G, & \text{otherwise.} \end{cases}$$

Here "G is a real symplectic group" means that $(\epsilon, \dot\epsilon) = (-1,-1)$, and similar terminologies will be used later on. In the case of a real symplectic group, we use ε_G to denote the nontrivial element in the kernel of the covering homomorphism $\widetilde{G} \to G$. In general, we use "$\sim$" over a subgroup of G to indicate its inverse image under the covering map $\widetilde{G} \to G$. For example, $\widetilde{K}$ is the a maximal compact subgroup of $\widetilde{G}$ with a covering map $\widetilde{K} \to K$ of degree 1 or 2. Similar notation will be used for other covering groups similar to $\widetilde{G}$.

When there is a need to indicate the dependence of various objects introduced on the $(\epsilon, \dot\epsilon)$-space $\mathbf{V}$, we will add the subscript $\mathbf{V}$ in various notations (e.g. $\widetilde{G}_{\mathbf{V}}$ and $\widetilde{K}_{\mathbf{V}}$).

7.4. **Young diagrams and complex nilpotent orbits.** Let $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ be the set of nilpotent $\mathbf{G}$-orbits in $\mathfrak{g}$. For $n \in \mathbb{N} := \{0, 1, 2, \cdots\}$, let $\mathcal{P}_\epsilon(n)$ be the set of Young diagrams of size n such that

i) rows of even length appear in even times if $\epsilon = 1$;
ii) rows of odd length appear in even times if $\epsilon = -1$.

In this paper, a Young diagram $\mathbf{d}$ will be labeled by a sequence $[c_0, c_1, \cdots, c_k]$ of integers enumerating its columns, where $k \geqslant -1$. By convention, $[c_0, c_1, \cdots, c_k]$ denotes the empty sequence $\varnothing$ which labels the empty Young diagram when $k = -1$. It is easy to see that $\mathcal{P}_\epsilon(n)$ consists of sequences of the form $[c_0, c_1, \cdots, c_k]$ such that

$$\begin{cases} c_0 \geqslant c_1 \geqslant \cdots \geqslant c_k > 0, \\ \sum_{l=0}^k c_l = n, \text{ and} \\ \sum_{l=i}^k c_l \text{ is even, when } i \equiv \frac{1+\epsilon}{2} \pmod 2 \text{ and } 0 \leqslant i \leqslant k, \end{cases}$$

and $\mathcal{P}_\epsilon(0) := \{\varnothing\}$ by convention.

Definition 7.6. *For a nilpotent element X in $\mathfrak{g}$, set*

$$\mathrm{depth}(X) := \begin{cases} \max\{\, l \in \mathbb{N} \mid X^l \neq 0 \,\}, & \text{if } \mathbf{V} \neq \{0\}; \\ -1, & \text{if } \mathbf{V} = \{0\}. \end{cases} \tag{7.1}$$

Given $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$, pick any $X \in \mathcal{O}$. Set $\mathrm{depth}(\mathcal{O}) := \mathrm{depth}(X)$ *and define the Young diagram*

$$\mathbf{d}_{\mathcal{O}} := [c_0, c_1, \cdots, c_k], \qquad (k := \mathrm{depth}(\mathcal{O}) \geqslant -1)$$

where

$$c_l := \dim(\mathrm{Ker}(X^{l+1})/\,\mathrm{Ker}(X^l)), \quad \textit{for all } 0 \leqslant l \leqslant k.$$

We have the one-one correspondence: ([20, Chapter 5])

$$\begin{aligned} \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g}) &\longrightarrow \mathcal{P}_\epsilon(\dim \mathbf{V}), \\ \mathcal{O} &\longmapsto \mathbf{d}_{\mathcal{O}}. \end{aligned} \tag{7.2}$$

Let

$$\mathrm{Nil}_\epsilon := \left(\bigsqcup_{\mathbf{V}} \mathrm{Nil}_{\mathbf{G}_{\mathbf{V}}}(\mathfrak{g}_{\mathbf{V}}) \right) \Big/ \sim$$

where $\mathbf{V}$ runs over all ϵ-symmetric bilinear spaces, and $\sim$ denotes the equivalence relation induced by isomorphisms of ϵ-symmetric bilinear spaces. Let

$$\mathcal{P}_\epsilon := \bigsqcup_{n \geqslant 0} \mathcal{P}_\epsilon(n).$$

The correspondence (7.2) induces a bijection

$$\mathbf{D}\colon\ \mathrm{Nil}_\epsilon \longrightarrow \mathcal{P}_\epsilon.$$

Definition 7.7. *Define the descent of a Young diagram by*

$$\begin{aligned} \boldsymbol{\nabla}\colon \mathcal{P}_\epsilon &\longrightarrow \mathcal{P}_{-\epsilon} \\ [c_0, c_1, \cdots, c_k] &\longmapsto [c_1, \cdots, c_k], \end{aligned}$$

namely by removing the left most column of the diagram. By convention, $\boldsymbol{\nabla}(\varnothing) := \varnothing$.

Fix $\mathbb{p} \in \mathbb{Z}/2\mathbb{Z}$ as in the Introduction. As highlighted in the Introduction, we will consider the following set of partitions/Young diagrams/nilpotent orbits.

Definition 7.8. *Denote $\epsilon_l := \epsilon(-1)^l$, for $l \geqslant 0$. Let*

$$\mathcal{P}_\epsilon^{\mathbb{p}} := \left\{ \mathbf{d} = [c_0, \cdots, c_k] \in \mathcal{P}_\epsilon \;\middle|\; \begin{array}{l} \bullet \text{ all } c_i\text{'s have parity } \mathbb{p}; \\ \bullet\ c_l \geqslant c_{l+1} + 2, \text{ if } 0 \leqslant l \leqslant k-1 \text{ and} \\ \quad \epsilon_l = 1. \end{array} \right\}.$$

By convention, $\varnothing \in \mathcal{P}_\epsilon^{\mathbb{p}}$. Denote by $\mathrm{Nil}_{\mathbf{G}}^{\mathbb{p}}(\mathfrak{g})$ the subset of $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ corresponding to partitions in $\mathcal{P}_{\mathbf{G}}^{\mathbb{p}} := \mathcal{P}_\epsilon^{\mathbb{p}} \cap \mathcal{P}_\epsilon(\dim \mathbf{V})$.

7.5. **Signed Young diagrams and rational nilpotent orbits.** Recall that $(\mathbf{V}, J, L)$ is an $(\epsilon, \dot\epsilon)$-space. Under conjugation by L, the complex Lie algebra $\mathfrak{g}$ decomposes into ± 1-eigenspaces:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k}_{\mathbf{V}} \oplus \mathfrak{p}_{\mathbf{V}}. \tag{7.3}$$

Let $\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ be the set of nilpotent $\mathbf{K}$-orbits in $\mathfrak{p}$.

7.5.1. *Parametrization.* In this section, we explain the parameterization of $\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$.

Definition 7.9. *Let $\mathbb{Z}^2_{\geqslant 0}$ be the set of pairs of non-negative integers, whose elements are called* signatures. *For a signature $n = (n^+, n^-)$, its* dual signature *is defined to be $\check{n} := (n^-, n^+)$. Define a partial order on $\mathbb{Z}^2_{\geqslant 0}$ by*

$$(n_1^+, n_1^-) \succeq (n_2^+, n_2^-) \quad \textit{if and only if} \quad n_1^+ \geqslant n_2^+ \textit{ and } n_1^- \geqslant n_2^-.$$

Recall that a signed Young diagram is a Young diagram in which every box is labeled with a $+$ or $-$ sign in such a way that signs alternate across rows. For a signed Young diagram which contains n_i^+ number of "$+$" signs and n_i^- number of "$-$" signs in the i-th column, we will label it by the sequence of signatures $[d_0, d_1, \cdots, d_k]$, where $k \geqslant -1$ and $d_i = (n_i^+, n_i^-)$. By convention $[d_0, d_1, \cdots, d_k]$ labels the empty diagram when $k = -1$.

The set of all signed Young diagrams will then correspond to the set

$$\ddot{\mathcal{P}} := \bigsqcup_{k \geqslant -1} \{ [d_0, \cdots, d_k] \in (\mathbb{Z}^2_{\geqslant 0} \backslash \{(0,0)\})^{k+1} \mid d_l \succeq \check{d}_{l+1} \text{ for all } 0 \leqslant l \leqslant k-1 \}.$$

Similar to Definition 7.7, we define the descent map

$$\nabla\colon \ddot{\mathcal{P}} \longrightarrow \ddot{\mathcal{P}}, \quad [d_0, d_1, \cdots, d_k] \mapsto [d_1, \cdots, d_k]. \tag{7.4}$$

Definition 7.10. *Given $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, pick any $X \in \mathscr{O}$. Set* $\mathrm{depth}(\mathscr{O}) := \mathrm{depth}(X)$ *(see* (7.1)*) and define the signed Young diagram*

$$\ddot{\mathbf{d}}_{\mathscr{O}} := [d_0, \cdots, d_k], \qquad (k := \mathrm{depth}(\mathscr{O}) \geqslant -1)$$

where

$$d_l := \mathrm{Sign}(\mathrm{Ker}(X^{l+1}) / \mathrm{Ker}(X^l)), \quad \textit{for all } 0 \leqslant l \leqslant k.$$

We parameterize $\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ via the injective map (*cf.* [22])

$$\ddot{\mathbf{D}}\colon \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p}) \longrightarrow \ddot{\mathcal{P}}, \quad \mathscr{O} \mapsto \ddot{\mathbf{d}}_{\mathscr{O}}.$$

7.5.2. *Stabilizers.* Let $\mathfrak{sl}_2(\mathbb{C})$ be the complex Lie algebra consisting of 2×2 complex matrices of trace zero. Write

$$\mathring{L} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathring{X} := \begin{pmatrix} 1/2 & \mathbf{i}/2 \\ \mathbf{i}/2 & -1/2 \end{pmatrix}.$$

Let $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ with $\ddot{\mathbf{D}}(\mathscr{O}) = [d_0, \cdots, d_k]$ and pick any element $X \in \mathscr{O}$. By [69] (also see [77, Section 6]), there is a Lie algebra homomorphism (unique up to $\mathbf{K}$-conjugation)

$$\phi_{\mathfrak{k}}\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$$

such that

- $\phi_{\mathfrak{k}}$ intertwines the conjugation of $\mathring{L}$ on $\mathfrak{sl}_2(\mathbb{C})$ and the conjugation of L on $\mathfrak{g}$; and
- $\phi_{\mathfrak{k}}(\mathring{X}) = X$;

We call $\phi_{\mathfrak{k}}$ an L-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple attached to X.

For each $l \geqslant 0$, let $\phi_l\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{l+1}(\mathbb{C})$ denote an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ realized on $\mathbb{C}^{l+1}$. Fix an $\mathrm{SL}_2(\mathbb{C})$-invariant $(-1)^l$-symmetric non-degenerate bilinear form $\langle\, ,\, \rangle_l$ on $\mathbb{C}^{l+1}$. As an $\mathfrak{sl}_2(\mathbb{C})$-module via $\phi_{\mathfrak{k}}$, we have

$$\mathbf{V} = \bigoplus_{l=0}^{k} {}^l\mathbf{V} \otimes_{\mathbb{C}} \mathbb{C}^{l+1}, \qquad (k := \mathrm{depth}(\mathscr{O}) \geqslant -1) \tag{7.5}$$

where $\mathbb{C}^{l+1}$ is viewed as an $\mathfrak{sl}_2(\mathbb{C})$-module via the differential of ϕ_l, and

$${}^l\mathbf{V} := \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V})$$

is the multiplicity space.

Let

$$L_l := (-1)^{\lfloor \frac{l}{2} \rfloor} \phi_l(\mathring{L}).$$

Then $(\mathbb{C}^{l+1}, L_l)$ is a $((-1)^l, (-1)^l)$-space and the L_l-stable subspace

$$(\mathbb{C}^{l+1})^{\mathring{X}} := \{ v \in \mathbb{C}^{l+1} \mid \mathring{X} \cdot v = 0 \}$$

has signature $(1, 0)$.

Define the $(-1)^l\epsilon$-symmetric bilinear form $\langle\, ,\, \rangle_{{}^l\mathbf{V}}$ on ${}^l\mathbf{V}$ by requiring that

$$\langle\, ,\, \rangle_{\mathbf{V}} = \bigoplus_l \langle\, ,\, \rangle_{{}^l\mathbf{V}} \otimes \langle\, ,\, \rangle_l .$$

Define

$${}^lL(T) := L \circ T \circ L_l^{-1}, \qquad \text{for all } T \in {}^l\mathbf{V} = \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V}).$$

It is routine to check that $({}^l\mathbf{V}, {}^lL)$ is a $((-1)^l\epsilon, (-1)^l\dot{\epsilon})$-space and it has signature $d_l - \check{d}_{l+1}$ if l is even and has signature $\check{d}_l - d_{l+1}$ if l is odd.

Let $\mathbf{K}_X := \mathrm{Stab}_{\mathbf{K}}(X)$ be the stabilizer of X in $\mathbf{K}$, and $\mathbf{R}_X$ the stabilizer of $\phi_{\mathfrak{k}}$ in $\mathbf{K}$. Then $\mathbf{K}_X = \mathbf{R}_X \ltimes \mathbf{U}_X$, where $\mathbf{U}_X$ denotes the unipotent radical of $\mathbf{K}_X$.

Set ${}^l\mathbf{K} := (\mathbf{G}_{{}^l\mathbf{V}})^{{}^lL}$. Using the decomposition (7.5), we get the following lemma from the discussion above.

Lemma 7.11. *There is a canonical isomorphism*

$$\mathbf{R}_X \cong \prod_{l=0}^{k} {}^l\mathbf{K}.$$

Let $A := G/G^\circ$ be the component group of G, where G° denotes the identity connected component of G. It is isomorphic to the component group of $\mathbf{K}$ and is trivial unless G is a real orthogonal group.

Suppose we are in the case when $G = \mathrm{O}(p, q)$ $(p, q \geqslant 0)$ is a real orthogonal group. Then $\mathbf{K} = \mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C})$ is a product of two complex orthogonal groups. For every pair $\eta := (\eta^+, \eta^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, let sgn^η denote the character $\det^{\eta^+} \boxtimes \det^{\eta^-}$ of $\mathrm{O}(p, \mathbb{C}) \times \mathrm{O}(q, \mathbb{C})$, where det denotes the sign character of an orthogonal group. It obviously induces characters on $\mathrm{O}(p) \times \mathrm{O}(q)$, $\mathrm{O}(p, q)$ and A, which are still denoted by sgn^η. Then

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2 \to \widehat{A}, \quad \eta \mapsto \mathrm{sgn}^\eta$$

is a surjective homomorphism. Here and henceforth, "$\widehat{\ }$" indicates the group of characters of a finite abelian group.

We record the following lemma.

Lemma 7.12. *Suppose* $G = \mathrm{O}(p,q)$ $(p, q \geqslant 0)$. *Write* $\mathrm{Sign}({}^{l}\mathbf{V}) = ({}^{l}p^{+}, {}^{l}p^{-})$ *for* $0 \leqslant l \leqslant k$. *Then there is an obvious isomorphism*

$$
{}^{l}\mathbf{K} \cong \begin{cases} \mathrm{O}({}^{l}p^{+}, \mathbb{C}) \times \mathrm{O}({}^{l}p^{-}, \mathbb{C}), & \text{if } l \text{ is even;} \\ \mathrm{GL}({}^{l}p^{+}, \mathbb{C}), & \text{if } l \text{ is odd.} \end{cases} \tag{7.6}
$$

Moreover for $\eta := (\eta^{+}, \eta^{-}) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, *the character* sgn^{η} *of* $\mathbf{K}$ *has trivial restriction to* $\mathbf{U}_X$, *and with respect to the isomorphism* (7.6),

$$
\mathrm{sgn}^{\eta}|_{{}^{l}\mathbf{K}} = \begin{cases} \det^{\eta^{+}} \boxtimes \det^{\eta^{-}}, & \text{if } l \text{ is even;} \\ 1, & \text{if } l \text{ is odd.} \end{cases}
$$

Proof. The first assertion follows by using Table 3. Since $\mathbf{U}_X$ is unipotent, it has no nontrivial algebraic character. Thus sgn^{η} of $\mathbf{K}$ has trivial restriction to $\mathbf{U}_X$. The last assertion of the lemma is routine to check, which we omit. ☐

7.6. **Dual pairs, descent and lift of nilpotent orbits.**

Definition 7.13 (Dual pair)**.** *(i) A* complex dual pair *is a pair consisting of an ϵ-symmetric bilinear space and an ϵ'-symmetric bilinear space, where $\epsilon, \epsilon' \in \{\pm 1\}$ with $\epsilon\epsilon' = -1$.*

(ii) A rational dual pair *is a pair consisting of an $(\epsilon, \dot\epsilon)$-space and an $(\epsilon', \dot\epsilon')$-space, where $\epsilon, \epsilon', \dot\epsilon, \dot\epsilon' \in \{\pm 1\}$ with $\epsilon\epsilon' = \dot\epsilon\dot\epsilon' = -1$.*

Let $(\mathbf{V}, \mathbf{V}')$ be a complex dual pair. Define a (-1)-symmetric bilinear space

$$
\mathbf{W} := \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}'),
$$

with the form

$$
\langle T_1, T_2 \rangle_{\mathbf{W}} := \mathrm{tr}(T_1^{\star} T_2), \qquad \text{for all } T_1, T_2 \in \mathbf{W}.
$$

Here $\star\colon \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}') \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}', \mathbf{V})$ is the adjoint map induced by the non-degenerate forms on $\mathbf{V}$ and $\mathbf{V}'$:

$$
\langle Tv, v' \rangle_{\mathbf{V}'} = \langle v, T^{\star} v' \rangle_{\mathbf{V}}, \qquad \text{for all } v \in \mathbf{V}, v' \in \mathbf{V}', T \in \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}').
$$

The pair of groups $(\mathbf{G}, \mathbf{G}') := (\mathbf{G}_{\mathbf{V}}, \mathbf{G}_{\mathbf{V}'})$ is a *complex reductive dual pair* in $\mathrm{Sp}(\mathbf{W})$ in the sense of Howe [34].

Further suppose that $(\mathbf{V}, J, L)$ and $(\mathbf{V}', J', L')$ form a rational dual pair. Then the complex symplectic space $\mathbf{W} = \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}')$ is naturally a $(-1,-1)$-space by defining

$$
J_{\mathbf{W}}(T) := J' \circ T \circ J^{-1} \quad \text{and} \quad L_{\mathbf{W}}(T) := \dot\epsilon\, L' \circ T \circ L^{-1}, \qquad \text{for all } T \in \mathbf{W}.
$$

The pair of groups $(G, G') = (\mathbf{G}_{\mathbf{V}}^{J}, \mathbf{G}_{\mathbf{V}'}^{J'})$ is then a (real) reductive dual pair in $\mathrm{Sp}(W)$, where $W := \mathbf{W}^{J_{\mathbf{W}}}$.

For the rest of this section, we work in the setting of rational dual pairs and introduce a notion of descent and lift of nilpotent orbits in this context.

7.6.1. *Moment maps.* We define the following *moment maps* $\mathbf{M}$ and $\mathbf{M}'$ with respect to a complex dual pair $(\mathbf{V}, \mathbf{V}')$:

$$
\begin{array}{ccccc}
\mathfrak{g} & \xleftarrow{\ \mathbf{M}\ } & \mathbf{W} & \xrightarrow{\ \mathbf{M}'\ } & \mathfrak{g}', \\
T^{\star}T & \longleftarrow\!\shortmid & T & \longmapsto & TT^{\star}.
\end{array}
$$

When we have a rational dual pair, we decompose

$$
\mathbf{W} = \mathcal{X} \oplus \mathcal{Y}
$$

where $\mathcal{X}$ and $\mathcal{Y}$ are $+\mathbf{i}$ and $-\mathbf{i}$ eigenspaces of $L_{\mathbf{W}}$, respectively. Restriction on $\mathcal{X}$ induces a pair of maps (see (7.3) for the definition of $\mathfrak{p}$ and $\mathfrak{p}'$):

$$\mathfrak{p} \xleftarrow{M:=\mathbf{M}|_{\mathcal{X}}} \mathcal{X} \xrightarrow{M':=\mathbf{M}'|_{\mathcal{X}}} \mathfrak{p}',$$

which are also called *moment maps* (with respect to the rational dual pair). By classical invariant theory (see [82] or [58, Lemma 2.1]), the image $M(\mathcal{X})$ is Zariski closed in $\mathfrak{p}$, and the moment map M induces an isomorphism

$$\mathbf{K}'\backslash\backslash\mathcal{X} \cong M(\mathcal{X}) \tag{7.7}$$

of affine algebraic varieties, where $\mathbf{K}'\backslash\backslash\mathcal{X}$ denotes the affine quotient.[5] Thus [64, Corollary 4.7] implies that M maps every $\mathbf{K}'$-stable Zariski closed subset of $\mathcal{X}$ onto a Zariski closed subset of $\mathfrak{p}$. Similar statement holds for M'. We will use these basic facts freely.

7.6.2. *Lifts and descents of nilpotent orbits.* Let

$$\mathbf{W}^\circ := \{ T \in \mathbf{W} \mid T \text{ is a surjective map from } \mathbf{V} \text{ onto } \mathbf{V}' \}.$$

Clearly $\mathbf{W}^\circ \neq \varnothing$ only if $\dim \mathbf{V} \geqslant \dim \mathbf{V}'$.

Suppose $\mathbf{e} \in \mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathbf{e}' \in \mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. We call $\mathbf{e}'$ (resp. $\mathcal{O}'$) a descent of $\mathbf{e}$ (resp. $\mathcal{O}$), if there exits $T \in \mathbf{W}^\circ$ such that

$$\mathbf{M}(T) = \mathbf{e} \quad \text{and} \quad \mathbf{M}'(T) = \mathbf{e}'.$$

Put

$$\mathcal{X}^\circ := \mathbf{W}^\circ \cap \mathcal{X},$$

and write $\mathbf{K} := \mathbf{K}_{\mathbf{V}}$ and $\mathbf{K}' := \mathbf{K}_{\mathbf{V}'}$. Suppose $X \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ and $X' \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. We call X' (resp. $\mathscr{O}'$) a descent of X (resp. $\mathscr{O}$), if there exits $T \in \mathcal{X}^\circ$ such that

$$M(T) = X \quad \text{and} \quad M'(T) = X'.$$

In all cases, we will say that T realizes the descent, and $\mathcal{O}$ (resp. $\mathscr{O}$) is the lift of $\mathcal{O}'$ (resp. $\mathscr{O}'$). In the notation of Definition 7.7, we then have

$$\boldsymbol{\nabla}(\mathbf{d}_{\mathcal{O}}) = \mathbf{d}_{\mathcal{O}'} \quad \text{and} \quad \nabla(\ddot{\mathbf{d}}_{\mathscr{O}}) = \ddot{\mathbf{d}}_{\mathscr{O}'}.$$

Hence the notion of descent (for nilpotent orbits) defined here agrees with that of Definition 7.7 and (7.4) (for Young diagrams). We will thus write

$$\mathcal{O}' = \boldsymbol{\nabla}(\mathcal{O}) = \boldsymbol{\nabla}_{\mathbf{V},\mathbf{V}'}(\mathcal{O}) \quad \text{and} \quad \mathscr{O}' = \nabla(\mathscr{O}) = \nabla_{\mathbf{V},\mathbf{V}'}(\mathscr{O}).$$

We record a key property on descent and lift:

$$\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}} \quad \text{and} \quad M(M'^{-1}(\overline{\mathscr{O}'})) = \overline{\mathscr{O}}, \tag{7.8}$$

where " $\overline{}$ " means taking Zariski closure. This is checked by using explicit formulas in [21, 41] (for complex dual pairs) and [60, Lemma 14] (for rational dual pairs).

In fact by [21, Theorem 1.1], the notion of lift can be extended to an arbitrary complex dual pair and an arbitrary complex nilpotent orbit: for any $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$, $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'}))$ equals to the closure of a unique nilpotent orbit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$. We call $\mathcal{O}$ the *theta lift* of $\mathcal{O}'$, written as

$$\mathcal{O} = \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}(\mathcal{O}'). \tag{7.9}$$

[5] See [64, Section 4.4] for the definition of affine quotient.

7.6.3. *Generalized descent of nilpotent orbits.* Let

$$\mathbf{W}^{\mathrm{gen}} := \{\, T \in \mathbf{W} \mid \text{the image of } T \text{ is a non-degenerate subspace of } \mathbf{V}' \,\}$$

and

$$\mathcal{X}^{\mathrm{gen}} := \mathbf{W}^{\mathrm{gen}} \cap \mathcal{X}.$$

Suppose $X \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ and $X' \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. We call X' (resp. $\mathscr{O}'$) a generalized descent of X (resp. $\mathscr{O}$), if there exits $T \in \mathcal{X}^{\mathrm{gen}}$ such that

$$M(T) = X \quad \text{and} \quad M'(T) = X'.$$

As before, we say that T realizes the generalized descent.

It is easy to see that for each nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, the following three assertions are equivalent (*cf.* [22, Table 4]).

- The orbit $\mathscr{O}$ has a generalized descent.
- The orbit $\mathscr{O}$ is contained in the image of the moment map M.
- Write $\ddot{\mathbf{d}}_{\mathscr{O}} = [d_0, d_1, \cdots, d_k] \in \ddot{\mathcal{P}}$, then

$$\mathrm{Sign}(\mathbf{V}') \succeq \sum_{i=1}^{k} d_i.$$

When this is the case, $\mathscr{O}$ has a unique generalized descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$, and

$$\ddot{\mathbf{d}}_{\mathscr{O}'} = [d_1 + s, d_2, \cdots, d_k], \quad \text{where } s := \mathrm{Sign}(\mathbf{V}') - \sum_{i=1}^{k} d_i. \tag{7.10}$$

We write $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})$. On the other hand, different nilpotent orbits may map to a same nilpotent orbit under $\nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}$.

Analogously, suppose $\mathbf{e} \in \mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathbf{e}' \in \mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. We call $\mathbf{e}'$ (resp. $\mathcal{O}'$) a generalized descent of $\mathbf{e}$ (resp. $\mathcal{O}$), if there exits an element $T \in \mathbf{W}^{\mathrm{gen}}$ such that

$$\mathbf{M}(T) = \mathbf{e} \quad \text{and} \quad \mathbf{M}'(T) = \mathbf{e}'.$$

When this is the case, $\mathcal{O}'$ is determined by $\mathcal{O}$ and we write $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$.

Lemma 7.14. *Assume that $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ has a generalized descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Let $X \in \mathscr{O}$ and $T \in \mathcal{X}^{\mathrm{gen}}$ such that $M(T) = X$. Then $\mathbf{K}' \cdot T$ is the unique closed $\mathbf{K}'$-orbit in $M^{-1}(X)$. Moreover,*

$$\mathscr{O}' = M'(M^{-1}(\mathscr{O})) \cap \mathcal{O}' = M'(M^{-1}(\mathscr{O})) \cap \overline{\mathcal{O}'}, \tag{7.11}$$

where $\mathcal{O}' := \mathbf{G}' \cdot \mathscr{O}'$, which is the generalized descent of $\mathcal{O} := \mathbf{G} \cdot \mathscr{O}$.

Proof. It is elementary to check that $\mathbf{K}' \cdot T$ is Zariski closed in $\mathcal{X}$. Then the first assertion follows by using the isomorphism (7.7). Note that $\mathscr{O}'$ is the only $\mathbf{K}'$-orbit in $\overline{\mathcal{O}'} \cap \mathfrak{p}'$ whose Zariski closure contains $\mathscr{O}'$. Thus the first assertion implies the second one. ☐

In this article, we will need to consider the following special types of nilpotent orbits.

Definition 7.15. *A nilpotent oribit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ with $\mathrm{d}_{\mathcal{O}} = [c_0, c_1 \cdots, c_k] \in \mathcal{P}_\epsilon$ is said to be good for generalized descent if $k \geqslant 1$ and $c_0 = c_1$. A nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is said to be* good *for generalized descent if the nilpotent orbit $\mathbf{G} \cdot \mathscr{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is good for generalized descent.*

The following lemma exhibits a certain maximality property of nilpotent orbits which are good for generalized descent.

Lemma 7.16. *If $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is good for generalized descent, and $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$, then $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}$. Consequently, if $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is good for generalized descent and $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})$, then $\mathscr{O}$ is an open $\mathbf{K}$-orbit in $M(M'^{-1}(\overline{\mathscr{O}'}))$.*

Proof. This is easy to check using the explicit description of $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'}))$ in [21, Theorem 5.2 and 5.6]. ☐

7.6.4. *Map between isotropy groups.* Suppose $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ admits a descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. According to [41, Proposition 11.1] and [60, Lemmas 13 and 14], $M^{-1}(\mathscr{O})$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit contained in $\mathcal{X}^{\circ}$ and $M'(M^{-1}(\mathscr{O})) = \mathscr{O}'$. Moreover $\mathbf{K}'$ acts on $M^{-1}(\mathscr{O})$ freely.

Fix $T \in M^{-1}(\mathscr{O})$ which realizes the descent from $X := M(T) \in \mathscr{O}$ to $X' := M'(T) \in \mathscr{O}'$. Denote the respective isotropy subgroups by

$$\mathbf{S}_T := \mathrm{Stab}_{\mathbf{K}\times\mathbf{K}'}(T), \quad \mathbf{K}_X := \mathrm{Stab}_{\mathbf{K}}(X) \quad \text{and} \quad \mathbf{K}'_{X'} := \mathrm{Stab}_{\mathbf{K}'}(X').$$

Then there is a unique homomorphism

$$\alpha\colon \mathbf{K}_X \mapsto \mathbf{K}'_{X'} \tag{7.12}$$

such that $\mathbf{S}_T$ is the graph of α:

$$\mathbf{S}_T = \{\, (k, \alpha(k)) \in \mathbf{K}_X \times \mathbf{K}'_{X'} \mid k \in \mathbf{K}_X \,\}.$$

The homomorphism α is uniquely determined by the requirement that

$$\alpha(k)(Tv) = T(kv) \quad \text{ for all } v \in \mathbf{V},\ k \in \mathbf{K}_X.$$

We recall the notation in Section 7.5.2 where $\phi_{\mathfrak{k}}$ is an L-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple attached to X. Let $\mathring{H} := \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$. Then there is a unique L'-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple $\phi_{\mathfrak{k}'}$ attached to X' such that (see [25, Section 5.2])

$$\phi_{\mathfrak{k}'}(\mathring{H}) \circ T - T \circ \phi_{\mathfrak{k}}(\mathring{H}) = T.$$

As an $\mathfrak{sl}_2(\mathbb{C})$-module via $\phi_{\mathfrak{k}'}$,

$$\mathbf{V}' = \bigoplus_{l\geqslant 0}^{k-1} {}^{l}\mathbf{V}' \otimes \mathbb{C}^{l+1}.$$

We adopt notations in Section 7.5.2 to $X' \in \mathscr{O}'$, via $\phi_{\mathfrak{k}'}$. In particular, we have $\mathbf{K}_X = \mathbf{R}_X \ltimes \mathbf{U}_X$ and $\mathbf{K}'_{X'} = \mathbf{R}_{X'} \ltimes \mathbf{U}_{X'}$. Here $\mathbf{U}_X$ and $\mathbf{U}'_{X'}$ are the unipotent radicals, and $\mathbf{R}_X = \prod_{l=0}^{k} {}^{l}\mathbf{K}$ and $\mathbf{R}_{X'} = \prod_{l=0}^{k-1} {}^{l}\mathbf{K}'$ are Levi factors of $\mathbf{K}_X$ and $\mathbf{K}'_{X'}$, respectively.

For each irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}^{l+1}$ fix a nonzero vector $v_l \in (\mathbb{C}^{l+1})^{\mathring{X}}$. For each $l \geqslant 0$, the map $\nu \mapsto \nu(v_l)$ identifies ${}^{l}\mathbf{V} = \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V})$ (resp. ${}^{l}\mathbf{V}'$) with a subspace ${}^{l}\mathbf{V}_0$ of $\mathbf{V}$ (resp. ${}^{l}\mathbf{V}'_0$ of $\mathbf{V}'$). For $1 \leqslant l \leqslant k$, $T^{\star}$ induces a vector space isomorphism[6]

$$\tau_l\colon\ {}^{l-1}\mathbf{V}' = {}^{l-1}\mathbf{V}'_0 \xrightarrow{v \mapsto T^{\star}(v)} {}^{l}\mathbf{V}_0 = {}^{l}\mathbf{V}.$$

This results in an isomorphism (which is independent of the choices of v_l and v_{l-1})

$$\alpha_l : {}^{l}\mathbf{K} \xrightarrow{\cong} {}^{l-1}\mathbf{K}', \qquad h_l \mapsto (\tau_l)^{-1} \circ h_l \circ \tau_l. \tag{7.13}$$

[6]In fact, it is a similitude between the two formed spaces and satisfis $L \circ \tau_l = \mathbf{i}\, \tau_l \circ L'$.

Lemma 7.17. *The homomorphism α maps $\mathbf{R}_X$ into $\mathbf{R}_{X'}$ and maps $\mathbf{U}_X$ into $\mathbf{U}_{X'}$. Moreover, the map $\alpha|_{\mathbf{R}_X}$ is given by*

$$\begin{aligned} \alpha|_{\mathbf{R}_X}\colon \prod_{l=0}^{k} {}^{l}\mathbf{K} &\longrightarrow \prod_{l=0}^{k-1} {}^{l}\mathbf{K}', \\ (h_0, h_1, \cdots h_k) &\longmapsto (\alpha_1(h_1), \cdots, \alpha_k(h_k)), \qquad h_l \in {}^{l}\mathbf{K}, \end{aligned}$$

where α_l ($1 \leqslant l \leqslant k$) is given in (7.13).

Proof. Note that α is a surjection since $M^{-1}(X)$ is a $\mathbf{K}'$-orbit (see Lemma B.1 (ii) or [60, Lemma 13]). So $\alpha(\mathbf{U}_X)$ is a unipotent normal subgroup in $\mathbf{K}'_{X'}$ which must be contained in $\mathbf{U}_{X'}$. Note that (see [20, Lemma 3.4.4])

$$\mathbf{R}_X = \mathrm{Stab}_{\mathbf{K}}(\phi_{\mathfrak{k}}(\mathring{X})) \cap \mathrm{Stab}_{\mathbf{K}}(\phi_{\mathfrak{k}}(\mathring{H})).$$

By the defintion of α and $\phi_{\mathfrak{k}'}$, we see

$$\alpha(\mathbf{R}_X) \subset \mathrm{Stab}_{\mathbf{K}'}(\phi_{\mathfrak{k}'}(\mathring{X})) \cap \mathrm{Stab}_{\mathbf{K}'}(\phi_{\mathfrak{k}'}(\mathring{H})) = \mathbf{R}_{X'}.$$

The rest follows from the discussions before the lemma. □

Let A, A_X and $A'_{X'}$ be the component groups of G, $\mathbf{K}_X$ and $\mathbf{K}'_{X'}$ respectively. We will identify A with the component group of $\mathbf{K}$ and let $\chi|_{A_X}$ denote the pullback of $\chi \in \widehat{A}$ via the natural map $A_X \to A$. The homomorphism $\alpha\colon \mathbf{K}_X \mapsto \mathbf{K}'_{X'}$ induces a homomorphism $\alpha\colon A_X \mapsto A'_{X'}$, which further yields a homomorphism

$$\widehat{A'_{X'}} \to \widehat{A_X}, \quad \rho' \mapsto \rho' \circ \alpha.$$

Lemma 7.18. *The following map is surjective:*

$$\begin{aligned} \widehat{A} \times \widehat{A'_{X'}} &\longrightarrow\!\!\!\!\rightarrow \widehat{A_X}, \\ (\chi, \rho') &\longmapsto \chi|_{A_X} \cdot (\rho' \circ \alpha). \end{aligned}$$

Proof. This follows easily from Lemma 7.17 and Lemma 7.12. □

7.7. **Lifting of equivariant vector bundles and admissible orbit data.** Recall from the Introduction the complexification $\widetilde{\mathbf{K}}$ of $\widetilde{K}$, which is $\mathbf{K}$ except when G is a real symplectic group. For a nilpotent $\mathbf{K}$-orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, let $\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ denote the Grothendieck group of $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaves on $\mathscr{O}$. This is a free abelian group with a free basis consisting of isomorphism classes of irreducible $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Taking the isotropy representation at a point $X \in \mathscr{O}$ yields an identification

$$\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) = \mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}_X), \tag{7.14}$$

where the right hand side denotes the Grothendieck group of the category of $\mathbb{p}$-genuine algebraic representations of the stabilizer group $\widetilde{\mathbf{K}}_X$.

For $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$, let $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ denote the Grothendieck group of $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaves on $\mathcal{O} \cap \mathfrak{p}$. Since $\mathcal{O} \cap \mathfrak{p}$ is the finite union of its $\mathbf{K}$-orbits, we have

$$\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) = \bigoplus_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{7.15}$$

There is a natural partial order $\geq$ on $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ and $\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$: we say $c_1 \geq c_2$ (or $c_2 \preceq c_1$) if $c_1 - c_2$ is represented by a $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaf. The same notation obviously applies to other Grothendieck groups.

7.7.1. *An algebraic character.* Attached to a rational dual pair $(\mathbf{V}, \mathbf{V}')$, there is a distinguished character $\varsigma_{\mathbf{V},\mathbf{V}'}$ of $\widetilde{\mathbf{K}} \times \widetilde{\mathbf{K}}'$ arising from the oscillator representation, which we shall describe.

When G is a real symplectic group, let $\mathcal{X}_{\mathbf{V}}$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}}$ is identified with

$$\{\, (g, c) \in \mathrm{GL}(\mathcal{X}_{\mathbf{V}}) \times \mathbb{C}^\times \mid \det(g) = c^2 \,\}.$$

It has a character $(g, c) \mapsto c$, which is denoted by $\det_{\mathcal{X}_{\mathbf{V}}}^{\frac{1}{2}}$.

When G is a quaternionic orthogonal group, still let $\mathcal{X}_V$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}} = \mathrm{GL}(\mathcal{X}_{\mathbf{V}})$. Let $\det_{\mathcal{X}_{\mathbf{V}}}$ denote its determinant character.

Write $\mathrm{Sign}(\mathbf{V}') = (n'^+, n'^-)$. Then the character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}}$ is given by the following formula:

$$\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}} := \begin{cases} \left(\det_{\mathcal{X}_{\mathbf{V}}}^{\frac{1}{2}}\right)^{n'^+ - n'^-}, & \text{if } G \text{ is a real symplectic group;} \\ \det_{\mathcal{X}_{\mathbf{V}}}^{\frac{n'^+ - n'^-}{2}}, & \text{if } G \text{ is a quaternionic orthogonal group;} \\ \text{the trivial character,} & \text{otherwise.} \end{cases}$$

The character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'}$ is given by a similar formula with $n'^+ - n'^-$ replaced by $n^- - n^+$, where $(n^+, n^-) = \mathrm{Sign}(\mathbf{V})$.

7.7.2. *Lift of algebraic vector bundles.* In the rest of this section, we assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$. Then $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}}$ and $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'}$ are $\mathbb{p}$-genuine.

Suppose $T \in \mathcal{X}^\circ$ realizes the descent from $X = M(T) \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ to $X' = M'(T) \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Let $\alpha\colon \mathbf{K}_X \to \mathbf{K}'_{X'}$ be the homomorphism as in (7.12).

Let ρ' be a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}'_{X'}$. Then the representation $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}'}_{X'}} \otimes \rho'$ of $\widetilde{\mathbf{K}}'_{X'}$ descends to a representation of $\mathbf{K}'_{X'}$. Define

$$\vartheta_T(\rho') := \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}_X} \otimes (\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}'}_{X'}} \otimes \rho') \circ \alpha, \tag{7.16}$$

which is a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}_X$.

Clearly ϑ_T induces a homomorphism from $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}'_{X'})$ to $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}_X)$. In view of (7.14), we thus have a homomorphism

$$\vartheta_{\mathscr{O}',\mathscr{O}}\colon \mathcal{K}^{\mathbb{p}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{7.17}$$

This is independent of the choice of T.

Suppose $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}' = \nabla(\mathcal{O}) \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. Using decomposition (7.15), we define a homomorphism

$$\vartheta_{\mathcal{O}',\mathcal{O}} := \sum_{\substack{\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p} \\ \mathscr{O}' = \nabla(\mathscr{O}) \subset \mathfrak{p}'}} \vartheta_{\mathscr{O}',\mathscr{O}}\colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}'}(\widetilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) \tag{7.18}$$

where the summation is over all pairs $(\mathscr{O}, \mathscr{O}')$ such that $\mathscr{O}' \subset \mathfrak{p}'$ is the descent of $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}$.

Now suppose $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the generalized decent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. From the discussion in Section 7.6.3, there is an $(\epsilon', \dot{\epsilon}')$-space decomposition $\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$ and an element

$$T \in \mathcal{X}_1^\circ := \{w \in \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_1) \mid w \text{ is surjective}\} \cap \mathcal{X} \subseteq \mathcal{X}^{\mathrm{gen}}$$

such that $\mathscr{O}'_1 := \mathbf{K}'_1 \cdot X' \in \mathrm{Nil}_{\mathbf{K}'_1}(\mathfrak{p}'_1)$ is the descent of $\mathscr{O}$. Here $X' := M'(T) \in \mathscr{O}'$, $\mathbf{K}'_i := \mathbf{G}^{L'}_{\mathbf{V}'_i}$ is a subgroup of $\mathbf{K}'$ for $i = 1, 2$, and $\mathfrak{p}'_1 := \mathfrak{p}_{\mathbf{V}'_1}$.

Let $X := M(T) \in \mathscr{O}$ and

$$\alpha_1 \colon \mathbf{K}_X \to \mathbf{K}'_{1,X'} \tag{7.19}$$

be the (surjective) homomorphism defined in (7.12) with respect to the descent from $\mathscr{O}$ to $\mathscr{O}'_1$. Then the stabilizer $\mathbf{S}_T := \mathrm{Stab}_{\mathbf{K}\times\mathbf{K}'}(T)$ of T is given by

$$\mathbf{S}_T = \{\, (k, \alpha_1(k)k'_2) \in \mathbf{K} \times \mathbf{K}' \mid k \in \mathbf{K} \text{ and } k'_2 \in \mathbf{K}'_2 \,\}. \tag{7.20}$$

For a $\mathbb{p}$-genuine algebraic representation ρ' of $\widetilde{\mathbf{K}}'_{X'}$, define a representation

$$\vartheta^{\mathrm{gen}}_T(\rho') := \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}_X} \otimes \left((\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'_{X'}} \otimes \rho')^{\mathbf{K}'_2} \right) \circ \alpha_1. \tag{7.21}$$

Clearly, (7.21) is a generalization of (7.16), as $\mathbf{K}'_2$ is the trivial group in the latter case. As in the descent case, $\vartheta^{\mathrm{gen}}_T$ induces a homomorphism

$$\vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}'}(\mathscr{O}') \to \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}}(\mathscr{O}).$$

Furthermore, (7.18) is extended to the generalized descent case: for every pair $(\mathcal{O}, \mathcal{O}') \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g}) \times \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ with $\mathcal{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$, we define a homomorhism

$$\vartheta_{\mathcal{O}',\mathcal{O}} := \sum_{\substack{\mathscr{O}\subset\mathcal{O}\cap\mathfrak{p},\\ \mathscr{O}'=\nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})\subset\mathfrak{p}'}} \vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}'}(\mathcal{O}') \to \mathcal{K}^{\mathbb{p}}_{\widetilde{\mathbf{K}}}(\mathcal{O}).$$

7.7.3. *Lift of admissible orbit data.* Now let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$, where $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{g})$. Let $X \in \mathscr{O}$. The component group $A_X := \mathbf{K}_X/\mathbf{K}^\circ_X$ is an elementary abelian 2-group by Section 7.5.2. Let $\mathfrak{k}_X$ be the Lie algebra of $\mathbf{K}_X$. Denote by $\widetilde{\mathbf{K}}_X \to \mathbf{K}_X$ the covering map induced by the covering $\widetilde{\mathbf{K}} \to \mathbf{K}$.

We make the following definition.

Definition 7.19 ([77, Definition 7.13])**.** *Let γ_X denote the one-dimensional $\mathbf{K}_X$-module $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$ and let $\mathrm{d}\gamma_X$ be its differential.* [7] *An irreducible representation ρ of $\widetilde{\mathbf{K}}_X$ is called* admissible *if*

(i) its differential $\mathrm{d}\rho$ is isomorphic to a multiple of $\frac{1}{2}\mathrm{d}\gamma_X$, equivalently,

$$\rho(\exp(x)) = \gamma_X(\exp(x/2)) \cdot \mathrm{id}, \quad \textit{for all } x \in \mathfrak{k}_X, \textit{ and}$$

(ii) it is $\mathbb{p}$-genuine.

Let Φ_X denote the set of all isomorphism classes of admissible irreducible representations of $\widetilde{\mathbf{K}}_X$.

Definition 7.19 is obviously consistent with Definition 3.3, since a representation $\rho \in \Phi_X$ determines an admissible orbit datum $\mathcal{E} \in \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$, where $\mathcal{E}$ is a $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundle on $\mathscr{O}$ whose isotropy representation $\mathcal{E}_X$ at X is isomorphic to ρ. We therefore have an identification

$$\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) = \Phi_X. \tag{7.22}$$

From the structure of $\mathbf{K}_X$ in Lemma 7.11 and Lemma 7.12, it is easy to see that Φ_X consists of one-dimensional representations.

Lemma 7.20. *The tensor product yields a simply transitive action of the character group $\widehat{A_X}$ on the set Φ_X.*

[7] $\mathrm{d}\gamma_X$ is the same as $\mathrm{d}\gamma_{\mathfrak{k}}$ in [77, Theorem 7.11] since $\mathfrak{k}$ is reductive.

Proof. It is clear that we only need to show that Φ_X is nonempty. Consider a rational dual pair $(\mathbf{V}, \mathbf{V}')$ such that $\mathscr{O}' = \nabla(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the descent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. Suppose that $T \in \mathcal{X}^\circ$ realizes the descent from X to $X' \in \mathscr{O}'$. The proof of [48, Proposition 6.1] shows that if $\rho' \in \Phi_{X'}$ is admissible, then $\vartheta_T(\rho')$ is admissible. Therefore, we may do reductions and eventually reduce the problem to the case when $\mathbf{V}$ is the zero space. It is clear that Φ_X is a singleton in this case. □

Remark. By Lemma 7.20, the set $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ of admissible orbit data over $\mathscr{O}$ has 2^r elements, where r is the number of orthogonal groups appearing in the decomposition of $\mathbf{R}_X$ in Lemma 7.11.

As in the proof of Lemma 7.20, the homomorphism (7.17) restricts to a map

$$\vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \to \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{7.23}$$

Lemma 7.21. *Let A be the component group of G which is identified with the component group of $\mathbf{K}$. Then the following map is surjective:*[8]

$$\begin{aligned} \widehat{A} \times \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') &\longrightarrow \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}), \\ (\chi, \mathcal{E}') &\longmapsto \chi \otimes \vartheta_{\mathscr{O}',\mathscr{O}}(\mathcal{E}'). \end{aligned}$$

In particular, $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ is a singleton if G is a quaternionic group.

Proof. This follows from Lemma 7.18 and Lemma 7.20. □

8. Combinatoral Convention on the local system

An irreducible local system is represented by a *marked Young diagram.* A general local system is understand as the union of irreducible ones. By our computation later, it would be clear that the local systems appeared in our cases are always *multiplicity one.*

8.1. **Commutative free monoid.** Given a set A, the commutative free monoid $\mathsf{C}(\mathbb{A})$ on $\mathbf{A}$ consists all finite multisets with elements drawn from A. It is equipped with the binary operation, say $\bullet$ such that

$$\{ a_1, \cdots, a_{k_1} \} \bullet \{ b_1, \cdots, b_{k_2} \} := \{ a_1, \cdots, a_{k_1}, b_1, \cdots, b_{k_2} \}.$$

In the following, we always identify A with the subset of $\mathsf{C}(\mathsf{A})$ consists of singletons.

Suppose M is a commutative monoid, a map $f \colon \mathsf{A} \to \mathsf{M}$ is uniquelly extends to a morphism $\mathsf{C}(\mathsf{A}) \to \mathsf{M}$ which is still called f by abuse of notations.

In particular a map $f \colon \mathsf{A} \to \mathsf{B}$ between sets uniquely extends to an morphism $f \colon \mathsf{C}(\mathsf{A}) \to \mathsf{C}(\mathsf{B})$ by

$$f(\{ a_1, \cdots, a_k \}) = \{ f(a_1), \cdots, f(a_k) \}.$$

8.2. **Marked Young diagram.** We now explain the combinatiorial objects parameterizing local systems. Basically ,we use $+/-$ (resp. $*/=$) to denote the associated character on the corresponding factor of the isotropic group. Let $\star = B, C, D, M$.

- **Young diagram** Let YD be the commutative free monoid $\mathsf{C}(\mathbb{N}^+)$ on the set of positive integers $\mathbb{Z}_{\geqslant 1}$. An element $\mathcal{O} = \{ R_1, \cdots, R_k \}$ is identified with the Young diagram whose set of row lengths is $\mathcal{O}$. Therefore, we identify YD with the set of Young diagrams. Let
$$| \ | \colon \mathsf{YD} \longrightarrow \mathbb{N}$$

[8] Under the identification (7.22), the tensor product of a character $\chi \in \widehat{A}$ on $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ is identified with the tensor product of $\chi|_{A_X}$ with the isotropy representation.

be the map sending $\{ R_1, \cdots R_k \}$ to $\sum_{i=1}^{k} R_i$.

Let $\mathsf{YD}(\star)$ to denote the Young diagrams corresponding to partitions of type $\star$. The set $\mathsf{YD}(\star)$ parameterizes the set of nilpotent orbits of type $\star$. Note that $\mathsf{YD}(C) = \mathsf{YD}(M)$.

- **Signed Young diagram** Let S be the set of non-empty finite length strings of alternating signs of the form $+-+\cdots$ or $-+-\cdots$. Let $\mathsf{SYD} := \mathsf{C}(\mathsf{S})$ be the monoid on S.

  The monoid SYD is identified with the set of all signed Young diagrams: $\mathscr{O} = \{ \mathbf{r}_1, \cdots, \mathbf{r}_k \} \in \mathsf{SYD}$ is identified with the signed Young diagram whose rows are filled by $\mathbf{r}_j$. The map $\mathsf{S} \to \mathbb{Z}_{\geqslant 1}$ sending a string to its length extends to the morphism

$$\begin{aligned} \mathcal{Y} \colon \mathsf{SYD} &\longrightarrow \mathsf{YD} \\ \{ \mathbf{r}_1, \cdots, \mathbf{r}_k \} &\longmapsto \{ |\mathbf{r}_1|, \cdots, |\mathbf{r}_k| \} \end{aligned}$$

  where $|\mathbf{r}_j|$ denote the length of the string $\mathbf{r}_j$. By abused of notation, we also denote $|\ \ | \circ \mathcal{Y}$ by $|\ \ |$.

  We define $\mathsf{SYD}(\star)$ to be a subset of the preimage of $\mathsf{YD}(\star)$ under the above map: When $\star = C, M$ (resp. $\star = D, B$),

$$\mathsf{SYD}(\star) = \left\{ \mathscr{O} \in \mathsf{SYD} \;\middle|\; \begin{array}{l} \mathcal{Y}(\mathscr{O}) \in \mathsf{YD}(\star) \\ \text{For each positive odd number (resp.} \\ \text{even number) } r\text{, the strings } +-\cdots \\ \text{and } -+\cdots \text{ of the length } r \text{ ap-} \\ \text{pears in } \mathscr{O} \text{ with the same multiplic-} \\ \text{ity (could be 0).} \end{array} \right\}$$

  The set $\mathsf{SYD}(\star)$ parameterizes the real nilpotent orbits of type $\star$ and $\mathbf{K}$-orbits in $\mathfrak{p}$.

  We define two signature maps $\mathsf{S} \to \mathbb{N} \times \mathbb{N}$ as the following. For $\mathbf{r} \in \mathsf{S}$, define

$$\begin{aligned} \mathrm{Sign}(\mathbf{r}) &:= (\# + (\mathbf{r}), \# - (\mathbf{r})) \quad \text{and} \\ {}^{l}\mathrm{Sign}(\mathbf{r}) &:= \begin{cases} (1,0) & \text{if } \mathbf{r} = +\cdots \\ (0,1) & \text{if } \mathbf{r} = -\cdots \end{cases}. \end{aligned}$$

  The above defined signature maps naturally extends to signature maps $\mathrm{Sign} \colon \mathsf{SYD} \to \mathbb{N} \times \mathbb{N}$ and $\mathrm{Sign} \colon \mathsf{SYD} \to \mathbb{N} \times \mathbb{N}$.

- **Marked Young diagram** Let M be the set of finite length strings of alternating marks of the form $+-+\cdots$,$-+-\cdots$, $\ast{=}\ast\cdots$, or $={\ast}=\cdots$.

  Let $\mathsf{MYD} := \mathsf{C}(\mathsf{M})$ which is identified with the set of Young diagrams marked with alternating marks $+/-$, $\ast/=$. Define $\mathscr{F} \colon \mathsf{M} \to \mathsf{S}$ by the formula

$$\mathscr{F}(\mathbf{r}) := \begin{cases} +-+\cdots & \text{if } \mathbf{r} = +-+\cdots \text{ or } \ast{=}\ast\cdots \\ -+-\cdots & \text{if } \mathbf{r} = -+-\cdots \text{ or } ={\ast}=\cdots \end{cases} \qquad \forall \mathbf{r} \in \mathsf{M}$$

  such that $\mathscr{F}(\mathbf{r})$ and $\mathbf{r}$ have the same length. $\mathscr{F}$ extends to the map $\mathscr{F} : \mathsf{MYD} \to \mathsf{SYD}$.

  By abused of notation, we also denote $|\ \ | \circ \mathcal{Y}$ by $|\ \ |$.

  For $\mathcal{L} \in \mathsf{MYD}$, we define

$$\mathrm{Sign}(\mathcal{L}) = \mathrm{Sign}(\mathscr{F}(\mathcal{L})) \quad \text{and} \quad {}^{l}\mathrm{Sign}(\mathcal{L}) = {}^{l}\mathrm{Sign}(\mathscr{F}(\mathcal{L})).$$

We define $\mathsf{MYD}(\star)$ to be a subset of the preimage of the above map:

$$\mathsf{MYD}(\star) = \left\{ \mathcal{L} \in \mathsf{MYD}(\star) \;\middle|\; \begin{array}{l} \mathscr{F}(\mathcal{L}) \in \mathsf{SYD}(\star); \\ \mathbf{r}_1 = \mathbf{r}_2 \text{ if } \mathbf{r}_1, \mathbf{r}_2 \in \mathcal{L} \text{ and } \mathscr{F}(\mathbf{r}_1) = \mathscr{F}(\mathbf{r}_2); \\ \mathbf{r} = + - + \cdots \text{ or } - + - \cdots \text{ if the length} \\ \text{of } \mathbf{r} \text{ is odd (resp. even) when } \star \in \{ C, M \} \\ \text{(resp. } \star \in \{ B, D \}). \end{array} \right\}$$

We will use $\mathsf{MYD}(\star)$ to parameterize irreducible local systems attached to the nilpotent orbits of type $\star$.

For monoid YD, SYD or MYD, the identity element is the empty set $\varnothing$, and the monoid binary operator is denoted by "$\cdot$". The $\varnothing$ is always in $\mathsf{MYD}(\star)$ by convention. According to the definition of $\mathsf{MYD}(\star)$. it could happen that $\mathcal{L}_1 \cdot \mathcal{L}_2 \notin \mathsf{MYD}(\star)$ for $\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{MYD}(\star)$.

We define an involution

$$\overline{} \colon \mathsf{M} \to \mathsf{M} \tag{8.1}$$

on M by switching the symbols $+$ with $\ast$ and $-$ with $=$. For example $\overline{+-+} = \ast{=}\ast$ and $\overline{=\ast} = -+$.

We define the map

$$\dagger \colon \mathsf{M} \to \mathsf{M} \tag{8.2}$$

by extending the length of the strings to the left by one. For example $\dagger(+-+) = -+-+$ and $\dagger(=\ast) = \ast{=}\ast$.

For $\mathcal{L}, \mathcal{C} \in \mathsf{MYD}$, the expression $\mathcal{L} \succ \mathcal{C}$ means that there is $\mathcal{B} \in \mathsf{MYD}$ such that the factorization $\mathcal{L} = \mathcal{B} \cdot \mathcal{C}$ holds.

In the following, we will represent an element $\mathcal{L} \in \mathsf{MYD}$ by the Young diagram whose rows are filled with the strings in $\mathcal{L}$. The terminology "l-row" means a row of length l. If we do not care whether the mark is $\ast$ or $+$ (resp. $=$ or $=$), we will use $\divideontimes$ (resp. $\dot{-}$) to mark the corresponding position.

Example 8.1. *The following two diagrams $\ddagger_{p,q}$ and $\dagger_{p,q}$ are in* $\mathsf{MYD}(B) \cup \mathsf{MYD}(D)$.

$$\ddagger_{p,q} := \begin{array}{|c|} \hline + \\ \hline \vdots \\ \hline + \\ \hline = \\ \hline \vdots \\ \hline = \\ \hline \end{array}$$

represents the local system on the trivial orbit of $\mathrm{O}(p,q)$ *which has the trivial character on* $\mathrm{O}(p)$ *and* $\det$ *on* $\mathrm{O}(q)$. *Similarly,*

$$\dagger_{p,q} := \begin{array}{|c|} \hline + \\ \hline \vdots \\ \hline + \\ \hline - \\ \hline \vdots \\ \hline - \\ \hline \end{array}$$

represents the trivial local system on the trivial orbit.

8.2.1. *Local systems.* We define $\mathcal{K}_{\mathsf{M}}$ to be the free commutative monoid generated by MYD. The binary operation in $\mathcal{K}_{\mathsf{M}}$ is denoted by "$+$".

The operation "$\cdot$" naturally extends to $\mathcal{K}_{\mathsf{M}}$:

$$\Big(\sum_i m_i \mathcal{L}_i\Big) \cdot \Big(\sum_j m_j \mathcal{L}'_j\Big) := \sum_{i,j} m_i m_j (\mathcal{L}_i \cdot \mathcal{L}'_j).$$

where $\mathcal{L}_i, \mathcal{L}'_j \in \mathsf{MYD}$, $m_i, m_j \in \mathbb{Z}_{\geqslant 0}$.

Now $(\mathcal{K}_{\mathsf{M}}, +, \cdot)$ is in fact a commutative semiring. We let 0 be the additive identity element in $\mathcal{K}_{\mathsf{M}}$ and $\varnothing$ is the multiplicative identify. Note that $\mathcal{L} \cdot 0 = 0$ for any $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$ by definition.

For $\mathcal{L}, \mathcal{D} \in \mathcal{K}_{\mathsf{M}}$, $\mathcal{L} \succ \mathcal{D}$ means that there exists $\mathcal{B} \in \mathcal{K}_{\mathsf{M}}$ such that $\mathcal{L} = \mathcal{B} \cdot \mathcal{D}$. In addition, we write $\mathcal{L} \supset \mathcal{D}$ if there is $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{K}_{\mathsf{M}}$ such that $\mathcal{L}_1 \succ \mathcal{D}$ and $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}$.

Let $\mathcal{K}_{\mathsf{M}}(\star)$ be the sub-monoid of $\mathcal{K}_{\mathsf{M}}$ generated by $\mathsf{MYD}(\star)$. Note that $\mathcal{K}_{\mathsf{M}}(\star)$ is not a sub semi-ring of $\mathcal{K}_{\mathsf{M}}$.

For $\mathcal{L} = \sum_{i=1}^{k} \mathcal{L}_i \in \mathcal{K}_{\mathsf{M}}$ with $\mathcal{L}_i \in \mathsf{MYD}$, we let

$$\mathrm{Sign}(\mathcal{L}) = \{\, \mathrm{Sign}(\mathcal{L}_i) \mid i = 1, \cdots, k \,\} \text{ and } {}^{l}\mathrm{Sign}(\mathcal{L}) = \left\{\, {}^{l}\mathrm{Sign}(\mathcal{L}_i) \;\middle|\; i = 1, \cdots, k \,\right\}$$

8.3. **Operations on $\mathcal{K}_{\mathsf{M}}(\star)$.** The operation $\dagger$ defined in (8.2) extends to an operation

$$\dagger \colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$$

on $\mathcal{K}_{\mathsf{M}}$ which adding a column on the left of the original marked Young diagram.

8.3.1. *Character twists on $\mathcal{K}_{\mathsf{M}}(D)$ and $\mathcal{K}_{\mathsf{M}}(B)$.* Recall that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ paramterizes the characters of a indefinite orthogonal group $\mathrm{O}(p, q)$:

$$\chi = (\chi^+, \chi^-) \longleftrightarrow (\mathbf{1}^{-,+})^{\chi^+} \otimes (\mathbf{1}^{+,-})^{\chi^-}.$$

We define the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on M by:

$$\mathbf{r} \otimes \chi := \begin{cases} \bar{\mathbf{r}} & \text{if } p + q \equiv 1 \text{ and } \chi^+ p + \chi^- q \equiv 1 \pmod 2 \\ \mathbf{r} & \text{otherwise} \end{cases}$$

where $\mathbf{r} \in \mathsf{M}$, $\chi = (\chi^+, \chi^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $(p, q) = \mathrm{Sign}(\mathbf{r})$.

The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action extends to actions on $\mathcal{K}_{\mathsf{M}}(D)$ and $\mathcal{K}_{\mathsf{M}}(B)$.

8.3.2. *Twists on $\mathcal{K}_{\mathsf{M}}(C)$ and $\mathcal{K}_{\mathsf{M}}(M)$.* We define an involution $\maltese_C \colon \mathsf{M} \to \mathsf{M}$ on M by

$$\maltese(\mathbf{r}) := \begin{cases} \bar{\mathbf{r}} & \text{if } |\mathbf{r}| \equiv 2 \pmod 4 \\ \mathbf{r} & \text{otherwise.} \end{cases}$$

The involution $\maltese$ extends to an involution on $\mathcal{K}_{\mathsf{M}}$.We will only apply $\maltese$ on $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}(C)$ or $\mathcal{K}_{\mathsf{M}}(M)$.

8.3.3. *Truncations.* We define the Truncation maps

$$\mathsf{T}^+ \colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$$

such that for $\mathcal{L} \in \mathsf{MYD}$

$$\mathsf{T}^+(\mathcal{L}) = \begin{cases} \mathcal{L}_1 & \text{if there is } \mathcal{L}_1 \in \mathsf{MYD} \text{ such that } \mathcal{L}_1 \cdot + = \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathsf{T}^- \colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$ similarly such that $\mathcal{L} = \mathsf{T}^-(\mathcal{L}) \cdot -$ if $\mathcal{L} \succ -$ and $\mathsf{T}^-(\mathcal{L}) = 0$ otherwise for $\mathcal{L} \in \mathsf{MYD}$.

To symplify the notation, we write $\mathscr{L}^{\pm} := \mathsf{T}^{\pm}(\mathscr{L})$ for $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$.

8.4. **Theta lifts of local systems.**

8.4.1. *Lift from type C to D.* For each signature $(p,q)\in\mathbb{N}\times\mathbb{N}$ such that $p+q$ is even, we define a map

$$\vartheta_{CD,(p,q)}\colon \mathcal{K}_{\mathsf{M}}(C)\to\mathcal{K}_{\mathsf{M}}(D)$$

as the following: Let $\mathcal{L}\in\mathsf{MYD}(C)$. Let $(p_1,q_1)={}^l\mathrm{Sign}(\mathcal{L})$ and

$$(p_0,q_0)=(p,q)-\mathrm{Sign}(\mathcal{L})-(q_1,p_1).$$

We define

$$\vartheta_{CD,(p,q)}(\mathcal{L})=\begin{cases}(\dagger\maltese^{\frac{p-q}{2}}(\mathcal{L}))\cdot\dagger_{p_0,q_0} & \text{if } p_0\geqslant 0 \text{ and } q_0\geqslant 0\\ 0 & \text{otherwise}\end{cases}$$

8.4.2. *Lift from type D to type C.* For each non-zero integer $n\in\mathbb{N}$, we define a map

$$\vartheta_{DC,n}\colon \mathcal{K}_{\mathsf{M}}(D)\to\mathcal{K}_{\mathsf{M}}(C)$$

as the following: Let $\mathcal{L}\in\mathsf{MYD}(D)$. Let $(p_1,q_1)={}^l\mathrm{Sign}(\mathcal{L})$ and $(p,q)=\mathrm{Sign}(\mathcal{L})$. Let

$$n_0=n-(p+q)/2-(p_1+q_1)/2$$

We define

$$\vartheta_{DC,n}(\mathcal{L})=\begin{cases}\maltese^{\frac{p-q}{2}}((\dagger\mathcal{L})\cdot\dagger_{n_0,n_0}) & \text{if } n_0\in\mathbb{Z}_{\geqslant 0}\\ \maltese^{\frac{p-q}{2}}(\dagger\mathcal{L}^+ + \dagger\mathcal{L}^-) & \text{if } n_0=-1\\ 0 & \text{otherwise}\end{cases}$$

We remark that $\maltese^{\frac{p-q}{2}}((\dagger\mathcal{L})\cdot\dagger_{n_0,n_0})=(\maltese^{\frac{p-q}{2}}(\dagger\mathcal{L}))\cdot\dagger_{n_0,n_0}$.

[We take a splitting $\mathrm{O}(p,q)\times\mathrm{Sp}(n,\mathbb{R})$ in to the big metaplectic group Mp such that $\mathrm{O}(p,\mathbb{C})\times\mathrm{O}(q,\mathbb{C})$ acts on the Fock model linearly. The maximal compact $K_{\mathrm{Sp}(n,\mathbb{R})}=\mathrm{U}(2n)$ acts on the Fock model by the character $\zeta=\det^{(p-q)/2}$. For the component of the rational nilpotent orbit $\mathrm{Sp}(2n,\mathbb{R})$, the factor of the component group is Sp if the corresponding row has odd length. Otherwise the component group is $\mathrm{O}(p_{2k})\times\mathrm{O}(q_{2k})$ where (p_{2k},q_{2k}) is the signature corresponding to the $2k$-rows. $\zeta|_{\mathrm{O}(p_{2k})}=\det^{(p-q)k/2}_{\mathrm{O}(p_{2k})}$ which is nontrivial if and only if $p-q\equiv 2 \pmod 4$ and $2k\equiv 2\pmod 4$. This gives the formula above.]

In the proof later, we don't care above the marks for rows with length longer than 1 in the most of the case. So we will omit $\maltese$ sometimes. When there is no confusion, we will simply write ϑ for $\vartheta_{CD,(p,q)}$ and $\vartheta_{CD,(p,q)}$.

8.4.3. *Lift from type M to B.* For each signature $(p,q)\in\mathbb{N}\times\mathbb{N}$ such that $p+q$ is odd, we define a map

$$\vartheta_{MB,(p,q)}\colon \mathcal{K}_{\mathsf{M}}(M)\to\mathcal{K}_{\mathsf{M}}(B)$$

as the following: Let $\mathcal{L}\in\mathsf{MYD}(M)$. Let $(p_1,q_1)={}^l\mathrm{Sign}(\mathcal{L})$ and

$$(p_0,q_0)=(p,q)-\mathrm{Sign}(\mathcal{L})-(q_1,p_1).$$

We define

$$\vartheta_{MB,(p,q)}(\mathcal{L})=\begin{cases}(\dagger\maltese^{\frac{p-q+1}{2}}(\mathcal{L}))\cdot\dagger_{p_0,q_0} & \text{if } p_0\geqslant 0 \text{ and } q_0\geqslant 0\\ 0 & \text{otherwise}\end{cases}$$

8.4.4. *Lift from type B to type M.* For each non-zero integer $n \in \mathbb{N}$, we define a map

$$\vartheta_{BM,n} \colon \mathcal{K}_{\mathsf{M}}(B) \to \mathcal{K}_{\mathsf{M}}(M)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(D)$. Let $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and $(p, q) = \mathrm{Sign}(\mathcal{L})$. Let

$$n_0 = n - (p+q)/2 - (p_1 + q_1)/2$$

We define

$$\vartheta_{BM,n}(\mathcal{L}) = \begin{cases} \maltese^{\frac{p-q-1}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geqslant 0} \\ \maltese^{\frac{p-q-1}{2}}(\dagger\mathcal{L}^+ + \dagger\mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that $\maltese^{\frac{p-q-1}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) = (\maltese^{\frac{p-q-1}{2}}(\dagger\mathcal{L})) \cdot \dagger_{n_0,n_0}$.

[For odd orthogonal-metaplectic group case, we still take the splitting such that $\mathrm{O}(p, q)$ acts linearly.

The maximal compact $\widetilde{K}_{\mathrm{Sp}(n,\mathbb{R})} = \widetilde{\mathrm{U}(2n)}$ acts on the Fock model by the character $\zeta = \det^{(p-q)/2}$.

Then $\det^{(p-q)/2}|_{\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})} = \det_{\mathrm{O}(p_{2k})}^{\frac{(p-q)k}{2}} \boxtimes \det_{\mathrm{O}(q_{2k})}^{\frac{(p-q)k}{2}}$. Note that $p - q$ is an odd number. When k is even the character on $\widetilde{\mathrm{O}(p_{2k})}/\widetilde{\mathrm{O}(q_{2k})}$ are $\mathbf{1}$ or det. When k is odd the character would be $\det^{\frac{1}{2}}$ or $\det^{\frac{1}{2}+1}$.

We assume the default character on $2k$-rows is $\det^{\frac{k}{2}}$, and we use $+/-$ to mark the row. Otherwise, we use $\ast/=$ to mark the rows.]

8.5. **Examples and remarks.**

8.5.1. *Example.*

Example 8.2. *Let* $\mathcal{T} := \dagger\dagger(\dagger_{2,1}) + \dagger\dagger(\dagger_{1,2})$ *and* $\mathcal{P} = \ddagger_{1,3}$. *Then*

$$\mathcal{T} \cdot \mathcal{P} = \begin{array}{|c|c|c|} \hline \divideontimes & \dot{-} & \divideontimes \\ \hline \divideontimes & \dot{-} & \divideontimes \\ \hline \dot{-} & \divideontimes & \dot{-} \\ \hline + \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} \end{array} \cup \begin{array}{|c|c|c|} \hline \divideontimes & \dot{-} & \divideontimes \\ \hline \dot{-} & \divideontimes & \dot{-} \\ \hline \dot{-} & \divideontimes & \dot{-} \\ \hline + \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} \end{array}.$$

8.5.2. *Signs of the local systems obtained by iterated lifting.* Suppose $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$ is obtained by iterated theta lifting and character twisting in Section 8.3.1. Then the set ${}^l\mathrm{Sign}(\mathcal{L})$ has restricted possibilities. First, $\mathrm{Sign}(\mathcal{L})$ is always a singleton.

When $\mathcal{O}$ is of type B/D, ${}^l\mathrm{Sign}(\mathcal{L})$ is a singleton $\{ (p_1, q_1) \}$ where

$$(p_1, q_1) = \mathrm{Sign}(\mathcal{L}) - (n_0, n_0) \quad \text{and} \quad 2n_0 = \left|\overline{\nabla}(\mathcal{O})\right|.$$

When $\mathcal{O}$ is of type C/M, ${}^l\mathrm{Sign}(\mathcal{L}) = \{ (p_1, q_1) \}$ is a singleton in the most of the case. The ${}^l\mathrm{Sign}(\mathcal{L})$ may have two elements if $\mathcal{L}$ is obtained by good generalized lifting, and it has the form ${}^l\mathrm{Sign}(\mathcal{L}) = \{ (p_1 + 1, q_1), (p_1, q_1 + 1) \}$.

9. Proof of main theorem in the type C/D case

We do induction on the number of columns of the nilpotent orbits.

9.1. **Notation.** In this section, $\mathcal{O}$ is always an nilpotent orbit of type D in $\mathrm{Nil}^{\mathrm{dpe}}(D)$. We always let

$$\begin{aligned}\mathcal{O} &= (C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0),\\ \mathcal{O}' &= \overline{\nabla}(\mathcal{O}) = (C_{2k}, C_{2k-1}, \cdots, C_1, C_0).\end{aligned}$$

When $k \geqslant 1$, we let

$$\mathcal{O}'' = \overline{\nabla}^2(\mathcal{O}) = \begin{cases}(C_{2k-1}, \cdots, C_1, C_0) & \text{when } C_{2k} \text{ is even,}\\ (C_{2k-1}+1, \cdots, C_1, C_0) & \text{when } C_{2k} \text{ is odd.}\end{cases}$$

For a $\tau \in \mathrm{DRC}(\mathcal{O})$, we let

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \quad \text{and} \quad (\tau'', \epsilon') = \overline{\nabla}(\tau') \text{ (when } k \geqslant 1).$$

9.1.1. *Main proposition.* The aim of this section is to prove the following main proposition holds for $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(D)$.

Proposition 9.1. *We have:*

i) $\mathcal{L}_\tau$ *is non-zero. In particular,* $\pi_\tau \neq 0$.
ii) Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$*, then* $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.
iii) Suppose $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$*, then* $\pi_{\tau_1} \neq \pi_{\tau_2}$.
iv) The local system $\mathcal{L}_\tau$ *is disjoint with their determinant twist:*

$$\{\, \mathcal{L}_\tau \mid \tau \in \mathrm{DRC}(\mathcal{O}) \,\} \cap \{\, \mathcal{L}_\tau \otimes \det \mid \tau \in \mathrm{DRC}(\mathcal{O}) \,\} = \varnothing. \tag{9.1}$$

v) We have
(i) $x_\tau = s$*, then* $\mathcal{L}_\tau^+ = \mathcal{L}_\tau^- = 0$.
(ii) $x_\tau = r/c$*, then* $\mathcal{L}_\tau^+ \neq 0$ *and* $\mathcal{L}_\tau^- = 0$.
(iii) $x_\tau = d$*, then* $\mathcal{L}_\tau^+ \neq 0$ *and* $\mathcal{L}_\tau^- \neq 0$.
vi) If $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ *is a usual descent, we have a simpler factorization:*

$$\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_{\mathbf{x}_\tau}. \tag{9.2}$$

vii) When $\mathcal{O}'$ *is noticed, the map* $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}') \longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O}')$ *is a bijection.*
viii) When $\mathcal{O}$ *is noticed, the map* $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}) \longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O})$ *is a bijection.*
ix) When $\mathcal{O}$ *is +noticed, the maps*

$$\begin{aligned}\Upsilon^+_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau^+ \neq 0 \,\} &\longrightarrow \{\, \mathcal{L}_\tau^+ \,\}\\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^+ \qquad\qquad \textit{and}\\ \Upsilon^-_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau^- \neq 0 \,\} &\longrightarrow \{\, \mathcal{L}_\tau^- \,\}\\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^-\end{aligned}$$

are injective.
x) When $\mathcal{O}$ *is noticed, the map*

$$\begin{aligned}\Upsilon_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau^+ \neq 0 \,\} &\longrightarrow \{\, (\mathcal{L}_\tau^+, \mathcal{L}_\tau^-) \mid \mathcal{L}_\tau^+ \neq 0 \,\}\\ \mathcal{L}_\tau &\longmapsto (\mathcal{L}_\tau^+, \mathcal{L}_\tau^-)\end{aligned} \tag{9.3}$$

is injective.[9]

[9]In fact, we only need to know whether $\mathcal{L}_\tau^-$ is non-zero or not.

Note that, $\mathrm{DRC}(\mathcal{O})$ counts the unipotent representations of special orthogonal groups a priori. But Proposition 9.1 (iv)), implies that the representation π_τ and $\pi_\tau \otimes \det$ are two different representations of the orthogonal groups. Therefore $\mathrm{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z}$ counts the set of unipotent representations of orthogonal groups.

The main theorem would be a consequence of Proposition 9.1.

9.1.2. *How to think about the factorization of local systems.* When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is the usual descent, the local system $\mathcal{L}_\tau$ could be compared to a water hydra, see in Figure 1.

The tentacles part $\mathcal{T}_\tau = \dagger\mathcal{L}_{\tau'}$ could be reducible. The peduncle part $\mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}$ and $\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau$.

When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a generalized descent, the situation is more complicated as the local system $\mathcal{L}_\tau$ may consists of two hydras, see (9.8).

9.2. **The initial case:** When $k = 0$, $\mathcal{O} = (C_1 = 2c_1, C_0 = 2c_0)$ has at most two columns. Now $\pi_{\tau'}$ is the trivial representation of $\mathrm{Sp}(2c_0, \mathbb{R})$. The lift $\pi_{\tau'} \mapsto \bar{\Theta}(\pi_{\tau'})$ is in fact a stable range theta lift.For $\tau \in \mathrm{DRC}(\mathcal{O})$, let $(p_1, q_1) = \mathrm{Sign}(\mathbf{x}_\tau)$ and x_τ be the foot of τ. It is easy to see that (using associated character formula)

$$\mathcal{L}_{\pi_\tau} = \mathcal{T}_\tau \cdot \mathcal{P}_\tau, \text{ where } \mathcal{T}_\tau := \dagger\dagger_{c_0,c_0}, \text{ and } \mathcal{P}_\tau := \begin{cases} \ddagger_{p_1,q_1} & x_\tau \neq d, \\ \dagger_{p_1,q_1} & x_\tau = d. \end{cases}$$

Note that $\mathcal{L}_{\pi_\tau}$ is irreducible.

Let $\mathcal{O}_1 = (2(c_1 - c_0)) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. Using the above formula, one can check that the following maps are bijections

$$\begin{array}{ccccc} \mathrm{DRC}(\mathcal{O}_1) & \longleftarrow & \mathrm{DRC}(\mathcal{O}) & \longrightarrow & {}^{\ell}\mathrm{LS}(\mathcal{O}) \\ \mathbf{x}_\tau & \longleftarrow\!\!\!\shortmid & \tau & \longmapsto & \mathcal{L}_\tau. \end{array}$$

To see that $\mathrm{DRC}(\mathcal{O}) \ni \tau \mapsto \mathcal{L}_\tau$ is an injection, one check the following lemma holds:

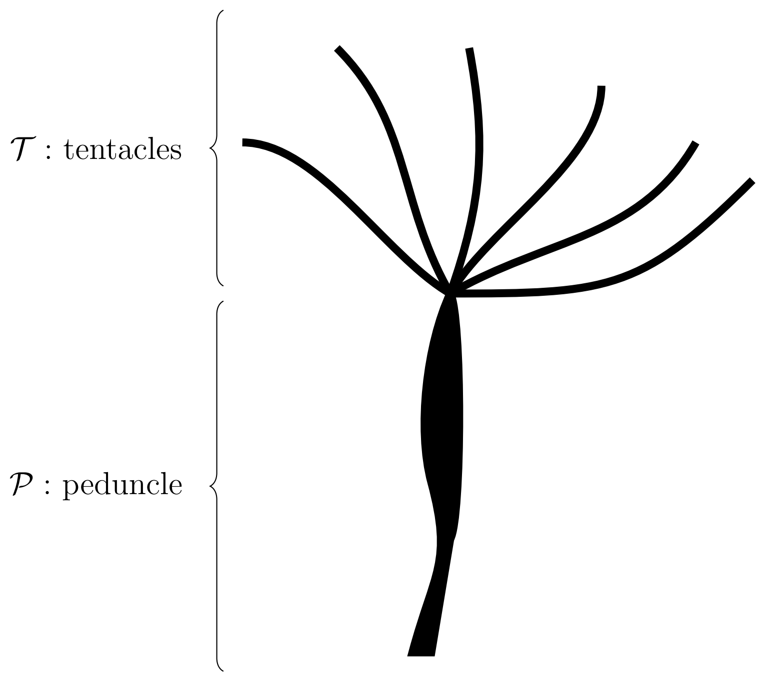


FIGURE 1. Freshwater hydra

Lemma 9.2. *The map* $\mathrm{DRC}(\mathcal{O}_1) \longrightarrow \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective (see Section 6.2.4 for the definition of* ϵ_τ*). Moreover,* $\epsilon_\tau = 0$ *only if* $\mathrm{Sign}(\tau) \geqslant (0,1)$.

Moreover, $\mathcal{L}_\tau \otimes \det \notin {}^{\ell}\mathrm{LS}(\mathcal{O})$ for any $\tau \in \mathrm{DRC}(\mathcal{O})$. In fact, if $\mathrm{Sign}(\mathbf{x}_\tau) \succeq (1,0)$, $\mathcal{L}_\tau$ on the 1-rows with +-signs is trivial and $\mathcal{L}_\tau \otimes \det$ has non-trivial restriction. When $\mathrm{Sign}(\mathbf{x}_\tau) \not\succeq (1,0)$, $\mathbf{x}_\tau = s \cdots s$, all 1-rows of $\mathcal{L}_\tau$ are marked by "=" and all 1-rows of $\mathcal{L}_\tau \otimes \det$ are marked by "$-$".

Therefore our main Theorem 4.4 and main proposition Proposition 9.1 holds for $\mathcal{O}$.

9.3. **The descent case.** Now we assume $k \geqslant 1$ and C_{2k} is even. In this case $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a usual descent.

9.3.1. *Unipotent representations attached to* $\mathcal{O}'$. Therefore, $\mathrm{LS}(\mathcal{O}'') \longrightarrow \mathrm{LS}(\mathcal{O}')$ given by $\mathcal{L} \mapsto \vartheta(\mathcal{L})$ is an injection between abelian groups.

For $\mathcal{L}'' \in {}^{\ell}\mathrm{LS}(\mathcal{O}'') \sqcup {}^{\ell}\mathrm{LS}(\mathcal{O}'') \otimes \det$, let $(p_1, q_1) = {}^{l}\mathrm{Sign}(\mathcal{L})$ and $(p_2, q_2) = (C_{2k} - q_1, C_{2k} - p_1)$. Then

$$\vartheta(\mathcal{L}'') = \dagger\mathcal{L}'' \cdot \dagger_{p_2,q_2}.$$

By (9.1), we have a injection

$$\begin{aligned} {}^{\ell}\mathrm{LS}(\mathcal{O}'') \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O}') \\ (\mathcal{L}'', \epsilon') &\longmapsto \vartheta(\mathcal{L}'' \otimes \det{}^{\epsilon'}). \end{aligned} \tag{9.4}$$

In particular, we have $\mathcal{L}_{\tau'} \neq 0$ and $\pi_{\tau'} \neq 0$ for every $\tau' \in \mathrm{DRC}(\mathcal{O}')$.

9.3.2. *Unipotent representations attached to* $\mathcal{O}$. Now suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (9.4), $\epsilon'_1 = \epsilon'_2$ and $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$. In particular, τ''_1 and τ''_2 have the same signature. By Lemma 6.4 and the induction hypothesis, $\tau''_1 \neq \tau''_2$ and so $\pi_{\tau''_1} \otimes \det^{\epsilon'_1} \neq \pi_{\tau''_2} \otimes \det^{\epsilon'_2}$. Now the injectivity of theta lifting yields

$$\pi_{\tau'_2} = \bar{\Theta}(\pi_{\tau''_1} \otimes \det{}^{\epsilon'_1}) \neq \bar{\Theta}(\pi_{\tau''_2} \otimes \det{}^{\epsilon'_2}) = \pi_{\tau'_2}$$

Suppose $\mathcal{O}'$ is noticed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is noticed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is an injection into $\mathrm{LS}(\mathcal{O}'')$. Now (9.4) implies $\tau' \mapsto \mathcal{L}_{\tau'}$ is also an injection.

Suppose $\mathcal{O}'$ is quasi-distingushed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is quasi-distingushed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is a bijection with $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}'')$. Since the component group of each K-nilpotent orbit in $\mathcal{O}'$ is naturally isomorphic to the component group of its descent. So we deduce that $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection onto $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}')$.

This proves the main proposition for $\mathcal{O}'$.

Now let $\tau \in \mathrm{DRC}(\mathcal{O})$. By (6.3), we have

$$\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau \neq 0, \text{ where } \mathcal{T}_\tau := \dagger\mathcal{L}_{\tau'}, \text{ and } \mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}. \tag{9.5}$$

In particular, $\pi_\tau \neq 0$ and we could determine $\mathbf{x}_\tau$ from the marks on the 1-rows of $\mathcal{L}_\tau$.

Now suppose $\tau_1 \neq \tau_2$ and $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (9.5), the result in Section 9.2 and Lemma 6.6 (ii)), we have $\mathbf{x}_{\tau_1} = \mathbf{x}_{\tau_2}$, $\epsilon_1 = \epsilon_2$ and $\tau''_1 \neq \tau''_2$. By (9.5), we have $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. Applying the main proposition of $\mathcal{O}'$, we have $\epsilon'_1 = \epsilon'_2$ and $\tau'_1 \neq \tau'_2$. Now by the injectivity of theta lifing, we conclude that

$$\pi_{\tau_2} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (\mathbf{1}^{+,-})^{\epsilon_1} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (\mathbf{1}^{+,-})^{\epsilon_2} = \pi_{\tau'_2}$$

9.3.3. Note that $\mathcal{L}_\tau$ is obtained from $\mathcal{L}_{\tau'}$ by attaching the peduncle determined by $\mathbf{x}_\tau$. The claims about noticed and quasi-distinguished orbit are easy to verify. We leave them to the reader.

9.4. **The general descent case.** We assume $k \geqslant 1$ and all the properties are satisfied by $\tau'' \in \mathrm{DRC}(\mathcal{O}'')$. We retain the notation in Section 6.2.9.

9.4.1. *Local systems.* We now describe the local systems $\mathcal{L}_{\tau'}$ and $\mathcal{L}_\tau$ more precisely using our formula of the associated character and the induction hypothesis. Note that these local systems are always non-zero which implies that $\pi_{\tau'}$ and π_τ are unipotent representations attached to $\mathcal{O}'$ and $\mathcal{O}$ respectively.

First note that in τ' and τ'', $x_{\tau''} \neq s$ by the definition of dot-r-c diagram. [10] By the induction hypothesis, $\mathcal{L}^+_{\tau''} \neq 0$.

Let $(p''_1, q''_1) := {}^l\mathrm{Sign}(\mathcal{L}_{\tau''})$. Then

$$ {}^l\mathrm{Sign}(\mathcal{L}^+_{\tau''}) = (p_0 - 1, q_0), \text{ and } {}^l\mathrm{Sign}(\mathcal{L}^-_{\tau''}) = (p_0, q_0 - 1) \text{ if } \mathcal{L}^-_{\tau''} \neq \varnothing. \tag{9.6}$$

$$ \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''}) = \maltese^{\frac{|\mathrm{Sign}(\tau'')|}{2}}(\dagger\mathcal{L}^+_{\tau''} + \dagger\mathcal{L}^-_{\tau''}) \neq 0 \tag{9.7}$$

where ${}^l\mathrm{Sign}(\dagger\mathcal{L}^+_{\tau''}) = (q''_1, p''_1 - 1)$ and ${}^l\mathrm{Sign}(\dagger\mathcal{L}^-_{\tau''}) = (q''_1 - 1, p''_1)$ (if $\dagger\mathcal{L}^-_{\tau''} \neq 0$).

Let $(p_1, q_1) := {}^l\mathrm{Sign}(\mathcal{L}_\tau)$ and $(e, f) := (p_1 - p''_1 + 1, q_1 - q''_1 + 1)$. Using Lemma 6.8 one can show that

$$(e, f) = \mathrm{Sign}(\mathbf{u}_\tau).$$

[It suffice to consider the most left three columns of the peduncle part: this part has signature ${}^l\mathrm{Sign}(\mathcal{P}_\tau) + (1,1) = \mathrm{Sign}(\mathbf{u}_\tau x_{\tau''}) = \mathrm{Sign}(\mathbf{u}_\tau) + {}^l\mathrm{Sign}(\mathcal{D}_{\tau''})$. Therfore,

$$\mathrm{Sign}(\mathbf{u}_\tau) = {}^l\mathrm{Sign}(\mathcal{P}_\tau) - {}^l\mathrm{Sign}(\mathcal{D}_{\tau''}) + (1,1) = {}^l\mathrm{Sign}(\mathcal{L}_\tau) - {}^l\mathrm{Sign}(\mathcal{L}_{\tau''}) + (1,1).$$

]

Now

$$\mathcal{L}_\tau = \begin{cases} (\mathbf{1}^{+,-} \otimes \dagger\maltese^t\dagger\mathcal{L}^+_{\tau''}) \cdot \mathcal{P}^+_\tau + (\mathbf{1}^{+,-} \otimes \dagger\maltese^t\dagger\mathcal{L}^-_{\tau''}) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau \neq d \\ (\dagger\maltese^t\dagger\mathcal{L}^+_{\tau''}) \cdot \mathcal{P}^+_\tau + (\dagger\maltese^t\dagger\mathcal{L}^-_{\tau''}) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau = d \end{cases} \tag{9.8}$$

where

$$t = \frac{|\mathrm{Sign}(\tau) - \mathrm{Sign}(\tau'')|}{2} = \frac{|\mathrm{Sign}(\mathbf{u}_\tau)|}{2}$$

$$\mathcal{P}^+_\tau = \begin{cases} \ddagger_{e,f-1} & \text{when } f \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e,f-1} & \text{when } f \geqslant 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{P}^-_\tau = \begin{cases} \ddagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 9.3. *The local system $\mathcal{L}_\tau \neq 0$.*

Proof. Since $x_{\tau''} \neq s$, $\mathcal{L}^+_{\tau''} \neq 0$. When $\mathrm{Sign}(\mathbf{u}_\tau) \succeq (0,1)$, the first term in (9.8) dose not vanish. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau) \nsucceq (0,1)$. We have $x_{\tau''} = d$ by Lemma 6.7 and $\mathrm{Sign}(\mathbf{u}_\tau) \succ (1,0)$. Therefore, $\mathcal{L}^-_{\tau''} \neq 0$ by the induction hypothesis and the second term in (9.8) dose not vanish. Hence $\mathcal{L}_\tau \neq 0$ in all cases. □

Lemma 9.4. *The local system $\mathcal{L}_\tau$ satisfies the following properties:*

(i) *When $x_\tau = s$, then $\mathcal{L}^+_\tau = \mathcal{L}^-_\tau = 0$.*
(ii) *When $x_\tau = r/c$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau = 0$.*
(iii) *When $x_\tau = d$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau \neq 0$.*

[10] Suppose $\tau'' \in \mathrm{DRC}(\mathcal{O}'')$ has the shape in (6.4) and $x_{\tau''} = s$. By the induction hypothesis, $\mathcal{L}^+_{\tau''} = \mathcal{L}^-_{\tau''} = 0$ and so the theta lift of $\pi_{\tau''}$ vanishes.

Proof. If $x_\tau = s/r/c$, "$-$" mark dose not appear as a 1-row in $\mathcal{L}_\tau$ by (9.8). So $\mathcal{L}_\tau^- = 0$ in these cases.

(i) Suppose $x_\tau = s$. We have $\mathbf{u}_\tau = s \cdots s$. Therefore, only the first term in (9.8) is non-zero and $\mathcal{P}_\tau^+$ consists of $=$ marks. Hence $\mathcal{L}_\tau^+ = 0$.

(ii) Suppose $x_\tau = r$. If $\mathrm{Sign}(\mathbf{u}_\tau) \succ (1,1)$, the first term in (9.8) is non-zero and $\mathcal{L}_\tau^+ \succ \dagger_{(1,0)}$. So $\mathcal{L}_\tau^+ \neq 0$. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau) \not\succ (1,1)$. Then $\mathbf{u}_\tau = r \cdots r$ and $x_{\tau''} = d$ by Lemma 6.7. Now the second term of (9.8) is non-zero and $\mathcal{L}_\tau^- \succ \dagger_{(1,0)}$. So $\mathcal{L}_\tau^+ \neq 0$.

(iii) Suppose $x_\tau = d$. Suppose $\mathrm{Sign}(\mathbf{u}_\tau) \succ (1,2)$. Then the first term in (9.8) is non-zero and $\mathcal{P}_\tau^+ \succ \dagger_{(1,1)}$. So $\mathcal{L}_\tau^+ \neq 0$ and $\mathcal{L}_\tau^- \neq 0$. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau) \not\succ (1,2)$. Then $\mathbf{u}_\tau = r \cdots rd$ [11] and $x_{\tau''} = d$ by Lemma 6.7. So the both terms in (9.8) are non-zero. Since $\mathcal{P}_\tau^+ \succ \dagger_{(1,0)}$ and $\mathcal{P}_\tau^- \succ \dagger_{(0,1)}$, we get the conclusion.

$\square$

9.4.2. *Unipotent representations attached to $\mathcal{O}'$.* In this section, we let $\tau' \in \mathrm{DRC}(\mathcal{O}')$ and $\tau'' = \overline{\nabla}_1(\tau') \in \mathrm{DRC}(\mathcal{O}'')$. First note that $x_{\tau''} \neq s$ and so $\mathcal{L}_{\tau''}^+$ always non-zero.

Recall (9.7). We claim that we can recover $\mathcal{L}_{\tau''}^+$ and $\mathcal{L}_{\tau''}^-$ from $\mathcal{L}_{\tau'}$:

Lemma 9.5. *The map* $\Omega_{\mathcal{O}'} \colon \mathcal{L}_{\tau'} \mapsto (\mathcal{L}_{\tau''}^+, \mathcal{L}_{\tau''}^-)$ *is a well defined map.*

Proof. When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has two elements, $x_{\tau''} = d$ and ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'}) = \{\, (q_1'', p_1''-1), (q_1''-1, p_1'')\,\}$. In (9.7), $\dagger\mathcal{L}_{\tau''}^+$ (resp. $\dagger\mathcal{L}_{\tau''}^-$) consists of components whose first column has siginature $(q_1'', p_1''-1)$ (resp. $(q_1''-1, p_1'')$). When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has only one elements, $x_\tau = r/c$, ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'}) = \{\, (q_1'', p_1''-1)\,\}$, $\mathcal{L}_{\tau'} = \dagger\mathcal{L}_{\tau''}^+$ and $\mathcal{L}_{\tau''}^- = 0$

In any case, we get

$$\mathrm{Sign}(\tau'') = (n_0, n_0) + (p_1'', q_1'') \quad \text{where } 2n_0 = |\nabla(\mathcal{O}'')|\,. \tag{9.9}$$

Using the signature $\mathrm{Sign}(\tau'')$, we could recover the twisting characters in the theta lifting of local system. Therefore $\mathcal{L}_{\tau''}^+$ and $\mathcal{L}_{\tau''}^-$ can be recovered from $\mathcal{L}_{\tau'}$. $\square$

Lemma 9.6. *Suppose* $\tau_1' \neq \tau_2' \in \mathrm{DRC}(\mathcal{O}')$. *Then* $\pi_{\tau_1'} \neq \pi_{\tau_2'}$.

Proof. It suffice to consider the case when $\mathcal{L}_{\tau_1'} = \mathcal{L}_{\tau_2'}$. By (9.9) in the proof of Lemma 9.5, $\mathrm{Sign}(\tau_1'') = \mathrm{Sign}(\tau_2'')$. On the other hand, $\epsilon_1 = \epsilon_2 = 0$ and $\tau_1'' \neq \tau_2''$ by Lemma 6.5. So

$$\pi_{\tau_1'} = \bar{\Theta}(\pi_{\tau_1''}) \neq \bar{\Theta}(\pi_{\tau_2''}) = \pi_{\tau_2'}$$

by the injectivity of theta lift. $\square$

9.4.3. *Unipotent representations attached to $\mathcal{O}$.*

Lemma 9.7. *Suppose* $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$. *Then* $\pi_{\tau_1} \neq \pi_{\tau_2}$.

Proof. It suffice to consider the case that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. On the other hand, the twisting $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ since it is determined by the local system: $\epsilon_{\tau_i} = 0$ if and only if $\mathcal{L}_{\tau_i} \supset \boxed{-}$.

We conclude that $\tau_1'' \neq \tau_2''$ and $\tau_1' \neq \tau_2'$ by Lemma 6.8 and Lemma 6.5. By Lemma 9.6 and the injectivity of theta lift,

$$\pi_{\tau_1} = \bar{\Theta}(\pi_{\tau_1'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_1}} \neq \bar{\Theta}(\pi_{\tau_2'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_2}} = \pi_{\tau_2}$$

$\square$

[11] $\mathbf{u}_\tau = d$ if it has length 1.

9.4.4. *Noticed orbits.* In this section, we prove the claims about noticed and +noticed orbits.

Lemma 9.8. *Suppose $\mathcal{O}'$ is noticed. Then*

(i) The map $\{\mathcal{L}_{\tau''} \mid x_{\tau''} \neq s\} \longrightarrow \{\mathcal{L}_{\tau'}\} = {}^{\ell}\mathrm{LS}(\mathcal{O}')$ *given by* $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''})$ *is a bijection.*

(ii) The map $\mathrm{DRC}(\mathcal{O}') \to {}^{\ell}\mathrm{LS}(\mathcal{O}')$ *given by* $\tau' \mapsto \mathcal{L}_{\tau'}$ *is a bijection.*

Proof. (i) Note that $\mathcal{O}''$ is noticed by definition. Hence $\Upsilon_{\mathcal{O}''}$ is invertible. Recall Lemma 9.5, we see that

$$(\Upsilon_{\mathcal{O}''})^{-1} \circ \Omega_{\mathcal{O}'} \colon \mathcal{L}_{\tau'} \mapsto (\mathcal{L}^+_{\tau''}, \mathcal{L}^-_{\tau''}) \mapsto \mathcal{L}_{\tau''}$$

gives the inverse of the map in the claim.

(ii) Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$. By Lemma 6.5, $\tau''_1 \neq \tau''_2$. Hence $\mathcal{L}_{\tau''_1} \neq \mathcal{L}_{\tau''_2}$ by the induction hypothesis. Therefore, $\mathcal{L}_{\tau'_1} \neq \mathcal{L}_{\tau'_2}$ by (i) of the claim. □

Lemma 9.9. *Suppose $\mathcal{O}$ is noticed,* $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}) \mapsto {}^{\ell}\mathrm{LS}(\mathcal{O})$ *given by* $\tau \mapsto \mathcal{L}_\tau$ *is bijective.*

Proof. We prove the claim by contradiction. We assume $\tau_1 \neq \tau_2$ such that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$.

Recall (9.8): there is an pair of non-negative integers $(e_i, f_i) = \mathrm{Sign}(\mathbf{u}_{\tau_i})$ such that

$$\mathcal{L}_{\tau_i} = \dagger\dagger\mathcal{L}^+_{\tau''_i} \cdot \mathcal{P}^+_{\tau_i} + \dagger\dagger\mathcal{L}^-_{\tau''} \cdot \mathcal{P}^-_{\tau_i}.$$

(i) Suppose that $\mathcal{L}_{\tau_i}$ contains a 1-row marked by $-$ or $=$. Then we have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$, which can be read from the mark $-/=$. Moreover, the term $\dagger\dagger\mathcal{L}^+_{\tau''_i} \cdot \mathcal{P}^+_{\tau_i} \neq 0$ and we can recover it from $\mathcal{L}_{\tau_i}$. So $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$. Since $\mathcal{O}''$ is +noticed, $\tau''_1 = \tau''_2$ by (9.11) and the induction hypothesis. Note that $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by Lemma 6.8 which is contradict to our assumption.

(ii) Now we assume that $\mathcal{L}_{\tau_i}$ does not contains a 1-row marked by $-/=$. By (9.8), we conclude that $x_{\tau''} \neq d$ and $\mathbf{u}_\tau = r\cdots r$ or $r\cdots rc$. Therefore $\mathbf{p}_{\tau_i}$ must be one of the following form by Lemma 6.7:

$$\mathbf{p}_1 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} r & \\ \cline{1-1}\end{array}, \quad \mathbf{p}_2 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} c & \\ \cline{1-1}\end{array}, \quad \text{and} \quad \mathbf{p}_3 = \begin{array}{|c|c|}\hline r & d \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} r & \\ \cline{1-1}\end{array}$$

We consider case by case according to the set $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\}$:

(a) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1\}$ or $\{\mathbf{p}_2\}$. In these cases, $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by Lemma 6.8. On the other hand, $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ by (9.8). Since $\mathcal{O}''$ is +noticed, we get $\tau''_1 = \tau''_2$ a contradiction.

(b) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_3\}$. We still have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by Lemma 6.8. On the other hand, $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$ by (9.8). This again contradict to the injectivity of $\mathcal{L}_{\tau''} \mapsto \mathcal{L}^-_{\tau''}$.

(c) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1, \mathbf{p}_2\}$. By the argument above, we have $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ Now $\tau''_1 \neq \tau''_2$ since $\{x_{\tau''_1}, x_{\tau''_2}\} = \{r, c\}$. This contradict to that $\mathcal{O}''$ is +noticed again.

(d) Suppose $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ or $\mathbf{p}_2, \mathbf{p}_{\tau_2} = \mathbf{p}_3$. Then

$$\mathcal{L}_{\tau_1} = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_1}|}{2}}\dagger\mathcal{L}^+_{\tau''_1} \cdot \mathcal{P}^+_{\tau''_1} \quad \text{and} \quad \mathcal{L}_{\tau_2} = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_2}|}{2}}\dagger\mathcal{L}^-_{\tau''_2} \cdot \mathcal{P}^-_{\tau''_2}.$$

This implies $|\mathrm{Sign}(\tau''_1)| \equiv |\mathrm{Sign}(\tau''_2)| + 2 \pmod 4$ and $\maltese\dagger\mathcal{L}^+_{\tau''_1} = \dagger\mathcal{L}^-_{\tau''_2}$.

Claim 9.10. *We have* $\mathcal{L}^-_{\tau''_2} \supset \boxed{+}$.

Proof. If $\mathcal{O}''$ is obtained by the usual descent, we have $\mathcal{L}_\tau \geq \begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array}$ and we are done. Now assume $\mathcal{O}''$ is obtained by generalized descent and $\mathcal{O}''$ is $+$noticed, i.e. $\mathbf{u}_{\tau''}$ has at least length 2. Suppose $\mathrm{Sign}(\mathbf{u}_{\tau''}) \succ (1,2)$. Applying (9.8) to τ'', we see that the first term is non-zero and $\mathcal{L}_{\tau''} \supseteq \dagger_{(1,1)}$.
Otherwise, $\mathrm{Sign}(\mathbf{u}_{\tau''}) = (2n_0-1,1) \geq (3,1)$ where n_0 is the length of $\mathbf{u}_{\tau''}$. By (9.8), the second term is non-zero and $\mathcal{L}_{\tau''} \succ \dagger_{(2n_0-1,1)} \succ +$. □
The claim leads to a contradiction: Thanks to the character twist in the theta lifting formula of the local system, we see that the the associated character restricted on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$ of $\maltese\dagger\mathcal{L}^+_{\tau''_1}$ and $\dagger\mathcal{L}^-_{\tau''_2}$ must be are different, a contradiction.

We finished the proof of the lemma. □

9.4.5. *The maps $\Upsilon^+_{\mathcal{O}}$, $\Upsilon^-_{\mathcal{O}}$ and $\Upsilon_{\mathcal{O}}$.*

Lemma 9.11. *Suppose $\mathcal{O}$ is $+$noticed. The map $\Upsilon^+_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0 \,\} \to \{\, \mathcal{L}^+_\tau \neq 0 \,\}$ is injective.*

Proof. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$. This is equivalent to $\mathcal{L}^+_\tau$ has an irreducible component $\succ \boxed{+}$. Now all irreducible components of $\mathcal{L}_\tau \succ \boxed{+}$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^+_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^+_\tau$.

(ii) Suppose that $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ do not contain a component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$.

By Lemma 9.9 $\tau_1 \neq \tau_2$.[12] We now show that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2} \neq 0$ leads to a contradiction. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. By (9.8) and checking the properties in Lemma 6.7, we have

$$\mathbf{p}_{\tau_i} = \begin{array}{|c|c|}\hline s & x_{\tau''} \\ \hline \vdots & \\ \cline{1-1} s & \\ \cline{1-1} x_{\tau_i} & \\ \cline{1-1}\end{array} \qquad \text{where } x_{\tau''_i} = c/d, x_{\tau_i} = c/d.$$

On the other hand, when $\mathbf{p}_{\tau_i}$ have the above form, the first term of (9.8) is non-zero. So we conclude that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$ implies

$$\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}. \tag{9.10}$$

Moreover, we see that every components of $\mathcal{L}_{\tau_i}$ has a 1-row of mark "$-/=$" and we can determine ϵ_{τ_i} by the mark $-/=$ of the 1-row appeared in $\mathcal{L}^+_{\tau_i}$. In particular, we have $\epsilon_1 = \epsilon_2$. Now Lemma 6.8 implies $\tau''_1 \neq \tau''_2$. This is contradict to (9.10) and our induction hypothesis since $\mathcal{O}''$ is also $+$noticed.

□

Lemma 9.12. *Suppose $\mathcal{O}$ is noticed. Then the map $\Upsilon_{\mathcal{O}}\colon \mathcal{L}_\tau \mapsto (\mathcal{L}^+_\tau, \mathcal{L}^-_\tau)$ is injective.*

Proof. It suffice to consider the case where $\mathcal{O}$ is noticed but not $+$noticed, i.e. $C_{2k+1} = C_{2k}+1$. In this case, $n = 1$. The peduncle $\mathbf{p}_\tau$ have four possible cases:

$$\begin{array}{|c|c|}\hline r & c \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline c & c \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline c & d \\ \hline\end{array}, \quad \text{or} \quad \begin{array}{|c|c|}\hline d & d \\ \hline\end{array}.$$

By (9.8), $\mathcal{L}^+_\tau = \dagger\dagger\mathcal{L}^+_{\tau''}$.

Now suppose $\tau_1 \neq \tau_2$ such that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$. Since $\mathcal{O}''$ is $+$noticed, we have $\tau''_1 = \tau''_2$ by the induction hypothesis (see Lemma 9.11). By Lemma 6.8, this only happens when $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \begin{array}{|c|c|}\hline c & d \\ \hline\end{array}, \begin{array}{|c|c|}\hline d & d \\ \hline\end{array} \,\}$. Since one of $\mathcal{L}^-_{\tau_i}$ is zero and the other is non-zero, we conclude that $\Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_1}) \neq \Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_2})$. □

[12] $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ clearly implies $\tau_1 \neq \tau_2$.

Lemma 9.13. *Suppose $\mathcal{O}$ is +noticed. The map $\Upsilon^-_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0 \,\} \to \{\, \mathcal{L}^-_\tau \neq 0 \,\}$ is injective.*

Proof. The proof is similar to that of Lemma 9.11. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $> \begin{array}{|c|}\hline - \\ \hline - \\ \hline \end{array}$. This is equivalent to $\mathcal{L}^-_\tau$ has an irreducible component $> \boxed{-}$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^-_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^-_\tau$.

(ii) Now we make a weaker assumption that $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$ such that $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ do not contain a component $> \begin{array}{|c|}\hline - \\ \hline - \\ \hline \end{array}$.

By (9.8) and the properties in Lemma 6.7, we see that $\mathbf{p}_{\tau_i}$ must be one of the following

$$\mathbf{p}_1 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} d & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \text{or} \qquad \mathbf{p}_2 = \begin{array}{|c|c|}\hline r & d \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} d & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} .$$

Without of loss of generality, we have the following possibilities:

(a) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_1$. We have $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ and obtain the contradiction.

(b) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_2$. We have $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$ and obtain the contradiction.

(c) $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ and $\mathbf{p}_{\tau_2} = \mathbf{p}_2$. Now we apply the same argument in the case (d) of the proof of Lemma 9.9. We get $\maltese\dagger\mathcal{L}^+_{\tau''_1} = \dagger\mathcal{L}^-_{\tau''_2}$, $\mathcal{L}^-_{\tau''_2} \supset +$ and a contradiction by looking at the associated character on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline \end{array}$.

This finished the proof. $\square$

10. Type B/M case

10.1. "Special" and "non-special" diagrams.

10.1.1. *type M.*

Definition 10.1. *Let $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_1, C_0 = 0) \in \mathrm{Nil}^{\mathrm{dpe}}(M)$ with $k \geqslant 1$. Let*

$$\tau = (\tau_{2k}, \cdots, \tau_2) \times (\tau_{2k-1}, \cdots, \tau_1).$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(M)$. We say τ has

- special shape *if $\tau_{2k} \geqslant \tau_{2k-1}$;*
- non-special shape *if $\tau_{2k} < \tau_{2k-1}$.*

Clearly, this gives a partition of the set of Weyl group representations attached to $\mathcal{O}$.

Suppose $k \geqslant 1$, $C_{2k} = 2c_{2k} - 1$ and $C_{2k-1} = 2c_{2k-1} + 1$. The special shape representation τ^s

$$\tau^s = \tau^s_L \times \tau^s_R = (c_{2k}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k-1}, \tau_{2k-3} \cdots, \tau_1).$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau^{ns}_L \times \tau^{ns}_R = (c_{2k-1}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k}, \tau_{2k-3}, \cdots, \tau_1)$$

In the paring $\tau^s \leftrightarrow \tau^{ns}$, τ_j is unchanged for $j \leqslant 2k-2$. We call τ^s the special shape *and τ^{ns} the corresponding* non-special shape

Definition 10.2. *Suppose C_{2k} is odd. We retain the notation in Definition 10.1 where the shapes τ^s and τ^{ns} correspond with each other. We define a bijection between* $\mathrm{DRC}(\tau^s)$ *and* $\mathrm{DRC}(\tau^{ns})$*:*

$$\begin{array}{ccc} \mathrm{DRC}(\tau^s) & \longleftrightarrow & \mathrm{DRC}(\tau^{ns}) \\ \boldsymbol{\tau}^s & \longleftrightarrow & \boldsymbol{\tau}^{ns} \end{array}$$

Without loss of generality, we can assume that $\uptau^s \in \mathrm{DRC}(\tau^s)$ *and* $\uptau^{ns} \in \mathrm{DRC}(\tau^{ns})$ *have the following shapes:*

$$
\uptau^s : \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline y_1 & \cdots & \cdots \\ \hline s \\ \cline{1-1} \vdots \\ \cline{1-1} s \\ \cline{1-1} y_2 \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline x_0 & \cdots & \cdots \\ \hline \\ \cline{1-1} \\ \cline{1-1} \\ \cline{1-1} \\ \cline{1-1} \end{array} \longleftrightarrow \uptau^{ns} : \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline y_0 & \cdots & \cdots \\ \hline \\ \cline{1-1} \\ \cline{1-1} \\ \cline{1-1} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline x_1 & \cdots & \cdots \\ \hline r \\ \cline{1-1} \vdots \\ \cline{1-1} r \\ \cline{1-1} x_2 \\ \cline{1-1} \end{array} \tag{10.1}
$$

Here the grey parts have length $(C_{2k} - C_{2k-1})/2$ *(could be zero length). The entries* x_0, x_1, y_0 *and* y_1 *are either all empty or all non-empty. The correspondence between* (x_0, x_1, x_2), *with* (y_0, y_1, y_2) *is given by switching* r *and* s, d *and* c. *i.e.*

$$
\begin{cases} y_i = s \Leftrightarrow x_i = r \\ y_i = c \Leftrightarrow x_i = d \end{cases} \qquad \text{for } i = 0, 1, 2.
$$

We define

$$
\mathrm{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \mathrm{DRC}(\tau^s) \qquad \text{and} \qquad \mathrm{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \mathrm{DRC}(\tau^{ns})
$$

where τ^s *(resp.* τ^{ns}*)runs over all specail (resp. non-special) shape representations attached to* $\mathcal{O}$. *Clearly* $\mathrm{DRC}(\mathcal{O}) = \mathrm{DRC}^s(\mathcal{O}) \sqcup \mathrm{DRC}^{ns}(\mathcal{O})$.

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

10.1.2. *Special and non-special shapes of type B.* Now we define the notion of special and non-special shape of type B.

Definition 10.3. *Let*

$$
\tau = (\tau_{2k}, \cdots, \tau_2) \times (\tau_{2k+1}, \tau_{2k-1}, \cdots, \tau_1)
$$

be a Weyl group representation attached to $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(B)$. *We say* τ *has*

- special shape *if* $\tau_{2k} \geqslant \tau_{2k-1}$;
- non-special shape *if* $\tau_{2k} < \tau_{2k-1}$.

Suppose $k \geqslant 1$ *and*

$$
\mathcal{O} = (C_{2k+1} = 2c_{2k+1} + 1, C_{2k} = 2c_{2k} - 1, C_{2k-1} = 2c_{2k-1} + 1, \cdots, C_1 > 0, C_0 = 0). \tag{10.2}
$$

A representation τ^s *attached to* $\mathcal{O}$ *has the shape*

$$
\tau^s = \tau^s_L \times \tau^s_R = (c_{2k}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \cdots, \tau_1)
$$

is paired with

$$
\tau^{ns} = \tau^{ns}_L \times \tau^{ns}_R = (c_{2k-1}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k+1}, c_{2k}, \tau_{2k-3}, \cdots, \tau_1).
$$

In the paring $\tau^s \leftrightarrow \tau^{ns}$ *described as the above,* τ_j *are unchanged for* $j \leqslant 2k-2$.

10.2. Definition of $\overline{\nabla}$ for type B and M.

10.2.1. *The defintion of* ϵ.

(1). When $\uptau \in \mathrm{DRC}(B)$, ϵ is determined by the "basal disk" of $\uptau$ (see (10.3) for the definition of $x_\uptau$):

$$
\epsilon_\uptau := \begin{cases} 0, & \text{if } x_\uptau = d; \\ 1, & \text{otherwise.} \end{cases}
$$

(2). When $\uptau \in \mathrm{DRC}(M)$, the ϵ is determined by the lengths of $\uptau_{L,0}$ and $\uptau_{R,0}$:

$$
\epsilon_\uptau := \begin{cases} 0, & \text{if } |\uptau_{L,0}| - |\uptau_{R,0}| \geqslant 0; \\ 1, & \text{if } |\uptau_{L,0}| - |\uptau_{R,0}| < 0. \end{cases}
$$

10.2.2. *Initial cases.*

(3). Suppose $\mathcal{O} = (2c_1 + 1)$ has only one column. Then $\mathcal{O}'_0 := \overline{\nabla}(\mathcal{O})$ is the trivial orbit of $\mathrm{Mp}(0, \mathbb{R})$. $\mathrm{DRC}(\mathcal{O})$ consists of diagrams of shape $\tau_L \times \tau_R = \varnothing \times (c_1,)$. The set $\mathrm{DRC}(\mathcal{O}'_0)$ is a singleton $\{\, \tau'_0 = \varnothing \times \varnothing \,\}$, and every element $\tau \in \mathrm{DRC}(\mathcal{O})$ maps to τ_0 (see (5.1)):

$$\mathrm{DRC}(\mathcal{O}) \ni \tau := \boxed{\varnothing} \times \begin{array}{|c|}\hline x_1 \\ \vdots \\ \hline x_n \\ \hline m_\tau \\ \hline \end{array} \mapsto \tau'_0 := \varnothing \times \varnothing$$

Here $m_\tau = a$ or b is the mark of τ. We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := x_1 \cdots x_n m_\tau.$$

We call $\mathbf{p}_\tau = \mathbf{x}_\tau$ the "peduncle" part of τ.

10.2.3. *The descent from M to B.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

(4). Suppose $|\tau_{L,0}| \geqslant |\tau_{R,0}|$, i.e. τ has special shape. Keep r, d unchanged, delete $\tau_{L,0}$ and fill the remaining part with "$\bullet$" and "s" by $\overline{\nabla}_{\mathrm{R}}(\tau^\bullet)$ using the dot-s switching algorithm. The mark $m_{\tau'}$ of $\tau' := \overline{\nabla}_1(\tau)$ is given by the following formula:

$$m_{\tau'} := \begin{cases} a & \text{if } \tau_{L,0} \text{ ends with } s \text{ or } \bullet, \\ b & \text{if } \tau_{L,0} \text{ ends with } c. \end{cases}$$

(5). Suppose $|\tau_{L,0}| < |\tau_{R,0}|$, i.e. τ has non-special shape. We define

$$\overline{\nabla}_1(\tau) := \overline{\nabla}_1(\tau^s)$$

where τ^s is the special diagram corresponding to τ defined in Definition 10.2.

Lemma 10.4. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} is odd. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1}, \cdots, C_0)$. Then*

$$\mathrm{DRC}^s(\mathcal{O}) \xrightarrow{\quad \overline{\nabla}_1 \quad} \mathrm{DRC}(\mathcal{O}')$$

$$\mathrm{DRC}^{ns}(\mathcal{O}) \xrightarrow{\quad \overline{\nabla}_1 \quad} \mathrm{DRC}(\mathcal{O}')$$

are a bijections.

Proof. The claim for $\mathrm{DRC}^s(\mathcal{O})$ is clear by the definition of descent. The claim for $\mathrm{DRC}^{ns}(\mathcal{O})$ reduces to that of $\mathrm{DRC}^s(\mathcal{O})$ using Definition 10.2. $\square$

Lemma 10.5. *Suppose $\mathcal{O} = (C_{2k}, C_{2k-1}, \cdots, C_0)$ with $k \geqslant 1$ and C_{2k} even. Let $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1} + 1, \cdots, C_0)$. Then the following map is a bijection*

$$\mathrm{DRC}(\mathcal{O}) \xrightarrow{\quad \overline{\nabla}_1 \quad} \left\{ \tau' \in \mathrm{DRC}(\mathcal{O}') \;\middle|\; \begin{array}{l} m_{\tau'} \neq b \textit{ and} \\ \tau'_{R,0} \textit{ dose not ends with } s. \end{array} \right\}.$$

Proof. This is clear by the descent algorithm. $\square$

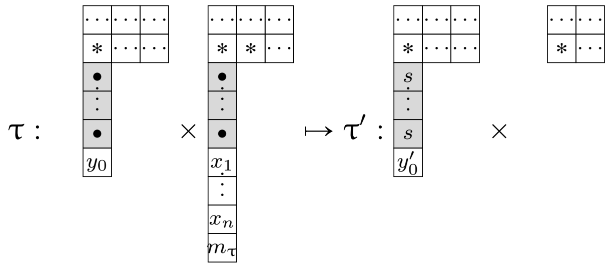
10.2.4. *Descent from B to M.* Now we define the general case of the descent from type B to type M. We assume that $k \geqslant 1$ i.e. $\mathcal{O}$ has at least 3 columns. First note that the shape of $\tau' = \overline{\nabla}(\tau)$ is the shape of τ deleting the most left column on the right diagram.

The definition splits in cases below. In all these cases we define

$$\tilde{\mathbf{x}}_\tau := x_1 \cdots x_n m_\tau \text{ and } \tilde{x}_\tau := x_n \tag{10.3}$$

which is marked by $s/r/c/d$. For the part marked by $*/\cdots$, $\overline{\nabla}$ keeps r, c, d and maps the rest part consisting of $\bullet$ and s by dot-s switching algorithm.

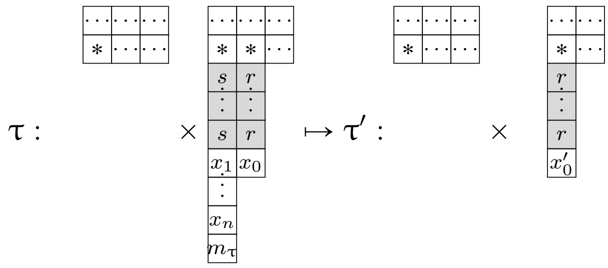
(6). When C_{2k} is odd and τ has special shape, the descent is given by the following diagram



Here the grey columns has length $(C_{2k} - C_{2k-1})/2$ and $n = (C_{2k+1} - C_{2k})/2$. The "$*/\cdots$" part of τ' is given by keeping the corresponding entries marked by $r/d/c$ in τ unchange and filling $s/\bullet$ accordingly in the rest of the entries. The entry y_0' of τ' is given by the following formula:

$$y_0' := \begin{cases} s & \text{if } x_1 = \bullet \text{ or } (x_1, m_\tau) = (r/d, a) \\ c & \text{if } x_1 = s \text{ or } (x_1, m_\tau) = (r/d, b). \end{cases}$$

(7). When C_{2k} is odd and τ has non-special shape, then

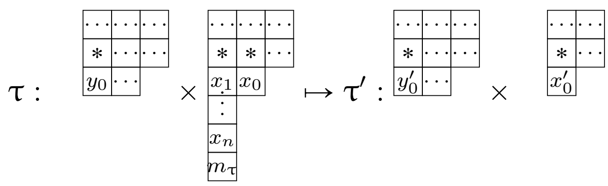


Here the grey columns has length $(C_{2k} - C_{2k-1})/2$ and $n = (C_{2k+1} - C_{2k})/2$. The "$*/\cdots$" part of τ' is given by keeping the corresponding entries marked by $r/d/c$ in τ unchange and filling $s/\bullet$ accordingly in the rest of the entries. The entry x_0' of τ' is given by the following formula:

$$x_0' := \begin{cases} r & \text{if } (x_1, m_\tau) = (r/d, a), \\ d & \text{if } (x_1, m_\tau) = (r/d, b), \\ x_0 & \text{if } x_1 = s. \end{cases}$$

Note that

(8). When $C_{2k} = C_{2k-1}$ is even, we could assume τ and τ' hase the following forms with $n = (C_{2k+1} - C_{2k} + 1)/2$.



In the most of the case, τ' is obtained from τ by deleting the first column of τ_R, keeping $c/r/d$ and filling $s/\bullet$ accordingly. The exceptional cases are when $x_1 = r/d$. In the exceptional case we always have $x_0 = d$. We define

$$x_0' := d,$$

$$y_0' := \begin{cases} s & \text{if } m_\tau = a, \\ c & \text{if } m_\tau = b. \end{cases}$$

Note that this definition is similar to that of the special shape diagrams in the descent case. We define the "peduncle" of τ to be

$$\mathbf{p}_\tau = \begin{array}{|c|c|}\hline x_1 & x_0 \\ \hline \vdots & \\ \cline{1-1} x_n & \\ \cline{1-1} m_\tau & \\ \cline{1-1}\end{array}.$$

In all the cases, let $\mathcal{O}_1 = (2n,)$ is the trivial nilpotent of type D. For each $\tau \in \mathrm{DRC}(\mathcal{O})$, we define $\mathbf{x}_\tau \in \mathrm{DRC}(\mathcal{O}_1)$ by require the following identity of multisets:

$$\{\text{entry of } \mathbf{x}_\tau\} = \begin{cases} \{c, x_2, \cdots, x_n\} & \text{if } (x_1, m_\tau) = (\bullet/s, a), \\ \{s, x_2, \cdots, x_n\} & \text{if } (x_1, m_\tau) = (\bullet/s, b), \\ \{x_1, x_2, \cdots, x_n\} & \text{otherwise } (x_1 = r/d). \end{cases}$$

In the above definition we include the initial case where $\mathcal{O}$ is a trivial orbit of type B. We define $\mathbf{x}_\tau = \varnothing$ when $\mathcal{O}$ is the trivial orbit of $\mathrm{O}(1, \mathbb{C})$.

We define the "basal disk" of τ as the following:

$$x_\tau := \begin{cases} \mathbf{x}_\tau & \text{if } C_{2k+1} = C_{2k} + 1. \\ c & \text{if } x_n m_\tau = sa \text{ or } \bullet a \\ x_n & \text{otherwise.} \end{cases}$$

10.2.5. *A key proposition for descent case.* Now we assume $\mathcal{O}$ is given by (10.2) Let $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$.

The following lemma is the key property satisfied by our definition

Lemma 10.6. *Suppose $k \geqslant 1$ and τ^s and τ^{ns} are two representations attached to $\mathcal{O}$ as in Definition 10.3. Then*

i) For every $\tau \in \mathrm{DRC}(\tau^s) \sqcup \mathrm{DRC}(\tau^{ns})$, the shape of $\overline{\nabla}^2(\tau)$ is

$$\tau'' = (\tau_{2k-2}, \cdots, \tau_2) \times (c_{2k-1}, \tau_{2k-3}, \tau_1)$$

ii) We define $\delta \colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ by $\delta(\tau) = (\overline{\nabla}^2(\tau), \mathbf{x}_\tau)$. The following maps are bijective

$$\begin{tikzcd} \mathrm{DRC}(\tau^s) \arrow[r, "\delta"] & \mathrm{DRC}(\tau'') \times \mathrm{DRC}(\mathcal{O}_1) & \mathrm{DRC}(\tau^{ns}) \arrow[l, "\delta"'] \\ \tau^s \arrow[r, mapsto] & (\overline{\nabla}^2(\tau^s), \mathbf{x}_{\tau^s}) & \\ & (\overline{\nabla}^2(\tau^{ns}), \mathbf{x}_{\tau^{ns}}) & \tau^{ns} \arrow[l, mapsto] \end{tikzcd}$$

In particular, we obtain an one-one correspondence $\tau^s \leftrightarrow \tau^{ns}$ such that $\delta(\tau^s) = \delta(\tau^{ns})$.

iii) Suppose τ^s and τ^{ns} correspond as the above such that $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$. Then

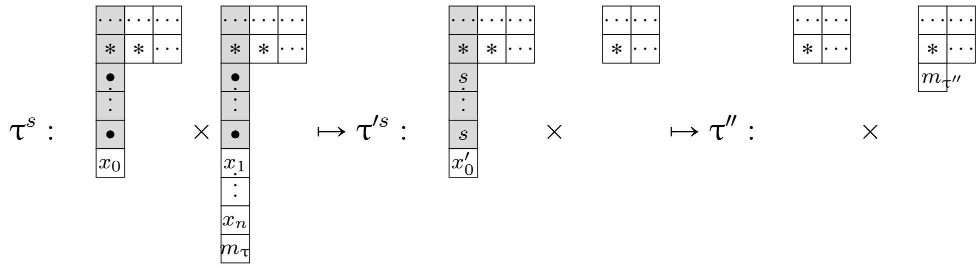
$$\mathrm{Sign}(\tau^s) = \mathrm{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \mathrm{Sign}(\tau'') + \mathrm{Sign}(\mathbf{x}). \tag{10.4}$$

Note that $\mathrm{Sign}(\mathbf{x}) \in \mathbb{N} \times \mathbb{N}$.

Proof. By our assumption, we have $c_{2k+1} \geqslant c_{2k} > c_{2k-1}$ and

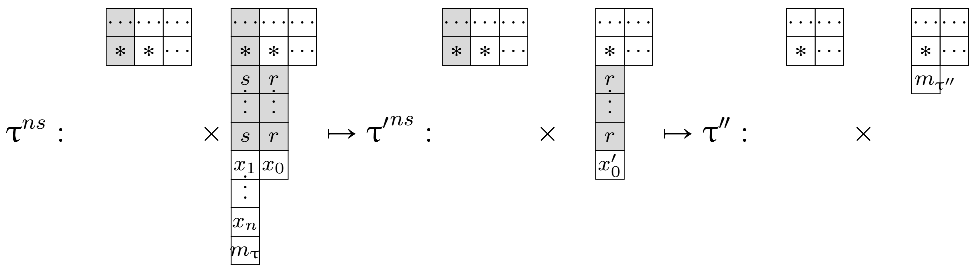
$$\tau^s = (c_{2k}, \tau_{2k-2}, \cdots, \tau_2) \times (c_{2k+1}, c_{2k-1}).$$

The the behavior of $\boldsymbol{\tau}^s$ under the descent map $\overline{\nabla}$ is illustrated as the following:

(10.5)


The grey part consists of totally $2(c_{2k}-1) = C_{2k}-1$ dots. Hence the total signature of the gray part is $(C_{2k}-1, C_{2k}-1)$. Note that $\boldsymbol{\tau}^s$ is obtained from the "$* \cdots$" part of τ'' by attaching the grey part and $x_0, \cdots, x_n, m_{\boldsymbol{\tau}}$.

The descent of a non-special diagram $\boldsymbol{\tau}^{ns}$:

(10.6)


Here the gray parts in $\boldsymbol{\tau}$ consists of $C_{2k}-1$ marks of $\bullet$, s, or r and s and r occur with the same multiplicity. Hence the total signature of the gray part is $(C_{2k}-1, C_{2k}-1)$.

To prove the lemma, it suffice to verify the following claims which we leave to the reader:

- In both the special shape and non-special shape cases, the following map is bijective:
$$x_0 x_1 \cdots x_n m_{\boldsymbol{\tau}^{s/ns}} \mapsto (\mathbf{x}_{\boldsymbol{\tau}^{s/ns}}, m_{\boldsymbol{\tau}''}).$$
- The following equation of signatures holds
$$\mathrm{Sign}(x_0 x_1 \cdots x_n m_{\boldsymbol{\tau}}) - \mathrm{Sign}(m_{\boldsymbol{\tau}''}) = \mathrm{Sign}(\mathbf{x}_{\boldsymbol{\tau}}).$$

In the verification, one can use the fact that $m_{\boldsymbol{\tau}} = m_{\boldsymbol{\tau}''}$ when $x_1 = r/d$.

[Suppose $x_1 = r/d$. Then $x_0 = c/d$ and
$$\mathrm{Sign}(x_0 \cdots x_n m_{\boldsymbol{\tau}}) - \mathrm{Sign}(m_{\boldsymbol{\tau}''}) = (1,1) + \mathrm{Sign}(x_1 \cdots x_n).$$

Now we consider the generic cases, i.e. $x_1 = \bullet/s$: In this case, we clain that $\mathrm{Sign}(x_0 x_1) - \mathrm{Sign}(m_{\boldsymbol{\tau}''}) = (1,2)$. Now
$$\begin{aligned}
&\mathrm{Sign}(x_0 x_1 \cdots x_n m_{\boldsymbol{\tau}}) - \mathrm{Sign}(m_{\boldsymbol{\tau}''}) \\
=&(1,1) + \mathrm{Sign}(x_2 \cdots x_n m_{\boldsymbol{\tau}}) + (0,1) \\
=&\begin{cases} (1,1) + \mathrm{Sign}(x_2 \cdots x_n) + \mathrm{Sign}(c) & \text{if } m_{\boldsymbol{\tau}} = a \\ (1,1) + \mathrm{Sign}(x_2 \cdots x_n) + \mathrm{Sign}(s) & \text{if } m_{\boldsymbol{\tau}} = b \end{cases}
\end{aligned}$$

First consider the special shape diagram $\boldsymbol{\tau} = \boldsymbol{\tau}^s$.

(i) Suppose $m_{\boldsymbol{\tau}''} = a$. Then $x_0 \times x_1 = \bullet \times \bullet$.
(ii) Suppose $m_{\boldsymbol{\tau}''} = b$. Then $x_0 \times x_1 = c \times s$.

Now consider the non-special shape diagram $\boldsymbol{\tau} = \boldsymbol{\tau}^{ns}$.

(i) Suppose $m_{\boldsymbol{\tau}''} = a$. Then $\varnothing \times x_1 x_0 = \varnothing \times sr$.
(ii) Suppose $m_{\boldsymbol{\tau}''} = b$. Then $\varnothing \times x_1 x_0 = \varnothing \times sd$.

The macthing between $\boldsymbol{\tau}^s$ and $\boldsymbol{\tau}^{ns}$ is also clear by the above listing of cases.] $\square$

10.2.6. *A key proposition for the generalized descent case.* Now assume C_{2k} is even. By our assumption, we have $c_{2k+1} \geqslant c_{2k} = c_{2k-1}$.

$$
\tau : \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline y_0 & \cdots & \cdots \\ \hline \end{array} \times \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline x_1 & x_0 & \cdots \\ \hline x_2 \\ \cline{1-1} \vdots \\ \cline{1-1} x_n \\ \cline{1-1} m_\tau \\ \cline{1-1} \end{array} \mapsto \tau' : \begin{array}{|c|c|c|}\hline \cdots & \cdots & \cdots \\ \hline y'_0 & \cdots & \cdots \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \cdots & \cdots \\ \hline x'_0 & \cdots \\ \hline \end{array} \mapsto \tau'' : \begin{array}{|c|c|}\hline \cdots & \cdots \\ \hline \cdots & \cdots \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \cdots & \cdots \\ \hline x'' & \cdots \\ \hline m_{\tau''} \\ \cline{1-1} \end{array} \tag{10.7}
$$

Here the gray parts in τ are two columns of $\bullet$ of length $C_{2k}/2 - 1$.

Lemma 10.7. *In* (10.7)*,* $x'' = x_0$*. The map* $\delta \colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathrm{DRC}(\mathcal{O}_1)$ *given by* $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathbf{x}_\tau)$ *is injective. Moreover,*

$$
\mathrm{Sign}(\tau) = \mathrm{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \mathrm{Sign}(\mathbf{x}_\tau).
$$

The map $\tilde{\delta} \colon \mathrm{DRC}(\mathcal{O}) \to \mathrm{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\overline{\nabla}_1^2(\tau), \mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective.*

Proof. The claim $x'' = x_0$, the injectivity of δ and the siginature formula follows directly from our algorithm.

Now the injectivity of $\tilde{\delta}$ follows from $\mathbf{x}_\tau \mapsto (\mathrm{Sign}(\mathbf{x}_\tau), \epsilon)$ is injective by Lemma 9.2. [We can fully recover τ from $\mathbf{x}_\tau$ and τ''.

Suppose $\mathbf{x}_\tau$ dose not contains s or c. Then $x_1 = r/d$, $m_\tau = m_{\tau''}$, $y_0 = c$. Hence the claim holds.

$$
\mathrm{Sign}(\tau) = \mathrm{Sign}(\tau'') + (C_{2k} - 2, C_{2k} - 2) + \mathrm{Sign}(c) + \mathrm{Sign}(\mathbf{x}_\tau)
$$

Now assume $\mathbf{x}_\tau$ contain at least one s or c. Now

$$
m_\tau = \begin{cases} a & \text{if } \mathbf{x}_\tau \text{ contains } c \\ b & \text{otherwise} \end{cases}
$$

Therefore, $\mathrm{Sign}(x_2 \cdots x_n m_\tau) = \mathrm{Sign}(\mathbf{x}_\tau) - (0, 1)$.

Clearly, we can recover $x_2 \cdots x_n$ from $\mathbf{x}_\tau$ by deleting the c (if it exists) or a s. On the other hand, (y_0, x_1) is completely determined by $m_{\tau''}$:

$$
(y_0, x_1) = \begin{cases} (\bullet, \bullet) & \text{if } m_{\tau''} = a \\ (c, s) & \text{if } m_{\tau''} = b \end{cases}
$$

Hence $\mathrm{Sign}(y_0 x_1) = \mathrm{Sign}(m_{\tau''}) + (1, 2)$.

Now $\mathrm{Sign}(y_0 x_0 x_1 \cdots x_n m_\tau) = \mathrm{Sign}(\mathbf{x}_\tau) + \mathrm{Sign}(x'' m_{\tau''}) + (1, 2) - (0, 1)$. This yields the signature identity.] $\square$

10.3. **Proof of the type BM case.** We will reduce the proof to the type CD case.

In this section, $k \geqslant 1$ and $\mathcal{O} = (C_{2k+1}, C_{2k}, \cdots, C_1, C_0 = 0) \in \mathrm{Nil}^{\mathrm{dpe}}(B)$. $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$ and $\mathcal{O}'' := \overline{\nabla}(\mathcal{O}')$.

Take $\tau \in \mathrm{DRC}(\mathcal{O})$, we marks the entries in τ as in (10.5) to (10.7).

10.3.1. *Usual decent case.* Thanks to Lemma 10.6, the proof in the descent case is exactly the same as that in Section 9.3.

10.3.2. *Reduction to type CD case in the general descent case.* Suppose C_{2k} is even. We define $\tilde{\mathcal{O}} = (C_{2k+1} - C_{2k} + 1, 1, 1) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$ and $\tilde{\mathcal{O}}'' := \overline{\nabla}_1(\tilde{\mathcal{O}}) = (2)$.

The key proposition of reduction.

Proposition 10.8. *We have the following properties:*

i) When $\tau'' = \overline{\nabla}_1^2(\tau)$ for certain $\tau \in \mathrm{DRC}(\mathcal{O})$, we have $x_{\tau''} \neq s$;

ii) For each τ'' such that $x_{\tau''} \neq s$, we have a bijection

$$\bar{\delta}\colon \{\, \tau \in \mathrm{DRC}(\mathcal{O}) \mid \overline{\nabla}_1^2(\tau) = \tau'' \,\} \longrightarrow \{\, (x_{\tilde{\tau}}, \mathbf{u}_{\tilde{\tau}}) \mid \tilde{\tau} \in \mathrm{DRC}(\tilde{\mathcal{O}}) \text{ s.t. } \overline{\nabla}_1^2(\tilde{\tau}) = x_{\tau''} \,\}$$
$$\tau \longmapsto (x_{\tau''}, \mathbf{x}_\tau)$$

where $\mathbf{u}_{\tilde{\tau}}$ is defined in (6.5)*.*

iii) We have the following properties:

a) $x_{\tilde{\tau}} = s$ if and only if $x_n m_\tau = sb$ or $\bullet b$.

b) $x_{\tilde{\tau}} = d$ if and only if $x_n = d$.

The situation could be summarized in the following commutative diagram:

$$\begin{tikzcd}
 & \tau \arrow[r, mapsto, "\delta"] \arrow[d, mapsto] & (\tau'', \mathbf{x}_\tau) \arrow[r, mapsto] \arrow[d, mapsto] & \tau'' \arrow[d, mapsto] \\
x_\tau \arrow[d, leftrightarrow, dashed] & \mathbf{p}_\tau \arrow[l, mapsto] \arrow[r, mapsto] \arrow[d, leftrightarrow] & (x_{\tau''}, \mathbf{x}_\tau) \arrow[r, mapsto] \arrow[d, equal] & x_{\tau''} \arrow[d, equal] \\
x_{\tilde{\tau}} & \tilde{\tau} \arrow[l, mapsto] \arrow[r, mapsto, "\delta"] & (\tilde{\tau}'', \mathbf{u}_{\tilde{\tau}}) \arrow[r, mapsto] & \tilde{\tau}''
\end{tikzcd}$$

We remark that x_τ and $x_{\tilde{\tau}}$ are equal in the most of the case, especially when $n = 1$. But there are exceptional cases.

Proof. This follows from a case by case verification according to our algorithm. We list all possible cases below:

When $n = 1$, see Table 4.

Now assume $n \geqslant 1$. We let $\mathbf{x} = x_1 x_2 \cdots x_n$ and define

$$\mathbf{x}' = \begin{cases} \mathbf{x} \text{ deleteing ``}c\text{''} & \text{if } c \in \mathbf{x} \\ \mathbf{x} \text{ deleteing ``}s\text{''} & \text{if } c \notin \mathbf{x}, s \in \mathbf{x} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now all the cases are listed in Table 5.

□

Thanks to $\bar{\delta}$ defined in the above lemma, we can use (9.8). The rest of the proof is similar to that in Section 9.4. We leave the details to the reader.

11. Examples

11.1. **Regular orbit of type C.** Consider the case that $\mathcal{O} = (2, \underbrace{1, \cdots, 1}_{(2n)\text{-terms}}) \in \mathrm{Nil}(D)$.

Now $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$ is the regular orbit of $\mathrm{Sp}(2n)$.

$\mathcal{O}'$ is the opposite case compared to the noticed oribts. It is very "degenerate".

TBA

12. Matrix coefficient integrals and degenerate principal series

In this section, we define the notion of a ν-bounded representation and investigate the matrix coefficient integrals of such representations against the oscillator representation as well as certain degenerate principle series.

12.1. Matrix coefficients: growth and positivity.

In this subsection, let G be an arbitrary real reductive group.

Definition 12.1. *A (complex valued) function ℓ on G is said to be of* logarithmic growth *if there is a continuous homomorphism $\sigma\colon G \to \mathrm{GL}_n(\mathbb{R})$ for some $n \geqslant 1$, and an integer $d \geqslant 0$ such that*

$$|\ell(g)| \leqslant \left(\log(1 + \mathrm{tr}((\sigma(g))^{\mathrm{t}} \cdot \sigma(g)))\right)^d \quad \text{for all } g \in G.$$

Assume that a maximal compact subgroup K of G is given. Let Ξ_G denote Harish-Chandra's Ξ-function on G associated to K.

Definition 12.2. *Let $\nu \in \mathbb{R}$. A function f on G is said to be ν-*bounded *if there is a positive function ℓ on G of logarithmic growth such that*

$$|f(g)| \leqslant \ell(g) \cdot (\Xi_G(g))^\nu \quad \text{for all } g \in G.$$

We remark that the definition is independent of the choice of K.

Lemma 12.3. *Assume that the identity connected component of G has a compact center. Then for all $\nu > 2$, every ν-bounded continuous function on G is integrable (with respect to a Haar measure).*

Proof. This follows easily from the well-known estimate of Harish-Chandra's Ξ-function, see [80, Theorem 4.5.3]. □

Definition 12.4. *Let $\nu \in \mathbb{R}$. A Casselman-Wallach representation π of G is said to be ν-*bounded *if there is a ν-bounded positive function f on G, and continuous seminorms $|\ |_\pi$ on π and $|\ |_{\pi^\vee}$ on $\pi^\vee$ such that*

$$|\langle g \cdot u, v\rangle| \leqslant f(g) \cdot |u|_\pi \cdot |v|_{\pi^\vee},$$

for all $g \in G$, $u \in \pi$, $v \in \pi^\vee$.

We also recall the following positivity result on diagonal matrix coefficients, which is a special case of [28, Theorem A. 5].

Proposition 12.5. *Let G be a real reductive group with a maximal compact subgroup K. Let π_1 and π_2 be two unitary representations of G such that π_2 is weakly contained in the regular representation. Let*

$$u := \sum_{i=1}^{s} u_i \otimes v_i \in \pi_1 \otimes \pi_2.$$

Assume that

(a) for all $i, j = 1, 2, \cdots, s$, the function $g \mapsto \langle g \cdot u_i, u_j\rangle \Xi_G(g)$ on G is absolutely integrable with respect to a Haar measure $\mathrm{d}g$ on G;

(b) $v_1, v_2, \cdots, v_s$ are all K-finite.

Then the integral

$$\int_G \langle g \cdot u, u\rangle \, \mathrm{d}g$$

absolutely converges to a nonnegative real number.

12.2. **Theta lifting via matrix coefficient integrals.** We return to the setting of Section 7.6 so that $\mathbf{V}$ is an $(\epsilon, \dot\epsilon)$-space, $\mathbf{V}'$ is a $(-\epsilon, -\dot\epsilon)$-space, and $(\mathbf{W}, J_{\mathbf{W}}, L_{\mathbf{W}})$ is a $(-1,-1)$-space with $\mathbf{W} := \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}')$. Recall that $(G, G') = (\mathbf{G}^{J}_{\mathbf{V}}, \mathbf{G}^{J'}_{\mathbf{V}'})$ is a dual pair in $\mathrm{Sp}(W)$, where $W = \mathbf{W}^{J_{\mathbf{W}}}$.

Let $\mathrm{H}(W) := W \times \mathbb{R}$ denote the Heisenberg group attached to W, with group multiplication

$$(u,t)\cdot(u',t') := (u+u', t+t'+\langle u,u'\rangle_{\mathbf{W}}), \qquad u,u' \in W,\ t,t' \in \mathbb{R}.$$

Note that $\mathbf{W}$ is naturally identified with a subspace of the complexified Lie algebra of $\mathrm{H}(W)$. There is a totally complex polarization $\mathbf{W} = \mathcal{X} \oplus \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are the $+\mathbf{i}$ and $-\mathbf{i}$ eigenspaces of $L_{\mathbf{W}}$ respectively. We have a Cartan decomposition:

$$G_{\mathbf{W}} = \mathrm{Sp}(W) = K_{\mathbf{W}} \times S_{\mathbf{W}}.$$

where $S_{\mathbf{W}} := \left\{ \exp(X) \;\middle|\; X \in \mathfrak{g}_{\mathbf{W}}^{J_{\mathbf{W}}},\ XL_{\mathbf{W}} + L_{\mathbf{W}}X = 0 \right\}$.

12.2.1. *The oscillator representation.* The group $\widetilde{G} \times \widetilde{G}'$ acts on $\mathrm{H}(W)$ as group automorphisms through the natural action of $G \times G'$ on W. Using this, we form the semidirect product $(\widetilde{G} \times \widetilde{G}') \ltimes \mathrm{H}(W)$. As in Section 7.7, we assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$.

Definition 12.6. *A smooth oscillator representation associated to* $(\mathbf{V}, \mathbf{V}')$ *is a smooth Fréchet representation* $\omega_{\mathbf{V},\mathbf{V}'}$ *of* $(\widetilde{G} \times \widetilde{G}') \ltimes \mathrm{H}(W)$ *of moderate growth such that*

- *as a representation of* $\mathrm{H}(W)$*, it is irreducible with central character* $t \mapsto e^{\mathbf{i}t}$*;*
- $\widetilde{K} \times \widetilde{K}'$ *acts on* $\omega^{\mathcal{X}}_{\mathbf{V},\mathbf{V}'}$ *(the space of vectors in* $\omega_{\mathbf{V},\mathbf{V}'}$ *which are annihilated by* $\mathcal{X}$*) through the character* $\varsigma_{\mathbf{V},\mathbf{V}'}$*.*

Note that the first condition in Definition 12.6 implies that the space $\omega^{\mathcal{X}}_{\mathbf{V},\mathbf{V}'}$ is one-dimensional. If G is a nontrivial real symplectic group, a quaternionic symplectic group, or a quaternionic orthogonal group which is not isomorphic to $\mathrm{O}^*(2)$, then the first condition also implies that $\widetilde{K}$ acts on $\omega^{\mathcal{X}}_{\mathbf{V},\mathbf{V}'}$ through the character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}}$. Similar result holds for $\widetilde{K}'$. The smooth oscillator representation always exists and is unique up to isomorphism. From now on we fix a smooth oscillator representation $\omega_{\mathbf{V},\mathbf{V}'}$ for each rational dual pair $(\mathbf{V}, \mathbf{V}')$.

Let $\Psi_{\mathbf{W}}$ be the positive function on $G_{\mathbf{W}}$ which is bi-$K_{\mathbf{W}}$-invariant such that

$$\Psi_{\mathbf{W}}(g) = \prod_a (1+a)^{-\frac{1}{2}}, \quad \text{for every } g \in S_{\mathbf{W}},$$

where a runs over all eigenvalues of g, counted with multiplicities. By abuse of notation, we still use $\Psi_{\mathbf{W}}$ to denote the pullback function through the covering map $\widetilde{G}_{\mathbf{W}} \to G_{\mathbf{W}}$.

Lemma 12.7. *Extend the irreducible representation* $\omega_{\mathbf{V},\mathbf{V}'}|_{\mathrm{H}(W)}$ *to the group* $\widetilde{G}_{\mathbf{W}} \ltimes \mathrm{H}(W)$*. Then there exists continuous seminorms* $|\cdot|_1$ *on* $\omega_{\mathbf{V},\mathbf{V}'}$ *and* $|\cdot|_2$ *on* $\omega^{\vee}_{\mathbf{V},\mathbf{V}'}$ *such that*

$$|\langle g\cdot\phi, \phi'\rangle| \leqslant \Psi_{\mathbf{W}}(g)\cdot|\phi|_1\cdot|\phi'|_2, \quad \text{for all } g \in \widetilde{G}_{\mathbf{W}},\ \phi \in \omega_{\mathbf{V},\mathbf{V}'},\ \phi' \in \omega^{\vee}_{\mathbf{V},\mathbf{V}'}.$$

Proof. This is in the proof of [46, Theorem 3.2]. □

Define

$$\dim^{\circ} \mathbf{V} := \begin{cases} \dim \mathbf{V}, & \text{if } G \text{ is real symplectic;} \\ \dim \mathbf{V} - 1, & \text{if } G \text{ is quaternionic symplectic;} \\ \dim \mathbf{V} - 2, & \text{if } G \text{ is real orthogonal;} \\ \dim \mathbf{V} - 3, & \text{if } G \text{ is quaternionic orthogonal.} \end{cases}$$

Let $\Psi_{\mathbf{W}}|_{\widetilde{G}\times\widetilde{G}'}$ denote the pull back of $\Psi_{\mathbf{W}}$ through the natural homomorphism $\widetilde{G}\times\widetilde{G}' \to \widetilde{G}_{\mathbf{W}}$.

Lemma 12.8. *Assume that both* $\dim^\circ \mathbf{V}$ *and* $\dim^\circ \mathbf{V}'$ *are positive. Then the pointwise inequality*

$$\Psi_{\mathbf{W}}|_{\widetilde{G}\times\widetilde{G}'} \leqslant (\Xi_{\widetilde{G}})^{\frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}} \cdot (\Xi_{\widetilde{G}'})^{\frac{\dim \mathbf{V}}{\dim^\circ \mathbf{V}'}}$$

holds.

Proof. This is easy to check, by using the well-known estimate of Harish-Chandra's Ξ-function (*cf.* [80, Theorem 4.5.3].) □

12.2.2. *Matrix coefficient integrals against oscillator representations.* Let π be a Casselman-Wallach representation of $\widetilde{G}$.

Definition 12.9. *Assume that* $\dim^\circ \mathbf{V} > 0$*. The pair* $(\pi, \mathbf{V}')$ *is said to be in the* convergent range *if*

- π *is* ν_π*-bounded for some* $\nu_\pi > 2 - \frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}$*;*
- *if* G *is a real symplectic group, then* ε_G *acts on* π *through the scalar multiplication by* $(-1)^{\dim \mathbf{V}'}$*.*

In the rest of this section, assume that $\dim^\circ \mathbf{V} > 0$ and the pair $(\pi, \mathbf{V}')$ is in the convergent range. Consider the following integral:

$$\begin{aligned} (\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}) \times (\pi^\vee\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}^\vee) &\longrightarrow \mathbb{C}, \\ (u,v) &\longmapsto \textstyle\int_{\widetilde{G}} \langle g\cdot u, v\rangle \,\mathrm{d}g. \end{aligned} \tag{12.1}$$

Lemma 12.10. *The integrals in* (12.1) *are absolutely convergent and yield a continuous bilinear map.*

Proof. This is implied by Lemma 12.8 and Lemma 12.3. □

In view of Lemma 12.10, we define

$$\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi) := \frac{\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}}{\text{the left kernel of (12.1)}}. \tag{12.2}$$

This is a Casselman-Wallach representation of $\widetilde{G}'$, since it is a quotient of the full theta lift $(\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'})_G$ (the Hausdorff coinvariant space).

Lemma 12.11. *Assume that* $\dim^\circ \mathbf{V}' > 0$*. Then the representation* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is* $\frac{\dim \mathbf{V}}{\dim^\circ \mathbf{V}'}$*-bounded.*

Proof. This is also implied by Lemma 12.8 and Lemma 12.3. □

12.2.3. *Unitarity.*

Theorem 12.12. *Assume that* $\dim \mathbf{V}' \geqslant \dim^\circ \mathbf{V}$ *and* π *is* ν_π*-bounded for some*

$$\nu_\pi > \begin{cases} 2 - \frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}, & \textit{if } \dim^\circ \mathbf{V} \textit{ is even,} \\ 2 - \frac{\dim \mathbf{V}'-1}{\dim^\circ \mathbf{V}}, & \textit{if } \dim^\circ \mathbf{V} \textit{ is odd.} \end{cases}$$

Then $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is unitarizable if* π *is unitarizable. Moreover, when* π *is irreducible and unitarizable, and* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is nonzero, then* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is irreducible unitarizable.*

Proof. Assume that π is unitarizable. For the first claim, it suffices to show that

$$\int_{\widetilde{G}} \langle g \cdot u, u \rangle \, \mathrm{d}g \geqslant 0 \quad \text{for all } u \text{ in a dense subspace of } \pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}. \tag{12.3}$$

Here $\langle \, , \, \rangle$ denotes a Hermitian inner product on $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}$ which is invariant under $\widetilde{G} \times ((\widetilde{G} \times \widetilde{G}') \ltimes \mathrm{H}(W))$.

Take an orthogonal decomposition

$$\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$$

which is stable under both J' and L' such that

$$\dim \mathbf{V}'_2 = \begin{cases} \dim^\circ \mathbf{V}, & \text{if } \dim^\circ \mathbf{V} \text{ is even.} \\ \dim^\circ \mathbf{V} + 1, & \text{if } \dim^\circ \mathbf{V} \text{ is odd.} \end{cases}$$

(When $\dim^\circ \mathbf{V}$ is odd, $\dim \mathbf{V}'$ is always even.) Write

$$\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'} = (\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'_1})\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'_2}.$$

Note that the Hilbert space completions of $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'_1}$ and $\omega_{\mathbf{V},\mathbf{V}'_2}$, viewed as unitary representations of $\widetilde{G}$, satisfy the hypothesis for π_1 and π_2 in Proposition 12.5. For the latter, see [46, Theorem 3.2]. Thus (12.3) follows by Proposition 12.5.

The second claim follows from the fact that $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is a quotient of $(\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\pi)_G$ which is of finite length and has a unique irreducible quotient [36]. □

12.3. **Degenerate principal series.** Let $(\mathbf{U}, J_{\mathbf{U}}, L_{\mathbf{U}})$ be a nonzero $(\epsilon, \dot{\epsilon})$-space which is split in the sense that there is a $J_{\mathbf{U}}$-stable totally isotropic subspace $\mathbf{E}$ of $\mathbf{U}$ whose dimension equals half of that of $\mathbf{U}$. Then we have a complete polarization

$$\mathbf{U} = \mathbf{E} \oplus \mathbf{E}', \quad \text{where } \mathbf{E}' := L_{\mathbf{U}}(\mathbf{E}) \text{ is also totally isotropic.} \tag{12.4}$$

Define notations as in Section 7.3, for example $G_{\mathbf{U}} := \mathbf{G}_{\mathbf{U}}^{J_{\mathbf{U}}}$. The parabolic subgroup $P_{\mathbf{E}}$ of $G_{\mathbf{U}}$ stabilizing $\mathbf{E}$, has a Levi decomposition

$$P_{\mathbf{E}} = \mathrm{GL}_{\mathbf{E}} \ltimes N_{\mathbf{E}},$$

where $N_{\mathbf{E}}$ is the unipotent radical and $\mathrm{GL}_{\mathbf{E}} \cong \mathrm{GL}(\mathbf{E})^{J_{\mathbf{U}}}$ is the stabilizer of the polarization (12.4) in $G_{\mathbf{U}}$. Since the covering $\widetilde{P_{\mathbf{E}}} \to P_{\mathbf{E}}$ uniquely splits over $N_{\mathbf{E}}$, we view $N_{\mathbf{E}}$ as a subgroup of $\widetilde{P_{\mathbf{E}}}$. Thus we have a decomposition

$$\widetilde{P_{\mathbf{E}}} = \widetilde{\mathrm{GL}_{\mathbf{E}}} \ltimes N_{\mathbf{E}}.$$

In this section, we assume that $\mathbb{p}$ is the parity of $\dim \mathbf{E} + 1$ if $G_{\mathbf{U}}$ is real orthogonal or real symplectic, and the parity of $\dim \mathbf{E}$ if $G_{\mathbf{U}}$ is quaternionic. Let $\chi_{\mathbf{E}}$ be a character of $\widetilde{\mathrm{GL}_{\mathbf{E}}}$ as in Table 6. In the real symplectic case, $\varepsilon_{\mathbf{U}}$ denotes the nontrivial element in the kernel of the covering $\widetilde{G}_{\mathbf{U}} \to G_{\mathbf{U}}$, and there are two choices of $\chi_{\mathbf{E}}$ if $\mathbf{U}$ is moreover nonzero. In the quaternionic case, $\det : \mathrm{GL}_{\mathbf{E}} \to \mathbb{R}_+^\times$ denotes the reduced norm.

For each character χ of $\widetilde{\mathrm{GL}_{\mathbf{E}}}$, define the degenerate principal series

$$\mathrm{I}(\chi) := \mathrm{Ind}_{\widetilde{P_{\mathbf{E}}}}^{\widetilde{G}_{\mathbf{U}}} \chi.$$

We are particularly interested in $\mathrm{I}(\chi_{\mathbf{E}})$, which plays a critical role in the study of theta correspondence.

Define a positive function on $\widetilde{G}_{\mathbf{U}}$ by setting

$$\Xi_{\mathbf{U}}(g) := \int_{\widetilde{K}_{\mathbf{U}}} f_0(xg) \, \mathrm{d}x, \qquad g \in \widetilde{G}_{\mathbf{U}},$$

where f_0 denotes the element of $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}}}^{\widetilde{G}_{\mathbf{U}}} |\chi_{\mathbf{E}}|$ whose restriction to $\widetilde{K}_{\mathbf{U}}$ is the constant function 1, and $\mathrm{d}x$ denotes the normalized Haar measure on $\widetilde{K}_{\mathbf{U}}$. This function is independent of $\mathbf{E}$.

Identify the representation $\mathrm{I}(\chi_{\mathbf{E}})^\vee$ with $\mathrm{I}(\chi_{\mathbf{E}}^{-1})$ via the paring

$$(f_1, f_2) \mapsto \langle f_1, f_2\rangle := \int_{\widetilde{K}_{\mathbf{U}}} f_1(x) f_2(x)\mathrm{d}x, \quad f_1 \in \mathrm{I}(\chi_{\mathbf{E}}),\ f_2 \in \mathrm{I}(\chi_{\mathbf{E}}^{-1}).$$

The following lemma is clear from the definition of $\Xi_{\mathbf{U}}$.

Lemma 12.13. *For all $f_1 \in \mathrm{I}(\chi_{\mathbf{E}})$, $f_2 \in \mathrm{I}(\chi_{\mathbf{E}}^{-1})$ and $g \in \widetilde{G}_{\mathbf{U}}$,*

$$|\langle g \cdot f_1, f_2\rangle| \leqslant \Xi_{\mathbf{U}}(g) \cdot \max_{x \in \widetilde{K}_{\mathbf{U}}} |f_1(x)| \cdot \max_{x \in \widetilde{K}_{\mathbf{U}}} |f_2(x)|. \quad \square$$

12.4. **Matrix coefficient integrals against degenerate principal series.** Now we assume that $(\mathbf{V}, J, L)$ is a non-degenerate $(\epsilon, \dot{\epsilon})$-subspace of $\mathbf{U}$ so that

$$J = J_{\mathbf{U}}|_{\mathbf{V}} \quad \text{and} \quad L = L_{\mathbf{U}}|_{\mathbf{V}}.$$

Then $\widetilde{G} := \widetilde{G}_{\mathbf{V}}$ is naturally a closed subgroup of $\widetilde{G}_{\mathbf{U}}$. Further assume that

$$\dim \mathbf{V} \leqslant \dim \mathbf{E}.$$

We are interested in the growth of $\Xi_{\mathbf{U}}|_{\widetilde{G}}$.

Lemma 12.14. *Assume that $\dim^\circ \mathbf{V} > 0$. Then $\Xi_{\mathbf{U}}|_{\widetilde{G}}$ is $\nu_{\mathbf{U},\mathbf{V}}$-bounded, where*

$$\nu_{\mathbf{U},\mathbf{V}} := \begin{cases} \frac{\dim \mathbf{U} - 2}{2 \dim^\circ \mathbf{V}}, & \text{if } G \text{ is real orthogonal;} \\ \frac{\dim \mathbf{U} + 2}{2 \dim^\circ \mathbf{V}}, & \text{if } G \text{ is real symplectic;} \\ \frac{\dim \mathbf{U}}{2 \dim^\circ \mathbf{V}}, & \text{if } G \text{ is quaternionic.} \end{cases} \tag{12.5}$$

Proof. In view of the estimate of Harish-Chandra's Ξ-function as in [80, Theorem 4.5.3], and the estimate of $\Xi_{\mathbf{U}}$ using [80, Corollary 3.6.8], the lemma is routine to check. $\square$

Let π be a $\mathbb{p}$-genuine Casselman-Wallach representation of $\widetilde{G}$. Further assume that $\dim^\circ \mathbf{V} > 0$ and

$$\pi \text{ is } \nu_\pi\text{-bounded for some } \nu_\pi > 2 - \nu_{\mathbf{U},\mathbf{V}}, \tag{12.6}$$

where $\nu_{\mathbf{U},\mathbf{V}}$ is as in (12.5).

Let $\mathcal{J}$ be a subquotient of the degenerate principal series $\mathrm{I}(\chi_{\mathbf{E}})$.

Lemma 12.15. *The integrals in*

$$\begin{array}{rcl} (\pi \widehat{\otimes} \mathcal{J}) \times (\pi^\vee \widehat{\otimes} \mathcal{J}^\vee) & \to & \mathbb{C} \\ (u, v) & \mapsto & \int_{\widetilde{G}} \langle g \cdot u, v\rangle \mathrm{d}g, \end{array} \tag{12.7}$$

are absolutely convergent and define a continuous bilinear map.

Proof. By Lemma 12.13, there is a continuous seminorm $|\cdot|_{\mathcal{J}}$ on $\mathcal{J}$ and a continuous seminorm $|\cdot|_{\mathcal{J}^\vee}$ on $\mathcal{J}^\vee$ such that

$$|\langle g \cdot v_1, v_2\rangle| \leqslant \Xi_{\mathbf{U}}(g) \cdot |v_1|_{\mathcal{J}} \cdot |v_2|_{\mathcal{J}^\vee}, \quad \text{for all } v_1 \in \mathcal{J},\ v_2 \in \mathcal{J}^\vee,\ g \in \widetilde{G}_{\mathbf{U}}.$$

Take a positive function ℓ on $\widetilde{G}$ of logarithmic growth, and continuous seminorms $|\ |_\pi$ on π and $|\ |_{\pi^\vee}$ on $\pi^\vee$ such that

$$|\langle g \cdot u_1, u_2\rangle| \leqslant \ell(g) \cdot (\Xi_{\widetilde{G}}(g))^{\nu_\pi} \cdot |u_1|_\pi \cdot |u_2|_{\pi^\vee},$$

for all $g \in \widetilde{G}$, $u_1 \in \pi$, $u_2 \in \pi^\vee$.

Let $|\cdot|_{\pi\widehat{\otimes}\mathcal{J}}$ denote the continuous seminorm of $\pi\widehat{\otimes}\mathcal{J}$ which is the projective product of $|\cdot|_\pi$ and $|\cdot|_{\mathcal{J}}$. Likewise let $|\cdot|_{\pi^\vee\widehat{\otimes}\mathcal{J}^\vee}$ denote the continuous seminorm of $\pi^\vee\widehat{\otimes}\mathcal{J}^\vee$ which is the projective product of $|\cdot|_{\pi^\vee}$ and $|\cdot|_{\mathcal{J}^\vee}$. Then it is easy to see that

$$|\langle g\cdot w_1, w_2\rangle| \leqslant \ell(g)\cdot(\Xi_{\widetilde{G}}(g))^{\nu_\pi}\cdot\Xi_{\mathbf{U}}(g)\cdot|w_1|_{\pi\widehat{\otimes}\mathcal{J}}\cdot|w_2|_{\pi^\vee\widehat{\otimes}\mathcal{J}^\vee}, \tag{12.8}$$

for all $g\in\widetilde{G}$, $w_1\in\pi\widehat{\otimes}\mathcal{J}$, $w_2\in\pi^\vee\widehat{\otimes}\mathcal{J}^\vee$. Thus the lemma follows by Lemma 12.3 and Lemma 12.14. □

The orthogonal complement $\mathbf{V}^\perp$ of $\mathbf{V}$ in $\mathbf{U}$ is naturally an $(\epsilon,\dot\epsilon)$-space with $J_{\mathbf{V}^\perp} := J_{\mathbf{U}}|_{\mathbf{V}^\perp}$ and $L_{\mathbf{V}^\perp} := L_{\mathbf{U}}|_{\mathbf{V}^\perp}$. Define

$$\mathcal{R}_{\mathcal{J}}(\pi) := \frac{\pi\widehat{\otimes}\mathcal{J}}{\text{the left kernel of the bilinear map (12.7)}}. \tag{12.9}$$

It is a smooth Fréchet representation of $\widetilde{G}_{\mathbf{V}^\perp}$ of moderate growth.

Let $(\mathbf{V}^-, J_{\mathbf{V}^-}, L_{\mathbf{V}^-})$ denote the $(\epsilon,\dot\epsilon)$-space which equals $\mathbf{V}$ as a vector space, and is equipped with the form $-\langle\ ,\ \rangle_{\mathbf{V}}$, the conjugate linear map $J_{\mathbf{V}^-} = J$, and the linear map $L_{\mathbf{V}^-} = -L$. Then we have an obvious identification $G_{\mathbf{V}^-} = G_{\mathbf{V}}$. We identify $\mathbf{V}^-$ with an $(\epsilon,\dot\epsilon)$-subspace of $\mathbf{V}^\perp$ and fix a $J_{\mathbf{U}}$-stable subspace $\mathbf{E}_0$ in $\mathbf{E}$ such that

$$\mathbf{V}^\perp = \mathbf{V}^-\oplus(\mathbf{E}_0\oplus\mathbf{E}_0'),\quad \mathbf{E} = \mathbf{V}^\vartriangle\oplus\mathbf{E}_0 \quad\text{and}\quad \mathbf{E}' = \mathbf{V}^\triangledown\oplus\mathbf{E}_0', \tag{12.10}$$

where $\mathbf{E}_0' = L_{\mathbf{V}^\perp}(\mathbf{E}_0)$,

$$\mathbf{V}^\vartriangle := \{(v,v)\in\mathbf{V}\oplus\mathbf{V}^- \mid v\in\mathbf{V}\}\quad\text{and}\quad \mathbf{V}^\triangledown := \{(v,-v)\in\mathbf{V}\oplus\mathbf{V}^- \mid v\in\mathbf{V}\}.$$

In particular, $G := G_{\mathbf{V}}$ is identified with a subgroup of $G_{\mathbf{V}^\perp}$. Let $\mathrm{GL}_{\mathbf{E}_0} := \mathrm{GL}(\mathbf{E}_0)^{J_{\mathbf{V}^\perp}|_{\mathbf{E}_0}}$ and let

$$P_{\mathbf{E}_0} = M_{\mathbf{E}_0}\ltimes N_{\mathbf{E}_0}$$

denote the parabolic subgroup of $G_{\mathbf{V}^\perp}$ stabilizing $\mathbf{E}_0$, where $N_{\mathbf{E}_0}$ denotes the unipotent radical, and $M_{\mathbf{E}_0} = G\times\mathrm{GL}_{\mathbf{E}_0}$ is the Levi subgroup stabilizing the first decomposition in (12.10).

Let $\chi_{\mathbf{E}_0}$ denotes the restriction of $\chi_{\mathbf{E}}$ to $\widetilde{\mathrm{GL}_{\mathbf{E}_0}}$. Then $\pi\otimes\chi_{\mathbf{E}_0}$, which is preliminarily a representation of $\widetilde{G}\times\widetilde{\mathrm{GL}_{\mathbf{E}_0}}$, descends to a representation of $\widetilde{M}_{\mathbf{E}_0}$.

The rest of this section is devoted to a proof of the following proposition.

Proposition 12.16. *There is an isomorphism*

$$\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi)\cong\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})$$

of representations of $\widetilde{G}_{\mathbf{V}^\perp}$.

Identify $\mathrm{I}(\chi_{\mathbf{E}})$ with the space of smooth sections of the line bundle

$$\mathcal{L}(\chi_{\mathbf{E}}) := \widetilde{P}_{\mathbf{E}}\backslash(\widetilde{G}_{\mathbf{U}}\times(\chi_{\mathbf{E}}\otimes\rho_{\mathbf{E}}))$$

over $\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}_{\mathbf{U}}$, where $\rho_{\mathbf{E}}$ denotes the positive character of $\widetilde{P}_{\mathbf{E}}$ whose square is the modular character.

Note that $\widetilde{G}^\circ_{\mathbf{U}} := \widetilde{P}_{\mathbf{E}}\widetilde{G}\widetilde{G}_{\mathbf{V}^\perp}$ is open in $\widetilde{G}_{\mathbf{U}}$. Put

$$\mathrm{I}^\circ(\chi_{\mathbf{E}}) := \left\{ f\in\mathrm{I}(\chi_{\mathbf{E}}) \;\middle|\; \begin{array}{l}(Df)|_{\widetilde{G}_{\mathbf{U}}\backslash\widetilde{G}^\circ_{\mathbf{U}}} = 0 \text{ for all left invariant}\\ \text{differential operators } D \text{ on } \widetilde{G}_{\mathbf{U}}\end{array}\right\}.$$

It is identical to the space of Schwartz sections of $\mathcal{L}(\chi_{\mathbf{E}})$ over $\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}^\circ_{\mathbf{U}}$. Similarly we define a subspace $\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$ of $\mathrm{I}(\chi_{\mathbf{E}}^{-1})$.

As in (12.7), we define a continuous bilinear map

$$(\pi\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}))\times(\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1}))\to\mathbb{C}. \tag{12.11}$$

Using this map, we define a representation $\mathcal{R}_{\mathrm{I}^\circ(\chi_{\mathbf{E}})}(\pi)$ of $\widetilde{G}_{\mathbf{V}^\perp}$ as in (12.9).

Put

$$P'_{\mathbf{E}_0}:=\mathrm{GL}_{\mathbf{E}_0}\ltimes N_{\mathbf{E}_0}. \tag{12.12}$$

Lemma 12.17. *There is a canonical isomorphism*

$$\pi\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}})\cong\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}) \tag{12.13}$$

of representations of $\widetilde{G}\times\widetilde{G}_{\mathbf{V}^\perp}$, where $\widetilde{G}$ acts diagonally on the left hand side of (12.13)*, and it acts on the right hand side of* (12.13) *by*

$$(g\cdot f)(x)=g\cdot(f(g^{-1}x)),\quad f\in\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}),\ g\in\widetilde{G},\ x\in\widetilde{G}_{\mathbf{V}^\perp}.$$

Proof. Note that

$$\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}^\circ_{\mathbf{U}}=\widetilde{P}'_{\mathbf{E}_0}\backslash\widetilde{G}_{\mathbf{V}^\perp}$$

and thus

$$\mathrm{I}^\circ(\chi_{\mathbf{E}})\cong\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}\chi_{\mathbf{E}_0} \tag{12.14}$$

as representations of $\widetilde{G}\times\widetilde{G}_{\mathbf{V}^\perp}$, where $\widetilde{G}$ acts on the right hand side of (12.14) by

$$(g\cdot f)(x)=f(g^{-1}x),\quad f\in\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}\chi_{\mathbf{E}_0},\ g\in\widetilde{G},\ x\in\widetilde{G}_{\mathbf{V}^\perp}.$$

The lemma then easily follows by (12.14). □

Lemma 12.18. *The representation $\mathcal{R}_{\mathrm{I}^\circ(\chi_{\mathbf{E}})}(\pi)$ is isomorphic to $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})$.*

Proof. Similar to (12.13), we have a canonical isomorphism

$$\pi^\vee\widetilde{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})\cong\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1}). \tag{12.15}$$

It is clear that the continuous linear map

$$\begin{array}{rcl}\xi:\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}) & \longrightarrow & \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}),\\ f & \mapsto & \left(x\mapsto\int_{\widetilde{P}_{\mathbf{E}_0}/\widetilde{P}'_{\mathbf{E}_0}}(\rho_{\mathbf{E}_0}(g))\cdot(g\cdot(f(g^{-1}x)))\,\mathrm{d}g\right),\end{array}$$

is $\widetilde{G}_{\mathbf{V}^\perp}$-intertwining and surjective, where $\rho_{\mathbf{E}_0}$ denotes the positive character of $\widetilde{P}_{\mathbf{E}_0}$ whose square is the modular character, and $\mathrm{d}g$ denotes a $\widetilde{P}_{\mathbf{E}_0}$-invariant positive Borel measure on $\widetilde{\mathrm{P}}(\mathbf{E}_0)/\widetilde{\mathrm{P}}'(\mathbf{E}_0)$. Similarly, we define a $\widetilde{G}_{\mathbf{V}^\perp}$-intertwining surjective continuous linear map

$$\xi':\ \mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1})\to\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1}). \tag{12.16}$$

It is elementary to see that, when the invariant measures are suitably normalized,

$$\langle\xi(f),\xi'(f')\rangle=\int_{\widetilde{G}}\langle g\cdot f,f'\rangle\,\mathrm{d}g, \tag{12.17}$$

for all f and f' in the domains of ξ and ξ', respectively. Note that under isomorphisms (12.13) and (12.15), the paring between the domains of ξ and ξ' as in the right hand side of (12.17) agrees with the paring (12.11) between $\pi\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}})$ and $\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$. Thus the lemma follows as the natural pairing between $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})$ and $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1})$ is non-degenerate. □

Lemma 12.19. *Let* $u\in\pi\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}})$. *If*

$$\int_{\widetilde{G}}\langle g\cdot u,v\rangle\,\mathrm{d}g=0 \tag{12.18}$$

for all $v\in\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$, *then* (12.18) *also holds for all* $v\in\pi^\vee\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}}^{-1})$.

Proof. Take a sequence $(\eta_1,\eta_2,\eta_3,\cdots)$ of real valued smooth functions on $\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}_{\mathbf{U}}$ such that

- for all $i\geqslant 1$, $\mathrm{supp}(\eta_i)\subset\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}_{\mathbf{U}}^\circ$ and $0\leqslant\eta_i(x)\leqslant\eta_{i+1}(x)\leqslant 1$ for all $x\in\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}_{\mathbf{U}}$;
- $\bigcup_{i=1}^\infty\eta_i^{-1}(1)=\widetilde{P}_{\mathbf{E}}\backslash\widetilde{G}_{\mathbf{U}}^\circ$.

Let $v\in\pi^\vee\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}}^{-1})$. Note that $\eta_i\mathrm{I}(\chi_{\mathbf{E}}^{-1})\subset\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$. Thus $\eta_i v\in\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$.

In the proof of Lemma 12.15, take $\mathcal{J}=\mathrm{I}(\chi_{\mathbf{E}})$ and

$$|f_1|_{\mathcal{J}}=\max_{x\in\widetilde{K}_{\mathbf{U}}}\{|f_1(x)|\mid x\in\widetilde{K}_{\mathbf{U}}\}\quad\text{and}\quad|f_2|_{\mathcal{J}^\vee}=\max_{x\in\widetilde{K}_{\mathbf{U}}}\{|f_2(x)|\mid x\in\widetilde{K}_{\mathbf{U}}\},$$

for all $f_1\in\mathrm{I}(\chi_{\mathbf{E}})$, $f_2\in\mathrm{I}(\chi_{\mathbf{E}}^{-1})$. Then

$$|\eta_i v|_{\pi^\vee\widehat{\otimes}\mathcal{J}^\vee}\leqslant|v|_{\pi^\vee\widehat{\otimes}\mathcal{J}^\vee}.$$

Thus by (12.8) and Lebesgue's dominated convergence theorem,

$$\int_{\widetilde{G}}\langle g\cdot u,v\rangle\,\mathrm{d}g=\lim_{i\to+\infty}\int_{\widetilde{G}}\langle g\cdot u,\eta_i v\rangle\,\mathrm{d}g=0.$$

□

Lemma 12.20. *Let* $u\in\pi\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}})$. *Then there is a vector* $u'\in\pi\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}})$ *such that*

$$\int_{\widetilde{G}}\langle g\cdot u,v\rangle\,\mathrm{d}g=\int_{\widetilde{G}}\langle g\cdot u',v\rangle\,\mathrm{d}g$$

for all $v\in\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$.

Proof. Similar to Lemma 12.19, we know that for every $v\in\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$, if

$$\int_{\widetilde{G}}\langle g\cdot u',v\rangle\,\mathrm{d}g=0\quad\text{for all } u'\in\pi\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}), \tag{12.19}$$

then (12.19) also holds for all $u'\in\pi\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}})$. The proof of Lemma 12.18 shows that v satisfies (12.19) if and only if it is in the kernel of ξ'. Here ξ' is as in (12.16), and $\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})$ is identified with $\mathrm{ind}_{\widetilde{P}'_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1})$ as in (12.15). Thus the continuous linear functional

$$\begin{aligned}\pi^\vee\widehat{\otimes}\mathrm{I}^\circ(\chi_{\mathbf{E}}^{-1})&\longrightarrow\mathbb{C},\\ v&\longmapsto\textstyle\int_{\widetilde{G}}\langle g\cdot u,v\rangle\,\mathrm{d}g\end{aligned}$$

descents to a continuous linear functional

$$\varphi_u:\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee\otimes\chi_{\mathbf{E}_0}^{-1})\longrightarrow\mathbb{C}.$$

Using the theorem of Dixmier-Malliavin [23, Theorem 3.3], we write

$$u=\sum_{i=1}^s\int_{\widetilde{G}_{\mathbf{V}^\perp}}\phi_i(g)g\cdot u_i\,\mathrm{d}g,$$

where ϕ_i's are compactly supported smooth functions on $\widetilde{G}_{\mathbf{V}^\perp}$, and u_i's are elements of $\pi\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}})$. Here $\widetilde{G}_{\mathbf{V}^\perp}$ acts on $\pi\widehat{\otimes}\mathrm{I}(\chi_{\mathbf{E}})$ through the restriction of the action of $\widetilde{G}_{\mathbf{U}}$ on $\mathrm{I}(\chi_{\mathbf{E}})$. Then it is easily checked that for every $v' \in \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi^\vee \otimes \chi_{\mathbf{E}_0}^{-1})$,

$$\varphi_u(v') = \sum_{i=1}^{s} \varphi_{u_i}\left(\int_{\widetilde{G}_{\mathbf{V}^\perp}} \phi_i(g^{-1})g\cdot v'\,\mathrm{d}g\right).$$

This equals the evaluation at v' of the linear functional

$$\sum_{i=1}^{s}\int_{\widetilde{G}_{\mathbf{V}^\perp}} \phi_i(g)g\cdot\varphi_{u_i}\,\mathrm{d}g.$$

By [73, Lemma 3.5], the above functional equals the pairing with an element of $\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes \chi_{\mathbf{E}_0})$, and the lemma then follows by the proof of Lemma 12.18. □

Now Proposition 12.16 follows by Lemmas 12.18, 12.19 and 12.20.

12.5. **Degenerate principal series and Rallis quotients.** We are in the setting of Section 12.2 and Section 12.3. Assume that $\dim^\circ \mathbf{V}' > 0$. Recall $\mathbf{U} = \mathbf{E}\oplus\mathbf{E}'$ from (12.4) and assume that

$$\dim\mathbf{E} = \dim\,\mathbf{V}' + \delta, \quad \text{where } \delta := \begin{cases} 1, & \text{if } G_{\mathbf{U}} \text{ is real orthogonal;}\\ -1, & \text{if } G_{\mathbf{U}} \text{ is real symplectic;}\\ 0, & \text{if } G_{\mathbf{U}} \text{ is quaternionic.}\end{cases} \tag{12.20}$$

The Rallis quotient $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}$ is an irreducible unitarizable representation of $\widetilde{G}_{\mathbf{U}}$ (*cf.* [42, 67, 83]). Let $1_{\mathbf{V}'}$ denote the trivial representation of $\widetilde{G}'$. Then $(1_{\mathbf{V}'}, \mathbf{U})$ is in the convergent range.

Lemma 12.21. *One has that*

$$\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'}) \cong (\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}.$$

Proof. Since $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}$ is irreducible and $\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})$ is obviously a quotient of $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}$, it suffices to show that $\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})$ is nonzero.

Write V_0' for the standard real or quaternionic representation of G'. As usual, realize both $\omega_{\mathbf{V}',\mathbf{U}}$ and $\omega_{\mathbf{V}',\mathbf{U}}^\vee$ on the space of Schwartz functions on $V_0'^d$, where $d = \dim\mathbf{E}$ if G' is real orthogonal or real symplectic, and $d = \frac{1}{2}\dim\mathbf{E}$ if G' is quaternionic. Take a positive valued Schwartz function ϕ on $V_0'^d$. Then

$$\langle g\cdot\phi,\phi\rangle = \int_{V_0'^d}\phi(g^{-1}\cdot x)\cdot\phi(x)\,\mathrm{d}x > 0, \quad \text{for all } g\in\widetilde{G}'.$$

Thus

$$\int_{\widetilde{G}'}\langle g\cdot\phi,\phi\rangle\,\mathrm{d}g \neq 0,$$

and the lemma follows. □

Let $\chi_{\mathbf{E}}$ be as in Section 12.3. The relationship between the degenerate principle series representation $\mathrm{I}(\chi_{\mathbf{E}})$ and Rallis quotients is summarized in the following lemma (see [43, Theorem 2.4], [44, Introduction], [45, Theorem 6.1] and [83, Sections 9 and 10]).

Lemma 12.22. *(a) If $G_{\mathbf{U}}$ is real orthogonal, then*

$$\mathrm{I}(\chi_{\mathbf{E}}) \cong (\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}\oplus((\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}\otimes\mathrm{sgn}_{\mathbf{U}}),$$

where $\mathrm{sgn}_{\mathbf{U}}$ *denotes the sign character of the orthogonal group* $\widetilde{G}_{\mathbf{U}}$.

(b) If $G_{\mathbf{U}}$ is real symplectic, then

$$\mathrm{I}(\chi_{\mathbf{E}}) \oplus \mathrm{I}(\chi'_{\mathbf{E}}) \cong \bigoplus_{\mathbf{V}''} (\omega_{\mathbf{V}'',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}''}},$$

where $\chi'_{\mathbf{E}} \neq \chi_{\mathbf{E}}$ is the character which also satisfies the condition in Table 6, and $\mathbf{V}''$ runs over the isomorphism classes of $(1,1)$-spaces whose dimension equals that of $\mathbf{V}'$.

(c) If $G_{\mathbf{U}}$ is quaternionic symplectic, then

$$\mathrm{I}(\chi_{\mathbf{E}}) \cong (\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}.$$

(d) If $G_{\mathbf{U}}$ is quaternionic orthogonal, then there is an exact sequence

$$0 \longrightarrow \bigoplus\nolimits_{\mathbf{V}''} (\omega_{\mathbf{V}'',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}''}} \longrightarrow \mathrm{I}(\chi_{\mathbf{E}}) \longrightarrow \bigoplus\nolimits_{\mathbf{V}'''} (\omega_{\mathbf{V}''',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}'''}} \longrightarrow 0,$$

where $\mathbf{V}''$ runs over the isomorphism classes of $(-1,1)$-spaces whose dimension equals that of $\mathbf{V}'$, and $\mathbf{V}'''$ runs over the isomorphism classes of $(-1,1)$-spaces whose dimension equals $\dim \mathbf{V}' - 2$. ☐

Thanks to Lemma 12.21, $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}'}}$ and $(\omega_{\mathbf{V}'',\mathbf{U}})_{\widetilde{G}_{\mathbf{V}''}}$ in Lemma 12.22 can be replaced by $\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})$ and $\bar{\Theta}_{\mathbf{V}'',\mathbf{U}}(1_{\mathbf{V}''})$ respectively.

13. Associated characters of theta lifts

In this section we will prove Theorem 13.4 which gives a formula for the associated character of a certain theta lift in the convergent range.

Throughout this section, we fix a rational dual pair $(\mathbf{V}, \mathbf{V}')$ and we retain the notation in Section 7.6 and Section 12.2. As in Section 7.7, we also assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$.

13.1. An upper bound on the associated character of certain theta lift.

13.1.1. *Associated characters and associated cycles.* We recall fundamental results of Vogan [77]. Suppose $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is a nilpotent orbit. A finite length $(\mathfrak{g}, \widetilde{K})$-module π is said to be *$\mathcal{O}$-bounded* (or *bounded by $\mathcal{O}$*) if the associated variety of the annihilator ideal $\mathrm{Ann}(\pi)$ is contained in $\overline{\mathcal{O}}$. It follows from [77, Theorem 8.4] that π is $\mathcal{O}$-bounded if and only if its associated variety $\mathrm{AV}(\pi)$ is contained in $\overline{\mathcal{O}} \cap \mathfrak{p}$. Let $\mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K})$ denote the category of $\mathbb{p}$-genuine $\mathcal{O}$-bounded finite length $(\mathfrak{g}, \widetilde{K})$-modules, and write $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K})$ for its Grothendieck group. Recall the group $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ from Section 7.7. From [77, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{Ch}_{\mathcal{O}} \colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K}) \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

For a $\mathbb{p}$-genuine $\mathcal{O}$-bounded $(\mathfrak{g}, \widetilde{K})$-module π of finite length, we call $\mathrm{Ch}_{\mathcal{O}}(\pi)$ the associated character of π.

Let $\mu \colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) \to \mathbb{Z}[(\mathcal{O} \cap \mathfrak{p})/\mathbf{K}]$ be the map of taking dimensions of the isotropy representations. Post-composing μ with $\mathrm{Ch}_{\mathcal{O}}$ gives the associated cycle map:

$$\mathrm{AC}_{\mathcal{O}} \colon \mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K}) \longrightarrow \mathbb{Z}[(\mathcal{O} \cap \mathfrak{p})/\mathbf{K}].$$

For any π in $\mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g}, \widetilde{K})$ and a nilpotent $\mathbf{K}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}$, let $\mathrm{c}_{\mathscr{O}}(\pi)$ denote the multiplicity of $\mathscr{O}$ in $\mathrm{AC}_{\mathcal{O}}(\pi)$. For a $\mathbb{p}$-genuine Casselman-Wallach representation of $\widetilde{G}$, the notion of $\mathcal{O}$-boundedness, and its associated character is defined by using its Harish-Chandra module.

13.1.2. *Algebraic theta lifting.* Write $\mathscr{Y}_{\mathbf{V},\mathbf{V}'}$ for the subspace of $\omega_{\mathbf{V},\mathbf{V}'}$ consisting of all the vectors which are annihilated by some powers of $\mathcal{X}$. This is a dense subspace of $\omega_{\mathbf{V},\mathbf{V}'}$ and is naturally a $((\mathfrak{g}\times\mathfrak{g}')\ltimes\mathfrak{h}(W),\ \widetilde{K}\times\widetilde{K}')$-module. Here $\mathfrak{h}(W)$ denotes the complexified Lie algebra of $\mathrm{H}(W)$.

Definition 13.1. *For each* $\mathbb{p}$*-genuine* $(\mathfrak{g}',\ \widetilde{K}')$*-module* π' *of finite length, define*

$$\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi') := (\mathscr{Y}_{\mathbf{V},\mathbf{V}'}\otimes\pi')_{\mathfrak{g}',\widetilde{K}'}, \qquad \text{(the coinvariant space).}$$

For each $\mathbb{p}$-genuine finite length $(\mathfrak{g}',\ \widetilde{K}')$-module π', the $(\mathfrak{g},\ \widetilde{K})$-module $\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi')$, also abbreviated as $\check{\Theta}(\pi')$, is $\mathbb{p}$-genuine and of finite length [36]. Recall the notations in Section 7.6.2, in particular the map $\boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}$ in (7.9). We have the following estimate of the size of $\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi')$.

Lemma 13.2 ([48, Theorem B and Corollary E])**.** *For each* $\mathbb{p}$*-genuine* $(\mathfrak{g}',\ \widetilde{K}')$*-module* π' *of finite length,*

$$\mathrm{AV}(\check{\Theta}(\pi')) \subset M(M'^{-1}(\mathrm{AV}(\pi'))).$$

Consequently, if π *is* $\mathcal{O}'$*-bounded for a nilpotent orbit* $\mathcal{O}'\in\mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$*, then* $\check{\Theta}(\pi')$ *is* $\boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}(\mathcal{O}')$*-bounded.*

13.1.3. *The bound in the descent and good generalized descent cases.* We will prove an upper bound of associated characters in the descent and good generalized descent cases.

Theorem 13.3. *Let* $\mathcal{O}\in\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ *and* $\mathcal{O}'\in\mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ *. Suppose that*

(a) $\mathcal{O}'$ *is a descent of* $\mathcal{O}$*, that is,* $\mathcal{O}'=\boldsymbol{\nabla}(\mathcal{O})$*, or*

(b) $\mathcal{O}$ *is good for generalized descent (see Definition 7.15) and* $\mathcal{O}'=\boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$*.*

Then for every $\mathbb{p}$*-genuine* $\mathcal{O}'$*-bounded* $(\mathfrak{g}',\ \widetilde{K}')$*-module* π' *of finite length,* $\check{\Theta}(\pi')$ *is* $\mathcal{O}$*-bounded and*

$$\mathrm{Ch}_{\mathcal{O}}\,\check{\Theta}(\pi') \preceq \vartheta_{\mathcal{O}',\mathcal{O}}(\mathrm{Ch}_{\mathcal{O}'}(\pi')).$$

Proof. With the geometric properties investigated in Appendix B.1, the proof to be given is similar to that of [48, Theorem C].

We recall some results in [48]. There are natural good filtrations on π' and $\check{\Theta}(\pi')$ generated by the minimal degree $\widetilde{K}'$-types and $\widetilde{K}$-types, respectively. Define

$$\mathscr{A} := \mathrm{Gr}\,\pi'\otimes\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}'} \quad\text{and}\quad \mathscr{B} := \mathrm{Gr}\,\check{\Theta}(\pi')\otimes\varsigma^{-1}_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}}.$$

We view $\mathscr{A}$ and $\mathscr{B}$ as a $\mathbf{K}'$-equivariant coherent sheaf on $\mathfrak{p}'^*\cong\mathfrak{p}'$ and a $\mathbf{K}$-equivariant coherent sheaf on $\mathfrak{p}^*\cong\mathfrak{p}$, respectively. Moreover, $\mathrm{AV}(\pi')=\mathrm{Supp}(\mathscr{A})$ and $\mathrm{AV}(\check{\Theta}(\pi'))=\mathrm{Supp}(\mathscr{B})$.

Recall the moment maps defined in Section 7.6.1. Define the following right exact functor

$$\mathcal{L}\colon \mathscr{F}\mapsto (M_*(M'^*(\mathscr{F})))^{\mathbf{K}'}$$

from the category of $\mathbf{K}'$-equivariant quasi-coherent sheaves on $\mathfrak{p}'$ to the category of $\mathbf{K}$-equivariant quasi-coherent sheaves on $\mathfrak{p}$, where $\mathscr{F}$ is a $\mathbf{K}'$-equivariant quasi-coherent sheave on $\mathfrak{p}'$, and M'^* and M_* denote the pull-back and push-forward functors respectively. There is a canonical surjective morphism of $\mathbf{K}$-equivariant sheaves on $\mathfrak{p}$ as follows: (*cf.* [48, Equation (16)])

$$\mathcal{Q}\colon \mathcal{L}(\mathscr{A}) \longrightarrow\!\!\!\!\!\rightarrow \mathscr{B}.$$

Note that the first equality of (7.8) and the first assertion of Lemma 7.16 imply that $\mathrm{Supp}(\mathcal{L}(\mathscr{A}))$ is contained in $M(M'^{-1}(\mathrm{Supp}(\mathscr{A})))\subset\overline{\mathcal{O}}\cap\mathfrak{p}$. In particular, $\check{\Theta}(\pi)$ is $\mathcal{O}$-bounded.

According to [48, Proposition 4.3], we may fix a finite filtration

$$0 = \mathscr{A}_0 \subset \cdots \subset \mathscr{A}_l \subset \cdots \subset \mathscr{A}_f = \mathscr{A} \qquad (f \geqslant 0)$$

of $\mathscr{A}$ by $\mathbf{K}'$-equivariant coherent sheaves on $\mathfrak{p}'$ such that for any $1 \leqslant l \leqslant f$, the space of global sections of $\mathscr{A}^l := \mathscr{A}_l/\mathscr{A}_{l-1}$ is an irreducible $(\mathbb{C}[\overline{\mathscr{O}'_l}], \mathbf{K}')$-module for a nilpotent $\mathbf{K}'$-orbit $\mathscr{O}'_l$ in $\overline{\mathcal{O}'} \cap \mathfrak{p}'$.

Let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$. If $\mathscr{O}$ does not admit a generalized descent, then

$$\mathscr{O} \cap M(\mathcal{X}) = \varnothing,$$

and $\mathscr{O}$ does not appear in the support of $\mathrm{Ch}_{\mathcal{O}}(\check{\Theta}(\pi))$.

Now suppose that $\mathscr{O}$ admits a generalized descent $\mathscr{O}' := \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Retain the notation in Section 7.7.2 where an $(\epsilon', \dot{\epsilon}')$-space decomposition $\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$ and an element

$$T \in \mathcal{X}_1^{\circ} := \{\, w \in \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_1) \mid w \text{ is surjective} \,\} \cap \mathcal{X} \subseteq \mathcal{X}^{\mathrm{gen}}$$

is fixed such that $X := M(T) \in \mathscr{O}$ and $X' := M'(T) \in \mathscr{O}'$.

For $1 \leqslant l \leqslant f$, if $\mathscr{O}'_l = \mathscr{O}'$, then Lemma B.2 and Lemma B.4 ensure that there exists a Zariski open set U in $\mathfrak{p}$ such that $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$ and the quasi-coherent sheaf $\mathcal{L}(\mathscr{A}^l)|_U$ descends to a quasi-coherent sheaf on the closed subvariety $\mathscr{O}$ of U. (Thus $\mathcal{L}(\mathscr{A}^l)$ is generically reduced around X in the sense of [77, Proposition 2.9].)

Claim. *Assume that $\mathscr{O}'_l = \mathscr{O}'$ $(1 \leqslant l \leqslant f)$. Then we have the following isomorphism of algebraic representations of $\mathbf{K}_X$:*

$$i_X^*(\mathcal{L}(\mathscr{A}^l)) \cong (i_{X'}^*(\mathscr{A}^l))^{\mathbf{K}'_2} \circ \alpha_1.$$

Here α_1 is defined in (7.19), *$i_X\colon \{\, X \,\} \hookrightarrow \mathfrak{p}$ and $i_{X'}\colon \{\, X' \,\} \hookrightarrow \mathfrak{p}'$ are the inclusion maps, and i_X^* and $i_{X'}^*$ are the associated pull-back functors of the quasi-coherent sheaves.*

Let $\mathrm{m}_{\mathfrak{p}}(X)$ be the maximal ideal of $\mathbb{C}[\mathfrak{p}]$ associated to X, and $\kappa(X) := \mathbb{C}[\mathfrak{p}]/\mathrm{m}_{\mathfrak{p}}(X)$ be the residual field at X. Let

$$Z_{X,\overline{\mathscr{O}'}} := \left\{\, w \in \mathcal{X} \;\middle|\; M(w) = X, M'(w) \in \overline{\mathscr{O}'} \,\right\}.$$

Then $T \in Z_{X,\overline{\mathscr{O}'}}$ and

$$\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}] = \mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}]\otimes\mathbb{C}[\mathfrak{p}']} (\kappa(X) \otimes \mathbb{C}[\overline{\mathscr{O}'}])$$

by Lemma B.1 and Lemma B.3.

Let A^l be the module of global sections of $\mathscr{A}^l$. We have

$$\begin{aligned} i_X^*(\mathcal{L}(\mathscr{A}^l)) &= \kappa(X) \otimes_{\mathbb{C}[\mathfrak{p}]} \left(\mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}']} \mathbb{C}[\overline{\mathscr{O}'}] \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l\right)^{\mathbf{K}'} \\ &= \left(\mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}]\otimes\mathbb{C}[\mathfrak{p}']} (\kappa(X) \otimes \mathbb{C}[\overline{\mathscr{O}'}]) \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l\right)^{\mathbf{K}'} \\ &= (\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}] \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l)^{\mathbf{K}'}. \end{aligned}$$

Let $\rho' := i_{X'}^*(\mathscr{A}^l)$ be the isotropy representation of $\mathscr{A}^l$ at X'. Since $Z_{X,\overline{\mathscr{O}'}}$ is the $\mathbf{K}' \times \mathbf{K}_X$-orbit of T by Lemma B.1 (iii) and Lemma B.3 (ii), by considering the diagram (*cf.* [48, Section 4.2])

$$\begin{tikzcd} \{\, X' \,\} \arrow[d, hook] & \{\, T \,\} \arrow[l] \arrow[d, hook] \arrow[dr] & \\ \overline{\mathscr{O}'} & Z_{X,\overline{\mathscr{O}'}} \arrow[l, "M'"'] \arrow[r, "M"] & \{\, X \,\}, \end{tikzcd}$$

we know that $\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}]\otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l$ is isomorphic to the algebraically induced representation $\mathrm{Ind}_{\mathbf{S}_T}^{\mathbf{K}'\times\mathbf{K}_X}\rho'$, where $\mathbf{S}_T$ is the stabilizer group as in (7.20), and ρ' is viewed as an $\mathbf{S}_T$-module via the natural projection $\mathbf{S}_T\to\mathbf{K}'_{X'}$. Now by Frobenius reciprocity, we have

$$i_X^*(\mathcal{L}(\mathscr{A}^l))\cong(\mathrm{Ind}_{\mathbf{S}_T}^{\mathbf{K}'\times\mathbf{K}_X}\rho')^{\mathbf{K}'}=(\mathrm{Ind}_{\mathbf{K}'_2\times(\mathbf{K}'_{1,X'_1}\times_{\alpha_1}\mathbf{K}_X)}^{\mathbf{K}'\times\mathbf{K}_X}\rho')^{\mathbf{K}'}=(\rho')^{\mathbf{K}'_2}\circ\alpha_1.$$

This finishes the proof of the claim.

On the other hand, if $\mathscr{O}'_l\neq\mathscr{O}'$, then (7.11) implies that

$$\mathscr{O}\cap M(M'^{-1}(\overline{\mathscr{O}'_l}))=\varnothing,$$

and hence $\mathscr{O}$ is not contained in $\mathrm{Supp}(\mathcal{L}(\mathscr{A}^l))$. In conlcusion, the following inequality holds in the Grothedieck group of the category of algebraic representations of $\mathbf{K}_X$:

$$\chi(X,\mathscr{B})\preceq\chi(X,\mathcal{L}(\mathscr{A}))\preceq\sum_{l=1}^{f}\chi(X,\mathcal{L}(\mathscr{A}^l))=\sum_{l=1}^{f}(i_{X'}^*(\mathscr{A}^l))^{\mathbf{K}'_2}\circ\alpha_1=\chi(X',\mathscr{A})^{\mathbf{K}'_2}\circ\alpha_1,$$

where $\chi(X,\,\cdot\,)$ indicates the virtual character of $\mathbf{K}_X$ attached to a $\mathbf{K}$-equivariant coherent sheaf on $\mathfrak{p}$ whose support is contained in $\overline{\mathcal{O}}\cap\mathfrak{p}$, and similarly for $\chi(X',\,\cdot\,)$ (cf. [77, Definition 2.12]). This finishes the proof of the theorem. □

13.2. **An equality of associated characters in the convergent range.** In this subsection, we assume that $\dim^\circ\mathbf{V}>0$. The purpose of this section is to prove the following theorem.

Theorem 13.4. *Let $\mathcal{O}\in\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}'\in\mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ such that $\mathcal{O}$ is the descent of $\mathcal{O}'$. Write $\mathbf{D}(\mathcal{O}')=[c_0,c_1,\cdots,c_k]$ ($k\geqslant1$). Assume that $c_0>c_1$ when G is a real symplectic group. Then for every $\mathcal{O}$-bounded Casselman-Wallach representation π of $\widetilde{G}$ such that $(\pi,\,\mathbf{V}')$ is in the convergent range (see Definition 12.9), $\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi)$ is $\mathcal{O}'$-bounded and*

$$\mathrm{Ch}_{\mathcal{O}'}(\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi))=\vartheta_{\mathcal{O},\mathcal{O}'}(\mathrm{Ch}_{\mathcal{O}}(\pi)).\tag{13.1}$$

Remark. When $(\mathbf{V},\,\mathbf{V}')$ is in the stable range, Theorem 13.4 is proved for unitary representations π in [48].

We keep the setting of Theorem 13.4. First observe that the Harish-Chandra module of $\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi)$ is isomorphic to a quotient of $\check\Theta(\pi^{\mathrm{al}})$, where π^{al} denotes the Harish-Chandra module of π. Thus Theorem 13.3 implies that $\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi)$ is $\mathcal{O}'$-bounded and

$$\mathrm{Ch}_{\mathcal{O}'}\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi)\preceq\vartheta_{\mathcal{O},\mathcal{O}'}(\mathrm{Ch}_{\mathcal{O}}(\pi)).\tag{13.2}$$

We will devote the rest of this section to prove that the equality in (13.2) holds.

13.2.1. *Doubling.* We consider a two-step theta lifting. Let $\mathbf{U}$ be as in (12.20). We realize $\mathbf{V}$ as a non-genenerate $(\epsilon,\dot\epsilon)$-subspace of $\mathbf{U}$ and write $\mathbf{V}^\perp$ for the orthogonal complement of $\mathbf{V}$ in $\mathbf{U}$, which is also an $(\epsilon,\dot\epsilon)$-space.

Note that $\dim^\circ\mathbf{V}'>0$ and Lemma 12.11 implies that $(\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi),\,\mathbf{V}^\perp)$ is in the convergent range. Comparing (12.5) with (12.20), we have $\nu_{\mathbf{U},\mathbf{V}}=\frac{\dim\mathbf{V}'}{\dim^\circ\mathbf{V}}$. Thus (12.6) holds and $\mathcal{R}_{\bar\Theta_{\mathbf{U}}(1_{\mathbf{V}'})}(\pi)$ is defined by Lemma 12.15.

Lemma 13.5. *One has that*

$$\bar\Theta_{\mathbf{V}',\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi))\cong\mathcal{R}_{\bar\Theta_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})}(\pi).\tag{13.3}$$

Proof. Note that the integral in

$$(13.4)\qquad \begin{aligned} (\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega_{\mathbf{V}',\mathbf{V}^\perp}) \times (\pi^\vee\widehat{\otimes}\omega^\vee_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega^\vee_{\mathbf{V}',\mathbf{V}^\perp}) &\longrightarrow \mathbb{C} \\ (u,v) &\longmapsto \int_{\widetilde{G}\times\widetilde{G}'} \langle g\cdot u, v\rangle \,\mathrm{d}g \end{aligned}$$

is absolutely convergent and defines a continuous bilinear map. In view of Fubini's theorem and Lemma 12.21, the lemma follows as both sides of (13.3) are isomorphic to the quotient of $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega_{\mathbf{V}',\mathbf{V}^\perp}$ by the left radical of the pairing (13.4). ☐

We will use (13.3) freely in the rest of this section.

13.2.2. *On certain induced orbits.* The main step in the proof of Theorem 13.4 consists of comparison of bound of the associated cycle of $\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))$ with the formula (due to Barbasch) of the wavefront cycle of a certain parabolically induced representation.

In this section, let $\mathfrak{g}_\mathbb{R} = \mathfrak{g}^J$ be the Lie algebra of G, which is identified with its dual space $\mathfrak{g}_\mathbb{R}^*$ under the trace from. Let $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R})$ denote the set of nilpotent G-orbits in $\mathbf{i}\mathfrak{g}_\mathbb{R}$. Similar notation will be used without further explanation. For example, for every Levi subgroup M of G, $\mathfrak{m}_\mathbb{R}$ denotes its Lie algebra, which is identified with the dual space $\mathfrak{m}_\mathbb{R}^*$ by using the trace form on $\mathfrak{g}$.

Let

$$\mathsf{KS}\colon\ \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R}) \to \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$$

be the natural bijection given by the Kostant-Sekiguchi correspondence (*cf.* [70, Equation (6.7)]). By abuse of notation, we also let $\ddot{\mathbf{D}}$ denote the map $\ddot{\mathbf{D}}\circ\mathsf{KS}\colon\ \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R}) \to \ddot{\mathcal{P}}$. See Section 7.5 for the parametrization map $\ddot{\mathbf{D}}\colon\ \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p}) \to \ddot{\mathcal{P}}$.

Theorem 13.6 (*cf.* Barbasch [10, Corollary 5.0.10])**.** *Let G_1 be an arbitrary real reductive group and let P_1 be a real parabolic subgroup of G_1, namely a closed subgroup of G_1 whose Lie algebra is a parabolic subalgebra of $\mathfrak{g}_{1,\mathbb{R}}$. Let N_1 be the unipotent radical of P_1 and $M_1 := P_1/N_1$. Write*

$$r_1 : \mathbf{i}(\mathfrak{g}_{1,\mathbb{R}}/\mathfrak{n}_{1,\mathbb{R}})^* \longrightarrow \mathbf{i}\,\mathfrak{m}^*_{1,\mathbb{R}}$$

for the natural map. Let π_1 be a Casselman-Wallach representation of M_1 with the wavefront cycle

$$\mathrm{WF}(\pi_1) = \sum_{\mathscr{O}_\mathbb{R}\in\mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}^*_{1,\mathbb{R}})} \mathrm{c}_{\mathscr{O}_\mathbb{R}}[\mathscr{O}_\mathbb{R}].$$

Then

$$\mathrm{WF}(\mathrm{Ind}_{P_1}^{G_1}\pi) = \sum_{(\mathscr{O}_\mathbb{R},\mathscr{O}'_\mathbb{R})} \mathrm{c}_{\mathscr{O}_\mathbb{R}} \frac{\#C_{G_1}(v')}{\#C_{P_1}(v')}[\mathscr{O}'_\mathbb{R}],$$

where the summation runs over all pairs $(\mathscr{O}_\mathbb{R},\mathscr{O}'_\mathbb{R})$ such that $\mathscr{O}_\mathbb{R}\in\mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}^*_{1,\mathbb{R}})$ and $\mathscr{O}'_\mathbb{R}\in\mathrm{Nil}_{G_1}(\mathbf{i}\mathfrak{g}^*_{1,\mathbb{R}})$ is an induced orbit of $\mathscr{O}_\mathbb{R}$, v' is an element of $\mathscr{O}'_\mathbb{R}\cap r_1^{-1}(\mathscr{O}_\mathbb{R})$, and $C_{G_1}(v')$ and $C_{P_1}(v')$ are the component groups of the centralizers of v' in G_1 and P_1, respectively.

Remarks. 1. While Barbasch proved the theorem when G_1 is the real points of a connected reductive algebraic group, his proof still works in the slightly more general setting of Theorem 13.6.
2. The notion of induced nilpotent orbits was introduced by Lusztig and Spaltenstein for complex reductive groups [49]. For a real reductive group G_1, a nilpotent orbit $\mathscr{O}'_\mathbb{R}\in\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}^*_{1,\mathbb{R}})$ is called an induced orbit of a nilpotent orbit $\mathscr{O}_\mathbb{R}\in\mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}^*_{1,\mathbb{R}})$ if $\mathscr{O}'_\mathbb{R}\cap r_1^{-1}(\mathscr{O}_\mathbb{R})$ is open in $r_1^{-1}(\mathscr{O}_\mathbb{R})$ (see [10, Definition 5.0.7] or [61]). We write $\mathrm{Ind}_{P_1}^{G_1}\mathscr{O}_\mathbb{R}$ for the set of all induced orbits of $\mathscr{O}_\mathbb{R}$. Similar notation applies for a complex reductive group.

3. According to the fundamental result of Schimd-Vilonen [70], the wave front cycle and the associated cycle agree under the Kostant-Sekiguchi correspondence. We thank Professor Vilonen for confirming that their result extends to nonlinear groups. Therefore the associated cycle of a parabolically induced representation as in the above theorem of Barbasch is also determined.

We now consider induced orbits appearing in our cases. Retain the setting in Section 12.4 (see (12.10) and onwards), where

$$\mathbf{V}^\perp = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}_0',$$

$M_{\mathbf{E}_0} = G \times \mathrm{GL}_{\mathbf{E}_0}$ and the parabolic subgroup of $G_{\mathbf{V}^\perp}$ stabilizing $\mathbf{E}_0$ is $P_{\mathbf{E}_0} = M_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$.

Recall that $\mathbf{D}(\mathcal{O}') = [c_0, \cdots, c_k]$ and $\mathbf{D}(\mathcal{O}) = [c_1, \cdots, c_k]$. Put $l := \dim \mathbf{E}_0$. In the notation of (12.20), we have

$$l = c_0 + \delta \geqslant c_1. \tag{13.5}$$

Note that if $\mathbf{G}$ is an orthogonal group, then $c_0 - c_1$ is even. Hence $l - c_1$ is odd if G is real orthogonal and $l - c_1$ is even if G is quaternionic orthogonal. View $\mathcal{O}$ as a nilpotent orbit in $\mathfrak{m}_{\mathbf{E}_0}$ via inclusion. The following lemma is clear (*cf.* [20, Section 7.3]).

Lemma 13.7. *(i) If G is a real orthogonal group, then*

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}) = [l+1, l-1, c_1, \cdots, c_k].$$

(ii) Otherwise,

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}) = [l, l, c_1, \cdots, c_k].$$

By [21, Theorem 5.2 and 5.6], the complex nilpotent orbit

$$\mathcal{O}^\perp := \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}^\perp}(\mathcal{O}') = \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'}(\mathcal{O}))$$

is given by

$$\mathbf{D}(\mathcal{O}^\perp) = \begin{cases} [c_0+2, c_0, c_1, \cdots, c_k], & \text{if } G \text{ is a real orthogonal group;} \\ [c_0-1, c_0-1, c_1, \cdots, c_k], & \text{if } G \text{ is a real symplectic group;} \\ [c_0, c_0, c_1, \cdots, c_k], & \text{otherwise.} \end{cases}$$

This implies that

$$\mathcal{O}^\perp = \mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}.$$

View each orbit in $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R})$ as an orbit in $\mathrm{Nil}_{M_{\mathbf{E}_0}}(\mathbf{i}\,\mathfrak{m}_{\mathbf{E}_0}^J)$ via inclusion. We state the result for the induction of real nilpotent orbits in the following lemma. The proof will be given in Appendix B.2, which is by elementary matrix manipulations.

Lemma 13.8. *Let $\mathscr{O}_\mathbb{R} \in \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R})$ be a real nilpotent orbit in $\mathcal{O}$ with $\ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}) = [d_1, \cdots, d_k]$.*

(i) Suppose G is a real orthogonal group. Then $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$ consists of a single orbit $\mathscr{O}_\mathbb{R}^\perp$ with

$$\ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}^\perp) = [d_1 + s + (1,1), \check{d}_1 + \check{s}, d_1, \cdots, d_k],$$

where $s := (\frac{l-c_1-1}{2}, \frac{l-c_1-1}{2}) \in \mathbb{Z}^2_{\geqslant 0}$. Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is injective and its image has index 2 in $C_{G_{\mathbf{V}^\perp}}(X)$ for $X \in \mathscr{O}_\mathbb{R}^\perp \cap (\mathscr{O}_\mathbb{R} + \mathbf{i}\,\mathfrak{n}_{\mathbf{E}_0}^J)$.

(ii) Otherwise, $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$ equals the set

$$\left\{ \mathscr{O}_\mathbb{R}^\perp \in \mathrm{Nil}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\mathfrak{g}_{\mathbf{V}^\perp}^J) \;\middle|\; \begin{array}{l} \ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}^\perp) = [d_1 + s, \check{d}_1 + \check{s}, d_1, \cdots, d_k] \textit{ for a signature} \\ s \textit{ of a } (-\epsilon, -\dot{\epsilon})\textit{-space of dimension } l - c_1 \end{array} \right\}.$$

Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is an isomorphism for $X \in \mathscr{O}_{\mathbb{R}}^{\perp} \cap (\mathscr{O}_{\mathbb{R}} + \mathbf{i}\,\mathfrak{n}_{\mathbf{E}_0}^{J})$.

Here $C_{P_{\mathbf{E}_0}}(X)$ and $C_{G_{\mathbf{V}^\perp}}(X)$ are the component groups as in Theorem 13.6. Under the setting of Lemma 13.8, we see that $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}}$ consists of a single orbit when G is a real orthogonal group or a quaternionic symplectic group.

For each $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, let

$$\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O} := \left\{ \mathsf{KS}(\mathscr{O}_{\mathbb{R}}) \,\middle|\, \mathscr{O}_{\mathbb{R}} \in \mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}}(\mathsf{KS}^{-1}(\mathscr{O})) \right\}.$$

13.2.3. *Finishing the proof when G is a real orthogonal group.* Let $\mathrm{sgn}_{\mathbf{V}}$ and $\mathrm{sgn}_{\mathbf{V}^\perp}$ be the sign character of the orthogonal groups G and $G_{\mathbf{V}^\perp}$ respectively. By Proposition 12.16, Lemma 12.22 (a) and Lemma 12.21, we have

$$\begin{aligned}&\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi \otimes \chi_{\mathbf{E}_0})\\ &\cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \cong \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})}(\pi) \oplus \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})\otimes \mathrm{sgn}_{\mathbf{U}}}(\pi)\\ &\cong \bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \oplus (\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi\otimes \mathrm{sgn}_{\mathbf{V}}))) \otimes \mathrm{sgn}_{\mathbf{V}^\perp}.\end{aligned} \tag{13.6}$$

By Theorem 13.6, Lemma 13.7, Lemma 13.8 (i) and [70, Theorem 1.4], the representation $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi \otimes \chi_{\mathbf{E}_0}))$ is $\mathcal{O}^\perp$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi \otimes \chi_{\mathbf{E}_0})) = 2\sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{13.7}$$

where the summation runs over all $\mathbf{K}$-orbits $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^\perp$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}$.

On the other hand, by the explicit formula for the descents of nilpotent orbits, we have

$$\nabla_{\mathbf{V}',\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'}(\mathscr{O}^\perp)) = \mathscr{O}.$$

Applying Theorem 13.3 twice, we have

$$\begin{aligned}\mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))) &\leq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and}\\ \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi \otimes \mathrm{sgn}_{\mathbf{V}})))\otimes \mathrm{sgn}_{\mathbf{V}^\perp}) &\leq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp]\end{aligned} \tag{13.8}$$

where the summations run over the same set as the right hand side of (13.7).

In view of (13.6) to (13.8), we conclude that both inequalities in (13.8) are equalities. Thus (13.1) follows.

13.2.4. *Finishing the proof when G is a real symplectic group.* Let $\mathbf{V}_1'$, $\mathbf{V}_2', \cdots, \mathbf{V}_s'$ be a list of representatives of the isomorphic classes of all $(1,1)$-spaces with dimension $\dim \mathbf{V}'$.

By Proposition 12.16, Lemma 12.22 (b) and Lemma 12.21, we have

$$\begin{aligned}&\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi \otimes \chi_{\mathbf{E}_0}) \oplus \mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}\left(\pi \otimes \chi_{\mathbf{E}_0}'\right)\\ &\cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \oplus \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}}')}(\pi)\\ &\cong \bigoplus_{i=1}^{s} \mathcal{R}_{\bar{\Theta}_{\mathbf{V}_i',\mathbf{U}}(1_{\mathbf{V}_i'})}(\pi) \cong \bigoplus_{i=1}^{s} \bar{\Theta}_{\mathbf{V}_i',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}_i'}(\pi)).\end{aligned} \tag{13.9}$$

By Theorem 13.6, Lemma 13.7, Lemma 13.8 (ii) and [70, Theorem 1.4], $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})$ and $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}\left(\pi\otimes\chi'_{\mathbf{E}_0}\right)$ are $\mathcal{O}^\perp$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}))=\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}\left(\pi\otimes\chi'_{\mathbf{E}_0}\right))=\sum_{\mathscr{O},\mathscr{O}^\perp}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{13.10}$$

where the summation runs over all pairs $(\mathscr{O},\mathscr{O}^\perp)$, where $\mathscr{O}$ is a $\mathbf{K}$-orbit in $\mathcal{O}\cap\mathfrak{p}_{\mathbf{V}}$ and $\mathscr{O}^\perp\in\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}}\mathscr{O}$.

Applying Theorem 13.3 twice, we see that $\bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi))$ is $\mathcal{O}^\perp$-bounded $(1\leqslant i\leqslant s)$, and

$$\sum_{i=1}^{s}\mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi)))\preceq\sum_{i=1}^{s}\sum_{\mathscr{O},\mathscr{O}^\perp}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],$$

where the inner summation runs over all pairs of orbits $(\mathscr{O},\mathscr{O}^\perp)$ such that

$$\nabla_{\mathbf{V}'_i,\mathbf{V}}(\nabla^{\mathrm{gen}}_{\mathbf{V}^\perp,\mathbf{V}'_i}(\mathscr{O}^\perp))=\mathscr{O}. \tag{13.11}$$

By Lemma 13.8 (ii) and (7.10), such kind of pairs $(\mathscr{O},\mathscr{O}^\perp)$ are the same as those of the right hand side of (13.10). Suppose $\ddot{\mathbf{D}}(\mathscr{O}^\perp)=[d,\check{d},d_1,\cdots,d_k]$, then $\ddot{\mathbf{D}}(\mathscr{O})=[d_1,\cdots,d_k]$ and (13.11) holds for exactly two $\mathbf{V}'_i$, having signature $\check{d}+\mathrm{Sign}(\mathbf{V})+(1,0)$ and $\check{d}+\mathrm{Sign}(\mathbf{V})+(0,1)$ respectively. Hence we have

$$\sum_{i=1}^{s}\mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi)))\preceq\sum_{\mathscr{O},\mathscr{O}^\perp}2\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{13.12}$$

where the summation is as in the right hand side of (13.10).

In view of (13.9), (13.10) and (13.12), the equality holds in (13.12). Thus (13.1) holds.

13.2.5. *Finishing the proof when G is a quaternionic symplectic group.* Using Proposition 12.16, Lemma 12.22 (c), Lemma 12.21, Theorem 13.6, Lemma 13.8 (ii), [70, Theorem 1.4] and Theorem 13.3, and a similar argument as in Section 13.2.3 shows that

$$\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})\cong\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi)\cong\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \tag{13.13}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0}))=\sum_{\mathscr{O}}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],\quad\text{and} \tag{13.14}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)))\preceq\sum_{\mathscr{O}}\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{13.15}$$

Here the summations run over all $\mathbf{K}_{\mathbf{V}}$-orbit $\mathscr{O}$ in $\mathcal{O}\cap\mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^\perp$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}}\mathscr{O}$. Note that

$$\nabla_{\mathbf{V}',\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'}(\mathscr{O}^\perp))=\mathscr{O}.$$

Now (13.13), (13.14) and (13.15) imply that (13.15) is an equality and so (13.1) holds.

13.2.6. *Finishing the proof when G is a quaternionic orthogonal group.* Let $\mathbf{V}'_1,\mathbf{V}'_2,\cdots,\mathbf{V}'_s$ be a list of representatives of the isomorphic classes of all $(-1,1)$-spaces with dimension $\dim\mathbf{V}'$.

Lemma 13.9. *One has that*

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi))=\sum_{i=1}^{s}\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{\bar{\Theta}_{\mathbf{V}'_i,\mathrm{U}}(1_{\mathbf{V}'_i})}(\pi)).$$

Proof. For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in Lemma 12.22 (d). Note that $\mathcal{R}_{I_2}(\pi)$ is $\mathcal{O}^\perp$-bounded by Proposition 12.16, Theorem 13.6 and [70, Theorem 1.4]. By the definition in (12.9), we could view $\mathcal{R}_{I_1}(\pi)$ as a subrepresentation of $\mathcal{R}_{I_2}(\pi)$ which is also $\mathcal{O}^\perp$-bounded.

Since G acts on $\mathcal{R}_{I_2}(\pi)$ trivially, the natural homomorphism

$$\pi\widehat{\otimes} I_3 \cong \pi\widehat{\otimes} I_2/\pi\widehat{\otimes} I_1 \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)$$

descents to a surjective homomorphism

$$(\pi\widehat{\otimes} I_3)_G \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi).$$

For each $(-1,1)$-space $\mathbf{V}'''$ of dimension $\dim \mathbf{V}' - 2$, by applying Lemma 13.2 twice, we see that

$$(\pi\widehat{\otimes}(\omega_{\mathbf{V}''',\mathbf{U}})_{G_{\mathbf{V}'''}})_G \cong ((\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'''})_G\widehat{\otimes}\omega_{\mathbf{V}''',\mathbf{V}^\perp})_{G_{\mathbf{V}'''}}$$

is bounded by $\boldsymbol{\vartheta}_{\mathbf{V}''',\mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'''}(\mathcal{O}))$. Using formulas in [21, Theorem 5.2 and 5.6], one checks that the latter set is contained in the boundary of $\mathcal{O}^\perp$. Hence

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)) = 0$$

and the lemma follows. □

Using Theorem 13.6, Lemma 13.8 (ii), [70, Theorem 1.4], Theorem 13.3 and a similar argument as in Section 13.2.4, we have

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})) = \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and} \tag{13.16}$$

$$\bigoplus_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi))) \leq \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp]. \tag{13.17}$$

Here the summations run over all pairs $(\mathscr{O},\mathscr{O}^\perp)$ such that $\mathscr{O}^\perp$ is a $\mathbf{K}_{\mathbf{V}^\perp}$-orbit in $\mathcal{O}^\perp\cap\mathfrak{p}_{\mathbf{V}^\perp}$ and $\mathscr{O} = \nabla_{\mathbf{V}'_i,\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'_i}(\mathscr{O}^\perp))$ for a unique $(-1,1)$-space $\mathbf{V}'_i$.

In view of Proposition 12.16, Lemma 13.9, (13.16) and (13.17), we see that the equality holds in (13.17). Thus (13.1) holds.

14. The unipotent representations construction and iterated theta lifting

In this section, we will prove Theorem 3.4. Recall that $\mathbf{V}$ is an $(\epsilon,\dot\epsilon)$-space and $\mathcal{O} \in \mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{g})$. Since Theorem 3.4 is obvious when $\mathcal{O}$ is the zero orbit, we assume without loss of generality that $\mathcal{O}$ is not the zero orbit.

14.1. **The construction.** Suppose there is a $\mathbf{K}$-orbit $\mathscr{O} \in \mathcal{O}\cap\mathfrak{p}$. Define a sequence $(\mathbf{V}_0,\mathscr{O}_0),(\mathbf{V}_1,\mathscr{O}_1),\cdots,(\mathbf{V}_k,\mathscr{O}_k)$ $(k \geqslant 1)$ such that

- $\mathbf{V}_j$ is a nonzero $((-1)^j\epsilon,(-1)^j\dot\epsilon)$-space for all $0 \leqslant j \leqslant k$;
- $(\mathbf{V}_0,\mathscr{O}_0) = (\mathbf{V},\mathscr{O})$, and $\mathscr{O}_j = \nabla_{\mathbf{V}_{j-1},\mathbf{V}_j}(\mathscr{O}_{j-1}) \in \mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_j}}(\mathfrak{p}_{\mathbf{V}_j})$ for all $1 \leqslant j \leqslant k$;
- $\mathscr{O}_k$ is the zero orbit.

Let $\mathcal{O}_j$ denote the nilpotent $G_{\mathbf{V}_j}$-orbit containing $\mathscr{O}_j$ $(0 \leqslant j \leqslant k)$. To ease the notation, we also let $\mathbf{V}_{k+1}$ be the zero $((-1)^{k+1}\epsilon,(-1)^{k+1}\dot\epsilon)$-space and let $\mathscr{O}_{k+1} \in \mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_{k+1}}}(\mathfrak{p}_{\mathbf{V}_{k+1}})$ and $\mathcal{O}_{k+1} \in \mathrm{Nil}_{\mathbf{G}_{\mathbf{V}_{k+1}}}(\mathfrak{g}_{\mathbf{V}_{k+1}})$ be $\{0\}$.

Let

$$\eta = \chi_0 \boxtimes \chi_1 \boxtimes \cdots \boxtimes \chi_k \tag{14.1}$$

be a character of $G_{\mathbf{V}_0}\times G_{\mathbf{V}_1}\times\cdots\times G_{\mathbf{V}_k}$. For $0 \leqslant j \leqslant k$, put

$$\eta_j := \chi_j \boxtimes \chi_{j+1} \boxtimes \cdots \boxtimes \chi_k.$$

For $0 \leqslant j < k$, write

$$\omega_{\mathscr{O}_j} := \omega_{\mathbf{V}_j,\mathbf{V}_{j+1}} \widehat{\otimes} \omega_{\mathbf{V}_{j+1},\mathbf{V}_{j+2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{\mathbf{V}_{k-1},\mathbf{V}_k},$$

and one checks that the integrals in

$$\begin{aligned} (14.2) \qquad (\omega_{\mathscr{O}_j} \otimes \eta_j) \times (\omega_{\mathscr{O}_j}^{\vee} \otimes \eta_j^{-1}) &\longrightarrow \mathbb{C}, \\ (u,v) &\longmapsto \int_{\widetilde{G}_{\mathbf{V}_{j+1}} \times \widetilde{G}_{\mathbf{V}_{j+2}} \times \cdots \times \widetilde{G}_{\mathbf{V}_k}} \langle g \cdot u, v \rangle \, \mathrm{d}g, \end{aligned}$$

are absolutely convergent and define a continuous bilinear map using Lemma 12.7. Define

$$\pi_{\mathscr{O}_j,\eta_j} := \frac{\omega_{\mathscr{O}_j} \otimes \eta_j}{\text{the left kernel of (14.2)}},$$

which is a Casselman-Wallach representation of $\widetilde{G}_{\mathbf{V}_j}$, as in (12.2). Set $\pi_{\mathscr{O}_k,\eta_k} := \chi_k$ by convention.

For $0 \leqslant j \leqslant k$, let $\chi_i|_{\widetilde{\mathbf{K}}_{\mathbf{V}_i}}$ denote the algebraic character whose restriction to $\widetilde{K}_{\mathbf{V}_i}$ equals the pullback of χ_i through the natural homomorphism $\widetilde{K}_{\mathbf{V}_i} \to G_{\mathbf{V}_i}$. Let $\mathcal{E}_{\mathscr{O}_k,\chi_k}$ denote the $\widetilde{\mathbf{K}}_{\mathbf{V}_k}$-equivariant algebraic line bundle on the zero orbit $\mathscr{O}_k$ corresponding to $\chi_k|_{\widetilde{\mathbf{K}}_{\mathbf{V}_k}}$. Inductively define

$$\mathcal{E}_{\mathscr{O}_j,\eta_j} := \chi_j|_{\widetilde{\mathbf{K}}_{\mathbf{V}_j}} \otimes \vartheta_{\mathscr{O}_{j+1},\mathscr{O}_j}(\mathcal{E}_{\mathscr{O}_{j+1},\eta_{j+1}}), \quad 0 \leqslant j < k.$$

This is an admissible orbit datum over $\mathscr{O}_i$ by (7.23).

Theorem 14.1. *For each $0 \leqslant j \leqslant k$, $\pi_{\mathscr{O}_j,\eta_j}$ is an irreducible, unitarizable, $\mathcal{O}_j$-unipotent representation whose associated character*

$$\mathrm{Ch}_{\mathcal{O}_j}(\pi_{\mathscr{O}_j,\eta_j}) = \mathcal{E}_{\mathscr{O}_j,\eta_j}. \tag{14.3}$$

Moreover,

$$\bar{\Theta}_{\mathbf{V}_{j+1},\mathbf{V}_j}(\pi_{\mathscr{O}_{j+1},\eta_{j+1}}) \otimes \chi_j = \pi_{\mathscr{O}_j,\eta_j} \quad \textit{for all } 0 \leqslant j < k-1. \tag{14.4}$$

Proof. Since $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}^{\mathbb{p}}(\mathfrak{p})$, one verifies that

- $\dim^{\circ} \mathbf{V}_j > 0$ for $0 \leqslant j < k$,
- $\dim \mathbf{V}_{j+1} + \dim \mathbf{V}_{j-1} > 2 \dim^{\circ} \mathbf{V}_j$ for $1 \leqslant j \leqslant k$, and
- $\pi_{\mathscr{O}_j,\eta_j}$ is $\mathbb{p}$-genuine for $0 \leqslant j \leqslant k$.

The representation $\pi_{\mathcal{O}_k,\eta_k}$ clearly satisfies all claims in the theorem. For $j = k-1$, $\pi_{\mathscr{O}_{k-1},\eta_{k-1}}$ is the twist by χ_{k-1} of the theta lift of character χ_k in the stable range. It is known that the statement of Theorem 14.1 holds in this case (*cf.* [46, Section 2] and [48, Section 1.8]).

We now prove the theorem by induction. Assume the theorem holds for $j+1$ with $1 \leqslant j+1 \leqslant k-1$. Applying Lemma 12.10 and Lemma 12.11, we see that $(\pi_{\mathscr{O}_{j+1},\eta_{j+1}}, \mathbf{V}_j)$ is in the convergent range. Thus (14.4) holds by Fubini's theorem. By [65, Theorem 1.19], $\mathrm{U}(\mathfrak{g}_{\mathbf{V}_j})^{\mathbf{G}_{\mathbf{V}_j}}$ acts on $\pi_{\mathscr{O}_j,\eta_j}$ through the character $\lambda_{\mathcal{O}_j}$. By Theorem 13.4, $\pi_{\mathscr{O}_j,\eta_j}$ is $\mathcal{O}_j$-bounded and (14.3) holds. The unitarity and irreducibility of $\pi_{\mathscr{O}_j,\eta_j}$ follows from Theorem 12.12.

By Definition 3.2, $\pi_{\mathscr{O}_j,\eta_j}$ is thus $\mathcal{O}_j$-unipotent. This finishes the proof of the theorem. $\square$

15. Concluding remarks

We record the following result, which is a form of automatic continuity.

Proposition 15.1. *Retain the setting in Section 14.1. The representation $\pi_{\mathscr{O},\eta}$ is isomorphic to*

$$(\omega_{\mathbf{V}_0,\mathbf{V}_1}\widehat{\otimes}\omega_{\mathbf{V}_1,\mathbf{V}_2}\widehat{\otimes}\cdots\widehat{\otimes}\omega_{\mathbf{V}_{k-1},\mathbf{V}_k}\otimes\eta)_{\tilde{G}_{\mathbf{V}_1}\times\tilde{G}_{\mathbf{V}_2}\times\cdots\times\tilde{G}_{\mathbf{V}_k}}, \tag{15.1}$$

and its underlying Harish-Chandra module is isomorphic to

$$(\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1}\otimes\mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2}\otimes\cdots\otimes\mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k}\otimes\eta)_{(\mathfrak{g}_{\mathbf{V}_1}\times\mathfrak{g}_{\mathbf{V}_2}\times\cdots\times\mathfrak{g}_{\mathbf{V}_k},\tilde{\mathbf{K}}_{\mathbf{V}_1}\times\tilde{\mathbf{K}}_{\mathbf{V}_2}\times\cdots\times\tilde{\mathbf{K}}_{\mathbf{V}_k})}. \tag{15.2}$$

Proof. Note that the representation $\pi_{\mathscr{O},\eta}$ is a quotient of the representation (15.1), and the underlying Harish-Chandra module of (15.1) is a quotient of (15.2). Thus it suffices to show that (15.2) is irreducible. We prove by induction on k. Assume that $k \geqslant 1$ and

$$\pi_1 := (\mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2}\otimes\cdots\otimes\mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k}\otimes\eta_1)_{(\mathfrak{g}_{\mathbf{V}_2}\times\cdots\times\mathfrak{g}_{\mathbf{V}_k},\tilde{\mathbf{K}}_{\mathbf{V}_2}\times\cdots\times\tilde{\mathbf{K}}_{\mathbf{V}_k})}$$

is irreducible. Then π_1 is isomorphic to the Harish-Chandra module of $\pi_{\mathscr{O}_1,\eta_1}$. The representation (15.2) is isomorphic to

$$\pi_0 := \chi_0\otimes(\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1}\otimes\pi_1)_{(\mathfrak{g}_{\mathbf{V}_1},\tilde{\mathbf{K}}_{\mathbf{V}_1})}.$$

We know that $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ acts on $\pi_{\mathscr{O},\eta}$ through the character $\lambda_{\mathcal{O}}$ (*cf.* [65]), and by Lemma 13.2, π_0 is $\mathcal{O}$-bounded. Thus every irreducible subquotient of π_0 is $\mathcal{O}$-unipotent. Theorem 13.3 implies that $\mathrm{AC}_{\mathcal{O}}(\pi_0)$ is bounded by $1\cdot[\mathscr{O}]$. Now Theorem 3.4 implies that π_0 must be irreducible. $\square$

Finally, we remark that the Whittaker cycles (attached to G-orbits in $\mathcal{O}\cap\mathbf{i}\mathfrak{g}_{\mathbb{R}}$) of all unipotent representations in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ can be calculated by using [25, Theorem 1.1]. These agree with the associated cycles under the Kostant-Sekiguchi correspondence.

Appendix A. Combinatorial parameterization of special unipotent representations

A.1. **Parameterize of Unipotent representations.** We fix an abstract complex Cartan subgroup $\mathbf{H}_a$ and $\mathfrak{h}_a$ in $\mathbf{G}$ and a set of simple roots Π_a. Let $\mathcal{P}(\mathbf{G})$ be the set of all Langlands parameters of G-modules with character ρ (i.e. the infinitesimal character of the trivial representation). For $\gamma\in\mathcal{P}(\mathbf{G})$, let $\mathcal{L}(\gamma)$, $\mathcal{S}(\gamma)$ and Φ_γ be the corresponding Langlands quotient, standard module and coherent family such that $\Phi_\gamma(\rho)=\mathcal{L}(\gamma)$. Let $\mathcal{M}(\mathbf{G})$ be the span of $\mathcal{L}(\gamma)$. Let $\{\,\mathbb{B}\,\}$ be the set of all blocks. Then $\mathcal{P}(\mathbf{G})=\bigsqcup_{\mathcal{B}}\mathcal{B}$. The Weyl group $W=W(G)$ acts on $\mathcal{M}(\mathbf{G})$ by coherent continuation. Let $\mathcal{M}_{\mathcal{B}}$ be the submodule of $\mathcal{M}(\mathbf{G})$ spand by $\gamma\in\mathcal{B}$, then

$$\mathcal{M}(\mathbf{G})=\bigoplus_{\mathcal{B}}\mathcal{M}_{\mathcal{B}}$$

Let $\tau(\gamma)\subset\Pi_a$ be the τ-invariant of γ.

Let $\check{\mathcal{O}}$ be even orbit. $\lambda=\frac{1}{2}\check{h}$. Define

$$S(\lambda)=\{\,\alpha\in\Pi_a\mid\langle\alpha,\lambda\rangle=0\,\}.$$

Let $\mathcal{P}_\lambda(\mathbf{G})$ be the set of all Langlands parameters with infinitesimal character λ. Let $T_{\lambda,\rho}$ be the translation functor. Let

$$\mathcal{B}(S)=\{\,\gamma\in\mathcal{B}\mid S\cap\tau(\gamma)=\varnothing\,\}$$

and

$$\mathcal{P}(\mathbf{G},S)=\bigsqcup_{\mathcal{B}}\mathcal{B}(S)$$

Then

$$\begin{aligned} \mathcal{P}(\mathbf{G}, S) &\longrightarrow \mathcal{P}_\lambda(\mathbf{G}) \\ \gamma &\longmapsto T_{\lambda,\rho}(\gamma) \end{aligned}$$

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Let

$$\mathcal{B}(S, \mathcal{O}) = \{ \gamma \in \mathcal{B}(S) \mid \mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}} \}$$

Let

$$\begin{aligned} m_S(\sigma) &= [\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] \\ m_{\mathcal{B}}(\sigma) &= [\sigma : \mathcal{M}_{\mathcal{B}}] \end{aligned}$$

Barbasch [11, Theorem 9.1] established the following theorem.

Theorem A.1.

$$|\mathcal{B}(S, \mathcal{O})| = \sum_{\sigma} m_{\mathcal{B}}(\sigma) m_S(\sigma)$$

Here $\sigma \times \sigma$ running over the $W \times W$ appears in the double cell $\mathcal{C}(\mathcal{O})$.

Proof. We need to take the graded module of $\mathcal{M}(\mathbf{G})$ with respect to the $\overset{LR}{\leqslant}$. By abuse of notation, we identify the basis $\mathcal{P}(\mathbf{G})$ with its image in the graded module. Note that $S \cap \tau(\lambda) = \varnothing$ if and only if $W(S)$ acts on γ trivially by [75, Lemma 14.7]. On the other hand, by [75, Theorem 14.10, and page 58], $\mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}}$ only if γ generate a W-module in the double cell of $\mathcal{O}$. □

Now assume $S = S(\lambda)$. By [14, Cor 5.30 b) and c)], $[\sigma : \mathrm{Ind}_{W(S)}^{W} \mathbf{1}] = [\mathbf{1}|_{W(S)} : \sigma] \leqslant 1$.

A.2. **Combinatorics of Weyl group representations.** The irreducible representations of S_n are parameterized by Young diagrams. We use the notation that the trivial representation corresponds to a row and the sign representation corresponds to a column.

$$\mathit{triv} \leftrightarrow \begin{array}{|c|c|c|c|c|c|c|}\hline n & & b & o & x & e & s \\ \hline\end{array} \qquad \mathrm{sgn} \leftrightarrow \begin{array}{|c|}\hline n \\ \hline \\ \hline b \\ \hline o \\ \hline x \\ \hline e \\ \hline s \\ \hline\end{array}$$

[This notation coincide with the Springer correspondence, where a row of boxes represents the regular nilpotent orbit and the corresponding representation is the trivial representation. The trivial orbit yeilds the sign representation.]

Under the above paramterization, $\tau \mapsto \tau \otimes \mathrm{sgn}$ is described by the transpose of Young diagram. The branching rule is given by Littlewoods-Richardson. Let $\mathcal{K} = \bigoplus_n \mathrm{Groth}(S_n)$ be the graded ring of the Grothendieck groups of S_n. This yeilds well defined ring structure on the graded algebra

$$\mu\nu := \mathrm{Ind}_{S_{|\mu|} \times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu \otimes \nu.$$

Notation, we will use $[r_1, r_2, \cdots, r_k]$ or $(c_1, c_2, \cdots, c_k)$ denote the Young diagram, where $\{ r_i \}$ denote the lengthes of its rows and $\{ c_i \}$ denote the lengthes of its columns.

Let W_n denote the Weyl group of type BC: $W_n = S_n \ltimes \{ \pm 1 \}^n$. Now $\mathrm{Irr}(W_n)$ is paramterized by a pair of Young diagram, we use the notation $\mu \times \nu$.

As the tensor algebra, $\bigoplus_n \mathrm{Groth}(W_n) \cong \mathcal{K} \otimes \mathcal{K}$ via $\mu \times \nu \mapsto \mu \otimes \nu$.

The branching rule is given by: $(\mu_1 \times \nu_1)(\mu_2 \times \nu_2) = (\mu_1\mu_2) \times (\nu_1\nu_2)$. The sgn repersentation of W_n is parameterized by $\varnothing \times (n)$. From the definitioin of the identification of bipartition with $\mathrm{Irr}(W_n)$, we also have

$$(\mu \times \nu) \otimes \mathrm{sgn} = \nu^t \times \mu^t.$$

We also will use the following branching formula:

$$\mathrm{Ind}_{S_n}^{W_n} \mathrm{triv} = \sum_{\substack{a,b,\\ a+b=n}} [a] \times [b],$$

and

$$\mathrm{Ind}_{S_n}^{W_n} \mathrm{sgn} = \sum_{\substack{a,b,\\ a+b=n}} (a) \times (b). \tag{A.1}$$

[Sketch of the proof of the first formula, the dimension of LHS is $n!2^n/n! = 2^n$. On the other hand, by Mackey theory, $[LHS : \mathrm{Ind}_{W_a\times W_b}^{W_n} \mathrm{triv} \otimes \chi] \geqslant 1$ since $S_n \cap W_a \times W_b = S_a \times S_b$. Therefore, $LHS \supset RHS$. On the other hand, dimension of RHS is $\sum_a n!/a!b! = 2^n$. This finished the proof. The proof of the second formula is similar. One also can obtain it by tensoring with sgn of the first formula.]

A.2.1. *Coherent continuation representations.* TBA

Appendix B. A few geometric facts

B.1. **Geometry of moment maps.** In this section, let $(\mathbf{V}, \mathbf{V}')$ be a rational dual pair. We use the notation of Section 7.6. Write $\breve{\mathfrak{p}} := \mathfrak{p} \oplus \mathfrak{p}'$ and

$$\breve{M} := M \times M' \colon \mathcal{X} \longrightarrow \breve{\mathfrak{p}} = \mathfrak{p} \times \mathfrak{p}'.$$

When there is a morphism $f\colon A \to B$ between smooth algebraic varieties, let $\mathrm{d}f_a\colon \mathrm{T}_a A \to \mathrm{T}_{f(a)} B$ denote the tangent map at a closed point $a \in A$, where $\mathrm{T}_a A$ and $\mathrm{T}_{f(a)} B$ are the tangent spaces. For a sub-scheme S of B, let $A \times_f S$ denote the scheme theoretical inverse image of S under f, which is a subscheme of A. Along the proof, we will use the Jacobian criterion for regularity in various places (see [47, Theorem 2.19]).

B.1.1. *Scheme theoretical results on descents.*

Lemma B.1 (*cf.* [60, Lemma 13 and Lemma 14])**.** *Suppose $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the descent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. Fix $X \in \mathscr{O}$ and let*

$$\mathcal{Z}_X := \mathcal{X} \times_M \{ X \}$$

be the scheme theoretical inverse image of X under the moment map $M : \mathcal{X} \to \mathfrak{p}$. Then

(i) *$\mathcal{Z}_X$ is smooth (and hence reduced);*
(ii) *$\mathcal{Z}_X$ is a single free $\mathbf{K}'$-orbit in $\mathcal{X}^\circ$;*
(iii) *$M'(\mathcal{Z}_X) = \mathscr{O}'$ and $\mathcal{Z}_X = \mathcal{X} \times_{\breve{M}} (\{ X \} \times \overline{\mathscr{O}'})$.*

Proof. Fix a point $w \in \mathcal{X}^\circ$ such that $M(w) = X$. It is elementary to see that

$$M^{-1}(X) = \mathbf{K}' \cdot w$$

is a single $\mathbf{K}'$-orbit. This orbit is clearly Zariski closed and the stabilizer $\mathrm{Stab}_{\mathbf{K}'}(w)$ of w under the $\mathbf{K}'$-action is trivial. This proves part (ii) and the first assertion of part (iii).

We identify the tangent space $\mathrm{T}_w\mathcal{X}$ with $\mathcal{X}$. Let $N \subset \mathcal{X}$ be a complement of $\mathrm{T}_w(\mathbf{K}'\cdot w)$ in $\mathrm{T}_w\mathcal{X}$. By [64, Section 6.3], there is an affine open subvariety N° of N containing 0 such that the diagram

$$\begin{tikzcd} \mathbf{K}' \times N^\circ \arrow[r, "(k',n)\mapsto k'\cdot(w+n)"] \arrow[d, "\text{projection}"'] & \mathcal{X} \arrow[d, "M"] \\ N^\circ \arrow[r, "n\mapsto M(w+n)"] & M(\mathcal{X}) \end{tikzcd}$$

is Cartesian and the horizontal morphisms are étale. Therefore the scheme $\mathcal{X} \times_M \{X\}$ is a smooth algebraic variety. This proves part (i), and the second assertion of (iii) then easily follows. $\square$

Remark. Retain the setting of Lemma B.1. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from X to $X' \in \mathfrak{p}'$. Let $\check{X} := (X, X') \in \mathfrak{p} \oplus \mathfrak{p}'$ and $\mathbf{K}'_{X'}$ be the stabilizer of X' under the $\mathbf{K}'$-action. By the $\mathbf{K}'$-equivariance, the map $\mathcal{Z}_X \to \mathscr{O}'$ is a smooth morphism between smooth schemes. Hence the scheme theoretical fibre

$$\mathcal{Z}_{\check{X}} := \mathcal{X} \times_{\check{M}} \{\check{X}\} = \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{X'\}$$

is smooth and equals the orbit $\mathbf{K}'_{X'}\cdot w$. Consequently, we have an exact sequence by the Jacobian criterion for regularity:

$$0 \longrightarrow \mathrm{T}_w\mathcal{Z}_{\check{X}} \longrightarrow \mathrm{T}_w\mathcal{X} \xrightarrow{\mathrm{d}\check{M}_w} \mathrm{T}_X\mathfrak{p} \oplus \mathrm{T}_{X'}\mathfrak{p}'. \tag{B.1}$$

Lemma B.2. *Retain the setup in Lemma B.1. Let*

$$U := \mathfrak{p}\backslash((\overline{\mathcal{O}} \cap \mathfrak{p})\backslash\mathscr{O})$$

and $\mathcal{Z}_{U,\overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$. *Then*

(i) U *is a Zariski open subset of* $\mathfrak{p}$ *such that* $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$;

(ii) $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ *is smooth and it is a single* $\mathbf{K}\times\mathbf{K}'$*-orbit.*

Proof. Part (i) is clear by (7.8). It is also clear that the underlying set of $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is a single $\mathbf{K}\times\mathbf{K}'$-orbit by Lemma B.1 (iii).

Note that $M'|_{\mathcal{X}^\circ}$ is smooth because $\mathrm{d}M'_w\colon \mathrm{T}_w\mathcal{X}^\circ \longrightarrow \mathrm{T}_{M'(w)}\mathfrak{p}'$ is surjective for each $w \in \mathcal{X}^\circ$ (*cf.* [29, Proposition 10.4]). Since smooth morphisms are stable under base changes and compositions, the scheme $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$ is a smooth algebraic variety. Also note that $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is an open subscheme of $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$. Thus it is smooth. $\square$

Remark. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from $X \in \mathscr{O}$ to $X' \in \mathscr{O}' \subset \mathfrak{p}'$. By the proof of Lemma B.2,

$$\mathcal{Z}_{U,X'} := \mathcal{X} \times_{\check{M}} (U \times \{X'\}) = \mathbf{K}\mathbf{K}'_{X'}\cdot w$$

is smooth. Similar to the remark of Lemma B.1, this yields an exact sequence:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A,A')\mapsto A'w-wA]{} \mathcal{X} \xrightarrow[b\mapsto bw^\star+wb^\star]{\mathrm{d}M'_w} \mathfrak{p}'. \tag{B.2}$$

B.1.2. *Scheme theoretical results on generalized descents of good orbits.*

Lemma B.3. *Suppose* $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ *is good for generalized descent (see Definition 7.15) and* $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. *Fix* $X \in \mathscr{O}$ *and let* $\mathcal{Z}_X := \mathcal{X} \times_{\check{M}} (\{X\} \times \overline{\mathscr{O}'})$. *Then*

(i) $\mathcal{Z}_X$ *is smooth;*

(ii) $\mathcal{Z}_X$ *is a single* $\mathbf{K}'$*-orbit;*

(iii) $\mathcal{Z}_X$ *is contained in* $\mathcal{X}^{\mathrm{gen}}$ *and* $M'(\mathcal{Z}_X) = \mathscr{O}'$.

Proof. Part (ii) follows from [22, Table 4]. Part (iii) follows from part (ii). By the generic smoothness, in order to prove part (i), it suffices to show that $\mathcal{Z}_X$ is reduced. Note that the $\mathbf{K}'$-equivariant morphism $\mathcal{Z}_X \xrightarrow{M'} \mathscr{O}'$ is flat by generic flatness [26, Théorème 6.9.1]. By [27, Proposition 11.3.13], to show that $\mathcal{Z}_X$ is reduced, it suffices to show that

$$\mathcal{Z}_{\check{X}} := \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{ X' \} = \mathcal{X} \times_{\check{M}} \{ \check{X} \}$$

is reduced, where $X' \in \mathscr{O}'$ and $\check{X} := (X, X')$. Let $w \in \mathcal{Z}_{\check{X}}$ be a closed point. Then $\mathbf{K}' \cdot w$ is the underlying set of $\mathcal{Z}_X$, and $Z_{\check{X}} := \mathbf{K}'_{X'} \cdot w$ is the underlying set of $\mathcal{Z}_{\check{X}}$. By the Jacobian criterion for regularity, to complete the proof of the lemma, it remains to prove the following claim.

Claim. *The following sequence is exact:*

$$0 \longrightarrow \mathrm{T}_w Z_{\check{X}} \longrightarrow \mathrm{T}_w \mathcal{X} \xrightarrow{\mathrm{d}\check{M}} \mathrm{T}_X \mathfrak{p} \oplus \mathrm{T}_{X'} \mathfrak{p}'.$$

We prove the above claim in what follows. Identify $\mathrm{T}_w \mathcal{X}$, $\mathrm{T}_X \mathfrak{p}$ and $\mathrm{T}_{X'}\mathfrak{p}'$ with $\mathcal{X}, \mathfrak{p}$ and $\mathfrak{p}'$ respectively and view $\mathrm{T}_w Z_{\check{X}}$ as a quotient of $\mathfrak{k}'_{X'}$. It suffices to show that the following sequence is exact:

$$\mathfrak{k}'_{X'} \xrightarrow[A' \mapsto A'w]{} \mathcal{X} \xrightarrow[b \mapsto (b^\star w + w^\star b, b w^\star + w b^\star)]{\mathrm{d}\check{M}_w} \mathfrak{p} \oplus \mathfrak{p}' = \check{\mathfrak{p}}. \tag{B.3}$$

In fact, we will show that the following sequence

$$\mathfrak{g}'_{X'} \longrightarrow \mathbf{W} \longrightarrow \mathfrak{g} \oplus \mathfrak{g}' \tag{B.4}$$

is exact, where $\mathfrak{g}'_{X'}$ is the Lie algebra of the stabilizer group $\mathrm{Stab}_{\mathbf{G}'}(X')$ and the arrows are defined by the same formulas in (B.3). Then the exactness of (B.3) will follow since (B.4) is compatible with the natural (L, L')-actions.

We now prove the exactness of (B.4). Let $\mathbf{V}'_1 := w\,\mathbf{V}$ and $\mathbf{V}'_2$ be the orthogonal complement of $\mathbf{V}'_1$ so that

$$\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2.$$

Let $\mathbf{W}_i := \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_i)$ and $\mathfrak{g}'_i := \mathfrak{g}_{\mathbf{V}'_i}$ for $i = 1, 2$. Let $*\colon \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \to \mathrm{Hom}(\mathbf{V}'_2, \mathbf{V}'_1)$ denote the adjoint map. We have $\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2$ and $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxslash}$, where

$$\mathfrak{g}^{\boxslash} := \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \right\}.$$

Now w and X' are naturally identified with an element w_1 in $\mathbf{W}_1$ and an element X'_1 in $\mathfrak{g}'_1$, respectively. We have $\mathfrak{g}'_{X'} = \mathfrak{g}'_{1,X'_1} \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxslash}_{X'_1}$, where

$$\mathfrak{g}^{\boxslash}_{X'_1} = \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \,\middle|\, E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2),\ EX'_1 = 0 \right\}.$$

Now maps in (B.4) have the following forms respectively:

$$\mathfrak{g}'_{X'} \ni A' = (A'_1, A'_2, \left(\begin{smallmatrix} 0 & -E^* \\ E & 0 \end{smallmatrix}\right)) \mapsto A'w = (A'_1 w_1, E w_1) \in \mathbf{W} \qquad \text{and}$$

$$\mathbf{W} \ni (b_1, b_2) \mapsto \left(b_1^\star w_1 + w_1^\star b_1, \begin{pmatrix} b_1 w_1^\star + w_1 b_1^\star & w_1 b_2^\star \\ b_2 w_1^\star & 0 \end{pmatrix} \right) \in \mathfrak{g} \oplus \mathfrak{g}'.$$

Applying (B.1) to the complex dual pair $(\mathbf{V}, \mathbf{V}'_1)$, we see that the sequence

$$\mathfrak{g}'_{1,X'_1} \xrightarrow{A'_1 \mapsto A'_1 w_1} \mathbf{W}_1 \xrightarrow{b_1 \mapsto (b_1^\star w_1 + w_1^\star b_1, b_1 w_1^\star + w_1 b_1^\star)} \mathfrak{g} \oplus \mathfrak{g}'_1$$

is exact. The task is then reduced to show that the following sequence is exact:

$$\mathfrak{g}^{\boxslash}_{X_1'} \xrightarrow[\text{①}]{\begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \mapsto Ew_1} \mathbf{W}_2 \xrightarrow[\text{②}]{b_2 \mapsto b_2 w_1^\star} \mathrm{Hom}(\mathbf{V}_1', \mathbf{V}_2'). \tag{B.5}$$

Note that w_1 is a surjection, hence ① is an injection. We have

$$\dim \mathfrak{g}^{\boxslash}_{X_1'} = \dim \mathbf{V}_2' \cdot \dim \mathrm{Ker}(X_1')$$

and

$$\dim \mathrm{Ker}(\text{②}) = \dim \mathbf{V}_2' \cdot (\dim \mathbf{V} - \dim \mathrm{Im}(w_1)) = \dim \mathbf{V}_2' \cdot (\dim \mathbf{V} - \dim \mathbf{V}_1').$$

Note that $\dim \mathrm{Ker}(X_1') = c_1$, and $\dim \mathbf{V} - \dim \mathbf{V}_1' = c_0$. Since we are in the setting of good generalized descent (Definition 7.15), we have $c_0 = c_1$. Now the exactness of (B.5) follows by dimension counting. □

Remark. Suppose $\mathscr{O}$ is not good for generalized descent. Then the underlying set of $\mathcal{Z}_X$ may not be a single $\mathbf{K}'$-orbit. Even if it is a single orbit, the scheme $\mathcal{Z}_X$ may not be reduced.

Lemma B.4. *Retain the notation in Lemma B.3. Let*

$$U := \mathfrak{p} \backslash ((\overline{\mathcal{O}} \cap \mathfrak{p}) \backslash \mathscr{O})$$

and $\mathcal{Z}_{U,\overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$*. Then*

(i) U *is a Zariski open subset of* $\mathfrak{p}$ *such that* $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$*;*
(ii) $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ *is smooth and it is a single* $\mathbf{K} \times \mathbf{K}'$*-orbit.*

Proof. Part (i) is follows from Lemma 7.16. By Lemma B.3 (ii), the underlying set of $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit whose image under M' is contained in $\mathscr{O}'$. By generic flatness [26, Théorème 6.9.1], the morphism $\mathcal{Z}_{U,\overline{\mathscr{O}'}} \longrightarrow \mathscr{O}'$ is flat. To prove the scheme theoretical claim in part (ii), it suffices to show that

$$\mathcal{Z}_{U,X'} := \mathcal{X} \times_{\check{M}} (U \times \{ X' \})$$

is reduced, where $X' \in \mathscr{O}'$.

Let $w \in \mathcal{Z}_{U,X'}$ be a closed point. As in the proof of Lemma B.3, by the Jacobian criterion for regularity, it suffices to show that the following sequence is the exact:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A,A') \mapsto A'w - wA]{} \mathcal{X} \xrightarrow[b \mapsto bw^\star + wb^\star]{\mathrm{d}M'_w} \mathfrak{p}'.$$

The proof of the exactness follows the same line as the proof of Lemma B.3 utilizing (B.5) and the established exact sequence (B.2) in the descent case. We leave the details to the reader. □

B.2. **Proof of Lemma 13.8.** We will use notations of Section 12.3 and Section 13.2.2. Let Hom_J denote the space of homomorphisms commuting with the $\dot{\epsilon}$-real from J, and $\mathfrak{n}_{\mathbb{R}}$ denote the Lie algebra of $N_{\mathbf{E}_0}$.

We first explicitly construct some elements in the induced orbits in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}}$. Fix an element $\mathbf{i}\, X \in \mathscr{O}_{\mathbb{R}}$ and a J-invariant complement $\mathbf{V}_+$ of $\mathrm{Im}(X)$ in the vector space $\mathbf{V}^- = \mathbf{V}$ so that

$$\mathbf{V}^- = \mathbf{V}_+ \oplus \mathrm{Im}(X).$$

Fix any J-invariant decompositions

$$\mathbf{E}_0 = \mathbf{L}_1 \oplus \mathbf{L}_0 \quad \text{and} \quad \mathbf{E}_0' = \mathbf{L}_1' \oplus \mathbf{L}_0'$$

such that $\dim \mathbf{L}_1 = \dim \mathbf{V}_+$, and $(\mathbf{L}_1 \oplus \mathbf{L}_1') \perp (\mathbf{L}_0 \oplus \mathbf{L}_0')$. (This is possible due to the dimension inequality in (13.5).)

Fix a linear isomorphism

$$S \in \mathrm{Hom}_J(\mathbf{L}_1', \mathbf{V}_+),$$

and view it as an element of $\mathrm{Hom}_J(\mathbf{E}_0', \mathbf{V}^-)$ via the aforementioned decompositions. Put

$$\begin{aligned} \mathscr{T} &:= \{\, T \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0) \mid T^* + T = 0 \,\} \quad \text{and} \\ \mathscr{T}^\circ &:= \{\, T \in \mathscr{T} \mid T \text{ has maximal possible rank} \,\} \end{aligned}$$

to be viewed as subsets of $\mathrm{Hom}_J(\mathbf{E}_0', \mathbf{E}_0)$ as before, where $T^* \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0)$ is specified by requiring that

$$\text{(B.6)} \qquad \langle T \cdot u, v \rangle_{\mathbf{V}^\perp} = \langle u, T^* v \rangle_{\mathbf{V}^\perp}, \quad \text{for all } u, v \in \mathbf{L}_0',$$

and $\langle\, ,\, \rangle_{\mathbf{V}^\perp}$ denotes the ϵ-symmetric bilinear form on $\mathbf{V}^\perp$.

Each $T \in \mathscr{T}$ defines a $(-\epsilon)$-symmetric bilinear form $\langle\, ,\, \rangle_T$ on $\mathbf{L}_0'$ where

$$\langle v_1, v_2 \rangle_T := -\langle v_1, T v_2 \rangle, \quad \text{for all } v_1, v_2 \in \mathbf{L}_0'.$$

Put

$$\text{(B.7)} \qquad X_{S,T} := \begin{pmatrix} 0 & S^* & T \\ & X & S \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_{\mathbb{R}}$$

according to the decomposition $\mathbf{V}^\perp = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}_0'$, where $S^* \in \mathrm{Hom}_J(\mathbf{V}^-, \mathbf{E}_0)$ is similarly defined as in (B.6).

Now suppose that $T \in \mathscr{T}^\circ$.

In case (ii) of Lemma 13.8, the form $\langle\, ,\, \rangle_T$ must be non-degenerate and $((\mathbf{L}_0', \langle\, ,\, \rangle_T), J|_{\mathbf{L}_0'})$ becomes a $(-\epsilon, -\dot\epsilon)$-space. Conversely, up to isomorphism, every $(-\epsilon, -\dot\epsilon)$-space of dimension $l - c_1$ is isomorphic to $((\mathbf{L}_0', \langle\, ,\, \rangle_T), J|_{\mathbf{L}_0'})$ for some $T \in \mathscr{T}$.

Let s be the signature of $((\mathbf{L}_0', \langle\, ,\, \rangle_T), J|_{\mathbf{L}_0'})$ and let $G_T := \mathbf{G}^J_{\mathbf{L}_0'}$ denote the corresponding real group. Then $\mathbf{i}\, X_{S,T}$ generates an induced orbit of $\mathscr{O}_{\mathbb{R}}$ with signed Young diagram $[d_1 + s, \check d_1 + \check s, d_1, \cdots, d_k]$.[13] One checks that the reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T})$ and $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\, X_{S,T})$ are both canonically isomorphic to $R \times G_T$, where R is the reductive quotient of $\mathrm{Stab}_G(X)$. Hence $C_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T}) \cong C_{G_{\mathbf{V}^\perp}}(\mathbf{i}\, X_{S,T})$.

In case (i) of Lemma 13.8, $\langle\, ,\, \rangle_T$ is skew symmetric, and T has rank $\dim \mathbf{L}_0 - 1 = l - c_1 - 1$ since $l - c_1$ is odd. One checks that $\mathbf{i}\, X_{S,T}$ generates an induced orbit of $\mathscr{O}_{\mathbb{R}}$ with the signed Young diagram prescribed in (i) of Lemma 13.8.

Fix decompositions

$$\mathbf{L}_0 = \mathbf{L}_2 \oplus \mathbf{L}_3 \quad \text{and} \quad \mathbf{L}_0' = \mathbf{L}_2' \oplus \mathbf{L}_3'$$

which are dual to each other such that $\mathrm{Ker}(T) = \mathbf{L}_3'$ and $\langle\, ,\, \rangle_T |_{\mathbf{L}_2' \times \mathbf{L}_2'}$ is a non-degenerate skew symmetric bilinear form on $\mathbf{L}_2'$. Let $G_T := \mathbf{G}^J_{\mathbf{L}_2'}$. Then G_T is a real symplectic group. The reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i} X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathrm{GL}(\mathbf{L}_3)^J$ and the reductive quotient of $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i} X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathbf{G}^J_{\mathbf{L}_3 \oplus \mathbf{L}_3'}$, where R is the reductive quotient of $\mathrm{Stab}_G(\mathbf{i} X)$ and $\mathbf{G}^J_{\mathbf{L}_3 \oplus \mathbf{L}_3'} \cong \mathrm{O}(1,1)$. The homomorphism

$$C_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T}) \longrightarrow C_{G_{\mathbf{V}^\perp}}(\mathbf{i} X_{S,T})$$

is therefore an injection whose image has index 2.

[13] The signed Young diagram may be computed using the explicit description of Kostant-Sekiguchi correspondence in [22, Propositions 6.2 and 6.4].

To finish the proof of the lemma, it suffices to show that

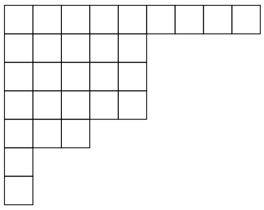
$$\operatorname{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}} = \{\, G_{\mathbf{V}^\perp} \cdot \mathbf{i} X_{S,T} \mid T \in \mathscr{T}^\circ \,\}. \tag{B.8}$$

Consider the set

$$\mathfrak{A} := \left\{ \begin{pmatrix} 0 & B^* & C \\ & X & B \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_{\mathbb{R}} \;\middle|\; X \oplus B \in \operatorname{Hom}_J(\mathbf{V}^- \oplus \mathbf{E}'_0, \mathbf{V}^-) \text{ is surjective} \right\}.$$

Clearly $P'_{\mathbf{E}_0} := \mathrm{GL}_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$ acts on $\mathfrak{A}$ (*cf.* (12.12)). By suitable matrix manipulations, one sees that every element in $\mathfrak{A}$ is conjugated to an element in $\{\, X_{S,T} \mid T \in \mathscr{T} \,\}$ under the $P'_{\mathbf{E}_0}$-action. Hence $P'_{\mathbf{E}_0} \cdot \{\, X_{S,T} \mid T \in \mathscr{T}^\circ \,\}$ is open dense in $\mathfrak{A}$. Now (B.8) follows since $P_{\mathbf{E}_0} \cdot \mathbf{i}\mathfrak{A}$ is open dense in $\mathscr{O}_{\mathbb{R}} + \mathbf{i}\mathfrak{n}_{\mathbb{R}}$, $\square$

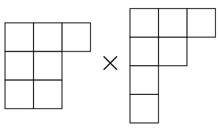
Example. Suppose that $\star = C$, and $\check{\mathcal{O}}$ is the following Young diagram which is $\star$-even.

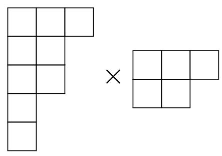


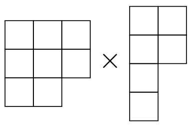
Then

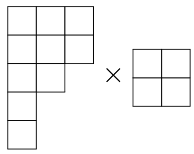
$$\mathrm{DP}_\star(\check{\mathcal{O}}) = \{(1,2),(5,6)\}.$$

and $(\imath_\star(\check{\mathcal{O}}, \wp), \jmath_\star(\check{\mathcal{O}}, \wp)) \in \mathrm{BP}_\star(\check{\mathcal{O}})$ has the following form.

$\wp = \varnothing$: 

$\wp = \{(1,2)\}$: 

$\wp = \{(5,6)\}$: 

$\wp = \{(1,2),(5,6)\}$: 

Appendix A. The case of general orbits

Let G, $\check{\mathfrak{g}}$ and $\star \in \{B, C, \widetilde{C}, D, C^*, D^*\}$ be as in Section 1.2. Suppose that G has type $\star$. Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$, to be considered as a Young diagram as usual. By counting the row lengths, we identify Young diagrams with finite multisets of positive integers. We have a unique pair $(\check{\mathcal{O}}_{\star-\mathrm{good}}, \check{\mathcal{O}}_{\star-\mathrm{bad}})$ of Young diagrams such that

- if $\star \in \{B, \widetilde{C}\}$, then all nonzero row lengths of $\check{\mathcal{O}}_{\star-\mathrm{good}}$ are even and all nonzero row lengths of $\check{\mathcal{O}}_{\star-\mathrm{bad}}$ are odd;
- if $\star \in \{C, D, C^*, D^*\}$, then all nonzero row lengths of $\check{\mathcal{O}}_{\star-\mathrm{good}}$ are odd and all nonzero row lengths of $\check{\mathcal{O}}_{\star-\mathrm{bad}}$ are even;
- as multisets, $\check{\mathcal{O}}$ is the union of $\check{\mathcal{O}}_{\star-\mathrm{good}}$ and $\check{\mathcal{O}}_{\star-\mathrm{bad}}$.

Note that $\check{\mathcal{O}}_{\star-\mathrm{good}}$ has $\star$-good parity as defined in Section 1.3. Also note that every positive integer occurs in the multiset $\check{\mathcal{O}}_{\star-\mathrm{bad}}$ with even multiplicity. Thus there is a Young diagram $\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}$ such that the union of it with itself equals $\check{\mathcal{O}}_{\star-\mathrm{bad}}$.

Put

$$\mathbb{K} := \begin{cases} \mathbb{R}, & \text{if } \star = B, C, D \text{ or } \widetilde{C}; \\ \text{the quaternions algebra } \mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}, & \text{if } \star = C^* \text{ or } D^*, \end{cases}$$

and put

$$l := l_{\check{\mathcal{O}}} := \begin{cases} \frac{1}{2}\left|\check{\mathcal{O}}_{\star-\mathrm{bad}}\right|, & \text{if } \star = B, C, D \text{ or } \widetilde{C}; \\ \frac{1}{4}\left|\check{\mathcal{O}}_{\star-\mathrm{bad}}\right|, & \text{if } \star = C^* \text{ or } D^*. \end{cases}$$

Special unipotent representations of general linear groups are well-understood. Similar to the situation in Section 1.2, write $\mathrm{Unip}_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}}(\mathrm{GL}_l(\mathbb{K}))$ for the set of all isomorphism classes of irreducible Casselman-Wallach representations of $\mathrm{GL}_l(\mathbb{K})$ that are attached to $\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}$. The row lengths of the Young diagram $\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}$ forms a partition of $\left|\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}\right|$, and hence corresponds to a standard parabolic subgroup $P_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}}$ of $\mathrm{GL}_l(\mathbb{K})$. It is well-known that

$$\mathrm{Unip}_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}}(\mathrm{GL}_l(\mathbb{K})) = \left\{ \mathrm{Ind}_{P_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}}}^{\mathrm{GL}_l(\mathbb{K}))} \chi \mid \chi \text{ is a quadratic character of } P_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}} \right\}.$$

Note that this set is a singleton in the quaternionic case.

When the real rank of G is at least l, respectively define

$$G_l := \mathrm{O}(p-l, q-l),\ \mathrm{Sp}_{2n-2l}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n-2l}(\mathbb{R}),\ \mathrm{Sp}(p-l, q-l),\ \text{or}\ \mathrm{O}^*(2n-2l)$$

if

$$G = \mathrm{O}(p, q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p, q),\ \text{or}\ \mathrm{O}^*(2n).$$

If $\star = \widetilde{C}$ so that $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. Write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. The group $\mathrm{Sp}_{2n}(\mathbb{R})$ has a Levi subgroup of the form $\mathrm{GL}_l(\mathbb{R}) \times \mathrm{Sp}_{2n-2l}(\mathbb{R})$. Write $\widetilde{\mathrm{GL}}_l(\mathbb{R})$ for the inverse images of $\mathrm{GL}_l(\mathbb{R})$ under the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then

$$L_l := \frac{\widetilde{\mathrm{GL}}_l(\mathbb{R}) \times G_l}{\text{the diagonal group } \ \{1, \varepsilon_G\}}$$

is a Levi subgroup of G. We fix a finite order genuine character ζ_l of $\widetilde{\mathrm{GL}}_l(\mathbb{R})$.

If $\star \neq \widetilde{C}$, then the group G has a Levi subgroup of the form

$$L_l := \mathrm{GL}_l(\mathbb{K}) \times G_l.$$

In all case, let $P_l = L_l \ltimes N_l$ be a parabolic subgroup of G such that L_l is a Levi factor of it and N_l is the unipotent radical. Every representation of L_l is viewed as a representation of P_l with the trivial action of N_l. Given $\sigma \in \mathrm{Unip}_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}}(\mathrm{GL}_l(\mathbb{K}))$ and $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}_{\star-\mathrm{good}}}(G_l)$, put

$$\sigma \ltimes \pi := \begin{cases} \mathrm{Ind}_{P_l}^G((\sigma \otimes \zeta_l)\widehat{\otimes}\pi), & \text{if } \star = \widetilde{C}; \\ \mathrm{Ind}_{P_l}^G(\sigma\widehat{\otimes}\pi), & \text{otherwise.} \end{cases}$$

Theorem A.1. *(a) If the real rank of G is smaller than l, then the set* $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ *is empty.*
(b) Suppose that the real rank of G is greater than or equal to l. Then for every $\sigma \in \mathrm{Unip}_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}}(\mathrm{GL}_l(\mathbb{K}))$ *and* $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}_{\star-\mathrm{good}}}(G_l)$*, the representation $\sigma \ltimes \pi$ is irreducible and belongs to the set* $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$*. Moreover, the map*

$$\mathrm{Unip}_{\frac{1}{2}\check{\mathcal{O}}_{\star-\mathrm{bad}}}(\mathrm{GL}_l(\mathbb{K})) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\star-\mathrm{good}}}(G_l) \to \mathrm{Unip}_{\check{\mathcal{O}}}(G), \qquad (\sigma, \pi) \mapsto \sigma \ltimes \pi$$

is bijective.

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x_τ	$\begin{smallmatrix}\tilde{x}_{\tau''}\\ m_{\tau''}\end{smallmatrix}$	$\mathbf{p}_\tau$	$(x_{\tau''}, \mathbf{x}_\tau)$	$\tilde{\tau}_L$
r	$\begin{smallmatrix}r\\ a\end{smallmatrix}$	$\begin{smallmatrix}\bullet & r\\ b\end{smallmatrix}$	(r, s)	$\begin{smallmatrix}s & r\end{smallmatrix}$
		$\begin{smallmatrix}\bullet & r\\ a\end{smallmatrix}$	(r, c)	$\begin{smallmatrix}r & c\end{smallmatrix}$
r	$\begin{smallmatrix}r\\ b\end{smallmatrix}$	$\begin{smallmatrix}s & r\\ b\end{smallmatrix}$	(r, s)	$\begin{smallmatrix}s & r\end{smallmatrix}$
		$\begin{smallmatrix}s & r\\ a\end{smallmatrix}$	(r, c)	$\begin{smallmatrix}r & c\end{smallmatrix}$
c	$\begin{smallmatrix}s\\ a\end{smallmatrix}$	$\begin{smallmatrix}\bullet & s\\ b\end{smallmatrix}$	(c, s)	$\begin{smallmatrix}s & c\end{smallmatrix}$
		$\begin{smallmatrix}\bullet & s\\ a\end{smallmatrix}$	(c, c)	$\begin{smallmatrix}c & c\end{smallmatrix}$
d	$\begin{smallmatrix}d\\ a\end{smallmatrix}$	$\begin{smallmatrix}\bullet & d\\ b\end{smallmatrix}$	(d, s)	$\begin{smallmatrix}s & d\end{smallmatrix}$
		$\begin{smallmatrix}\bullet & d\\ a\end{smallmatrix}$	(d, c)	$\begin{smallmatrix}c & d\end{smallmatrix}$
		$\begin{smallmatrix}r & d\\ a\end{smallmatrix}$	(d, r)	$\begin{smallmatrix}r & d\end{smallmatrix}$
		$\begin{smallmatrix}d & d\\ a\end{smallmatrix}$	(d, d)	$\begin{smallmatrix}d & d\end{smallmatrix}$
	$\begin{smallmatrix}d\\ b\end{smallmatrix}$	$\begin{smallmatrix}s & d\\ b\end{smallmatrix}$	(d, s)	$\begin{smallmatrix}s & d\end{smallmatrix}$
		$\begin{smallmatrix}s & d\\ a\end{smallmatrix}$	(d, c)	$\begin{smallmatrix}c & d\end{smallmatrix}$
		$\begin{smallmatrix}r & d\\ b\end{smallmatrix}$	(d, r)	$\begin{smallmatrix}r & d\end{smallmatrix}$
		$\begin{smallmatrix}d & d\\ b\end{smallmatrix}$	(d, d)	$\begin{smallmatrix}d & d\end{smallmatrix}$

TABLE 4. Reduction when $n = 1$

x_τ	$\begin{smallmatrix} \tilde{x}_{\tau''} \\ m_{\tau''} \end{smallmatrix}$	$\mathbf{p}_\tau$	$x_{\tau''}, \mathbf{x}_\tau$	$\tilde{\tau}_L$	conditions	
r	$\begin{smallmatrix} r \\ a \end{smallmatrix}$	$\begin{smallmatrix} \bullet & r \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$\boxed{r}\,,\,\boxed{\mathbf{x}}$	$\begin{smallmatrix} \mathbf{x} & r \end{smallmatrix}$	$x_1 \neq r$	$c \in \mathbf{x}$
				$\begin{smallmatrix} r & c \\ \mathbf{x}' & \end{smallmatrix}$	$x_1 = r$	
		$\begin{smallmatrix} \bullet & r \\ \mathbf{x}' & \\ b & \end{smallmatrix}$	$\boxed{r}\,,\,\boxed{\mathbf{x}}$	$\begin{smallmatrix} \mathbf{x} & r \end{smallmatrix}$	$c \notin \mathbf{x}, s \in \mathbf{x}$	
r	$\begin{smallmatrix} r \\ b \end{smallmatrix}$	$\begin{smallmatrix} s & r \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$\boxed{r}\,,\,\boxed{\mathbf{x}}$	$\begin{smallmatrix} \mathbf{x} & r \end{smallmatrix}$	$x_1 \neq r$	$c \in \mathbf{x}$
				$\begin{smallmatrix} r & c \\ \mathbf{x}' & \end{smallmatrix}$	$x_1 = r$	
		$\begin{smallmatrix} s & r \\ \mathbf{x}' & \\ b & \end{smallmatrix}$	$\boxed{r}\,,\,\boxed{\mathbf{x}}$	$\begin{smallmatrix} \mathbf{x} & r \end{smallmatrix}$	$c \notin \mathbf{x}, s \in \mathbf{x}$	
c	$\begin{smallmatrix} s \\ a \end{smallmatrix}$	$\begin{smallmatrix} \bullet & s \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$\boxed{c}\,,\,\boxed{\mathbf{x}}$	$\begin{smallmatrix} \mathbf{x} & c \end{smallmatrix}$	$c \in \mathbf{x}$	$s \in \mathbf{x}$ is automatic
		$\begin{smallmatrix} \bullet & s \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$c \notin \mathbf{x}$	
d	$\begin{smallmatrix} d \\ a \end{smallmatrix}$	$\begin{smallmatrix} \bullet & d \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$\boxed{d}\,,\,\boxed{\mathbf{x}}$	$\begin{smallmatrix} \mathbf{x} & d \end{smallmatrix}$	$c \in \mathbf{x}$	
		$\begin{smallmatrix} \bullet & d \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$x_1 = s$	$c \notin \mathbf{x}$
		$\begin{smallmatrix} \mathbf{x} & d \\ a & \end{smallmatrix}$			$x_1 \neq s$	$c \notin \mathbf{x}$
d	$\begin{smallmatrix} d \\ b \end{smallmatrix}$	$\begin{smallmatrix} s & d \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$\boxed{d}\,,\,\boxed{\mathbf{x}}$	$\begin{smallmatrix} \mathbf{x} & d \end{smallmatrix}$	$c \in \mathbf{x}$	
		$\begin{smallmatrix} s & d \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$x_1 = s$	$c \notin \mathbf{x}$
		$\begin{smallmatrix} \mathbf{x} & d \\ b & \end{smallmatrix}$			$x_1 \neq s$	$c \notin \mathbf{x}$

TABLE 5. Reduction when $n \geqslant 1$

$G_{\mathbf{U}}$	$\chi_{\mathbf{E}}$
real orthogonal	1
real symplectic	$\chi_{\mathbf{E}}^4 = 1, \ \ \chi_{\mathbf{E}}(\varepsilon_{\mathbf{U}}) = (-1)^{\mathbb{p}}$
quaternionic orthogonal	$\det^{\frac{1}{2}}$
quaternionic symplectic	$\det^{-\frac{1}{2}}$

TABLE 6. The character $\chi_{\mathbf{E}}$

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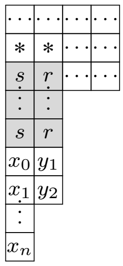

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