Unipotent representations of real classical groups

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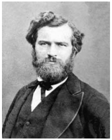
Jordan-Chevalley decomposition

- For each $g \in G := \mathrm{GL}_n(\mathbb{C})$, $\exists!$ pair (s, u) such that

$$g = su = us,$$

- s is semisimple, and u is unipotent ($u - 1$ is nilpotent.)

Camille Jordan   Claude Chevalley

-  

- Classification of conjugation classes in G:

$$G/\sim = \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \{\, su \mid u \in G^s \,\}/\sim \xleftrightarrow{bij.} \bigsqcup_{s \in G_{\mathrm{s.s.}}/\sim} \mathrm{unip}(G^s)$$

Reductive groups and its representation

- $\mathbf{G}$: a linear algebraic group: $\mathrm{SL}_n, \mathrm{Sp}_{2n}, \mathrm{O}_n, G_2, F_4, E_6, E_7, E_8 \cdots$.
- *Question:* How to classify $\mathrm{Irr}(G)$ where $G = \mathbf{G}(k)$?
- What is the structure of $\mathrm{Irr}(G)$?
- When G is abelian:

  Lev S. Pontryagin

  $$G \leftrightsquigarrow \mathrm{Irr}_{\text{unit.}}(G) = \widehat{G}, \text{ an abelian group}$$

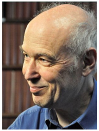
- Examples: $\mathbb{R} \leftrightsquigarrow \mathbb{R}$, $\mathbb{Z} \leftrightsquigarrow S^1$, $\mathbb{Z}/n\mathbb{Z} \leftrightsquigarrow \mu_n$

Finite group of Lie type: $G := \mathbf{G}(\mathbb{F}_q)$

- Deligne-Lusztig and Lusztig (1970s–1980s)
- *Jordan decomposition*:

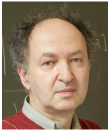
  Define dual group $\widehat{\mathbf{G}}$. E.g. $\mathrm{Sp}_{2n} = \mathbf{G} \leftrightsquigarrow \widehat{\mathbf{G}} = \mathrm{SO}_{2n+1}$

Deligne



$$\mathrm{Irr}(G) = \bigsqcup_{s \in \widehat{G}_{\text{s.s.}}/\sim} \mathcal{E}(G,s).$$

Lusztig



Lusztig's map to the unipotent packet.

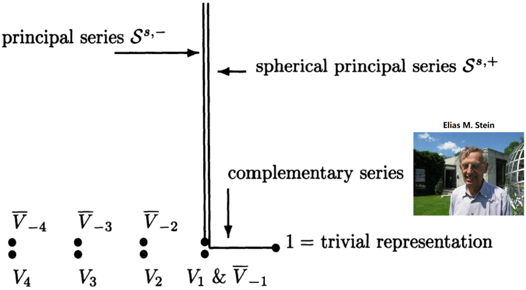
$$\mathcal{E}(G,s) \xleftrightarrow{bij.} \mathcal{E}(\widehat{G}_s, 1)$$

Representations of Real Lie groups

- $G = \mathrm{GL}_n(\mathbb{R}), \mathrm{U}(p,q), \mathrm{O}(p,q), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{Mp}(2n,\mathbb{R}), \cdots$

$$\{\text{ discrete series }\} \subset \{\text{ tempered }\} \subset \{\text{ unitary }\} \subset \mathrm{Irr}(G)$$

- Unitary dual of $\mathrm{SL}_2(\mathbb{R})$:



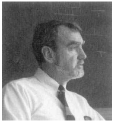
- *Open problem:* Structure of the unitary dual!

Langlands and Arthur

- $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\,\mathbb{C}^\times$ such that $jzj^{-1} = \bar{z}, j^2 = -1 \in \mathbb{C}^\times$.
- Langlands dual group ${}^L G = \widehat{G} \rtimes \langle j \rangle$. Eg:

$$\mathrm{Sp}_{2n}(\mathbb{R}) = G \longleftrightarrow \widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$$

- *Langlands*



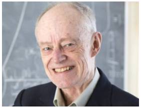
$$\mathrm{Irr}(G) \xrightarrow{\text{finite to one}} \{\, \phi\colon W_{\mathbb{R}} \to {}^L G \,\} \,/\, \widehat{G}.$$

- *Arthur: (Classical group, 2013)* $\mathrm{Irr}_{\mathrm{temp}}(G) \subset \mathrm{Irr}_A(G) \subset \mathrm{Irr}_{\mathrm{unit}}(G)$

$$\mathrm{Irr}_A(G) \xrightarrow{\text{finite to finite}} \{\, \psi\colon W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G \,\} \,/\, \widehat{G}.$$

Arthur's unipotent representations

- $\psi : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$
- Unipotent parameter: $\psi|_{\mathbb{C}^\times} = \text{trivial}$

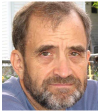
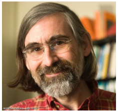
  $\Leftrightarrow$ nilpotent orbit of a real from of $\widehat{G}$.
-  *Conjecture:* $\exists$ unipotent Arthur packets

  "On some problems suggested by the trace formula" 1980's

  Mœglin Renard
- `Moeglin.jpg`  *Reduction to unipotent Arthur packet*

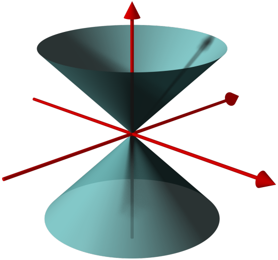
  "Sur Les paquets d'Arthur des groupes classiques réels" (2020 JEMS)

Barbasch-Vogan's definition of special unip. repn.

- Barbasch  Vogan  $\mathrm{Unip}(G) \approx$ repn. has smallest "size"
- Wavefront cycle/Associated cycle:

$$\mathrm{Irr}(G) \longrightarrow \mathcal{K}(\text{equiv. sheaves on nil-cone})$$

- E.g.: $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R})) \cong \mathbb{R}^3$.



Barbasch-Vogan's definition of special unip. repn. II

G: a real reductive group.

- $\widehat{\mathcal{O}}$: a nilpotent orbit in $\widehat{G}$.

  $\rightsquigarrow$ the maximal primitive ideal $\mathcal{I}_{\widehat{\mathcal{O}}} \subset \mathcal{U}(\mathfrak{g})$.

Duflo

Joseph

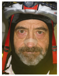
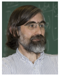
- *Definition* (Barbasch-Vogan):

  An irr. admissible G-repn. is called *special unipotent* if

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) = \mathcal{I}_{\widehat{\mathcal{O}}}.$$

$\Longleftrightarrow$ π has inf. char. $\chi_{\widehat{\mathcal{O}}}$ and $\mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}}$

- $\mathcal{O} \in \mathrm{Nil}^{\mathrm{special}}(\mathbf{G})$:

  the  Spaltenstein    dual of $\widehat{\mathcal{O}}$.

- $\mathrm{Unip}_{\widehat{\mathcal{O}}}(G) :=$ { special unipotent repn. attached to $\widehat{\mathcal{O}}$}.

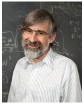
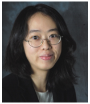
Conjecture/Open problems

- *Major open problem:* Classify the unitary dual:

$$\mathrm{Irr}_{\mathrm{unit}}(G) = \{ \text{ irr. unitary repn. of } G\}.$$

- *Philosophy:* $\mathrm{Unip}_{\widehat{\mathcal{O}}}(G) =$ the building blocks of the unitary dual.
- *Conjecture (1980s)*: $\mathrm{Unip}_{\widehat{\mathcal{O}}}(G)$ consists of unitary repn.
- Question 1: Size of $\mathrm{Unip}_{\widehat{\mathcal{O}}}(G)$?
- Question 2: Construction of unipotent repn.
- Question 3: Character formula?
- *Barbasch-Vogan* 1985: Complete answers for complex groups.
- *Barbasch* 1989: Proved the conj. for complex classical groups.

Atlas of Lie groups

- De Cloux, Adams  Leeuwen  Paul  Trapa  Vogan  Yee 

```
atlas> set G = Sp(2,2)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> void: for v in up do for r in v do print(r,infinitesimal_character (r)) od od
(final parameter(x=12,lambda=[1,0,1,0]/1,nu=[0,0,0,0]/1),[ 1, 1, 1, 1 ]/2)
(final parameter(x=25,lambda=[2,1,0,-1]/1,nu=[1,0,0,-1]/2),[ 2, 1, 1, 0 ]/2)
(final parameter(x=36,lambda=[3,3,2,2]/1,nu=[1,1,1,1]/1),[ 3, 3, 1, 1 ]/2)
(final parameter(x=39,lambda=[4,3,1,0]/1,nu=[3,3,0,0]/2),[ 4, 2, 1, 1 ]/2)
(final parameter(x=35,lambda=[4,2,3,-1]/1,nu=[3,1,3,-1]/2),[ 2, 1, 1, 0 ]/1)
(final parameter(x=39,lambda=[4,3,1,0]/1,nu=[5,5,1,-1]/2),[ 3, 2, 1, 0 ]/1)
(final parameter(x=41,lambda=[4,3,2,1]/1,nu=[7,7,3,3]/2),[ 4, 3, 2, 1 ]/1)
atlas>
```

$\rightsquigarrow$ complete answer for *exceptional groups*.

Classical groups and special unipotent representations

	G	$\mathbf{G}$	$\widehat{\mathbf{G}}$	
D_n	$\mathrm{O}(p,2n-p)$	$\mathrm{O}(2n,\mathbb{C})$	$\mathrm{O}(2n,\mathbb{C})$	D_n
C_n	$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{SO}(2n+1,\mathbb{C})$	B_n
B_n	$\mathrm{O}(p,2n+1-p)$	$\mathrm{O}(2n+1,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$	C_n
$\tilde{C}_n$	$\mathrm{Mp}(2n,\mathbb{R})$	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{Sp}(2n,\mathbb{C})$	C_n
D_n	$\mathrm{O}^*(n)$	$\mathrm{SO}(2n,\mathbb{C})$	$\mathrm{SO}(2n,\mathbb{C})$	D_n
C_n	$\mathrm{Sp}(p,n-p)$	$\mathrm{Sp}(2n,\mathbb{C})$	$\mathrm{SO}(2n+1,\mathbb{C})$	B_n
A_n	$\mathrm{U}(p,n-p)$	$\mathrm{GL}(n,\mathbb{C})$	$\mathrm{GL}(n,\mathbb{C})$	A_n
A_m	$\mathrm{U}(r,m-r)$	$\mathrm{GL}(m,\mathbb{C})$	$\mathrm{GL}(m,\mathbb{C})$	A_m

Theorem (Barbasch-M.-Sun-Zhu 2021)

Arthur-Barbasch-Vogan's conj. on special unipotent repn. holds for G: All *specail unipotent repn.* of G are *unitarizable*.

Theta lifting

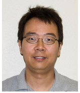
- Howe
  
  $\exists\theta$-correspondence. (1970s)

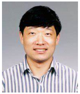
  E.g. a subset of $\mathrm{Irr}(\mathrm{O}(p,q)) \overset{bij.}{\longleftrightarrow}$ a subset of $\mathrm{Irr}(\mathrm{Sp}(2n,\mathbb{R}))$

- θ-lifting $\approx \sqrt{\text{induction}}$: a effective tool for *"small"* repn.
- Towards the construction of unipotent repn.:

Adams

Barbasch

He

Huang

Li

Loke


Nishiyama

Mœglin

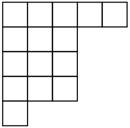
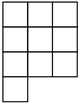
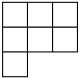
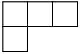
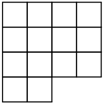
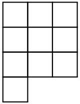
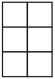
Paul

Przebinda

Trapa

. . .

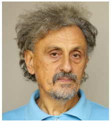
Inductive structure of nilpotent orbits

$\widehat{\mathbf{G}}_i$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$
$\widehat{\mathcal{O}}_i$				
$\mathcal{O}_i$				
$\mathbf{G}_i$	$\mathrm{Sp}(14,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{Sp}(6,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

Relate to

Kraft

Procesi



Takuya Ohta 太田 琢也 's study of singularities of a nilpotent orbit closure.

About the proof

- θ-lifting $\approx \sqrt{\text{parabolic induction}}$
- Structure of degenerate principle series (Algebra).
- Estimation of matrix coeff. integrals (Analysis) $\rightsquigarrow$ unitarity
- Sharp formula of Asso. Cycles: lower bound=upper bound
  - *Algebra+Analysis:* $\rightsquigarrow$ lower bound.
  - *Algebraic Geometry:* double fibration of moment maps
    $\rightsquigarrow$ upper bound
- Exhaustion: lower bound=upper bound (Combinatorics)
  - Character theory (Kazhdan-Lusztig-Vogan theory)
    $\rightsquigarrow$ upper bound by counting "tableaux'.
  - Asso. Cycle+Injectivity of θ $\rightsquigarrow$ lower bound.

Preprints (Barbasch, M. , Sun and Zhu)

- *Definition for metaplectic groups*

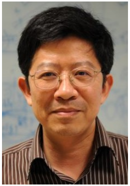
  On the notion of metaplectic Barbasch-Vogan duality
  https://arxiv.org/abs/2010.16089
- *Construction using θ-lifting, 76 pages*

  Special unipotent representations: orthogonal and symplectic groups
  https://arxiv.org/abs/1712.05552v2
- *Counting unipotent representations (in preparation)*

Barbasch

Sun

Zhu



Thank you for your attention!