SPECIAL UNIPOTENT REPRESENTATIONS : ORTHOGONAL AND SYMPLECTIC GROUPS

DAN M. BARBASCH, JIA-JUN MA, BINYONG SUN, AND CHEN-BO ZHU

Contents

1. Introduction and the main results

1.1. **Unitary representations and the orbit method.** A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [43, 44]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [6, 43]. For more general Lie groups, work of Mackey and Duflo [28, 56] suggest that one should focus attention on reductive Lie groups. As expounded by Vogan in his writings (see for example [82, 84, 85]), the problem finally is to

2000 *Mathematics Subject Classification.* 22E45, 22E46.

Key words and phrases. orbit method, unitary dual, unipotent representation, classical group, theta lifting, moment map.

quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We mention two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [4, 5], Adams, Barbasch and Vogan established some important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [2]. The problem of classifying (integral) special unipotent representations for complex semisimple groups was solved earlier by Barbasch and Vogan [16] and the unitarity of these representations was established by Barbasch for complex classical groups [7, Section 10]. Shortly after, Barbasch outlined a proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [8]. The second major development is Vogan's theory of associated varieties [83] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a representation "attached" to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [39], the description of discrete spectrum by Adams [1] and Li [51], and the preservation of unitarity in stable range theta lifting by Li [50]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [12, 17, 34, 36, 37, 68, 71, 74, 80]. We would also like to mention the work of Przebinda [71] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [34] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [29, 53, 62, 64], which among other things demonstrate the tight link between the orbit method and Howe's theory.

In the present article we will demonstrate that the orbit method and Howe's theory in fact have superb synergy when it comes to special unipotent representations. (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [12, 34, 59, 80].) We will restrict our attention to a real classical group G of type B, C or D (which in our terminology includes a real metaplectic group), and we will classify all special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. Here $\check{\mathcal{O}}$ is a nilpotent adjoint orbit of $\check{G}$, the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group [13]). When $\check{\mathcal{O}}$ is of good parity [60], we will construct all special unipotent representations of G attached to $\check{\mathcal{O}}$ via the method of theta lifting. As a direct consequence of the construction and the classification, we conclude that all special unipotent representations of G are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([2, Introduction]). Here we wish to emphasize, that while there have been extensive investigations of unipotent representations for real reductive groups, by Vogan and his collaborators (see e.g. [2, 82, 83]), it is only for unitary groups complete results are known (*cf.* [15, 80]), in which case all such representations may also be described in terms of cohomological induction.

1.2. **Special unipotent representations of classical groups of type B, C or D.** In this article, we aim to classify special unipotent representations of classical groups of type B, C or D. As the cases of complex orthogonal groups and complex symplectic groups are well-understood (see [16] and [12]), we will focus on the following groups:

$$\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p,q),\ \mathrm{O}^*(2n), \tag{1.1}$$

where $p,q,n \in \mathbb{N} := \{0,1,2,\cdots\}$. Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the real metaplectic group, namely the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n=0$.

Let G be one of the groups in (1.1). As usual, we view G as a real form of $G_{\mathbb{C}}$ (or a double cover of a real form of $G_{\mathbb{C}}$ in the metaplecitic case), where

$$G_{\mathbb{C}} := \begin{cases} \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \text{ or } \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2p+2q}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ for the Lie algebras of G and $G_{\mathbb{C}}$, respectively, and view $\mathfrak{g}_{\mathbb{R}}$ as a real form of $\mathfrak{g}$.

Denote $r_{\mathfrak{g}}$ the rank of $\mathfrak{g}$. Let $W_{r_{\mathfrak{g}}}$ be the subgroup of $\mathrm{GL}_{r_{\mathfrak{g}}}(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. Then as usual, Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} = (\mathrm{S}(\mathbb{C}^{r_{\mathfrak{g}}}))^{W_{r_{\mathfrak{g}}}}. \tag{1.2}$$

Here and henceforth, "U" indicates the universal enveloping algebra of a Lie algebra, a superscript group indicate the space of invariant vectors under the group action, and "S" indicates the symmetric algebra. Unless $G_{\mathbb{C}}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. By (1.2), we have the following parameterization of characters of $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$:

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}, \mathbb{C}) = W_{r_{\mathfrak{g}}}\backslash(\mathbb{C}^{r_{\mathfrak{g}}})^* = W_{r_{\mathfrak{g}}}\backslash\mathbb{C}^{r_{\mathfrak{g}}}$$

Here "$\mathrm{Hom}_{\mathrm{alg}}$" indicates the set of $\mathbb{C}$-algebra homomorphisms, and a superscript " $*$ " over a vector space indicates the dual space.

We define the Langlands dual of G to be the complex group

$$\check{G} := \begin{cases} \mathrm{Sp}_{p+q-1}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is odd}; \\ \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is even}; \\ \mathrm{O}_{2n+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{O}_{2p+2q+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$.

Remark. The authors have defined the notion of metaplectic dual for a real metaplectic group [13]. In this article, we have chosen to use the uniform terminology of Langlands dual (rather than metaplectic dual in the case of a real metaplectic group).

Denote by $\mathrm{Nil}(\check{\mathfrak{g}})$ the set of nilpotent $\check{G}$-orbits in $\check{\mathfrak{g}}$. When no confusion is possible, we will not distinguish a nilpotent orbit in $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ with its corresponding Young diagram. In particular, the zero orbit is represented by the Young diagram consisting of one nonempty column.

Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. It determines a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$ as in what follows. For every $a \in \mathbb{N}$, write

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd;} \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even;} \end{cases}$$

By convention, $\rho(1)$ and $\rho(0)$ are the empty sequence. Write $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ ($s \geqslant 0$) for the row lengths of $\check{\mathcal{O}}$. Define

$$\chi(\check{\mathcal{O}}) := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0), \tag{1.3}$$

to be viewed as a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$. Here the number of 0's is

$$\left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor.$$

Recall the following well-known result of Dixmier ([18, Section 3]): for every algebraic character χ of $\mathrm{Z}(\mathfrak{g})$, there exists a unique maximal ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of χ. As an easy consequence, there will be a unique maximal $G_{\mathbb{C}}$-stable ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$, for $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.

Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation ([20, 87]) if its Harish-Chandra module has finite length. When $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a metaplectic group, write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then a representation of G is said to be genuine if ε_G acts via the scalar multiplication by -1. The notion of "genuine" will be used in similar situations without further explanation. Following Barbasch and Vogan ([2, 16]), we make the following definition.

Definition 1.1. *Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. An irreducible Casselman-Wallach representation π of G is attached to $\check{\mathcal{O}}$ if*

- *$I_{\check{\mathcal{O}}}$ annihilates π; and*
- *π is genuine if G is a metaplectic group.*

Write $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ for the set of isomorphism classes of irreducible Casselman-Wallach representations of G that are attached to $\check{\mathcal{O}}$. We say that an irreducible Casselman-Wallach representation of G is special unipotent if it is attached to $\check{\mathcal{O}}$, for some $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. As mentioned earlier, we will construct all special unipotent representations of G, and will show that all of them are unitarizable, as predicted by the Arthur-Barbasch-Vogan conjecture ([2, Introduction]).

1.3. **Combinatorial construct: painted bipartitions.** We introduce a symbol $\star$, taking values in $\{B, C, D, \widetilde{C}, C^*, D^*\}$, to specify the type of the groups that we are considering as in (1.1), namely odd real orthogonal groups, real symplectic groups, even real orthogonal groups, real metaplectic groups, quaternionic symplectic groups and quaternionic orthogonal groups, respectively.

For a Young diagram $\imath$, write

$$\mathbf{r}_1(\imath) \geqslant \mathbf{r}_2(\imath) \geqslant \mathbf{r}_3(\imath) \geqslant \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty} \mathbf{r}_i(\imath)$ the total size of $\imath$.

For any Young diagram $\imath$, we introduce the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as the following subset of $\mathbb{N}^+ \times \mathbb{N}^+$ ($\mathbb{N}^+$ denotes the set of positive integers):

$$\mathrm{Box}(\imath) := \left\{ (i, j) \in \mathbb{N}^+ \times \mathbb{N}^+ \;\middle|\; j \leqslant \mathbf{r}_i(\imath) \right\}. \tag{1.4}$$

We also introduce five symbols $\bullet$, s, r, c and d, and make the following definition.

Definition 1.2. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *when $S = \{s\}$ or $\{r\}$, every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$;*
- *when $S = \{c\}$ or $\{d\}$, every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$.*

A painted Young diagram is then a pair $(\imath, \mathcal{P})$, consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$.

Example. Suppose that $\imath = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$, then there are $25 + 12 + 6 + 2 = 45$ paintings on $\imath$ in total as listed below.

$$\begin{array}{|c|c|}\hline \bullet & \alpha \\ \hline \beta \\ \cline{1-1}\end{array} \quad \alpha, \beta \in \{\bullet, s, r, c, d\} \qquad \begin{array}{|c|c|}\hline s & \alpha \\ \hline \beta \\ \cline{1-1}\end{array} \quad \alpha \in \{r, c, d\}, \beta \in \{s, r, c, d\}$$

$$\begin{array}{|c|c|}\hline r & \alpha \\ \hline \beta \\ \cline{1-1}\end{array} \quad \alpha \in \{c, d\}, \beta \in \{r, c, d\} \qquad \begin{array}{|c|c|}\hline c & \alpha \\ \hline d \\ \cline{1-1}\end{array} \quad \alpha \in \{c, d\}$$

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.3. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- *$\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$;*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, r, c, d\}, & \text{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \text{if } \alpha = D; \\ \{\bullet, s, c\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \text{if } \alpha = C^*; \\ \{\bullet, s\}, & \text{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, s\}, & \text{if } \alpha = C; \\ \{\bullet\}, & \text{if } \alpha = D; \\ \{\bullet, r, d\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \text{if } \alpha = C^*; \\ \{\bullet, r\}, & \text{if } \alpha = D^*. \end{cases}$$

For any painted bipartition τ as in Definition 1.3, we write

$$\imath_\tau := \imath, \ \mathcal{P}_\tau := \mathcal{P}, \ \jmath_\tau := \jmath, \ \mathcal{Q}_\tau := \mathcal{Q}, \ \alpha_\tau := \alpha,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

We further attach some objects to τ in what follows:

$$|\tau|, \ (p_\tau, \ q_\tau), \ G_\tau, \ \dim \tau, \ \varepsilon_\tau.$$

$|\tau|$**:** This is the natural number

$$|\tau| := |\imath| + |\jmath|\,.$$

(p_τ, q_τ)**:** If $\star_\tau \in \{B, D, C^*\}$, this is a pair of natural numbers given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := \#\bullet + 2\#r + \#c + \#d + \#B^+; \\ q_\tau := \#\bullet + 2\#s + \#c + \#d + \#B^-. \end{cases}$$

If $\star_\tau \in \{C, \widetilde{C}, D^*\}$, we let $p_\tau := q_\tau := |\tau|$.

G_τ**:** This is a classical group given by

$$G_\tau := \begin{cases} \mathrm{O}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

$\dim \tau$**:** This is the dimension of the standard representation of the complexification of G_τ, or equivalently,

$$\dim \tau := \begin{cases} 2\,|\tau| + 1, & \text{if } \star_\tau = B; \\ 2\,|\tau|\,, & \text{otherwise.} \end{cases}$$

ε_τ**:** This is the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\tau = 0 \Leftrightarrow \text{the symbol } d \text{ occurs in the first column of } (\imath, \mathcal{P}) \text{ or } (\jmath, \mathcal{Q}).$$

The triple $\mathsf{s}_\tau = (\star_\tau, p_\tau, q_\tau) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ will also be referred to as the classical signature attached to τ.

Example. Suppose that

$$\tau = \begin{ytableau} \bullet & c \\ \bullet \\ c \end{ytableau} \times \begin{ytableau} \bullet & s & r \\ \bullet \\ r \\ r \end{ytableau} \times B^+.$$

Then

$$\begin{cases} |\tau| = 10; \\ p_\tau = 4 + 6 + 2 + 0 + 1 = 13; \\ q_\tau = 4 + 2 + 2 + 0 + 0 = 8; \\ G_\tau = \mathrm{O}(13, 8); \\ \dim \tau = 21; \\ \varepsilon_\tau = 1. \end{cases}$$

1.4. **Counting special unipotent representations by painted bipartitions.** We fix a classical group G which has type $\star$. Following [60, Definition 4.1], we say that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity if

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \check{G} \text{ is a complex symplectic group; and} \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \check{G} \text{ is a complex orthogonal group.} \end{cases}$$

In general, the study of the special unipotent representations attached to $\check{\mathcal{O}}$ will be reduced to the case when $\check{\mathcal{O}}$ has $\star$-good parity. We refer the reader to [14].

For the rest of this subsection, assume that $\check{\mathcal{O}}$ has $\star$-good parity. Equivalently we consider $\check{\mathcal{O}}$ as a Young diagram that has $\star$-good parity in the following sense:

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \star \in \{B, \widetilde{C}\}; \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \star \in \{C, D, C^*, D^*\}; \text{ and} \\ \text{the total size } \left|\check{\mathcal{O}}\right| \text{ is odd if and only if } \star \in \{C, C^*\}. \end{cases}$$

Definition 1.4. *A $\star$-pair is a pair $(i, i+1)$ of consecutive positive integers such that*

$$u\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be

- *vacant in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*
- *tailed in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and even.*

Denote $\mathrm{PP}_\star(\check{\mathcal{O}})$ the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$.

For any $\check{\mathcal{O}}$, we attach a pair of Young diagrams

$$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) := (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})),$$

as follows.

The case when $\star = B$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}).$$

The case when $\star = \widetilde{C}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}).$$

The case when $\star \in \{C, C^*\}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

The case when $\star \in \{D, D^*\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0, \end{cases}$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

Define

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) := \left\{ \tau \text{ is a painted bipartition} \mid \star_\tau = \star \text{ and } (\imath_\tau, \jmath_\tau) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})) \right\}.$$

We also define the following extended parameter set:

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) := \begin{cases} \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})\}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\varnothing\}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

We use $\boldsymbol{\tau} = (\tau, \wp)$ to denote an element in $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$.

Put

$$\mathrm{Unip}_\star(\check{\mathcal{O}}) := \begin{cases} \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|+1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = B; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{|\check{\mathcal{O}}|-1}(\mathbb{R})), & \text{if } \star = C; \\ \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = D; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{Sp}}_{|\check{\mathcal{O}}|}(\mathbb{R})), & \text{if } \star = \widetilde{C}; \\ \bigsqcup_{p,q\in\mathbb{N}, 2p+2q=|\check{\mathcal{O}}|-1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q)), & \text{if } \star = C^*; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}^*(|\check{\mathcal{O}}|)), & \text{if } \star = D^*. \end{cases}$$

The main result of [14], on counting of special unipotent representations, is as follows.

Theorem 1.5. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then*

$$\#(\mathrm{Unip}_\star \check{\mathcal{O}}) = \begin{cases} 2\#(\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})), & \textit{if } \star \in \{B, D\} \textit{ and } (\star, \check{\mathcal{O}}) \neq (D, \varnothing); \\ \#(\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})), & \textit{otherwise}. \end{cases}$$

1.5. **Descending painted bipartitions and constructing representations by induction.** Let $\star$ and $\check{\mathcal{O}}$ be as before. For every $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, in what follows we will construct a representation $\pi_{\boldsymbol{\tau}}$ of G_τ by the method of theta lifting. For the initial case when $\check{\mathcal{O}}$ is the empty Young diagram, define

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \text{the one dimensional genuine representation,} & \text{if } \star = \widetilde{C} \text{ so that } G_\tau = \widetilde{\mathrm{Sp}}_0(\mathbb{R}); \\ \text{the one dimensional trivial representation,} & \text{if } \star \neq \widetilde{C}. \end{cases}$$

Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or } \ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or } \ D^*.$$

We call $\star'$ the Howe dual of $\star$. Assume now that $\check{\mathcal{O}}$ is nonempty, and define its dual descent to be

$$\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}}) := \text{the Young diagram obtained from } \check{\mathcal{O}} \text{ by removing the first row.}$$

Note that $\check{\mathcal{O}}'$ has $\star'$-good parity, and

$$\mathrm{PP}_{\star'}(\check{\mathcal{O}}') = \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \mathrm{PP}_\star(\check{\mathcal{O}})\}.$$

Define the dual descent of $\wp$ to be

$$\wp' := \check{\nabla}(\wp) := \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \wp\} \subset \mathrm{PP}_{\star'}(\check{\mathcal{O}}'). \tag{1.5}$$

In Section 2, we will define the descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

We then define the descent of $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ to be the element

$$\boldsymbol{\tau}' := (\tau', \wp') := \nabla(\boldsymbol{\tau}) := (\nabla(\tau), \check{\nabla}(\wp)) \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

Let $(W_{\tau,\tau'}, \langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\tau'})$ be a real symplectic space of dimension $\dim\tau\cdot\dim\tau'$. As usual, there are continuous homomorphisms $G_\tau \to \mathrm{Sp}(W_{\tau,\tau'})$ and $G_{\tau'} \to \mathrm{Sp}(W_{\tau,\tau'})$ whose images form a reductive dual pair in $\mathrm{Sp}(W_{\tau,\tau'})$. We form the semidirect product

$$J_{\tau,\tau'} := (G_\tau \times G_{\tau'}) \ltimes \mathrm{H}(W_{\tau,\tau'}),$$

where

$$\mathrm{H}(W_{\tau,\tau'}) := W_{\tau,\tau'} \times \mathbb{R}$$

is the Heisenberg group with group multiplication

$$(w,t)(w',t') := (w+w', t+t'+\langle w,w'\rangle_{\tau,\tau'}), \quad w,w' \in W_{\tau,\tau'},\ t,t' \in \mathbb{R}.$$

Let $\omega_{\tau,\tau'}$ be a suitably normalized smooth oscillator representation of $J_{\tau,\tau'}$ such that every $t \in \mathbb{R} \subset J_{\tau,\tau'}$ acts on it through the scalar multiplication by $e^{2\pi\sqrt{-1}\,t}$ (the letter π often denotes a representation, but here it stands for the circumference ratio). See Section 4.1 for details.

For any Casselman-Wallach representation π' of $G_{\tau'}$, write

$$\check{\Theta}_{\tau'}^{\tau}(\pi') := (\omega_{\tau,\tau'}\widehat{\otimes}\pi')_{G_{\tau'}} \qquad \text{(the Hausdorff coinvariant space)},$$

where $\widehat{\otimes}$ indicates the complete projective tensor product. This is a Casselman-Wallach representation of G_τ. Now we define the representation $\pi_\uptau$ of G_τ by induction on the number of nonempty rows of $\check{\mathcal{O}}$:

$$\pi_\uptau := \begin{cases} \check{\Theta}_{\tau'}^{\tau}(\pi_{\uptau'}) \otimes (1_{p_\tau,q_\tau}^{+,-})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D;\\ \check{\Theta}_{\tau'}^{\tau}(\pi_{\uptau'} \otimes \det^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C};\\ \check{\Theta}_{\tau'}^{\tau}(\pi_{\uptau'}), & \text{if } \star = C^* \text{ or } D^*. \end{cases} \tag{1.6}$$

Here $1_{p_\tau,q_\tau}^{+,-}$ denotes the character of $\mathrm{O}(p_\tau,q_\tau)$ whose restriction to $\mathrm{O}(p_\tau)\times\mathrm{O}(q_\tau)$ equals $1\otimes\det$ (1 stands for the trivial character), and $\varepsilon_\wp$ denote the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\wp := 1 \Leftrightarrow (1,2) \in \wp.$$

In the sequel, we will use $\varnothing$ to denote the empty set (in the usual way), as well as the empty Young diagram or the painted Young diagram whose underlying Young diagram is empty. As always, we let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$.

We are now ready to state our first main theorem.

Theorem 1.6. *Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For every $\uptau = (\tau,\wp) \in \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}})$, the representation $\pi_\uptau$ of G_τ in (1.6) *is irreducible and attached to $\check{\mathcal{O}}$.*
(b) If $\star \in \{B,D\}$ and $(\star,\check{\mathcal{O}}) \neq (D,\varnothing)$, then the map

$$\begin{array}{rcl} \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}),\\ (\uptau,\epsilon) & \mapsto & \pi_\uptau \otimes \det^\epsilon \end{array}$$

is bijective.
(c) In all other cases, the map

$$\begin{array}{rcl} \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}) & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}),\\ \uptau & \mapsto & \pi_\uptau \end{array}$$

is bijective.

By the above theorem, we have explicitly constructed all special unipotent representations in $\mathrm{Unip}_\star(\check{\mathcal{O}})$, when $\check{\mathcal{O}}$ has $\star$-good parity. The method of matrix coefficient integrals (Section 15) will imply the following

Corollary 1.7. *Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity. Then all special unipotent representations in $\mathrm{Unip}_\star(\check{\mathcal{O}})$ are unitarizable.*

The reduction of [14] allows us to classify all special unipotent presentations attached to a general $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ from those which have $\star$-good parity. We thus conclude

Theorem 1.8. *All special unipotent representations of the classical groups in* (1.1) *are unitarizable.*

1.6. **Computing associated cycles of the constructed representations.** Let $G, G_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{R}}, \mathfrak{g}, \check{\mathfrak{g}}$ and $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ be as in Section 1.2. Fix a maximal compact subgroup K of G whose Lie algebra is denoted by $\mathfrak{k}_{\mathbb{R}}$. We equipping $\mathfrak{g}$ with the trace form. Then we have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the complexification of $\mathfrak{k}_{\mathbb{R}}$. By taking the dual spaces, we also have a decomposition

$$\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*,$$

Write $K_{\mathbb{C}}$ for the complexification of the compact group K. It is a complex algebraic group with an obvious algebraic action on $\mathfrak{p}^*$.

For any $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$, write $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}^*)$ for the Barbarsch-Vogan dual of $\check{\mathcal{O}}$ so that its Zariski closure in $\mathfrak{g}^*$ equals the associated variety of the ideal $I_{\check{\mathcal{O}}} \subset \mathrm{U}(\mathfrak{g})$. See [13, 16]. The algebraic variety $\mathcal{O} \cap \mathfrak{p}^*$ is a finite union of $K_{\mathbb{C}}$-orbits. Given any such orbit $\mathscr{O}$, write $\mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O})$ for the Grothendieck group of the category of $K_{\mathbb{C}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathcal{K}_{K_{\mathbb{C}}}(\mathscr{O}).$$

We say that a Casselman-Wallach representation of G is $\mathcal{O}$-bounded if the associated variety of its annihilator ideal is contained in the Zariski closure of $\mathcal{O}$. Note that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are $\mathcal{O}$-bounded. Write $\mathcal{K}(G)_{\mathcal{O}-\text{bounded}}$ for the Grothendieck group of the category of all such representations. From [83, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{AC}_{\mathcal{O}} \colon \mathcal{K}(G)_{\mathcal{O}-\text{bounded}} \longrightarrow \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}).$$

We call $\mathrm{AC}_{\mathcal{O}}(\pi)$ the associated cycle of π, where π is an $\mathcal{O}$-bounded Casselman-Wallach representation of G. This is a fundamental invariant attached to π.

Following Vogan [83, Section 8], we make the following definition.

Definition 1.9. *Let $\mathscr{O}$ be a $K_{\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}^*$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $K_{\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_{\mathbf{e}}$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}})^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_{\mathbf{e}}$;*
- *if $\star = \widetilde{C}$, then ε_G acts on $\mathcal{E}$ by the scalar multiplication by -1.*

Here $\mathbf{e} \in \mathscr{O}$, $\mathcal{E}_{\mathbf{e}}$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_{\mathbf{e}}$ denotes the Lie algebra of the stabilizer of $\mathbf{e}$ in $K_{\mathbb{C}}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}})^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_{\mathbf{e}}$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_{\mathbf{e}}$.

Note that in the situation of the classical groups we consider in this article, all admissible orbit data are line bundles. Denote by $\mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O})$. Put

$$\mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}_{K_{\mathbb{C}}}(\mathscr{O}) \subset \mathcal{K}_{K_{\mathbb{C}}}(\mathcal{O}). \tag{1.7}$$

For the rest of this subsection, we suppose that G has type $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and $\check{\mathcal{O}}$ has $\star$-good parity. Recall that a nilpotent orbit in $\check{\mathfrak{g}}$ is said to be distinguished if it is has no nontrivial intersection with any proper Levi subalgebra of $\check{\mathfrak{g}}$. Combinatorially, this is equivalent to saying that no pair of rows of the Young diagram have equal nonzero length. Note that all distinguished nilpotent orbits in $\check{\mathfrak{g}}$ has $\star$-good parity.

Definition 1.10. *(a) The orbit $\check{\mathcal{O}}$ (which has $\star$-good parity) is said to be quasi-distinguished if there is no $\star$-pair that is balanced in $\check{\mathcal{O}}$.*
(b) If $\star \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be weakly-distinguished if there is no positive even integer i such that $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = \mathbf{r}_{i+2}(\check{\mathcal{O}}) = \mathbf{r}_{i+3}(\check{\mathcal{O}}) > 0$.
(c) If $\star \in \{C, \widetilde{C}, C^*\}$ so that its Howe dual $\star' \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be weakly-distinguished if either it is the empty Young diagram or it is nonempty and its dual descent $\check{\mathcal{O}}'$ (which is a Young diagram that has $\star'$-good parity) is weakly-distinguished.

We will compute the associated cycle of π_τ for every painted bipartition τ associated to $\check{\mathcal{O}}$ which has $\star$-good parity. The computation will be a key ingredient in the proof of Theorem 1.6. Our second main theorem is the following result concerning associated cycles of the special unipotent representations.

Theorem 1.11. *Suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For any $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the associated cycle $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{K_\mathbb{C}}(\mathcal{O})$ is a nonzero sum of pairwise distinct elements of $\mathrm{AOD}_{K_\mathbb{C}}(\mathcal{O})$.
(b) If $\check{\mathcal{O}}$ is weakly-distinguished, then the map

$$\mathrm{AC}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathcal{K}_{K_\mathbb{C}}(\mathcal{O})$$

is injective.
(c) If $\check{\mathcal{O}}$ is quasi-distinguished, then the map $\mathrm{AC}_{\mathcal{O}}$ induces a bijection

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}_{K_\mathbb{C}}(\mathcal{O}).$$

Remark. Suppose that $\star \in \{C^*, D^*\}$ so that G is quaternionic. Then there is precisely one admissible orbit datum over $\mathscr{O}$ for each $K_\mathbb{C}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}^*$. Thus

$$\mathrm{AOD}_{K_\mathbb{C}}(\mathscr{O}) = K_\mathbb{C} \backslash (\mathcal{O} \cap \mathfrak{p}^*).$$

If $\check{\mathcal{O}}$ is not quasi-distinghuished, then $\mathcal{O} \cap \mathfrak{p}^*$ is empty (see [23, Theorems 9.3.4 and 9.3.5]), and hence $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is also empty.

2. Descents of painted bipartitions: combinatorics

In this section, we define the descent of a painted bipartition, as alluded to in Section 1.5. As before, let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.

For a diagram $\imath$, its naive descent, which is denoted by $\nabla_{\mathrm{naive}}(\imath)$, is defined to be the Young diagram obtained from $\imath$ by removing the first column. By convention, $\nabla_{\mathrm{naive}}(\emptyset) = \emptyset$.

In the rest of this section, we assume that $\check{\mathcal{O}} \neq \emptyset$. Recall that we have its dual descent $\check{\mathcal{O}}'$, and $\check{\mathcal{O}}'$ has $\star'$-good parity, where $\star'$ is the Howe dual of $\star$.

2.1. Naive descents of painted bipartitions.

In this subsection, let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ be a painted bipartition such that $\star_\tau = \star$. Put

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ does not occur in the leading column of } \tau; \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ occurs in the leading column of } \tau; \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{2.1}$$

Lemma 2.1. *If $\star \in \{\, B, C, C^* \,\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\imath, \nabla_{\mathrm{naive}}(\jmath))$;
- *for all* $(i,j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}(i,j), & \text{if } \mathcal{P}(i,j) \notin \{\bullet, s\}; \end{cases}$$

- *for all* $(i,j) \in \mathrm{Box}(\jmath')$,

$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j+1), & \text{if } \mathcal{Q}(i,j+1) \notin \{\bullet, s\}. \end{cases}$$

Proof. First assume that the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$. Then the image of $\mathcal{P}$ is in fact contained in $\{\bullet\}$, and $(\imath, \jmath)$ is right interlaced in the sense that

$$\mathbf{c}_1(\jmath) \geqslant \mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\jmath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\jmath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots .$$

Hence $(\imath', \jmath') := (\imath, \nabla(\jmath))$ is left interlaced in the sense that

$$\mathbf{c}_1(\imath') \geqslant \mathbf{c}_1(\jmath') \geqslant \mathbf{c}_2(\imath') \geqslant \mathbf{c}_2(\jmath') \geqslant \mathbf{c}_3(\imath') \geqslant \mathbf{c}_3(\jmath') \geqslant \cdots .$$

Then it is clear that there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ such that images of $\mathcal{P}'$ and $\mathcal{Q}'$ are both contained in $\{\bullet, s\}$. This proves the lemma in the special case when the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$.

The proof of the lemma in the general case is easily reduced to this special case. □

Lemma 2.2. *If* $\star \in \{\widetilde{C}, D, D^*\}$, *then there is a unique painted bipartition of the form* $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ *with the following properties:*

- $(\imath', \jmath') = (\nabla_{\mathrm{naive}}(\imath), \jmath)$;
- *for all* $(i,j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{P}(i,j+1), & \text{if } \mathcal{P}(i,j+1) \notin \{\bullet, s\}; \end{cases}$$

- *for all* $(i,j) \in \mathrm{Box}(\jmath')$,

$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j), & \text{if } \mathcal{Q}(i,j) \notin \{\bullet, s\}. \end{cases}$$

Proof. The proof is similar to that of Lemma 2.1. □

Definition 2.3. *In the notation of Lemma 2.1 and 2.2, we call* τ' *the naive descent of* τ, *to be denoted by* $\nabla_{\mathrm{naive}}(\tau)$.

Example. If

$$\tau = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & \bullet & c \\ \hline \bullet & s & c \\ \cline{1-3} s \\ \cline{1-1} c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & r & d \\ \hline d & d \\ \cline{1-2} \end{array} \times \widetilde{C},$$

then

$$\nabla_{\mathrm{naive}}(\tau) = \begin{array}{|c|c|c|} \hline \bullet & \bullet & c \\ \hline \bullet & c \\ \cline{1-2} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & \bullet & s \\ \hline \bullet & r & d \\ \hline d & d \\ \cline{1-2} \end{array} \times B^-.$$

2.2. **Descents of painted bipartitions.** Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ and write

$$\tau'_{\mathrm{naive}} = (\imath', \mathcal{P}'_{\mathrm{naive}}) \times (\jmath', \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'$$

for the naive descent of τ. This is clearly an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$.

The following two lemmas are easily verified and we omit the proofs. We will give an example for each case.

Lemma 2.4. *Suppose that*

$$\begin{cases} \alpha = B^+; \\ \mathbf{r}_2(\check{\mathcal{O}}) > 0; \\ \mathcal{Q}(\mathbf{c}_1(\jmath), 1) \in \{ r, d \}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} s, & \text{if } (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & c \\ \hline c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|}\hline \bullet & r \\ \hline r & d \\ \hline \end{array} \times B^+,$$

then

$$\tau'_{\mathrm{naive}} = \begin{array}{|c|c|}\hline s & c \\ \hline c \\ \cline{1-1} \end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C} \qquad \text{and} \qquad \tau' = \begin{array}{|c|c|}\hline s & c \\ \hline s \\ \cline{1-1} \end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C}.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $4, 4, 4, 2$.

Lemma 2.5. *Suppose that*

$$\begin{cases} \alpha = D; \\ \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0; \\ \big(\mathcal{P}(\mathbf{c}_2(\imath), 1), \mathcal{P}(\mathbf{c}_2(\imath), 2)\big) = (r, c); \\ \mathcal{P}(\mathbf{c}_1(\imath), 1) \in \{ r, d \}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} r, & \text{if } \ (i,j) = (\mathbf{c}_1(\imath'), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & s \\ \hline \bullet & s \\ \hline r & c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet \\ \cline{1-1} \bullet \\ \cline{1-1} \end{array} \times D,$$

then

$$\tau'_{\text{naive}} = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times C, \qquad \text{and} \qquad \tau' = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline r \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times C.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $7, 7, 7, 3$.

Definition 2.6. *We define the descent of τ to be*

$$\nabla(\tau) := \begin{cases} \tau', & \textit{if either of the condition of Lemma 2.4 or 2.5 holds;} \\ \nabla_{\text{naive}}(\tau), & \textit{otherwise,} \end{cases}$$

which is an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$. *Here τ' is as in Lemmas 2.4 and 2.5.*

In conclusion, we have by now a well-defined descent map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

The following injectivity result will be important for us.

Proposition 2.7. *If $\star \in \{ B, D, C^* \}$, then the map*

$$\begin{array}{rcl} \mathrm{PBP}_{\star}(\check{\mathcal{O}}) & \to & \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array} \tag{2.2}$$

is injective. If $\star \in \{ C, \widetilde{C}, D^* \}$, then the map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \tag{2.3}$$

is injective.

3. Associated cycles of painted bipartitions: geometry

3.1. **Classical spaces.** Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ as before. Put

$$(\epsilon, \dot{\epsilon}) := (\epsilon_\star, \dot{\epsilon}_\star) := \begin{cases} (1, 1), & \text{if } \star \in \{B, D\}; \\ (-1, -1), & \text{if } \star \in \{C, \widetilde{C}\}; \\ (-1, 1), & \text{if } \star = C^*; \\ (1, -1), & \text{if } \star = D^*. \end{cases}$$

A classical signature is defined to be a triple $\mathsf{s} = (\star, p, q) \in \{B, C, D, \widetilde{C}, C^*, D^*\} \times \mathbb{N} \times \mathbb{N}$ such that

$$\begin{cases} p + q \text{ is odd }, & \text{if } \star = B; \\ p + q \text{ is even }, & \text{if } \star = D; \\ p = q, & \text{if } \star \in \{C, \widetilde{C}, D^*\}; \\ \text{both } p \text{ and } q \text{ are even}, & \text{if } \star = C^*. \end{cases}$$

Suppose that $\mathsf{s} = (\star, p, q)$ is a classical signature in the rest of this section.

We omit the proof of the following lemma (*cf.* [65, Section 1.3]).

Lemma 3.1. *Then there is quadruple $(V, \langle\, , \,\rangle, J, L)$ satisfying the following conditions:*

- *V is a complex vector space of dimension $p + q$;*
- *$\langle\, , \,\rangle$ is an ϵ-symmetric non-degenerate bilinear form on V;*
- *$J : V \to V$ is a conjugate linear automorphism of V such that $J^2 = \epsilon \cdot \dot{\epsilon}$;*
- *$L : V \to V$ is a linear automorphism of V such that $L^2 = \dot{\epsilon}$;*

- *$\langle Ju, Jv\rangle = \overline{\langle u, v\rangle}$, for all $u, v \in V$;*
- *$\langle Lu, Lv\rangle = \langle u, v\rangle$, for all $u, v \in V$;*
- *$LJ = JL$;*
- *the Hermitian form $(u, v) \mapsto \langle Lu, Jv\rangle$ on V is positive definite;*
- *if $\dot\epsilon = 1$, then $\dim\{v \in V \mid Lv = v\} = p$ and $\dim\{v \in V \mid Lv = -v\} = q$.*

Moreover, such a quadruple is unique in the following sense: if $(V', \langle\,,\,\rangle', J', L')$ is another quadruple satisfying the analogous conditions, then there is a linear isomorphism $\phi : V \to V'$ that respectively transforms $\langle\,,\,\rangle$, J and L to $\langle\,,\,\rangle'$, J' and L'.

In the notation of Lemma 3.1, we call $(V, \langle\,,\,\rangle, J, L)$ a classical space of signature s, and denote it by $(V_{\mathsf{s}}, \langle\,,\,\rangle_{\mathsf{s}}, J_{\mathsf{s}}, L_{\mathsf{s}})$.

Denote by $G_{\mathsf{s},\mathbb{C}}$ the isometry group of $(V_{\mathsf{s}}, \langle\,,\,\rangle_{\mathsf{s}})$, which is an complex orthogonal group if $\epsilon = 1$ and a complex symplectic group if $\epsilon = -1$. Respectively denote by $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ and $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ the centralizes of J_{s} and L_{s} in $G_{\mathsf{s},\mathbb{C}}$. Then $G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}$ is a real classical group isomorphic with

$$\begin{cases} \mathrm{O}(p,q), & \text{if } \star = B \text{ or } D; \\ \mathrm{Sp}_{2p}(\mathbb{R}), & \text{if } \star \in \{C, \widetilde{C}\}; \\ \mathrm{Sp}(\frac{p}{2}, \frac{q}{2}), & \text{if } \star = C^*; \\ \mathrm{O}^*(2p), & \text{if } \star_\tau = D^*. \end{cases}$$

Put

$$G_{\mathsf{s}} := \begin{cases} \text{the metaplectic double cover of } G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{if } \star = \widetilde{C}; \\ G_{\mathsf{s},\mathbb{C}}^{J_{\mathsf{s}}}, & \text{otherwise};. \end{cases}$$

Denote by K_{s} the inverse image of $G_{\mathsf{s},\mathbb{C}}^{L_{\mathsf{s}}}$ under the natural homomorphism $G_{\mathsf{s}} \to G_{\mathsf{s},\mathbb{C}}$, which is a maximal compact subgroup of G_{s}. Write $K_{\mathsf{s},\mathbb{C}}$ for the complexification of K_{s}, which is a reductive complex linear algebraic group.

For every $\lambda \in \mathbb{C}$, write $V_{\mathsf{s},\lambda}$ for the eigenspace of L_{s} with eigenvalue λ. Write

$$\det\nolimits_\lambda : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$(3.1) \qquad K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s},\lambda}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times.$$

If $\star = \widetilde{C}$, then the natural homomorphism $K_{\mathsf{s},\mathbb{C}} \to \mathrm{GL}(V_{\mathsf{s},\sqrt{-1}})$ is a double cover, and there is a unique genuine algebraic character $\det_{\sqrt{-1}}^{\frac{1}{2}} : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$ whose square equals $\det_{\sqrt{-1}}$.

Write

$$\det : K_{\mathsf{s},\mathbb{C}} \to \mathbb{C}^\times$$

for the composition of

$$(3.2) \qquad K_{\mathsf{s},\mathbb{C}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(V_{\mathsf{s}}) \xrightarrow{\text{the determinant character}} \mathbb{C}^\times.$$

This is the trivial character unless $\star = B$ or D.

Denote by $\mathfrak{g}_{\mathsf{s}}$ the Lie algebra of $G_{\mathsf{s},\mathbb{C}}$. Then we have a decomposition

$$\mathfrak{g}_{\mathsf{s}} = \mathfrak{k}_{\mathsf{s}} \oplus \mathfrak{p}_{\mathsf{s}},$$

where $\mathfrak{k}_{\mathsf{s}}$ is the Lie algebra of $K_{\mathsf{s},\mathbb{C}}$, and $\mathfrak{p}_{\mathsf{s}}$ is the orthogonal complement of $\mathfrak{k}_{\mathsf{s}}$ in $\mathfrak{g}_{\mathsf{s}}$ under the trace form. Using the trace form, we also identify $\mathfrak{g}_{\mathsf{s}}^*$ with $\mathfrak{g}_{\mathsf{s}}$. Denote by $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ the set of nilpotent $G_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{g}_{\mathsf{s}}$, and by $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$ the set of nilpotent $K_{\mathsf{s},\mathbb{C}}$-orbits in $\mathfrak{p}_{\mathsf{s}}$.

Given a $K_{\mathsf{s},\mathbb{C}}$-orbit $\mathscr{O}$ in $\mathfrak{p}_{\mathsf{s}}$, write $\mathcal{K}_{\mathsf{s}}(\mathscr{O})$ for the Grothendieck group of the categogy of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundles over $\mathscr{O}$. Denote by $\mathcal{K}_{\mathsf{s}}^+(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}(\mathscr{O})$ the submonoid generated by all these equivariant algebraic vector bundles. The notion of admissible orbit datum over $\mathscr{O}$ is defined as in Definition 1.9. Write

$$\mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}^{+}(\mathscr{O})$$

for the set of isomorphism classes of admissible orbit data over $\mathscr{O}$.

Let $\mathcal{O}$ be a $G_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s},\mathbb{C}}$. It is well-known that $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ has only finitely many $K_{\mathsf{s},\mathbb{C}}$-orbits. Put

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}_{\mathsf{s}}(\mathscr{O}),$$

and

$$\mathcal{K}_{\mathsf{s}}^{+}(\mathcal{O}) := \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathcal{K}_{\mathsf{s}}^{+}(\mathscr{O}).$$

Put

$$\mathrm{AOD}_{\mathsf{s}}(\mathcal{O}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}} \mathrm{AOD}_{\mathsf{s}}(\mathscr{O}) \subset \mathcal{K}_{\mathsf{s}}^{+}(\mathcal{O}).$$

Define a partial order $\preceq$ on $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$ such that

$$\mathcal{E}_1 \preceq \mathcal{E}_2 \Leftrightarrow \mathcal{E}_2 - \mathcal{E}_1 \in \mathcal{K}_{\mathsf{s}}^{+}(\mathcal{O}) \qquad (\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})).$$

For every algebraic character χ of $K_{\mathsf{s},\mathbb{C}}$, the twisting map

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \qquad \mathcal{E} \mapsto \mathcal{E} \otimes \chi$$

is obviously defined.

3.2. **The moment maps.** Recall that $\star'$ is the Howe dual of $\star$. Suppose that $\mathsf{s}' = (\star', p', q')$ is another classical signature. Put

$$W_{\mathsf{s},\mathsf{s}'} := \mathrm{Hom}_{\mathbb{C}}(V_{\mathsf{s}}, V_{\mathsf{s}'}).$$

Then we have the adjoint map

$$W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s}',\mathsf{s}}, \qquad \phi \mapsto \phi^*$$

that is specified by requiring

$$\langle \phi v, v' \rangle_{\mathsf{s}'} = \langle v, \phi^* v' \rangle_{\mathsf{s}}, \qquad \text{for all } v \in V_{\mathsf{s}},\ v' \in V_{\mathsf{s}'},\ \phi \in W_{\mathsf{s},\mathsf{s}'}.$$

Define three maps

$$\langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \times W_{\mathsf{s},\mathsf{s}'} \to \mathbb{C}, \quad (\phi_1, \phi_2) \mapsto \mathrm{tr}(\phi_1^* \phi_2),$$
$$J_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto J_{\mathsf{s}'} \circ \phi \circ J_{\mathsf{s}}^{-1};$$

and

$$L_{\mathsf{s},\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to W_{\mathsf{s},\mathsf{s}'}, \quad \phi \mapsto \dot{\epsilon} L_{\mathsf{s}'} \circ \phi \circ L_{\mathsf{s}}^{-1}.$$

It is routine to check that

$$(W_{\mathsf{s},\mathsf{s}'}, \langle\, ,\, \rangle_{\mathsf{s},\mathsf{s}'}, J_{\mathsf{s},\mathsf{s}'}, L_{\mathsf{s},\mathsf{s}'})$$

is a classical space of signature $(C, \frac{(p+q)(p'+q')}{2}, \frac{(p+q)(p'+q')}{2})$.

Write

$$W_{\mathsf{s},\mathsf{s}'} = \mathcal{X}_{\mathsf{s},\mathsf{s}'} \oplus \mathcal{Y}_{\mathsf{s},\mathsf{s}'},$$

where $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ and $\mathcal{Y}_{\mathsf{s},\mathsf{s}'}$ are the eigenspaces of $L_{\mathsf{s},\mathsf{s}'}$ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Then we have the following two well-defined algebraic maps:

$$\begin{array}{ccccc} \mathfrak{p}_{\mathsf{s}} & \xleftarrow{M_{\mathsf{s}}} & \mathcal{X}_{\mathsf{s},\mathsf{s}'} & \xrightarrow{M_{\mathsf{s}'}} & \mathfrak{p}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array} \tag{3.3}$$

These two maps M_{s} and $M_{\mathsf{s}'}$ are called the moment maps. They are both $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-equivariant. Here $K_{\mathsf{s}',\mathbb{C}}$ acts trivially on $\mathfrak{p}_{\mathsf{s}}$, $K_{\mathsf{s},\mathbb{C}}$ acts trivially on $\mathfrak{p}'_{\mathsf{s}}$, and all the other actions are the obvious ones.

Put

$$W^{\circ}_{\mathsf{s},\mathsf{s}'} := \{\phi \in W_{\mathsf{s},\mathsf{s}'} \mid \text{the image of } \phi \text{ is non-degenerate with respect to } \langle\,,\,\rangle_{\mathsf{s}'}\}$$

and

$$\mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} := \mathcal{X}_{\mathsf{s},\mathsf{s}'} \cap W^{\circ}_{\mathsf{s},\mathsf{s}'}.$$

Lemma 3.2 (*cf.* [65, Lemma 13])**.** *Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}. Then the set*

$$M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'} \tag{3.4}$$

is a single $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$-orbit. Moreover, for any element ϕ in $M_{\mathsf{s}}^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'}$, there is an exact sequence of algebraic groups:

$$1 \to K_{\mathsf{s}_0,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \xrightarrow{\textit{the projection to the first factor}} (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1,$$

where $\mathbf{e} := M_{\mathsf{s}}(\phi) \in \mathscr{O}$, s_0 is a certain classical signature of form $(\star', p_0, q_0)$ ($p_0, q_0 \in \mathbb{N}$), $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$ is the stabilizer of ϕ in $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$, and $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$ is the stabilizer of $\mathbf{e}$ in $K_{\mathsf{s},\mathbb{C}}$.

In the notation of Lemma 3.2, write

$$\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) := \text{the image of the set (3.4) under the moment map } M_{\mathsf{s}'},$$

which is a $K_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}'}$. This is called the descent of $\mathscr{O}$. It is an element of $\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}'})$ if $\mathscr{O} \in \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$.

3.3. **Geometric theta lift.** Let $\zeta_{\mathsf{s},\mathsf{s}'}$ denote the algebraic character on $K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}}$ such that

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s},\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{B, D, C^*\}; \\ (\det_{\sqrt{-1}})^{\frac{q'-p'}{2}}, & \text{if } \star \in \{C, D^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{q'-p'}, & \text{if } \star = \widetilde{C}, \end{cases}$$

and

$$(\zeta_{\mathsf{s},\mathsf{s}'})|_{K_{\mathsf{s}',\mathbb{C}}} = \begin{cases} 1, & \text{if } \star \in \{C, \widetilde{C}, D^*\}; \\ (\det_{\sqrt{-1}})^{\frac{p-q}{2}}, & \text{if } \star \in \{D, C^*\}; \\ (\det^{\frac{1}{2}}_{\sqrt{-1}})^{p-q}, & \text{if } \star = B. \end{cases}$$

Here and henceforth, when no confusion is possible, we use 1 to indicate the trivial representation of a group (we also use 1 to denote the identity element of a group). Let $\mathscr{O}$ be a $K_{\mathsf{s},\mathbb{C}}$-orbit in $\mathfrak{p}_{\mathsf{s}}$ as before. Suppose that $\mathscr{O}$ is contained in the image of the moment map M_{s}, and write $\mathscr{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O})$. Let $\phi, \mathbf{e}$ be as in Lemma 3.2 and let $\mathbf{e}' := M_{\mathsf{s}'}(\phi)$. We have an exact sequence

$$1 \to K_{\mathsf{s}_0,\mathbb{C}} \to (K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \to (K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}} \to 1$$

as in Lemma 3.2.

Let $\mathcal{E}'$ be a $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}'$. Its fibre $\mathcal{E}'_{\mathbf{e}}$ at $\mathbf{e}'$ is an algebraic representation of the stabilizer group $(K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}$. We also view it as a representation of the group $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi}$ by the pull-back through the homomorphism

$$(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_{\phi} \xrightarrow{\text{the projection to the second factor}} (K_{\mathsf{s}',\mathbb{C}})_{\mathbf{e}'}.$$

Then $\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'}$ is a representation of $(K_{\mathsf{s},\mathbb{C}} \times K_{\mathsf{s}',\mathbb{C}})_\phi$, and the coinvariant space

$$(\mathcal{E}'_{\mathbf{e}'} \otimes \zeta_{\mathsf{s},\mathsf{s}'})_{K_{\mathsf{s}_0,\mathbb{C}}}$$

is an algebraic representation of $(K_{\mathsf{s},\mathbb{C}})_{\mathbf{e}}$. Write $\mathcal{E} := \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}')$ for the $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$ whose fibre at $\mathbf{e}$ equals this coinvariant space representation. In this way, we get an exact functor $\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}$ from the category of $K_{\mathsf{s}',\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}'$ to the category of $K_{\mathsf{s},\mathbb{C}}$-equivariant algebraic vector bundle over $\mathscr{O}$. This exact functor induces a homomorphism of the Grothendieck groups:

$$\check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'} : \mathcal{K}_{\mathsf{s}'}(\mathscr{O}') \to \mathcal{K}_{\mathsf{s}}(\mathscr{O}).$$

The above homomorphism is independent of the choice of ϕ.

Similar to (3.3), we have the following two well-defined algebraic maps:

$$\begin{array}{ccccc} \mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array} \tag{3.5}$$

These two maps are also called the moment maps. Similar to the maps in (3.3), they are both $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-equivariant.

Now we suppose that $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Similar to the first assertion of Lemma 3.2, the set

$$\tilde{M}_{\mathsf{s}}^{-1}(\mathcal{O}) \cap W^{\circ}_{\mathsf{s},\mathsf{s}'} \tag{3.6}$$

is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit. Write

$$\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) := \text{the image of the set (3.6) under the moment map } \tilde{M}_{\mathsf{s}'},$$

which is a $G_{\mathsf{s}',\mathbb{C}}$-orbit in $\mathfrak{g}_{\mathsf{s}'}$. This is called the descent of $\mathcal{O}$. It is an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ if $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$.

Finally, we define the geometric theta lift to be the homomorphism

$$\check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'} : \mathcal{K}_{\mathsf{s}'}(\mathcal{O}') \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

such that

$$\check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathcal{E}') = \sum_{\mathscr{O} \text{ is a } K_{\mathsf{s},\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}_{\mathsf{s}},\ \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathscr{O}) = \mathscr{O}'} \check{\vartheta}^{\mathscr{O}}_{\mathscr{O}'}(\mathcal{E}'),$$

for all $K_{\mathsf{s}',\mathbb{C}}$-orbit $\mathscr{O}'$ in $\mathcal{O}' \cap \mathfrak{p}_{\mathsf{s}'}$, and all $\mathcal{E}' \in \mathcal{K}_{\mathsf{s}'}(\mathscr{O}')$.

3.4. **Associated cycles of painted bipartitions.** As before, let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity. Suppose that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is its Barbasch-Vogan dual, where $\mathsf{s} = (\star, p, q)$ is a classical signature. Put

$$\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) := \{(\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}) \mid (p_\tau, q_\tau) = (p, q)\}.$$

Let $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$. In what follows we will define the associated cycle $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ of $\boldsymbol{\tau}$.

First assume that $\check{\mathcal{O}} = \varnothing$. Then $\mathcal{O} \cap \mathfrak{p}_{\mathsf{s}}$ is a singleton. If $\star = \widetilde{C}$, we define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ to be the element that corresponds to the one dimensional genuine representation of $K_{\mathsf{s},\mathbb{C}} = \{\pm 1\}$. In all other cases, we define $\mathrm{AC}(\boldsymbol{\tau}) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ to be the element that corresponds to the one dimensional trivial representation of $K_{\mathsf{s},\mathbb{C}}$.

Now we assume that $\check{\mathcal{O}} \neq \varnothing$. As before, let $\check{\mathcal{O}}'$ be its dual descent. Write $\boldsymbol{\tau}' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$ for the descent of $\boldsymbol{\tau}$, and assume that $\boldsymbol{\tau}' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}', \mathsf{s}')$.

Lemma 3.3. *The orbit $\mathcal{O}$ is in the image of the moment map $\tilde{M}_{\mathsf{s}}$, and*

$$\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) = \textit{the Barbasch-Vogan dual of } \check{\mathcal{O}}'.$$

By Lemma 3.3, $\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ equals the Barbasch-Vogan dual of $\check{\mathcal{O}}'$.

Similar to the definition of π_τ in the introductory section, we inductively define $\mathrm{AC}(\tau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ by

$$\mathrm{AC}(\tau) := \begin{cases} \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\tau')) \otimes (\det{}_{-1})^{\varepsilon_\tau}, & \text{if } \star = B \text{ or } D; \\ \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\tau') \otimes \det{}^{\varepsilon_\wp}), & \text{if } \star = C \text{ or } \widetilde{C}; \\ \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}(\tau')), & \text{if } \star = C^* \text{ or } D^*. \end{cases}$$

We have the following three analogous results of Theorem 1.11.

Proposition 3.4. *For every $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, the associated cycle $\mathrm{AC}(\tau) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ is a nonzero sum of pairwise distinct elements of $\mathrm{AOD}_{\mathcal{O}}(K_{\mathsf{s},\mathbb{C}})$.*

Proposition 3.5. *Suppose that $\check{\mathcal{O}}$ is weakly-distinguished. If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}), \quad (\tau, \epsilon) \mapsto \mathrm{AC}(\tau) \otimes \det{}^\epsilon$$

is injective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

is injective.

Proposition 3.6. *Suppose that $\check{\mathcal{O}}$ is quasi-distinguished. If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$, then the map*

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{AOD}_{K_{\mathbb{C}}}(\mathcal{O}), \quad (\tau, \epsilon) \mapsto \mathrm{AC}(\tau) \otimes \det{}^\epsilon$$

is well-defined and bijective. In all other cases, the map

$$\mathrm{AC} : \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O})$$

is well-defined and bijective.

The following two propositions will also be important for us.

Proposition 3.7. *Suppose that $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \varnothing)$. Let $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$ and $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ ($i = 1, 2$). If*

$$\mathrm{AC}(\tau_1) \otimes \det{}^{\epsilon_1} = \mathrm{AC}(\tau_2) \otimes \det{}^{\epsilon_2},$$

then

$$\epsilon_1 = \epsilon_2 \qquad \textit{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Proposition 3.8. *Suppose that $\star \in \{C, \widetilde{C}, D^*\}$ and $\check{\mathcal{O}} \neq \varnothing$. Let $\tau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$, and write $\tau'_i = (\tau'_i, \wp'_i)$ for its descent ($i = 1, 2$). If*

$$\mathrm{AC}(\tau_1) = \mathrm{AC}(\tau_2),$$

then

$$(p_{\tau'_1}, q_{\tau'_1}) = (p_{\tau'_2}, q_{\tau'_2}) \qquad \textit{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

3.5. **Distinguishing the constructed representations.** Let $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}, \mathsf{s})$. Recall that π_τ is the Casselman-Wallach representation of G_{s}, defined in the introductory section. We will prove the following theorem in Section 7.

Theorem 3.9. *The representation π_τ is irreducible, unitarizable and attached to $\check{\mathcal{O}}$. Moreover,*

$$\mathrm{AC}_{\mathcal{O}}(\pi_\tau) = \mathrm{AC}(\tau)$$

as elements of $\mathcal{K}_{\mathsf{s}}(\mathcal{O})$.

Recall from the introductory section, $\mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}})$, which is the set of isomorphism classes of irreducible Casselman-Wallach representations of G_{s} that are attached to $\check{\mathcal{O}}$.

Theorem 3.10. *If $\star \in \{B, D\}$ and $(\star, \check{\mathcal{O}}) \neq (D, \emptyset)$, then the map*

$$\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \times \mathbb{Z}/2\mathbb{Z} \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}}), \quad (\boldsymbol{\tau}, \epsilon) \mapsto \pi_{\boldsymbol{\tau}} \otimes \det{}^{\epsilon} \tag{3.7}$$

is bijective. In all other cases, the map

$$\mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s}) \to \mathrm{Unip}_{\check{\mathcal{O}}}(G_{\mathsf{s}}), \quad \boldsymbol{\tau} \mapsto \pi_{\tau} \tag{3.8}$$

is bijective.

Proof. In view of Theorem 1.5, we only need to show the injectivity of the two maps in the statement of the theorem. We prove by induction on the number of nonempty rows of $\check{\mathcal{O}}$. The theorem is trivially true in the case when $\check{\mathcal{O}} = \emptyset$. So we assume that $\check{\mathcal{O}} \neq \emptyset$ and the theorem has been proved for the dual descent $\check{\mathcal{O}}'$. Suppose that $\boldsymbol{\tau}_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}}, \mathsf{s})$, $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$. Write $\boldsymbol{\tau}'_i = (\tau'_i, \wp'_i)$ for the descent of $\boldsymbol{\tau}_i$.

First assume that $\star \in \{B, D\}$, and

$$\pi_{\boldsymbol{\tau}_1} \otimes \det{}^{\epsilon_1} \cong \pi_{\boldsymbol{\tau}_2} \otimes \det{}^{\epsilon_2}.$$

Theorem 3.9 implies that

$$\mathrm{AC}(\boldsymbol{\tau}_1) \otimes \det{}^{\epsilon_1} = \mathrm{AC}(\boldsymbol{\tau}_2) \otimes \det{}^{\epsilon_2}.$$

By Proposition 3.7, we know that

$$\epsilon_1 = \epsilon_2 \qquad \text{and} \qquad \varepsilon_{\tau_1} = \varepsilon_{\tau_2}.$$

Then the definition of $\pi_{\boldsymbol{\tau}_1}$ and $\pi_{\boldsymbol{\tau}_2}$ implies that

$$\check{\Theta}^{\tau_1}_{\tau'_1}(\pi_{\boldsymbol{\tau}'_1}) \cong \check{\Theta}^{\tau_2}_{\tau'_2}(\pi_{\boldsymbol{\tau}'_2}).$$

Consequently,

$$\pi_{\boldsymbol{\tau}'_1} \cong \pi_{\boldsymbol{\tau}'_2}$$

by the injectivity property of the theta correspondence. Hence $\boldsymbol{\tau}'_1 = \boldsymbol{\tau}'_2$ by the induction hypothesis, and Proposition 2.7 finally implies $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. This proves that the map (3.7) is injective.

A slightly simplified argument shows that the map (3.8) is injective when $\star = C^*$.

Now assume that $\star \in \{C, \widetilde{C}, D^*\}$. Suppose that

$$\pi_{\boldsymbol{\tau}_1} \cong \pi_{\boldsymbol{\tau}_2}.$$

Theorem 3.9 implies that

$$\mathrm{AC}(\boldsymbol{\tau}_1) = \mathrm{AC}(\boldsymbol{\tau}_2).$$

By Proposition 3.8, we know that

$$(p_{\tau'_1}, q_{\tau'_1}) = (p_{\tau'_2}, q_{\tau'_2}) \qquad \text{and} \qquad \varepsilon_{\wp_1} = \varepsilon_{\wp_2}.$$

Then the definition of $\pi_{\boldsymbol{\tau}_1}$ and $\pi_{\boldsymbol{\tau}_2}$ implies that

$$\check{\Theta}^{\tau_1}_{\tau'_1}(\pi_{\boldsymbol{\tau}'_1} \otimes \det{}^{\varepsilon_{\wp_1}}) \cong \check{\Theta}^{\tau_2}_{\tau'_2}(\pi_{\boldsymbol{\tau}'_2} \otimes \det{}^{\varepsilon_{\wp_1}}).$$

The injectivity property of the theta correspondence then implies that

$$\pi_{\boldsymbol{\tau}'_1} \otimes \det{}^{\varepsilon_{\wp_1}} \cong \pi_{\boldsymbol{\tau}'_2} \otimes \det{}^{\varepsilon_{\wp_2}},$$

which further implies that $\pi_{\boldsymbol{\tau}'_1} \cong \pi_{\boldsymbol{\tau}'_2}$. Hence $\boldsymbol{\tau}'_1 = \boldsymbol{\tau}'_2$ by the induction hypothesis, and Proposition 2.7 finally implies $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$. This proves that the map (3.8) is injective.

A slightly simplified argument shows that the map (3.8) is injective when $\star = D^*$. This finishes the proof of the theorem.

□

Finally, our first main theorem (Theorem 1.6) follows from Theorems 3.9 and 3.10. Our second main theorem (Theorem 1.11) follows from Propositions 3.4 and 3.5, and Theorems 3.9 and 3.10.

4. Theta lifts via matrix coefficient integrals

In this section, we study theta lift via matrix coefficient integrals. Let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$.

4.1. The oscillator representation.

We use the notation of Section 3.2. Write $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \subset W_{\mathsf{s},\mathsf{s}'}$ for the fixed point set of $J_{\mathsf{s},\mathsf{s}'}$. It is a real symplectic space under the restriction of the form $\langle\,,\,\rangle_{\mathsf{s},\mathsf{s}'}$. Let $H_{\mathsf{s},\mathsf{s}'} := W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}} \times \mathbb{R}$ denote the Heisenberg group attached to $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$, with group multiplication

$$(u,t)\cdot(u',t') := (u+u', t+t'+\langle u,u'\rangle_{\mathsf{s},\mathsf{s}'}), \qquad u,u' \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}, \quad t,t' \in \mathbb{R}.$$

Denote by $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$ the complexified Lie algebra of $H_{\mathsf{s},\mathsf{s}'}$. Then $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$ is an abelian Lie subalgebra of $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$.

Lemma 4.1. *Up to isomorphism, there exists a unique irreducible smooth Fréchet representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ of moderate growth with central charater*

$$\mathbb{R} \to \mathbb{C}^\times, \; t \mapsto e^{\sqrt{-1}\,t}.$$

Moreover, the space

$$\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}} := \{v \in \omega_{\mathsf{s},\mathsf{s}'} \mid x.v = 0 \quad \text{for all } x \in \mathcal{X}_{\mathsf{s},\mathsf{s}'}\}$$

is one-dimensional.

The group $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ acts on $H_{\mathsf{s},\mathsf{s}'}$ as group automorphisms through the following action of $G_{\mathsf{s}} \times G_{\mathsf{s}'}$ on $W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}$:

$$(g,g').\phi := g' \circ \phi \circ g^{-1}, \qquad (g,g') \in G_{\mathsf{s}} \times G_{\mathsf{s}'}, \; \phi \in W_{\mathsf{s},\mathsf{s}'}^{J_{\mathsf{s},\mathsf{s}'}}.$$

Using this action, we form the semidirect product $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Lemma 4.2. *The representation $\omega_{\mathsf{s},\mathsf{s}'}$ of $H_{\mathsf{s},\mathsf{s}'}$ in Lemma 4.1 uniquely extends to a smooth representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ such that $K_{\mathsf{s}} \times K_{\mathsf{s}'}$ acts on $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$ through the scalar multiplication by $\zeta_{\mathsf{s},\mathsf{s}'}$.*

Let $\omega_{\mathsf{s},\mathsf{s}'}$ denote the representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ as in Lemma 4.2, which is called the smooth oscillator representation. This representation is unitarizable. Fix an invariant continuous Hermitian inner product $\langle\,,\,\rangle$ on it, which is unique up to a positive scalar multiplication. Denote by $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ the completion of $\omega_{\mathsf{s},\mathsf{s}'}$ with respect to this inner product, which is a unitary representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$.

Let $\omega_{\mathsf{s},\mathsf{s}'}^\vee$ denote the contragredient of the oscillator representation $\omega_{\mathsf{s},\mathsf{s}'}$. It is the smooth Fréchet representation of $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$ of moderate growth specified by the following conditions:

- there is given a $(G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'}$-invariant, non-degenerate, continuous bilinear form
$$\langle\,,\,\rangle : \omega_{\mathsf{s},\mathsf{s}'} \times \omega_{\mathsf{s},\mathsf{s}'}^\vee \to \mathbb{C};$$
- $\omega_{\mathsf{s},\mathsf{s}'}^\vee$ is irreducible as a representation of $H_{\mathsf{s},\mathsf{s}'}$.

Since $\omega_{\mathsf{s},\mathsf{s}'}$ is contained in the unitary representation $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$, $\omega^\vee_{\mathsf{s},\mathsf{s}'}$ is identified with the complex conjugation of $\omega_{\mathsf{s},\mathsf{s}'}$.

Put

$$\mathsf{s}'^- := (\star', q', p'),$$

Fix a linear isomorphism

$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^-}, \tag{4.1}$$

such that $(-\langle\,,\,\rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, -L_{\mathsf{s}'})$ corresponds to $(\langle\,,\,\rangle_{\mathsf{s}'^-}, J_{\mathsf{s}'^-}, L_{\mathsf{s}'^-})$ under this isomorphism. This induces an isomorphism

$$G_{\mathsf{s}'} \to G_{\mathsf{s}'^-}, \qquad g' \mapsto g'^-. \tag{4.2}$$

Then we have a group isomorphism

$$\begin{array}{rcl} (G_{\mathsf{s}} \times G_{\mathsf{s}'}) \ltimes H_{\mathsf{s},\mathsf{s}'} & \to & (G_{\mathsf{s}'^-} \times G_{\mathsf{s}}) \ltimes H_{\mathsf{s}'^-,\mathsf{s}}, \\ (g, g', (\phi, t)) & \mapsto & (g'^-, g, (\phi^* \circ \iota_{\mathsf{s}'}^{-1}, t)). \end{array} \tag{4.3}$$

It is easy to see that the irreducible representation $\omega_{\mathsf{s},\mathsf{s}'}$ corresponds to the irreducible representation $\omega_{\mathsf{s}'^-,\mathsf{s}}$ under this isomorphism.

For every Casselman-Wallach representation π' of $G_{\mathsf{s}'}$, put

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi') := (\omega_{\mathsf{s},\mathsf{s}'} \widehat{\otimes} \pi')_{G_{\mathsf{s}'}} \qquad \text{(the Hausdorff coinvariant space).}$$

This is a Casselman-Wallach representation of G_{s}.

4.2. **Growth of Casselman-Wallach representations.** Write $\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}}$ for the centralizer of J_{s} in $\mathfrak{p}_{\mathsf{s}}$, which is a real form of $\mathfrak{p}_{\mathsf{s}}$. The Cartan decomposition asserts that

$$G_{\mathsf{s}} = K_{\mathsf{s}} \cdot \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}}).$$

Denote by Ψ_{s} the function of G_{s} satisfying the following conditions:

- it is both left and right K_{s}-invariant;
- for all $g \in \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}})$,
$$\Psi_{\mathsf{s}} = \prod_a \left(\frac{1+a}{2}\right)^{-\frac{1}{2}},$$
where a runs over all eigenvalues of $g : V_{\mathsf{s}} \to V_{\mathsf{s}}$, counted with multiplicities.

Note that all the eigenvalues of $g \in \exp(\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}})$ are positive real numbers, and $0 < \Psi_{\mathsf{s}}(g) \leqslant 1$ for all $g \in G_{\mathsf{s}}$.

Denote by Ξ_{s} the bi-K_{s}-invariant Harish-Chandra's Ξ function on G_{s}. Set

$$|\mathsf{s}| := p + q,$$

and

$$\nu_{\mathsf{s}} := \begin{cases} |\mathsf{s}|, & \text{if } \star \in \{C, \widetilde{C}\}; \\ |\mathsf{s}| - 1, & \text{if } \star = C^*; \\ |\mathsf{s}| - 2, & \text{if } \star \in \{B, D\}; \\ |\mathsf{s}| - 3, & \text{if } \star = D^*. \end{cases}$$

Lemma 4.3. *There exists a real number $C_{\mathsf{s}} > 0$ such that*

$$\Psi_{\mathsf{s}}^{\nu_{\mathsf{s}}}(g) \leqslant C_{\mathsf{s}} \cdot \Xi_{\mathsf{s}}(g) \quad \text{for all } g \in G_{\mathsf{s}}.$$

Proof. This is implied by the well-known estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). ☐

Lemma 4.4. *Let f be a bi-K_{s}-invariant positive function on G_{s} such that*

$$\mathfrak{p}_{\mathsf{s}}^{J_{\mathsf{s}}} \to \mathbb{R}, \qquad x \mapsto f(\exp(x))$$

is a polynomial function. Then for every real number $\nu > 0$, the function $f \cdot \Psi_{\mathsf{s}}^{\nu} \cdot \Xi_{\mathsf{s}}^{2}$ is integrable with respect to a Haar measure on G_{s}.

Proof. This follows from the integral formula for G_{s} under the Cartan decomposition (see [86, Lemma 2.4.2]), as well as the estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

For every Casselman-Wallach representation π of G_{s}, write $\pi^{\vee}$ for its contragredient representation, which is a Casselman-Wallach representation of G_{s} equipped with a G_{s}-invariant, non-degenerate, continuous bilinear form

$$\langle\, ,\, \rangle : \pi \times \pi^{\vee} \to \mathbb{C}.$$

Definition 4.5. *Let $\nu \in \mathbb{R}$. A positive function Ψ on G_{s} is ν-bounded if there is a function f as in Lemma 4.4 such that*

$$\Psi(g) \leqslant f(g) \cdot \Psi_{\mathsf{s}}^{\nu}(g) \cdot \Xi_{\mathsf{s}}(g) \qquad \textit{for all } g \in G_{\mathsf{s}}.$$

A Casselman-Wallach representation π of G_{s} is said to be ν-bounded if there exist a ν-bounded positive function Ψ on G_{s}, and continuous seminorms $|\cdot|_{\pi}$ and $|\cdot|_{\pi^{\vee}}$ on π and $\pi^{\vee}$ respectively such that

$$|\langle g.u, v\rangle| \leqslant \Psi(g) \cdot |u|_{\pi} \cdot |v|_{\pi^{\vee}}$$

for all $u \in \pi$, $v \in \pi^{\vee}$, and $g \in G_{\mathsf{s}}$.

Let X_{s} be a maximal J_{s}-stable totally isotropic subspace of V_{s}, which is unique up to the action of K_{s}. Put $Y_{\mathsf{s}} := L_{\mathsf{s}}(X_{\mathsf{s}})$. Then $X_{\mathsf{s}} \cap Y_{\mathsf{s}} = \{0\}$. Write

$$P_{\mathsf{s}} = R_{\mathsf{s}} \ltimes N_{\mathsf{s}}$$

for the parabolic subgroup of G_{s} stabilizing X_{s}, where R_{s} is the Levi subgroup stabilizing both X_{s} and Y_{s}, and N_{s} is the unipotent radical. For every character $\chi : R_{\mathsf{s}} \to \mathbb{C}^{\times}$, view it as a character of P_{s} that is trivial on N_{s}. Write

$$I(\chi) := \mathrm{Ind}_{P_{\mathsf{s}}}^{G_{\mathsf{s}}} \chi \qquad \text{(normalized smooth induction)},$$

which is a Casselman-Wallach representation of G_{s} under the right translations. Note that the representations $I(\chi)$ and $I(\chi^{-1})$ are contragredients of each other with the G_{s}-invariant pairing

$$\langle\, ,\, \rangle : I(\chi) \times I(\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\mathsf{s}}} f(g) \cdot f'(g)\, \mathrm{d}g.$$

Let ν_{χ} be a real number such that $|\chi|$ equals the composition of

$$R_{\mathsf{s}} \xrightarrow{\text{the natural homomorphism}} \mathrm{GL}(X_{\mathsf{s}}) \xrightarrow{|\det|^{\nu_{\chi}}} \mathbb{C}^{\times}. \tag{4.4}$$

Definition 4.6. *A classical signature s is split if $V_{\mathsf{s}} = X_{\mathsf{s}} \oplus Y_{\mathsf{s}}$.*

When s is split, P_{s} is called a Siegel parabolic subgroup of G_{s}.

Lemma 4.7. *Suppose that s is split and let $\chi : P_{\mathsf{s}} \to \mathbb{C}^{\times}$ be a character. Then there is a positive function Ψ on G_{s} with the following properties:*

- *Ψ is $(1 - 2|\nu_{\chi}| - \frac{|\mathsf{s}|}{2})$-bounded if $\star \in \{B, C, D, \widetilde{C}\}$, and $(2 - 2|\nu_{\chi}| - \frac{|\mathsf{s}|}{2})$-bounded if $\star \in \{C^*, D^*\}$;*
- *for all $f \in I(\chi)$, $f' \in I(\chi^{-1})$ and $g \in G_{\mathsf{s}}$,*

$$|\langle g.f, f'\rangle| \leqslant \Psi(g) \cdot |f|_{K_{\mathsf{s}}}|_{\infty} \cdot |f'|_{K_{\mathsf{s}}}|_{\infty} \qquad \text{(} |\ |_{\infty} \textit{ stands for the suppernorm).} \tag{4.5}$$

Proof. Let f_0 denote the element in $I(|\chi|)$ such that $(f_0)|_{K_{\mathsf{s}}} = 1$, and likewise let f_0' denote the element in $I(|\chi^{-1}|)$ such that $(f_0')|_{K_{\mathsf{s}}} = 1$. Put

$$\Psi(g) := \langle g.f_0, f_0' \rangle, \qquad g \in G_{\mathsf{s}}.$$

Then it is easy to see that (4.5) holds. Note that Ψ is an elementary spherical function, and the lemma then follows by using the well-known estimate of the elementary spherical functions ([86, Lemma 3.6.7]). $\square$

4.3. **Matrix coefficient integrals against the oscillator representation.** We begin with the following lemma.

Lemma 4.8. *There exist continuous seminorms $|\cdot|_{\mathsf{s},\mathsf{s}'}$ and $|\cdot|^{\vee}_{\mathsf{s},\mathsf{s}'}$ on $\omega_{\mathsf{s},\mathsf{s}'}$ and $\omega^{\vee}_{\mathsf{s},\mathsf{s}'}$ respectively such that*

$$|\langle (g,g').u, v\rangle| \leqslant \Psi_{\mathsf{s}}^{|\mathsf{s}'|}(g) \cdot \Psi_{\mathsf{s}'}^{|\mathsf{s}|}(g') \cdot |u|_{\mathsf{s},\mathsf{s}'} \cdot |v|^{\vee}_{\mathsf{s},\mathsf{s}'}$$

for all $u \in \omega_{\mathsf{s},\mathsf{s}'}$, $v \in \omega^{\vee}_{\mathsf{s},\mathsf{s}'}$, and $(g,g') \in G_{\mathsf{s}} \times G_{\mathsf{s}'}$.

Proof. This follows from the proof of [50, Theorem 3.2]. $\square$

Definition 4.9. *A Casselman-Wallach representation of $G_{\mathsf{s}'}$ is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ if it is ν-bounded for some $\nu > \nu_{\mathsf{s}'} - |\mathsf{s}|$.*

Example. Suppose that $\star' \neq \widetilde{C}$. Then the trivial representation of $G_{\mathsf{s}'}$ is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$ if $|\mathsf{s}| > 2\nu_{\mathsf{s}'}$.

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is convergent for $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}$. Consider the integrals

$$\begin{array}{rcl} (\pi' \times \omega_{\mathsf{s},\mathsf{s}'}) \times (\pi'^{\vee} \times \omega^{\vee}_{\mathsf{s},\mathsf{s}'}) & \rightarrow & \mathbb{C}, \\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g.u, u'\rangle \cdot \langle g.v, v'\rangle \, \mathrm{d}g. \end{array} \tag{4.6}$$

Unless otherwise specified, all the measures on Lie groups occurring in this article are Haar measures.

Lemma 4.10. *The integrals in* (4.6) *are absolutely convergent and the map* (4.6) *is continuous and multi-linear.*

Proof. This is a direct consequence of Lemmas 4.3, 4.4 and 4.8. $\square$

By Lemma 4.10, the integrals in (4.6) yield a continuous bilinear form

$$(\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'}) \times (\pi'^{\vee} \widehat{\otimes} \omega^{\vee}_{\mathsf{s},\mathsf{s}'}) \rightarrow \mathbb{C}. \tag{4.7}$$

Put

$$\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi') := \frac{\pi' \widehat{\otimes} \omega_{\mathsf{s},\mathsf{s}'}}{\text{the left kernel of (4.7)}}. \tag{4.8}$$

Proposition 4.11. *The representation $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ of G_{s} is a quotient of $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$, and is $(|\mathsf{s}'| - \nu_{\mathsf{s}})$-bounded.*

Proof. Note that the bilinear form (4.7) is $(G_{\mathsf{s}'} \times G_{\mathsf{s}'})$-invariant, as well as G_{s}-invariant. Thus $\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$ is a quotient of $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')$, and is therefore a Casselman-Wallach representation. Its contragedient representation is identified with

$$(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi'))^{\vee} := \frac{\pi'^{\vee} \widehat{\otimes} \omega^{\vee}_{\mathsf{s},\mathsf{s}'}}{\text{the right kernel of (4.7)}}.$$

Lemmas 4.4 and 4.8 implies that there are continuous seminorms $|\cdot|_{\pi',\mathsf{s},\mathsf{s}'}$ and $|\cdot|_{\pi'^\vee,\mathsf{s},\mathsf{s}'}$ on $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$ and $\pi'^\vee\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}^\vee$ respectively such that

$$|\langle g.u, v\rangle| \leqslant \Psi_{\mathsf{s}}^{|\mathsf{s}'|}(g)\cdot |u|_{\pi',\mathsf{s},\mathsf{s}'}\cdot |v|_{\pi'^\vee,\mathsf{s},\mathsf{s}'}$$

for all $u\in \pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$, $v\in \pi'^\vee\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}^\vee$, and $g\in G_{\mathsf{s}}$. The proposition then easily follows in view of Lemma 4.3. □

4.4. **Unitarity.** For the notation of weakly containment of unitary representations, see [24] for example.

Lemma 4.12. *Suppose that $|\mathsf{s}|\geqslant \nu_{\mathsf{s}'}$. Then as a unitary representation of $G_{\mathsf{s}'}$, $\hat{\omega}_{\mathsf{s},\mathsf{s}'}$ is weakly contained in the regular representation.*

Proof. This has been known to experts (see [50, Theorem 3.2]). Lemmas 4.3, 4.4 and 4.8 implies that for a dense subspace of $\hat{\omega}_{\mathsf{s},\mathsf{s}'}|_{G_{\mathsf{s}'}}$, the diagonal matrix coefficients are almost square integrable. Thus the lemma follows form [24, Theorem 1]. □

However, if $\star'\in\{B, C^*, D^*\}$, $|\mathsf{s}|\neq \nu_{\mathsf{s}'}$ for the parity reason. For this reason, we introduce

$$\nu_{\mathsf{s}'}^\circ := \begin{cases}\nu_{\mathsf{s}'}, & \text{if } \star'\in\{C, D, \widetilde{C}\};\\ \nu_{\mathsf{s}'}+1, & \text{if } \star'\in\{B, C^*, D^*\}.\end{cases}$$

The following definition is a slight variation of definition 11.2.

Definition 4.13. *A Casselman-Wallach representation of $G_{\mathsf{s}'}$ is over-convergent for $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}$ if it is ν-bounded for some $\nu > \nu_{\mathsf{s}'}^\circ - |\mathsf{s}|$.*

We will prove the following unitarity result in the rest of this subsection.

Theorem 4.14. *Assume that $|\mathsf{s}|\geqslant \nu_{\mathsf{s}'}^\circ$. Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is over-convegent for $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}$. If π' is unitarizable, then so is $\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')$.*

Recall the following positivity result of matrix coefficient integrals, which is a special case of [32, Theorem A. 5].

Lemma 4.15. *Let G be a real reductive group with a maximal compact subgroup K. Let π_1 and π_2 be two unitary representations of G such that π_2 is weakly contained in the regular representation. Let $u_1, u_2, \cdots, u_r$ ($r\in\mathbb{N}$) be vectors in π_1 such that for all $i, j = 1, 2, \cdots, r$, the integral*

$$\int_G \langle g.u_i, u_j\rangle\, \Xi_G(g)\,\mathrm{d}g$$

is absolutely convergent, where Ξ_G is the bi-K-invariant Harish-Chandra's Ξ function on G. Let $v_1, v_2, \cdots, v_r$ be K-finite vectors in π_2. Put

$$u := \sum_{i=1}^r u_i\otimes v_i \in \pi_1\otimes\pi_2.$$

Then the integral

$$\int_G \langle g.u, u\rangle\,\mathrm{d}g$$

absolutely converges to a nonnegative real number.

Now we come to the proof of Theorem 4.14.

Proof of Theorem 4.14. Fix an invariant continuous Hermitian inner product on π', and write $\hat{\pi}'$ for the completion of π' with respect to this Hermitian inner product. The space $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$ is equipped with the inner product $\langle\,,\,\rangle$ that is the tensor product of the ones on π' and $\omega_{\mathsf{s},\mathsf{s}'}$. It suffices to show that

$$\int_{G_{\mathsf{s}'}} \langle g.u, u\rangle \,\mathrm{d}g \geqslant 0$$

for all u in a dense subspace of $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}$.

If $\nu^{\circ}_{\mathsf{s}'} < 0$, then $\star \in \{D, D^*\}$ and $|\mathsf{s}'| = 0$. The theorem is trivial in this case. Thus we assume that $\nu^{\circ}_{\mathsf{s}'} \geqslant 0$. We also assume that $\star = B$. The proof in the other cases is similar and is omitted.

Note that $\nu^{\circ}_{\mathsf{s}'} = \nu_{\mathsf{s}'} = |\mathsf{s}'|$ is even. Let $\mathsf{s}_1 = (B, p_1, q_1)$ and $\mathsf{s}_2 = (D, p_2, q_2)$ be two classical signatures such that

$$(p_1, q_1) + (p_2, q_2) = (p, q) \quad \text{and} \quad |\mathsf{s}_2| = \nu^{\circ}_{\mathsf{s}'}.$$

View $\hat{\omega}_{\mathsf{s}_2,\mathsf{s}'}$ as a unitary representation of the symplectic group $G_{(C,p',q')}$. Define π_2 to be its pull-back through the covering homomorphism $G_{\mathsf{s}'} \to G_{(C,p',q')}$. By Lemma 4.12, the representation π_2 of $G_{\mathsf{s}'}$ is weakly contained in the regular representation.

Put

$$\pi_1 := \hat{\pi}'\widehat{\otimes}_{\mathrm{h}}(\hat{\omega}_{\mathsf{s}_1,\mathsf{s}'}|_{G_{\mathsf{s}'}}) \qquad (\widehat{\otimes}_{\mathrm{h}} \text{ indicates the Hilbert space tensor product}).$$

Lemmas 4.4 and 4.8 imply that the integral

$$\int_{G_{\mathsf{s}'}} \langle g.u, v\rangle \cdot \Xi_{G_{\mathsf{s}'}}(g)\,\mathrm{d}g$$

is absolutely convergent for all $u, v \in \pi'\otimes\omega_{\mathsf{s}_1,\mathsf{s}'}$. The theorem then follows by Lemma 15.5. □

5. Double theta lifts via matrix coefficient integrals

In this section, we study double theta lifts via matrix coefficient integrals. Let

$$\dot{\mathsf{s}} = (\dot{\star}, \dot{p}, \dot{q}), \qquad \mathsf{s}' = (\star', p', q') \qquad \text{and} \qquad \mathsf{s}'' = (\star'', p'', q'')$$

be classical signatures such that

- $(q', p') + (p'', q'') = (\dot{p}, \dot{q})$;
- if $\dot{\star} \in \{B, D\}$, then $\star', \star'' \in \{B, D\}$;
- if $\dot{\star} \notin \{B, D\}$, then $\star' = \star'' = \dot{\star}$.

As in Section 4.1, put $\mathsf{s}'^{-} := (\star', q', p')$. We view $V_{\mathsf{s}'^{-}}$ and $V_{\mathsf{s}''}$ as subspaces of $V_{\dot{\mathsf{s}}}$ such that $(\langle\,,\,\rangle_{\dot{\mathsf{s}}}, J_{\dot{\mathsf{s}}}, L_{\dot{\mathsf{s}}})$ extends both $(\langle\,,\,\rangle_{\mathsf{s}'^{-}}, J_{\mathsf{s}'^{-}}, L_{\mathsf{s}'^{-}})$ and $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$, and $V_{\mathsf{s}'^{-}}$ and $V_{\mathsf{s}''}$ are perpendicular to each other under the form $\langle\,,\,\rangle_{\dot{\mathsf{s}}}$. Then we have an orthogonal decomposition

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^{-}} \oplus V_{\mathsf{s}''},$$

and both $G_{\mathsf{s}'^{-}}$ and $G_{\mathsf{s}''}$ are identified with subgroups of $G_{\dot{\mathsf{s}}}$.

Fix a linear isomorphism

$$\iota_{\mathsf{s}'} : V_{\mathsf{s}'} \to V_{\mathsf{s}'^{-}} \tag{5.1}$$

as in (4.1), which induces an isomorphism

$$G_{\mathsf{s}'} \to G_{\mathsf{s}'^{-}}, \quad g \mapsto g^{-}. \tag{5.2}$$

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$. Write π'^{-} for the representation of $G_{\mathsf{s}'^{-}}$ that corresponds to π' under the isomorphism (5.2).

5.1. **Matrix coefficient integrals and double theta lifts.** We begin with the following lemma.

Lemma 5.1. *The function $(\Xi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}}$ on $G_{\mathsf{s}'}$ is $|\mathsf{s}''|$-bounded.*

Proof. This follows from the estimate of Harish-Chandra's Ξ function ([86, Theorem 4.5.3]). □

Definition 5.2. *The Casselman-Wallach representation π'^{-} of $G_{\mathsf{s}'^{-}}$ is convergent for a Casselman-Wallach representation $\dot{\pi}$ of $G_{\dot{\mathsf{s}}}$ if there are real number ν' and $\dot{\nu}$ such that π' is ν'-bounded, $\dot{\pi}$ is $\dot{\nu}$-bounded, and*

$$\nu' + \dot{\nu} > -|\mathsf{s}''|\,.$$

Let $\dot{\pi}$ be a Casselman-Wallach representations of $G_{\dot{\mathsf{s}}}$ such that π'^{-} is convergent for $\dot{\pi}$. Consider the integrals

$$\begin{array}{rcl} (\pi'^{-}\times\dot{\pi})\times((\pi'^{-})^{\vee}\times\dot{\pi}^{\vee}) & \to & \mathbb{C}, \\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}}\langle g^{-}.u,u'\rangle\cdot\langle g^{-}.v,v'\rangle\,\mathrm{d}g. \end{array} \tag{5.3}$$

Lemma 5.3. *The integrals in* (5.3) *are absolutely convergent and the map* (5.3) *is continuous and multi-linear.*

Proof. Note that $(\Psi_{\dot{\mathsf{s}}})|_{G_{\mathsf{s}'}} = \Psi_{\mathsf{s}'}$. Thus the lemma follows from Lemmas 4.4 and 5.1. □

By Lemma 5.3, the integrals in (5.3) yield a continuous bilinear form

$$\langle\,,\,\rangle : (\pi'^{-}\widehat{\otimes}\dot{\pi})\times((\pi'^{-})^{\vee}\widehat{\otimes}\dot{\pi}^{\vee})\to\mathbb{C}. \tag{5.4}$$

Put

$$\pi'^{-} * \dot{\pi} := \frac{\pi'^{-}\widehat{\otimes}\dot{\pi}}{\text{the left kernel of (5.4)}}. \tag{5.5}$$

This is a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

For every classical signature $\mathsf{s} = (\star, p, q)$, put

$$[\mathsf{s}] := \begin{cases} (C,p,q), & \text{if } \star = \tilde{C}; \\ \mathsf{s}, & \text{if } \star \neq \tilde{C}. \end{cases}$$

Proposition 5.4. *Suppose that $\mathsf{s} = (\star, p, q)$ is a classical signature such that both $\star'$ and $\star''$ equals the Howe dual of $\star$, and $2\nu_{\mathsf{s}} < |\dot{\mathsf{s}}|$. Assume that π' is ν'-bounded for some $\nu' > \nu_{\mathsf{s}'} - |\mathsf{s}|$. Then*

- *π' is convergent for $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}$;*
- *$\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')$ is convergent for $\check{\Theta}_{\mathsf{s}}^{\mathsf{s}''}$;*
- *the trivial representation 1 of $G_{\bar{\mathsf{s}}}$ is convergent for $\check{\Theta}_{\bar{\mathsf{s}}}^{\dot{\mathsf{s}}}$;*
- *π'^{-} is convergent for $\bar{\Theta}_{[\mathsf{s}]}^{\dot{\mathsf{s}}}(1)$;*
- *as representations of $G_{\mathsf{s}''}$,*

$$\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \pi'^{-} * \bar{\Theta}_{[\mathsf{s}]}^{\dot{\mathsf{s}}}(1). \tag{5.6}$$

Proof. The first four claims in the proposition is obvious. We only need to prove the last one. Note that the integrals in

$$\begin{array}{rcl} (\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}})\times((\pi')^{\vee}\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}^{\vee}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}}^{\vee}) & \to & \mathbb{C}, \\ (u,v) & \mapsto & \int_{G_{\mathsf{s}'}\times G_{\mathsf{s}}}\langle(g',g).u,v\rangle\,\mathrm{d}g'\,\mathrm{d}g \end{array} \tag{5.7}$$

are absolutely convergent and defines a continuous bilinear map. Also note that

$$\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}_2,\mathsf{s}} \cong \omega_{\dot{\mathsf{s}},\mathsf{s}}.$$

In view of Fubini's theorem, the lemma follows as both sides of (5.6) are isomorphic to the quotient of $\pi'\widehat{\otimes}\omega_{\mathsf{s},\mathsf{s}'}\widehat{\otimes}\omega_{\mathsf{s}'',\mathsf{s}}$ by the left kernel of the pairing (5.7).

□

5.2. **Matrix coefficient integrals against degenerate principal series.** We are particularly interested in the case when $\dot{\pi}$ is a degenerate principal series representation. Suppose that $\dot{\mathsf{s}}$ is split, and

$$p' \leqslant p'' \qquad \text{and} \qquad q' \leqslant q''.$$

Then there is a split classical signature $\mathsf{s}_0 = (\star_0, p_0, q_0)$ such that

- $\star_0 = \dot{\star}$;
- $(p', q') + (p_0, q_0) = (p'', q'')$.

We view $V_{\mathsf{s}'}$ and V_{s_0} as subspaces of $V_{\mathsf{s}''}$ such that $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$ extends both $(\langle\,,\,\rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, L_{\mathsf{s}'})$ and $(\langle\,,\,\rangle_{\mathsf{s}_0}, J_{\mathsf{s}_0}, L_{\mathsf{s}_0})$, and $V_{\mathsf{s}'}$ and V_{s_0} are perpendicular to each other under the form $\langle\,,\,\rangle_{\mathsf{s}_2}$.

Put

$$V_{\mathsf{s}'}^{\triangle} := \{\iota_{\mathsf{s}'}(v) + v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\} \quad \text{and} \quad V_{\mathsf{s}'}^{\nabla} := \{\iota_{\mathsf{s}'}(v) - v \in V_{\dot{\mathsf{s}}} \mid v \in V_{\mathsf{s}'}\}.$$

As before, we have that

$$V_{\mathsf{s}_0} = X_{\mathsf{s}_0} \oplus Y_{\mathsf{s}_0},$$

where X_{s_0} is a maximal J_{s_0}-stable totally isotropic subspace of V_{s_0}, and $Y_{\mathsf{s}_0} := L_{\mathsf{s}_0}(X_{\mathsf{s}_0})$. Suppose that

$$X_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0} \quad \text{and} \quad Y_{\dot{\mathsf{s}}} = V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}.$$

In summary, we have decompositions

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}''} = V_{\mathsf{s}'^-} \oplus V_{\mathsf{s}'} \oplus V_{\mathsf{s}_0} = (V_{\mathsf{s}'}^{\triangle} \oplus X_{\mathsf{s}_0}) \oplus (V_{\mathsf{s}'}^{\nabla} \oplus Y_{\mathsf{s}_0}).$$

As before, $X_{\dot{\mathsf{s}}}$ and X_{s_0} yield the Siegel parabolic subgroups

$$P_{\dot{\mathsf{s}}} = R_{\dot{\mathsf{s}}} \ltimes N_{\dot{\mathsf{s}}} \subset G_{\dot{\mathsf{s}}} \qquad \text{and} \qquad P_{\mathsf{s}_0} = R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}_0} \subset G_{\mathsf{s}_0}.$$

Write

$$P_{\mathsf{s}'',\mathsf{s}_0} = R_{\mathsf{s}'',\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$$

for the parabolic subgroup of $G_{\mathsf{s}''}$ stabilizing X_{s_0}, where $R_{\mathsf{s}'',\mathsf{s}_0}$ is the Levi subgroup stabilizing both X_{s_0} and Y_{s_0}, and $N_{\mathsf{s}'',\mathsf{s}_0}$ is the unipotent radical. We have an obvious homomorphism

$$G_{\mathsf{s}'} \times R_{\mathsf{s}_0} \to R_{\mathsf{s}'',\mathsf{s}_0},$$

which is a two fold covering map when $\dot{\star} = \widetilde{C}$, and an isomorphism in the other cases. For every $g \in G_{\mathsf{s}'}$, write $g^{\triangle} := g^{-}g$, which is an element of $R_{\dot{\mathsf{s}}}$.

Let $\dot{\chi} : R_{\dot{\mathsf{s}}} \to \mathbb{C}^{\times}$ be a character. It yields a degenerate principal series representation

$$I(\dot{\chi}) := \operatorname{Ind}_{P_{\dot{\mathsf{s}}}}^{G_{\dot{\mathsf{s}}}} \dot{\chi}$$

of $G_{\dot{\mathsf{s}}}$. Write

$$\chi_0 := \dot{\chi}|_{R_{\mathsf{s}_0}}, \tag{5.8}$$

and define a character

$$\chi' : G_{\mathsf{s}'} \to \mathbb{C}^{\times}, \quad g \mapsto \dot{\chi}(g^{\triangle}) \qquad \text{(this is a quadratic character)}. \tag{5.9}$$

Recall the representations π' of $G_{\mathsf{s}'}$ and π'^- of $G_{\mathsf{s}'^-}$. If $\dot{\mathsf{s}} = \widetilde{C}$, we assume that both π' and $\dot{\chi}$ are genuine. Then $(\pi' \otimes \chi') \otimes \chi_0$ descends to a Casselman-Wallach representation of $R_{\mathsf{s}'',\mathsf{s}_0}$. View it as a representation of $P_{\mathsf{s}'',\mathsf{s}_0}$ via the trivial action of $N_{\mathsf{s}'',\mathsf{s}_0}$, and form the representation $\operatorname{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} ((\pi_1 \otimes \chi') \otimes \chi_0)$

Let $\nu_{\dot{\chi}} \in \mathbb{R}$ be as in (4.4). The rest of this subsection is devoted to a proof of the following theorem.

Theorem 5.5. *Assume that π' is ν' bounded for some*

$$\nu' > \begin{cases} 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 1, & \text{if } \dot\star \in \{B, C, D, \widetilde{C}\}; \\ 2\,|\nu_{\dot\chi}| - \frac{|\mathsf{s}_0|}{2} - 2, & \text{if } \dot\star \in \{C^*, D^*\}. \end{cases}$$

Then π'^{-} is convergent for $I_{\dot{\mathsf{s}}}(\dot\chi)$, and

$$\pi'^{-} * I(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Let the notation and assumptions be as in Theorem 5.5. Recall that the representations $I(\dot\chi)$ and $I(\dot\chi^{-1})$ are contragredients of each other with the $G_{\dot{\mathsf{s}}}$-invariant pairing

$$\langle\,,\,\rangle : I(\dot\chi) \times I(\dot\chi^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\dot{\mathsf{s}}}} f(g) \cdot f'(g)\,\mathrm{d}g,$$

The first assertion of Theorem 5.5 is a direct consequence of Lemma 4.7. As in (5.4), we have a continuous bilinear form

$$(5.10)\qquad \begin{array}{rcl} (\pi'^{-}\widehat\otimes I(\dot\chi)) \times ((\pi'^{-})^\vee \widehat\otimes I(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^{-}.u, v\rangle\,\mathrm{d}g \end{array}$$

so that

$$(5.11)\qquad \pi'^{-} * I(\dot\chi) := \frac{\pi'^{-}\widehat\otimes I(\dot\chi)}{\text{the left kernel of (5.10)}}.$$

Note that

$$G_{\dot{\mathsf{s}}}^\circ := P_{\dot{\mathsf{s}}} \cdot G_{\mathsf{s}''}$$

is open and dense in $G_{\dot{\mathsf{s}}}$, and its complement has measure zero in $G_{\dot{\mathsf{s}}}$. Moreover,

$$(5.12)\qquad P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^\circ = (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0})\backslash G_{\mathsf{s}''}.$$

Form the normalized Schwartz induction

$$I^\circ(\dot\chi) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0.$$

The reader is referred to [21, Section 6.2] for the general notion of Schwartz inductions (in a slightly different unnormalized setting). Similarly, put

$$I^\circ(\dot\chi^{-1}) := \mathrm{ind}_{R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}} \chi_0^{-1}.$$

Note that

(5.13) the modulus character of $P_{\dot{\mathsf{s}}}$ restricts to the modulus character of $R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0}$.

In view of (5.12) and (5.13), by extension by zero, $I^\circ(\dot\chi)$ is viewed as a closed subspace of $I(\dot\chi)$, and $I^\circ(\dot\chi^{-1})$ is viewed as a closed subspace of $I(\dot\chi^{-1})$. These two closed subspaces are $(G_{\mathsf{s}'} \times G_{\mathsf{s}''})$-stable.

Similar to (5.10), we have a continuous bilinear form

$$(5.14)\qquad \begin{array}{rcl} (\pi'^{-}\widehat\otimes I^\circ(\dot\chi)) \times ((\pi'^{-})^\vee \widehat\otimes I^\circ(\dot\chi^{-1})) & \to & \mathbb{C}, \\ (u, v) & \mapsto & \int_{G_{\mathsf{s}'}} \langle g^{-}.u, v\rangle\,\mathrm{d}g. \end{array}$$

Put

$$(5.15)\qquad \pi'^{-} * I^\circ(\dot\chi) := \frac{\pi'^{-}\widehat\otimes I^\circ(\dot\chi)}{\text{the left kernel of (5.14)}},$$

which is still a smooth Fréchet representation of $G_{\mathsf{s}''}$ of moderate growth.

Lemma 5.6. *As representations of $G_{\mathsf{s}''}$,*

$$\pi'^{-} * I^\circ(\dot\chi) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0).$$

Proof. As Fréchet spaces, π'^{-} is obviously identified with π'. Note that the natural map $G_{\mathsf{s}'} \to (R_{\mathsf{s}_0} \ltimes N_{\mathsf{s}'',\mathsf{s}_0})\backslash G_{\mathsf{s}''}$ is proper and hence the following integrals are absolutely convergent and yield a $G_{\mathsf{s}''}$-equivariant continuous linear map:

$$\begin{array}{rcl} \xi : \pi'^{-}\widehat{\otimes} I^\circ(\dot\chi) & \to & \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi' \otimes \chi') \otimes \chi_0), \\ v \otimes f & \mapsto & \left(h \mapsto \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f(g^{-1}h) \cdot g.v \,\mathrm{d}g\right). \end{array}$$

Moreover, this map is surjective (*cf.* [21, Section 6.2]). It is thus open by the open mapping theorem. Similarly, we have a open surjective $G_{\mathsf{s}''}$-equivariant continuous linear map

$$\begin{array}{rcl} \xi' : (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1}) & \to & \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi'^\vee \otimes \chi') \otimes \chi_0^{-1}), \\ v' \otimes f' & \mapsto & \left(h \mapsto \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f'(g^{-1}h) \cdot g.v' \,\mathrm{d}g\right). \end{array}$$

For all $v \otimes f \in \pi'^{-}\widehat{\otimes} I^\circ(\dot\chi)$ and $v' \otimes f' \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$, we have that

$$\begin{aligned} & \int_{G_{\mathsf{s}'}} \langle g^{-}.(v\otimes f), v' \otimes f'\rangle \,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-}.v, v'\rangle \cdot \langle g^{-}.f, f'\rangle \,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-}.v, v'\rangle \cdot \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} (g^{-}.f)(hx) \cdot f'(hx) \,\mathrm{d}h \,\mathrm{d}x \,\mathrm{d}g \\ = & \int_{G_{\mathsf{s}'}} \langle g^{-}.v, v'\rangle \cdot \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} \chi'(g) \cdot f(g^{-1}hx) \cdot f'(hx) \,\mathrm{d}h \,\mathrm{d}x \,\mathrm{d}g \\ = & \int_{K_{\mathsf{s}''}} \int_{G_{\mathsf{s}'}} \int_{G_{\mathsf{s}'}} \langle (h^{-}(g^{-})^{-1}).v, v'\rangle \cdot \chi'(hg^{-1}) \cdot f(gx) \cdot f'(hx) \,\mathrm{d}g \,\mathrm{d}h \,\mathrm{d}x \\ = & \int_{K_{\mathsf{s}''}} \langle \xi(v \otimes f)(x), \xi'(v' \otimes f')(x)\rangle \,\mathrm{d}x \\ = & \langle \xi(v \otimes f), \xi'(v' \otimes f')\rangle. \end{aligned}$$

This implies the lemma. ☐

Lemma 5.7. *Let $u \in \pi'^{-}\widehat{\otimes} I(\dot\chi)$. Assume that*

$$\int_{G_{\mathsf{s}'}} \langle g^{-}.u, v\rangle \,\mathrm{d}g = 0 \tag{5.16}$$

for all $v \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$. Then (5.16) *also holds for all $v \in (\pi'^{-})^\vee\widehat{\otimes} I(\dot\chi^{-1})$.*

Proof. Take a sequence $(\eta_1, \eta_2, \eta_3, \cdots)$ of real valued smooth functions on $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$ such that

- for all $i \geqslant 1$, the support of η_i is contained in $P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^\circ$;
- for all $i \geqslant 1$ and $x \in P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}$, $0 \leqslant \eta_i(x) \leqslant \eta_{i+1}(x) \leqslant 1$;
- $\bigcup_{i=1}^\infty \eta_i^{-1}(1) = P_{\dot{\mathsf{s}}}\backslash G_{\dot{\mathsf{s}}}^\circ$.

Let $v \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$. Note that $\eta_i I(\dot\chi^{-1}) \subset I^\circ(\dot\chi^{-1})$. Thus $\eta_i v \in (\pi'^{-})^\vee\widehat{\otimes} I^\circ(\dot\chi^{-1})$.

Lemma 4.7 and Lebesgue's dominated convergence theorem imply that

$$\int_{G_{\mathsf{s}'}} \langle g^{-}.u, v\rangle \,\mathrm{d}g = \lim_{i \to +\infty} \int_{G_{\mathsf{s}'}} \langle g^{-}.u, \eta_i v\rangle \,\mathrm{d}g = 0.$$

This proves the lemma. ☐

Lemma 5.7 implies that we have a natural continuous linear map

$$\mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}((\pi'\otimes\chi')\otimes\chi_0) = \pi'^{-} * I^{\circ}(\dot\chi) \to \pi'^{-} * I(\dot\chi).$$

This map is clearly injective and $G_{\mathsf{s}''}$-equivariant. Similarly to Lemma 5.7, we know that the natural paring

$$(\pi'^{-} * I(\dot\chi)) \times ((\pi'^{-})^{\vee} * I^{\circ}(\dot\chi^{-1})) \to \mathbb{C}$$

is well-defined and non-degenerate. Thus Theorem 5.5 follows by the following lemma.

Lemma 5.8. *Let G be a real reductive group. Let π be a Casselman-Wallach representation of G, and let $\tilde\pi$ be a smooth Fréchet representation of G with a G-equivariant injective continuous linear map*

$$\phi : \pi \to \tilde\pi.$$

Assume that there is a non-degenerate G-invariant continuous bilinear map

$$\langle\, ,\, \rangle : \tilde\pi \times \pi^{\vee} \to \mathbb{C},$$

such that the composition of

$$\pi \times \pi^{\vee} \xrightarrow{(u,v)\mapsto(\phi(u),v)} \tilde\pi \times \pi^{\vee} \xrightarrow{\langle\, ,\, \rangle} \mathbb{C}$$

is the natural paring. Then ϕ is a topological isomorphism.

Proof. Let $u \in \tilde\pi$. Using the theorem of Dixmier-Malliavin [27, Theorem 3.3], we write

$$u = \sum_{i=1}^{s} \int_G \varphi_i(g)\cdot g.u_i \,\mathrm{d}g \qquad (s \in \mathbb{N},\ u_i \in \tilde\pi),$$

where φ_i's are compactly supported smooth functions on G. As a continuous linear functional on $\pi^{\vee}$, we have that

$$\langle u, \cdot\rangle = \sum_{i=1}^{s} \int_G \varphi_i(g)\cdot \langle g.u_i, \cdot\rangle \,\mathrm{d}g.$$

By [79, Lemma 3.5], the right-hand side functional equals $\langle u_0, \cdot\rangle$ for a unique $u_0 \in \pi$. Thus $u = u_0$, and the lemma follows by the open mapping theorem.

□

5.3. **Double theta lifts and parabolic inductions.** In this subsection, we further assume that

$$\dot{\mathsf{s}} = (C, 2k-1, 2k-1),\ (D, 2k-1, 2k-1),\ (\widetilde{C}, 2k, 2k),\ (C^*, 2k, 2k), \text{or}\ \ (D^*, 2k, 2k),$$

where $k \in \mathbb{N}^+$. We also assume that the character $\dot\chi : R_{\dot{\mathsf{s}}} \to \mathbb{C}^{\times}$ satisfies the following conditions:

$$(5.17)\quad \begin{cases} \dot\chi = 1, & \text{if } \dot\star \in \{B, D\};\\ \dot\chi^2 = 1, & \text{if } \dot\star = C;\\ \dot\chi \text{ is genuine and } \dot\chi^4 = 1, & \text{if } \dot\star = \widetilde{C};\\ \dot\chi^2 \text{ equals the composition of } R_{\dot{\mathsf{s}}} \xrightarrow{\text{natural map}} \mathrm{GL}(Y_{\dot{\mathsf{s}}}) \xrightarrow{\det} \mathbb{C}^{\times}, & \text{if } \dot\star = C^*;\\ \dot\chi^2 \text{ equals the composition of } R_{\dot{\mathsf{s}}} \xrightarrow{\text{natural map}} \mathrm{GL}(X_{\dot{\mathsf{s}}}) \xrightarrow{\det} \mathbb{C}^{\times}, & \text{if } \dot\star = D^*. \end{cases}$$

If $\dot\star \notin \{C, \widetilde{C}\}$, the character $\dot\chi$ is uniquely determined by (5.17). If $\dot\star \in \{C, \widetilde{C}\}$, there are two characters satisfying (5.17), and we let $\dot\chi'$ denote the one other than $\dot\chi$.

The relationship between the degenerate principle series representation $I(\dot\chi)$ and Rallis quotients is summarized in the following lemma.

Lemma 5.9. *(a) If* $\dot{\mathsf{s}} = (C, 2k-1, 2k-1)$, *then*

$$I(\dot{\chi}) \oplus I(\dot{\chi}') \cong \bigoplus_{\mathsf{s}=(D,p,q),\, p+q=2k} \check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1).$$

(b) If $\dot{\mathsf{s}} = (\widetilde{C}, 2k, 2k)$, *then*

$$I(\dot{\chi}) \oplus I(\dot{\chi}') \cong \bigoplus_{\mathsf{s}=(B,p,q),\, p+q=2k+1} \check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1).$$

(c) If $\dot{\mathsf{s}} = (D, 2k-1, 2k-1)$, *then*

$$I(\dot{\chi}) \cong \check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) \oplus (\check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) \otimes \det), \qquad \text{where } \mathsf{s} = (C, k-1, k-1).$$

(d) If $\dot{\mathsf{s}} = (C^*, 2k, 2k)$, *then*

$$I(\dot{\chi}) \cong \check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1), \qquad \text{where } \mathsf{s} = (D^*, k, k).$$

(e) If $\dot{\mathsf{s}} = (D^*, 2k, 2k)$, *then there is an exact sequence of representations of* $G_{\dot{\mathsf{s}}}$*:*

$$0 \to \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) \to I(\dot{\chi}) \to \bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}_{\mathsf{s}_1}^{\dot{\mathsf{s}}}(1) \to 0.$$

(f) All the representations $\check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$*and* $\check{\Theta}_{\mathsf{s}_0}^{\dot{\mathsf{s}}}(1)$ *appearing in (a), (b), (c), (d), (e) are irreducible and unitarizable.*

Proof. See [47, Theorem 2.4], [48, Introduction], [49, Theorem 6.1] and [89, Sections 9 and 10]. □

Lemma 5.10. *For all* s *appearing in Lemma 5.9, the trivial representation* 1 *of* G_{s} *is over-convergent for* $\check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}$, *and*

$$\bar{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1) = \check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1).$$

Proof. Note that the trivial representation 1 of G_{s} is 0-bounded, and $0 > \nu_{\mathsf{s}} + \nu_{\mathsf{s}}^{\circ} - |\dot{\mathsf{s}}|$. This implies the first assertion. Since $\check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$ is irreducible and $\bar{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$ is a quotient of $\check{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$, for the proof of the second assertion, it suffices to show that $\bar{\Theta}_{\mathsf{s}}^{\dot{\mathsf{s}}}(1)$ is nonzero.

Put

$$V_{\mathsf{s}}(\mathbb{R}) := \begin{cases} \text{the fixed point set of } J_{\mathsf{s}} \text{ in } V_{\mathsf{s}}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ V_{\mathsf{s}}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

As usual, realize $\hat{\omega}_{\dot{\mathsf{s}},\mathsf{s}}$ on the space of square integrable functions on $(V_{\mathsf{s}}(\mathbb{R}))^{\dot{p}}$ so that $\omega_{\dot{\mathsf{s}},\mathsf{s}}$ is identified with the space of the Schwartz functions, and G_{s} acts on it through the obvious transformation.

Take a positive valued Schwartz function ϕ on $(V_{\mathsf{s}}(\mathbb{R}))^{\dot{p}}$. Then

$$\langle g.\phi, \phi\rangle = \int_{(V_{\mathsf{s}}(\mathbb{R}))^{\dot{p}}} \phi(g^{-1}.x) \cdot \phi(x)\,\mathrm{d}x > 0, \quad \text{for all } g \in G_{\mathsf{s}}.$$

Thus

$$\int_{G_{\mathsf{s}}} \langle g.\phi, \phi\rangle\,\mathrm{d}g \neq 0,$$

and the lemma follows. □

Lemma 5.11. *Assume that* π' *is* ν'*-bounded for some*

$$\nu' > \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \textit{if } \dot{\mathsf{s}} = D^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \textit{otherwise.} \end{cases}$$

Then for all s *appearing in Lemma 5.9 and all unitary character* α' *of* $G_{\mathsf{s}'}$, *the representation* $\pi' \otimes \alpha'$ *is convergent for* $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}$, *and* $\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi' \otimes \alpha')$ *is over-convergent for* $\check{\Theta}_{\mathsf{s}}^{\mathsf{s}''}$.

Proof. The first assertion follows by noting that

$$\nu_{\mathsf{s}'} - |\mathsf{s}| = \begin{cases} -\frac{|\mathsf{s}_0|}{2} - 3, & \text{if } \dot{\mathsf{s}} = D^*; \\ -\frac{|\mathsf{s}_0|}{2} - 1, & \text{otherwise.} \end{cases}$$

The proof of second assertion is similar to the proof of the first assertion of Lemma 5.10. $\square$

Recall the character $\chi_0 := \dot{\chi}|_{R_{\mathsf{s}_0}}$ from (5.8). Similarly we define $\chi_0' := \dot{\chi}'|_{R_{\mathsf{s}_0}}$. Recall that π' is assumed to be genuine when $\star = \widetilde{C}$.

Theorem 5.12. *Assume that π' is ν'-bounded for some $\nu' > -\frac{|\mathsf{s}_0|}{2} - 1$.*
(a) If $\dot{\mathsf{s}} = (C, 2k-1, 2k-1)$, then

$$\bigoplus_{\mathsf{s}=(D,p,q),\, p+q=2k} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \oplus \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0').$$

(b) If $\dot{\mathsf{s}} = (\widetilde{C}, 2k, 2k)$, then

$$\bigoplus_{\mathsf{s}=(B,p,q),\, p+q=2k+1} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \oplus \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0').$$

(c) If $\dot{\mathsf{s}} = (D, 2k-1, 2k-1)$, then

$$\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \oplus \Big((\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi' \otimes \det))) \otimes \det\Big) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s} = (C, k-1, k-1)$ if $|\mathsf{s}'|$ is even, and $\mathsf{s} = (\widetilde{C}, k-1, k-1)$ if $|\mathsf{s}'|$ is odd.
(d) If $\dot{\mathsf{s}} = (C^*, 2k, 2k)$, then

$$\bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \cong \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0),$$

where $\mathsf{s} = (D^*, k, k)$.
(e) If $\dot{\mathsf{s}} = (D^*, 2k, 2k)$, then there is an exact sequence

$$0 \to \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \bar{\Theta}_{\mathsf{s}}^{\mathsf{s}''}(\bar{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \to \mathrm{Ind}_{P_{\mathsf{s}'',\mathsf{s}_0}}^{G_{\mathsf{s}''}}(\pi' \otimes \chi_0) \to \mathcal{J} \to 0$$

of representations of $G_{\mathsf{s}''}$ such that $\mathcal{J}$ is a quotient of

$$\bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}_{\mathsf{s}_1}^{\mathsf{s}''}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}_1}(\pi')).$$

Proof. Using Lemma 4.7, we know that π'^- is convergent for $I(\dot{\chi})$ (and convergent for $I(\dot{\chi}')$ when $\star \in \{C, \widetilde{C}\}$). Also note that the character χ' (see (5.9)) is trivial.

If we are in the situation (a), (b), (c) or (d), by using Theorem 5.5, Lemma 5.10 and Proposition 5.4, the theorem follows by applying the operation $\pi'^- * (\,\cdot\,)$ to the isomorphism in Lemma 5.9.

Now assume that we are in the situation (e). For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in part (e) of Lemma 5.9. Since π'^- is nuclear as a Fréchet space, the sequence

$$0 \to \pi'^- \widehat{\otimes} I_1 \to \pi'^- \widehat{\otimes} I_2 \to \pi'^- \widehat{\otimes} I_3 \to 0$$

is also topologically exact. Note that the natural map

$$\pi'^- * I_1 \to \pi'^- * I_2$$

is injective, and the natural map

$$\pi'^- \widehat{\otimes} I_3 \cong \frac{\pi'^- \widehat{\otimes} I_2}{\pi'^- \widehat{\otimes} I_1} \longrightarrow \frac{\pi'^- * I_2}{\pi'^- * I_1}$$

descends to a surjective map

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'-}} \to \frac{\pi'^{-} * I_2}{\pi'^{-} * I_1}.$$

Note that

$$(\pi'^{-}\widehat{\otimes} I_3)_{G_{\mathsf{s}'-}} \cong \bigoplus_{\mathsf{s}_1=(C^*,2p_1,2q_1),\, p_1+q_1=k-1} \check{\Theta}^{\mathsf{s}''}_{\mathsf{s}_1}(\check{\Theta}^{\mathsf{s}_1}_{\mathsf{s}'}(\pi')).$$

Theorem 5.5 implies that

$$\pi'^{-} * I_2 \cong \mathrm{Ind}^{G_{\mathsf{s}''}}_{P_{\mathsf{s}'',\mathsf{s}_0}}(\pi' \otimes \chi_0).$$

Lemma 5.10 and Proposition 5.4 implies that.

$$\pi'^{-} * I_1 \cong \bigoplus_{\mathsf{s}=(C^*,2p,2q),\, p+q=k} \bar{\Theta}^{\mathsf{s}''}_{\mathsf{s}}(\bar{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\pi')).$$

Therefore the theorem follows. □

6. Bounding the associated cycles

In this section, let $\mathsf{s} = (\star, p, q)$ and $\mathsf{s}' = (\star', p', q')$ be classical signatures such that $\star'$ is the Howe dual of $\star$. Denote by $\omega^{\mathrm{alg}}_{\mathsf{s},\mathsf{s}'}$ the $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$-submodule of $\omega_{\mathsf{s},\mathsf{s}'}$ generated by $\omega^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}_{\mathsf{s},\mathsf{s}'}$. This is an $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}}) \times (\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module. For every $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module ρ' of finite length, put

$$\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho') := (\omega^{\mathrm{alg}}_{\mathsf{s},\mathsf{s}'} \otimes \rho')_{\mathfrak{g}_{\mathsf{s}'},K_{\mathsf{s}'}} \qquad \text{(the coinvariant space)}.$$

This is a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module of finite length.

Let $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$. We retain the notation of Section 3.

6.1. The main result.

Recall from (3.5) the moment maps

$$\begin{array}{ccccc} \mathfrak{g}_{\mathsf{s}} & \xleftarrow{\tilde{M}_{\mathsf{s}}} & W_{\mathsf{s},\mathsf{s}'} & \xrightarrow{\tilde{M}_{\mathsf{s}'}} & \mathfrak{g}_{\mathsf{s}'}, \\ \phi^*\phi & \longleftarrow\!\shortmid & \phi & \longmapsto & \phi\phi^*. \end{array}$$

Lemma 6.1. *The orbit $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$ if and only if*

$$\delta := |\mathsf{s}'| - |\nabla_{\mathrm{naive}}(\mathcal{O})| \geqslant 0.$$

When this is the case, the Young diagram of $\nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ is obtained from that of $\nabla_{\mathrm{naive}}(\mathcal{O})$ by adding δ boxes in the first column.

Proof. This is implied by [25, Theorem 3.6]. □

Definition 6.2. *The orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ if either*

$$|\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|,$$

or

$$|\mathsf{s}'| > |\nabla_{\mathrm{naive}}(\mathcal{O})| \qquad \textit{and} \qquad \mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O}).$$

In the rest of this section we suppose that $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Then $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$. Put $\mathcal{O}' := \nabla^{\mathsf{s}}_{\mathsf{s}'}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$. Let $\overline{\mathcal{O}}$ denote the Zariski closure of $\mathcal{O}$ in $\mathfrak{g}_{\mathsf{s}}$, and likewise let $\overline{\mathcal{O}'}$ denote the Zariski closure of $\mathcal{O}'$ in $\mathfrak{g}_{\mathsf{s}'}$.

Lemma 6.3. *As subsets of $\mathfrak{g}_{\mathsf{s}}$,*

$$\tilde{M}_{\mathsf{s}}(\tilde{M}^{-1}_{\mathsf{s}'}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}.$$

Proof. This is implied by [25, Theorems 5.2 and 5.6]. □

As in the Introduction, a finite length $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module ρ is said to be $\mathcal{O}$-bounded if the associated variety of its annihilator ideal in $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})$ is contained in the Zariski closure of $\mathcal{O}$. When this is the case, we have the associated $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ as in the Introduction.

Lemma 6.4. *For every $\mathcal{O}'$-bounded $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module ρ' of finite length, the $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module $\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho')$ is $\mathcal{O}$-bounded.*

Proof. In view of Lemma 6.3, this is implied by [53, Theorem B] and [83, Theorem 8.4]. □

The main goal of this section is to prove the following theorem.

Theorem 6.5. *Suppose that $\mathcal{O}$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$. Let ρ' be an $\mathcal{O}'$-bounded $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}^{\mathsf{s}}_{\mathsf{s}'}(\rho')) \preceq \check{\vartheta}^{\mathcal{O}}_{\mathcal{O}'}(\mathrm{AC}_{\mathcal{O}'}(\rho')).$$

6.2. **Geometric preparation.** Define a map

$$\tilde{M}_{\mathsf{s},\mathsf{s}'} := \tilde{M}_{\mathsf{s}} \times \tilde{M}_{\mathsf{s}'} : W_{\mathsf{s},\mathsf{s}'} \to \mathfrak{g}_{\mathsf{s}} \times \mathfrak{g}_{\mathsf{s}'}.$$

Lemma 6.6. *As subsets of $W_{\mathsf{s},\mathsf{s}'}$,*

$$\tilde{M}^{-1}_{\mathsf{s},\mathsf{s}'}(\mathcal{O} \times \mathcal{O}') = \tilde{M}^{-1}_{\mathsf{s},\mathsf{s}'}(\mathcal{O} \times \overline{\mathcal{O}'}) = \tilde{M}^{-1}(\mathcal{O}) \cap W^{\circ}_{\mathsf{s},\mathsf{s}'}.$$

Moreover, the above set is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit.

Definition 6.7. *The orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is regular for $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ if the scheme theoretic inverse image $\tilde{M}^{-1}_{\mathsf{s},\mathsf{s}'}(\mathcal{O} \times \mathcal{O}')$ is reduced as a scheme.*

Define a map

$$M_{\mathsf{s},\mathsf{s}'} := M_{\mathsf{s}} \times M_{\mathsf{s}'} : \mathcal{X}_{\mathsf{s},\mathsf{s}'} \to \mathfrak{p}_{\mathsf{s}} \times \mathfrak{p}_{\mathsf{s}'}.$$

Lemma 6.8. *As subsets of $\mathcal{X}_{\mathsf{s},\mathsf{s}'}$,*

$$M^{-1}_{\mathsf{s},\mathsf{s}'}(\mathscr{O} \times (\overline{\mathcal{O}'} \cap \mathfrak{p}_{\mathsf{s}'})) = M^{-1}(\mathscr{O}) \cap \mathcal{X}^{\circ}_{\mathsf{s},\mathsf{s}'}.$$

Moreover, the above set is a single $G_{\mathsf{s},\mathbb{C}} \times G_{\mathsf{s}',\mathbb{C}}$-orbit.

Proof. With the geometric properties investigated in Appendix B.1, the proof to be given is similar to that of [53, Theorem C].

We recall some results in [53]. There are natural good filtrations on π' and $\check{\Theta}(\pi')$ generated by the minimal degree $\widetilde{K}'$-types and $\widetilde{K}$-types, respectively. Define

$$\mathscr{A} := \mathrm{Gr}\, \pi' \otimes \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}'} \quad \text{and} \quad \mathscr{B} := \mathrm{Gr}\, \check{\Theta}(\pi') \otimes \varsigma^{-1}_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}}.$$

We view $\mathscr{A}$ and $\mathscr{B}$ as a $\mathbf{K}'$-equivariant coherent sheaf on $\mathfrak{p}'^{*} \cong \mathfrak{p}'$ and a $\mathbf{K}$-equivariant coherent sheaf on $\mathfrak{p}^{*} \cong \mathfrak{p}$, respectively. Moreover, $\mathrm{AV}(\pi') = \mathrm{Supp}(\mathscr{A})$ and $\mathrm{AV}(\check{\Theta}(\pi')) = \mathrm{Supp}(\mathscr{B})$.

Recall the moment maps defined in Section 12.2.1. Define the following right exact functor

$$\mathcal{L}\colon \mathscr{F} \mapsto (M_{*}(M'^{*}(\mathscr{F})))^{\mathbf{K}'}$$

from the category of $\mathbf{K}'$-equivariant quasi-coherent sheaves on $\mathfrak{p}'$ to the category of $\mathbf{K}$-equivariant quasi-coherent sheaves on $\mathfrak{p}$, where $\mathscr{F}$ is a $\mathbf{K}'$-equivariant quasi-coherent sheave on $\mathfrak{p}'$, and M'^{*} and M_{*} denote the pull-back and push-forward functors respectively. There is a canonical surjective morphism of $\mathbf{K}$-equivariant sheaves on $\mathfrak{p}$ as follows: (*cf.* [53, Equation (16)])

$$\mathcal{Q}\colon \mathcal{L}(\mathscr{A}) \twoheadrightarrow \mathscr{B}.$$

Note that the first equality of (12.18) and the first assertion of Lemma 12.25 imply that $\mathrm{Supp}(\mathcal{L}(\mathscr{A}))$ is contained in $M(M'^{-1}(\mathrm{Supp}(\mathscr{A}))) \subset \overline{\mathcal{O}} \cap \mathfrak{p}$. In particular, $\check{\Theta}(\pi)$ is $\mathcal{O}$-bounded.

According to [53, Proposition 4.3], we may fix a finite filtration

$$0 = \mathscr{A}_0 \subset \cdots \subset \mathscr{A}_l \subset \cdots \subset \mathscr{A}_f = \mathscr{A} \qquad (f \geqslant 0)$$

of $\mathscr{A}$ by $\mathbf{K}'$-equivariant coherent sheaves on $\mathfrak{p}'$ such that for any $1 \leqslant l \leqslant f$, the space of global sections of $\mathscr{A}^l := \mathscr{A}_l/\mathscr{A}_{l-1}$ is an irreducible $(\mathbb{C}[\overline{\mathscr{O}'_l}], \mathbf{K}')$-module for a nilpotent $\mathbf{K}'$-orbit $\mathscr{O}'_l$ in $\overline{\mathcal{O}'} \cap \mathfrak{p}'$.

Let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$. If $\mathscr{O}$ does not admit a generalized descent, then

$$\mathscr{O} \cap M(\mathcal{X}) = \varnothing,$$

and $\mathscr{O}$ does not appear in the support of $\mathrm{Ch}_{\mathcal{O}}(\check{\Theta}(\pi))$.

Now suppose that $\mathscr{O}$ admits a generalized descent $\mathscr{O}' := \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Retain the notation in Section 12.9.2 where an $(\epsilon', \dot{\epsilon}')$-space decomposition $\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$ and an element

$$T \in \mathcal{X}_1^{\circ} := \{\, w \in \mathrm{Hom}(\,\mathbf{V},\, \mathbf{V}'_1) \mid w \text{ is surjective} \,\} \cap \mathcal{X} \subseteq \mathcal{X}^{\mathrm{gen}}$$

is fixed such that $X := M(T) \in \mathscr{O}$ and $X' := M'(T) \in \mathscr{O}'$.

For $1 \leqslant l \leqslant f$, if $\mathscr{O}'_l = \mathscr{O}'$, then Lemma B.2 and Lemma B.4 ensure that there exists a Zariski open set U in $\mathfrak{p}$ such that $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$ and the quasi-coherent sheaf $\mathcal{L}(\mathscr{A}^l)|_U$ descends to a quasi-coherent sheaf on the closed subvariety $\mathscr{O}$ of U. (Thus $\mathcal{L}(\mathscr{A}^l)$ is generically reduced around X in the sense of [83, Proposition 2.9].)

Claim. *Assume that $\mathscr{O}'_l = \mathscr{O}'$ ($1 \leqslant l \leqslant f$). Then we have the following isomorphism of algebraic representations of $\mathbf{K}_X$:*

$$i_X^*(\mathcal{L}(\mathscr{A}^l)) \cong (i_{X'}^*(\mathscr{A}^l))^{\mathbf{K}'_2} \circ \alpha_1.$$

Here α_1 is defined in (12.29)*, $i_X\colon \{\, X \,\} \hookrightarrow \mathfrak{p}$ and $i_{X'}\colon \{\, X' \,\} \hookrightarrow \mathfrak{p}'$ are the inclusion maps, and i_X^* and $i_{X'}^*$ are the associated pull-back functors of the quasi-coherent sheaves.*

Let $\mathrm{m}_{\mathfrak{p}}(X)$ be the maximal ideal of $\mathbb{C}[\mathfrak{p}]$ associated to X, and $\kappa(X) := \mathbb{C}[\mathfrak{p}]/\mathrm{m}_{\mathfrak{p}}(X)$ be the residual field at X. Let

$$Z_{X,\overline{\mathscr{O}'}} := \left\{\, w \in \mathcal{X} \;\middle|\; M(w) = X, M'(w) \in \overline{\mathscr{O}'} \,\right\}.$$

Then $T \in Z_{X,\overline{\mathscr{O}'}}$ and

$$\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}] = \mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}]\otimes\mathbb{C}[\mathfrak{p}']} (\kappa(X) \otimes \mathbb{C}[\overline{\mathscr{O}'}])$$

by Lemma B.1 and Lemma B.3.

Let A^l be the module of global sections of $\mathscr{A}^l$. We have

$$\begin{aligned} i_X^*(\mathcal{L}(\mathscr{A}^l)) &= \kappa(X) \otimes_{\mathbb{C}[\mathfrak{p}]} \left(\mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}']} \mathbb{C}[\overline{\mathscr{O}'}] \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l\right)^{\mathbf{K}'} \\ &= \left(\mathbb{C}[\mathcal{X}] \otimes_{\mathbb{C}[\mathfrak{p}]\otimes\mathbb{C}[\mathfrak{p}']} (\kappa(X) \otimes \mathbb{C}[\overline{\mathscr{O}'}]) \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l\right)^{\mathbf{K}'} \\ &= (\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}] \otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l)^{\mathbf{K}'}. \end{aligned}$$

Let $\rho' := i_{X'}^*(\mathscr{A}^l)$ be the isotropy representation of $\mathscr{A}^l$ at X'. Since $Z_{X,\overline{\mathscr{O}'}}$ is the $\mathbf{K}' \times \mathbf{K}_X$-orbit of T by Lemma B.1 (iii) and Lemma B.3 (ii), by considering the diagram (*cf.* [53, Section 4.2])

$$\begin{tikzcd} \{\, X' \,\} \arrow[d, hook] & \{\, T \,\} \arrow[l] \arrow[d, hook] \arrow[dr] & \\ \overline{\mathscr{O}'} & Z_{X,\overline{\mathscr{O}'}} \arrow[l, "M'"'] \arrow[r, "M"] & \{\, X \,\}, \end{tikzcd}$$

we know that $\mathbb{C}[Z_{X,\overline{\mathscr{O}'}}]\otimes_{\mathbb{C}[\overline{\mathscr{O}'}]} A^l$ is isomorphic to the algebraically induced representation $\mathrm{Ind}_{\mathbf{S}_T}^{\mathbf{K}'\times\mathbf{K}_X}\rho'$, where $\mathbf{S}_T$ is the stabilizer group as in (12.30), and ρ' is viewed as an $\mathbf{S}_T$-module via the natural projection $\mathbf{S}_T\to\mathbf{K}'_{X'}$. Now by Frobenius reciprocity, we have

$$i_X^*(\mathcal{L}(\mathscr{A}^l))\cong(\mathrm{Ind}_{\mathbf{S}_T}^{\mathbf{K}'\times\mathbf{K}_X}\rho')^{\mathbf{K}'}=(\mathrm{Ind}_{\mathbf{K}'_2\times(\mathbf{K}'_{1,X'_1}\times_{\alpha_1}\mathbf{K}_X)}^{\mathbf{K}'\times\mathbf{K}_X}\rho')^{\mathbf{K}'}=(\rho')^{\mathbf{K}'_2}\circ\alpha_1.$$

This finishes the proof of the claim.

On the other hand, if $\mathscr{O}'_l\neq\mathscr{O}'$, then (12.21) implies that

$$\mathscr{O}\cap M(M'^{-1}(\overline{\mathscr{O}'_l}))=\varnothing,$$

and hence $\mathscr{O}$ is not contained in $\mathrm{Supp}(\mathcal{L}(\mathscr{A}^l))$. In conlcusion, the following inequality holds in the Grothedieck group of the category of algebraic representations of $\mathbf{K}_X$:

$$\chi(X,\mathscr{B})\leq\chi(X,\mathcal{L}(\mathscr{A}))\leq\sum_{l=1}^{f}\chi(X,\mathcal{L}(\mathscr{A}^l))=\sum_{l=1}^{f}(i_{X'}^*(\mathscr{A}^l))^{\mathbf{K}'_2}\circ\alpha_1=\chi(X',\mathscr{A})^{\mathbf{K}'_2}\circ\alpha_1,$$

where $\chi(X,\,\cdot\,)$ indicates the virtual character of $\mathbf{K}_X$ attached to a $\mathbf{K}$-equivariant coherent sheaf on $\mathfrak{p}$ whose support is contained in $\overline{\mathcal{O}}\cap\mathfrak{p}$, and similarly for $\chi(X',\,\cdot\,)$ (cf. [83, Definition 2.12]). This finishes the proof of the theorem. $\square$

7. A proof of Theorem 3.9

.

8. Nilpotent orbits: combinatorics

In this section, we describes our combinatorical parameterization of nilpotent orbits and the local systems in details.

8.1. **Young diagrams, signed Young diagrams and admissible orbit data.** The set of $\mathrm{Nil}(\mathfrak{g})$ nilpotent $G_{\mathbb{C}}$-orbits in $\mathfrak{g}$ are parameterized by Young diagrams, see [23, Section 5.1]. The nilpotent obit $G_{\mathbb{C}}\cdot\mathbf{e}$ generated by a nilpotent element $\mathbf{e}$ in $\mathfrak{g}$ is attached to the Young diagram $\mathcal{O}$ such that such that

$$\mathbf{c}_i(\mathcal{O})=\dim(\mathrm{Ker}(\mathbf{e}^{i+1})/\mathrm{Ker}(\mathbf{e}^i)),\quad\text{for all }\ i\in\mathbb{N}^+$$

By abuse of notaition, we identify $\mathcal{O}$ with the orbit $G_{\mathbb{C}}\cdot\mathbf{e}$.

Let $\mathfrak{sl}_2(\mathbb{C})$ be the complex Lie algebra consisting of 2×2 complex matrices of trace zero,

$$\mathring{\mathbf{e}}:=\begin{pmatrix}1/2 & \sqrt{-1}/2\\ \sqrt{-\mathbf{1}}/2 & -1/2\end{pmatrix},\quad\text{and}\quad\mathring{\mathbb{e}}:=\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

By the Jacobson-Morozov theorem, there is a Lie algebra homomorphism

$$\varphi\colon\mathfrak{sl}_2(\mathbb{C})\to\mathfrak{g}\subset\mathfrak{gl}(V)$$

such that $\varphi(\mathring{\mathbf{e}})=\mathbf{e}$.

Let $\mathrm{St}_i=\mathrm{Sym}^{i-1}(\mathbb{C}^2)$ denote the realization of the irreducible $\mathfrak{sl}_2(\mathbb{C})$-module φ_i on the $i-1$-symmetric power of the standard representation $\mathbb{C}^2$.

As an $\mathfrak{sl}_2(\mathbb{C})$-module via φ, we have

$$V=\bigoplus_{i\in\mathbb{N}^+}V_i\otimes_{\mathbb{C}}\mathrm{St}_i,\tag{8.1}$$

and

$${}^iV:=\mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathrm{St}_i,V)$$

is the multiplicity space and $\dim {}^iV = \mathbf{c}_i(\mathcal{O}) - \mathbf{c}_{i-1}(\mathcal{O})$. We view $\mathcal{O}$ as the function

$$\mathcal{O}\colon \mathbf{N}^+ \to \mathbb{N} \text{ such that } \mathcal{O}(i) = \dim {}^iV.$$

We equip St_i with a "standard" classical space structure $(\mathrm{St}_i, \langle\,,\,\rangle_{\mathrm{St}_i}, J_{\mathrm{St}_i}, L_{\mathrm{St}_i})$.Here

- the classical signature of St_i is $(B, (i+1)/2, (i-1)/2)$ if i is odd and $(C, i/2, i/2)$ if i is even,
- J_{St_i} is the complex conjugation on $\mathrm{St}_i = \mathrm{Sym}^{i-1}(\mathbb{C}^2)$,
- $L_{\mathrm{St}_i} = (-1)^{\lfloor \frac{i-1}{2} \rfloor} \mathrm{Sym}^{i-1}(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$,
- $\langle\,,\,\rangle_{\mathrm{St}_i}$ is uniquely determined by J_{St_i} and L_{St_i} up to a positive scalar (We fix a form $\langle\,,\,\rangle_{\mathrm{St}_i}$ for each i).

L_{St_i} should be the Lie group element action.

We define a equivalent relation on the set $\{ B, C, \widetilde{C}, D, C^*, D^* \}$ with the relation $B \sim D$ and $C \sim \widetilde{C}$.

Let $(V, \langle\,,\,\rangle J, L)$ be a classical group with signature s. Let $\mathscr{O} \in \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$. By [75] (also see [83, Section 6]), there is a Lie-algebra homomorphism $\varphi\colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g} \subset \mathfrak{gl}(V)$ such that $\varphi(\mathring{\mathbf{e}}) \in \mathscr{O}$ and compatible with the classical space structure on V. More precisely, the multiplicity space ${}^iV$ have a unique classical space structure $({}^iV, \langle\,,\,\rangle_{{}^iV}, J_i, L_i)$ with classical signature $\mathsf{s}_i = (\star_i, p_i, q_i)$ such that $\star_i \neq \widetilde{C}$ and restricted on the $\mathfrak{sl}_2(\mathbb{C})$-isotypic component ${}^iV \otimes \mathrm{St}_i$,

- $\langle\,,\,\rangle$ is given by $\langle\,,\,\rangle_{{}^iV} \otimes \langle\,,\,\rangle_{\mathrm{St}_i}$,
- J is given by $J_i \otimes J_{\mathrm{St}_i}$,
- L is given by $L_i \otimes L_{\mathrm{St}_i}$,
- the symbol $\star_i$ satisfies the following condition

$$\begin{cases} \star_i \sim \star_{\mathsf{s}} & \text{if } i \text{ is odd} \\ \star_i \sim \star'_{\mathsf{s}} & \text{if } i \text{ is even} \end{cases} \tag{8.2}$$

Here $\star'_{\mathsf{s}}$ is the Howe dual of $\star_{\mathsf{s}}$, and $\sim$ is the equivalent relation on $\{ B, C, \widetilde{C}, C^*, D, D^* \}$ such that $B \sim D$ and $C \sim \widetilde{C}$. Then the classical signature s_i are independent of the choice of φ and uniquely determined the orbit $\mathscr{O}$.

Let

$$\overline{\mathsf{CS}} := \{ (\star, p, q) \mid \star \neq \widetilde{C} \text{ and } (\star, p, q) \text{ is a classical signature} \}.$$

We identify $\mathscr{O}$ with the function

$$\mathscr{O}\colon \mathbb{N}^+ \to \overline{\mathsf{CS}}, \qquad i \mapsto \mathsf{s}_i.$$

A important feature of a compatible φ map is that the Kostant-Sekiguchi correspondence is given by

$$G_{\mathsf{s}} \cdot \varphi(\mathring{\mathbb{e}}) \leftrightarrow K_{\mathsf{s},\mathbb{C}} \cdot \varphi(\mathring{\mathbf{e}}).$$

Let $\mathsf{m}\colon \overline{\mathsf{CS}} \to \mathbb{N}$ be the map given by $(\star, p, q) \mapsto p+q$. Now the complexification $G_{\mathbb{C}} \cdot \mathscr{O}$ of a rational nilpotent orbit is given by $\mathcal{O} := \mathsf{m} \circ \mathscr{O}$. Define

$$\mathsf{SYD}_\star(\mathcal{O}) := \left\{ \mathscr{O}\colon \mathbb{N}^+ \to \overline{\mathsf{CS}} \;\middle|\; \begin{array}{l} \star_{\mathscr{O}(i)} \text{ satisfies (8.2)}, \\ \mathsf{m} \circ \mathscr{O} = \mathcal{O} \end{array} \right\}$$

$$\mathsf{SYD}_\star := \bigsqcup_{\mathcal{O} \in \mathrm{Nil}_\star} \mathsf{SYD}_\star(\mathcal{O}).$$

Now $\mathsf{SYD}_\star$ is naturally identified with the set $\bigsqcup_{\mathsf{s}} \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ (the set of real nilpotent orbits) and $\bigsqcup_{\mathsf{s}} \mathrm{Nil}(\mathfrak{p}_{\mathsf{s}})$. This is nothing but the signed Young diagram classification of rational nilpotent orbits, see [23].

For any $\mathscr{O} \in \mathsf{SYD}_\star$, let $\mathrm{Sign}(\mathscr{O}) = (p, q)$ if $\mathscr{F} \circ \mathscr{O}$ corresponds to a nilpotent orbit in $\mathrm{Nil}_{\mathsf{s}}(\mathfrak{p})$ with the classical signature $\mathsf{s} = (\star, p, q)$. When $\mathscr{O}(i) = (\mathsf{s}_i, p_i, q_i)$, the signature is computed by the following formula.

$$\begin{aligned}\mathrm{Sign}(\mathscr{O}) = \sum_{i \in \mathbb{N}^+} &\big(i \cdot p_{2i} + i \cdot q_{2i}, i \cdot p_{2i} + i \cdot q_{2i}\big) \\ &+ \sum_{i \in \mathbb{N}^+} \big(i \cdot p_{2i-1} + (i-1) \cdot q_{2i-1}, (i-1) \cdot p_{2i-1} + i \cdot q_{2i-1}\big).\end{aligned}$$

We write

$$\mathsf{SYD}_{\mathsf{s}}(\mathcal{O}) = \{\, \mathscr{O} \in \mathsf{SYD}_\star(\mathcal{O}) \mid \mathrm{Sign}(\mathscr{O}) = (p, q) \,\}.$$

which is identified with $\mathrm{Nil}_{\mathsf{s}}(\mathcal{O})$.

For a classical signature s, we define the complex group

$$K_{[\mathsf{s}],\mathbb{C}} := G_{\mathbb{C}}^{L_{\mathsf{s}}}.$$

Suppose $\mathscr{O} \in \mathrm{Nil}(\mathbf{p}_{\mathsf{s}})$, φ is compatible with $\mathscr{O}$ and $\mathbf{e} = \varphi(\mathring{\mathbf{e}}) \in \mathscr{O}$. Let

$$(K_{[\mathsf{s}],\mathbb{C}})_{\mathbf{e}} := \mathrm{Stab}_{K_{[\mathsf{s}],\mathbb{C}}}(\mathbf{e})$$

be the stabilizer of $\mathbf{e}$ in $K_{[\mathsf{s}],\mathbb{C}}$. Using the signed Young diagram classification, we have the following well known fact of the isotropic subgroup.

Lemma 8.1. *Then $(K_{[\mathsf{s}],\mathbb{C}})_{\mathbf{e}} = R_{\mathbf{e}} \ltimes U_{\mathbf{e}}$, where $U_{\mathbf{e}}$ is the unipotent radical of $(K_{[\mathsf{s}],\mathbb{C}})_{\mathbf{e}}$ and $R_{\mathbf{e}}$ is a reductive group canonically identified with*

$$\prod_{i \in \mathbb{N}^+} K_{[\mathsf{s}_{\mathscr{O}(i)}],\mathbb{C}}.$$

As a consequence, $R_{\mathbf{e}}$ is a products of complex general linear group and orthogonal groups

$$\prod_{\substack{\mathscr{O}(i) = (\star_i, p_i, q_i) \\ \star_i \in \{B, D\}, p_i + q_i > 0}} \mathrm{O}_{p_i}(\mathbb{C}) \times \mathrm{O}_{q_i}(\mathbb{C}).$$

Fix a one-dimensional character χ of the connected component $K^\circ_{\mathbf{e},\mathbb{C}}$ of $K_{\mathbf{e},\mathbb{C}}$, let

$$\mathsf{LB}_\chi(\mathscr{O}) := \{\, \text{line bundle } \mathcal{E} \text{ on } \mathscr{O} \text{ such that } K^\circ_{\mathbf{e},\mathbb{C}} \text{ acts on the fiber } \mathcal{E}_x \text{ by } \chi \,\}.$$

By the structure of $R_{\mathbf{e}}$, a line bundle $\mathcal{E} \in DCO\chi$ is completely determined by the restruction of the $K_{\mathbf{e},\mathbb{C}}$-module $\mathcal{E}_{\mathbf{e}}$ on the the orthogonal factors of $R_{\mathbf{e}}$.

Let

$$\mathsf{MK} = \overline{\mathsf{CS}} \cup \{\, (\star, r, s) \mid \star \in \{B, D\}, r, s \in \mathbb{Z} \,\}.$$

We identify $\mathcal{E} \in \mathsf{LB}_\chi(\mathscr{O})$ with the map

$$\mathcal{E} \colon \mathbb{N}^+ \to \mathsf{MK}$$

such that, for $\mathscr{O}(i) = (\star_i, p_i q_i)$,

- $\mathcal{E}(i) := \mathscr{O}(i)$ if $\star_i \notin \{B, D\}$;
- $\mathcal{E}(i) := (\star_i, (-1)^{\epsilon_i^+} p_i, (-1)^{\epsilon_i^-} q_i)$ if $\star_i \in \{B, D\}$, and the factor $K_{[\mathcal{E}(i)],\mathbb{C}} = \mathrm{O}_{p_i}(\mathbb{C}) \times \mathrm{O}_{q_i}(\mathbb{C})$ acts by $\det^{\epsilon_i^+} \otimes \det^{\epsilon_i^-}$ on

$$\begin{cases} \mathcal{E}, & \text{when } \star = D, \\ \mathcal{E} \otimes \det_{\sqrt{-1}}^{-1}, & \text{when } \star = B. \end{cases}$$

.

Let $\mathscr{F}\colon \mathsf{MK} \to \overline{\mathsf{CS}}$ be the map given by $(\star, r, s) \mapsto (\star, |r|, |s|)$ and

$$\mathsf{MYD}(\mathscr{O}) := \left\{ \mathcal{E}\colon \mathbb{N}^+ \to \mathsf{MK} \;\middle|\; \mathscr{F} \circ \mathcal{E} = \mathscr{O} \right\}.$$

Thanks to the simple structure of $R_{\mathbf{e}}$, the above recipe allows us to identify $\mathsf{MYD}(\mathscr{O})$ with the set $\mathsf{LB}_\chi(\mathscr{O})$ when the later set is non-empty.

We wreite $\mathrm{Sign}(\mathcal{E}) := \mathrm{Sign}$

Let

$$\mathsf{MYD}_{\mathsf{s}}(\mathcal{O}) := \bigcup_{\mathscr{O} \in \mathrm{Nil}_{\mathsf{s}}(\mathcal{O})} \mathsf{MYD}(\mathscr{O}), \quad \text{and} \quad \mathsf{MYD}_\star(\mathcal{O}) := \bigcup_{\mathscr{O} \in \mathrm{Nil}_\star(\mathcal{O})} \mathsf{MYD}(\mathscr{O})$$

In particular, we identify $\mathrm{AOD}_{\mathsf{s}}(\mathcal{O})$ with a subset of $\mathsf{MYD}_\star(\mathcal{O})$, see Definition 1.9.

Set

$$\mathsf{MYD}_\star := \bigcup_{\mathcal{O} \in \mathrm{Nil}_\star} \mathsf{MYD}_\star(\mathcal{O}),$$

In the next section, we will define some operations on the free abelian groups $\mathbb{Z}[\mathsf{SYD}_\star]$ and $\mathbb{Z}[\mathsf{MYD}_\star]$.

Suppose $\mathcal{E} \in \mathsf{MYD}_\star$ such that $\mathcal{E}(i) = (\mathsf{s}_i, p_i, q_j)$. To ease the notation, we will use

$$(i_1^{(p_{i_1}, q_{i_1})} i_2^{(p_{i_2}, q_{i_2})} \cdots i_k^{(p_{i_k}, q_{i_k})})_\star$$

to represent $\mathcal{E}$ where $i_1, \cdots, i_k$ are all the indices such that $(p_i, q_i) \neq (0, 0)$. A Young diagram in $\mathsf{YD}_\star$ is represented similarly.

We adopt the usual convention to draw a singed Young diagram where the signature (p_i, q_i) equals to the total signature for the most right parts of all length i rows. We use "$\ast$" and "$=$" instants "$+$" and "$-$" to mark the rows when p_i or q_i are negative.

Example 8.2. *The expression $(1^{(r,s)})_\star$ represents the marked Young diagram in $\mathsf{MYD}_\star$ such that*

$$(1^{(r,s)})_\star(i) := \begin{cases} (\star, r, s) & \text{if } i = 1 \\ (\star_i, 0, 0) & \text{if } i > 0 \end{cases}$$

Here the symbols $\star_i$ are uniquely determined by (8.2).

The marked Young diagram $\mathcal{E} := (1^{(1,-1)} 2^{(2,0)} 3^{(0,-1)})_B$ is drew as the following.

$$\begin{array}{|c|c|c|} \hline = & \ast & = \\ \hline - & + \\ \cline{1-2} - & + \\ \cline{1-2} + \\ \cline{1-1} = \\ \cline{1-1} \end{array}$$

The singed Young diagram $\mathscr{O} := \mathscr{F} \circ \mathcal{E}$ is $(1^{(1,1)} 2^{(2,0)} 3^{(0,1)})_B$ and the complexification of $\mathscr{O}$ is $(1^2\, 2^2\, 3^1)_B$.

8.2. **Operations on diagrams.** In this section, we define some operations on the signed Young diagrams and marked Young diagram.

Suppose $\star \in \{B, D\}$. We define a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on $\mathsf{MYD}_\star$. Let $(\epsilon^+, \epsilon^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathcal{E} \in \mathsf{MYD}_\star$ such that $(\star_i, p_i, q_i) := \mathcal{E}(i)$, we define $\mathcal{E} \otimes (\epsilon^+, \epsilon^-) \in \mathsf{MYD}_\star$ by

$$(\mathcal{E} \otimes (\epsilon^+, \epsilon^-))(i) := \begin{cases} (\star_i, (-1)^{\frac{\epsilon^+(i+1)+\epsilon^-(i-1)}{2}} p_i, (-1)^{\frac{\epsilon^+(i-1)+\epsilon^-(i+1)}{2}} q_i) & \text{if } i \text{ is odd,} \\ (\star_i, p_i, q_i) & \text{otherwise.} \end{cases}$$

For each $\mathcal{E} \in \mathrm{AOD}_\star$, its twist $\mathcal{E} \otimes (1^{-,+})^{\epsilon^+} \otimes (1^{+,1})^{\epsilon^-}$ is also in $\mathrm{AOD}_\star$. As elements in $\mathsf{MYD}_\star$, we have

$$\mathcal{E} \otimes (1^{-,+})^{\epsilon^+} \otimes (1^{+,-})^{\epsilon^-} = \mathcal{E} \otimes (\epsilon^+, \epsilon^-).$$

Suppose $\star \in \{ C, \widetilde{C}, D^* \}$.

For $\mathcal{E} \in \mathsf{MYD}_\star$ such that $(\star_i, p_i, q_i) := \mathcal{E}(i)$, we define $\maltese\mathcal{E} \in \mathsf{MYD}_\star$ by

$$(\maltese\mathcal{E})(i) := \begin{cases} (\star_i, -p_i, -q_i) & \text{if } i \equiv 2 \pmod 4 \text{ and } \star_i \in \{ B, D \}, \\ (\star_i, p_i, q_i) & \text{otherwise.} \end{cases}$$

Clearly, the a operation $\maltese$ defines an involution on $\mathsf{MYD}_\star$. For each $\mathcal{E} \in \mathsf{LB}_\chi(\mathscr{O})$, its twist $\mathcal{E} \otimes \det^{-1}_{\sqrt{-1}}$ is a line bundle in $\mathsf{LB}_{\chi'}(\mathscr{O})$ where χ' is a certain character. As elements in $\mathsf{MYD}_\star$, we have

$$\mathcal{E} \otimes \det_{\sqrt{-1}} = \maltese(\mathcal{E}).$$

We define the argumentation of a marked Young diagram. Suppose that $(p_0, q_0) \in \mathbb{Z} \times \mathbb{Z}$ and there is a $\star'_0 \sim \star'$ such that $\mathsf{s}_0 := (\star'_0, p_0, q_0) \in \overline{\mathsf{CS}}$. with $\star'_0 \sim \star'$. For each $\mathcal{E} \in \mathsf{MYD}_\star$, let $\mathcal{E} \cdot (p_0, q_0)$ be the marked Young diagram in $\mathsf{MYD}_{\star'}$ given by

$$(\mathcal{E} \cdot (p_0, q_0))(i) := \begin{cases} \mathsf{s}_0 & \text{if } i = 1, \\ \mathcal{E}(i-1) & \text{if } i > 1. \end{cases}$$

We will use the same notation to denote the natural extension of above operations to $\mathbb{Z}[\mathsf{MYD}_\star]$.

We define the partial order $\succeq$ on $\mathbb{Z} \times \mathbb{Z}$ by

$$(a, b) \succeq (c, d) \Leftrightarrow \begin{cases} a \geqslant c \geqslant 0 \\ b \geqslant d \geqslant 0 \end{cases} \quad \text{or} \quad \begin{cases} a \leqslant c \leqslant 0 \\ b \leqslant d \leqslant 0 \end{cases}.$$

For a $\mathcal{E} \in \mathsf{MYD}$ and $(p_0, q_0) \in \mathbb{Z} \times \mathbb{Z}$, we write

$$\mathcal{E} \sqsupseteq (p_0, q_0) \Leftrightarrow (p_1, q_1) \succeq (p_0, q_0) \text{ where } \mathcal{E}(1) = (\star_1, p_1, q_1).$$

Morever, we define an involution on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b) \mapsto (a, b)^- := (b, a)$.

Suppose $\mathcal{E} \sqsupseteq (p_0, q_0)$, we define the truncation of $\mathcal{E}$ by

$$\Lambda_{(p_0,q_0)}(\mathcal{E}) = \begin{cases} (\star_1, p_1 - p_0, q_1 - q_0) & \text{if } i = 1, \\ \mathcal{E}(i) & \text{if } i > 1. \end{cases}$$

We extend the truncation operation $\Lambda_{(p_0,q_0)}$ to $\mathbb{Z}[\mathsf{MYD}_\star]$ by setting $\Lambda_{p_0,q_0}(\mathcal{E}) := 0$ when $(p_0, q_0) \not\sqsubseteq \mathcal{E} \in \mathsf{MYD}_\star$. Note that $\Lambda_{(0,0)}$ is the identity map.

The augmentation and truncation of signed Young diagram are defined by the same formula.

8.3. **Theta lifts of local systems.** Let $\mathcal{O} \in \mathrm{Nil}_\star$ and $\mathcal{O}' \in \mathrm{Nil}_{\star'}$ such that $\mathcal{O}'$ is a regular descent of $\mathcal{O}'$. Let s and s' be two fixed classical signature, the theta lifting of marked Young diagrams is a homomorphism between

$$\vartheta_{\mathsf{s}',\mathsf{s}} \colon \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}'}(\mathcal{O}')] \to \mathbb{Z}[\mathsf{MYD}_{\mathsf{s}}(\mathcal{O})]$$

It suffice to define $\vartheta_{\mathsf{s}',\mathsf{s}}(\mathcal{E}')$ for each $\mathcal{E}' \in \mathsf{MYD}_{\mathsf{s}'}$ case by case. We set

$$t := \mathbf{c}_1(\mathcal{O}') - \mathbf{c}_2(\mathcal{O}).$$

Suppose $\mathsf{s}' = (\star', n, n)$ and $\mathsf{s} = (\star, p, q)$ such that $\star' \in \{ C, \widetilde{C}, D^* \}$ and $\star \in \{ D, B, C^* \}$. In this case, t is always an even integer. We define

$$\vartheta_{\mathsf{s}',\mathsf{s}}(\mathcal{E}') := \begin{cases} \left(✠^{\frac{p-q}{2}} (\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}')) \right) \cdot (p_0, q_0) & \text{if } (p_0, q_0) \succeq (0,0) \text{ and } \star = D, \\ \left(✠^{\frac{p-q+1}{2}} (\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}')) \right) \cdot (p_0, q_0) & \text{if } (p_0, q_0) \succeq (0,0) \text{ and } \star = B, \\ \left(\Lambda_{(\frac{t}{2},\frac{t}{2})}(\mathcal{E}') \right) \cdot (p_0, q_0) & \text{if } (p_0, q_0) \succeq (0,0) \text{ and } \star = C^*, \\ 0 & \text{otherwise} \end{cases} \tag{8.3}$$

where $(p_0, q_0) := (p,q) - (n,n) - {}^l\mathrm{Sign}(\mathcal{E}')^\sharp + (\frac{t}{2}, \frac{t}{2})$.

Suppose $\mathsf{s}' = (\star', p, q)$ and $\mathsf{s} = (\star, n, n)$ such that $\star' \in \{ D, B, C^* \}$ and $\star \in \{ C, \widetilde{C}, D^* \}$. We define

$$\vartheta_{\mathsf{s}',\mathsf{s}}(\mathcal{E}') := \begin{cases} ✠^{\frac{p-q}{2}} \left(\sum_{j=0}^{t} \Lambda_{(j,t-j)}(\mathcal{E}') \cdot (n_0, n_0) \right) & \text{if } \star = C, \\ ✠^{\frac{p-q-1}{2}} \left(\sum_{j=0}^{t} \Lambda_{(j,t-j)}(\mathcal{E}') \cdot (n_0, n_0) \right) & \text{if } \star = \widetilde{C}, \\ \sum_{j=0}^{\frac{t}{2}} \Lambda_{(2j,t-2j)}(\mathcal{E}') \cdot (n_0, n_0) & \text{if } \star = D^*, \end{cases} \tag{8.4}$$

where $2n_0 = \mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O})$.

9. Properties of painted bipartitions

9.1. **Vanishing propotition.** Because of the following proposition, we assume in the rest of this paper that $\check{\mathcal{O}}$ is quasi-distinghuished when $\star \in \{C^*, D^*\}$.

Proposition 9.1. *Suppose that $\star \in \{C^*, D^*\}$. If the set $\mathrm{PBP}_\star(\check{\mathcal{O}})$ is nonempty, then $\check{\mathcal{O}}$ is quasi-distinguished.*

Proof. Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. If $\star = C^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_i(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots.$$

This forces that $\check{\mathcal{O}}$ is quasi-distinguished.

If $\star = D^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_{i+1}(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots.$$

This also forces that $\check{\mathcal{O}}$ is quasi-distinguished. □

9.2. **Tails of painted bipartitions.** In the rest of this subsection, we assume that $\star \in \{B, D, C^*\}$. Let $(\imath, \jmath) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}}))$. Put

$$\star_{\mathbf{t}} := \begin{cases} D, & \text{if } \star \in \{B, D\}; \\ C^*, & \text{if } \star = C^*. \end{cases} \quad \text{and} \quad k := \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} + 1 & \text{if } \star \in \{B, D\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} - 1 & \text{if } \star = C^*. \end{cases}$$

Here $k = \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, $\mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath)$, and $\mathbf{c}_1(\imath) - \mathbf{c}_1(\jmath)$ when $\star = B, C^*, D^*$, respectively.

Let $\check{\mathcal{O}}_{\mathbf{t}}$ be the following Young diagram that is determined by the pair $(\star, \check{\mathcal{O}})$.

- If $\star \in \{ B, D \}$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of two rows with lengths $2k - 1$ and 1.
- If $\star = C^*$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of one row with length $2k + 1$.

Note that in all these three cases $\check{\mathcal{O}}_{\mathbf{t}}$ has $\star_{\mathbf{t}}$-good parity and every element in $\mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ has the form

$$\text{(9.1)}\qquad \begin{array}{|c|}\hline x_1\\\hline x_2\\\hline \vdots\\ \vdots\\\hline x_k\\\hline\end{array}\times\varnothing\times D,\qquad \begin{array}{|c|}\hline x_1\\\hline x_2\\\hline \vdots\\ \vdots\\\hline x_k\\\hline\end{array}\times\varnothing\times D\qquad\text{or}\qquad \varnothing\times\begin{array}{|c|}\hline x_1\\\hline x_2\\\hline \vdots\\ \vdots\\\hline x_k\\\hline\end{array}\times C^*,$$

respectively if $\star = B, D$ or C^*.

Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ be as before.

The case when $\star = B$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the first painted bipartition in (12.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{\, \mathcal{Q}(j,1) \mid \mathbf{c}_1(\imath) + 1 \leqslant j \leqslant \mathbf{c}_1(\jmath) \,\}$$

with the set

$$\begin{cases} \{\, c \,\}, & \text{if } \alpha = B^+, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{\, \bullet, s \,\};\\ \{\, s \,\}, & \text{if } \alpha = B^-, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{\, \bullet, s \,\};\\ \{\, \mathcal{Q}(\mathbf{c}_1(\imath), 1) \,\}, & \text{if } \mathbf{c}_1(\imath) > 0 \text{ and } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{r, d\}. \end{cases}$$

The case when $\star = D$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the second painted bipartition in (12.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{\, \mathcal{P}(j,1) \mid \mathbf{c}_1(\jmath) + 2 \leqslant j \leqslant \mathbf{c}_1(\imath) \,\}$$

with the set

$$\begin{cases} \{\, c \,\}, & \begin{array}{l}\text{if } \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}),\\ \quad (\mathcal{P}(\mathbf{c}_1(\jmath)+1, 1), \mathcal{P}(\mathbf{c}_1(\jmath)+1, 2)) = (r, c)\\ \quad \text{and } \mathcal{P}(l, 1) \in \{\, r, d \,\};\end{array}\\ \{\, \mathcal{P}(\mathbf{c}_1(\jmath)+1, 1) \,\}, & \text{otherwise.} \end{cases}$$

The case $\star = C^*$. When $k = 0$, we define the tail $\tau_{\mathbf{t}}$ of τ to be $\varnothing \times \varnothing \times C^*$. When $k > 0$, we define the tail $\tau_{\mathbf{t}}$ of τ to be the third painted bipartition in (12.1) such that

$$(x_1, x_2, \cdots, x_k) = (\mathcal{Q}(\mathbf{c}_1(\imath)+1, 1), \mathcal{Q}(\mathbf{c}_1(\imath)+2, 1), \cdots, \mathcal{Q}(\mathbf{c}_1(\jmath), 1)).$$

When $\star \in \{\, B, D \,\}$, the symbol in the last box of the tail $\tau_{\mathbf{t}} \in \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ will be impotent for us. We write x_τ for it, namely

$$x_\tau := \mathcal{P}_{\tau_{\mathbf{t}}}(k, 1).$$

The following lemma is easy to check.

Lemma 9.2. *If $\star = B$, then*

$$x_\tau = s \Longleftrightarrow \begin{cases} \alpha = B^-;\\ \mathcal{Q}(\mathbf{c}_1(\jmath), 1) \in \{\bullet, s\}, \end{cases}$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{Q}(\mathbf{c}_1(\jmath), 1) = d.$$

If $\star = D$, then

$$x_\tau = s \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath), 1) = s,$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath), 1) = d.$$

□

9.3. **Some properties of the descent maps.** The key properties of the descent map when $\star \in \{ C, \widetilde{C}, D^* \}$ are summarized in the following proposition.

Proposition 9.3. *Suppose that $\star \in \{ C, \widetilde{C}, D^* \}$ and cosider the descent map*

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}'). \tag{9.2}$$

(a) If $\star = D^*$ or $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$, then the map (12.2) is bijective.

(b) If $\star \in \{C, \widetilde{C}\}$ and $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$, then the map (12.2) is injective and its image equals

$$\left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, x_{\tau'} \neq s \right\}.$$

Proof. We give the detailed proof of part (b) when $\star = \widetilde{C}$. The proofs in the other cases are similar and are left to the reader.

By the definition of descent map (see (2.1)), we have a well defined map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\}. \tag{9.3}$$

Suppose that τ' is an element in the codomain of the map (12.3). Similar to the proof of Lemma 2.1, there is a unique element in $\tau := \nabla^{-1}(\tau') \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ such that for all $i = 1, 2, \cdots, \mathbf{c}_1(\imath)$,

$$\mathcal{P}_\tau(i, 1) \in \{\bullet, s\},$$

and for all $(i, j) \in \mathrm{Box}(\tau')$,

$$\mathcal{P}_\tau(i, j+1) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}_{\tau'}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}_{\tau'}(i,j), & \text{if } \mathcal{P}_{\tau'}(i,j) \notin \{\bullet, s\}, \end{cases}$$

and

$$\mathcal{Q}_\tau(i, j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}_{\tau'}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}_{\tau'}(i,j), & \text{if } \mathcal{Q}_{\tau'}(i,j) \notin \{\bullet, s\}. \end{cases}$$

Note that τ is in the domain of (12.3). It is then routine to check that the map

$$\nabla^{-1} : \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\} \to \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq c \right\}$$

and the map (12.3) are inverse to each other. Hence the map (12.3) is bijective.

Similarly, the map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) = c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^-, \mathcal{Q}_{\tau'}(\mathbf{c}_1(\imath), 1) \in \{r, d\} \right\}$$

is well-defined, and we show that it is bijective by explicitly constructing its inverse. In view of Lemma 12.7, this proves the proposition in the case we are considering.

[Suppose $\star = \widetilde{C}$ and $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$. Then $\mathbf{c}_1(\imath) > \mathbf{c}_1(\jmath)$ and the "inverse" of descent is constructed in an obvious way such that $\mathcal{P}(\mathbf{c}_1(\imath), 1) = c \Leftrightarrow \alpha = B^-$.

Suppose $\star = C$ and $\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})$. Then $\mathbf{c}_1(\imath) = \mathbf{c}_1(\jmath) + 1$. So $\mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq s$. Hence $\mathcal{P}_{\tau'}(\mathbf{c}_1(\imath), 1) = \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq s$. The "inverse" of descent is constructed in an obvious way.

Suppose $\star = D^*$. Then $\mathbf{c}_1(\imath) \geqslant \mathbf{c}_1(\jmath) + 1$. The "inverse" of descent is constructed in an obvious way.] □

As in the Introduction, $\mathcal{O}$ denotes the Barbasch-Vogan dual of $\check{\mathcal{O}}$. The following proposition is easy to verify:

Lemma 9.4. *Suppose $k \in \mathbb{N}^+$, $\star \in \{ B, D, C^* \}$. Let $\check{\mathcal{O}}$ be the $\star$-good parity orbit such that $\mathbf{r}_3(\check{\mathcal{O}}) = 0$ and*

$$(\mathbf{r}_1(\check{\mathcal{O}}), \mathbf{r}_2(\check{\mathcal{O}})) = \begin{cases} (2k-2, 0) & \text{if } \star = B, \\ (2k-1, 1) & \text{if } \star = D, \\ (2k-1, 0) & \text{if } \star = C^*. \end{cases}$$

Then $\check{\mathcal{O}}$ is the regular nilpotent orbit and $\mathcal{O} = (1^{\mathbf{r}_1(\check{\mathcal{O}})+1})_\star$.

When $\star \in \{ B, D \}$ the following map is bijective:

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \longrightarrow & \{ (p,q,\varepsilon) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z} \mid \varepsilon = 1 \textit{ if } pq = 0, \textit{ and } p+q = |\mathcal{O}| \}, \\ \tau & \mapsto & (p_\tau, q_\tau, \varepsilon_\tau). \end{array}$$

When $\star = C^*$, the following map is bijective:

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \{ (p,q) \mid p,q \in 2\mathbb{N} \textit{ and } p+q = |\mathcal{O}| \}, \quad \tau \mapsto (p_\tau, q_\tau).$$

□

For every painted bipartition τ, write

$$\mathrm{Sign}(\tau) := (p_\tau, q_\tau).$$

When $\mathbf{r}_2(\check{\mathcal{O}}) > 0$, the double descent $\nabla^2(\tau) := \nabla(\nabla(\tau))$ is well-defined whenever $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$.

The key properties of the descent map when $\star \in \{ D, B, C^* \}$ are summarized in the following two propositions.

Lemma 9.5. *Assume that $\star \in \{ D, B, C^* \}$ and $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Write $\check{\mathcal{O}}'' := \check{\nabla}(\check{\mathcal{O}}')$ and consider the map*

$$\delta \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}), \qquad \tau \mapsto (\nabla^2(\tau), \tau_{\mathbf{t}}). \tag{9.4}$$

(a) Suppose that $\star = C^*$ or $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then the map (12.7) *is a bijective, and for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,*

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}), \mathbf{c}_2(\mathcal{O})) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{9.5}$$

(b) Suppose that $\star \in \{ B, D \}$ and $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$. Then the map (12.7) *is injection and its image equals*

$$\left\{ (\tau'', \tau_0) \in \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}}) \;\middle|\; \begin{array}{l} \textit{either } x_{\tau''} = d, \textit{ or} \\ x_{\tau''} \in \{ r, c \} \textit{ and } \mathcal{P}_{\tau_0}^{-1}(\{ s, c \}) \neq \varnothing \end{array} \right\}. \tag{9.6}$$

Moreover, for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{9.7}$$

Proof. We give the detailed proof of part (b) when $\star = B$. The proofs in the other cases are similar and are left to the reader. Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ and $\tau'' = \nabla^2(\tau)$.

According to the descent algorithm, we have

$$\begin{aligned} \mathcal{P}(i,j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{P}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 1; \end{cases} \\ \mathcal{Q}(i,j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{Q}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 2; \end{cases} \end{aligned} \tag{9.8}$$

We label the boxes which are not considered in (9.8) as the following:

$$\tau : \begin{array}{|c|}\hline x_0 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline x_1 & x'' \\ \hline x_2 \\ \cline{1-1} \vdots \\ \vdots \\ \cline{1-1} x_k \\ \cline{1-1} \end{array} \times \alpha \quad \mapsto \quad \tau'' : \varnothing \times \begin{array}{|c|}\hline y'' \\ \hline \end{array} \times \alpha'', \tag{9.9}$$

where $k := \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, x_0, x_1 and y'' has coordinate $(\mathbf{c}_1(\imath), 1)$ in the corresponding painted Young diagram. By the descent algorithm, $x'' = y''$. Also note that $\mathbf{c}_2(\mathcal{O}) = 2\mathbf{c}_1(\imath)$.

We now discuss case by case according to the symbol x_1. When $x_1 = s$, we have $(x_0, \alpha'') = (c, B^-)$. Now

$$\begin{aligned} &\mathrm{Sign}(\tau) - \mathrm{Sign}(\tau'') \\ &= (2\mathbf{c}_1(\imath) - 2, 2\mathbf{c}_1(\imath) - 2) + (1,1) + (0,2) - (0,1) + \mathrm{Sign}(\alpha) + \mathrm{Sign}(x_2 \cdots x_k) \\ &= (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\tau_{\mathbf{t}}). \end{aligned} \tag{9.10}$$

Here Sign counts the signature of various symbols in the same way of (1.3). In second line of the above formula, the terms count the signatures of extra bullets, x_0, x_1, α'', α and $x_1 \cdots x_k$ respectively.

When $x_1 = \bullet$, we have $(x_0, \alpha'') = (\bullet, B^+)$. When $x_1 \in \{ r, d \}$, we have $x'' = y'' = d$ $(x_0, \alpha'') = (c, \alpha)$. In these two cases, the signature formula can be checked similarly. It also clear that the image of δ is in (9.6).

It is not hard to construct the inverse of δ from (9.6) to $\mathrm{PBP}_\star(\check{\mathcal{O}})$, and then the proposition follows. To illustrate the idea, we give the detail construction of $\delta^{-1}(\tau'', \tau_0)$ when $\mathcal{P}_{\tau_0}(k,1) = c$. Most of the entries in $\mathcal{P}$ and $\mathcal{Q}$ are defined by (9.8). We refer to (9.9) for the labeling of the rest of the entries, and set $\alpha := B^-$, $x'' := \mathcal{Q}_{\tau''}(\mathbf{c}_1(\imath), 1)$,

$$(x_0, x_1) := \begin{cases} (\bullet, \bullet) & \text{if } \alpha'' = B^+, \\ (c, s) & \text{if } \alpha'' = B^-, \end{cases} \quad \text{and} \quad x_i := \mathcal{P}_{\tau_0}(i-1, 1) \quad \text{for } i = 2, 3, \cdots, k.$$

[In fact, it suffice to check the proposition for the orbit $\check{\mathcal{O}}$ consists of three rows with lengths $2k$, 2, and 2.]

[Suppose $\star = C^*$. Then τ is obtained by attaching a column of $\mathbf{c}_1(\imath)$ bullets on the left of $\mathcal{P}$, and a column of $\mathbf{c}_1(\imath)$ bullets concatenated with $\tau_{\mathbf{t}}$ on the left of $\mathcal{Q}$. The bijectivity is clear. The signature formula follows from $2\mathbf{c}_1(\imath) = \mathbf{c}_2(\mathcal{O})$.

Suppose $\star = D$ and $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then τ is obtained by attaching a column of $\mathbf{c}_1(\jmath)$ bullets concatenated with $\tau_{\mathbf{t}}$ on the left of $\mathcal{P}$, and a column of $\mathbf{c}_1(\jmath)$ bullets on the left of $\mathcal{Q}$. The bijectivity is clear. The signature formula follows from $2\mathbf{c}_1(\jmath) = \mathbf{c}_2(\mathcal{O})$.
] □

Proposition 9.6. *Suppose that* $\star \in \{ D, B, C^* \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. *Then the map*

$$\delta' \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}) \qquad \tau \mapsto (\nabla(\tau), \tau_{\mathbf{t}}) \tag{9.11}$$

is injective. Moreover, (9.11) *is bijective unless* $\star \in \{ B, D \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$.

Proof. It follows from the injectivity results in Lemma 9.5 and Proposition 9.3. □

Combining the above propotition with Lemma 9.4, we get the following.

Corollary 9.7. *If* $\star \in \{ B, D, C^* \}$, *then the map*

$$\begin{aligned} \mathrm{PBP}_\star(\check{\mathcal{O}}) &\to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau &\mapsto (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{aligned} \tag{9.12}$$

is injective. □

10. Marked Young diagrams attached to the painted bipartitions: combinatorics

We fist define the combinatorical companion of $\mathrm{AC}(\boldsymbol{\tau})$. Suppose $\left|\check{\mathcal{O}}\right| = 0$. For $\boldsymbol{\tau} \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, we define

$$\mathcal{L}_{\boldsymbol{\tau}} := \begin{cases} (1^{\mathrm{Sign}(\boldsymbol{\tau})})_\star \otimes (0,1) & \text{if } \star = B \\ (1^{(0,0)})_\star & \text{otherwise.} \end{cases}$$

Suppose $\left|\check{\mathcal{O}}\right| > 0$. For $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, let $\boldsymbol{\tau}' = \nabla(\boldsymbol{\tau})$ and we define

$$\mathcal{L}_{\boldsymbol{\tau}} := \begin{cases} \vartheta_{\mathsf{s}_{\boldsymbol{\tau}'},\mathsf{s}_{\boldsymbol{\tau}}}(\mathcal{L}_{\boldsymbol{\tau}'} \otimes (\varepsilon_\wp, \varepsilon_\wp)) & \text{if } \star \in \{\, C, \widetilde{C} \,\}, \\ \vartheta_{\mathsf{s}_{\boldsymbol{\tau}'},\mathsf{s}_{\boldsymbol{\tau}}}(\mathcal{L}_{\boldsymbol{\tau}'}) & \text{if } \star = D^*, \\ \vartheta_{\mathsf{s}_{\boldsymbol{\tau}'},\mathsf{s}_{\boldsymbol{\tau}}}(\mathcal{L}_{\boldsymbol{\tau}'}) \otimes (0, \varepsilon_\tau) & \text{if } \star \in \{\, B, D \,\}, \\ \vartheta_{\mathsf{s}_{\boldsymbol{\tau}'},\mathsf{s}_{\boldsymbol{\tau}}}(\mathcal{L}_{\boldsymbol{\tau}'}) & \text{if } \star = C^*. \end{cases}$$

See (8.3) and (8.4) for the definition of $\vartheta_{\mathsf{s}',\mathsf{s}}$.

For $\star \in \{\, B, D \,\}$ and $\mathcal{E} \in \mathbb{Z}[\mathsf{MYD}_\star]$, we write

$$\mathcal{E}^+ := \Lambda_{(1,0)}(\mathcal{E}) \quad \text{and} \quad \mathcal{E}^- := \Lambda_{(0,1)}(\mathcal{E}).$$

We adopt the convention that $\mathcal{E} \cdot \varnothing = 0$ for $\mathcal{E} \in \mathbb{Z}[\mathsf{MYD}]$.

Using the signature formulas in Lemma 9.5, we have the following more explicit formulas on $\mathcal{L}_{\boldsymbol{\tau}}$.

Lemma 10.1. *Suppose $\check{\mathcal{O}}$ such that $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Let $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ and $\boldsymbol{\tau}' = \nabla(\boldsymbol{\tau})$.*

(a) Suppose $\star \in \{\, C, \widetilde{C} \,\}$. Then

$$\mathcal{L}_{\boldsymbol{\tau}} = \begin{cases} \maltese^y\big((\mathcal{L}_{\boldsymbol{\tau}'} \otimes (\varepsilon_\wp, \varepsilon_\wp)) \cdot (n_0, n_0)\big) & \textit{if } \mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}) \\ \maltese^y\big((\mathcal{L}^+_{\boldsymbol{\tau}'} + \mathcal{L}^-_{\boldsymbol{\tau}'}) \cdot (0,0)\big) & \textit{if } \mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}) \end{cases} \tag{10.1}$$

where y is an integer determined by $\mathrm{Sign}(\boldsymbol{\tau}')$ and $n_0 = (\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O}))/2$.

(b) Suppose $\star \in \{\, B, D \,\}$ and $\mathbf{r}_2(\mathcal{O}) > \mathbf{r}_3(\mathcal{O})$. Then

$$\mathcal{L}_{\boldsymbol{\tau}} = \big((\maltese^y \mathcal{L}_{\boldsymbol{\tau}'}) \cdot (p_{\boldsymbol{\tau}_{\mathbf{t}}}, q_{\boldsymbol{\tau}_{\mathbf{t}}})\big) \otimes (0, \varepsilon_\tau) \tag{10.2}$$

where y is an integer determined by $\mathrm{Sign}(\boldsymbol{\tau})$. In particular,

$$\mathcal{E}(1) = \mathcal{L}_{(\boldsymbol{\tau}_{\mathbf{t}}, \varnothing)}(1) \tag{10.3}$$

for each $\mathcal{E} \in \mathsf{MYD}_\star$ has nonzero multiplicity in $\mathcal{L}_{\boldsymbol{\tau}}$.

(c) When $\mathbf{r}_2(\mathcal{O}) = \mathbf{r}_3(\mathcal{O})$,

$$\mathcal{L}_{\boldsymbol{\tau}} = \big((\maltese^t(\mathcal{L}^+_{\boldsymbol{\tau}''} \cdot (0,0)) \cdot \mathcal{T}^+_{\boldsymbol{\tau}} + (\maltese^t(\mathcal{L}^-_{\boldsymbol{\tau}''} \cdot (0,0))) \cdot \mathcal{T}^-_{\boldsymbol{\tau}}\big) \otimes (0, \varepsilon_\tau) \tag{10.4}$$

where $\boldsymbol{\tau}'' = \nabla^2(\boldsymbol{\tau})$ and

$$t = \frac{|\mathrm{Sign}(\boldsymbol{\tau}_{\mathbf{t}})|}{2},$$

$$\mathcal{T}^+_{\boldsymbol{\tau}} = \begin{cases} (p_{\boldsymbol{\tau}_{\mathbf{t}}}, q_{\boldsymbol{\tau}_{\mathbf{t}}} - 1) & \textit{when } q_{\boldsymbol{\tau}_{\mathbf{t}}} - 1 \geqslant 0, \\ \varnothing & \textit{otherwise,} \end{cases}$$

$$\mathcal{T}^-_{\boldsymbol{\tau}} = \begin{cases} (p_{\boldsymbol{\tau}_{\mathbf{t}}} - 1, q_{\boldsymbol{\tau}_{\mathbf{t}}}) & \textit{when } p_{\boldsymbol{\tau}_{\mathbf{t}}} - 1 \geqslant 0, \\ \varnothing & \textit{otherwise.} \end{cases}$$

(d) Suppose $\star \in \{\, D^* \,\}$. Then

$$\mathcal{L}_{\boldsymbol{\tau}} = \mathcal{L}_{\boldsymbol{\tau}'} \cdot (n_0, n_0),$$

where $n_0 = (\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O}))/2$.

(e) *Suppose* $\star \in \{ C^* \}$. *Then*

$$\mathcal{L}_\uptau = \mathcal{L}_{\uptau'} \cdot (p_{\tau_\mathbf{t}}, q_{\tau_\mathbf{t}}).$$

□

[Hint of the proof of (c). It suffice to consider the most left three columns of the peduncle part: this part has signature ${}^l\mathrm{Sign}(\mathcal{T}_\uptau) + (1,1) = \mathrm{Sign}(\tau_\mathbf{t} x_{\uptau''}) = \mathrm{Sign}(\tau_\mathbf{t}) + {}^l\mathrm{Sign}(\mathcal{D}_{\uptau''})$. Therfore,

$$\mathrm{Sign}(\tau_\mathbf{t}) = {}^l\mathrm{Sign}(\mathcal{T}_\uptau) - {}^l\mathrm{Sign}(\mathcal{D}_{\uptau''}) + (1,1) = {}^l\mathrm{Sign}(\mathcal{L}_\uptau) - {}^l\mathrm{Sign}(\mathcal{L}_{\uptau''}) + (1,1).$$

]

10.1. **Properties of $\mathcal{L}_\uptau$.** In this section, we summarizes the properties of $\mathcal{L}_\uptau$ in the following lemmas. Their proofs are given in the next three sections.

Lemma 10.2. *For each* $\uptau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, $\mathcal{L}_\uptau \neq 0$.

Lemma 10.3. *Suppose* $\star \in \{ C, \widetilde{C}, D^* \}$ *and* $\check{\mathcal{O}} \neq \emptyset$.

(a) *Let* $\uptau_1 = (\tau_1, \wp_1)$ *and* $\uptau_2 = (\tau_2, \wp_2)$ *be two elements in* $\mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$ *such that* $\mathcal{L}_{\uptau_1} = \mathcal{L}_{\uptau_2}$. *Then*
$$\mathrm{Sign}(\tau_1') = \mathrm{Sign}(\tau_2') \quad \textit{and} \quad \varepsilon_{\wp_1} = \varepsilon_{\wp_2},$$
where $\tau_i' := \nabla(\tau_i)$ *for* $i = 1, 2$.

(b) *When* $\check{\mathcal{O}}$ *is weakly-distinguished, the following map is injective.*
$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \longrightarrow \mathbb{Z}[\mathsf{MYD}_\star(\mathcal{O})], \quad \uptau \mapsto \mathcal{L}_\uptau.$$

(c) *When* $\check{\mathcal{O}}$ *is quasi-distinguished, the image of the above map is in* $\mathsf{MYD}_\star(\mathcal{O})$.

Lemma 10.4. *Suppose* $\star \in \{ B, D \}$ *and* $(\star, \check{\mathcal{O}}) \neq (D, \emptyset)$.

(a) *Let* $\uptau_i = (\tau_i, \wp_i) \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ *and* $\epsilon_i \in \mathbb{Z}/2\mathbb{Z}$ $(i = 1, 2)$ *such that*
$$\mathcal{L}_{\uptau_1} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{\uptau_2} \otimes (\epsilon_2, \epsilon_2).$$
Then
$$\epsilon_1 = \epsilon_2, \quad \varepsilon_{\tau_1} = \varepsilon_{\tau_2} \quad \textit{and} \quad \mathcal{L}_{\uptau_1} = \mathcal{L}_{\uptau_2}.$$

(b) *When* $\check{\mathcal{O}}$ *is weakly-distinguished, the following map is injective.*
$$\mathrm{PBP}^{\mathrm{ext}}_\star(\mathcal{O}') \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}[\mathsf{MYD}_\star(\mathcal{O})], \quad (\uptau, \epsilon) \mapsto \mathcal{L}_\uptau \otimes (\epsilon, \epsilon).$$

(c) *When* $\check{\mathcal{O}}$ *is quasi-distinguished, the image of the above map is in* $\mathsf{MYD}_\star(\mathcal{O})$.

Lemma 10.5. *Suppose* $\star \in \{ B, D \}$ *and* $\check{\mathcal{O}} \neq \emptyset$.

(a) *If* $x_\uptau = s$, *then* $\mathcal{L}^+_\uptau = 0$ *and* $\mathcal{L}^-_\uptau = 0$.

(b) *If* $x_\uptau \in \{ r, c \}$, *then* $\mathcal{L}^+_\uptau \neq 0$ *and* $\mathcal{L}^-_\uptau = 0$.

(c) *If* $x_\uptau = d$, *then* $\mathcal{L}^+_\uptau \neq 0$ *and* $\mathcal{L}^-_\uptau \neq 0$.

Lemma 10.6. *Suppose* $\star \in \{ B, D \}$, $\check{\mathcal{O}} \neq \emptyset$ *and* $\check{\mathcal{O}}$ *is weakly-distinguished.*

(a) *The map*
$$\begin{array}{rccl} \Upsilon_{\check{\mathcal{O}}}\colon & \{ \mathcal{L}_\uptau \mid \uptau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \textit{ s.t. } \mathcal{L}^+_\uptau \neq 0 \} & \to & \mathbb{Z}[\mathsf{MYD}] \times \mathbb{Z}[\mathsf{MYD}] \\ & \mathcal{L}_\uptau & \mapsto & (\mathcal{L}^+_\uptau, \mathcal{L}^-_\uptau) \end{array} \tag{10.5}$$
is injective.

(b) *When* $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$, *the maps*
$$\begin{array}{rccl} \Upsilon^+_{\check{\mathcal{O}}}\colon & \{ \mathcal{L}_\uptau \mid \uptau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \textit{ s.t. } \mathcal{L}^+_\uptau \neq 0 \} & \to & \{ \mathcal{L}^+_\uptau \} \\ & \mathcal{L}_\uptau & \mapsto & \mathcal{L}^+_\uptau \\ \Upsilon^-_{\check{\mathcal{O}}}\colon & \{ \mathcal{L}_\uptau \mid \uptau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \textit{ s.t. } \mathcal{L}^+_\uptau \neq 0 \} & \to & \{ \mathcal{L}^-_\uptau \} \\ & \mathcal{L}_\uptau & \mapsto & \mathcal{L}^-_\uptau \end{array}$$
are injective.

(c) *When* $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$, *and* $x_\tau = d$. *Then there is* $\mathcal{E} \in \mathsf{MYD}$ *with non-zero coefficient in* $\mathcal{L}_\tau^-$ *such that* $\mathcal{E} \sqsupseteq (1,0)$.

All the claims for $\star \in \{ C^*, D^* \}$ in the above lemmas are straight forward to verify. We will focus on the non-quaternionic cases in the next two sections.

10.2. **The initial cases.** Suppose $\mathbf{c}_1(\mathcal{O}) = 0$. Then the group $G_{\mathbb{C}}$ is the trivial group, everything are clear.

We now assume $\mathbf{c}_1(\mathcal{O}) > 0$ and $\mathbf{c}_2(\mathcal{O}) = 0$.

Suppose $\star \in \{ C, \widetilde{C}, D^* \}$. There there is $k \in \mathbb{N}^+$ such that $\mathcal{O} = (1^{2k})_\star$, and $\check{\mathcal{O}}$ is the regular nilpotent orbit in $\check{\mathfrak{g}}$. Now $\mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}})$ has only one element, say τ_0, which maps to $\mathcal{L}_{\tau_0} = (1^{(k,k)})_\star$.

Suppose $\star \in \{ B, D, C^* \}$. There is $k \in \mathbb{N}^+$ such that $\mathcal{O}$ and $\check{\mathcal{O}}$ are given by Lemma 9.4. When $\star \in \{ B, D \}$,

$$\mathcal{L}_\tau = (1^{(p_\tau, (-1)^{\varepsilon_\tau} q_\tau)})_\star \quad \text{for all } \tau \in \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}).$$

Thanks to Lemma 9.4, it is easy to see that

$$\begin{array}{rcl} \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \to & \mathsf{MYD}_\star(\mathcal{O}) \\ (\tau, \epsilon) & \mapsto & \mathcal{L}_\tau \otimes (\epsilon, \epsilon) \end{array}$$

is a bijection.

When $\star = C^*$, for $\tau \in \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}})$,

$$\mathcal{L}_\tau = (1^{(p_\tau, q_\tau)})_\star \quad \text{for all } \tau \in \mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}).$$

Now

$$\mathrm{PBP}_\star^{\mathrm{ext}}(\check{\mathcal{O}}) \to \mathsf{MYD}_\star(\mathcal{O}), \quad \tau \mapsto \mathcal{L}_\tau$$

is a bijection by Lemma 9.4.

In all the cases, the orbit $\check{\mathcal{O}}$ is quasi-distinghuished and all the lemmas in Section 10.1 are clear now.

10.3. **The general case for $\star \in \{ C, \widetilde{C} \}$.** We now assume $\mathbf{c}_2(\mathcal{O}) > 0$ and retain the notation in Lemma 10.3.

For $\sum_{j=1}^{b} m_i \mathcal{E}_i \in \mathbb{Z}[\mathsf{MYD}_\star]$ with $m_j > 0$ and $\mathcal{E}_i \in \mathsf{MYD}_\star$, we define

$${}^d\mathrm{Sign}(\sum_{j=1}^{b} m_i \mathcal{E}_i) := \{ \mathrm{Sign}(\mathscr{O}'_j) \mid j = 1, 2, \cdots, b \} .$$

where $\mathscr{O}'_j \in \mathsf{SYD}_{\star'}$ is given by $\mathscr{O}'_j(i) := \mathscr{F} \circ \mathcal{E}_j(i+1)$ for all $i \in \mathbb{N}^+$.

We first consider the case when $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$.

Since $\mathcal{L}_{\tau'} \neq 0$, $\mathcal{L}_\tau \neq 0$ by (10.1). This proves Lemma 10.2.

Suppose $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (10.1), $\{ \mathrm{Sign}(\tau'_1) \} = {}^d\mathrm{Sign}(\mathcal{L}_{\tau_1}) = {}^d\mathrm{Sign}(\mathcal{L}_{\tau_2}) = \{ \mathrm{Sign}(\tau'_2) \}$. In particular, $\mathrm{Sign}(\tau'_1) = \mathrm{Sign}(\tau'_2)$ and the integer "y" in (10.1) are the same for τ_1 and τ_2. We get $\mathcal{L}_{\tau'_1} \otimes (\varepsilon_{\wp_1}) = \mathcal{L}_{\tau'_2} \otimes (\varepsilon_{\wp_2})$. So $\varepsilon_{\wp_1} = \varepsilon_{\wp_2}$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$ by Lemma 10.4 (a). This proves Lemma 10.3 (a).

When $\check{\mathcal{O}}$ is weakly-distinguished, $\check{\mathcal{O}}'$ is also weakly-distinguished and we get $\tau'_1 = \tau'_2$ by Lemma 10.4 (b). Hence $\tau_1 = \tau_2$ by Proposition 9.3. This proves Lemma 10.3 (b).

When $\check{\mathcal{O}}$ is quasi-distinguished, $\check{\mathcal{O}}'$ is quasi-distinguished $\mathcal{L}_{\tau'} \in \mathsf{MYD}_{\star'}(\mathcal{O}')$ by Lemma 10.4 (c). Hence $\mathcal{L}_\tau \in \mathsf{MYD}_\star(\mathcal{O})$ by (10.1). This proves Lemma 10.3 (c).

Now we consider the rest case, where $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}) > 0$.

According to Proposition 9.3, $x_{\tau'} \neq s$. Thanks to Lemma 10.5 and (10.1), we have $\mathcal{L}^+_{\tau'} \neq 0$ and

$$\tag{10.6} {}^d\mathrm{Sign}(\mathcal{L}_\tau) = \begin{cases} \{\, \mathrm{Sign}(\tau') - (1,0) \,\} & \text{if } x_{\tau'} \in \{\, r, c \,\}, \\ \{\, \mathrm{Sign}(\tau') - (1,0), \mathrm{Sign}(\tau') - (0,1) \,\} & \text{if } x_{\tau'} = d, \end{cases}$$

In particular, $\mathcal{L}_\tau \neq 0$ and Lemma 10.2 is proven.

Suppose $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (10.6), $\mathrm{Sign}(\tau'_1) = \mathrm{Sign}(\tau'_2)$. Since we have $\varepsilon_{\wp_1} = \varepsilon_{\wp_2} = 0$, Lemma 10.3 (a) is proven.

Now (10.1) implies $\mathcal{L}^+_{\tau'_1} = \mathcal{L}^+_{\tau'_2}$ and $\mathcal{L}^-_{\tau'_1} = \mathcal{L}^-_{\tau'_2}$. By the injectivity of $\Upsilon_{\check{\mathcal{O}}'}$ in Lemma 10.6 (a), we get $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. Suppose $\check{\mathcal{O}}$ is weakly-distinghuished, then $\check{\mathcal{O}}'$ is weakly-distinghuished. So $\tau'_1 = \tau'_2$ by Lemma 10.4 (b). Now Proposition 9.3 implies $\tau_1 = \tau_2$, which proves Lemma 10.3 (b).

Note that $\check{\mathcal{O}}$ is not quasi-distinguished, Lemma 10.3 (c) is vacant.

10.4. **The general case for $\star \in \{\, B, D \,\}$.** To ease the following discussion, we adopt the following convension. Suppose $\mathcal{A} \in \mathbb{Z}[\mathsf{MYD}]$. We write $\mathcal{A} \rhd (p_0, q_0)$ if $\mathcal{E} \sqsupset (p_0, q_0)$ for all $\mathcal{E} \in \mathsf{MYD}$ with non-zero coefficient in $\mathcal{A}$, and write $\mathcal{A} \sqsupset (p_0, q_0)$ if there is an $\mathcal{E} \in \mathsf{MYD}$ with non-zero coefficient in $\mathcal{A}$ such that $\mathcal{E} \sqsupseteq (p_0, q_0)$.

We now assume $\mathbf{c}_2(\mathcal{O}) > 0$ and retain the notation in Lemma 10.4 to 10.6.

We first consider the case when $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Lemma 10.2, i.e. $\mathcal{L}_\tau \neq 0$, follows from (10.2) directly.

Suppose $\mathcal{L}_{\tau_1} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{\tau_2} \otimes (\epsilon_2, \epsilon_2)$. By (10.3), $\mathcal{L}_{(\tau_1)_\mathbf{t}} \otimes (\epsilon_1, \epsilon_1) = \mathcal{L}_{(\tau_2)_\mathbf{t}} \otimes (\epsilon_2, \epsilon_2)$. By the result in the initial case, we get $\epsilon_1 = \epsilon_2$ and $\varepsilon_{\tau_1} = \varepsilon_{(\tau_1)_\mathbf{t}} = \varepsilon_{(\tau_2)_\mathbf{t}} = \varepsilon_{\tau_2}$. This proves Lemma 10.4 (a).

Now assume $\check{\mathcal{O}}$ is weakly-distinguished in addition. Then $\check{\mathcal{O}}'$ is also weakly-distinghuished. We get $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$ by (10.3) and so $\tau'_1 = \tau'_2$. By Corollary 9.7, we conclude that $\tau_1 = \tau_2$. This proves Lemma 10.4 (b).

Suppose $\check{\mathcal{O}}$ is quasi-distinghuished. Then $\mathcal{L}_\tau \in \mathsf{MYD}_\star$ since $\mathcal{L}_{\tau'} \in \mathsf{MYD}_{\star'}$ by the quasi-distinguishness of $\check{\mathcal{O}}'$. This proves Lemma 10.4 (c).

Using (10.3), all the claims in Lemma 10.5 and 10.6 follows from that of $\check{\mathcal{O}}_\mathbf{t}$, see Section 9.2. (Note that we always have $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. So Lemma 10.5 (b) is not vacant.)

We now assume that $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$.

Let $\tau'' := (\tau'', \wp'') := \nabla^2(\tau)$. By Lemma 9.5, $x_{\tau''} \neq s$.

Proof of Lemma 10.2. When $\mathrm{Sign}(\tau_\mathbf{t}) \succeq (0,1)$, the first term in (10.4) dose not vanish since $\mathcal{L}^+_{\tau''} \neq 0$ by Lemma 10.5. When $\mathrm{Sign}(\tau_\mathbf{t}) \not\succeq (0,1)$, $\mathrm{Sign}(\tau_\mathbf{t}) = (2k, 0)$ and $\mathcal{P}^{-1}_{\tau_\mathbf{t}}(\{\, s, c \,\}) = \emptyset$. Hence $x_{\tau''} = d$ by Lemma 9.5. Now $\mathcal{L}^-_{\tau''} \neq 0$ by Lemma 10.5 and the second term in (10.4) dose not vanish. This proves Lemma 10.2

Proof of Lemma 10.5. It is clear that $\mathcal{L}^-_\tau = 0$ when $x_\tau \in \{\, s, r, c \,\}$ because of the twist $\otimes(0,1)$ in Equation (10.4).

(i) Suppose $x_\tau = s$. We have $\mathrm{Sign}(\tau_\mathbf{t}) = (0, 2k)$. where $2k = \left|\check{\mathcal{O}}_\mathbf{t}\right|$. Therefore, only the first term in (10.4) is non-zero and $\mathcal{E}(1) = (B, 0, -(2k-1))$ for every $\mathcal{E} \in \mathsf{MYD}_\star$ with non-zero coefficient in $\mathcal{L}_\tau$. Hence $\mathcal{L}^+_\tau = 0$.

(ii) Suppose $x_\tau \in \{\, r, c \,\}$. If $\mathrm{Sign}(\tau_\mathbf{t}) \succ (1,1)$, the first term in (10.4) is non-zero and $\mathcal{L}_\tau \sqsupseteq (1,0)$. If $\mathrm{Sign}(\tau_\mathbf{t}) \not\succ (1,1)$, then $\mathcal{P}^{-1}_{\tau_\mathbf{t}}(\{\, s, c \,\}) = \emptyset$, $\mathrm{Sign}(\tau_\mathbf{t}) = (2k, 0)$ and $x_{\tau''} = d$ by Lemma 9.5. Now the second term of (10.4) is non-zero and $\mathcal{T}^-_\tau \succ (1,0)$. In both cases, we conclude that $\mathcal{L}^+_\tau \neq 0$.

(iii) Suppose $x_\tau = d$. If $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (1,2)$, then the first term in (10.4) is non-zero and so $\mathcal{L}_\tau \sqsupseteq (1,1)$. Hence $\mathcal{L}_\tau^+ \neq 0$ and $\mathcal{L}_\tau^- \neq 0$. If $\mathrm{Sign}(\tau_{\mathbf{t}}) \nsucc (1,2)$, then $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k-1,1)$ [1] and $x_{\tau''} = d$ by Lemma 9.5. So the both terms in (10.4) are non-zero, which leads to the conclusion.

Proof of Lemma 10.4 (a). The following proof only depends on Lemma 10.5. Suppose $\mathcal{L}' = \mathcal{L}_\tau \otimes (\epsilon, \epsilon)$. Let

$$\begin{aligned} \mathcal{L}'^+ &:= \Lambda_{(1,0)}(\mathcal{L}'), & \mathcal{L}'^- &:= \Lambda_{(0,1)}(\mathcal{L}'), \\ \widetilde{\mathcal{L}'}^+ &:= \Lambda_{(1,0)}(\mathcal{L}' \otimes (1,1)), \text{ and} & \widetilde{\mathcal{L}'}^- &:= \Lambda_{(0,1)}(\mathcal{L}' \otimes (1,1)). \end{aligned}$$

Note that $\mathbf{c}_1(\mathcal{O}) - \mathbf{c}_2(\mathcal{O}) > 0$. So $\mathcal{L}_\tau^+$, $\mathcal{L}_\tau^-$, $\widetilde{\mathcal{L}'}^+$ and $\widetilde{\mathcal{L}'}^-$ can not all vanish. Using Lemma 10.5, we have the following criteria for $(\varepsilon_\tau, \epsilon)$:

$$(\varepsilon_\tau, \epsilon) = (0,0) \Leftrightarrow \begin{cases} \mathcal{L}'^+ \neq 0 \\ \mathcal{L}'^- \neq 0 \end{cases};$$

$$(\varepsilon_\tau, \epsilon) = (0,1) \Leftrightarrow \begin{cases} \widetilde{\mathcal{L}'}^+ \neq 0 \\ \widetilde{\mathcal{L}'}^- \neq 0 \end{cases};$$

$$(\varepsilon_\tau, \epsilon) = (1,0) \Leftrightarrow \begin{cases} \mathcal{L}'^+ \neq 0 \\ \mathcal{L}'^- = 0 \end{cases} \text{ or } \begin{cases} \widetilde{\mathcal{L}'}^+ = 0 \\ \widetilde{\mathcal{L}'}^- \neq 0 \end{cases};$$

$$(\varepsilon_\tau, \epsilon) = (1,1) \Leftrightarrow \begin{cases} \widetilde{\mathcal{L}'}^+ \neq 0 \\ \widetilde{\mathcal{L}'}^- = 0 \end{cases} \text{ or } \begin{cases} \mathcal{L}'^+ = 0 \\ \mathcal{L}'^- \neq 0 \end{cases}.$$

This implies Lemma 10.4 (a).

Proof of Lemma 10.4 (b). First not that $\mathbf{r}_1(\check{\mathcal{O}}'') = \mathbf{r}_3(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_5(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}'')$.

Suppose that $\mathcal{L}_{\tau_i} \sqsupseteq (0,1)$ or $\mathcal{L}_{\tau_i} \sqsupseteq (0,-1)$. The first term in (10.4) for both τ_1 and τ_2 must be non-zero and equal. This implies $\mathcal{L}_{\tau_1''}^+ = \mathcal{L}_{\tau_2''}^+$. By Lemma 10.5 (b) and Lemma 10.4 (b), $\mathcal{L}_{\tau_1''} = \mathcal{L}_{\tau_2''}$ and $\tau_1'' = \tau_2''$. Since $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by Lemma 9.5.

Now consider the case when $\mathcal{L}_{\tau_i} \not\sqsupseteq (0,1)$ and $\mathcal{L}_{\tau_i} \not\sqsupseteq (0,-1)$. In partcular, $x_\tau \neq d$.

By (10.4), we conclude that $\mathrm{Sign}((\tau_i)_{\mathbf{t}}) \in \{(2k,0), (2k-1,1)\}$.

We consider case by case according to the set $\{\mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i = 1,2\}$:

(i) Suppose $\{\mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i = 1,2\} = \{(2k-1,1)\}$. We have $\mathcal{L}_{\tau_1''}^+ = \mathcal{L}_{\tau_2''}^+$, and deduces $\tau_1'' = \tau_2''$.

(ii) Suppose $\{\mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i = 1,2\} = \{(2k,0)\}$. We have $\mathcal{L}_{\tau_1''}^- = \mathcal{L}_{\tau_2''}^-$, and deduces $\tau_1'' = \tau_2''$ by Lemma 10.5 (b) and Lemma 10.4 (b).

(iii) Without loss of generality, we assume that $\mathrm{Sign}(\tau_{1\mathbf{t}}) = (2k-1,1)$ and $\mathrm{Sign}(\tau_{2\mathbf{t}}) = (2k,0)$. By (10.4), we get

$$\mathcal{L}_{\tau_1''}^+ \cdot (0,0) = \maltese(\mathcal{L}_{\tau_2''}^- \cdot (0,0)),$$

which is contradict to Lemma 10.5 (c).

We finished the proof of Lemma 10.4 (b).

Lemma 10.4 (c) is vacant in the current case.

---

[1] $\tau_{\mathbf{t}} = d$ if it has length 1.

Proof of Lemma 10.6 (b). Note that the condition $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$ implies $2k = |\check{\mathcal{O}}_{\mathbf{t}}| \geqslant 4$.

We first prove the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$. Note that $\mathcal{L}_\tau \sqsupseteq (2,0)$ is equivalent to $\mathcal{L}^+_\tau \sqsupseteq (1,0)$. We clearly have the following bijection

$$\{\, \mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}_\tau \sqsupseteq (2,0) \,\} \to \{\, \mathcal{L}^+_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^+_\tau \sqsupseteq (1,0) \,\},$$

since $\mathcal{L}_\tau) \lhd (1,0)$ for each element in the domain by (10.4).

Now we consider the set $\{\, \mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^+_\tau \neq 0, \mathcal{L}_\tau \not\sqsupseteq (2,0) \,\}$. Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in this set such that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$.

By (10.4), we have $\mathrm{Sign}(\tau_{i\mathbf{t}}) = (1, 2k-1)$ and the first term of (10.4) non-vanish.

Note that $\mathcal{E}(1) = (B, 2k-2, (-1)^{\varepsilon_{\tau_i}} 1)$ for each $\mathcal{E} \in \mathsf{MYD}$ having non-zero coefficient in the first term of (10.4). So $x_{\tau_1} = x_{\tau_2}$. Compairing the first term of (10.4) gives $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$. Note that $\check{\mathcal{O}}''$ is weakly-distinghuished and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$, we deduce that $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}''}$. Now we conclude that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by (10.4) and finished the proof of the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$.

The injectivity of $\Upsilon^-_{\check{\mathcal{O}}}$ is proved similarly.

Note that $\mathcal{L}_\tau \sqsupseteq (0,2)$ is equivalent to $\mathcal{L}^-_\tau) \sqsupseteq (0,1)$. We get the bijection

$$\{\, \mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}_\tau \sqsupseteq (0,2) \,\} \to \{\, \mathcal{L}^-_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^-_\tau \sqsupseteq (0,1) \,\}.$$

Now we consider the set $\{\, \mathcal{L}_\tau \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}), \mathcal{L}^-_\tau \neq 0, \mathcal{L}_\tau \not\sqsupseteq (0,2) \,\}$. Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in it such that $\mathcal{L}^-_{\tau_1} = \mathcal{L}^-_{\tau_2}$. By (10.4), we have $\mathrm{Sign}((\tau_i)_{\mathbf{t}}) \in \{\, (2k-1,1), (2k-2,2) \,\}$. We consider case by case according to the set $\{\, \mathrm{Sign}((\tau_i)_{\mathbf{t}}) \mid i = 1,2 \,\}$:

(i) Suppose $\{\, \mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i = 1,2 \,\} = \{\, (2k-1,1) \,\}$. We have $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$, and so $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ by the injectivity of $\Upsilon^-_{\check{\mathcal{O}}''}$.

(ii) Suppose $\{\, \mathrm{Sign}(\tau_{i\mathbf{t}}) \mid i = 1,2 \,\} = \{\, (2k-2,2) \,\}$. We get $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$, and so $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$ by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}''}$.

(iii) Without loss of generality, we assume that $\mathrm{Sign}(\tau_{1\mathbf{t}}) = (2k-1,1)$ and $\mathrm{Sign}(\tau_{2\mathbf{t}}) = (2k-2,2)$. But this will leads to $\maltese(\mathcal{L}^-_{\tau''_1} \cdot (0,0)) = \mathcal{L}^+_{\tau''_2} \cdot (0,0)$ which is contradict to Lemma 10.6 (c).

Now we conclude that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by (10.4) and finished the proof of the injectivity of $\Upsilon^-_{\check{\mathcal{O}}}$.

Proof of Lemma 10.6 (c). Note that we have $k \geqslant 2$ and $x_\tau = d$ under the stated condition. Suppose $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (1,2)$, the first term of (10.4) is non-vanish.

Otherwise, $\mathrm{Sign}(\tau_{\mathbf{t}}) = (2k-1,1) \succeq (3,1)$. Since $\mathcal{L}^-_\tau \neq 0$, the second term of (10.4) must non-vanish.

In all the cases, $\mathcal{L}_\tau \sqsupseteq (1,1)$ and so $\mathcal{L}^-_\tau \sqsupseteq (1,0)$.

Proof of Lemma 10.6 (a). Suppose $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$. The injectivity followes from the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$.

It left to consider the case when $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$. Note that we have $k = 1$ and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$.

Suppose $\Upsilon_{\check{\mathcal{O}}}(\mathcal{L}_{\tau_1}) = \Upsilon_{\check{\mathcal{O}}}(\mathcal{L}_{\tau_2})$.

When $\mathcal{L}^-_{\tau_i} \neq 0$, we have $x_{\tau_1} = x_{\tau_2} = d$ and $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ by (10.4).

Suppose $\mathcal{L}^-_{\tau_i} = 0$. By the similar argument in the proof of Lemma 10.4 (b), we conclude that there are only two possibilities:

- $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ and $x_{\tau_1} = x_{\tau_2} = c$;
- $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$ and $x_{\tau_1} = x_{\tau_2} = r$.

In all the cases, we get $\mathcal{L}_{\tau_1''} = \mathcal{L}_{\tau_2''}$ by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}''}$ or $\Upsilon^-_{\check{\mathcal{O}}''}$. Hence $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by (10.4).

Lemma 10.7. *Suppose $\star \in \{B, D\}$, $\check{\mathcal{O}}$ is weakly-distinguished and $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Let $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ such that $\mathcal{L}^-_\tau \neq 0$. Then*

$$\mathcal{L}^+_{\tau_1} \neq \maltese\mathcal{L}^-_\tau \quad \textit{for all} \quad \tau_1 \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}). \tag{10.7}$$

Proof. Note that the condtion $\mathcal{L}^-_\tau \neq 0$ is equivalent to $x_\tau = d$. We claim that $\mathcal{L}^-_{\tau_2''} \supset 1^{(1,0)}$.

If $\mathbf{r}_2(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$, we have $\mathcal{L}_\tau \succeq 1^{(1,1)}$ and we are done.

Now assume $\mathbf{r}_2(\mathcal{O}'') = \mathbf{r}_3(\check{\mathcal{O}})$. Then $\tau''_{i\,\mathbf{t}}$ has at least length 2. Suppose $\mathrm{Sign}(\tau''_{i\,\mathbf{t}}) \succ (1,2)$. Applying (14.8) to τ'', we see that the first term is non-zero and $\mathcal{L}_{\tau''} \supseteq 1^{(1,1)}$.

Otherwise, $\mathrm{Sign}(\tau_{2\mathbf{t}}) = (2k''-1,1) \succeq (3,1)$ where $k'' = (\mathbf{r}_1(\check{\mathcal{O}}'') - \mathbf{r}_2(\check{\mathcal{O}}''))/2 + 1$. By (14.8), the second term is non-zero and $\mathcal{L}_{\tau''} \supset 1^{(2k''-1,1)}$.

The claim leads to (10.7): the RHS of (10.7) $\succeq 1^{(-1,0)}$, while each component $\mathcal{L}$ of the LHS (10.7) satisfies $\tilde{p}\mathcal{L} \geqslant 0$. $\square$

10.4.1. *The maps $\Upsilon^+_{\mathcal{O}}$, $\Upsilon^-_{\mathcal{O}}$ and $\Upsilon_{\mathcal{O}}$.*

Lemma 10.8. *Suppose $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$. The map $\Upsilon^+_{\check{\mathcal{O}}}\colon \{\mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0\} \to \{\mathcal{L}^+_\tau \neq 0\}$ is injective.*

Proof. Suppose $\mathcal{L}_\tau \supset 1^{(2,0)}$. This is equivalent to $\mathcal{L}^+_\tau) \supset 1^{(1,0)}$.

Then we get the bijectivity of

$$\{\mathcal{L}_\tau \mid \mathcal{L}_\tau \supset 1^{(2,0)}\} \to \{\mathcal{L}^+_\tau \mid \mathcal{L}^+_\tau \supset 1^{(1,0)}\}.$$

$\mathcal{L}_\tau := \mathcal{L}^+_\tau \cdot 1^{(1,0)}$.

Consider the set $\{\mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0, \mathcal{L}_\tau \not\supset 1^{(2,0)}\}$. Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in it such that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$.

Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. By (14.8) we have

$$\tau_{i\mathbf{t}} = s\cdots s x_{\tau_i} \quad \text{and} \quad x_{\tau_i} \in \{c, d\}$$

Since $\mathrm{AC}^{\mathrm{weak}}(\tau_i) \succeq 1^{(0,1)}$, we have $x_{\tau_1} = x_{\tau_2}$, i.e. $\tau_{1\mathbf{t}} = \tau_{2\mathbf{t}}$.

Since the first term of (14.8) is non-zero, we conclude that

$$\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}. \tag{10.8}$$

Note that $\check{\mathcal{O}}''$ is weakly-distinghuished and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$, we deduce that $\mathcal{L}_{\tau_1''} = \mathcal{L}_{\tau_2''}$ by induction hypothesis. Now (14.8) yields $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. $\square$

Lemma 10.9. *Suppose $\mathcal{O}$ is noticed. Then the map $\Upsilon_{\mathcal{O}}\colon \mathcal{L}_\tau \mapsto (\Lambda_+(\mathrm{AC}(\tau)), \mathcal{L}^-_\tau)$ is injective.*

Proof. Suppose $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$. The claim followed by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}}$.

It suffice to consider the case where $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$. In this case, $k = 1$, $\tau_{i\mathbf{t}} = x_{\tau_i}$ and $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$.

Let $\tau_i \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ such that $\mathcal{L}^+_{\tau_i} \neq 0$ for $i = 1, 2$. Suppose $\Upsilon_{\check{\mathcal{O}}}(\tau_1) = \Upsilon_{\check{\mathcal{O}}}(\tau_2)$.

When $\mathcal{L}^-_{\tau_i} \neq 0$, we have $x_{\tau_1} = x_{\tau_2} = d$ and $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$. Now by the injectivity of $\Upsilon^+_{\check{\mathcal{O}}''}$, we deduce $\mathrm{AC}(\tau_1'') = \mathrm{AC}(\tau_2'')$ and so $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by (14.8).

Suppose $\mathcal{L}^-_{\tau_i} = 0$. Thanks to Lemma 10.7, we must in one of the following two cases

- $\mathcal{L}^+_{\tau_1''} = \mathcal{L}^+_{\tau_2''}$ and $x_{\tau_1} = x_{\tau_2} = c$;
- $\mathcal{L}^-_{\tau_1''} = \mathcal{L}^-_{\tau_2''}$ and $x_{\tau_1} = x_{\tau_2} = r$.

By the injectivity of $\Upsilon^\pm_{\check{\mathcal{O}}''}$ and (14.8), we conclude that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. $\square$

Lemma 10.10. *Suppose* $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. *The map* $\Upsilon^-_{\check{\mathcal{O}}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0 \,\} \to \{\, \mathcal{L}^-_\tau \neq 0 \,\}$ *is injective.*

Proof. The proof is similar to that of Lemma 14.11.

Suppose $\mathcal{L}_\tau \supset 1^{(0,2)}$. This is equivalent to $\mathcal{L}^+_\tau) \supset 1^{(0,1)}$.

Then we get the bijectivity of

$$\{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau \supset 1^{(0,2)} \,\} \to \{\, \mathcal{L}^-_\tau \mid \mathcal{L}^-_\tau \supset 1^{(0,1)} \,\}.$$

$\mathcal{L}_\tau := \mathcal{L}^+_\tau \cdot 1^{(0,1)}$.

Consider the set $\{\, \mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0, \mathcal{L}_\tau \not\supset 1^{(0,2)} \,\}$. Let $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ be two elements in it such that $\mathcal{L}^-_{\tau_1} = \mathcal{L}^-_{\tau_2}$.

Thanks to Lemma 10.7, we must in one of the following two cases

- $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$, $\tau_{i\mathbf{t}} = r\cdots rd$ and $x_{\tau''_i} = d$ for $i = 1,2$;
- $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$, $\tau_{i\mathbf{t}} = r\cdots rcd$ and $x_{\tau''_i} \in \{\, r, c \,\}$ for $i = 1,2$.

Since $\mathbf{r}_1(\check{\mathcal{O}}'') > \mathbf{r}_3(\check{\mathcal{O}}'')$, we get $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ by the injectivity of $\Upsilon^\pm_{\check{\mathcal{O}}''}$ and (14.8).

This finished the proof. □

Proposition 10.11. *Suppose that* $\star \in \{\, C, \widetilde{C}, D^* \,\}$ *and cosider the descent map*

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}'). \tag{10.9}$$

(a) If $\star = D^*$ *or* $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$, *then the map* (12.2) *is bijective.*

(b) If $\star \in \{C, \widetilde{C}\}$ *and* $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$, *then the map* (12.2) *is injective and its image equals*

$$\left\{\, \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \;\middle|\; x_{\tau'} \neq s \,\right\}.$$

10.5. **Tails of painted bipartitions.** Because of the following proposition, we assume in the rest of this paper that $\check{\mathcal{O}}$ is quasi-distinghuished when $\star \in \{C^*, D^*\}$.

Proposition 10.12. *Suppose that* $\star \in \{C^*, D^*\}$. *If the set* $\mathrm{PBP}_\star(\check{\mathcal{O}})$ *is nonempty, then* $\check{\mathcal{O}}$ *is quasi-distinguished.*

Proof. Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. If $\star = C^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_i(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1,2,3,\cdots.$$

This forces that $\check{\mathcal{O}}$ is quasi-distinguished.

If $\star = D^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_{i+1}(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1,2,3,\cdots.$$

This also forces that $\check{\mathcal{O}}$ is quasi-distinguished. □

11. Combinatorics

Definition 11.1. *The orbit* $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ *is good for* $\nabla^{\mathsf{s}}_{\mathsf{s}'}$ *if it is contained in the either*

$$p' + q' - |\nabla_{\mathrm{naive}}(\mathcal{O})| = 0,$$

or

$$p' + q' - |\nabla_{\mathrm{naive}}(\mathcal{O})| > 0 \quad \textit{and} \quad \mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O}).$$

There is a totally complex polarization $\mathbf{W} = \mathcal{X} \oplus \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are the $+\mathbf{i}$ and $-\mathbf{i}$ eigenspaces of $L_{\mathbf{W}}$ respectively. We have a Cartan decomposition:

$$G_{\mathbf{W}} = \mathrm{Sp}(W) = K_{\mathbf{W}} \times S_{\mathbf{W}}.$$

where $S_{\mathbf{W}} := \left\{\, \exp(X) \;\middle|\; X \in \mathfrak{g}^{J_{\mathbf{W}}}_{\mathbf{W}},\ XL_{\mathbf{W}} + L_{\mathbf{W}}X = 0 \,\right\}$.

Note that the first condition in **??** implies that the space $\omega^{\mathcal{X}}_{\mathbf{V},\mathbf{V}'}$ is one-dimensional. If G is a nontrivial real symplectic group, a quaternionic symplectic group, or a quaternionic orthogonal group which is not isomorphic to $\mathrm{O}^*(2)$, then the first condition also implies that $\widetilde{K}$ acts on $\omega^{\mathcal{X}}_{\mathbf{V},\mathbf{V}'}$ through the character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{K}}$. Similar result holds for $\widetilde{K}'$. The smooth oscillator representation always exists and is unique up to isomorphism. From now on we fix a smooth oscillator representation $\omega_{\mathbf{V},\mathbf{V}'}$ for each rational dual pair $(\mathbf{V}, \mathbf{V}')$.

Let

$$\star,\quad \check{\mathcal{O}},\quad \mathcal{O},\quad \mathsf{s}=(\star,p,q),\quad \star',\quad \check{\mathcal{O}}',\quad \mathcal{O}',\quad \mathsf{s}'=(\star',p',q')$$

be as in Section 3.4. Thus we have that

- $\star'$ is the Howe dual of $\star$;
- the Young diagram $\check{\mathcal{O}}$ has $\star$-good parity, and the Young diagram $\check{\mathcal{O}}'$ has $\star'$-good parity;
- $\check{\mathcal{O}} \neq \emptyset$, and $\check{\mathcal{O}}'$ is the dual descent of $\check{\mathcal{O}}$;
- $\mathcal{O}$ and $\mathcal{O}'$ are respectively the Barbasch-Vogan duals of $\check{\mathcal{O}}$ and $\check{\mathcal{O}}'$;
- $p+q = |\mathcal{O}|$ and $p'+q' = |\mathcal{O}'|$.

In particular,, $\mathcal{O}$ and $\mathcal{O}'$ are respectively the Barbasch-Vogan duals of $\check{\mathcal{O}}$ and $\check{\mathcal{O}}'$, and

$$p+q=|\mathcal{O}|\,,\qquad p'+q'=|\mathcal{O}'|\,,$$

and view $\mathcal{O}$ as an element of $\mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$. Similar to (1.7), we define the following set of admissible orbit data in the sum of the Grothedieck groups:

$$\mathrm{AOD}_{\mathcal{O}}(K_{\mathsf{s},\mathbb{C}}) := \bigsqcup_{\mathscr{O}\in\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}}),\mathscr{O}\subset\mathcal{O}} \mathrm{AOD}_{\mathscr{O}}(K_{\mathsf{s},\mathbb{C}}) \subset \mathcal{K}_{\mathsf{s}}(\mathcal{O}) := \bigsqcup_{\mathscr{O}\in\mathrm{Nil}(\mathfrak{p}_{\mathsf{s}}),\mathscr{O}\subset\mathcal{O}} \mathcal{K}_{\mathsf{s}}(\mathscr{O}).$$

For every algebraic character χ of $K_{\mathsf{s},\mathbb{C}}$, the twisting map

$$\mathcal{K}_{\mathsf{s}}(\mathcal{O}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}),\qquad \mathcal{E} \mapsto \mathcal{E}\otimes\chi$$

is obviously defined.

In this section we will prove Theorem 15.12 which gives a formula for the associated character of a certain theta lift in the convergent range.

Throughout this section, we fix a rational dual pair $(\mathbf{V}, \mathbf{V}')$ and we retain the notation in Section 12.8 and Section 15.2. As in Section 12.9, we also assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$.

11.1. **An upper bound on the associated character of certain theta lift.** Put

$$\delta_{\dot\star} := \begin{cases} 1, & \text{if } \dot\star \in \{C, \widetilde{C}\};\\ 0, & \text{if } \dot\star = C^*;\\ -1, & \text{if } \dot\star = D;\\ -2, & \text{if } \dot\star = D^*.\end{cases}$$

Definition 11.2. *A Casselman-Wallach representation of $G_{\mathsf{s}'}$ is convergent for $\dot{\mathsf{s}}$ if it is ν-bounded for some $\nu > 2\nu_{\mathsf{s}'} - \dot p - \delta_{\dot\star}$.*

Let π' be a Casselman-Wallach representation of $G_{\mathsf{s}'}$ that is convergent for $\dot{\mathsf{s}}$. Let I be a subqutient of $I(\dot\chi)$. Consider the integrals

$$\begin{array}{rcl}(\pi'\times I)\times(\pi'^{\vee}\times I^{\vee}) & \to & \mathbb{C},\\ ((u,v),(u',v')) & \mapsto & \int_{G_{\mathsf{s}'}}\langle g.u,u'\rangle\cdot\langle g.v,v'\rangle\,\mathrm{d}g.\end{array}\tag{11.1}$$

For each $(-1,1)$-space $\mathbf{V}'''$ of dimension $\dim \mathbf{V}' - 2$, by applying Lemma 12.5 twice, we see that

$$(\pi\widehat{\otimes}(\omega_{\mathbf{V}''',\mathbf{U}})_{G_{\mathbf{V}'''}})_G \cong ((\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'''})_G\widehat{\otimes}\omega_{\mathbf{V}''',\mathbf{V}^\perp})_{G_{\mathbf{V}'''}}$$

is bounded by $\boldsymbol{\vartheta}_{\mathbf{V}''',\mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'''}(\mathcal{O}))$. Using formulas in [25, Theorem 5.2 and 5.6], one checks that the latter set is contained in the boundary of $\mathcal{O}^\perp$. Hence

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)) = 0$$

and the lemma follows.

Lemma 5.9 **??**. Note that $\mathcal{R}_{I_2}(\pi)$ is $\mathcal{O}^\perp$-bounded by Proposition 11.4, Theorem 15.14 and [76, Theorem 1.4]. By the definition in (11.6), we could view $\mathcal{R}_{I_1}(\pi)$ as a subrepresentation of $\mathcal{R}_{I_2}(\pi)$ which is also $\mathcal{O}^\perp$-bounded.

Since G acts on $\mathcal{R}_{I_2}(\pi)$ trivially, the natural homomorphism

$$\pi\widehat{\otimes}I_3 \cong \pi\widehat{\otimes}I_2/\pi\widehat{\otimes}I_1 \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)$$

descents to a surjective homomorphism

$$(\pi\widehat{\otimes}I_3)_G \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi).$$

For each $(-1,1)$-space $\mathbf{V}'''$ of dimension $\dim \mathbf{V}' - 2$, by applying Lemma 12.5 twice, we see that

$$(\pi\widehat{\otimes}(\omega_{\mathbf{V}''',\mathbf{U}})_{G_{\mathbf{V}'''}})_G \cong ((\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'''})_G\widehat{\otimes}\omega_{\mathbf{V}''',\mathbf{V}^\perp})_{G_{\mathbf{V}'''}}$$

is bounded by $\boldsymbol{\vartheta}_{\mathbf{V}''',\mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'''}(\mathcal{O}))$. Using formulas in [25, Theorem 5.2 and 5.6], one checks that the latter set is contained in the boundary of $\mathcal{O}^\perp$. Hence

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)) = 0$$

and the lemma follows.

Lemma 11.3. *The integrals in* (11.1) *are absolutely convergent and the map* (11.1) *is continuous and multi-linear.*

Proof. First assume that $\dot\star \neq D^*$. Lemma 5.9, Lemma 5.10 and Proposition 4.11 implies that I is $\dot p + \delta_{\dot\star}$-bounded. Note that $\Psi_{\dot{\mathsf{s}}}|_{G_{\mathsf{s}'}} = \Psi_{\mathsf{s}'}$, and the lemma then easily follow by Lemmas 4.4.

Now we assume that $\dot\star = D^*$. Let $\mathbb{H}$ denote the quaternion division algebra of the Hamiltonians. As usual, identify $X_{\dot{\mathsf{s}}}$ with $\mathbb{H}^k$, and hence $M_{\dot{\mathsf{s}}}$ is identified with $\mathrm{GL}_k(\mathbb{H})$. Write $\mathrm{B}_k(\mathbb{H})$ for the standard minimal parabolic subgroup of $\mathrm{GL}_k(\mathbb{H})$ consisting of the upper triangular matrices. Then $(\mathbb{H}^\times)^k$ is a Levi subgroup of $\mathrm{B}_k(\mathbb{H})$, and $\mathrm{B}_k(\mathbb{H}) \ltimes N_{\dot{\mathsf{s}}}$ is a minima parabolic subgroup of $G_{\dot{\mathsf{s}}}$.

Put

$$\nu_0 := (3-2k, 7-2k, \cdots, 2k-1) \in \mathbb{R}^k.$$

It determines a positive character $\chi_0 : \mathrm{B}_k(\mathbb{H}) \ltimes N_{\dot{\mathsf{s}}} \to \mathbb{C}^\times$ whose restriction to $(\mathbb{H}^\times)^k$ equals

$$|\cdot|_{\mathbb{H}}^{3-2k} \otimes |\cdot|_{\mathbb{H}}^{7-2k} \otimes \cdots \otimes |\cdot|_{\mathbb{H}}^{2k-1},$$

where $|\cdot|_{\mathbb{H}} : \mathbb{H}^\times \to \mathbb{C}^\times$ stands for the quaternion absolute value that extends the absolute value on $\mathbb{R}^\times$.

Note that $I(\dot\chi)$ is a subrepresentation of the spherical principal series

$$I(\chi_0) := \mathrm{Ind}_{\mathrm{B}_k(\mathbb{H})\ltimes N_{\dot{\mathsf{s}}}}^{G_{\dot{\mathsf{s}}}} \chi_0.$$

The representations $I(\chi_0)$ and $I(\chi_0^{-1})$ are contragredients of each other with the invariant paring

$$\langle\,,\,\rangle : I(\chi_0) \times I(\chi_0^{-1}) \to \mathbb{C}, \quad (f, f') \mapsto \int_{K_{\dot{\mathsf{s}}}} f(g)\cdot f'(g)\,\mathrm{d}g,$$

where the Haar measure $\mathrm{d}g$ is normalized so that $K_{\dot{\mathsf{s}}}$ has total volume 1. Define the elementary spherical function Ξ_{ν_0} on $G_{\dot{\mathsf{s}}}$ by

$$\Xi_{\nu_0}(g) := \langle g.f_0, f_0'\rangle, \qquad g \in G_{\dot{\mathsf{s}}},$$

where $f_0 \in I(\chi_0)$ is the normalized spherical vector so that $f_0|_{K_{\dot{\mathsf{s}}}} = 1$, and likewise $f_0' \in I(\chi_0^{-1})$ is the normalized spherical vector so that $f_0'|_{K_{\dot{\mathsf{s}}}} = 1$.

It is easy to see that

$$|\langle g.f, f'\rangle| \leqslant \Xi_{\nu_0}(g) \cdot |(f|_{K_{\dot{\mathsf{s}}}})|_\infty \cdot |(f|_{K_{\dot{\mathsf{s}}}})|_\infty \qquad (|\cdot|_\infty \text{ indicates the super-norm}),$$

for all $g \in G_{\dot{\mathsf{s}}}$, $f \in I(\chi_0)$ and $f' \in I(\chi_0^{-1})$. Therefore, there exist continuous seminorms $|\cdot|_I$ on I and $|\cdot|_{I^\vee}$ on $I^\vee$ such that

$$|\langle g.v, v'\rangle| \leqslant \Xi_{\nu_0}(g) \cdot |v|_I \cdot |v'|_{I^\vee}$$

for all $g \in G_{\dot{\mathsf{s}}}$, $v \in I$ and $v' \in I^\vee$.

Suppose that π' is ν-bounded for some $\nu > 2\nu_{\mathsf{s}'} - \dot{p} - \delta_\star$. Using the well-known estimate of the elementary spherical functions ([86, Theorem 4.5.3 and Lemma 3.6.7]), as well as the integral formula for $G_{\mathsf{s}'}$ under the Cartan decomposition ([86, Lemma 2.4.2]), it is routine to check that the function

$$\Psi_{\mathsf{s}'}^\nu \cdot (\Xi_{\nu_0})|_{G_{\mathsf{s}'}}$$

is integrable with respect to the Haar measure. This implies the lemma as before. □

11.2. **The representation $\pi' * I(\dot{\chi})$.** In this subsection, we further assume that

$$|\mathsf{s}'| \leqslant \dot{p}.$$

Put

$$\dot{\mathsf{s}}_0 := (\dot{\star}, \dot{p} - |\mathsf{s}'|, \dot{p} - |\mathsf{s}'|),$$

which is also a classical signature. As in (**??**), we have a complete polarization

$$V_{\dot{\mathsf{s}}_0} = X_{\dot{\mathsf{s}}_0} \oplus Y_{\dot{\mathsf{s}}_0}, \tag{11.2}$$

such that $X_{\dot{\mathsf{s}}_0}$ and $Y_{\dot{\mathsf{s}}_0}$ are $J_{\dot{\mathsf{s}}_0}$-stable and totally isotropic, and $Y_{\dot{\mathsf{s}}_0} = L_{\dot{\mathsf{s}}_0}(X_{\dot{\mathsf{s}}_0})$.

View $V_{\dot{\mathsf{s}}_0}$ as a subspaces of $V_{\mathsf{s}''}$ such that $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$ extends $(\langle\,,\,\rangle_{\dot{\mathsf{s}}_0}, J_{\dot{\mathsf{s}}_0}, L_{\dot{\mathsf{s}}_0})$. Likewise, fix a linear embedding

$$V_{\mathsf{s}'} \to V_{\mathsf{s}''}, \quad v \mapsto v^- \tag{11.3}$$

such that $(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$ extends $(-\langle\,,\,\rangle_{\mathsf{s}'}, J_{\mathsf{s}'}, -L_{\mathsf{s}'})$ via this embedding. Let $V_{\mathsf{s}'}^-$ denote the image of the embedding (11.3).

Write

$$P_{\mathsf{s}''} = M_{\mathsf{s}''} \ltimes N_{\mathsf{s}''}$$

for the parabolic subgroup of $G_{\mathsf{s}''}$ stabilizing $X_{\dot{\mathsf{s}}_0}$, where $M_{\mathsf{s}''}$ is the Levi subgroup stabilizing both $X_{\dot{\mathsf{s}}_0}$ and $Y_{\dot{\mathsf{s}}_0}$, and $N_{\mathsf{s}''}$ is the unipotent radical. We have an obvious homomorphism

$$G_{\mathsf{s}'} \times M_{\dot{\mathsf{s}}_0} \to M_{\mathsf{s}''},$$

where $M_{\dot{\mathsf{s}}_0}$ is the subgroup of $G_{\dot{\mathsf{s}}_0}$ stabilizing both $X_{\dot{\mathsf{s}}_0}$ and $Y_{\dot{\mathsf{s}}_0}$, which is a general linear group.

Let $\dot{\chi} : M_{\dot{\mathsf{s}}} \to \mathbb{C}^\times$ be a character satisfying the following conditions: (5.17)

Write

$$P_{\dot{\mathsf{s}},\dot{\mathsf{s}}_0} = M_{\dot{\mathsf{s}},\dot{\mathsf{s}}_0} \ltimes N_{\dot{\mathsf{s}},\dot{\mathsf{s}}_0}$$

for the parabolic subgroup of $G_{\dot{\mathsf{s}}}$ stabilizing $X_{\dot{\mathsf{s}}_0}$, where $M_{\dot{\mathsf{s}},\dot{\mathsf{s}}_0}$ is the Levi subgroup stabilizing both $X_{\dot{\mathsf{s}}_0}$ and $Y_{\dot{\mathsf{s}}_0}$, and $N_{\dot{\mathsf{s}},\dot{\mathsf{s}}_0}$ is the unipotent radical. Let $\dot{\chi} : M_{\dot{\mathsf{s}}} \to \mathbb{C}^\times$ be a character satisfying the following conditions: (5.17)

$(\langle\,,\,\rangle_{\mathsf{s}''}, J_{\mathsf{s}''}, L_{\mathsf{s}''})$, and $V_{\mathsf{s}'}$ and $V_{\mathsf{s}''}$ are perpendicular to each other under the form $\langle\,,\,\rangle_{\dot{\mathsf{s}}}$. Then we have an orthogonal decomposition

$$V_{\dot{\mathsf{s}}} = V_{\mathsf{s}'} \oplus V_{\mathsf{s}''},$$

and both $G_{\mathsf{s}'}$ and $G_{\mathsf{s}''}$ are identified with subgroups of $G_{\dot{\mathsf{s}}}$.

$$\mathbf{V}^\perp = \mathbf{V}^- \oplus (\mathbf{E}_0 \oplus \mathbf{E}_0'), \quad \mathbf{E} = \mathbf{V}^\triangle \oplus \mathbf{E}_0 \quad \text{and} \quad \mathbf{E}' = \mathbf{V}^\nabla \oplus \mathbf{E}_0', \tag{11.4}$$

where $\mathbf{E}_0' = L_{\mathbf{V}^\perp}(\mathbf{E}_0)$,

$$\mathbf{V}^\triangle := \{(v,v) \in \mathbf{V} \oplus \mathbf{V}^- \mid v \in \mathbf{V}\} \quad \text{and} \quad \mathbf{V}^\nabla := \{(v,-v) \in \mathbf{V} \oplus \mathbf{V}^- \mid v \in \mathbf{V}\}.$$

In particular, $G := G_{\mathbf{V}}$ is identified with a subgroup of $G_{\mathbf{V}^\perp}$. Let $\mathrm{GL}_{\mathbf{E}_0} := \mathrm{GL}(\mathbf{E}_0)^{J_{\mathbf{V}^\perp}|_{\mathbf{E}_0}}$ and let

$$P_{\mathbf{E}_0} = M_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$$

denote the parabolic subgroup of $G_{\mathbf{V}^\perp}$ stabilizing $\mathbf{E}_0$, where $N_{\mathbf{E}_0}$ denotes the unipotent radical, and $M_{\mathbf{E}_0} = G \times \mathrm{GL}_{\mathbf{E}_0}$ is the Levi subgroup stabilizing the first decomposition in (11.4).

Let $\chi_{\mathbf{E}_0}$ denotes the restriction of $\chi_{\mathbf{E}}$ to $\widetilde{\mathrm{GL}_{\mathbf{E}_0}}$. Then $\pi \otimes \chi_{\mathbf{E}_0}$, which is preliminarily a representation of $\widetilde{G} \times \widetilde{\mathrm{GL}_{\mathbf{E}_0}}$, descends to a representation of $\widetilde{M}_{\mathbf{E}_0}$.

The rest of this section is devoted to a proof of the following proposition.

Proposition 11.4. *There is an isomorphism*

$$\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \cong \mathrm{Ind}_{\widetilde{P}_{\mathbf{E}_0}}^{\widetilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})$$

of representations of $\widetilde{G}_{\mathbf{V}^\perp}$.

$$p' \leqslant q'' \qquad \text{and} \qquad q' \leqslant p''.$$

Now we assume that $(\mathbf{V}, J, L)$ is a non-degenerate $(\epsilon, \dot\epsilon)$-subspace of $\mathbf{U}$ so that

$$J = J_{\mathbf{U}}|_{\mathbf{V}} \quad \text{and} \quad L = L_{\mathbf{U}}|_{\mathbf{V}}.$$

Then $\widetilde{G} := \widetilde{G}_{\mathbf{V}}$ is naturally a closed subgroup of $\widetilde{G}_{\mathbf{U}}$. Further assume that

$$\dim \mathbf{V} \leqslant \dim \mathbf{E}.$$

We are interested in the growth of $\Xi_{\mathbf{U}}|_{\widetilde{G}}$.

Let π be a $\mathbb{p}$-genuine Casselman-Wallach representation of $\widetilde{G}$. Further assume that $\dim^\circ \mathbf{V} > 0$ and

$$\pi \text{ is } \nu_\pi\text{-bounded for some } \nu_\pi > 2 - \nu_{\mathbf{U},\mathbf{V}}, \tag{11.5}$$

where $\nu_{\mathbf{U},\mathbf{V}}$ is as in (**??**).

Let $\mathcal{J}$ be a subquotient of the degenerate principal series $\mathrm{I}(\chi_{\mathbf{E}})$.

The orthogonal complement $\mathbf{V}^\perp$ of $\mathbf{V}$ in $\mathbf{U}$ is naturally an $(\epsilon, \dot\epsilon)$-space with $J_{\mathbf{V}^\perp} := J_{\mathbf{U}}|_{\mathbf{V}^\perp}$ and $L_{\mathbf{V}^\perp} := L_{\mathbf{U}}|_{\mathbf{V}^\perp}$. Define

$$\mathcal{R}_{\mathcal{J}}(\pi) := \frac{\pi \widehat{\otimes} \mathcal{J}}{\text{the left kernel of the bilinear map (??)}}. \tag{11.6}$$

It is a smooth Fréchet representation of $\widetilde{G}_{\mathbf{V}^\perp}$ of moderate growth.

Let $(\mathbf{V}^-, J_{\mathbf{V}^-}, L_{\mathbf{V}^-})$ denote the $(\epsilon, \dot\epsilon)$-space which equals $\mathbf{V}$ as a vector space, and is equipped with the form $-\langle\, ,\, \rangle_{\mathbf{V}}$, the conjugate linear map $J_{\mathbf{V}^-} = J$, and the linear map $L_{\mathbf{V}^-} = -L$. Then we have an obvious identification $G_{\mathbf{V}^-} = G_{\mathbf{V}}$. We identify $\mathbf{V}^-$ with an $(\epsilon, \dot\epsilon)$-subspace of $\mathbf{V}^\perp$ and fix a $J_{\mathbf{U}}$-stable subspace $\mathbf{E}_0$ in $\mathbf{E}$ such that

11.3. **Degenerate principal series and Rallis quotients.** We are in the setting of Section 15.2 and **??**. Assume that $\dim^\circ \mathbf{V}' > 0$. Recall $\mathbf{U} = \mathbf{E} \oplus \mathbf{E}'$ from (**??**) and assume that

$$\dim \mathbf{E} = \dim \mathbf{V}' + \delta, \quad \text{where } \delta := \begin{cases} 1, & \text{if } G_{\mathbf{U}} \text{ is real orthogonal;} \\ -1, & \text{if } G_{\mathbf{U}} \text{ is real symplectic;} \\ 0, & \text{if } G_{\mathbf{U}} \text{ is quaternionic.} \end{cases} \tag{11.7}$$

The Rallis quotient $(\omega_{\mathbf{V}',\mathbf{U}})_{\widetilde{G}'}$ is an irreducible unitarizable representation of $\widetilde{G}_{\mathbf{U}}$ (*cf.* [46, 72, 89]). Let $1_{\mathbf{V}'}$ denote the trivial representation of $\widetilde{G}'$. Then $(1_{\mathbf{V}'}, \mathbf{U})$ is in the convergent range.

Let $\chi_{\mathbf{E}}$ be as in **??**.

12. Bounding the associated cycles

Denote by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}}$ the $\mathfrak{h}_{\mathsf{s},\mathsf{s}'}$-submodule of $\omega_{\mathsf{s},\mathsf{s}'}$ generated by $\omega_{\mathsf{s},\mathsf{s}'}^{\mathcal{X}_{\mathsf{s},\mathsf{s}'}}$. This is an $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}}) \times (\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module. Given a $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module ρ' of finite length, put

$$\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\rho') := (\omega_{\mathsf{s},\mathsf{s}'}^{\mathrm{alg}} \otimes \pi')_{\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'}} \qquad \text{(the coinvariant space).}$$

This is a $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module of finite length.

12.1. **Good descents of nilpotent orbits.** Let $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$. As in the Introduction, a finite length $(\mathfrak{g}_{\mathsf{s}}, K_{\mathsf{s}})$-module π is said to be $\mathcal{O}$-bounded if the associated variety of its annihilator ideal in $\mathrm{U}(\mathfrak{g}_{\mathsf{s}})$ is contained in the Zariski closure of $\mathcal{O}$. When this is the case, we have the associated $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{\mathsf{s}}(\mathcal{O})$ as in the Introduction.

Lemma 12.1. *The orbit $\mathcal{O}$ is contained in the image of the moment map $\tilde{M}_{\mathsf{s}}$ if and only if*

$$\delta := |\mathsf{s}'| - |\nabla_{\mathrm{naive}}(\mathcal{O})| \geqslant 0.$$

When this is the case, the Young diagram of $\nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$ is obtained from that of $\nabla_{\mathrm{naive}}(\mathcal{O})$ by adding δ boxes in the first column.

Definition 12.2. *The orbit $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is good for $\nabla_{\mathsf{s}'}^{\mathsf{s}}$ if either*

$$|\mathsf{s}'| = |\nabla_{\mathrm{naive}}(\mathcal{O})|,$$

or

$$|\mathsf{s}'| > |\nabla_{\mathrm{naive}}(\mathcal{O})| \quad \textit{and} \quad \mathbf{c}_1(\mathcal{O}) = \mathbf{c}_2(\mathcal{O}).$$

The rest of this section is denoted to a proof of the following theorem.

Theorem 12.3. *Suppose that $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}})$ is good for $\nabla_{\mathsf{s}'}^{\mathsf{s}}$, and write $\mathcal{O}' := \nabla_{\mathsf{s}'}^{\mathsf{s}}(\mathcal{O}) \in \mathrm{Nil}(\mathfrak{g}_{\mathsf{s}'})$. Let π' be an $\mathcal{O}'$-bounded $(\mathfrak{g}_{\mathsf{s}'}, K_{\mathsf{s}'})$-module of finite length. Then $\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')$ is $\mathcal{O}$-bounded, and*

$$\mathrm{AC}_{\mathcal{O}}(\check{\Theta}_{\mathsf{s}'}^{\mathsf{s}}(\pi')) \preceq \check{\vartheta}_{\mathcal{O}'}^{\mathcal{O}}(\mathrm{AC}_{\mathcal{O}'}(\pi')).$$

12.1.1. *Associated characters and associated cycles.* We recall fundamental results of Vogan [83]. Suppose $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is a nilpotent orbit. A finite length $(\mathfrak{g}, \widetilde{K})$-module π is said to be *$\mathcal{O}$-bounded* (or *bounded by $\mathcal{O}$*) if the associated variety of the annihilator ideal $\mathrm{Ann}(\pi)$ is contained in $\overline{\mathcal{O}}$. It follows from [83, Theorem 8.4] that π is $\mathcal{O}$-bounded if and only if its associated variety $\mathrm{AV}(\pi)$ is contained in $\overline{\mathcal{O}} \cap \mathfrak{p}$. Let $\mathcal{M}_{\mathcal{O}}^{\mathbb{p}}(\mathfrak{g}, \widetilde{K})$ denote the category of $\mathbb{p}$-genuine $\mathcal{O}$-bounded finite length $(\mathfrak{g}, \widetilde{K})$-modules, and write $\mathcal{K}_{\mathcal{O}}^{\mathbb{p}}(\mathfrak{g}, \widetilde{K})$ for its Grothendieck group. Recall the group $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ from Section 12.9. From [83, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{Ch}_{\mathcal{O}}\colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g},\, \widetilde{K}) \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}).$$

For a $\mathbb{p}$-genuine $\mathcal{O}$-bounded $(\mathfrak{g},\, \widetilde{K})$-module π of finite length, we call $\mathrm{Ch}_{\mathcal{O}}(\pi)$ the associated character of π.

Let $\mu\colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) \to \mathbb{Z}[(\mathcal{O}\cap\mathfrak{p})/\mathbf{K}]$ be the map of taking dimensions of the isotropy representations. Post-composing μ with $\mathrm{Ch}_{\mathcal{O}}$ gives the associated cycle map:

$$\mathrm{AC}_{\mathcal{O}}\colon \mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g},\, \widetilde{K}) \longrightarrow \mathbb{Z}[(\mathcal{O}\cap\mathfrak{p})/\mathbf{K}].$$

For any π in $\mathcal{M}^{\mathbb{p}}_{\mathcal{O}}(\mathfrak{g},\, \widetilde{K})$ and a nilpotent $\mathbf{K}$-orbit $\mathscr{O} \subset \mathcal{O}\cap\mathfrak{p}$, let $\mathrm{c}_{\mathscr{O}}(\pi)$ denote the multiplicity of $\mathscr{O}$ in $\mathrm{AC}_{\mathcal{O}}(\pi)$. For a $\mathbb{p}$-genuine Casselman-Wallach representation of $\widetilde{G}$, the notion of $\mathcal{O}$-boundedness, and its associated character is defined by using its Harish-Chandra module.

12.1.2. *Algebraic theta lifting.* Write $\mathscr{Y}_{\mathbf{V},\mathbf{V}'}$ for the subspace of $\omega_{\mathbf{V},\mathbf{V}'}$ consisting of all the vectors which are annihilated by some powers of $\mathcal{X}$. This is a dense subspace of $\omega_{\mathbf{V},\mathbf{V}'}$ and is naturally a $((\mathfrak{g}\times\mathfrak{g}')\ltimes\mathfrak{h}(W),\, \widetilde{K}\times\widetilde{K}')$-module. Here $\mathfrak{h}(W)$ denotes the complexified Lie algebra of $\mathrm{H}(W)$.

Definition 12.4. *For each $\mathbb{p}$-genuine $(\mathfrak{g}',\, \widetilde{K}')$-module π' of finite length, define*

$$\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi') := (\mathscr{Y}_{\mathbf{V},\mathbf{V}'}\otimes\pi')_{\mathfrak{g}',\widetilde{K}'}, \qquad \textit{(the coinvariant space).}$$

For each $\mathbb{p}$-genuine finite length $(\mathfrak{g}',\, \widetilde{K}')$-module π', the $(\mathfrak{g},\, \widetilde{K})$-module $\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi')$, also abbreviated as $\check{\Theta}(\pi')$, is $\mathbb{p}$-genuine and of finite length [40]. Recall the notations in Section 12.8.1, in particular the map $\boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}$ in (12.19). We have the following estimate of the size of $\check{\Theta}_{\mathbf{V}',\mathbf{V}}(\pi')$.

Lemma 12.5 ([53, Theorem B and Corollary E])**.** *For each $\mathbb{p}$-genuine $(\mathfrak{g}',\, \widetilde{K}')$-module π' of finite length,*

$$\mathrm{AV}(\check{\Theta}(\pi')) \subset M(M'^{-1}(\mathrm{AV}(\pi'))).$$

Consequently, if π is $\mathcal{O}'$-bounded for a nilpotent orbit $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$, then $\check{\Theta}(\pi')$ is $\boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}(\mathcal{O}')$-bounded.

12.1.3. *The bound in the descent and good generalized descent cases.* We will prove an upper bound of associated characters in the descent and good generalized descent cases.

Theorem 12.6. *Let $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. Suppose that*

(a) $\mathcal{O}'$ is a descent of $\mathcal{O}$, that is, $\mathcal{O}' = \boldsymbol{\nabla}(\mathcal{O})$, or

(b) $\mathcal{O}$ is good for generalized descent (see Definition 12.24) and $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$.

Then for every $\mathbb{p}$-genuine $\mathcal{O}'$-bounded $(\mathfrak{g}',\, \widetilde{K}')$-module π' of finite length, $\check{\Theta}(\pi')$ is $\mathcal{O}$-bounded and

$$\mathrm{Ch}_{\mathcal{O}}\,\check{\Theta}(\pi') \preceq \vartheta_{\mathcal{O}',\mathcal{O}}(\mathrm{Ch}_{\mathcal{O}'}(\pi')).$$

In the rest of this subsection, we assume that $\star \in \{B, D, C^*\}$. Note that $l \geqslant l'$ if $\star \in \{B, C^*\}$, and $l \geqslant l'+1$ if $\star = D$. Put

$$\star_{\mathbf{t}} := \begin{cases} D, & \text{if } \star \in \{B, D\};\\ C^*, & \text{if } \star = C^*. \end{cases}$$

Let $\check{\mathcal{O}}_{\mathbf{t}}$ be the following Young diagram that is determined by the pair $(\star, \check{\mathcal{O}})$.

- If $\star = B$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of two rows with lengths $2(l-l')+1$ and 1.
- If $\star = D$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of two rows with lengths $2(l-l')-1$ and 1.
- If $\star = C^*$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of one row with length $2(l-l')+1$.

Note that in all these three cases $\check{\mathcal{O}}_{\mathbf{t}}$ has $\star_{\mathbf{t}}$-good parity and every element in $\mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ has the form

$$\begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline\end{array} \times \varnothing \times D, \qquad \begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline\end{array} \times \varnothing \times D \qquad \text{or} \qquad \varnothing \times \begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline\end{array} \times C^*, \tag{12.1}$$

respectively if $\star = B, D$ or C^*. Here $k = l-l'+1, l-l'$ or $l-l'$ respectively.

Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ be as before.

The case when $\star = B$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the first painted bipartition in (12.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{\, \mathcal{Q}(l'+1,1), \mathcal{Q}(l'+2,1), \cdots, \mathcal{Q}(l,1) \,\}$$

with the set

$$\begin{cases} \{\, c \,\}, & \text{if } \alpha = B^+, \text{ and either } l' = 0 \text{ or } \mathcal{Q}(l',1) \in \{\, \bullet, s \,\}; \\ \{\, s \,\}, & \text{if } \alpha = B^-, \text{ and either } l' = 0 \text{ or } \mathcal{Q}(l',1) \in \{\, \bullet, s \,\}; \\ \{\, \mathcal{Q}(l',1) \,\}, & \text{if } l' > 0 \text{ and } \mathcal{Q}(l',1) \in \{r, d\}. \end{cases}$$

The case when $\star = D$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the second painted bipartition in (12.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{\, \mathcal{P}(l'+2,1), \mathcal{P}(l'+3,1), \cdots, \mathcal{P}(l,1) \,\}$$

with the set

$$\begin{cases} \{\, c \,\}, & \text{if } \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}),\ \mathcal{P}(l'+1,1) = r,\ \mathcal{P}(l'+1,2) = c \text{ and } \mathcal{P}(l,1) \in \{\, r, d \,\}; \\ \{\, \mathcal{P}_\tau(l'+1,1) \,\}, & \text{otherwise.} \end{cases}$$

The case $\star = C^*$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the third painted bipartition in (12.1) such that

$$(x_1, x_2, \cdots, x_k) = (\mathcal{Q}(l'+1,1), \mathcal{Q}(l'+2,1), \cdots, \mathcal{Q}(l,1)).$$

When $\star \in \{\, B, D \,\}$, the symbol in the last box of the tail $\tau_{\mathbf{t}} \in \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ will be impotent for us. We write x_τ for it, namely

$$x_\tau := \mathcal{P}_{\tau_{\mathbf{t}}}(\mathbf{c}_1(\imath_{\tau_{\mathbf{t}}}), 1).$$

The following lemma is easy to check.

Lemma 12.7. *If $\star = B$, then*

$$x_\tau = s \Longleftrightarrow \begin{cases} \alpha = B^-; \\ \mathcal{Q}(l,1) \in \{\bullet, s\}, \end{cases}$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{Q}(l,1) = d.$$

If $\star = D$, then

$$x_\tau = s \Longleftrightarrow \mathcal{P}(l,1) = s,$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{P}(l,1) = d.$$

□

12.2. **Some properties of the descent maps.** The key properties of the descent map when $\star \in \{ C, \widetilde{C}, D^* \}$ are summarized in the following proposition.

Proposition 12.8. *Suppose that $\star \in \{ C, \widetilde{C}, D^* \}$ and cosider the descent map*

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}'). \tag{12.2}$$

(a) If $\star = D^*$ or $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$, then the map (12.2) *is bijective.*
(b) If $\star \in \{C, \widetilde{C}\}$ and $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$, then the map (12.2) *is injective and its image equals*

$$\left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, x_{\tau'} \neq s \right\}.$$

Proof. We assume that $\star = \widetilde{C}$ and $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$. The proofs in the other cases are similar and are left to the reader.

Note that the map (12.2) induces a map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(l,1) \neq c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\}. \tag{12.3}$$

Suppose that τ' is an element in the codomain of the map (12.3). Similar to the proof of Lemma 2.1, there is a unique element in $\tau := \nabla^{-1}(\tau') \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ such that for all $i = 1, 2, \cdots, l$,

$$\mathcal{P}_\tau(i,1) \in \{\bullet, s\},$$

and for all $(i,j) \in \mathrm{Box}(\tau')$,

$$\mathcal{P}_\tau(i,j+1) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}_{\tau'}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}_{\tau'}(i,j), & \text{if } \mathcal{P}_{\tau'}(i,j) \notin \{\bullet, s\}, \end{cases} \tag{12.4}$$

and

$$\mathcal{Q}_\tau(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}_{\tau'}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}_{\tau'}(i,j), & \text{if } \mathcal{Q}_{\tau'}(i,j) \notin \{\bullet, s\}. \end{cases} \tag{12.5}$$

Note that τ is in the domain of (12.3). It is then routine to check that the map

$$\nabla^{-1} : \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\} \to \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(l,1) \neq c \right\}$$

and the map (12.3) are inverse to each other. Hence the map (12.3) is bijective.

Similarly, the map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(l,1) = c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^-, \mathcal{Q}_{\tau'}(l,1) \in \{r, d\} \right\}$$

is well-defined, and we show that it is bijective by explicitly constructing its inverse. In view of Lemma 12.7, this proves the proposition in the case we are considering.

□

The key properties of the descent map when $\star \in \{ D, B, C^* \}$ are summarized in the following two propositions.

Proposition 12.9. *Suppose that $\star \in \{ D, B, C^* \}$. Then the map*

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}) \qquad \tau \mapsto (\nabla(\tau), \tau_{\mathbf{t}}) \tag{12.6}$$

is injective. Moreover, (12.6) *is bijective unless $\star \in \{ B, D \}$ and $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$.*

Proof. □

For every painted bipartition τ, write

$$\mathrm{Sign}(\tau) := (p_\tau, q_\tau).$$

When $\mathbf{r}_2(\check{\mathcal{O}}) > 0$, the double descent $\nabla^2(\tau) := \nabla(\nabla(\tau))$ is well-defined whenever $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. As in the Introduction, $\mathcal{O}$ denotes the Barbasch-Vogan dual of $\check{\mathcal{O}}$. We also consider it as a Young diagram.

Proposition 12.10. *Assume that* $\star \in \{ D, B, C^* \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. *Write* $\check{\mathcal{O}}'' := \check{\nabla}(\check{\mathcal{O}}')$ *and consider the map*

$$\delta\colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}), \qquad \tau \mapsto (\nabla^2(\tau), \tau_{\mathbf{t}}). \tag{12.7}$$

(a) Suppose that $\star = C^*$ *or* $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. *Then the map* (12.7) *is a bijective, and for every* $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}), \mathbf{c}_2(\mathcal{O})) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}).$$

(b) Suppose that $\star \in \{ B, D \}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$. *Then the map* (12.7) *is injection and its image equals*

$$\left\{ (\tau'', \tau_0) \in \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}}) \;\middle|\; x_{\tau''} = d \;\; \textit{or} \;\; \mathcal{P}_{\tau_0}^{-1}(\{ s, c \}) \neq \varnothing \right\}.$$

Moreover, for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}).$$

Proof. □

Suppose $T \in \mathcal{X}^\circ$ realizes the descent from $X = M(T) \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ to $X' = M'(T) \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Let $\alpha\colon \mathbf{K}_X \to \mathbf{K}'_{X'}$ be the homomorphism as in (12.22).

Let ρ' be a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}'_{X'}$. Then the representation $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}'}_{X'}} \otimes \rho'$ of $\widetilde{\mathbf{K}}'_{X'}$ descends to a representation of $\mathbf{K}'_{X'}$. Define

$$\vartheta_T(\rho') := \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}_X} \otimes (\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}'}_{X'}} \otimes \rho') \circ \alpha, \tag{12.8}$$

which is a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}_X$.

Clearly ϑ_T induces a homomorphism from $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}'_{X'})$ to $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}_X)$. In view of (12.24), we thus have a homomorphism

$$\vartheta_{\mathscr{O}',\mathscr{O}}\colon \mathcal{K}^{\mathbb{p}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{12.9}$$

This is independent of the choice of T.

Attached to a rational dual pair $(\mathbf{V}, \mathbf{V}')$, there is a distinguished character $\varsigma_{\mathbf{V},\mathbf{V}'}$ of $\widetilde{\mathbf{K}} \times \widetilde{\mathbf{K}}'$ arising from the oscillator representation, which we shall describe.

When G is a real symplectic group, let $\mathcal{X}_{\mathbf{V}}$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}}$ is identified with

$$\{ (g, c) \in \mathrm{GL}(\mathcal{X}_{\mathbf{V}}) \times \mathbb{C}^\times \mid \det(g) = c^2 \}.$$

It has a character $(g, c) \mapsto c$, which is denoted by $\det_{\mathcal{X}_{\mathbf{V}}}^{\frac{1}{2}}$.

When G is a quaternionic orthogonal group, still let $\mathcal{X}_V$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}} = \mathrm{GL}(\mathcal{X}_{\mathbf{V}})$. Let $\det_{\mathcal{X}_{\mathbf{V}}}$ denote its determinant character.

Write $\mathrm{Sign}(\mathbf{V}') = (n'^+, n'^-)$. Then the character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\tilde{\mathbf{K}}}$ is given by the following formula:

$$\varsigma_{\mathbf{V},\mathbf{V}'}|_{\tilde{\mathbf{K}}} := \begin{cases} \left(\det^{\frac{1}{2}}_{\mathcal{X}_{\mathbf{V}}}\right)^{n'^+ - n'^-}, & \text{if } G \text{ is a real symplectic group;} \\ \det_{\mathcal{X}_{\mathbf{V}}}^{\frac{n'^+ - n'^-}{2}}, & \text{if } G \text{ is a quaternionic orthogonal group;} \\ \text{the trivial character}, & \text{otherwise.} \end{cases}$$

The character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\tilde{\mathbf{K}}'}$ is given by a similar formula with $n'^+ - n'^-$ replaced by $n^- - n^+$, where $(n^+, n^-) = \mathrm{Sign}(\mathbf{V})$.

Let $(\mathbf{V}, \mathbf{V}')$ be a complex dual pair. Define a (-1)-symmetric bilinear space

$$\mathbf{W} := \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}'),$$

with the form

$$\langle T_1, T_2 \rangle_{\mathbf{W}} := \mathrm{tr}(T_1^\star T_2), \qquad \text{for all } T_1, T_2 \in \mathbf{W}.$$

Here $\star\colon \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}') \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}', \mathbf{V})$ is the adjoint map induced by the non-degenerate forms on $\mathbf{V}$ and $\mathbf{V}'$:

$$\langle Tv, v' \rangle_{\mathbf{V}'} = \langle v, T^\star v' \rangle_{\mathbf{V}}, \qquad \text{for all } v \in \mathbf{V}, v' \in \mathbf{V}', T \in \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}').$$

The pair of groups $(\mathbf{G}, \mathbf{G}') := (\mathbf{G}_{\mathbf{V}}, \mathbf{G}_{\mathbf{V}'})$ is a *complex reductive dual pair* in $\mathrm{Sp}(\mathbf{W})$ in the sense of Howe [38].

Further suppose that $(\mathbf{V}, J, L)$ and $(\mathbf{V}', J', L')$ form a rational dual pair. Then the complex symplectic space $\mathbf{W} = \mathrm{Hom}_{\mathbb{C}}(\mathbf{V}, \mathbf{V}')$ is naturally a $(-1,-1)$-space by defining

$$J_{\mathbf{W}}(T) := J' \circ T \circ J^{-1} \quad \text{and} \quad L_{\mathbf{W}}(T) := \dot{\epsilon}\, L' \circ T \circ L^{-1}, \qquad \text{for all } T \in \mathbf{W}.$$

The pair of groups $(G, G') = (\mathbf{G}_{\mathbf{V}}^{J}, \mathbf{G}_{\mathbf{V}'}^{J'})$ is then a (real) reductive dual pair in $\mathrm{Sp}(W)$, where $W := \mathbf{W}^{J_{\mathbf{W}}}$.

For the rest of this section, we work in the setting of rational dual pairs and introduce a notion of descent and lift of nilpotent orbits in this context.

12.2.1. *Moment maps.* We define the following *moment maps* $\mathbf{M}$ and $\mathbf{M}'$ with respect to a complex dual pair $(\mathbf{V}, \mathbf{V}')$:

$$\begin{array}{ccccc} \mathfrak{g} & \xleftarrow{\quad \mathbf{M} \quad} & \mathbf{W} & \xrightarrow{\quad \mathbf{M}' \quad} & \mathfrak{g}', \\ T^\star T & \longleftarrow\!\!\shortmid & T & \longmapsto & TT^\star. \end{array}$$

When we have a rational dual pair, we decompose

$$\mathbf{W} = \mathcal{X} \oplus \mathcal{Y}$$

where $\mathcal{X}$ and $\mathcal{Y}$ are $+\mathbf{i}$ and $-\mathbf{i}$ eigenspaces of $L_{\mathbf{W}}$, respectively. Restriction on $\mathcal{X}$ induces a pair of maps (see (12.14) for the definition of $\mathfrak{p}$ and $\mathfrak{p}'$):

$$\mathfrak{p} \xleftarrow{M := \mathbf{M}|_{\mathcal{X}}} \mathcal{X} \xrightarrow{M' := \mathbf{M}'|_{\mathcal{X}}} \mathfrak{p}',$$

which are also called *moment maps* (with respect to the rational dual pair). By classical invariant theory (see [88] or [63, Lemma 2.1]), the image $M(\mathcal{X})$ is Zariski closed in $\mathfrak{p}$, and the moment map M induces an isomorphism

$$\mathbf{K}' \backslash\backslash \mathcal{X} \cong M(\mathcal{X}) \tag{12.10}$$

of affine algebraic varieties, where $\mathbf{K}'\backslash\backslash\mathcal{X}$ denotes the affine quotient.[2] Thus [69, Corollary 4.7] implies that M maps every $\mathbf{K}'$-stable Zariski closed subset of $\mathcal{X}$ onto a Zariski closed subset of $\mathfrak{p}$. Similar statement holds for M'. We will use these basic facts freely.

In the rest of this section, we assume that $\left|\check{\mathcal{O}}\right| > 0$ unless otherwise mentioned. As before, write $\check{\mathcal{O}}'$ for the dual descent of $\check{\mathcal{O}}$ which has $\star'$-good parity. Let $\mathcal{O}'$ be the Barbarsch-Vogan dual of $\check{\mathcal{O}}'$.

Write $\mathsf{s}' := (\star', p', q')$, where (p', q') is a signature of type $\star'$. In what follows we will define a homomorphism

$$\check{\Theta} : \mathcal{K}_{\mathcal{O}'}(K_{\mathsf{s}',\mathbb{C}}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}),$$

to be called the geometric theta lift.

12.2.2. *An algebraic character.*

12.2.3. *Lift of algebraic vector bundles.* In the rest of this section, we assume that $\mathbb{p}$ is the parity of dim $\mathbf{V}$ if $\epsilon = 1$, and the parity of dim $\mathbf{V}'$ if $\epsilon' = 1$. Then $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\tilde{\mathbf{K}}}$ and $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\tilde{\mathbf{K}}'}$ are $\mathbb{p}$-genuine.

Suppose $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}' = \nabla(\mathcal{O}) \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. Using decomposition (12.25), we define a homomorphism

$$\vartheta_{\mathcal{O}',\mathcal{O}} := \sum_{\substack{\mathscr{O}\subset\mathcal{O}\cap\mathfrak{p} \\ \mathscr{O}'=\nabla(\mathscr{O})\subset\mathfrak{p}'}} \vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}'}(\tilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\tilde{\mathbf{K}}) \tag{12.11}$$

where the summation is over all pairs $(\mathscr{O}, \mathscr{O}')$ such that $\mathscr{O}' \subset \mathfrak{p}'$ is the descent of $\mathscr{O} \subset \mathcal{O}\cap\mathfrak{p}$.

12.3. **Geometric theta lift.** Recall that $\star'$ is the Howe dual of $\star$. In the rest of this section, we assume that $\left|\check{\mathcal{O}}\right| > 0$ unless otherwise mentioned. As before, write $\check{\mathcal{O}}'$ for the dual descent of $\check{\mathcal{O}}$ which has $\star'$-good parity. Let $\mathcal{O}'$ be the Barbarsch-Vogan dual of $\check{\mathcal{O}}'$.

Write $\mathsf{s}' := (\star', p', q')$, where (p', q') is a signature of type $\star'$. In what follows we will define a homomorphism

$$\check{\Theta} : \mathcal{K}_{\mathcal{O}'}(K_{\mathsf{s}',\mathbb{C}}) \to \mathcal{K}_{\mathsf{s}}(\mathcal{O}),$$

to be called the geometric theta lift.

For every $\dot{\epsilon}$-real form J of $\mathbf{V}$, up to conjugation by $\mathbf{G}^J$, there exists a unique $\dot{\epsilon}$-Cartan form L of $\mathbf{V}$ such that

Let $\epsilon \in \{\pm 1\}$. Let V $\mathbf{V}$ be a finite dimensional complex vector space equipped with an ϵ-symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle_{\mathbf{V}}$, to be called an ϵ-symmetric bilinear space. Denote its isometry group by

$$\mathbf{G} := \mathbf{G}_{\mathbf{V}} := \{\, g \in \mathrm{End}_{\mathbb{C}}(\mathbf{V}) \mid \langle g\cdot v_1, g\cdot v_2\rangle_{\mathbf{V}} = \langle v_1, v_2\rangle_{\mathbf{V}}\,, \quad \text{for all } v_1, v_2 \in \mathbf{V}\,\}\,.$$

The Lie algebra of $\mathbf{G}$ is given by

$$\mathfrak{g} := \mathfrak{g}_{\mathbf{V}} := \{\, X \in \mathrm{End}_{\mathbb{C}}(\mathbf{V}) \mid \langle X\cdot v_1, v_2\rangle_{\mathbf{V}} + \langle v_1, X\cdot v_2\rangle_{\mathbf{V}} = 0, \quad \text{for all } v_1, v_2 \in \mathbf{V}\,\}\,.$$

Note that $\mathbf{G}_{\mathbf{V}}$ is a trivial group if and only if $\mathbf{V} = \{0\}$.

12.4. **Real form, Cartan involution, and $(\epsilon, \dot{\epsilon})$-space.** We recall some facts about real structures and Cartan involutions on the complex classical group $\mathbf{G}$. See [65, Section 1.2-1.3]. Let $\dot{\epsilon} \in \{\pm 1\}$.[3]

Definition 12.11. *An $\dot{\epsilon}$-real form of $\mathbf{V}$ is a conjugate linear automorphism J of $\mathbf{V}$ such that*

$$J^2 = \epsilon\dot{\epsilon} \quad \text{and} \quad \langle Jv_1, Jv_2\rangle_{\mathbf{V}} = \overline{\langle v_1, v_2\rangle_{\mathbf{V}}}, \quad \text{for all } v_1, v_2 \in \mathbf{V}.$$

[2]See [69, Section 4.4] for the definition of affine quotient.

[3]In [65], $\dot{\epsilon}$ is denoted by ω.

For J as in Definition 12.11, write $\mathbf{G}^J$ for the centralizer of J in $\mathbf{G}$ (we will use similar notation without further explanation). It is a real form of $\mathbf{G}$ as in Table 1.

$\epsilon \backslash \dot\epsilon$	1	-1
1	real orthogonal group	quaternionic orthogonal group
-1	quaternionic symplectic group	real symplectic group

Table 1. The real classical group $\mathbf{G}^J$

Remarks. 1. The real form $\mathbf{G}^J$ may also be described as the isometry group of a $\dot\epsilon$-Hermitian space over a division algebra D, where D is $\mathbb{R}$ if $\epsilon\dot\epsilon = 1$ and the quaternion algebra $\mathbb{H}$ if $\epsilon\dot\epsilon = -1$.
2. Every real form of $\mathbf{G}$ is of the form $\mathbf{G}^J$ for some J.
3. The conjugate linear map J is an equivalent formulation of Adams-Barbasch-Vogan's notion of strong real forms [2, Definition 2.13].

Definition 12.12. *An $\dot\epsilon$-Cartan form of $\mathbf{V}$ is a linear automorphism L of $\mathbf{V}$ such that*

$$L^2 = \dot\epsilon \quad \text{and} \quad \langle Lv_1, Lv_2\rangle_{\mathbf{V}} = \langle v_1, v_2\rangle_{\mathbf{V}}, \quad \text{for all } v_1, v_2 \in \mathbf{V}.$$

Given an $\dot\epsilon$-Cartan form L of $\mathbf{V}$, conjugation by L yields an involution of the algebraic group $\mathbf{G}$. Thus we have a symmetric subgroup $\mathbf{G}^L$ of $\mathbf{G}$.

For L as in Lemma 3.1, conjugation by L induces a Cartan involution on $\mathbf{G}^J$.

Definition 12.13. *An ϵ-symmetric bilinear space $\mathbf{V}$ with a pair (J, L) as in Lemma 3.1 is called an $(\epsilon, \dot\epsilon)$*-space.

In view of Lemma 3.1 and when no confusion is possible, we also name the pairs $(\mathbf{V}, L)$ and $(\mathbf{V}, J)$ as $(\epsilon, \dot\epsilon)$-spaces. We define the notion of a non-degenerate $(\epsilon, \dot\epsilon)$-subspace and isomorphisms of $(\epsilon, \dot\epsilon)$-spaces in an obvious way.

Definition 12.14. *The signature of an $(\epsilon, \dot\epsilon)$-space $(\mathbf{V}, J, L)$ is defined by the following recipe:*

$$\mathrm{Sign}(\mathbf{V}) := \mathrm{Sign}(\mathbf{V}, L) := (n^+, n^-),$$

where n^+ and n^- are respectively the dimensions of $+1$ and -1 eigenspaces of L if $\dot\epsilon = 1$, and the dimensions of $+\mathbf{i}$ and $-\mathbf{i}$ eigenspaces of L if $\dot\epsilon = -1$. For every L-stable subquotient $\mathbf{E}$ of $\mathbf{V}$, the signature $\mathrm{Sign}(\mathbf{E})$ *is defined analogously.*

For a fixed pair $(\epsilon, \dot\epsilon)$, the isomorphism class of an $(\epsilon, \dot\epsilon)$-space $(\mathbf{V}, J, L)$ is determined by $\mathrm{Sign}(\mathbf{V})$. In view of Lemma 3.1, we define the signatures of the pairs $(\mathbf{V}, L)$ and $(\mathbf{V}, J)$ as that of $(\mathbf{V}, J, L)$. The value of $\mathrm{Sign}(\mathbf{V})$ is listed in Table 2 for the real classical group $\mathbf{G}^J$.

$\mathbf{G}^J$	$\mathrm{Sign}(\mathbf{V})$
$\mathrm{O}(p, q)$	(p, q)
$\mathrm{Sp}(2n, \mathbb{R})$	(n, n)
$\mathrm{O}^*(2n)$	(n, n)
$\mathrm{Sp}(p, q)$	$(2p, 2q)$

Table 2. Signature of $(\epsilon, \dot\epsilon)$-spaces

12.5. **Metaplectic cover.** Let (J, L) be as in Lemma 3.1. Put $G := \mathbf{G}^J$ and $\mathbf{K} := \mathbf{G}^L$. Then $K := G \cap \mathbf{K}$ is a maximal compact subgroup of both G and $\mathbf{K}$. As a slight modification of G, we define

$$\widetilde{G} := \begin{cases} \text{the metaplectic double cover of } G, & \text{if } G \text{ is a real symplectic group;} \\ G, & \text{otherwise.} \end{cases}$$

Here "G is a real symplectic group" means that $(\epsilon, \dot{\epsilon}) = (-1, -1)$, and similar terminologies will be used later on. In the case of a real symplectic group, we use ε_G to denote the nontrivial element in the kernel of the covering homomorphism $\widetilde{G} \to G$. In general, we use "$\widetilde{\ }$" over a subgroup of G to indicate its inverse image under the covering map $\widetilde{G} \to G$. For example, $\widetilde{K}$ is the a maximal compact subgroup of $\widetilde{G}$ with a covering map $\widetilde{K} \to K$ of degree 1 or 2. Similar notation will be used for other covering groups similar to $\widetilde{G}$.

When there is a need to indicate the dependence of various objects introduced on the $(\epsilon, \dot{\epsilon})$-space $\mathbf{V}$, we will add the subscript $\mathbf{V}$ in various notations (e.g. $\widetilde{G}_{\mathbf{V}}$ and $\widetilde{K}_{\mathbf{V}}$).

12.6. **Young diagrams and complex nilpotent orbits.** Let $\mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ be the set of nilpotent $\mathbf{G}$-orbits in $\mathfrak{g}$. For $n \in \mathbb{N} := \{0, 1, 2, \cdots\}$, let $\mathcal{P}_\epsilon(n)$ be the set of Young diagrams of size n such that

i) rows of even length appear in even times if $\epsilon = 1$;
ii) rows of odd length appear in even times if $\epsilon = -1$.

In this paper, a Young diagram $\mathbf{d}$ will be labeled by a sequence $[c_0, c_1, \cdots, c_k]$ of integers enumerating its columns, where $k \geqslant -1$. By convention, $[c_0, c_1, \cdots, c_k]$ denotes the empty sequence $\varnothing$ which labels the empty Young diagram when $k = -1$. It is easy to see that $\mathcal{P}_\epsilon(n)$ consists of sequences of the form $[c_0, c_1, \cdots, c_k]$ such that

$$\begin{cases} c_0 \geqslant c_1 \geqslant \cdots \geqslant c_k > 0, \\ \sum_{l=0}^k c_l = n, \text{ and} \\ \sum_{l=i}^k c_l \text{ is even, when } i \equiv \frac{1+\epsilon}{2} \pmod 2 \text{ and } 0 \leqslant i \leqslant k, \end{cases}$$

and $\mathcal{P}_\epsilon(0) := \{\varnothing\}$ by convention.

Definition 12.15. *For a nilpotent element X in $\mathfrak{g}$, set*

$$\mathrm{depth}(X) := \begin{cases} \max\{\, l \in \mathbb{N} \mid X^l \neq 0 \,\}, & \textit{if } \mathbf{V} \neq \{0\}; \\ -1, & \textit{if } \mathbf{V} = \{0\}. \end{cases} \tag{12.12}$$

Given $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$, pick any $X \in \mathcal{O}$. Set $\mathrm{depth}(\mathcal{O}) := \mathrm{depth}(X)$ *and define the Young diagram*

$$\mathbf{d}_{\mathcal{O}} := [c_0, c_1, \cdots, c_k], \qquad (k := \mathrm{depth}(\mathcal{O}) \geqslant -1)$$

where

$$c_l := \dim(\mathrm{Ker}(X^{l+1})/\mathrm{Ker}(X^l)), \quad \textit{for all } 0 \leqslant l \leqslant k.$$

We have the one-one correspondence: ([23, Chapter 5])

$$\begin{aligned} \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g}) &\longrightarrow \mathcal{P}_\epsilon(\dim \mathbf{V}), \\ \mathcal{O} &\longmapsto \mathbf{d}_{\mathcal{O}}. \end{aligned} \tag{12.13}$$

Let

$$\mathrm{Nil}_\epsilon := \left(\bigsqcup_{\mathbf{V}} \mathrm{Nil}_{\mathbf{G}_{\mathbf{V}}}(\mathfrak{g}_{\mathbf{V}}) \right) \Big/ \sim$$

where $\mathbf{V}$ runs over all ϵ-symmetric bilinear spaces, and $\sim$ denotes the equivalence relation induced by isomorphisms of ϵ-symmetric bilinear spaces. Let

$$\mathcal{P}_\epsilon := \bigsqcup_{n\geqslant 0} \mathcal{P}_\epsilon(n).$$

The correspondence (12.13) induces a bijection

$$\mathbf{D}\colon \operatorname{Nil}_\epsilon \longrightarrow \mathcal{P}_\epsilon.$$

Definition 12.16. *Define the descent of a Young diagram by*

$$\begin{aligned} \boldsymbol{\nabla}\colon \mathcal{P}_\epsilon &\longrightarrow \mathcal{P}_{-\epsilon} \\ [c_0, c_1, \cdots, c_k] &\longmapsto [c_1, \cdots, c_k], \end{aligned}$$

namely by removing the left most column of the diagram. By convention, $\boldsymbol{\nabla}(\varnothing) := \varnothing$.

Fix $\mathbb{p} \in \mathbb{Z}/2\mathbb{Z}$ as in the Introduction. As highlighted in the Introduction, we will consider the following set of partitions/Young diagrams/nilpotent orbits.

Definition 12.17. *Denote $\epsilon_l := \epsilon(-1)^l$, for $l \geqslant 0$. Let*

$$\mathcal{P}_\epsilon^{\mathbb{p}} := \left\{ \mathbf{d} = [c_0, \cdots, c_k] \in \mathcal{P}_\epsilon \;\middle|\; \begin{array}{l} \bullet \text{ all } c_i\text{'s have parity } \mathbb{p}; \\ \bullet\ c_l \geqslant c_{l+1} + 2, \text{ if } 0 \leqslant l \leqslant k-1 \text{ and} \\ \quad \epsilon_l = 1. \end{array} \right\}.$$

By convention, $\varnothing \in \mathcal{P}_\epsilon^{\mathbb{p}}$. Denote by $\operatorname{Nil}_{\mathbf{G}}^{\mathbb{p}}(\mathfrak{g})$ the subset of $\operatorname{Nil}_{\mathbf{G}}(\mathfrak{g})$ corresponding to partitions in $\mathcal{P}_{\mathbf{G}}^{\mathbb{p}} := \mathcal{P}_\epsilon^{\mathbb{p}} \cap \mathcal{P}_\epsilon(\dim \mathbf{V})$.

12.7. **Signed Young diagrams and rational nilpotent orbits.** Recall that $(\mathbf{V}, J, L)$ is an $(\epsilon, \dot\epsilon)$-space. Under conjugation by L, the complex Lie algebra $\mathfrak{g}$ decomposes into ± 1-eigenspaces:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k}_{\mathbf{V}} \oplus \mathfrak{p}_{\mathbf{V}}. \tag{12.14}$$

Let $\operatorname{Nil}_{\mathbf{K}}(\mathfrak{p})$ be the set of nilpotent $\mathbf{K}$-orbits in $\mathfrak{p}$.

12.7.1. *Parametrization.* In this section, we explain the parameterization of $\operatorname{Nil}_{\mathbf{K}}(\mathfrak{p})$.

Definition 12.18. *Let $\mathbb{Z}^2_{\geqslant 0}$ be the set of pairs of non-negative integers, whose elements are called* signatures. *For a signature $n = (n^+, n^-)$, its* dual signature *is defined to be $\check n := (n^-, n^+)$. Define a partial order on $\mathbb{Z}^2_{\geqslant 0}$ by*

$$(n_1^+, n_1^-) \succeq (n_2^+, n_2^-) \quad \text{if and only if} \quad n_1^+ \geqslant n_2^+ \text{ and } n_1^- \geqslant n_2^-.$$

Recall that a signed Young diagram is a Young diagram in which every box is labeled with a $+$ or $-$ sign in such a way that signs alternate across rows. For a signed Young diagram which contains n_i^+ number of "$+$" signs and n_i^- number of "$-$" signs in the i-th column, we will label it by the sequence of signatures $[d_0, d_1, \cdots, d_k]$, where $k \geqslant -1$ and $d_i = (n_i^+, n_i^-)$. By convention $[d_0, d_1, \cdots, d_k]$ labels the empty diagram when $k = -1$.

The set of all signed Young diagrams will then correspond to the set

$$\ddot{\mathcal{P}} := \bigsqcup_{k\geqslant -1} \{\, [d_0, \cdots, d_k] \in (\mathbb{Z}^2_{\geqslant 0}\backslash\{(0,0)\})^{k+1} \mid d_l \succeq \check d_{l+1} \text{ for all } 0 \leqslant l \leqslant k-1 \,\}.$$

Similar to Definition 12.16, we define the descent map

$$\nabla\colon \ddot{\mathcal{P}} \longrightarrow \ddot{\mathcal{P}}, \quad [d_0, d_1, \cdots, d_k] \mapsto [d_1, \cdots, d_k]. \tag{12.15}$$

Definition 12.19. *Given $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, pick any $X \in \mathscr{O}$. Set* $\mathrm{depth}(\mathscr{O}) := \mathrm{depth}(X)$ *(see* (12.12)*) and define the signed Young diagram*

$$\ddot{\mathbf{d}}_{\mathscr{O}} := [d_0, \cdots, d_k], \qquad (k := \mathrm{depth}(\mathscr{O}) \geqslant -1)$$

where

$$d_l := \mathrm{Sign}(\mathrm{Ker}(X^{l+1})/\,\mathrm{Ker}(X^l)), \quad \textit{for all } 0 \leqslant l \leqslant k.$$

We parameterize $\mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ via the injective map (*cf.* [26])

$$\ddot{\mathbf{D}} \colon \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p}) \longrightarrow \ddot{\mathcal{P}}, \quad \mathscr{O} \mapsto \ddot{\mathbf{d}}_{\mathscr{O}}.$$

12.7.2. *Stabilizers.* Let $\mathfrak{sl}_2(\mathbb{C})$ be the complex Lie algebra consisting of 2×2 complex matrices of trace zero. Write

$$\mathring{L} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathring{\mathbf{e}} := \begin{pmatrix} 1/2 & \mathbf{i}/2 \\ \mathbf{i}/2 & -1/2 \end{pmatrix}.$$

Let $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ with $\ddot{\mathbf{D}}(\mathscr{O}) = [d_0, \cdots, d_k]$ and pick any element $X \in \mathscr{O}$. By [75] (also see [83, Section 6]), there is a Lie algebra homomorphism (unique up to $\mathbf{K}$-conjugation)

$$\phi_{\mathfrak{k}} \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{g}$$

such that

- $\phi_{\mathfrak{k}}$ intertwines the conjugation of $\mathring{L}$ on $\mathfrak{sl}_2(\mathbb{C})$ and the conjugation of L on $\mathfrak{g}$; and
- $\phi_{\mathfrak{k}}(\mathring{\mathbf{e}}) = X$;

We call $\phi_{\mathfrak{k}}$ an L-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple attached to X.

For each $l \geqslant 0$, let $\phi_l \colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{l+1}(\mathbb{C})$ denote an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ realized on $\mathbb{C}^{l+1}$. Fix an $\mathrm{SL}_2(\mathbb{C})$-invariant $(-1)^l$-symmetric non-degenerate bilinear form $\langle\, ,\, \rangle_l$ on $\mathbb{C}^{l+1}$. As an $\mathfrak{sl}_2(\mathbb{C})$-module via $\phi_{\mathfrak{k}}$, we have

$$\mathbf{V} = \bigoplus_{l=0}^{k} {}^l\mathbf{V} \otimes_{\mathbb{C}} \mathbb{C}^{l+1}, \qquad (k := \mathrm{depth}(\mathscr{O}) \geqslant -1) \tag{12.16}$$

where $\mathbb{C}^{l+1}$ is viewed as an $\mathfrak{sl}_2(\mathbb{C})$-module via the differential of ϕ_l, and

$${}^l\mathbf{V} := \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V})$$

is the multiplicity space.

Let

$$L_l := (-1)^{\lfloor \frac{l}{2} \rfloor} \phi_l(\mathring{L}).$$

Then $(\mathbb{C}^{l+1}, L_l)$ is a $((-1)^l, (-1)^l)$-space and the L_l-stable subspace

$$(\mathbb{C}^{l+1})^{\mathring{\mathbf{e}}} := \{ v \in \mathbb{C}^{l+1} \mid \mathring{\mathbf{e}} \cdot v = 0 \}$$

has signature $(1, 0)$.

Define the $(-1)^l\epsilon$-symmetric bilinear form $\langle\, ,\, \rangle_{{}^l\mathbf{V}}$ on ${}^l\mathbf{V}$ by requiring that

$$\langle\, ,\, \rangle_{\mathbf{V}} = \bigoplus_l \langle\, ,\, \rangle_{{}^l\mathbf{V}} \otimes \langle\, ,\, \rangle_l\,.$$

Define

$${}^lL(T) := L \circ T \circ L_l^{-1}, \qquad \text{for all } T \in {}^l\mathbf{V} = \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V}).$$

It is routine to check that $({}^l\mathbf{V}, {}^lL)$ is a $((-1)^l\epsilon, (-1)^l\dot{\epsilon})$-space and it has signature $d_l - \check{d}_{l+1}$ if l is even and has signature $\check{d}_l - d_{l+1}$ if l is odd.

Let $\mathbf{K}_X := \mathrm{Stab}_{\mathbf{K}}(X)$ be the stabilizer of X in $\mathbf{K}$, and $\mathbf{R}_X$ the stabilizer of $\phi_{\mathfrak{k}}$ in $\mathbf{K}$. Then $\mathbf{K}_X = \mathbf{R}_X \ltimes \mathbf{U}_X$, where $\mathbf{U}_X$ denotes the unipotent radical of $\mathbf{K}_X$.

Set ${}^{l}\mathbf{K} := (\mathbf{G}_{{}^{l}\mathbf{V}})^{{}^{l}L}$. Using the decomposition (12.16), we get the following lemma from the discussion above.

Lemma 12.20. *There is a canonical isomorphism*

$$\mathbf{R}_X \cong \prod_{l=0}^{k} {}^{l}\mathbf{K}.$$

Let $A := G/G^\circ$ be the component group of G, where G° denotes the identity connected component of G. It is isomorphic to the component group of $\mathbf{K}$ and is trivial unless G is a real orthogonal group.

Suppose we are in the case when $G = \mathrm{O}(p,q)$ $(p,q \geqslant 0)$ is a real orthogonal group. Then $\mathbf{K} = \mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C})$ is a product of two complex orthogonal groups. For every pair $\eta := (\eta^+,\eta^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, let sgn^η denote the character $\det^{\eta^+} \boxtimes \det^{\eta^-}$ of $\mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C})$, where det denotes the sign character of an orthogonal group. It obviously induces characters on $\mathrm{O}(p) \times \mathrm{O}(q)$, $\mathrm{O}(p,q)$ and A, which are still denoted by sgn^η. Then

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2 \to \widehat{A}, \quad \eta \mapsto \mathrm{sgn}^\eta$$

is a surjective homomorphism. Here and henceforth, "$\widehat{\ }$" indicates the group of characters of a finite abelian group.

We record the following lemma.

Lemma 12.21. *Suppose $G = \mathrm{O}(p,q)$ $(p,q \geqslant 0)$. Write $\mathrm{Sign}({}^{l}\mathbf{V}) = ({}^{l}p^+, {}^{l}p^-)$ for $0 \leqslant l \leqslant k$. Then there is an obvious isomorphism*

$$ {}^{l}\mathbf{K} \cong \begin{cases} \mathrm{O}({}^{l}p^+,\mathbb{C}) \times \mathrm{O}({}^{l}p^-,\mathbb{C}), & \textit{if } l \textit{ is even;} \\ \mathrm{GL}({}^{l}p^+,\mathbb{C}), & \textit{if } l \textit{ is odd.} \end{cases} \tag{12.17}$$

Moreover for $\eta := (\eta^+,\eta^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the character sgn^η of $\mathbf{K}$ has trivial restriction to $\mathbf{U}_X$, and with respect to the isomorphism (12.17),

$$\mathrm{sgn}^\eta|_{{}^{l}\mathbf{K}} = \begin{cases} \det^{\eta^+} \boxtimes \det^{\eta^-}, & \textit{if } l \textit{ is even;} \\ 1, & \textit{if } l \textit{ is odd.} \end{cases}$$

Proof. The first assertion follows by using Table 2. Since $\mathbf{U}_X$ is unipotent, it has no nontrivial algebraic character. Thus sgn^η of $\mathbf{K}$ has trivial restriction to $\mathbf{U}_X$. The last assertion of the lemma is routine to check, which we omit. □

12.8. **Dual pairs, descent and lift of nilpotent orbits.**

Definition 12.22 (Dual pair). *(i) A* complex dual pair *is a pair consisting of an ϵ-symmetric bilinear space and an ϵ'-symmetric bilinear space, where $\epsilon, \epsilon' \in \{\pm 1\}$ with $\epsilon\epsilon' = -1$.*

(ii) A rational dual pair *is a pair consisting of an $(\epsilon,\dot\epsilon)$-space and an $(\epsilon',\dot\epsilon')$-space, where $\epsilon,\epsilon',\dot\epsilon,\dot\epsilon' \in \{\pm 1\}$ with $\epsilon\epsilon' = \dot\epsilon\dot\epsilon' = -1$.*

12.8.1. *Lifts and descents of nilpotent orbits.* Let

$$\mathbf{W}^\circ := \{ T \in \mathbf{W} \mid T \text{ is a surjective map from } \mathbf{V} \text{ onto } \mathbf{V}' \}.$$

Clearly $\mathbf{W}^\circ \neq \varnothing$ only if $\dim \mathbf{V} \geqslant \dim \mathbf{V}'$.

Suppose $\mathbf{e} \in \mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathbf{e}' \in \mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. We call $\mathbf{e}'$ (resp. $\mathcal{O}'$) a descent of $\mathbf{e}$ (resp. $\mathcal{O}$), if there exits $T \in \mathbf{W}^\circ$ such that

$$\mathbf{M}(T) = \mathbf{e} \quad \text{and} \quad \mathbf{M}'(T) = \mathbf{e}'.$$

Put
$$\mathcal{X}^\circ := \mathbf{W}^\circ \cap \mathcal{X},$$
and write $\mathbf{K} := \mathbf{K}_{\mathbf{V}}$ and $\mathbf{K}' := \mathbf{K}_{\mathbf{V}'}$. Suppose $X \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ and $X' \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. We call X' (resp. $\mathscr{O}'$) a descent of X (resp. $\mathscr{O}$), if there exits $T \in \mathcal{X}^\circ$ such that
$$M(T) = X \quad \text{and} \quad M'(T) = X'.$$
In all cases, we will say that T realizes the descent, and $\mathcal{O}$ (resp. $\mathscr{O}$) is the lift of $\mathcal{O}'$ (resp. $\mathscr{O}'$). In the notation of Definition 12.16, we then have
$$\boldsymbol{\nabla}(\mathbf{d}_{\mathcal{O}}) = \mathbf{d}_{\mathcal{O}'} \quad \text{and} \quad \nabla(\ddot{\mathbf{d}}_{\mathscr{O}}) = \ddot{\mathbf{d}}_{\mathscr{O}'}.$$
Hence the notion of descent (for nilpotent orbits) defined here agrees with that of Definition 12.16 and (12.15) (for Young diagrams). We will thus write
$$\mathcal{O}' = \boldsymbol{\nabla}(\mathcal{O}) = \boldsymbol{\nabla}_{\mathbf{V},\mathbf{V}'}(\mathcal{O}) \quad \text{and} \quad \mathscr{O}' = \nabla(\mathscr{O}) = \nabla_{\mathbf{V},\mathbf{V}'}(\mathscr{O}).$$
We record a key property on descent and lift:
$$\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}} \quad \text{and} \quad M(M'^{-1}(\overline{\mathscr{O}'})) = \overline{\mathscr{O}}, \tag{12.18}$$
where " $\overline{}$ " means taking Zariski closure. This is checked by using explicit formulas in [25, 45] (for complex dual pairs) and [65, Lemma 14] (for rational dual pairs).

In fact by [25, Theorem 1.1], the notion of lift can be extended to an arbitrary complex dual pair and an arbitrary complex nilpotent orbit: for any $\mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$, $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'}))$ equals to the closure of a unique nilpotent orbit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$. We call $\mathcal{O}$ the *theta lift* of $\mathcal{O}'$, written as
$$\mathcal{O} = \boldsymbol{\vartheta}_{\mathbf{V}',\mathbf{V}}(\mathcal{O}'). \tag{12.19}$$

12.8.2. *Generalized descent of nilpotent orbits.* Let
$$\mathbf{W}^{\mathrm{gen}} := \{\, T \in \mathbf{W} \mid \text{the image of } T \text{ is a non-degenerate subspace of } \mathbf{V}' \,\}$$
and
$$\mathcal{X}^{\mathrm{gen}} := \mathbf{W}^{\mathrm{gen}} \cap \mathcal{X}.$$
Suppose $X \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ and $X' \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. We call X' (resp. $\mathscr{O}'$) a generalized descent of X (resp. $\mathscr{O}$), if there exits $T \in \mathcal{X}^{\mathrm{gen}}$ such that
$$M(T) = X \quad \text{and} \quad M'(T) = X'.$$
As before, we say that T realizes the generalized descent.

It is easy to see that for each nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, the following three assertions are equivalent (*cf.* [26, Table 4]).

- The orbit $\mathscr{O}$ has a generalized descent.
- The orbit $\mathscr{O}$ is contained in the image of the moment map M.
- Write $\ddot{\mathbf{d}}_{\mathscr{O}} = [d_0, d_1, \cdots, d_k] \in \ddot{\mathcal{P}}$, then
$$\mathrm{Sign}(\mathbf{V}') \geq \sum_{i=1}^{k} d_i.$$

When this is the case, $\mathscr{O}$ has a unique generalized descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$, and
$$\ddot{\mathbf{d}}_{\mathscr{O}'} = [d_1 + s, d_2, \cdots, d_k], \quad \text{where } s := \mathrm{Sign}(\mathbf{V}') - \sum_{i=1}^{k} d_i. \tag{12.20}$$
We write $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})$. On the other hand, different nilpotent orbits may map to a same nilpotent orbit under $\nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}$.

Analogously, suppose $\mathbf{e} \in \mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathbf{e}' \in \mathcal{O}' \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. We call $\mathbf{e}'$ (resp. $\mathcal{O}'$) a generalized descent of $\mathbf{e}$ (resp. $\mathcal{O}$), if there exits an element $T \in \mathbf{W}^{\mathrm{gen}}$ such that

$$\mathbf{M}(T) = \mathbf{e} \quad \text{and} \quad \mathbf{M}'(T) = \mathbf{e}'.$$

When this is the case, $\mathcal{O}'$ is determined by $\mathcal{O}$ and we write $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$.

Lemma 12.23. *Assume that $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ has a generalized descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Let $X \in \mathscr{O}$ and $T \in \mathcal{X}^{\mathrm{gen}}$ such that $M(T) = X$. Then $\mathbf{K}' \cdot T$ is the unique closed $\mathbf{K}'$-orbit in $M^{-1}(X)$. Moreover,*

$$\mathscr{O}' = M'(M^{-1}(\mathscr{O})) \cap \mathcal{O}' = M'(M^{-1}(\mathscr{O})) \cap \overline{\mathcal{O}'}, \tag{12.21}$$

where $\mathcal{O}' := \mathbf{G}' \cdot \mathscr{O}'$, which is the generalized descent of $\mathcal{O} := \mathbf{G} \cdot \mathscr{O}$.

Proof. It is elementary to check that $\mathbf{K}' \cdot T$ is Zariski closed in $\mathcal{X}$. Then the first assertion follows by using the isomorphism (12.10). Note that $\mathscr{O}'$ is the only $\mathbf{K}'$-orbit in $\overline{\mathcal{O}'} \cap \mathfrak{p}'$ whose Zariski closure contains $\mathscr{O}'$. Thus the first assertion implies the second one. ☐

In this article, we will need to consider the following special types of nilpotent orbits.

Definition 12.24. *A nilpotent oribit $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ with $\mathrm{d}_{\mathcal{O}} = [c_0, c_1 \cdots, c_k] \in \mathcal{P}_\epsilon$ is said to be good for generalized descent if $k \geqslant 1$ and $c_0 = c_1$. A nilpotent orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is said to be* good *for generalized descent if the nilpotent orbit $\mathbf{G} \cdot \mathscr{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is good for generalized descent.*

The following lemma exhibits a certain maximality property of nilpotent orbits which are good for generalized descent.

Lemma 12.25. *If $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ is good for generalized descent, and $\mathcal{O}' = \boldsymbol{\nabla}^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$, then $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'})) = \overline{\mathcal{O}}$. Consequently, if $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ is good for generalized descent and $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})$, then $\mathscr{O}$ is an open $\mathbf{K}$-orbit in $M(M'^{-1}(\overline{\mathscr{O}'}))$.*

Proof. This is easy to check using the explicit description of $\mathbf{M}(\mathbf{M}'^{-1}(\overline{\mathcal{O}'}))$ in [25, Theorem 5.2 and 5.6]. ☐

12.8.3. *Map between isotropy groups.* Suppose $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ admits a descent $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. According to [45, Proposition 11.1] and [65, Lemmas 13 and 14], $M^{-1}(\mathscr{O})$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit contained in $\mathcal{X}^\circ$ and $M'(M^{-1}(\mathscr{O})) = \mathscr{O}'$. Moreover $\mathbf{K}'$ acts on $M^{-1}(\mathscr{O})$ freely.

Fix $T \in M^{-1}(\mathscr{O})$ which realizes the descent from $X := M(T) \in \mathscr{O}$ to $X' := M'(T) \in \mathscr{O}'$. Denote the respective isotropy subgroups by

$$\mathbf{S}_T := \mathrm{Stab}_{\mathbf{K}\times\mathbf{K}'}(T), \quad \mathbf{K}_X := \mathrm{Stab}_{\mathbf{K}}(X) \quad \text{and} \quad \mathbf{K}'_{X'} := \mathrm{Stab}_{\mathbf{K}'}(X').$$

Then there is a unique homomorphism

$$\alpha\colon \mathbf{K}_X \mapsto \mathbf{K}'_{X'} \tag{12.22}$$

such that $\mathbf{S}_T$ is the graph of α:

$$\mathbf{S}_T = \{\, (k, \alpha(k)) \in \mathbf{K}_X \times \mathbf{K}'_{X'} \mid k \in \mathbf{K}_X \,\}.$$

The homomorphism α is uniquely determined by the requirement that

$$\alpha(k)(Tv) = T(kv) \quad \text{for all } v \in \mathbf{V},\ k \in \mathbf{K}_X.$$

We recall the notation in Section 12.7.2 where $\phi_{\mathfrak{k}}$ is an L-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple attached to X. Let $\mathring{H} := \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$. Then there is a unique L'-compatible $\mathfrak{sl}_2(\mathbb{C})$-triple $\phi_{\mathfrak{k}'}$ attached to X' such that (see [29, Section 5.2])

$$\phi_{\mathfrak{k}'}(\mathring{H}) \circ T - T \circ \phi_{\mathfrak{k}}(\mathring{H}) = T.$$

As an $\mathfrak{sl}_2(\mathbb{C})$-module via $\phi_{\mathfrak{k}'}$,

$$\mathbf{V}' = \bigoplus_{l \geqslant 0}^{k-1} {}^{l}\mathbf{V}' \otimes \mathbb{C}^{l+1}.$$

We adopt notations in Section 12.7.2 to $X' \in \mathscr{O}'$, via $\phi_{\mathfrak{k}'}$. In particular, we have $\mathbf{K}_X = \mathbf{R}_X \ltimes \mathbf{U}_X$ and $\mathbf{K}'_{X'} = \mathbf{R}_{X'} \ltimes \mathbf{U}_{X'}$. Here $\mathbf{U}_X$ and $\mathbf{U}'_{X'}$ are the unipotent radicals, and $\mathbf{R}_X = \prod_{l=0}^{k} {}^{l}\mathbf{K}$ and $\mathbf{R}_{X'} = \prod_{l=0}^{k-1} {}^{l}\mathbf{K}'$ are Levi factors of $\mathbf{K}_X$ and $\mathbf{K}'_{X'}$, respectively.

For each irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}^{l+1}$ fix a nonzero vector $v_l \in (\mathbb{C}^{l+1})^{\mathring{\mathbf{e}}}$. For each $l \geqslant 0$, the map $\nu \mapsto \nu(v_l)$ identifies ${}^{l}\mathbf{V} = \mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(\mathbb{C}^{l+1}, \mathbf{V})$ (resp. ${}^{l}\mathbf{V}'$) with a subspace ${}^{l}\mathbf{V}_0$ of $\mathbf{V}$ (resp. ${}^{l}\mathbf{V}'_0$ of $\mathbf{V}'$). For $1 \leqslant l \leqslant k$, $T^\star$ induces a vector space isomorphism[4]

$$\tau_l \colon {}^{l-1}\mathbf{V}' = {}^{l-1}\mathbf{V}'_0 \xrightarrow{v \mapsto T^\star(v)} {}^{l}\mathbf{V}_0 = {}^{l}\mathbf{V}.$$

This results in an isomorphism (which is independent of the choices of v_l and v_{l-1})

$$\alpha_l : {}^{l}\mathbf{K} \xrightarrow{\cong} {}^{l-1}\mathbf{K}', \qquad h_l \mapsto (\tau_l)^{-1} \circ h_l \circ \tau_l. \tag{12.23}$$

Lemma 12.26. *The homomorphism α maps $\mathbf{R}_X$ into $\mathbf{R}_{X'}$ and maps $\mathbf{U}_X$ into $\mathbf{U}_{X'}$. Moreover, the map $\alpha|_{\mathbf{R}_X}$ is given by*

$$\begin{aligned} \alpha|_{\mathbf{R}_X} \colon \textstyle\prod_{l=0}^{k} {}^{l}\mathbf{K} &\longrightarrow \textstyle\prod_{l=0}^{k-1} {}^{l}\mathbf{K}', \\ (h_0, h_1, \cdots h_k) &\longmapsto (\alpha_1(h_1), \cdots, \alpha_k(h_k)), \qquad h_l \in {}^{l}\mathbf{K}, \end{aligned}$$

where α_l $(1 \leqslant l \leqslant k)$ is given in (12.23).

Proof. Note that α is a surjection since $M^{-1}(X)$ is a $\mathbf{K}'$-orbit (see Lemma B.1 (ii) or [65, Lemma 13]). So $\alpha(\mathbf{U}_X)$ is a unipotent normal subgroup in $\mathbf{K}'_{X'}$ which must be contained in $\mathbf{U}_{X'}$. Note that (see [23, Lemma 3.4.4])

$$\mathbf{R}_X = \mathrm{Stab}_{\mathbf{K}}(\phi_{\mathfrak{k}}(\mathring{\mathbf{e}})) \cap \mathrm{Stab}_{\mathbf{K}}(\phi_{\mathfrak{k}}(\mathring{H})).$$

By the defintion of α and $\phi_{\mathfrak{k}'}$, we see

$$\alpha(\mathbf{R}_X) \subset \mathrm{Stab}_{\mathbf{K}'}(\phi_{\mathfrak{k}'}(\mathring{\mathbf{e}})) \cap \mathrm{Stab}_{\mathbf{K}'}(\phi_{\mathfrak{k}'}(\mathring{H})) = \mathbf{R}_{X'}.$$

The rest follows from the discussions before the lemma. □

Let A, A_X and $A'_{X'}$ be the component groups of G, $\mathbf{K}_X$ and $\mathbf{K}'_{X'}$ respectively. We will identify A with the component group of $\mathbf{K}$ and let $\chi|_{A_X}$ denote the pullback of $\chi \in \widehat{A}$ via the natural map $A_X \to A$. The homomorphism $\alpha \colon \mathbf{K}_X \mapsto \mathbf{K}'_{X'}$ induces a homomorphism $\alpha \colon A_X \mapsto A'_{X'}$, which further yields a homomorphism

$$\widehat{A'_{X'}} \to \widehat{A_X}, \quad \rho' \mapsto \rho' \circ \alpha.$$

Lemma 12.27. *The following map is surjective:*

$$\begin{aligned} \widehat{A} \times \widehat{A'_{X'}} &\longrightarrow\!\!\!\!\rightarrow \widehat{A_X}, \\ (\chi, \rho') &\longmapsto \chi|_{A_X} \cdot (\rho' \circ \alpha). \end{aligned}$$

[4]In fact, it is a similitude between the two formed spaces and satisfis $L \circ \tau_l = \mathbf{i}\, \tau_l \circ L'$.

Proof. This follows easily from Lemma 12.26 and Lemma 12.21. □

12.9. **Lifting of equivariant vector bundles and admissible orbit data.** Recall from the Introduction the complexification $\widetilde{\mathbf{K}}$ of $\widetilde{K}$, which is $\mathbf{K}$ except when G is a real symplectic group. For a nilpotent $\mathbf{K}$-orbit $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, let $\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ denote the Grothendieck group of $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaves on $\mathscr{O}$. This is a free abelian group with a free basis consisting of isomorphism classes of irreducible $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Taking the isotropy representation at a point $X \in \mathscr{O}$ yields an identification

$$\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) = \mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}_X), \tag{12.24}$$

where the right hand side denotes the Grothendieck group of the category of $\mathbb{p}$-genuine algebraic representations of the stabilizer group $\widetilde{\mathbf{K}}_X$.

For $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$, let $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ denote the Grothendieck group of $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaves on $\mathcal{O} \cap \mathfrak{p}$. Since $\mathcal{O} \cap \mathfrak{p}$ is the finite union of its $\mathbf{K}$-orbits, we have

$$\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) = \bigoplus_{\mathscr{O} \text{ is a } \mathbf{K}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}} \mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{12.25}$$

There is a natural partial order $\geq$ on $\mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}})$ and $\mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$: we say $c_1 \geq c_2$ (or $c_2 \leq c_1$) if $c_1 - c_2$ is represented by a $\mathbb{p}$-genuine $\widetilde{\mathbf{K}}$-equivariant coherent sheaf. The same notation obviously applies to other Grothendieck groups.

12.9.1. *An algebraic character.* Attached to a rational dual pair $(\mathbf{V}, \mathbf{V}')$, there is a distinguished character $\varsigma_{\mathbf{V},\mathbf{V}'}$ of $\widetilde{\mathbf{K}} \times \widetilde{\mathbf{K}}'$ arising from the oscillator representation, which we shall describe.

When G is a real symplectic group, let $\mathcal{X}_{\mathbf{V}}$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}}$ is identified with

$$\{\, (g, c) \in \mathrm{GL}(\mathcal{X}_{\mathbf{V}}) \times \mathbb{C}^\times \mid \det(g) = c^2 \,\}.$$

It has a character $(g, c) \mapsto c$, which is denoted by $\det^{\frac{1}{2}}_{\mathcal{X}_{\mathbf{V}}}$.

When G is a quaternionic orthogonal group, still let $\mathcal{X}_V$ denote the $\mathbf{i}$-eigenspace of L. Then $\widetilde{\mathbf{K}} = \mathrm{GL}(\mathcal{X}_{\mathbf{V}})$. Let $\det_{\mathcal{X}_{\mathbf{V}}}$ denote its determinant character.

Write $\mathrm{Sign}(\mathbf{V}') = (n'^+, n'^-)$. Then the character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}}$ is given by the following formula:

$$\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}} := \begin{cases} \left(\det^{\frac{1}{2}}_{\mathcal{X}_{\mathbf{V}}}\right)^{n'^+ - n'^-}, & \text{if } G \text{ is a real symplectic group;} \\ \det_{\mathcal{X}_{\mathbf{V}}}^{\frac{n'^+ - n'^-}{2}}, & \text{if } G \text{ is a quaternionic orthogonal group;} \\ \text{the trivial character,} & \text{otherwise.} \end{cases}$$

The character $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'}$ is given by a similar formula with $n'^+ - n'^-$ replaced by $n^- - n^+$, where $(n^+, n^-) = \mathrm{Sign}(\mathbf{V})$.

12.9.2. *Lift of algebraic vector bundles.* In the rest of this section, we assume that $\mathbb{p}$ is the parity of $\dim \mathbf{V}$ if $\epsilon = 1$, and the parity of $\dim \mathbf{V}'$ if $\epsilon' = 1$. Then $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}}$ and $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'}$ are $\mathbb{p}$-genuine.

Suppose $T \in \mathcal{X}^\circ$ realizes the descent from $X = M(T) \in \mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ to $X' = M'(T) \in \mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$. Let $\alpha\colon \mathbf{K}_X \to \mathbf{K}'_{X'}$ be the homomorphism as in (12.22).

Let ρ' be a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}'_{X'}$. Then the representation $\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}'_{X'}} \otimes \rho'$ of $\widetilde{\mathbf{K}}'_{X'}$ descends to a representation of $\mathbf{K}'_{X'}$. Define

$$\vartheta_T(\rho') := \varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}}_X} \otimes (\varsigma_{\mathbf{V},\mathbf{V}'}|_{\widetilde{\mathbf{K}'}_{X'}} \otimes \rho') \circ \alpha, \tag{12.26}$$

which is a $\mathbb{p}$-genuine algebraic representation of $\widetilde{\mathbf{K}}_X$.

Clearly ϑ_T induces a homomorphism from $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}'_{X'})$ to $\mathcal{R}^{\mathbb{p}}(\widetilde{\mathbf{K}}_X)$. In view of (12.24), we thus have a homomorphism

$$\vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{12.27}$$

This is independent of the choice of T.

Suppose $\mathcal{O} \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}' = \nabla(\mathcal{O}) \in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$. Using decomposition (12.25), we define a homomorphism

$$\vartheta_{\mathcal{O}',\mathcal{O}} := \sum_{\substack{\mathscr{O}\subset\mathcal{O}\cap\mathfrak{p}\\ \mathscr{O}'=\nabla(\mathscr{O})\subset\mathfrak{p}'}} \vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\mathcal{O}'}(\widetilde{\mathbf{K}}') \longrightarrow \mathcal{K}^{\mathbb{p}}_{\mathcal{O}}(\widetilde{\mathbf{K}}) \tag{12.28}$$

where the summation is over all pairs $(\mathscr{O}, \mathscr{O}')$ such that $\mathscr{O}' \subset \mathfrak{p}'$ is the descent of $\mathscr{O} \subset \mathcal{O}\cap\mathfrak{p}$.

Now suppose $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the generalized decent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. From the discussion in Section 12.8.2, there is an $(\epsilon', \dot\epsilon')$-space decomposition $\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2$ and an element

$$T \in \mathcal{X}^{\circ}_1 := \{w \in \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_1) \mid w \text{ is surjective}\} \cap \mathcal{X} \subseteq \mathcal{X}^{\mathrm{gen}}$$

such that $\mathscr{O}'_1 := \mathbf{K}'_1 \cdot X' \in \mathrm{Nil}_{\mathbf{K}'_1}(\mathfrak{p}'_1)$ is the descent of $\mathscr{O}$. Here $X' := M'(T) \in \mathscr{O}'$, $\mathbf{K}'_i := \mathbf{G}^{L'}_{\mathbf{V}'_i}$ is a subgroup of $\mathbf{K}'$ for $i = 1, 2$, and $\mathfrak{p}'_1 := \mathfrak{p}_{\mathbf{V}'_1}$.

Let $X := M(T) \in \mathscr{O}$ and

$$\alpha_1 \colon \mathbf{K}_X \to \mathbf{K}'_{1,X'} \tag{12.29}$$

be the (surjective) homomorphism defined in (12.22) with respect to the descent from $\mathscr{O}$ to $\mathscr{O}'_1$. Then the stabilizer $\mathbf{S}_T := \mathrm{Stab}_{\mathbf{K}\times\mathbf{K}'}(T)$ of T is given by

$$\mathbf{S}_T = \{\, (k, \alpha_1(k)k'_2) \in \mathbf{K} \times \mathbf{K}' \mid k \in \mathbf{K} \text{ and } k'_2 \in \mathbf{K}'_2 \,\}. \tag{12.30}$$

For a $\mathbb{p}$-genuine algebraic representation ρ' of $\widetilde{\mathbf{K}}'_{X'}$, define a representation

$$\vartheta^{\mathrm{gen}}_T(\rho') := \varsigma_{\mathbf{V},\mathbf{V}'}|_{\tilde{\mathbf{K}}_X} \otimes \Big((\varsigma_{\mathbf{V},\mathbf{V}'}|_{\tilde{\mathbf{K}}'_{X'}} \otimes \rho')^{\mathbf{K}'_2}\Big) \circ \alpha_1. \tag{12.31}$$

Clearly, (12.31) is a generalization of (12.26), as $\mathbf{K}'_2$ is the trivial group in the latter case. As in the descent case, $\vartheta^{\mathrm{gen}}_T$ induces a homomorphism

$$\vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\tilde{\mathbf{K}}'}(\mathscr{O}') \to \mathcal{K}^{\mathbb{p}}_{\tilde{\mathbf{K}}}(\mathscr{O}).$$

Furthermore, (12.28) is extended to the generalized descent case: for every pair $(\mathcal{O}, \mathcal{O}') \in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g}) \times \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ with $\mathcal{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathcal{O})$, we define a homomorhism

$$\vartheta_{\mathcal{O}',\mathcal{O}} := \sum_{\substack{\mathscr{O}\subset\mathcal{O}\cap\mathfrak{p},\\ \mathscr{O}'=\nabla^{\mathrm{gen}}_{\mathbf{V},\mathbf{V}'}(\mathscr{O})\subset\mathfrak{p}'}} \vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathbb{p}}_{\tilde{\mathbf{K}}'}(\mathcal{O}') \to \mathcal{K}^{\mathbb{p}}_{\tilde{\mathbf{K}}}(\mathcal{O}).$$

12.9.3. *Lift of admissible orbit data.* Now let $\mathscr{O}$ be a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}$, where $\mathcal{O} \in \mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{g})$. Let $X \in \mathscr{O}$. The component group $A_X := \mathbf{K}_X/\mathbf{K}_X^\circ$ is an elementary abelian 2-group by Section 12.7.2. Let $\mathfrak{k}_X$ be the Lie algebra of $\mathbf{K}_X$. Denote by $\widetilde{\mathbf{K}}_X \to \mathbf{K}_X$ the covering map induced by the covering $\widetilde{\mathbf{K}} \to \mathbf{K}$.

We make the following definition.

Definition 12.28 ([83, Definition 7.13]). *Let γ_X denote the one-dimensional $\mathbf{K}_X$-module $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$ and let $\mathrm{d}\gamma_X$ be its differential.* [5] *An irreducible representation ρ of $\widetilde{\mathbf{K}}_X$ is called* admissible *if*

(i) its differential $\mathrm{d}\rho$ *is isomorphic to a multiple of $\frac{1}{2}\mathrm{d}\gamma_X$, equivalently,*
$$\rho(\exp(x)) = \gamma_X(\exp(x/2)) \cdot \mathrm{id}, \quad \textit{for all } x \in \mathfrak{k}_X, \textit{ and}$$
(ii) it is $\mathbb{p}$-genuine.

Let Φ_X denote the set of all isomorphism classes of admissible irreducible representations of $\widetilde{\mathbf{K}}_X$.

Definition 12.28 is obviously consistent with Definition 1.9, since a representation $\rho \in \Phi_X$ determines an admissible orbit datum $\mathcal{E} \in \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$, where $\mathcal{E}$ is a $\widetilde{\mathbf{K}}$-equivariant algebraic vector bundle on $\mathscr{O}$ whose isotropy representation $\mathcal{E}_X$ at X is isomorphic to ρ. We therefore have an identification

$$\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}) = \Phi_X. \tag{12.32}$$

From the structure of $\mathbf{K}_X$ in Lemma 12.20 and Lemma 12.21, it is easy to see that Φ_X consists of one-dimensional representations.

Lemma 12.29. *The tensor product yields a simply transitive action of the character group $\widehat{A_X}$ on the set Φ_X.*

Proof. It is clear that we only need to show that Φ_X is nonempty. Consider a rational dual pair $(\mathbf{V}, \mathbf{V}')$ such that $\mathscr{O}' = \nabla(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the descent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. Suppose that $T \in \mathcal{X}^\circ$ realizes the descent from X to $X' \in \mathscr{O}'$. The proof of [53, Proposition 6.1] shows that if $\rho' \in \Phi_{X'}$ is admissible, then $\vartheta_T(\rho')$ is admissible. Therefore, we may do reductions and eventually reduce the problem to the case when $\mathbf{V}$ is the zero space. It is clear that Φ_X is a singleton in this case. □

Remark. By Lemma 12.29, the set $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ of admissible orbit data over $\mathscr{O}$ has 2^r elements, where r is the number of orthogonal groups appearing in the decomposition of $\mathbf{R}_X$ in Lemma 12.20.

As in the proof of Lemma 12.29, the homomorphism (12.27) restricts to a map

$$\vartheta_{\mathscr{O}',\mathscr{O}} \colon \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') \to \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}). \tag{12.33}$$

Lemma 12.30. *Let A be the component group of G which is identified with the component group of $\mathbf{K}$. Then the following map is surjective:*[6]

$$\begin{aligned} \widehat{A} \times \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}'}(\widetilde{\mathbf{K}}') &\longrightarrow \mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}}), \\ (\chi, \mathcal{E}') &\longmapsto \chi \otimes \vartheta_{\mathscr{O}',\mathscr{O}}(\mathcal{E}'). \end{aligned}$$

In particular, $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ is a singleton if G is a quaternionic group.

[5] $\mathrm{d}\gamma_X$ is the same as $\mathrm{d}\gamma_{\mathfrak{k}}$ in [83, Theorem 7.11] since $\mathfrak{k}$ is reductive.

[6] Under the identification (12.32), the tensor product of a character $\chi \in \widehat{A}$ on $\mathcal{K}^{\mathrm{aod}}_{\mathscr{O}}(\widetilde{\mathbf{K}})$ is identified with the tensor product of $\chi|_{A_X}$ with the isotropy representation.

Proof. This follows from Lemma 12.27 and Lemma 12.29. $\square$

13. Combinatoral Convention on the local system

An irreducible local system is represented by a *marked Young diagram.* A general local system is understand as the union of irreducible ones. By our computation later, it would be clear that the local systems appeared in our cases are always *multiplicity one.*

13.1. **Commutative free monoid.** Given a set A, the commutative free monoid $\mathsf{C}(\mathbb{A})$ on $\mathbf{A}$ consists all finite multisets with elements drawn from A. It is equipped with the binary operation, say $\bullet$ such that

$$\{\, a_1, \cdots, a_{k_1} \,\} \bullet \{\, b_1, \cdots, b_{k_2} \,\} := \{\, a_1, \cdots, a_{k_1}, b_1, \cdots, b_{k_2} \,\}.$$

In the following, we always identify A with the subset of $\mathsf{C}(\mathsf{A})$ consists of singletons.

Suppose M is a commutative monoid, a map $f\colon \mathsf{A} \to \mathsf{M}$ is uniquelly extends to a morphism $\mathsf{C}(\mathsf{A}) \to \mathsf{M}$ which is still called f by abuse of notations.

In particular a map $f\colon \mathsf{A} \to \mathsf{B}$ between sets uniquely extends to an morphism $f\colon \mathsf{C}(\mathsf{A}) \to \mathsf{C}(\mathsf{B})$ by

$$f(\{\, a_1, \cdots, a_k \,\}) = \{\, f(a_1), \cdots, f(a_k) \,\}.$$

13.2. **Marked Young diagram.** We now explain the combinatiorial objects parameterizing local systems. Basically ,we use $+/-$ (resp. $*/=$) to denote the associated character on the corresponding factor of the isotropic group. Let $\star = B, C, D, M$.

- **Young diagram** Let YD be the commutative free monoid $\mathsf{C}(\mathbb{N}^+)$ on the set of positive integers $\mathbb{Z}_{\geqslant 1}$. An element $\mathcal{O} = \{\, R_1, \cdots, R_k \,\}$ is identified with the Young diagram whose set of row lengths is $\mathcal{O}$. Therefore, we identify YD with the set of Young diagrams. Let

  $$|\ \ | : \mathsf{YD} \longrightarrow \mathbb{N}$$

  be the map sending $\{\, R_1, \cdots R_k \,\}$ to $\sum_{i=1}^{k} R_i$.

  Let $\mathsf{YD}(\star)$ to denote the Young diagrams corresponding to partitions of type $\star$. The set $\mathsf{YD}(\star)$ parameterizes the set of nilpotent orbits of type $\star$. Note that $\mathsf{YD}(C) = \mathsf{YD}(M)$.
- **Signed Young diagram** Let S be the set of non-empty finite length strings of alternating signs of the form $+ - + \cdots$ or $- + - \cdots$. Let $\mathsf{SYD} := \mathsf{C}(\mathsf{S})$ be the monoid on S.

  The monoid SYD is identified with the set of all signed Young diagrams: $\mathscr{O} = \{\, \mathbf{r}_1, \cdots, \mathbf{r}_k \,\} \in \mathsf{SYD}$ is identified with the signed Young diagram whose rows are filled by $\mathbf{r}_j$. The map $\mathsf{S} \to \mathbb{Z}_{\geqslant 1}$ sending a string to its length extends to the morphism

  $$\begin{aligned} \mathcal{Y}\colon \mathsf{SYD} &\longrightarrow \mathsf{YD} \\ \{\, \mathbf{r}_1, \cdots, \mathbf{r}_k \,\} &\longmapsto \{\, |\mathbf{r}_1|, \cdots, |\mathbf{r}_k| \,\} \end{aligned}$$

  where $|\mathbf{r}_j|$ denote the length of the string $\mathbf{r}_j$. By abused of notation, we also denote $|\ \ | \circ \mathcal{Y}$ by $|\ \ |$.

  We define $\mathsf{SYD}(\star)$ to be a subset of the preimage of $\mathsf{YD}(\star)$ under the above map: When $\star = C, M$ (resp. $\star = D, B$),

  $$\mathsf{SYD}(\star) = \left\{\ \mathscr{O} \in \mathsf{SYD}\ \middle|\ \begin{array}{l} \mathcal{Y}(\mathscr{O}) \in \mathsf{YD}(\star) \\ \text{For each positive odd number (resp.} \\ \text{even number) } r\text{, the strings } + - \cdots \\ \text{and } - + \cdots \text{ of the length } r \text{ ap-} \\ \text{pears in } \mathscr{O} \text{ with the same multiplic-} \\ \text{ity (could be 0).} \end{array} \right\}$$

The set $\mathsf{SYD}(\star)$ parameterizes the real nilpotent orbits of type $\star$ and $\mathbf{K}$-orbits in $\mathfrak{p}$.

We define two signature maps $\mathsf{S} \to \mathbb{N} \times \mathbb{N}$ as the following. For $\mathbf{r} \in \mathsf{S}$, define

$$\mathrm{Sign}(\mathbf{r}) := (\# + (\mathbf{r}), \# - (\mathbf{r})) \quad \text{and}$$
$$^{l}\mathrm{Sign}(\mathbf{r}) := \begin{cases} (1,0) & \text{if } \mathbf{r} = + \cdots \\ (0,1) & \text{if } \mathbf{r} = - \cdots \end{cases}.$$

The above defined signature maps naturally extends to signature maps Sign: $\mathsf{SYD} \to \mathbb{N} \times \mathbb{N}$ and Sign: $\mathsf{SYD} \to \mathbb{N} \times \mathbb{N}$.

- **Marked Young diagram** Let M be the set of finite length strings of alternating marks of the form $+-+\cdots$, $-+-\cdots$, $\ast{=}\ast\cdots$, or $={\ast}=\cdots$.

  Let $\mathsf{MYD} := \mathsf{C}(\mathsf{M})$ which is identified with the set of Young diagrams marked with alternating marks $+/-$, $\ast/=$. Define $\mathscr{F} \colon \mathsf{M} \to \mathsf{S}$ by the formula

$$\mathscr{F}(\mathbf{r}) := \begin{cases} +-+\cdots & \text{if } \mathbf{r} = +-+\cdots \text{ or } \ast{=}\ast\cdots \\ -+-\cdots & \text{if } \mathbf{r} = -+-\cdots \text{ or } ={\ast}=\cdots \end{cases} \qquad \forall \mathbf{r} \in \mathsf{M}$$

  such that $\mathscr{F}(\mathbf{r})$ and $\mathbf{r}$ have the same length. $\mathscr{F}$ extends to the map $\mathscr{F} : \mathsf{MYD} \to \mathsf{SYD}$.

  By abused of notation, we also denote $|\ \ | \circ \mathcal{Y}$ by $|\ \ |$.

  For $\mathcal{L} \in \mathsf{MYD}$, we define

$$\mathrm{Sign}(\mathcal{L}) = \mathrm{Sign}(\mathscr{F}(\mathcal{L})) \quad \text{and} \quad {}^{l}\mathrm{Sign}(\mathcal{L}) = {}^{l}\mathrm{Sign}(\mathscr{F}(\mathcal{L})).$$

  We define $\mathsf{MYD}(\star)$ to be a subset of the preimage of the above map:

$$\mathsf{MYD}(\star) = \left\{ \mathcal{L} \in \mathsf{MYD}(\star) \;\middle|\; \begin{array}{l} \mathscr{F}(\mathcal{L}) \in \mathsf{SYD}(\star); \\ \mathbf{r}_1 = \mathbf{r}_2 \text{ if } \mathbf{r}_1, \mathbf{r}_2 \in \mathcal{L} \text{ and } \mathscr{F}(\mathbf{r}_1) = \mathscr{F}(\mathbf{r}_2); \\ \mathbf{r} = +-+\cdots \text{ or } -+-\cdots \text{ if the length} \\ \text{of } \mathbf{r} \text{ is odd (resp. even) when } \star \in \{C, M\} \\ \text{(resp. } \star \in \{B, D\}). \end{array} \right\}$$

  We will use $\mathsf{MYD}(\star)$ to parameterize irreducible local systems attached to the nilpotent orbits of type $\star$.

For monoid YD, SYD or MYD, the identity element is the empty set $\varnothing$, and the monoid binary operator is denoted by "$\cdot$". The $\varnothing$ is always in $\mathsf{MYD}(\star)$ by convention. According to the definition of $\mathsf{MYD}(\star)$. it could happen that $\mathcal{L}_1 \cdot \mathcal{L}_2 \notin \mathsf{MYD}(\star)$ for $\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{MYD}(\star)$.

We define an involution

$$\overline{} \colon \mathsf{M} \to \mathsf{M} \tag{13.1}$$

on M by switching the symbols $+$ with $\ast$ and $-$ with $=$. For example $\overline{+-+} = \ast{=}\ast$ and $\overline{={\ast}} = -+$.

We define the map

$$\dagger \colon \mathsf{M} \to \mathsf{M} \tag{13.2}$$

by extending the length of the strings to the left by one. For example $\dagger(+-+) = -+-+$ and $\dagger(={\ast}) = \ast{=}\ast$.

For $\mathcal{L}, \mathcal{C} \in \mathsf{MYD}$, the expression $\mathcal{L} \succ \mathcal{C}$ means that there is $\mathcal{B} \in \mathsf{MYD}$ such that the factorization $\mathcal{L} = \mathcal{B} \cdot \mathcal{C}$ holds.

In the following, we will represent an element $\mathcal{L} \in \mathsf{MYD}$ by the Young diagram whose rows are filled with the strings in $\mathcal{L}$. The terminology "l-row" means a row of length l. If we do not care whether the mark is $\ast$ or $+$ (resp. $=$ or $=$), we will use $\divideontimes$ (resp. $\dot{-}$) to mark the corresponding position.

Example 13.1. *The following two diagrams $\ddagger_{p,q}$ and $\dagger_{p,q}$ are in $\mathsf{MYD}(B) \cup \mathsf{MYD}(D)$.*

$$\ddagger_{p,q} := \begin{array}{|c|}\hline + \\ \hline \vdots \\ \hline + \\ \hline = \\ \hline \vdots \\ \hline = \\ \hline\end{array}$$

represents the local system on the trivial orbit of $\mathrm{O}(p,q)$ *which has the trivial character on* $\mathrm{O}(p)$ *and* det *on* $\mathrm{O}(q)$. *Similarly,*

$$\dagger_{p,q} := \begin{array}{|c|}\hline + \\ \hline \vdots \\ \hline + \\ \hline - \\ \hline \vdots \\ \hline - \\ \hline\end{array}$$

represents the trivial local system on the trivial orbit.

13.2.1. *Local systems.* We define $\mathcal{K}_{\mathsf{M}}$ to be the free commutative monoid generated by MYD. The binary operation in $\mathcal{K}_{\mathsf{M}}$ is denoted by "+".

The operation "$\cdot$" naturally extends to $\mathcal{K}_{\mathsf{M}}$:

$$(\sum_i m_i \mathcal{L}_i) \cdot (\sum_j m_j \mathcal{L}'_j) := \sum_{i,j} m_i m_j (\mathcal{L}_i \cdot \mathcal{L}'_j).$$

where $\mathcal{L}_i, \mathcal{L}'_j \in \mathsf{MYD}$, $m_i, m_j \in \mathbb{Z}_{\geqslant 0}$.

Now $(\mathcal{K}_{\mathsf{M}}, +, \cdot)$ is in fact a commutative semiring. We let 0 be the additive identity element in $\mathcal{K}_{\mathsf{M}}$ and $\varnothing$ is the multiplicative identify. Note that $\mathcal{L} \cdot 0 = 0$ for any $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$ by definition.

For $\mathcal{L}, \mathcal{D} \in \mathcal{K}_{\mathsf{M}}$, $\mathcal{L} \succ \mathcal{D}$ means that there exists $\mathcal{B} \in \mathcal{K}_{\mathsf{M}}$ such that $\mathcal{L} = \mathcal{B} \cdot \mathcal{D}$. In addition, we write $\mathcal{L} \supset \mathcal{D}$ if there is $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{K}_{\mathsf{M}}$ such that $\mathcal{L}_1 \succ \mathcal{D}$ and $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}$.

Let $\mathcal{K}_{\mathsf{M}}(\star)$ be the sub-monoid of $\mathcal{K}_{\mathsf{M}}$ generated by $\mathsf{MYD}(\star)$. Note that $\mathcal{K}_{\mathsf{M}}(\star)$ is not a sub semi-ring of $\mathcal{K}_{\mathsf{M}}$.

For $\mathcal{L} = \sum_{i=1}^k \mathcal{L}_i \in \mathcal{K}_{\mathsf{M}}$ with $\mathcal{L}_i \in \mathsf{MYD}$, we let

$$\mathrm{Sign}(\mathcal{L}) = \{\, \mathrm{Sign}(\mathcal{L}_i) \mid i = 1, \cdots, k \,\} \text{ and } {}^l\mathrm{Sign}(\mathcal{L}) = \left\{\, {}^l\mathrm{Sign}(\mathcal{L}_i) \,\middle|\, i = 1, \cdots, k \,\right\}$$

13.3. **Operations on $\mathcal{K}_{\mathsf{M}}(\star)$.** The operation $\dagger$ defined in (13.2) extends to an operation

$$\dagger \colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$$

on $\mathcal{K}_{\mathsf{M}}$ which adding a column on the left of the original marked Young diagram.

13.3.1. *Character twists on $\mathcal{K}_{\mathsf{M}}(D)$ and $\mathcal{K}_{\mathsf{M}}(B)$.* Recall that $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ paramterizes the characters of a indefinite orthogonal group $\mathrm{O}(p,q)$:

$$\chi = (\chi^+, \chi^-) \longleftrightarrow (\mathbf{1}^{-,+})^{\chi^+} \otimes (\mathbf{1}^{+,-})^{\chi^-}.$$

We define the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action on M by:

$$\mathbf{r} \otimes \chi := \begin{cases} \bar{\mathbf{r}} & \text{if } p+q \equiv 1 \text{ and } \chi^+ p + \chi^- q \equiv 1 \pmod 2 \\ \mathbf{r} & \text{otherwise} \end{cases}$$

where $\mathbf{r} \in \mathsf{M}$, $\chi = (\chi^+, \chi^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $(p,q) = \mathrm{Sign}(\mathbf{r})$.

The $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ action extends to actions on $\mathcal{K}_{\mathsf{M}}(D)$ and $\mathcal{K}_{\mathsf{M}}(B)$.

13.3.2. *Twists on $\mathcal{K}_{\mathsf{M}}(C)$ and $\mathcal{K}_{\mathsf{M}}(M)$.* We define an involution $✠_C \colon \mathsf{M} \to \mathsf{M}$ on M by

$$✠(\mathbf{r}) := \begin{cases} \bar{\mathbf{r}} & \text{if } |\mathbf{r}| \equiv 2 \pmod 4 \\ \mathbf{r} & \text{otherwise.} \end{cases}$$

The involution ✠ extends to an involution on $\mathcal{K}_{\mathsf{M}}$.We will only apply ✠ on $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}(C)$ or $\mathcal{K}_{\mathsf{M}}(M)$.

13.3.3. *Truncations.* We define the Truncation maps

$$\mathsf{T}^+ \colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$$

such that for $\mathcal{L} \in \mathsf{MYD}$

$$\mathsf{T}^+(\mathcal{L}) = \begin{cases} \mathcal{L}_1 & \text{if there is } \mathcal{L}_1 \in \mathsf{MYD} \text{ such that } \mathcal{L}_1 \cdot + = \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$

We define $\mathsf{T}^- \colon \mathcal{K}_{\mathsf{M}} \longrightarrow \mathcal{K}_{\mathsf{M}}$ similarly such that $\mathcal{L} = \mathsf{T}^-(\mathcal{L}) \cdot -$ if $\mathcal{L} \succ -$ and $\mathsf{T}^-(\mathcal{L}) = 0$ otherwise for $\mathcal{L} \in \mathsf{MYD}$.

To symplify the notation, we write $\mathscr{L}^{\pm} := \mathsf{T}^{\pm}(\mathscr{L})$ for $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$.

13.4. **Theta lifts of local systems.**

13.4.1. *Lift from type C to D.* For each signature $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that $p+q$ is even, we define a map

$$\vartheta_{CD,(p,q)} \colon \mathcal{K}_{\mathsf{M}}(C) \to \mathcal{K}_{\mathsf{M}}(D)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(C)$. Let $(p_1,q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and

$$(p_0,q_0) = (p,q) - \mathrm{Sign}(\mathcal{L}) - (q_1,p_1).$$

We define

$$\vartheta_{CD,(p,q)}(\mathcal{L}) = \begin{cases} (\dagger ✠^{\frac{p-q}{2}}(\mathcal{L})) \cdot \dagger_{p_0,q_0} & \text{if } p_0 \geqslant 0 \text{ and } q_0 \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

13.4.2. *Lift from type D to type C.* For each non-zero integer $n \in \mathbb{N}$, we define a map

$$\vartheta_{DC,n} \colon \mathcal{K}_{\mathsf{M}}(D) \to \mathcal{K}_{\mathsf{M}}(C)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(D)$. Let $(p_1,q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and $(p,q) = \mathrm{Sign}(\mathcal{L})$. Let

$$n_0 = n - (p+q)/2 - (p_1+q_1)/2$$

We define

$$\vartheta_{DC,n}(\mathcal{L}) = \begin{cases} ✠^{\frac{p-q}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geqslant 0} \\ ✠^{\frac{p-q}{2}}(\dagger\mathcal{L}^+ + \dagger\mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that $✠^{\frac{p-q}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) = (✠^{\frac{p-q}{2}}(\dagger\mathcal{L})) \cdot \dagger_{n_0,n_0}$.

[We take a splitting $\mathrm{O}(p,q) \times \mathrm{Sp}(n,\mathbb{R})$ in to the big metaplectic group Mp such that $\mathrm{O}(p,\mathbb{C}) \times \mathrm{O}(q,\mathbb{C})$ acts on the Fock model linearly. The maximal compact $K_{\mathrm{Sp}(n,\mathbb{R})} = \mathrm{U}(2n)$ acts on the Fock model by the character $\zeta = \det^{(p-q)/2}$. For the component of the rational nilpotent orbit $\mathrm{Sp}(2n,\mathbb{R})$, the factor of the component group is Sp if the corresponding row has odd length. Otherwise the component group is $\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})$ where (p_{2k},q_{2k}) is the signature corresponding to the $2k$-rows. $\zeta|_{\mathrm{O}(p_{2k})} = \det_{\mathrm{O}(p_{2k})}^{(p-q)k/2}$ which is nontrivial if and only if $p-q \equiv 2 \pmod 4$ and $2k \equiv 2 \pmod 4$. This gives the formula above.]

In the proof later, we don't care above the marks for rows with length longer than 1 in the most of the case. So we will omit ✠ sometimes. When there is no confusion, we will simply write ϑ for $\vartheta_{CD,(p,q)}$ and $\vartheta_{CD,(p,q)}$.

13.4.3. *Lift from type M to B.* For each signature $(p,q) \in \mathbb{N} \times \mathbb{N}$ such that $p+q$ is odd, we define a map

$$\vartheta_{MB,(p,q)} \colon \mathcal{K}_{\mathsf{M}}(M) \to \mathcal{K}_{\mathsf{M}}(B)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(M)$. Let $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and

$$(p_0, q_0) = (p,q) - \mathrm{Sign}(\mathcal{L}) - (q_1, p_1).$$

We define

$$\vartheta_{MB,(p,q)}(\mathcal{L}) = \begin{cases} (\dagger \maltese^{\frac{p-q+1}{2}}(\mathcal{L})) \cdot \dagger_{p_0,q_0} & \text{if } p_0 \geqslant 0 \text{ and } q_0 \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

13.4.4. *Lift from type B to type M.* For each non-zero integer $n \in \mathbb{N}$, we define a map

$$\vartheta_{BM,n} \colon \mathcal{K}_{\mathsf{M}}(B) \to \mathcal{K}_{\mathsf{M}}(M)$$

as the following: Let $\mathcal{L} \in \mathsf{MYD}(D)$. Let $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and $(p,q) = \mathrm{Sign}(\mathcal{L})$. Let

$$n_0 = n - (p+q)/2 - (p_1 + q_1)/2$$

We define

$$\vartheta_{BM,n}(\mathcal{L}) = \begin{cases} \maltese^{\frac{p-q-1}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geqslant 0} \\ \maltese^{\frac{p-q-1}{2}}(\dagger\mathcal{L}^+ + \dagger\mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that $\maltese^{\frac{p-q-1}{2}}((\dagger\mathcal{L}) \cdot \dagger_{n_0,n_0}) = (\maltese^{\frac{p-q-1}{2}}(\dagger\mathcal{L})) \cdot \dagger_{n_0,n_0}$.

[For odd orthogonal-metaplectic group case, we still take the splitting such that $\mathrm{O}(p,q)$ acts linearly.

The maximal compact $\widetilde{K}_{\mathrm{Sp}(n,\mathbb{R})} = \widetilde{\mathrm{U}(2n)}$ acts on the Fock model by the character $\zeta = \det^{(p-q)/2}$.

Then $\det^{(p-q)/2}|_{\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})} = \det_{\mathrm{O}(p_{2k})}^{\frac{(p-q)k}{2}} \boxtimes \det_{\mathrm{O}(q_{2k})}^{\frac{(p-q)k}{2}}$. Note that $p-q$ is an odd number. When k is even the character on $\widetilde{\mathrm{O}(p_{2k})}/\widetilde{\mathrm{O}(q_{2k})}$ are $\mathbf{1}$ or $\det$. When k is odd the character would be $\det^{\frac{1}{2}}$ or $\det^{\frac{1}{2}+1}$.

We assume the default character on $2k$-rows is $\det^{\frac{k}{2}}$, and we use $+/-$ to mark the row. Otherwise, we use $*/=$ to mark the rows.]

13.5. Examples and remarks.

13.5.1. *Example.*

Example 13.2. *Let* $\mathcal{T} := \dagger\dagger(\dagger_{2,1}) + \dagger\dagger(\dagger_{1,2})$ *and* $\mathcal{P} = \ddagger_{1,3}$. *Then*

$$\mathcal{T} \cdot \mathcal{P} = \begin{array}{|c|c|c|}\hline \divideontimes & \dotminus & \divideontimes \\ \hline \divideontimes & \dotminus & \divideontimes \\ \hline \dotminus & \divideontimes & \dotminus \\ \hline + \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} \end{array} \cup \begin{array}{|c|c|c|}\hline \divideontimes & \dotminus & \divideontimes \\ \hline \dotminus & \divideontimes & \dotminus \\ \hline \dotminus & \divideontimes & \dotminus \\ \hline + \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} = \\ \cline{1-1} \end{array}.$$

13.5.2. *Signs of the local systems obtained by iterated lifting.* Suppose $\mathcal{L} \in \mathcal{K}_{\mathsf{M}}$ is obtained by iterated theta lifting and character twisting in Section 13.3.1. Then the set ${}^{l}\mathrm{Sign}(\mathcal{L})$ has restricted possibilities. First, $\mathrm{Sign}(\mathcal{L})$ is always a singleton.

When $\mathcal{O}$ is of type B/D, ${}^{l}\mathrm{Sign}(\mathcal{L})$ is a singleton $\{ (p_1, q_1) \}$ where

$$(p_1, q_1) = \mathrm{Sign}(\mathcal{L}) - (n_0, n_0) \quad \text{and} \quad 2n_0 = \left|\overline{\nabla}(\mathcal{O})\right|.$$

When $\mathcal{O}$ is of type C/M, ${}^{l}\mathrm{Sign}(\mathcal{L}) = \{ (p_1, q_1) \}$ is a singleton in the most of the case. The ${}^{l}\mathrm{Sign}(\mathcal{L})$ may have two elements if $\mathcal{L}$ is obtained by good generalized lifting, and it has the form ${}^{l}\mathrm{Sign}(\mathcal{L}) = \{ (p_1 + 1, q_1), (p_1, q_1 + 1) \}$.

14. Proof of main theorem in the type C/D case

We do induction on the number of columns of the nilpotent orbits.

14.1. **Notation.** In this section, $\mathcal{O}$ is always an nilpotent orbit of type D in $\mathrm{Nil}^{\mathrm{dpe}}(D)$. We always let

$$\begin{aligned} \mathcal{O} &= (C_{2k+1}, C_{2k}, C_{2k-1}, \cdots, C_1, C_0), \\ \mathcal{O}' &= \overline{\nabla}(\mathcal{O}) = (C_{2k}, C_{2k-1}, \cdots, C_1, C_0). \end{aligned}$$

When $k \geqslant 1$, we let

$$\mathcal{O}'' = \overline{\nabla}^2(\mathcal{O}) = \begin{cases} (C_{2k-1}, \cdots, C_1, C_0) & \text{when } C_{2k} \text{ is even,} \\ (C_{2k-1} + 1, \cdots, C_1, C_0) & \text{when } C_{2k} \text{ is odd.} \end{cases}$$

For a $\tau \in \mathrm{DRC}(\mathcal{O})$, we let

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \quad \text{and} \quad (\tau'', \epsilon') = \overline{\nabla}(\tau') \text{ (when } k \geqslant 1).$$

14.1.1. *Main proposition.* The aim of this section is to prove the following main proposition holds for $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(D)$.

Proposition 14.1. *We have:*

i) *$\mathcal{L}_\tau$ is non-zero. In particular, $\pi_\tau \neq 0$.*

ii) *Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$, then $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.*

iii) *Suppose $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$, then $\pi_{\tau_1} \neq \pi_{\tau_2}$.*

iv) *The local system $\mathcal{L}_\tau$ is disjoint with their determinant twist:*

$$\{ \mathcal{L}_\tau \mid \tau \in \mathrm{DRC}(\mathcal{O}) \} \cap \{ \mathcal{L}_\tau \otimes \det \mid \tau \in \mathrm{DRC}(\mathcal{O}) \} = \varnothing. \tag{14.1}$$

v) *We have*

(i) $x_\tau = s$, then $\mathcal{L}^+_\tau = \mathcal{L}^-_\tau = 0$.

(ii) $x_\tau = r/c$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau = 0$.

(iii) $x_\tau = d$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau \neq 0$.

vi) *If $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a usual descent, we have a simpler factorization:*

$$\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_{\mathbf{x}_\tau}. \tag{14.2}$$

vii) *When $\mathcal{O}'$ is noticed, the map $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}') \longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O}')$ is a bijection.*

viii) *When $\mathcal{O}$ is noticed, the map $\mathcal{L}\colon \mathrm{DRC}(\mathcal{O}) \longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O})$ is a bijection.*

ix) *When $\mathcal{O}$ is +noticed, the maps*

$$\begin{aligned} \Upsilon^+_{\mathcal{O}}\colon \{ \mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0 \} &\longrightarrow \{ \mathcal{L}^+_\tau \} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}^+_\tau \qquad \textit{and} \\ \Upsilon^-_{\mathcal{O}}\colon \{ \mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0 \} &\longrightarrow \{ \mathcal{L}^-_\tau \} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}^-_\tau \end{aligned}$$

are injective.

x) When $\mathcal{O}$ is noticed, the map

$$\Upsilon_{\mathcal{O}}\colon \{\, \mathcal{L}_\tau \mid \mathcal{L}_\tau^+ \neq 0 \,\} \longrightarrow \{\, (\mathcal{L}_\tau^+, \mathcal{L}_\tau^-) \mid \mathcal{L}_\tau^+ \neq 0 \,\}$$
$$\mathcal{L}_\tau \longmapsto (\mathcal{L}_\tau^+, \mathcal{L}_\tau^-) \tag{14.3}$$

is injective.[7]

Note that, $\mathrm{DRC}(\mathcal{O})$ counts the unipotent representations of special orthogonal groups a priori. But Proposition 14.1 (iv)), implies that the representation π_τ and $\pi_\tau \otimes \det$ are two different representations of the orthogonal groups. Therefore $\mathrm{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z}$ counts the set of unipotent representations of orthogonal groups.

The main theorem would be a consequence of Proposition 14.1.

14.1.2. *How to think about the factorization of local systems.* When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is the usual descent, the local system $\mathcal{L}_\tau$ could be compared to a water hydra, see in Figure 1.

The tentacles part $\mathcal{T}_\tau = \dagger\mathcal{L}_{\tau'}$ could be reducible. The peduncle part $\mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}$ and $\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau$.

When $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a generalized descent, the situation is more complicated as the local system $\mathcal{L}_\tau$ may consists of two hydras, see (14.8).

14.2. **The initial case:** When $k = 0$, $\mathcal{O} = (C_1 = 2c_1, C_0 = 2c_0)$ has at most two columns. Now $\pi_{\tau'}$ is the trivial representation of $\mathrm{Sp}(2c_0, \mathbb{R})$. The lift $\pi_{\tau'} \mapsto \bar{\Theta}(\pi_{\tau'})$ is in fact a stable range theta lift.For $\tau \in \mathrm{DRC}(\mathcal{O})$, let $(p_1, q_1) = \mathrm{Sign}(\mathbf{x}_\tau)$ and x_τ be the foot of τ. It is easy to see that (using associated character formula)

$$\mathcal{L}_{\pi_\tau} = \mathcal{T}_\tau \cdot \mathcal{P}_\tau, \text{ where } \mathcal{T}_\tau := \dagger\dagger_{c_0,c_0}, \text{ and } \mathcal{P}_\tau := \begin{cases} \ddagger_{p_1,q_1} & x_\tau \neq d, \\ \dagger_{p_1,q_1} & x_\tau = d. \end{cases}$$

[7] In fact, we only need to know whether $\mathcal{L}_\tau^-$ is non-zero or not.

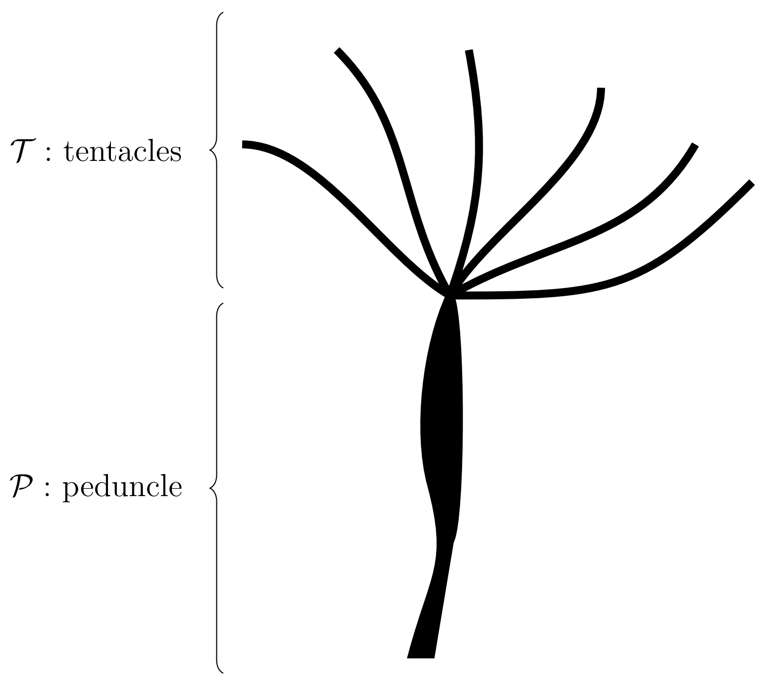

Figure 1. Freshwater hydra

Note that $\mathcal{L}_{\pi_\tau}$ is irreducible.

Let $\mathcal{O}_1 = (2(c_1 - c_0)) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. Using the above formula, one can check that the following maps are bijections

$$\begin{array}{ccccc} \mathrm{DRC}(\mathcal{O}_1) & \longleftarrow & \mathrm{DRC}(\mathcal{O}) & \longrightarrow & {}^\ell\mathrm{LS}(\mathcal{O}) \\ \mathbf{x}_\tau & \longleftarrow\!\shortmid & \tau & \longmapsto & \mathcal{L}_\tau. \end{array}$$

To see that $\mathrm{DRC}(\mathcal{O}) \ni \tau \mapsto \mathcal{L}_\tau$ is an injection, one check the following lemma holds:

Lemma 14.2. *The map* $\mathrm{DRC}(\mathcal{O}_1) \longrightarrow \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective (see* **??** *for the definition of* ϵ_τ*). Moreover,* $\epsilon_\tau = 0$ *only if* $\mathrm{Sign}(\tau) \geqslant (0,1)$.

Moreover, $\mathcal{L}_\tau \otimes \det \notin {}^\ell\mathrm{LS}(\mathcal{O})$ for any $\tau \in \mathrm{DRC}(\mathcal{O})$. In fact, if $\mathrm{Sign}(\mathbf{x}_\tau) \succeq (1,0)$, $\mathcal{L}_\tau$ on the 1-rows with $+$-signs is trivial and $\mathcal{L}_\tau \otimes \det$ has non-trivial restriction. When $\mathrm{Sign}(\mathbf{x}_\tau) \not\succeq (1,0)$, $\mathbf{x}_\tau = s \cdots s$, all 1-rows of $\mathcal{L}_\tau$ are marked by "$=$" and all 1-rows of $\mathcal{L}_\tau \otimes \det$ are marked by "$-$".

Therefore our main **??** and main proposition Proposition 14.1 holds for $\mathcal{O}$.

14.3. **The descent case.** Now we assume $k \geqslant 1$ and C_{2k} is even. In this case $\mathcal{O}' \rightsquigarrow \mathcal{O}''$ is a usual descent.

14.3.1. *Unipotent representations attached to* $\mathcal{O}'$. Therefore, $\mathrm{LS}(\mathcal{O}'') \longrightarrow \mathrm{LS}(\mathcal{O}')$ given by $\mathcal{L} \mapsto \vartheta(\mathcal{L})$ is an injection between abelian groups.

For $\mathcal{L}'' \in {}^\ell\mathrm{LS}(\mathcal{O}'') \sqcup {}^\ell\mathrm{LS}(\mathcal{O}'') \otimes \det$, let $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$ and $(p_2, q_2) = (C_{2k} - q_1, C_{2k} - p_1)$. Then

$$\vartheta(\mathcal{L}'') = \dagger\mathcal{L}'' \cdot \dagger_{p_2,q_2}.$$

By (14.1), we have a injection

$$\begin{array}{ccc} {}^\ell\mathrm{LS}(\mathcal{O}'') \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & {}^\ell\mathrm{LS}(\mathcal{O}') \\ (\mathcal{L}'', \epsilon') & \longmapsto & \vartheta(\mathcal{L}'' \otimes \det^{\epsilon'}). \end{array} \tag{14.4}$$

In particular, we have $\mathcal{L}_{\tau'} \neq 0$ and $\pi_{\tau'} \neq 0$ for every $\tau' \in \mathrm{DRC}(\mathcal{O}')$.

14.3.2. *Unipotent representations attached to* $\mathcal{O}$. Now suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$ and $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (14.4), $\epsilon'_1 = \epsilon'_2$ and $\mathcal{L}_{\tau''_1} = \mathcal{L}_{\tau''_2}$. In particular, τ''_1 and τ''_2 have the same signature. By **??** and the induction hypothesis, $\tau''_1 \neq \tau''_2$ and so $\pi_{\tau''_1} \otimes \det^{\epsilon'_1} \neq \pi_{\tau''_2} \otimes \det^{\epsilon'_2}$. Now the injectivity of theta lifting yields

$$\pi_{\tau'_2} = \bar{\Theta}(\pi_{\tau''_1} \otimes \det^{\epsilon'_1}) \neq \bar{\Theta}(\pi_{\tau''_2} \otimes \det^{\epsilon'_2}) = \pi_{\tau'_2}$$

Suppose $\mathcal{O}'$ is noticed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is noticed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is an injection into $\mathrm{LS}(\mathcal{O}'')$. Now (14.4) implies $\tau' \mapsto \mathcal{L}_{\tau'}$ is also an injection.

Suppose $\mathcal{O}'$ is quasi-distingushed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is quasi-distingushed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is a bijection with $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}'')$. Since the component group of each K-nilpotent orbit in $\mathcal{O}'$ is naturally isomorphic to the component group of its descent. So we deduce that $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection onto $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}')$.

This proves the main proposition for $\mathcal{O}'$.

Now let $\tau \in \mathrm{DRC}(\mathcal{O})$. By (**??**), we have

$$\mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau \neq 0, \text{ where } \mathcal{T}_\tau := \dagger\mathcal{L}_{\tau'}, \text{ and } \mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}. \tag{14.5}$$

In particular, $\pi_\tau \neq 0$ and we could determine $\mathbf{x}_\tau$ from the marks on the 1-rows of $\mathcal{L}_\tau$.

Now suppose $\tau_1 \neq \tau_2$ and $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. By (14.5), the result in Section 14.2 and **??** (**??**), we have $\mathbf{x}_{\tau_1} = \mathbf{x}_{\tau_2}$, $\epsilon_1 = \epsilon_2$ and $\tau''_1 \neq \tau''_2$. By (14.5), we have $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. Applying the main proposition of $\mathcal{O}'$, we have $\epsilon'_1 = \epsilon'_2$ and $\tau'_1 \neq \tau'_2$. Now by the injectivity of theta lifing, we conclude that

$$\pi_{\tau_2} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (\mathbf{1}^{+,-})^{\epsilon_1} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (\mathbf{1}^{+,-})^{\epsilon_2} = \pi_{\tau'_2}$$

14.3.3. Note that $\mathcal{L}_\tau$ is obtained from $\mathcal{L}_{\tau'}$ by attaching the peduncle determined by $\mathbf{x}_\tau$. The claims about noticed and quasi-distinguished orbit are easy to verify. We leave them to the reader.

14.4. **The general descent case.** We assume $k \geqslant 1$ and all the properties are satisfied by $\tau'' \in \mathrm{DRC}(\mathcal{O}'')$. We retain the notation in **??**.

14.4.1. *Local systems.* We now describe the local systems $\mathcal{L}_{\tau'}$ and $\mathcal{L}_\tau$ more precisely using our formula of the associated character and the induction hypothesis. Note that these local systems are always non-zero which implies that $\pi_{\tau'}$ and π_τ are unipotent representations attached to $\mathcal{O}'$ and $\mathcal{O}$ respectively.

First note that in τ' and τ'', $x_{\tau''} \neq s$ by the definition of dot-r-c diagram. [8] By the induction hypothesis, $\mathcal{L}^+_{\tau''} \neq 0$.

Let $(p''_1, q''_1) := {}^l\mathrm{Sign}(\mathcal{L}_{\tau''})$. Then

$$ {}^l\mathrm{Sign}(\mathcal{L}^+_{\tau''}) = (p_0 - 1, q_0), \text{ and } {}^l\mathrm{Sign}(\mathcal{L}^-_{\tau''}) = (p_0, q_0 - 1) \text{ if } \mathcal{L}^-_{\tau''} \neq \varnothing. \tag{14.6}$$

$$\mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''}) = \maltese^{\frac{|\mathrm{Sign}(\tau'')|}{2}} (\dagger\mathcal{L}^+_{\tau''} + \dagger\mathcal{L}^-_{\tau''}) \neq 0 \tag{14.7}$$

where ${}^l\mathrm{Sign}(\dagger\mathcal{L}^+_{\tau''}) = (q''_1, p''_1 - 1)$ and ${}^l\mathrm{Sign}(\dagger\mathcal{L}^-_{\tau''}) = (q''_1 - 1, p''_1)$ (if $\dagger\mathcal{L}^-_{\tau''} \neq 0$).

Let $(p_1, q_1) := {}^l\mathrm{Sign}(\mathcal{L}_\tau)$ and $(e, f) := (p_1 - p''_1 + 1, q_1 - q''_1 + 1)$. Using **??** one can show that

$$(e, f) = \mathrm{Sign}(\mathbf{u}_\tau).$$

[It suffice to consider the most left three columns of the peduncle part: this part has signature ${}^l\mathrm{Sign}(\mathcal{P}_\tau) + (1,1) = \mathrm{Sign}(\mathbf{u}_\tau x_{\tau''}) = \mathrm{Sign}(\mathbf{u}_\tau) + {}^l\mathrm{Sign}(\mathcal{D}_{\tau''})$. Therfore,

$$\mathrm{Sign}(\mathbf{u}_\tau) = {}^l\mathrm{Sign}(\mathcal{P}_\tau) - {}^l\mathrm{Sign}(\mathcal{D}_{\tau''}) + (1,1) = {}^l\mathrm{Sign}(\mathcal{L}_\tau) - {}^l\mathrm{Sign}(\mathcal{L}_{\tau''}) + (1,1).$$

]

Now

$$\mathcal{L}_\tau = \begin{cases} (\mathbf{1}^{+,-} \otimes \dagger\maltese^t\dagger\mathcal{L}^+_{\tau''}) \cdot \mathcal{P}^+_\tau + (\mathbf{1}^{+,-} \otimes \dagger\maltese^t\dagger\mathcal{L}^-_{\tau''}) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau \neq d \\ (\dagger\maltese^t\dagger\mathcal{L}^+_{\tau''}) \cdot \mathcal{P}^+_\tau + (\dagger\maltese^t\dagger\mathcal{L}^-_{\tau''}) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau = d \end{cases} \tag{14.8}$$

where

$$t = \frac{|\mathrm{Sign}(\tau) - \mathrm{Sign}(\tau'')|}{2} = \frac{|\mathrm{Sign}(\mathbf{u}_\tau)|}{2}$$

$$\mathcal{P}^+_\tau = \begin{cases} \ddagger_{e,f-1} & \text{when } f \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e,f-1} & \text{when } f \geqslant 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{P}^-_\tau = \begin{cases} \ddagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 14.3. *The local system $\mathcal{L}_\tau \neq 0$.*

[8]Suppose $\tau'' \in \mathrm{DRC}(\mathcal{O}'')$ has the shape in (**??**) and $x_{\tau''} = s$. By the induction hypothesis, $\mathcal{L}^+_{\tau''} = \mathcal{L}^-_{\tau''} = 0$ and so the theta lift of $\pi_{\tau''}$ vanishes.

Proof. Since $x_{\tau''} \neq s$, $\mathcal{L}^+_{\tau''} \neq 0$. When $\mathrm{Sign}(\mathbf{u}_\tau) \geq (0,1)$, the first term in (14.8) dose not vanish. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau) \not\succ (0,1)$. We have $x_{\tau''} = d$ by **??** and $\mathrm{Sign}(\mathbf{u}_\tau) \succ (1,0)$. Therefore, $\mathcal{L}^-_{\tau''} \neq 0$ by the induction hypothesis and the second term in (14.8) dose not vanish. Hence $\mathcal{L}_\tau \neq 0$ in all cases. □

Lemma 14.4. *The local system $\mathcal{L}_\tau$ satisfies the following properties:*

(i) When $x_\tau = s$, then $\mathcal{L}^+_\tau = \mathcal{L}^-_\tau = 0$.
(ii) When $x_\tau = r/c$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau = 0$.
(iii) When $x_\tau = d$, then $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau \neq 0$.

Proof. If $x_\tau = s/r/c$, "$-$" mark dose not appear as a 1-row in $\mathcal{L}_\tau$ by (14.8). So $\mathcal{L}^-_\tau = 0$ in these cases.

(i) Suppose $x_\tau = s$. We have $\mathbf{u}_\tau = s \cdots s$. Therefore, only the first term in (14.8) is non-zero and $\mathcal{P}^+_\tau$ consists of $=$ marks. Hence $\mathcal{L}^+_\tau = 0$.

(ii) Suppose $x_\tau = r$. If $\mathrm{Sign}(\mathbf{u}_\tau) \succ (1,1)$, the first term in (14.8) is non-zero and $\mathcal{L}^+_\tau \succ \dagger_{(1,0)}$. So $\mathcal{L}^+_\tau \neq 0$. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau) \not\succ (1,1)$. Then $\mathbf{u}_\tau = r \cdots r$ and $x_{\tau''} = d$ by **??**. Now the second term of (14.8) is non-zero and $\mathcal{L}^-_\tau \succ \dagger_{(1,0)}$. So $\mathcal{L}^+_\tau \neq 0$.

(iii) Suppose $x_\tau = d$. Suppose $\mathrm{Sign}(\mathbf{u}_\tau) \succ (1,2)$. Then the first term in (14.8) is non-zero and $\mathcal{P}^+_\tau \succ \dagger_{(1,1)}$. So $\mathcal{L}^+_\tau \neq 0$ and $\mathcal{L}^-_\tau \neq 0$. Now suppose $\mathrm{Sign}(\mathbf{u}_\tau) \not\succ (1,2)$. Then $\mathbf{u}_\tau = r \cdots rd$ [9] and $x_{\tau''} = d$ by **??**. So the both terms in (14.8) are non-zero. Since $\mathcal{P}^+_\tau \succ \dagger_{(1,0)}$ and $\mathcal{P}^-_\tau \succ \dagger_{(0,1)}$, we get the conclusion.

□

14.4.2. *Unipotent representations attached to $\mathcal{O}'$.* In this section, we let $\tau' \in \mathrm{DRC}(\mathcal{O}')$ and $\tau'' = \overline{\nabla}_1(\tau') \in \mathrm{DRC}(\mathcal{O}'')$. First note that $x_{\tau''} \neq s$ and so $\mathcal{L}^+_{\tau''}$ always non-zero.

Recall (14.7). We claim that we can recover $\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''}$ from $\mathcal{L}_{\tau'}$:

Lemma 14.5. *The map $\Omega_{\mathcal{O}'} \colon \mathcal{L}_{\tau'} \mapsto (\mathcal{L}^+_{\tau''}, \mathcal{L}^-_{\tau''})$ is a well defined map.*

Proof. When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has two elements, $x_{\tau''} = d$ and ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'}) = \{\, (q''_1, p''_1 - 1), (q''_1 - 1, p''_1) \,\}$. In (14.7), $\dagger\mathcal{L}^+_{\tau''}$ (resp. $\dagger\mathcal{L}^-_{\tau''}$) consists of components whose first column has siginature $(q''_1, p''_1 - 1)$ (resp. $(q''_1 - 1, p''_1)$). When ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'})$ has only one elements, $x_\tau = r/c$, ${}^l\mathrm{Sign}(\mathcal{L}_{\tau'}) = \{\, (q''_1, p''_1 - 1) \,\}$, $\mathcal{L}_{\tau'} = \dagger\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''} = 0$

In any case, we get

$$\mathrm{Sign}(\tau'') = (n_0, n_0) + (p''_1, q''_1) \quad \text{where } 2n_0 = |\nabla(\mathcal{O}'')|\,. \tag{14.9}$$

Using the signature $\mathrm{Sign}(\tau'')$, we could recover the twisting characters in the theta lifting of local system. Therefore $\mathcal{L}^+_{\tau''}$ and $\mathcal{L}^-_{\tau''}$ can be recovered from $\mathcal{L}_{\tau'}$. □

Lemma 14.6. *Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$. Then $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.*

Proof. It suffice to consider the case when $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$. By (14.9) in the proof of Lemma 14.5, $\mathrm{Sign}(\tau''_1) = \mathrm{Sign}(\tau''_2)$. On the other hand, $\epsilon_1 = \epsilon_2 = 0$ and $\tau''_1 \neq \tau''_2$ by **??**. So

$$\pi_{\tau'_1} = \bar{\Theta}(\pi_{\tau''_1}) \neq \bar{\Theta}(\pi_{\tau''_2}) = \pi_{\tau'_2}$$

by the injectivity of theta lift. □

[9] $\mathbf{u}_\tau = d$ if it has length 1.

14.4.3. *Unipotent representations attached to $\mathcal{O}$.*

Lemma 14.7. *Suppose $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$. Then $\pi_{\tau_1} \neq \pi_{\tau_2}$.*

Proof. It suffice to consider the case that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. On the other hand, the twisting $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ since it is determined by the local system: $\epsilon_{\tau_i} = 0$ if and only if $\mathcal{L}_{\tau_i} \supset \boxed{-}$.

We conclude that $\tau_1'' \neq \tau_2''$ and $\tau_1' \neq \tau_2'$ by **??** and **??**. By Lemma 14.6 and the injectivity of theta lift,

$$\pi_{\tau_1} = \bar{\Theta}(\pi_{\tau_1'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_1}} \neq \bar{\Theta}(\pi_{\tau_2'}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_2}} = \pi_{\tau_2}$$

□

14.4.4. *Noticed orbits.* In this section, we prove the claims about noticed and +noticed orbits.

Lemma 14.8. *Suppose $\mathcal{O}'$ is noticed. Then*

(i) The map $\{\, \mathcal{L}_{\tau''} \mid x_{\tau''} \neq s \,\} \longrightarrow \{\, \mathcal{L}_{\tau'} \,\} = {}^{\ell}\mathrm{LS}(\mathcal{O}')$ given by $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''})$ is a bijection.

(ii) The map $\mathrm{DRC}(\mathcal{O}') \to {}^{\ell}\mathrm{LS}(\mathcal{O}')$ given by $\tau' \mapsto \mathcal{L}_{\tau'}$ is a bijection.

Proof. (i) Note that $\mathcal{O}''$ is noticed by definition. Hence $\Upsilon_{\mathcal{O}''}$ is invertible. Recall Lemma 14.5, we see that

$$(\Upsilon_{\mathcal{O}''})^{-1} \circ \Omega_{\mathcal{O}'} \colon \mathcal{L}_{\tau'} \mapsto (\mathcal{L}^{+}_{\tau''}, \mathcal{L}^{-}_{\tau''}) \mapsto \mathcal{L}_{\tau''}$$

gives the inverse of the map in the claim.

(ii) Suppose $\tau_1' \neq \tau_2' \in \mathrm{DRC}(\mathcal{O}')$. By **??**, $\tau_1'' \neq \tau_2''$. Hence $\mathcal{L}_{\tau_1''} \neq \mathcal{L}_{\tau_2''}$ by the induction hypothesis. Therefore, $\mathcal{L}_{\tau_1'} \neq \mathcal{L}_{\tau_2'}$ by (i) of the claim.

□

Lemma 14.9. *Suppose $\mathcal{O}$ is noticed, $\mathcal{L} \colon \mathrm{DRC}(\mathcal{O}) \mapsto {}^{\ell}\mathrm{LS}(\mathcal{O})$ given by $\tau \mapsto \mathcal{L}_{\tau}$ is bijective.*

Proof. We prove the claim by contradiction. We assume $\tau_1 \neq \tau_2$ such that $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$.

Recall (14.8): there is an pair of non-negative integers $(e_i, f_i) = \mathrm{Sign}(\mathbf{u}_{\tau_i})$ such that

$$\mathcal{L}_{\tau_i} = \dagger\dagger\mathcal{L}^{+}_{\tau_i''} \cdot \mathcal{P}^{+}_{\tau_i} + \dagger\dagger\mathcal{L}^{-}_{\tau''} \cdot \mathcal{P}^{-}_{\tau_i}.$$

(i) Suppose that $\mathcal{L}_{\tau_i}$ contains a 1-row marked by $-$ or $=$. Then we have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$, which can be read from the mark $-/=$. Moreover, the term $\dagger\dagger\mathcal{L}^{+}_{\tau_i''} \cdot \mathcal{P}^{+}_{\tau_i} \neq 0$ and we can recover it from $\mathcal{L}_{\tau_i}$. So $\mathcal{L}^{+}_{\tau_1''} = \mathcal{L}^{+}_{\tau_2''}$. Since $\mathcal{O}''$ is +noticed, $\tau_1'' = \tau_2''$ by (14.11) and the induction hypothesis. Note that $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by **??** which is contradict to our assumption.

(ii) Now we assume that $\mathcal{L}_{\tau_i}$ does not contains a 1-row marked by $-/=$. By (14.8), we conclude that $x_{\tau''} \neq d$ and $\mathbf{u}_{\tau} = r \cdots r$ or $r \cdots rc$. Therefore $\mathbf{p}_{\tau_i}$ must be one of the following form by **??**:

$$\mathbf{p}_1 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \mathbf{p}_2 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} c & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,, \quad \text{and} \quad \mathbf{p}_3 = \begin{array}{|c|c|}\hline r & d \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

We consider case by case according to the set $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\}$:

(a) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_1 \,\}$ or $\{\, \mathbf{p}_2 \,\}$. In these cases, $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau_1'' \neq \tau_2''$ by **??**. On the other hand, $\mathcal{L}^{+}_{\tau_1''} = \mathcal{L}^{+}_{\tau_2''}$ by (14.8). Since $\mathcal{O}''$ is +noticed, we get $\tau_1'' = \tau_2''$ a contradiction.

(b) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_3 \,\}$. We still have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau_1'' \neq \tau_2''$ by **??**. On the other hand, $\mathcal{L}^{-}_{\tau_1''} = \mathcal{L}^{-}_{\tau_2''}$ by (14.8). This again contradict to the injectivity of $\mathcal{L}_{\tau''} \mapsto \mathcal{L}^{-}_{\tau''}$.

(c) Suppose $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1, \mathbf{p}_2\}$. By the argument above, we have $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ Now $\tau''_1 \neq \tau''_2$ since $\{x_{\tau''_1}, x_{\tau''_2}\} = \{r, c\}$. This contradict to that $\mathcal{O}''$ is +noticed again.

(d) Suppose $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ or $\mathbf{p}_2, \mathbf{p}_{\tau_2} = \mathbf{p}_3$. Then

$$\mathcal{L}_{\tau_1} = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_1}|}{2}}\dagger\mathcal{L}^+_{\tau''_1}\cdot\mathcal{P}^+_{\tau''_1} \quad\text{and}\quad \mathcal{L}_{\tau_2} = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_2}|}{2}}\dagger\mathcal{L}^-_{\tau''_2}\cdot\mathcal{P}^-_{\tau''_2}.$$

This implies $|\mathrm{Sign}(\tau''_1)| \equiv |\mathrm{Sign}(\tau''_2)| + 2 \pmod 4$ and $\maltese\dagger\mathcal{L}^+_{\tau''_1} = \dagger\mathcal{L}^-_{\tau''_2}$.

Claim 14.10. *We have* $\mathcal{L}^-_{\tau''_2} \supset \boxed{+}$.

Proof. If $\mathcal{O}''$ is obtained by the usual descent, we have $\mathcal{L}_\tau \succeq \begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array}$ and we are done. Now assume $\mathcal{O}''$ is obtained by generalized descent and $\mathcal{O}''$ is +noticed, i.e. $\mathbf{u}_{\tau''}$ has at least length 2. Suppose $\mathrm{Sign}(\mathbf{u}_{\tau''}) \succ (1, 2)$. Applying (14.8) to τ'', we see that the first term is non-zero and $\mathcal{L}_{\tau''} \supseteq \dagger_{(1,1)}$.

Otherwise, $\mathrm{Sign}(\mathbf{u}_{\tau''}) = (2n_0 - 1, 1) \geq (3, 1)$ where n_0 is the length of $\mathbf{u}_{\tau''}$. By (14.8), the second term is non-zero and $\mathcal{L}_{\tau''} \succ \dagger_{(2n_0-1,1)} \succ +$. □

The claim leads to a contradiction: Thanks to the character twist in the theta lifting formula of the local system, we see that the the associated character restricted on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$ of $\maltese\dagger\mathcal{L}^+_{\tau''_1}$ and $\dagger\mathcal{L}^-_{\tau''_2}$ must be are different, a contradiction.

We finished the proof of the lemma. □

14.4.5. *The maps* $\Upsilon^+_{\mathcal{O}}$, $\Upsilon^-_{\mathcal{O}}$ *and* $\Upsilon_{\mathcal{O}}$.

Lemma 14.11. *Suppose $\mathcal{O}$ is +noticed. The map* $\Upsilon^+_{\mathcal{O}}\colon \{\mathcal{L}_\tau \mid \mathcal{L}^+_\tau \neq 0\} \to \{\mathcal{L}^+_\tau \neq 0\}$ *is injective.*

Proof. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$. This is equivalent to $\mathcal{L}^+_\tau$ has an irreducible component $\succ \boxed{+}$. Now all irreducible components of $\mathcal{L}_\tau \succ \boxed{+}$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^+_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^+_\tau$.

(ii) Suppose that $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ do not contain a component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$.

By Lemma 14.9 $\tau_1 \neq \tau_2$.[10] We now show that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2} \neq 0$ leads to a contradiction. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. By (14.8) and checking the properties in **??**, we have

$$\mathbf{p}_{\tau_i} = \begin{array}{|c|c|}\hline s & x_{\tau''} \\ \hline \vdots & \\ \cline{1-1} s & \\ \cline{1-1} x_{\tau_i} & \\ \cline{1-1}\end{array} \qquad \text{where } x_{\tau''_i} = c/d, x_{\tau_i} = c/d.$$

On the other hand, when $\mathbf{p}_{\tau_i}$ have the above form, the first term of (14.8) is non-zero. So we conclude that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$ implies

$$\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}. \tag{14.10}$$

Moreover, we see that every components of $\mathcal{L}_{\tau_i}$ has a 1-row of mark "$-/=$" and we can determine ϵ_{τ_i} by the mark $-/=$ of the 1-row appeared in $\mathcal{L}^+_{\tau_i}$. In particular, we have $\epsilon_1 = \epsilon_2$. Now **??** implies $\tau''_1 \neq \tau''_2$. This is contradict to (14.10) and our induction hypothesis since $\mathcal{O}''$ is also +noticed.

□

Lemma 14.12. *Suppose $\mathcal{O}$ is noticed. Then the map* $\Upsilon_{\mathcal{O}}\colon \mathcal{L}_\tau \mapsto (\mathcal{L}^+_\tau, \mathcal{L}^-_\tau)$ *is injective.*

[10] $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$ clearly implies $\tau_1 \neq \tau_2$.

Proof. It suffice to consider the case where $\mathcal{O}$ is noticed but not +noticed, i.e. $C_{2k+1} = C_{2k} + 1$. In this case, $n = 1$. The peduncle $\mathbf{p}_\tau$ have four possible cases:

$$\begin{array}{|c|c|}\hline r & c \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline c & c \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline c & d \\ \hline\end{array}\,, \quad \text{or} \quad \begin{array}{|c|c|}\hline d & d \\ \hline\end{array}\,.$$

By (14.8), $\mathcal{L}^+_\tau = \dagger\dagger\mathcal{L}^+_{\tau''}$.

Now suppose $\tau_1 \neq \tau_2$ such that $\mathcal{L}^+_{\tau_1} = \mathcal{L}^+_{\tau_2}$. Since $\mathcal{O}''$ is +noticed, we have $\tau''_1 = \tau''_2$ by the induction hypothesis (see Lemma 14.11). By **??**, this only happens when $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \begin{array}{|c|c|}\hline c & d \\ \hline\end{array}, \begin{array}{|c|c|}\hline d & d \\ \hline\end{array} \,\}$. Since one of $\mathcal{L}^-_{\tau_i}$ is zero and the other is non-zero, we conclude that $\Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_1}) \neq \Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_2})$. ☐

Lemma 14.13. *Suppose $\mathcal{O}$ is +noticed. The map $\Upsilon^-_{\mathcal{O}} \colon \{\, \mathcal{L}_\tau \mid \mathcal{L}^-_\tau \neq 0 \,\} \to \{\, \mathcal{L}^-_\tau \neq 0 \,\}$ is injective.*

Proof. The proof is similar to that of Lemma 14.11. We have two cases:

(i) $\mathcal{L}_\tau$ contain an irreducible component $\succ \begin{array}{|c|}\hline - \\ \hline - \\ \hline\end{array}$. This is equivalent to $\mathcal{L}^-_\tau$ has an irreducible component $\succ \begin{array}{|c|}\hline - \\ \hline\end{array}$ and $\mathcal{L}_\tau \mapsto \mathcal{L}^-_\tau$ will not kill any irreducible components. Hence we could recover $\mathcal{L}_\tau$ from $\mathcal{L}^-_\tau$.

(ii) Now we make a weaker assumption that $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$ such that $\mathcal{L}_{\tau_1}$ and $\mathcal{L}_{\tau_2}$ do not contain a component $\succ \begin{array}{|c|}\hline - \\ \hline - \\ \hline\end{array}$.

By (14.8) and the properties in **??**, we see that $\mathbf{p}_{\tau_i}$ must be one of the following

$$\mathbf{p}_1 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} d & \\ \cline{1-1}\end{array} \qquad \text{or} \qquad \mathbf{p}_2 = \begin{array}{|c|c|}\hline r & d \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} d & \\ \cline{1-1}\end{array}\,.$$

Without of loss of generality, we have the following possibilities:

(a) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_1$. We have $\mathcal{L}^+_{\tau''_1} = \mathcal{L}^+_{\tau''_2}$ and obtain the contradiction.

(b) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_2$. We have $\mathcal{L}^-_{\tau''_1} = \mathcal{L}^-_{\tau''_2}$ and obtain the contradiction.

(c) $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ and $\mathbf{p}_{\tau_2} = \mathbf{p}_2$. Now we apply the same argument in the case (d) of the proof of Lemma 14.9. We get $\maltese\dagger\mathcal{L}^+_{\tau''_1} = \dagger\mathcal{L}^-_{\tau''_2}$, $\mathcal{L}^-_{\tau''_2} \supset +$ and a contradiction by looking at the associated character on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$.

This finished the proof. ☐

15. Matrix coefficient integrals and degenerate principal series

In this section, we define the notion of a ν-bounded representation and investigate the matrix coefficient integrals of such representations against the oscillator representation as well as certain degenerate principle series.

15.1. Matrix coefficients: growth and positivity.

In this subsection, let G be an arbitrary real reductive group.

Definition 15.1. *A (complex valued) function ℓ on G is said to be of* logarithmic growth *if there is a continuous homomorphism $\sigma \colon G \to \mathrm{GL}_n(\mathbb{R})$ for some $n \geqslant 1$, and an integer $d \geqslant 0$ such that*

$$|\ell(g)| \leqslant \big(\log(1 + \mathrm{tr}((\sigma(g))^{\mathrm{t}} \cdot \sigma(g)))\big)^d \quad \textit{for all } g \in G.$$

Assume that a maximal compact subgroup K of G is given. Let Ξ_G denote Harish-Chandra's Ξ-function on G associated to K.

Definition 15.2. *Let $\nu \in \mathbb{R}$. A function f on G is said to be ν-*bounded *if there is a positive function ℓ on G of logarithmic growth such that*

$$|f(g)| \leqslant \ell(g) \cdot (\Xi_G(g))^\nu \quad \textit{for all } g \in G.$$

We remark that the definition is independent of the choice of K.

Lemma 15.3. *Assume that the identity connected component of G has a compact center. Then for all $\nu > 2$, every ν-bounded continuous function on G is integrable (with respect to a Haar measure).*

Proof. This follows easily from the well-known estimate of Harish-Chandra's Ξ-function, see [86, Theorem 4.5.3]. □

Definition 15.4. *Let $\nu \in \mathbb{R}$. A Casselman-Wallach representation π of G is said to be* ν-bounded *if there is a ν-bounded positive function f on G, and continuous seminorms $|\ |_\pi$ on π and $|\ |_{\pi^\vee}$ on $\pi^\vee$ such that*

$$|\langle g \cdot u, v\rangle| \leqslant f(g) \cdot |u|_\pi \cdot |v|_{\pi^\vee},$$

for all $g \in G$, $u \in \pi$, $v \in \pi^\vee$.

We also recall the following positivity result on diagonal matrix coefficients, which is a special case of [32, Theorem A. 5].

Proposition 15.5. *Let G be a real reductive group with a maximal compact subgroup K. Let π_1 and π_2 be two unitary representations of G such that π_2 is weakly contained in the regular representation. Let*

$$u := \sum_{i=1}^{s} u_i \otimes v_i \in \pi_1 \otimes \pi_2.$$

Assume that

(a) for all $i, j = 1, 2, \cdots, s$, the function $g \mapsto \langle g \cdot u_i, u_j\rangle \Xi_G(g)$ on G is absolutely integrable with respect to a Haar measure dg *on G;*

(b) $v_1, v_2, \cdots, v_s$ are all K-finite.

Then the integral

$$\int_G \langle g \cdot u, u\rangle \, \mathrm{d}g$$

absolutely converges to a nonnegative real number.

15.2. **Theta lifting via matrix coefficient integrals.** Let $\Psi_{\mathbf{W}}$ be the positive function on $G_{\mathbf{W}}$ which is bi-$K_{\mathbf{W}}$-invariant such that

$$\Psi_{\mathbf{W}}(g) = \prod_a (1+a)^{-\frac{1}{2}}, \quad \text{for every } g \in S_{\mathbf{W}},$$

where a runs over all eigenvalues of g, counted with multiplicities. By abuse of notation, we still use $\Psi_{\mathbf{W}}$ to denote the pullback function through the covering map $\widetilde{G}_{\mathbf{W}} \to G_{\mathbf{W}}$.

Lemma 15.6. *Extend the irreducible representation $\omega_{\mathbf{V},\mathbf{V}'}|_{\mathrm{H}(W)}$ to the group $\widetilde{G}_{\mathbf{W}} \ltimes \mathrm{H}(W)$. Then there exists continuous seminorms $|\cdot|_1$ on $\omega_{\mathbf{V},\mathbf{V}'}$ and $|\cdot|_2$ on $\omega^\vee_{\mathbf{V},\mathbf{V}'}$ such that*

$$|\langle g \cdot \phi, \phi'\rangle| \leqslant \Psi_{\mathbf{W}}(g) \cdot |\phi|_1 \cdot |\phi'|_2, \quad \text{for all } g \in \widetilde{G}_{\mathbf{W}},\ \phi \in \omega_{\mathbf{V},\mathbf{V}'},\ \phi' \in \omega^\vee_{\mathbf{V},\mathbf{V}'}.$$

Proof. This is in the proof of [50, Theorem 3.2]. □

Define

$$\dim^\circ \mathbf{V} := \begin{cases} \dim \mathbf{V}, & \text{if } G \text{ is real symplectic;} \\ \dim \mathbf{V} - 1, & \text{if } G \text{ is quaternionic symplectic;} \\ \dim \mathbf{V} - 2, & \text{if } G \text{ is real orthogonal;} \\ \dim \mathbf{V} - 3, & \text{if } G \text{ is quaternionic orthogonal.} \end{cases}$$

Let $\Psi_{\mathbf{W}}|_{\widetilde{G}\times\widetilde{G}'}$ denote the pull back of $\Psi_{\mathbf{W}}$ through the natural homomorphism $\widetilde{G} \times \widetilde{G}' \to \widetilde{G}_{\mathbf{W}}$.

Lemma 15.7. *Assume that both* $\dim^\circ \mathbf{V}$ *and* $\dim^\circ \mathbf{V}'$ *are positive. Then the pointwise inequality*

$$\Psi_{\mathbf{W}}|_{\widetilde{G}\times\widetilde{G}'} \leqslant (\Xi_{\widetilde{G}})^{\frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}} \cdot (\Xi_{\widetilde{G}'})^{\frac{\dim \mathbf{V}}{\dim^\circ \mathbf{V}'}}$$

holds.

Proof. This is easy to check, by using the well-known estimate of Harish-Chandra's Ξ-function (*cf.* [86, Theorem 4.5.3].)

$\square$

15.2.1. *Matrix coefficient integrals against oscillator representations.* Let π be a Casselman-Wallach representation of $\widetilde{G}$.

Definition 15.8. *Assume that* $\dim^\circ \mathbf{V} > 0$*. The pair* $(\pi, \mathbf{V}')$ *is said to be in the* convergent range *if*

- π *is* ν_π*-bounded for some* $\nu_\pi > 2 - \frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}$*;*
- *if* G *is a real symplectic group, then* ε_G *acts on* π *through the scalar multiplication by* $(-1)^{\dim \mathbf{V}'}$*.*

In the rest of this section, assume that $\dim^\circ \mathbf{V} > 0$ and the pair $(\pi, \mathbf{V}')$ is in the convergent range. Consider the following integral:

$$\begin{aligned}(15.1) \qquad (\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}) \times (\pi^\vee\widehat{\otimes}\omega^\vee_{\mathbf{V},\mathbf{V}'}) &\longrightarrow \mathbb{C}, \\ (u,v) &\longmapsto \textstyle\int_{\widetilde{G}} \langle g\cdot u, v\rangle \,\mathrm{d}g.\end{aligned}$$

Lemma 15.9. *The integrals in* (15.1) *are absolutely convergent and yield a continuous bilinear map.*

Proof. This is implied by Lemma 15.7 and Lemma 15.3. $\square$

In view of Lemma 15.9, we define

$$\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi) := \frac{\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}}{\text{the left kernel of (15.1)}}. \tag{15.2}$$

This is a Casselman-Wallach representation of $\widetilde{G}'$, since it is a quotient of the full theta lift $(\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'})_G$ (the Hausdorff coinvariant space).

Lemma 15.10. *Assume that* $\dim^\circ \mathbf{V}' > 0$*. Then the representation* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is* $\frac{\dim \mathbf{V}}{\dim^\circ \mathbf{V}'}$*-bounded.*

Proof. This is also implied by Lemma 15.7 and Lemma 15.3. $\square$

15.2.2. *Unitarity.*

Theorem 15.11. *Assume that* $\dim \mathbf{V}' \geqslant \dim^\circ \mathbf{V}$ *and* π *is* ν_π*-bounded for some*

$$\nu_\pi > \begin{cases} 2 - \frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}, & \text{if } \dim^\circ \mathbf{V} \text{ is even,} \\ 2 - \frac{\dim \mathbf{V}'-1}{\dim^\circ \mathbf{V}}, & \text{if } \dim^\circ \mathbf{V} \text{ is odd.}\end{cases}$$

Then $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is unitarizable if* π *is unitarizable. Moreover, when* π *is irreducible and unitarizable, and* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is nonzero, then* $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ *is irreducible unitarizable.*

Proof. Assume that π is unitarizable. For the first claim, it suffices to show that

$$\int_{\widetilde{G}} \langle g\cdot u, u\rangle \,\mathrm{d}g \geqslant 0 \quad \text{for all } u \text{ in a dense subspace of } \pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}. \tag{15.3}$$

Here $\langle\,,\,\rangle$ denotes a Hermitian inner product on $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}$ which is invariant under $\widetilde{G}\times((\widetilde{G}\times\widetilde{G}')\ltimes \mathrm{H}(W))$.

Take an orthogonal decomposition

$$\mathbf{V}' = \mathbf{V}_1'\oplus\mathbf{V}_2'$$

which is stable under both J' and L' such that

$$\dim \mathbf{V}_2' = \begin{cases}\dim^\circ \mathbf{V}, & \text{if } \dim^\circ \mathbf{V} \text{ is even.}\\ \dim^\circ \mathbf{V}+1, & \text{if } \dim^\circ \mathbf{V} \text{ is odd.}\end{cases}$$

(When $\dim^\circ \mathbf{V}$ is odd, $\dim \mathbf{V}'$ is always even.) Write

$$\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'} = (\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}_1'})\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}_2'}.$$

Note that the Hilbert space completions of $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}_1'}$ and $\omega_{\mathbf{V},\mathbf{V}_2'}$, viewed as unitary representations of $\widetilde{G}$, satisfy the hypothesis for π_1 and π_2 in Proposition 15.5. For the latter, see [50, Theorem 3.2]. Thus (15.3) follows by Proposition 15.5.

The second claim follows from the fact that $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is a quotient of $(\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\pi)_G$ which is of finite length and has a unique irreducible quotient [40]. $\square$

15.3. **An equality of associated characters in the convergent range.** In this subsection, we assume that $\dim^\circ \mathbf{V}>0$. The purpose of this section is to prove the following theorem.

Theorem 15.12. *Let $\mathcal{O}\in \mathrm{Nil}_{\mathbf{G}}(\mathfrak{g})$ and $\mathcal{O}'\in \mathrm{Nil}_{\mathbf{G}'}(\mathfrak{g}')$ such that $\mathcal{O}$ is the descent of $\mathcal{O}'$. Write $\mathbf{D}(\mathcal{O}')=[c_0,c_1,\cdots,c_k]$ ($k\geqslant 1$). Assume that $c_0>c_1$ when G is a real symplectic group. Then for every $\mathcal{O}$-bounded Casselman-Wallach representation π of $\widetilde{G}$ such that $(\pi,\mathbf{V}')$ is in the convergent range (see Definition 15.8), $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is $\mathcal{O}'$-bounded and*

$$\mathrm{Ch}_{\mathcal{O}'}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) = \vartheta_{\mathcal{O},\mathcal{O}'}(\mathrm{Ch}_{\mathcal{O}}(\pi)). \tag{15.4}$$

Remark. When $(\mathbf{V},\mathbf{V}')$ is in the stable range, Theorem 15.12 is proved for unitary representations π in [53].

We keep the setting of Theorem 15.12. First observe that the Harish-Chandra module of $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is isomorphic to a quotient of $\check{\Theta}(\pi^{\mathrm{al}})$, where π^{al} denotes the Harish-Chandra module of π. Thus Theorem 12.6 implies that $\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)$ is $\mathcal{O}'$-bounded and

$$\mathrm{Ch}_{\mathcal{O}'}\,\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi) \preceq \vartheta_{\mathcal{O},\mathcal{O}'}(\mathrm{Ch}_{\mathcal{O}}(\pi)). \tag{15.5}$$

We will devote the rest of this section to prove that the equality in (15.5) holds.

15.3.1. *Doubling.* We consider a two-step theta lifting. Let $\mathbf{U}$ be as in (11.7). We realize $\mathbf{V}$ as a non-genenerate $(\epsilon,\dot\epsilon)$-subspace of $\mathbf{U}$ and write $\mathbf{V}^\perp$ for the orthogonal complement of $\mathbf{V}$ in $\mathbf{U}$, which is also an $(\epsilon,\dot\epsilon)$-space.

Note that $\dim^\circ \mathbf{V}'>0$ and Lemma 15.10 implies that $(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi),\mathbf{V}^\perp)$ is in the convergent range. Comparing (**??**) with (11.7), we have $\nu_{\mathbf{U},\mathbf{V}}=\frac{\dim \mathbf{V}'}{\dim^\circ \mathbf{V}}$. Thus (11.5) holds and $\mathcal{R}_{\bar{\Theta}_{\mathbf{U}}(1_{\mathbf{V}'})}(\pi)$ is defined by **??**.

Lemma 15.13. *One has that*

$$\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \cong \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})}(\pi). \tag{15.6}$$

Proof. Note that the integral in

$$\begin{array}{rcl}(\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega_{\mathbf{V}',\mathbf{V}^\perp})\times(\pi^\vee\widehat{\otimes}\omega^\vee_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega^\vee_{\mathbf{V}',\mathbf{V}^\perp}) & \longrightarrow & \mathbb{C}\\ (u,v) & \longmapsto & \int_{\widetilde{G}\times\widetilde{G}'}\langle g\cdot u,v\rangle\,\mathrm{d}g\end{array} \tag{15.7}$$

is absolutely convergent and defines a continuous bilinear map. In view of Fubini's theorem and Lemma 5.10, the lemma follows as both sides of (15.6) are isomorphic to the quotient of $\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'}\widehat{\otimes}\omega_{\mathbf{V}',\mathbf{V}^\perp}$ by the left radical of the pairing (15.7). □

We will use (15.6) freely in the rest of this section.

15.3.2. *On certain induced orbits.* The main step in the proof of Theorem 15.12 consists of comparison of bound of the associated cycle of $\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))$ with the formula (due to Barbasch) of the wavefront cycle of a certain parabolically induced representation.

In this section, let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}^J$ be the Lie algebra of G, which is identified with its dual space $\mathfrak{g}_{\mathbb{R}}^*$ under the trace from. Let $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}})$ denote the set of nilpotent G-orbits in $\mathbf{i}\mathfrak{g}_{\mathbb{R}}$. Similar notation will be used without further explanation. For example, for every Levi subgroup M of G, $\mathfrak{m}_{\mathbb{R}}$ denotes its Lie algebra, which is identified with the dual space $\mathfrak{m}_{\mathbb{R}}^*$ by using the trace form on $\mathfrak{g}$.

Let

$$\mathsf{KS}\colon \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}}) \to \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$$

be the natural bijection given by the Kostant-Sekiguchi correspondence (*cf.* [76, Equation (6.7)]). By abuse of notation, we also let $\ddot{\mathbf{D}}$ denote the map $\ddot{\mathbf{D}}\circ\mathsf{KS}\colon \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{\mathbb{R}}) \to \ddot{\mathcal{P}}$. See Section 12.7 for the parametrization map $\ddot{\mathbf{D}}\colon \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p}) \to \ddot{\mathcal{P}}$.

Theorem 15.14 (*cf.* Barbasch [10, Corollary 5.0.10])**.** *Let G_1 be an arbitrary real reductive group and let P_1 be a real parabolic subgroup of G_1, namely a closed subgroup of G_1 whose Lie algebra is a parabolic subalgebra of $\mathfrak{g}_{1,\mathbb{R}}$. Let N_1 be the unipotent radical of P_1 and $M_1 := P_1/N_1$. Write*

$$r_1 : \mathbf{i}(\mathfrak{g}_{1,\mathbb{R}}/\mathfrak{n}_{1,\mathbb{R}})^* \longrightarrow \mathbf{i}\,\mathfrak{m}_{1,\mathbb{R}}^*$$

for the natural map. Let π_1 be a Casselman-Wallach representation of M_1 with the wavefront cycle

$$\mathrm{WF}(\pi_1) = \sum_{\mathscr{O}_{\mathbb{R}}\in\mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}_{1,\mathbb{R}}^*)} \mathrm{c}_{\mathscr{O}_{\mathbb{R}}}[\mathscr{O}_{\mathbb{R}}].$$

Then

$$\mathrm{WF}(\mathrm{Ind}_{P_1}^{G_1}\pi) = \sum_{(\mathscr{O}_{\mathbb{R}},\mathscr{O}'_{\mathbb{R}})} \mathrm{c}_{\mathscr{O}_{\mathbb{R}}}\frac{\#C_{G_1}(v')}{\#C_{P_1}(v')}[\mathscr{O}'_{\mathbb{R}}],$$

where the summation runs over all pairs $(\mathscr{O}_{\mathbb{R}},\mathscr{O}'_{\mathbb{R}})$ such that $\mathscr{O}_{\mathbb{R}}\in\mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}_{1,\mathbb{R}}^*)$ and $\mathscr{O}'_{\mathbb{R}}\in\mathrm{Nil}_{G_1}(\mathbf{i}\mathfrak{g}_{1,\mathbb{R}}^*)$ is an induced orbit of $\mathscr{O}_{\mathbb{R}}$, v' is an element of $\mathscr{O}'_{\mathbb{R}}\cap r_1^{-1}(\mathscr{O}_{\mathbb{R}})$, and $C_{G_1}(v')$ and $C_{P_1}(v')$ are the component groups of the centralizers of v' in G_1 and P_1, respectively.

Remarks. 1. While Barbasch proved the theorem when G_1 is the real points of a connected reductive algebraic group, his proof still works in the slightly more general setting of Theorem 15.14.
2. The notion of induced nilpotent orbits was introduced by Lusztig and Spaltenstein for complex reductive groups [54]. For a real reductive group G_1, a nilpotent orbit $\mathscr{O}'_{\mathbb{R}}\in\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_{1,\mathbb{R}}^*)$ is called an induced orbit of a nilpotent orbit $\mathscr{O}_{\mathbb{R}}\in\mathrm{Nil}_{M_1}(\mathbf{i}\mathfrak{m}_{1,\mathbb{R}}^*)$ if $\mathscr{O}'_{\mathbb{R}}\cap r_1^{-1}(\mathscr{O}_{\mathbb{R}})$ is open in $r_1^{-1}(\mathscr{O}_{\mathbb{R}})$ (see [10, Definition 5.0.7] or [66]). We write $\mathrm{Ind}_{P_1}^{G_1}\mathscr{O}_{\mathbb{R}}$ for the set of all induced orbits of $\mathscr{O}_{\mathbb{R}}$. Similar notation applies for a complex reductive group.
3. According to the fundamental result of Schimd-Vilonen [76], the wave front cycle and the associated cycle agree under the Kostant-Sekiguchi correspondence. We thank Professor Vilonen for confirming that their result extends to nonlinear groups. Therefore the associated cycle of a parabolically induced representation as in the above theorem of Barbasch is also determined.

We now consider induced orbits appearing in our cases. Retain the setting in Section 5.2 (see (11.4) and onwards), where

$$\mathbf{V}^\perp = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}_0',$$

$M_{\mathbf{E}_0} = G \times \mathrm{GL}_{\mathbf{E}_0}$ and the parabolic subgroup of $G_{\mathbf{V}^\perp}$ stabilizing $\mathbf{E}_0$ is $P_{\mathbf{E}_0} = M_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$.

Recall that $\mathbf{D}(\mathcal{O}') = [c_0, \cdots, c_k]$ and $\mathbf{D}(\mathcal{O}) = [c_1, \cdots, c_k]$. Put $l := \dim \mathbf{E}_0$. In the notation of (11.7), we have

$$l = c_0 + \delta \geqslant c_1. \tag{15.8}$$

Note that if $\mathbf{G}$ is an orthogonal group, then $c_0 - c_1$ is even. Hence $l - c_1$ is odd if G is real orthogonal and $l - c_1$ is even if G is quaternionic orthogonal. View $\mathcal{O}$ as a nilpotent orbit in $\mathfrak{m}_{\mathbf{E}_0}$ via inclusion. The following lemma is clear (*cf.* [23, Section 7.3]).

Lemma 15.15. *(i) If G is a real orthogonal group, then*

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}) = [l+1, l-1, c_1, \cdots, c_k].$$

(ii) Otherwise,

$$\mathbf{D}(\mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}) = [l, l, c_1, \cdots, c_k].$$

By [25, Theorem 5.2 and 5.6], the complex nilpotent orbit

$$\mathcal{O}^\perp := \boldsymbol{\vartheta}_{\mathbf{V}', \mathbf{V}^\perp}(\mathcal{O}') = \boldsymbol{\vartheta}_{\mathbf{V}', \mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V}, \mathbf{V}'}(\mathcal{O}))$$

is given by

$$\mathbf{D}(\mathcal{O}^\perp) = \begin{cases} [c_0+2, c_0, c_1, \cdots, c_k], & \text{if } G \text{ is a real orthogonal group;} \\ [c_0-1, c_0-1, c_1, \cdots, c_k], & \text{if } G \text{ is a real symplectic group;} \\ [c_0, c_0, c_1, \cdots, c_k], & \text{otherwise.} \end{cases}$$

This implies that

$$\mathcal{O}^\perp = \mathrm{Ind}_{\mathbf{P}_{\mathbf{E}_0}}^{\mathbf{G}_{\mathbf{V}^\perp}} \mathcal{O}.$$

View each orbit in $\mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R})$ as an orbit in $\mathrm{Nil}_{M_{\mathbf{E}_0}}(\mathbf{i}\,\mathfrak{m}_{\mathbf{E}_0}^J)$ via inclusion. We state the result for the induction of real nilpotent orbits in the following lemma. The proof will be given in Appendix B.2, which is by elementary matrix manipulations.

Lemma 15.16. *Let $\mathscr{O}_\mathbb{R} \in \mathrm{Nil}_G(\mathbf{i}\mathfrak{g}_\mathbb{R})$ be a real nilpotent orbit in $\mathcal{O}$ with $\ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}) = [d_1, \cdots, d_k]$.*

(i) Suppose G is a real orthogonal group. Then $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$ consists of a single orbit $\mathscr{O}_\mathbb{R}^\perp$ with

$$\ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}^\perp) = [d_1 + s + (1,1), \check{d}_1 + \check{s}, d_1, \cdots, d_k],$$

where $s := (\frac{l-c_1-1}{2}, \frac{l-c_1-1}{2}) \in \mathbb{Z}_{\geqslant 0}^2$. Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is injective and its image has index 2 in $C_{G_{\mathbf{V}^\perp}}(X)$ for $X \in \mathscr{O}_\mathbb{R}^\perp \cap (\mathscr{O}_\mathbb{R} + \mathbf{i}\,\mathfrak{n}_{\mathbf{E}_0}^J)$.

(ii) Otherwise, $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$ equals the set

$$\left\{ \mathscr{O}_\mathbb{R}^\perp \in \mathrm{Nil}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\mathfrak{g}_{\mathbf{V}^\perp}^J) \;\middle|\; \begin{array}{l} \ddot{\mathbf{D}}(\mathscr{O}_\mathbb{R}^\perp) = [d_1 + s, \check{d}_1 + \check{s}, d_1, \cdots, d_k] \text{ for a signature} \\ s \text{ of a } (-\epsilon, -\dot{\epsilon})\text{-space of dimension } l - c_1 \end{array} \right\}.$$

Moreover, the natural map $C_{P_{\mathbf{E}_0}}(X) \to C_{G_{\mathbf{V}^\perp}}(X)$ is an isomorphism for $X \in \mathscr{O}_\mathbb{R}^\perp \cap (\mathscr{O}_\mathbb{R} + \mathbf{i}\,\mathfrak{n}_{\mathbf{E}_0}^J)$.

Here $C_{P_{\mathbf{E}_0}}(X)$ and $C_{G_{\mathbf{V}^\perp}}(X)$ are the component groups as in Theorem 15.14. Under the setting of Lemma 15.16, we see that $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}}$ consists of a single orbit when G is a real orthogonal group or a quaternionic symplectic group.

For each $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$, let

$$\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O} := \left\{ \mathsf{KS}(\mathscr{O}_{\mathbb{R}}) \,\middle|\, \mathscr{O}_{\mathbb{R}} \in \mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}}(\mathsf{KS}^{-1}(\mathscr{O})) \right\}.$$

15.3.3. *Finishing the proof when G is a real orthogonal group.* Let $\mathrm{sgn}_{\mathbf{V}}$ and $\mathrm{sgn}_{\mathbf{V}^\perp}$ be the sign character of the orthogonal groups G and $G_{\mathbf{V}^\perp}$ respectively. By Proposition 11.4, Lemma 5.9 **??** and Lemma 5.10, we have

$$\begin{aligned} &\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}) \\ &\cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \cong \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})}(\pi) \oplus \mathcal{R}_{\bar{\Theta}_{\mathbf{V}',\mathbf{U}}(1_{\mathbf{V}'})\otimes \mathrm{sgn}_{\mathbf{U}}}(\pi) \\ &\cong \bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi)) \oplus (\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi \otimes \mathrm{sgn}_{\mathbf{V}}))) \otimes \mathrm{sgn}_{\mathbf{V}^\perp}. \end{aligned} \tag{15.9}$$

By Theorem 15.14, Lemma 15.15, Lemma 15.16 (i) and [76, Theorem 1.4], the representation $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}))$ is $\mathcal{O}^\perp$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})) = 2 \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{15.10}$$

where the summation runs over all $\mathbf{K}$-orbits $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^\perp$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}$.

On the other hand, by the explicit formula for the descents of nilpotent orbits, we have

$$\nabla_{\mathbf{V}',\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'}(\mathscr{O}^\perp)) = \mathscr{O}.$$

Applying Theorem 12.6 twice, we have

$$\begin{aligned} \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi))) &\preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and} \\ \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}',\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'}(\pi \otimes \mathrm{sgn}_{\mathbf{V}}))) \otimes \mathrm{sgn}_{\mathbf{V}^\perp}) &\preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp] \end{aligned} \tag{15.11}$$

where the summations run over the same set as the right hand side of (15.10).

In view of (15.9) to (15.11), we conclude that both inequalities in (15.11) are equalities. Thus (15.4) follows.

15.3.4. *Finishing the proof when G is a real symplectic group.* Let $\mathbf{V}'_1, \mathbf{V}'_2, \cdots, \mathbf{V}'_s$ be a list of representatives of the isomorphic classes of all $(1,1)$-spaces with dimension $\dim \mathbf{V}'$.

By Proposition 11.4, Lemma 5.9 **??** and Lemma 5.10, we have

$$\begin{aligned} &\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}) \oplus \mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} \left(\pi \otimes \chi'_{\mathbf{E}_0}\right) \\ &\cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \oplus \mathcal{R}_{\mathrm{I}(\chi'_{\mathbf{E}})}(\pi) \\ &\cong \bigoplus_{i=1}^{s} \mathcal{R}_{\bar{\Theta}_{\mathbf{V}'_i,\mathbf{U}}(1_{\mathbf{V}'_i})}(\pi) \cong \bigoplus_{i=1}^{s} \bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi)). \end{aligned} \tag{15.12}$$

By Theorem 15.14, Lemma 15.15, Lemma 15.16 (ii) and [76, Theorem 1.4], $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})$ and $\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} \left(\pi \otimes \chi'_{\mathbf{E}_0}\right)$ are $\mathcal{O}^\perp$-bounded, and

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})) = \mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}} \left(\pi \otimes \chi'_{\mathbf{E}_0}\right)) = \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{15.13}$$

where the summation runs over all pairs $(\mathscr{O}, \mathscr{O}^\perp)$, where $\mathscr{O}$ is a $\mathbf{K}$-orbit in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$ and $\mathscr{O}^\perp \in \mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}$.

Applying Theorem 12.6 twice, we see that $\bar\Theta_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'_i}(\pi))$ is $\mathcal{O}^\perp$-bounded ($1 \leqslant i \leqslant s$), and

$$\sum_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^\perp}(\bar\Theta_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'_i}(\pi))) \preceq \sum_{i=1}^{s} \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp],$$

where the inner summation runs over all pairs of orbits $(\mathscr{O}, \mathscr{O}^\perp)$ such that

$$\nabla_{\mathbf{V}'_i,\mathbf{V}}(\nabla^{\mathrm{gen}}_{\mathbf{V}^\perp,\mathbf{V}'_i}(\mathscr{O}^\perp)) = \mathscr{O}. \tag{15.14}$$

By Lemma 15.16 (ii) and (12.20), such kind of pairs $(\mathscr{O}, \mathscr{O}^\perp)$ are the same as those of the right hand side of (15.13). Suppose $\ddot{\mathbf{D}}(\mathscr{O}^\perp) = [d, \check d, d_1, \cdots, d_k]$, then $\ddot{\mathbf{D}}(\mathscr{O}) = [d_1, \cdots, d_k]$ and (15.14) holds for exactly two $\mathbf{V}'_i$, having signature $\check d + \mathrm{Sign}(\mathbf{V}) + (1,0)$ and $\check d + \mathrm{Sign}(\mathbf{V}) + (0,1)$ respectively. Hence we have

$$\sum_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^\perp}(\bar\Theta_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'_i}(\pi))) \preceq \sum_{\mathscr{O},\mathscr{O}^\perp} 2\mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{15.15}$$

where the summation is as in the right hand side of (15.13).

In view of (15.12), (15.13) and (15.15), the equality holds in (15.15). Thus (15.4) holds.

15.3.5. *Finishing the proof when G is a quaternionic symplectic group.* Using Proposition 11.4, Lemma 5.9 **??**, Lemma 5.10, Theorem 15.14, Lemma 15.16 (ii), [76, Theorem 1.4] and Theorem 12.6, and a similar argument as in Section 15.3.3 shows that

$$\mathrm{Ind}_{\tilde P_{\mathbf{E}_0}}^{\tilde G_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0}) \cong \mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi) \cong \bar\Theta_{\mathbf{V}',\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi)) \tag{15.16}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde P_{\mathbf{E}_0}}^{\tilde G_{\mathbf{V}^\perp}} (\pi \otimes \chi_{\mathbf{E}_0})) = \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and} \tag{15.17}$$

$$\mathrm{AC}_{\mathcal{O}^\perp}(\bar\Theta_{\mathbf{V}',\mathbf{V}^\perp}(\bar\Theta_{\mathbf{V},\mathbf{V}'}(\pi))) \preceq \sum_{\mathscr{O}} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \tag{15.18}$$

Here the summations run over all $\mathbf{K}_{\mathbf{V}}$-orbit $\mathscr{O}$ in $\mathcal{O} \cap \mathfrak{p}_{\mathbf{V}}$, and $\mathscr{O}^\perp$ is the unique induced orbit in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}$. Note that

$$\nabla_{\mathbf{V}',\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'}(\mathscr{O}^\perp)) = \mathscr{O}.$$

Now (15.16), (15.17) and (15.18) imply that (15.18) is an equality and so (15.4) holds.

15.3.6. *Finishing the proof when G is a quaternionic orthogonal group.* Let $\mathbf{V}'_1, \mathbf{V}'_2, \cdots, \mathbf{V}'_s$ be a list of representatives of the isomorphic classes of all $(-1,1)$-spaces with dimension $\dim \mathbf{V}'$.

Lemma 15.17. *One has that*

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{\mathrm{I}(\chi_{\mathbf{E}})}(\pi)) = \sum_{i=1}^{s} \mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{\bar\Theta_{\mathbf{V}'_i,\mathbf{U}}(1_{\mathbf{V}'_i})}(\pi)).$$

Proof. For simplicity, write $0 \to I_1 \to I_2 \to I_3 \to 0$ for the exact sequence in Lemma 5.9 **??**. Note that $\mathcal{R}_{I_2}(\pi)$ is $\mathcal{O}^\perp$-bounded by Proposition 11.4, Theorem 15.14 and [76, Theorem 1.4]. By the definition in (11.6), we could view $\mathcal{R}_{I_1}(\pi)$ as a subrepresentation of $\mathcal{R}_{I_2}(\pi)$ which is also $\mathcal{O}^\perp$-bounded.

Since G acts on $\mathcal{R}_{I_2}(\pi)$ trivially, the natural homomorphism

$$\pi \widehat\otimes I_3 \cong \pi \widehat\otimes I_2 / \pi \widehat\otimes I_1 \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)$$

descents to a surjective homomorphism

$$(\pi\widehat{\otimes}I_3)_G \longrightarrow \mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi).$$

For each $(-1,1)$-space $\mathbf{V}'''$ of dimension $\dim \mathbf{V}'-2$, by applying Lemma 12.5 twice, we see that

$$(\pi\widehat{\otimes}(\omega_{\mathbf{V}''',\mathbf{U}})_{G_{\mathbf{V}'''}})_G \cong ((\pi\widehat{\otimes}\omega_{\mathbf{V},\mathbf{V}'''})_G\widehat{\otimes}\omega_{\mathbf{V}''',\mathbf{V}^\perp})_{G_{\mathbf{V}'''}}$$

is bounded by $\boldsymbol{\vartheta}_{\mathbf{V}''',\mathbf{V}^\perp}(\boldsymbol{\vartheta}_{\mathbf{V},\mathbf{V}'''}(\mathcal{O}))$. Using formulas in [25, Theorem 5.2 and 5.6], one checks that the latter set is contained in the boundary of $\mathcal{O}^\perp$. Hence

$$\mathrm{Ch}_{\mathcal{O}^\perp}(\mathcal{R}_{I_2}(\pi)/\mathcal{R}_{I_1}(\pi)) = 0$$

and the lemma follows. □

Using Theorem 15.14, Lemma 15.16 (ii), [76, Theorem 1.4], Theorem 12.6 and a similar argument as in Section 15.3.4, we have

$$\mathrm{AC}_{\mathcal{O}^\perp}(\mathrm{Ind}_{\tilde{P}_{\mathbf{E}_0}}^{\tilde{G}_{\mathbf{V}^\perp}}(\pi\otimes\chi_{\mathbf{E}_0})) = \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp], \quad \text{and} \tag{15.19}$$

$$\bigoplus_{i=1}^{s} \mathrm{AC}_{\mathcal{O}^\perp}(\bar{\Theta}_{\mathbf{V}'_i,\mathbf{V}^\perp}(\bar{\Theta}_{\mathbf{V},\mathbf{V}'_i}(\pi))) \preceq \sum_{\mathscr{O},\mathscr{O}^\perp} \mathrm{c}_{\mathscr{O}}(\pi)[\mathscr{O}^\perp]. \tag{15.20}$$

Here the summations run over all pairs $(\mathscr{O},\mathscr{O}^\perp)$ such that $\mathscr{O}^\perp$ is a $\mathbf{K}_{\mathbf{V}^\perp}$-orbit in $\mathcal{O}^\perp\cap\mathfrak{p}_{\mathbf{V}^\perp}$ and $\mathscr{O} = \nabla_{\mathbf{V}'_i,\mathbf{V}}(\nabla_{\mathbf{V}^\perp,\mathbf{V}'_i}(\mathscr{O}^\perp))$ for a unique $(-1,1)$-space $\mathbf{V}'_i$.

In view of Proposition 11.4, Lemma 15.17, (15.19) and (15.20), we see that the equality holds in (15.20). Thus (15.4) holds.

16. The unipotent representations construction and iterated theta lifting

In this section, we will prove **??**. Recall that $\mathbf{V}$ is an $(\epsilon,\dot\epsilon)$-space and $\mathcal{O}\in\mathrm{Nil}^{\mathbb{P}}_{\mathbf{G}}(\mathfrak{g})$. Since **??** is obvious when $\mathcal{O}$ is the zero orbit, we assume without loss of generality that $\mathcal{O}$ is not the zero orbit.

16.1. The construction.

Suppose there is a $\mathbf{K}$-orbit $\mathscr{O}\in\mathcal{O}\cap\mathfrak{p}$. Define a sequence $(\mathbf{V}_0,\mathscr{O}_0),(\mathbf{V}_1,\mathscr{O}_1),\cdots,(\mathbf{V}_k,\mathscr{O}_k)$ $(k\geqslant 1)$ such that

- $\mathbf{V}_j$ is a nonzero $((-1)^j\epsilon,(-1)^j\dot\epsilon)$-space for all $0\leqslant j\leqslant k$;
- $(\mathbf{V}_0,\mathscr{O}_0)=(\mathbf{V},\mathscr{O})$, and $\mathscr{O}_j=\nabla_{\mathbf{V}_{j-1},\mathbf{V}_j}(\mathscr{O}_{j-1})\in\mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_j}}(\mathfrak{p}_{\mathbf{V}_j})$ for all $1\leqslant j\leqslant k$;
- $\mathscr{O}_k$ is the zero orbit.

Let $\mathcal{O}_j$ denote the nilpotent $G_{\mathbf{V}_j}$-orbit containing $\mathscr{O}_j$ $(0\leqslant j\leqslant k)$. To ease the notation, we also let $\mathbf{V}_{k+1}$ be the zero $((-1)^{k+1}\epsilon,(-1)^{k+1}\dot\epsilon)$-space and let $\mathscr{O}_{k+1}\in\mathrm{Nil}_{\mathbf{K}_{\mathbf{V}_{k+1}}}(\mathfrak{p}_{\mathbf{V}_{k+1}})$ and $\mathcal{O}_{k+1}\in\mathrm{Nil}_{\mathbf{G}_{\mathbf{V}_{k+1}}}(\mathfrak{g}_{\mathbf{V}_{k+1}})$ be $\{0\}$.

Let

$$\eta = \chi_0\boxtimes\chi_1\boxtimes\cdots\boxtimes\chi_k \tag{16.1}$$

be a character of $G_{\mathbf{V}_0}\times G_{\mathbf{V}_1}\times\cdots\times G_{\mathbf{V}_k}$. For $0\leqslant j\leqslant k$, put

$$\eta_j := \chi_j\boxtimes\chi_{j+1}\boxtimes\cdots\boxtimes\chi_k.$$

For $0\leqslant j<k$, write

$$\omega_{\mathscr{O}_j} := \omega_{\mathbf{V}_j,\mathbf{V}_{j+1}}\widehat{\otimes}\omega_{\mathbf{V}_{j+1},\mathbf{V}_{j+2}}\widehat{\otimes}\cdots\widehat{\otimes}\omega_{\mathbf{V}_{k-1},\mathbf{V}_k},$$

and one checks that the integrals in

$$(16.2)\qquad \begin{aligned}(\omega_{\mathscr{O}_j}\otimes\eta_j)\times(\omega^\vee_{\mathscr{O}_j}\otimes\eta_j^{-1}) &\longrightarrow \mathbb{C},\\ (u,v) &\longmapsto \int_{\widetilde{G}_{\mathbf{V}_{j+1}}\times\widetilde{G}_{\mathbf{V}_{j+2}}\times\cdots\times\widetilde{G}_{\mathbf{V}_k}} \langle g\cdot u, v\rangle\,\mathrm{d}g,\end{aligned}$$

are absolutely convergent and define a continuous bilinear map using Lemma 15.6. Define

$$\pi_{\mathscr{O}_j,\eta_j} := \frac{\omega_{\mathscr{O}_j}\otimes\eta_j}{\text{the left kernel of (16.2)}},$$

which is a Casselman-Wallach representation of $\widetilde{G}_{\mathbf{V}_j}$, as in (15.2). Set $\pi_{\mathscr{O}_k,\eta_k} := \chi_k$ by convention.

For $0 \leqslant j \leqslant k$, let $\chi_i|_{\widetilde{\mathbf{K}}_{\mathbf{V}_i}}$ denote the algebraic character whose restriction to $\widetilde{K}_{\mathbf{V}_i}$ equals the pullback of χ_i through the natural homomorphism $\widetilde{K}_{\mathbf{V}_i} \to G_{\mathbf{V}_i}$. Let $\mathcal{E}_{\mathscr{O}_k,\chi_k}$ denote the $\widetilde{\mathbf{K}}_{\mathbf{V}_k}$-equivariant algebraic line bundle on the zero orbit $\mathscr{O}_k$ corresponding to $\chi_k|_{\widetilde{\mathbf{K}}_{\mathbf{V}_k}}$. Inductively define

$$\mathcal{E}_{\mathscr{O}_j,\eta_j} := \chi_j|_{\widetilde{\mathbf{K}}_{\mathbf{V}_j}} \otimes \vartheta_{\mathscr{O}_{j+1},\mathscr{O}_j}(\mathcal{E}_{\mathscr{O}_{j+1},\eta_{j+1}}), \quad 0 \leqslant j < k.$$

This is an admissible orbit datum over $\mathscr{O}_i$ by (12.33).

Theorem 16.1. *For each $0 \leqslant j \leqslant k$, $\pi_{\mathscr{O}_j,\eta_j}$ is an irreducible, unitarizable, $\mathcal{O}_j$-unipotent representation whose associated character*

$$(16.3)\qquad \mathrm{Ch}_{\mathcal{O}_j}(\pi_{\mathscr{O}_j,\eta_j}) = \mathcal{E}_{\mathscr{O}_j,\eta_j}.$$

Moreover,

$$(16.4)\qquad \bar{\Theta}_{\mathbf{V}_{j+1},\mathbf{V}_j}(\pi_{\mathscr{O}_{j+1},\eta_{j+1}})\otimes\chi_j = \pi_{\mathscr{O}_j,\eta_j} \quad \textit{for all } 0 \leqslant j < k-1.$$

Proof. Since $\mathcal{O} \in \mathrm{Nil}^{\mathbb{p}}_{\mathbf{G}}(\mathfrak{p})$, one verifies that

- $\dim^\circ \mathbf{V}_j > 0$ for $0 \leqslant j < k$,
- $\dim \mathbf{V}_{j+1} + \dim \mathbf{V}_{j-1} > 2\dim^\circ \mathbf{V}_j$ for $1 \leqslant j \leqslant k$, and
- $\pi_{\mathscr{O}_j,\eta_j}$ is $\mathbb{p}$-genuine for $0 \leqslant j \leqslant k$.

The representation $\pi_{\mathcal{O}_k,\eta_k}$ clearly satisfies all claims in the theorem. For $j = k-1$, $\pi_{\mathscr{O}_{k-1},\eta_{k-1}}$ is the twist by χ_{k-1} of the theta lift of character χ_k in the stable range. It is known that the statement of Theorem 16.1 holds in this case (*cf.* [50, Section 2] and [53, Section 1.8]).

We now prove the theorem by induction. Assume the theorem holds for $j+1$ with $1 \leqslant j+1 \leqslant k-1$. Applying Lemma 15.9 and Lemma 15.10, we see that $(\pi_{\mathscr{O}_{j+1},\eta_{j+1}}, \mathbf{V}_j)$ is in the convergent range. Thus (16.4) holds by Fubini's theorem. By [70, Theorem 1.19], $\mathrm{U}(\mathfrak{g}_{\mathbf{V}_j})^{\mathbf{G}_{\mathbf{V}_j}}$ acts on $\pi_{\mathscr{O}_j,\eta_j}$ through the character $\lambda_{\mathcal{O}_j}$. By Theorem 15.12, $\pi_{\mathscr{O}_j,\eta_j}$ is $\mathcal{O}_j$-bounded and (16.3) holds. The unitarity and irreducibility of $\pi_{\mathscr{O}_j,\eta_j}$ follows from Theorem 15.11.

By **??**, $\pi_{\mathscr{O}_j,\eta_j}$ is thus $\mathcal{O}_j$-unipotent. This finishes the proof of the theorem. □

17. Concluding remarks

We record the following result, which is a form of automatic continuity.

Proposition 17.1. *Retain the setting in Section 16.1. The representation $\pi_{\mathscr{O},\eta}$ is isomorphic to*

$$(17.1)\qquad (\omega_{\mathbf{V}_0,\mathbf{V}_1}\widehat{\otimes}\omega_{\mathbf{V}_1,\mathbf{V}_2}\widehat{\otimes}\cdots\widehat{\otimes}\omega_{\mathbf{V}_{k-1},\mathbf{V}_k}\otimes\eta)_{\widetilde{G}_{\mathbf{V}_1}\times\widetilde{G}_{\mathbf{V}_2}\times\cdots\times\widetilde{G}_{\mathbf{V}_k}},$$

and its underlying Harish-Chandra module is isomorphic to

$$(\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1} \otimes \mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2} \otimes \cdots \otimes \mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k} \otimes \eta)_{(\mathfrak{g}_{\mathbf{V}_1} \times \mathfrak{g}_{\mathbf{V}_2} \times \cdots \times \mathfrak{g}_{\mathbf{V}_k}, \tilde{\mathbf{K}}_{\mathbf{V}_1} \times \tilde{\mathbf{K}}_{\mathbf{V}_2} \times \cdots \times \tilde{\mathbf{K}}_{\mathbf{V}_k})}. \tag{17.2}$$

Proof. Note that the representation $\pi_{\mathscr{O},\eta}$ is a quotient of the representation (17.1), and the underlying Harish-Chandra module of (17.1) is a quotient of (17.2). Thus it suffices to show that (17.2) is irreducible. We prove by induction on k. Assume that $k \geqslant 1$ and

$$\pi_1 := (\mathscr{Y}_{\mathbf{V}_1,\mathbf{V}_2} \otimes \cdots \otimes \mathscr{Y}_{\mathbf{V}_{k-1},\mathbf{V}_k} \otimes \eta_1)_{(\mathfrak{g}_{\mathbf{V}_2} \times \cdots \times \mathfrak{g}_{\mathbf{V}_k}, \tilde{\mathbf{K}}_{\mathbf{V}_2} \times \cdots \times \tilde{\mathbf{K}}_{\mathbf{V}_k})}$$

is irreducible. Then π_1 is isomorphic to the Harish-Chandra module of $\pi_{\mathscr{O}_1,\eta_1}$. The representation (17.2) is isomorphic to

$$\pi_0 := \chi_0 \otimes (\mathscr{Y}_{\mathbf{V}_0,\mathbf{V}_1} \otimes \pi_1)_{(\mathfrak{g}_{\mathbf{V}_1}, \tilde{\mathbf{K}}_{\mathbf{V}_1})}.$$

We know that $\mathrm{U}(\mathfrak{g})^{\mathbf{G}}$ acts on $\pi_{\mathscr{O},\eta}$ through the character $\lambda_{\mathcal{O}}$ (*cf.* [70]), and by Lemma 12.5, π_0 is $\mathcal{O}$-bounded. Thus every irreducible subquotient of π_0 is $\mathcal{O}$-unipotent. Theorem 12.6 implies that $\mathrm{AC}_{\mathcal{O}}(\pi_0)$ is bounded by $1 \cdot [\mathscr{O}]$. Now **??** implies that π_0 must be irreducible. □

Finally, we remark that the Whittaker cycles (attached to G-orbits in $\mathcal{O} \cap \mathbf{i}\mathfrak{g}_{\mathbb{R}}$) of all unipotent representations in $\Pi^{\mathrm{unip}}_{\mathcal{O}}(\widetilde{G})$ can be calculated by using [29, Theorem 1.1]. These agree with the associated cycles under the Kostant-Sekiguchi correspondence.

Appendix A. Combinatorial parameterization of special unipotent representations

A.1. **Parameterize of Unipotent representations.** We fix an abstract complex Cartan subgroup $\mathbf{H}_a$ and $\mathfrak{h}_a$ in $\mathbf{G}$ and a set of simple roots Π_a. Let $\mathcal{P}(\mathbf{G})$ be the set of all Langlands parameters of G-modules with character ρ (i.e. the infinitesimal character of the trivial representation). For $\gamma \in \mathcal{P}(\mathbf{G})$, let $\mathcal{L}(\gamma)$, $\mathcal{S}(\gamma)$ and Φ_γ be the corresponding Langlands quotient, standard module and coherent family such that $\Phi_\gamma(\rho) = \mathcal{L}(\gamma)$. Let $\mathcal{M}(\mathbf{G})$ be the span of $\mathcal{L}(\gamma)$. Let $\{\mathbb{B}\}$ be the set of all blocks. Then $\mathcal{P}(\mathbf{G}) = \bigsqcup_{\mathcal{B}} \mathcal{B}$. The Weyl group $W = W(G)$ acts on $\mathcal{M}(\mathbf{G})$ by coherent continuation. Let $\mathcal{M}_{\mathcal{B}}$ be the submodule of $\mathcal{M}(\mathbf{G})$ spand by $\gamma \in \mathcal{B}$, then

$$\mathcal{M}(\mathbf{G}) = \bigoplus_{\mathcal{B}} \mathcal{M}_{\mathcal{B}}$$

Let $\tau(\gamma) \subset \Pi_a$ be the τ-invariant of γ.

Let $\check{\mathcal{O}}$ be even orbit. $\lambda = \frac{1}{2}\check{h}$. Define

$$S(\lambda) = \{\, \alpha \in \Pi_a \mid \langle \alpha, \lambda \rangle = 0 \,\}.$$

Let $\mathcal{P}_\lambda(\mathbf{G})$ be the set of all Langlands parameters with infinitesimal character λ. Let $T_{\lambda,\rho}$ be the translation functor. Let

$$\mathcal{B}(S) = \{\, \gamma \in \mathcal{B} \mid S \cap \tau(\gamma) = \varnothing \,\}$$

and

$$\mathcal{P}(\mathbf{G}, S) = \bigsqcup_{\mathcal{B}} \mathcal{B}(S)$$

Then

$$\begin{aligned} \mathcal{P}(\mathbf{G}, S) &\longrightarrow \mathcal{P}_\lambda(\mathbf{G}) \\ \gamma &\longmapsto T_{\lambda,\rho}(\gamma) \end{aligned}$$

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Let

$$\mathcal{B}(S, \mathcal{O}) = \{\, \gamma \in \mathcal{B}(S) \mid \mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}} \,\}$$

Let

$$m_S(\sigma) = [\sigma : \operatorname{Ind}_{W(S)}^{W} \mathbf{1}]$$
$$m_{\mathcal{B}}(\sigma) = [\sigma : \mathcal{M}_{\mathcal{B}}]$$

Barbasch [11, Theorem 9.1] established the following theorem.

Theorem A.1.

$$|\mathcal{B}(S, \mathcal{O})| = \sum_{\sigma} m_{\mathcal{B}}(\sigma) m_S(\sigma)$$

Here $\sigma \times \sigma$ running over the $W \times W$ appears in the double cell $\mathcal{C}(\mathcal{O})$.

Proof. We need to take the graded module of $\mathcal{M}(\mathbf{G})$ with respect to the $\overset{LR}{\leqslant}$. By abuse of notation, we identify the basis $\mathcal{P}(\mathbf{G})$ with its image in the graded module. Note that $S \cap \tau(\lambda) = \varnothing$ if and only if $W(S)$ acts on γ trivially by [81, Lemma 14.7]. On the other hand, by [81, Theorem 14.10, and page 58], $\mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}}$ only if γ generate a W-module in the double cell of $\mathcal{O}$. □

Now assume $S = S(\lambda)$. By [16, Cor 5.30 b) and c)], $[\sigma : \operatorname{Ind}_{W(S)}^{W} \mathbf{1}] = [\mathbf{1}|_{W(S)} : \sigma] \leqslant 1$.

A.2. **Combinatorics of Weyl group representations.** The irreducible representations of S_n are parameterized by Young diagrams. We use the notation that the trivial representation corresponds to a row and the sign representation corresponds to a column.

$$\textit{triv} \leftrightarrow \boxed{n}\boxed{}\boxed{b}\boxed{o}\boxed{x}\boxed{e}\boxed{s} \qquad \mathrm{sgn} \leftrightarrow \begin{array}{|c|}\hline n \\ \hline \\ \hline b \\ \hline o \\ \hline x \\ \hline e \\ \hline s \\ \hline \end{array}$$

[This notation coincide with the Springer correspondence, where a row of boxes represents the regular nilpotent orbit and the corresponding representation is the trivial representation. The trivial orbit yeilds the sign representation.]

Under the above paramterization, $\tau \mapsto \tau \otimes \mathrm{sgn}$ is described by the transpose of Young diagram. The branching rule is given by Littlewoods-Richardson. Let $\mathcal{K} = \bigoplus_n \mathrm{Groth}(S_n)$ be the graded ring of the Grothendieck groups of S_n. This yeilds well defined ring structure on the graded algebra

$$\mu\nu := \operatorname{Ind}_{S_{|\mu|} \times S_{|\nu|}}^{S_{|\mu|+|\nu|}} \mu \otimes \nu.$$

Notation, we will use $[r_1, r_2, \cdots, r_k]$ or $(c_1, c_2, \cdots, c_k)$ denote the Young diagram, where $\{ r_i \}$ denote the lengthes of its rows and $\{ c_i \}$ denote the lengthes of its columns.

Let W_n denote the Weyl group of type BC: $W_n = S_n \ltimes \{ \pm 1 \}^n$. Now $\mathrm{Irr}(W_n)$ is paramterized by a pair of Young diagram, we use the notation $\mu \times \nu$.

As the tensor algebra, $\bigoplus_n \mathrm{Groth}(W_n) \cong \mathcal{K} \otimes \mathcal{K}$ via $\mu \times \nu \mapsto \mu \otimes \nu$.

The branching rule is given by: $(\mu_1 \times \nu_1)(\mu_2 \times \nu_2) = (\mu_1\mu_2) \times (\nu_1\nu_2)$. The sgn repersentation of W_n is parameterized by $\varnothing \times (n)$. From the definitioin of the identification of bipartition with $\mathrm{Irr}(W_n)$, we also have

$$(\mu \times \nu) \otimes \mathrm{sgn} = \nu^t \times \mu^t.$$

We also will use the following branching formula:

$$\operatorname{Ind}_{S_n}^{W_n} \mathrm{triv} = \sum_{\substack{a,b,\\ a+b=n}} [a] \times [b],$$

and

$$\mathrm{Ind}_{S_n}^{W_n} \operatorname{sgn} = \sum_{\substack{a,b,\\ a+b=n}} (a) \times (b). \tag{A.1}$$

[Sketch of the proof of the first formula, the dimension of LHS is $n!2^n/n! = 2^n$. On the other hand, by Mackey theory, $[LHS : \mathrm{Ind}_{W_a\times W_b}^{W_n} \operatorname{triv} \otimes \chi] \geqslant 1$ since $S_n \cap W_a \times W_b = S_a \times S_b$. Therefore, $LHS \supset RHS$. On the other hand, dimension of RHS is $\sum_a n!/a!b! = 2^n$. This finished the proof. The proof of the second formula is similar. One also can obtain it by tensoring with sgn of the first formula.]

A.2.1. *Coherent continuation representations.* TBA

Appendix B. A few geometric facts

B.1. **Geometry of moment maps.** In this section, let $(\mathbf{V}, \mathbf{V}')$ be a rational dual pair. We use the notation of Section 12.8. Write $\breve{\mathfrak{p}} := \mathfrak{p} \oplus \mathfrak{p}'$ and

$$\breve{M} := M \times M' \colon \mathcal{X} \longrightarrow \breve{\mathfrak{p}} = \mathfrak{p} \times \mathfrak{p}'.$$

When there is a morphism $f\colon A \to B$ between smooth algebraic varieties, let $\mathrm{d}f_a\colon \mathrm{T}_a A \to \mathrm{T}_{f(a)} B$ denote the tangent map at a closed point $a \in A$, where $\mathrm{T}_a A$ and $\mathrm{T}_{f(a)} B$ are the tangent spaces. For a sub-scheme S of B, let $A \times_f S$ denote the scheme theoretical inverse image of S under f, which is a subscheme of A. Along the proof, we will use the Jacobian criterion for regularity in various places (see [52, Theorem 2.19]).

B.1.1. *Scheme theoretical results on descents.*

Lemma B.1 (*cf.* [65, Lemma 13 and Lemma 14])**.** *Suppose $\mathscr{O}' \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$ is the descent of $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$. Fix $X \in \mathscr{O}$ and let*

$$\mathcal{Z}_X := \mathcal{X} \times_M \{X\}$$

be the scheme theoretical inverse image of X under the moment map $M : \mathcal{X} \to \mathfrak{p}$. Then

(i) $\mathcal{Z}_X$ is smooth (and hence reduced);
(ii) $\mathcal{Z}_X$ is a single free $\mathbf{K}'$-orbit in $\mathcal{X}^\circ$;
(iii) $M'(\mathcal{Z}_X) = \mathscr{O}'$ and $\mathcal{Z}_X = \mathcal{X} \times_{\breve{M}} (\{X\} \times \overline{\mathscr{O}'})$.

Proof. Fix a point $w \in \mathcal{X}^\circ$ such that $M(w) = X$. It is elementary to see that

$$M^{-1}(X) = \mathbf{K}' \cdot w$$

is a single $\mathbf{K}'$-orbit. This orbit is clearly Zariski closed and the stabilizer $\mathrm{Stab}_{\mathbf{K}'}(w)$ of w under the $\mathbf{K}'$-action is trivial. This proves part (ii) and the first assertion of part (iii).

We identify the tangent space $\mathrm{T}_w\mathcal{X}$ with $\mathcal{X}$. Let $N \subset \mathcal{X}$ be a complement of $\mathrm{T}_w(\mathbf{K}' \cdot w)$ in $\mathrm{T}_w\mathcal{X}$. By [69, Section 6.3], there is an affine open subvariety N° of N containing 0 such that the diagram

$$\begin{tikzcd} \mathbf{K}' \times N^\circ \arrow[r, "(k',n)\mapsto k'\cdot(w+n)"] \arrow[d, "\text{projection}"'] & \mathcal{X} \arrow[d, "M"] \\ N^\circ \arrow[r, "n\mapsto M(w+n)"] & M(\mathcal{X}) \end{tikzcd}$$

is Cartesian and the horizontal morphisms are étale. Therefore the scheme $\mathcal{X} \times_M \{X\}$ is a smooth algebraic variety. This proves part (i), and the second assertion of (iii) then easily follows. $\square$

Remark. Retain the setting of Lemma B.1. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from X to $X' \in \mathfrak{p}'$. Let $\check{X} := (X, X') \in \mathfrak{p} \oplus \mathfrak{p}'$ and $\mathbf{K}'_{X'}$ be the stabilizer of X' under the $\mathbf{K}'$-action. By the $\mathbf{K}'$-equivariance, the map $\mathcal{Z}_X \to \mathscr{O}'$ is a smooth morphism between smooth schemes. Hence the scheme theoretical fibre

$$\mathcal{Z}_{\check{X}} := \mathcal{X} \times_{\check{M}} \{ \check{X} \} = \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{ X' \}$$

is smooth and equals the orbit $\mathbf{K}'_{X'} \cdot w$. Consequently, we have an exact sequence by the Jacobian criterion for regularity:

$$0 \longrightarrow \mathrm{T}_w \mathcal{Z}_{\check{X}} \longrightarrow \mathrm{T}_w \mathcal{X} \xrightarrow{\mathrm{d}\check{M}_w} \mathrm{T}_X \mathfrak{p} \oplus \mathrm{T}_{X'} \mathfrak{p}'. \tag{B.1}$$

Lemma B.2. *Retain the setup in Lemma B.1. Let*

$$U := \mathfrak{p} \backslash ((\overline{\mathcal{O}} \cap \mathfrak{p}) \backslash \mathscr{O})$$

and $\mathcal{Z}_{U, \overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$*. Then*

- *(i)* U *is a Zariski open subset of* $\mathfrak{p}$ *such that* $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$*;*
- *(ii)* $\mathcal{Z}_{U, \overline{\mathscr{O}'}}$ *is smooth and it is a single* $\mathbf{K} \times \mathbf{K}'$*-orbit.*

Proof. Part (i) is clear by (12.18). It is also clear that the underlying set of $\mathcal{Z}_{U, \overline{\mathscr{O}'}}$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit by Lemma B.1 (iii).

Note that $M'|_{\mathcal{X}^\circ}$ is smooth because $\mathrm{d}M'_w \colon \mathrm{T}_w \mathcal{X}^\circ \longrightarrow \mathrm{T}_{M'(w)} \mathfrak{p}'$ is surjective for each $w \in \mathcal{X}^\circ$ (*cf.* [33, Proposition 10.4]). Since smooth morphisms are stable under base changes and compositions, the scheme $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$ is a smooth algebraic variety. Also note that $\mathcal{Z}_{U, \overline{\mathscr{O}'}}$ is an open subscheme of $\mathcal{X}^\circ \times_{M'} \mathscr{O}'$. Thus it is smooth. □

Remark. Suppose $w \in \mathcal{X}^\circ$ realizes the descent from $X \in \mathscr{O}$ to $X' \in \mathscr{O}' \subset \mathfrak{p}'$. By the proof of Lemma B.2,

$$\mathcal{Z}_{U, X'} := \mathcal{X} \times_{\check{M}} (U \times \{X'\}) = \mathbf{K}\mathbf{K}'_{X'} \cdot w$$

is smooth. Similar to the remark of Lemma B.1, this yields an exact sequence:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A, A') \mapsto A'w - wA]{} \mathcal{X} \xrightarrow[b \mapsto bw^\star + wb^\star]{\mathrm{d}M'_w} \mathfrak{p}'. \tag{B.2}$$

B.1.2. *Scheme theoretical results on generalized descents of good orbits.*

Lemma B.3. *Suppose* $\mathscr{O} \in \mathrm{Nil}_{\mathbf{K}}(\mathfrak{p})$ *is good for generalized descent (see Definition 12.24) and* $\mathscr{O}' = \nabla^{\mathrm{gen}}_{\mathbf{V}, \mathbf{V}'}(\mathscr{O}) \in \mathrm{Nil}_{\mathbf{K}'}(\mathfrak{p}')$*. Fix* $X \in \mathscr{O}$ *and let* $\mathcal{Z}_X := \mathcal{X} \times_{\check{M}} (\{ X \} \times \overline{\mathscr{O}'})$*. Then*

- *(i)* $\mathcal{Z}_X$ *is smooth;*
- *(ii)* $\mathcal{Z}_X$ *is a single* $\mathbf{K}'$*-orbit;*
- *(iii)* $\mathcal{Z}_X$ *is contained in* $\mathcal{X}^{\mathrm{gen}}$ *and* $M'(\mathcal{Z}_X) = \mathscr{O}'$*.*

Proof. Part (ii) follows from [26, Table 4]. Part (iii) follows from part (ii). By the generic smoothness, in order to prove part (i), it suffices to show that $\mathcal{Z}_X$ is reduced. Note that the $\mathbf{K}'$-equivariant morphism $\mathcal{Z}_X \xrightarrow{M'} \mathscr{O}'$ is flat by generic flatness [30, Théorème 6.9.1]. By [31, Proposition 11.3.13], to show that $\mathcal{Z}_X$ is reduced, it suffices to show that

$$\mathcal{Z}_{\check{X}} := \mathcal{Z}_X \times_{M'|_{\mathcal{Z}_X}} \{ X' \} = \mathcal{X} \times_{\check{M}} \{ \check{X} \}$$

is reduced, where $X' \in \mathscr{O}'$ and $\check{X} := (X, X')$. Let $w \in \mathcal{Z}_{\check{X}}$ be a closed point. Then $\mathbf{K}' \cdot w$ is the underlying set of $\mathcal{Z}_X$, and $Z_{\check{X}} := \mathbf{K}'_{X'} \cdot w$ is the underlying set of $\mathcal{Z}_{\check{X}}$. By the Jacobian criterion for regularity, to complete the proof of the lemma, it remains to prove the following claim.

Claim. *The following sequence is exact:*

$$0 \longrightarrow \mathrm{T}_w Z_{\breve{X}} \longrightarrow \mathrm{T}_w \mathcal{X} \xrightarrow{\mathrm{d}\breve{M}} \mathrm{T}_X \mathfrak{p} \oplus \mathrm{T}_{X'} \mathfrak{p}'.$$

We prove the above claim in what follows. Identify $\mathrm{T}_w\mathcal{X}$, $\mathrm{T}_X\mathfrak{p}$ and $\mathrm{T}_{X'}\mathfrak{p}'$ with $\mathcal{X}, \mathfrak{p}$ and $\mathfrak{p}'$ respectively and view $\mathrm{T}_w Z_{\breve{X}}$ as a quotient of $\mathfrak{k}'_{X'}$. It suffices to show that the following sequence is exact:

$$\mathfrak{k}'_{X'} \xrightarrow[A' \mapsto A'w]{} \mathcal{X} \xrightarrow[b \mapsto (b^\star w + w^\star b, bw^\star + wb^\star)]{\mathrm{d}\breve{M}_w} \mathfrak{p} \oplus \mathfrak{p}' = \breve{\mathfrak{p}}. \tag{B.3}$$

In fact, we will show that the following sequence

$$\mathfrak{g}'_{X'} \longrightarrow \mathbf{W} \longrightarrow \mathfrak{g} \oplus \mathfrak{g}' \tag{B.4}$$

is exact, where $\mathfrak{g}'_{X'}$ is the Lie algebra of the stabilizer group $\mathrm{Stab}_{\mathbf{G}'}(X')$ and the arrows are defined by the same formulas in (B.3). Then the exactness of (B.3) will follow since (B.4) is compatible with the natural (L, L')-actions.

We now prove the exactness of (B.4). Let $\mathbf{V}'_1 := w\,\mathbf{V}$ and $\mathbf{V}'_2$ be the orthogonal complement of $\mathbf{V}'_1$ so that

$$\mathbf{V}' = \mathbf{V}'_1 \oplus \mathbf{V}'_2.$$

Let $\mathbf{W}_i := \mathrm{Hom}(\mathbf{V}, \mathbf{V}'_i)$ and $\mathfrak{g}'_i := \mathfrak{g}_{\mathbf{V}'_i}$ for $i = 1, 2$. Let $*\colon \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \to \mathrm{Hom}(\mathbf{V}'_2, \mathbf{V}'_1)$ denote the adjoint map. We have $\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2$ and $\mathfrak{g}' = \mathfrak{g}'_1 \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxslash}$, where

$$\mathfrak{g}^{\boxslash} := \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \;\middle|\; E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2) \right\}.$$

Now w and X' are naturally identified with an element w_1 in $\mathbf{W}_1$ and an element X'_1 in $\mathfrak{g}'_1$, respectively. We have $\mathfrak{g}'_{X'} = \mathfrak{g}'_{1,X'_1} \oplus \mathfrak{g}'_2 \oplus \mathfrak{g}'^{\boxslash}_{X'_1}$, where

$$\mathfrak{g}^{\boxslash}_{X'_1} = \left\{ \begin{pmatrix} 0 & -E^* \\ E & 0 \end{pmatrix} \;\middle|\; E \in \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2),\ EX'_1 = 0 \right\}.$$

Now maps in (B.4) have the following forms respectively:

$$\mathfrak{g}'_{X'} \ni A' = (A'_1, A'_2, \left(\begin{smallmatrix} 0 & -E^* \\ E & 0 \end{smallmatrix}\right)) \mapsto A'w = (A'_1 w_1, Ew_1) \in \mathbf{W} \qquad \text{and}$$

$$\mathbf{W} \ni (b_1, b_2) \mapsto \left(b_1^\star w_1 + w_1^\star b_1, \begin{pmatrix} b_1 w_1^\star + w_1 b_1^\star & w_1 b_2^\star \\ b_2 w_1^\star & 0 \end{pmatrix} \right) \in \mathfrak{g} \oplus \mathfrak{g}'.$$

Applying (B.1) to the complex dual pair $(\mathbf{V}, \mathbf{V}'_1)$, we see that the sequence

$$\mathfrak{g}'_{1,X'_1} \xrightarrow{A'_1 \mapsto A'_1 w_1} \mathbf{W}_1 \xrightarrow{b_1 \mapsto (b_1^\star w_1 + w_1^\star b_1, b_1 w_1^\star + w_1 b_1^\star)} \mathfrak{g} \oplus \mathfrak{g}'_1$$

is exact. The task is then reduced to show that the following sequence is exact:

$$\mathfrak{g}^{\boxslash}_{X'_1} \xrightarrow[①]{\left(\begin{smallmatrix} 0 & -E^* \\ E & 0 \end{smallmatrix}\right) \mapsto Ew_1} \mathbf{W}_2 \xrightarrow[②]{b_2 \mapsto b_2 w_1^\star} \mathrm{Hom}(\mathbf{V}'_1, \mathbf{V}'_2). \tag{B.5}$$

Note that w_1 is a surjection, hence ① is an injection. We have

$$\dim \mathfrak{g}^{\boxslash}_{X'_1} = \dim \mathbf{V}'_2 \cdot \dim \mathrm{Ker}(X'_1)$$

and

$$\dim \mathrm{Ker}(②) = \dim \mathbf{V}'_2 \cdot (\dim \mathbf{V} - \dim \mathrm{Im}(w_1)) = \dim \mathbf{V}'_2 \cdot (\dim \mathbf{V} - \dim \mathbf{V}'_1).$$

Note that $\dim \mathrm{Ker}(X_1') = c_1$, and $\dim \mathbf{V} - \dim \mathbf{V}_1' = c_0$. Since we are in the setting of good generalized descent (Definition 12.24), we have $c_0 = c_1$. Now the exactness of (B.5) follows by dimension counting. ☐

Remark. Suppose $\mathscr{O}$ is not good for generalized descent. Then the underlying set of $\mathcal{Z}_X$ may not be a single $\mathbf{K}'$-orbit. Even if it is a single orbit, the scheme $\mathcal{Z}_X$ may not be reduced.

Lemma B.4. *Retain the notation in Lemma B.3. Let*

$$U := \mathfrak{p}\backslash((\overline{\mathcal{O}} \cap \mathfrak{p})\backslash\mathscr{O})$$

and $\mathcal{Z}_{U,\overline{\mathscr{O}'}} := \mathcal{X} \times_{\check{M}} (U \times \overline{\mathscr{O}'})$. *Then*

(i) U is a Zariski open subset of $\mathfrak{p}$ such that $U \cap M(M'^{-1}(\overline{\mathscr{O}'})) = \mathscr{O}$;
(ii) $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is smooth and it is a single $\mathbf{K} \times \mathbf{K}'$-orbit.

Proof. Part (i) is follows from Lemma 12.25. By Lemma B.3 (ii), the underlying set of $\mathcal{Z}_{U,\overline{\mathscr{O}'}}$ is a single $\mathbf{K} \times \mathbf{K}'$-orbit whose image under M' is contained in $\mathscr{O}'$. By generic flatness [30, Théorème 6.9.1], the morphism $\mathcal{Z}_{U,\overline{\mathscr{O}'}} \longrightarrow \mathscr{O}'$ is flat. To prove the scheme theoretical claim in part (ii), it suffices to show that

$$\mathcal{Z}_{U,X'} := \mathcal{X} \times_{\check{M}} (U \times \{ X' \})$$

is reduced, where $X' \in \mathscr{O}'$.

Let $w \in \mathcal{Z}_{U,X'}$ be a closed point. As in the proof of Lemma B.3, by the Jacobian criterion for regularity, it suffices to show that the following sequence is the exact:

$$\mathfrak{k} \oplus \mathfrak{k}'_{X'} \xrightarrow[(A,A') \mapsto A'w - wA]{} \mathcal{X} \xrightarrow[b \mapsto bw^\star + wb^\star]{\mathrm{d}M'_w} \mathfrak{p}'.$$

The proof of the exactness follows the same line as the proof of Lemma B.3 utilizing (B.5) and the established exact sequence (B.2) in the descent case. We leave the details to the reader. ☐

B.2. **Proof of Lemma 15.16.** We will use notations of Section **??** and Section 15.3.2. Let Hom_J denote the space of homomorphisms commuting with the $\dot{\epsilon}$-real from J, and $\mathfrak{n}_\mathbb{R}$ denote the Lie algebra of $N_{\mathbf{E}_0}$.

We first explicitly construct some elements in the induced orbits in $\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_\mathbb{R}$. Fix an element $\mathbf{i}\, X \in \mathscr{O}_\mathbb{R}$ and a J-invariant complement $\mathbf{V}_+$ of $\mathrm{Im}(X)$ in the vector space $\mathbf{V}^- = \mathbf{V}$ so that

$$\mathbf{V}^- = \mathbf{V}_+ \oplus \mathrm{Im}(X).$$

Fix any J-invariant decompositions

$$\mathbf{E}_0 = \mathbf{L}_1 \oplus \mathbf{L}_0 \quad \text{and} \quad \mathbf{E}_0' = \mathbf{L}_1' \oplus \mathbf{L}_0'$$

such that $\dim \mathbf{L}_1 = \dim \mathbf{V}_+$, and $(\mathbf{L}_1 \oplus \mathbf{L}_1') \perp (\mathbf{L}_0 \oplus \mathbf{L}_0')$. (This is possible due to the dimension inequality in (15.8).)

Fix a linear isomorphism

$$S \in \mathrm{Hom}_J(\mathbf{L}_1', \mathbf{V}_+),$$

and view it as an element of $\mathrm{Hom}_J(\mathbf{E}_0', \mathbf{V}^-)$ via the aforementioned decompositions. Put

$$\mathscr{T} := \{ T \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0) \mid T^* + T = 0 \} \quad \text{and}$$
$$\mathscr{T}^\circ := \{ T \in \mathscr{T} \mid T \text{ has maximal possible rank} \}$$

to be viewed as subsets of $\mathrm{Hom}_J(\mathbf{E}_0', \mathbf{E}_0)$ as before, where $T^* \in \mathrm{Hom}_J(\mathbf{L}_0', \mathbf{L}_0)$ is specified by requiring that

$$\text{(B.6)} \qquad \langle T \cdot u, v \rangle_{\mathbf{V}^\perp} = \langle u, T^* v \rangle_{\mathbf{V}^\perp}, \quad \text{for all } u, v \in \mathbf{L}_0',$$

and $\langle\ ,\ \rangle_{\mathbf{V}^\perp}$ denotes the ϵ-symmetric bilinear form on $\mathbf{V}^\perp$.

Each $T \in \mathscr{T}$ defines a $(-\epsilon)$-symmetric bilinear form $\langle\ ,\ \rangle_T$ on $\mathbf{L}_0'$ where

$$\langle v_1, v_2\rangle_T := -\langle v_1, Tv_2\rangle, \quad \text{for all } v_1, v_2 \in \mathbf{L}_0'.$$

Put

$$X_{S,T} := \begin{pmatrix} 0 & S^* & T \\ & X & S \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_{\mathbb{R}} \tag{B.7}$$

according to the decomposition $\mathbf{V}^\perp = \mathbf{E}_0 \oplus \mathbf{V}^- \oplus \mathbf{E}_0'$, where $S^* \in \mathrm{Hom}_J(\mathbf{V}^-, \mathbf{E}_0)$ is similarly defined as in (B.6).

Now suppose that $T \in \mathscr{T}^\circ$.

In case (ii) of Lemma 15.16, the form $\langle\ ,\ \rangle_T$ must be non-degenerate and $((\mathbf{L}_0', \langle\ ,\ \rangle_T), J|_{\mathbf{L}_0'})$ becomes a $(-\epsilon, -\dot\epsilon)$-space. Conversely, up to isomorphism, every $(-\epsilon, -\dot\epsilon)$-space of dimension $l - c_1$ is isomorphic to $((\mathbf{L}_0', \langle\ ,\ \rangle_T), J|_{\mathbf{L}_0'})$ for some $T \in \mathscr{T}$.

Let s be the signature of $((\mathbf{L}_0', \langle\ ,\ \rangle_T), J|_{\mathbf{L}_0'})$ and let $G_T := \mathbf{G}^J_{\mathbf{L}_0'}$ denote the corresponding real group. Then $\mathbf{i}\, X_{S,T}$ generates an induced orbit of $\mathscr{O}_{\mathbb{R}}$ with signed Young diagram $[d_1 + s, \check{d}_1 + \check{s}, d_1, \cdots, d_k]$.[11] One checks that the reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T})$ and $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i}\, X_{S,T})$ are both canonically isomorphic to $R \times G_T$, where R is the reductive quotient of $\mathrm{Stab}_G(X)$. Hence $C_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T}) \cong C_{G_{\mathbf{V}^\perp}}(\mathbf{i}\, X_{S,T})$.

In case (i) of Lemma 15.16, $\langle\ ,\ \rangle_T$ is skew symmetric, and T has rank $\dim \mathbf{L}_0 - 1 = l - c_1 - 1$ since $l - c_1$ is odd. One checks that $\mathbf{i}\, X_{S,T}$ generates an induced orbit of $\mathscr{O}_{\mathbb{R}}$ with the signed Young diagram prescribed in (i) of Lemma 15.16.

Fix decompositions

$$\mathbf{L}_0 = \mathbf{L}_2 \oplus \mathbf{L}_3 \quad \text{and} \quad \mathbf{L}_0' = \mathbf{L}_2' \oplus \mathbf{L}_3'$$

which are dual to each other such that $\mathrm{Ker}(T) = \mathbf{L}_3'$ and $\langle\ ,\ \rangle_T|_{\mathbf{L}_2' \times \mathbf{L}_2'}$ is a non-degenerate skew symmetric bilinear form on $\mathbf{L}_2'$. Let $G_T := \mathbf{G}^J_{\mathbf{L}_2'}$. Then G_T is a real symplectic group. The reductive quotient of $\mathrm{Stab}_{P_{\mathbf{E}_0}}(\mathbf{i}X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathrm{GL}(\mathbf{L}_3)^J$ and the reductive quotient of $\mathrm{Stab}_{G_{\mathbf{V}^\perp}}(\mathbf{i}X_{S,T})$ is canonically isomorphic to $R \times G_T \times \mathbf{G}^J_{\mathbf{L}_3 \oplus \mathbf{L}_3'}$, where R is the reductive quotient of $\mathrm{Stab}_G(\mathbf{i}X)$ and $\mathbf{G}^J_{\mathbf{L}_3 \oplus \mathbf{L}_3'} \cong \mathrm{O}(1,1)$. The homomorphism

$$C_{P_{\mathbf{E}_0}}(\mathbf{i}\, X_{S,T}) \longrightarrow C_{G_{\mathbf{V}^\perp}}(\mathbf{i}X_{S,T})$$

is therefore an injection whose image has index 2.

To finish the proof of the lemma, it suffices to show that

$$\mathrm{Ind}_{P_{\mathbf{E}_0}}^{G_{\mathbf{V}^\perp}} \mathscr{O}_{\mathbb{R}} = \{\, G_{\mathbf{V}^\perp} \cdot \mathbf{i}X_{S,T} \mid T \in \mathscr{T}^\circ \,\}. \tag{B.8}$$

Consider the set

$$\mathfrak{A} := \left\{ \begin{pmatrix} 0 & B^* & C \\ & X & B \\ & & 0 \end{pmatrix} \in X + \mathfrak{n}_{\mathbb{R}} \;\middle|\; X \oplus B \in \mathrm{Hom}_J(\mathbf{V}^- \oplus \mathbf{E}_0', \mathbf{V}^-) \text{ is surjective} \right\}.$$

Clearly $P_{\mathbf{E}_0}' := \mathrm{GL}_{\mathbf{E}_0} \ltimes N_{\mathbf{E}_0}$ acts on $\mathfrak{A}$ (*cf.* **??**). By suitable matrix manipulations, one sees that every element in $\mathfrak{A}$ is conjugated to an element in $\{\, X_{S,T} \mid T \in \mathscr{T} \,\}$ under the $P_{\mathbf{E}_0}'$-action. Hence $P_{\mathbf{E}_0}' \cdot \{\, X_{S,T} \mid T \in \mathscr{T}^\circ \,\}$ is open dense in $\mathfrak{A}$. Now (B.8) follows since $P_{\mathbf{E}_0} \cdot \mathbf{i}\mathfrak{A}$ is open dense in $\mathscr{O}_{\mathbb{R}} + \mathbf{i}\mathfrak{n}_{\mathbb{R}}$, □

[11] The signed Young diagram may be computed using the explicit description of Kostant-Sekiguchi correspondence in [26, Propositions 6.2 and 6.4].

References

[1] J. Adams, *Discrete spectrum of the reductive dual pair $(O(p,q), Sp(2m))$*, Invent. Math. **74** (1983), no. 3, 449–475.

[2] J. Adams, B. Barbasch, and D. A. Vogan, *The Langlands classification and irreducible characters for real reductive groups*, Progress in Math., vol. 104, Birkhauser, 1991.

[3] Jeffrey Adams and Fokko du Cloux, *Algorithms for representation theory of real reductive groups*, Journal of the Institute of Mathematics of Jussieu **8** (2009), no. 2, 209-259, DOI 10.1017/S1474748008000352.

[4] J. Arthur, *On some problems suggested by the trace formula*, Lie group representations, II (College Park, Md.), Lecture Notes in Math. 1041 (1984), 1–49.

[5] ———, *Unipotent automorphic representations: conjectures*, Orbites unipotentes et représentations, II, Astérisque **171-172** (1989), 13–71.

[6] L. Auslander and B. Kostant, *Polarizations and unitary representations of solvable Lie groups*, Invent. Math. **14** (1971), 255–354.

[7] D. Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. **96** (1989), no. 1, 103–176.

[8] ———, *Unipotent representations for real reductive groups*, Proceedings of ICM (1990), Kyoto (2000), 769–777.

[9] Dan Barbasch and David Vogan, *Weyl Group Representations and Nilpotent Orbits*, Representation Theory of Reductive Groups: Proceedings of the University of Utah Conference 1982, 1983, pp. 21–33.

[10] D. Barbasch, *Orbital integrals of nilpotent orbits*, The mathematical legacy of Harish-Chandra, Proc. Sympos. Pure Math. **68** (2000), 97–110.

[11] ———, *The unitary spherical spectrum for split classical groups*, J. Inst. Math. Jussieu **9** (2010), 265–356.

[12] ———, *Unipotent representations and the dual pair correspondence*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 47–85.

[13] D. Barbasch, J.-J. Ma, B.-Y. Sun, and C.-B. Zhu, *On the notion of metaplectic Barbasch-Vogan duality*, arXiv:2010.16089.

[14] ———, *Counting special unipotent representations: orthogonal and symplectic groups*, in preparation.

[15] D. Barbasch and D. A. Vogan, *Weyl group representations and nilpotent orbits*, in Representation theory of reductive groups (Park City, Utah, 1982), Progress in Math. **40** (1983), 21–33.

[16] ———, *Unipotent representations of complex semisimple groups*, Annals of Math. **121** (1985), no. 1, 41–110.

[17] R. Brylinski, *Dixmier algebras for classical complex nilpotent orbits via Kraft-Procesi models. I*, The orbit method in geometry and physics (Marseille, 2000). Progress in Math. **213** (2003), 49–67.

[18] W. Borho, *Recent advances in enveloping algebras of semisimple Lie-algebras*, Séminaire Bourbaki, Exp. No. 489 (1976/77), 1–18.

[19] Roger W. Carter, *Finite groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Ltd., Chichester, 1993.

[20] W. Casselman, *Canonical extensions of Harish-Chandra modules to representations of G*, Canad. J. Math. **41** (1989), 385–438.

[21] Y. Chen and B. Sun, *Schwartz homologies of representations of almost linear Nash groups*, J. Funct. Anal. **280** (2021), 108817.

[22] F. Du Cloux, *Sur les représentations différentiables des groupes de Lie algébriques*, Ann. Sci. École Norm. Sup. **24** (1991), no. 3, 257–318.

[23] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebra: an introduction*, Van Nostrand Reinhold Co., 1993.

[24] M. Cowling, U. Haagerup, and R. Howe, *Almost L2 matrix coefficients*, J. Reine Angew. Math. **387** (1988), 97–110.

[25] A. Daszkiewicz, W. Kraśkiewicz, and T. Przebinda, *Nilpotent orbits and complex dual pairs*, J. Algebra **190** (1997), no. 2, 518 - 539.

[26] ———, *Dual pairs and Kostant-Sekiguchi correspondence. II. Classification of nilpotent elements*, Central European J. Math. **3** (2005), 430–474.

[27] J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Sci. Math. (2) **102** (1978), 307–330.

[28] M. Duflo, *Théorie de Mackey pour les groupes de Lie algébriques*, Acta Math. **149** (1982), no. 3-4, 153–213.

[29] R. Gomez and C.-B. Zhu, *Local theta lifting of generalized Whittaker models associated to nilpotent orbits*, Geom. Funct. Anal. **24** (2014), no. 3, 796–853.

[30] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algbrique IV: Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[31] ———, *Éléments de géométrie algbrique IV: Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966).

[32] M. Harris, J.-S. Li, and B. Sun, *Theta correspondences for close unitary groups*, Arithmetic Geometry and Automorphic Forms, Adv. Lect. Math. (ALM) **19** (2011), 265–307.

[33] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, 52. New York-Heidelberg-Berlin: Springer-Verlag, 1983.

[34] H. He, *Unipotent representations and quantum induction*, arXiv:math/0210372 (2002).

[35] ———, *Unitary representations and theta correspondence for type I classical groups*, J. Funct. Anal. **199** (2003), no. 1, 92–121.

[36] J.-S. Huang and J.-S. Li, *Unipotent representations attached to spherical nilpotent orbits*, Amer. J. Math. **121** (1999), no. 3, 497–517.

[37] J.-S. Huang and C.-B. Zhu, *On certain small representations of indefinite orthogonal groups*, Represent. Theory **1** (1997), 190–206.

[38] R. Howe, *θ-series and invariant theory*, Automorphic Forms, Representations and L-functions, Proc. Sympos. Pure Math, vol. 33, 1979, pp. 275-285.

[39] ———, *On a notion of rank for unitary representations of the classical groups*, Harmonic analysis and group representations, Liguori, Naples (1982), 223-331.

[40] ———, *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

[41] ———, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Piatetski-Shapiro, I. et al. (eds.), The Schur lectures (1992). Ramat-Gan: Bar-Ilan University, Isr. Math. Conf. Proc. 8, (1995), 1-182.

[42] D. Jiang, B. Liu, and G. Savin, *Raising nilpotent orbits in wave-front sets*, Represent. Theory **20** (2016), 419–450.

[43] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), 57–110.

[44] B. Kostant, *Quantization and unitary representations*, Lectures in Modern Analysis and Applications III, Lecture Notes in Math. **170** (1970), 87–208.

[45] H. Kraft and C. Procesi, *On the geometry of conjugacy classes in classical groups*, Comment. Math. Helv. **57** (1982), 539–602.

[46] S. S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.

[47] S. S. Kudla, *Some extensions of the Siegel-Weil formula*, In: Gan W., Kudla S., Tschinkel Y. (eds) Eisenstein Series and Applications. Progress in Mathematics, vol 258. Birkhäuser Boston (2008), 205–237.

[48] S. T. Lee and C.-B. Zhu, *Degenerate principal series and local theta correspondence II*, Israel J. Math. **100** (1997), 29–59.

[49] ———, *Degenerate principal series of metaplectic groups and Howe correspondence*, D. Prasad at al. (eds.), Automorphic Representations and L-Functions, Tata Institute of Fundamental Research, India, (2013), 379–408.

[50] J.-S. Li, *Singular unitary representations of classical groups*, Invent. Math. **97** (1989), no. 2, 237–255.

[51] ———, *Theta lifting for unitary representations with nonzero cohomology*, Duke Math. J. **61** (1990), no. 3, 913–937.

[52] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford University Press, 2006.

[53] H. Y. Loke and J. Ma, *Invariants and K-spectrums of local theta lifts*, Compositio Math. **151** (2015), 179–206.

[54] G. Lusztig and N. Spaltenstein, *Induced unipotent classes*, j. London Math. Soc. **19** (1979), 41–52.

[55] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272, DOI 10.1007/BF01388564. MR732546

[56] G. W. Mackey, *Unitary representations of group extentions*, Acta Math. **99** (1958), 265–311.

[57] W. M McGovern, *Cells of Harish-Chandra modules for real classical groups*, Amer. J. of Math. **120** (1998), 211–228.

[58] C. Mœglin, *Front d'onde des représentations des groupes classiques p-adiques*, Amer. J. Math. **118** (1996), 1313–1346.

[59] ———, *Paquets d'Arthur Spéciaux Unipotents aux Places Archimédiennes et Correspondance de Howe*, J. Cogdell et al. (eds.), Representation Theory, Number Theory, and Invariant Theory, In Honor of Roger Howe. Progress in Math. **323** (2017), 469–502.
[60] C. Mœglin and D. Renard, *Sur les paquets d'Arthur des groupes classiques réels*, J. Eur. Math. Soc. **22** (2020), 1827–1892.
[61] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, vol. 1291, Springer, 1987.
[62] K. Nishiyama, H. Ochiai, K. Taniguchi, H. Yamashita, and S. Kato, *Nilpotent orbits, associated cycles and Whittaker models for highest weight representations*, Astérisque **273** (2001), 1–163.
[63] K. Nishiyama, H. Ochiai, and C.-B. Zhu, *Theta lifting of nilpotent orbits for symmetric pairs*, Trans. Amer. Math. Soc. **358** (2006), 2713–2734.
[64] K. Nishiyama and C.-B. Zhu, *Theta lifting of unitary lowest weight modules and their associated cycles*, Duke Math. J. **125** (2004), 415–465.
[65] T. Ohta, *The closures of nilpotent orbits in the classical symmetric pairs and their singularities*, Tohoku Math. J. **43** (1991), no. 2, 161–211.
[66] ———, *Induction of nilpotent orbits for real reductive groups and associated varieties of standard representations*, Hiroshima Math. J. **29** (1999), no. 2, 347–360.
[67] ———, *Nilpotent orbits of $\mathbb{Z}_4$-graded Lie algebra and geometry of moment maps associated to the dual pair* $(\mathrm{U}(p,q),\mathrm{U}(r,s))$, Publ. RIMS **41** (2005), no. 3, 723–756.
[68] A. Paul and P. Trapa, *Some small unipotent representations of indefinite orthogonal groups and the theta correspondence*, University of Aarhus Publ. Series **48** (2007), 103–125.
[69] V. L. Popov and E. B. Vinberg, *Invariant Theory*, Algebraic Geometry IV: Linear Algebraic Groups, Invariant Theory, Encyclopedia of Mathematical Sciences, vol. 55, Springer, 1994.
[70] T. Przebinda, *The duality correspondence of infinitesimal characters*, Colloq. Math. **70** (1996), 93–102.
[71] ———, *Characters, dual pairs, and unitary representations*, Duke Math. J. **69** (1993), no. 3, 547–592.
[72] S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.
[73] David A. Renard and Peter E. Trapa, *Irreducible genuine characters of the metaplectic group: Kazhdan-Lusztig algorithm and Vogan duality*, Represent. Theory **4** (2000), 245–295, DOI 10.1090/S1088-4165-00-00105-9.
[74] S. Sahi, *Explicit Hilbert spaces for certain unipotent representations*, Invent. Math. **110** (1992), no. 2, 409–418.
[75] J. Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138.
[76] W. Schmid and K. Vilonen, *Characteristic cycles and wave front cycles of representations of reductive Lie groups*, Annals of Math. **151** (2000), no. 3, 1071–1118.
[77] E. Sommers, *Lusztig's canonical quotient and generalized duality*, J. Algebra **243** (2001), no. 2, 790–812.
[78] T. A. Springer and R. Steinberg, *Seminar on algebraic groups and related finite groups; Conjugate classes*, Lecture Notes in Math., vol. 131, Springer, 1970.
[79] B. Sun and C.-B. Zhu, *A general form of Gelfand-Kazhdan criterion*, Manuscripta Math. **136** (2011), 185–197.
[80] P. Trapa, *Special unipotent representations and the Howe correspondence*, University of Aarhus Publication Series **47** (2004), 210–230.
[81] D. A. Vogan, *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. **49** (1982), no. 4, 943–1073. MR683010
[82] ———, *Unitary representations of reductive Lie groups*, Ann. of Math. Stud., vol. 118, Princeton University Press, 1987.
[83] ———, *Associated varieties and unipotent representations*, Harmonic analysis on reductive groups, Proc. Conf., Brunswick/ME (USA) 1989, Prog. Math. **101** (1991), 315–388.
[84] ———, *The method of coadjoint orbits for real reductive groups*, Representation theory of Lie groups (Park City, UT, 1998). IAS/Park City Math. Ser. **8** (2000), 179–238.
[85] ———, *Unitary representations of reductive Lie groups*, Mathematics towards the Third Millennium (Rome, 1999). Accademia Nazionale dei Lincei, (2000), 147–167.
[86] N. R. Wallach, *Real reductive groups I*, Academic Press Inc., 1988.
[87] ———, *Real reductive groups II*, Academic Press Inc., 1992.
[88] H. Weyl, *The classical groups: their invariants and representations*, Princeton University Press, 1947.

[89] S. Yamana, *Degenerate principal series representations for quaternionic unitary groups*, Israel J. Math. **185** (2011), 77–124.

the Department of Mathematics, 310 Malott Hall, Cornell University, Ithaca, New York 14853
Email address: dmb14@cornell.edu

School of Mathematical Sciences, Shanghai Jiao Tong University, 800 Dongchuan Road, Shanghai, 200240, China
Email address: hoxide@sjtu.edu.cn

Institute for Advanced Study in Mathematics, Zhejiang University, Hangzhou, 310058, China
Email address: sunbinyong@zju.edu.cn

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076
Email address: matzhucb@nus.edu.sg