Special unipotent representations of real classical groups and theta correspondence

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Classical groups and special unipotent representations

	G	$\mathbf{G}$	$\mathbf{G}^\vee$	
D_n	$\mathrm{O}(p, 2n-p)$	$\mathrm{O}(2n, \mathbb{C})$	$\mathrm{O}(2n, \mathbb{C})$	D_n
C_n	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$	B_n
B_n	$\mathrm{O}(p, 2n+1-p)$	$\mathrm{O}(2n+1, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$	C_n
$\widetilde{C}_n$	$\mathrm{Mp}(2n, \mathbb{R})$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{C})$	C_n
D_n	$\mathrm{O}^*(n)$	$\mathrm{SO}(2n, \mathbb{C})$	$\mathrm{SO}(2n, \mathbb{C})$	D_n
C_n	$\mathrm{Sp}(p, q)$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{SO}(2n+1, \mathbb{C})$	B_n

$$\mathbf{Nil}(\mathbf{G}^\vee) := \{ \text{nilpotent oribt in } \mathbf{G}^\vee \}$$

Theorem (Barbasch-M.-Sun-Zhu)

Arthur-Barbasch-Vogan's conjecture on special unipotent repn. holds:

All elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are *unitarizable*.

Barbasch-Vogan's definition of unipotent representation

nilpotent orbit $\check{\mathcal{O}}$ in $\mathbf{G}^\vee$.

$\rightsquigarrow \phi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathbf{G}^\vee$ (Jacobson-Morozov)

$\rightsquigarrow$ an infinitesimal character $\frac{1}{2}d\phi(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \leftrightarrow \chi_{\mathcal{O}^\vee}$

$\rightsquigarrow$ the maximal primitive ideal $\mathcal{I}_{\check{\mathcal{O}}}$ with inf. char. $\chi_{\check{\mathcal{O}}}$

- *Definition* (Barbasch-Vogan):

  An irreducible admissible G-representation is called *unipotent* if

$$\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) = \mathcal{I}_{\check{\mathcal{O}}}.$$

- *Theorem* (Barbasch-Vogan):

  An irr. G-module has inf. char. $\chi_{\check{\mathcal{O}}}$ is uninpotent $\Longleftrightarrow$

  associated variety of $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\pi) = \overline{\mathcal{O}}$

  Here $\mathcal{O}$ is the Lusztig-Spaltenstein-Barbasch-Vogan dual of $\check{\mathcal{O}}$.

Conjecture

- *Conjecture*: $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ consists of unitary representations.
- Question: How to construct elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$?
- Question: How many elements are there in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$?

Nilpotent orbits with “good/bad parity”

- Bad parity (must occurs with even multiplicity in $\check{\mathcal{O}}$):

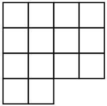
$$\begin{cases} \text{even number}, & \text{when } \mathbf{G}^\vee \text{ is type } B, D \\ \text{odd number}, & \text{when } \mathbf{G}^\vee \text{ is type } C \end{cases}$$

- $\check{\mathcal{O}}$ has *“good parity”* if $\check{\mathcal{O}}$ only contains

$$\begin{cases} \text{odd rows} & \text{when } \mathbf{G}^\vee \text{ is type } B, D \\ \text{even rows} & \text{when } \mathbf{G}^\vee \text{ is type } C \end{cases}$$

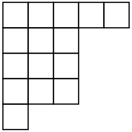
- Example of good parity:

$\mathcal{O}$



$\mathrm{Sp}(14, \mathbb{C})$

$\mathcal{O}^\vee$



$\mathrm{SO}(15, \mathbb{C})$

Reduction to the "good parity"

- Consider $G = \mathrm{Sp}(2n, \mathbb{R})$.
- $\check{\mathcal{O}}$ decompose into two parts $\check{\mathcal{O}}_g$ (good parity) and $\check{\mathcal{O}}_b$ (bad parity).
- Assume $\check{\mathcal{O}}_b = \{r_1, r_1, \cdots, r_k, r_k\}$, $\left|\check{\mathcal{O}}_b\right| = 2n_1$ and $|\check{\mathcal{O}}_g| = 2n_0$

  Theorem (Let $\check{\mathcal{O}}'_b = \{\, r_1, \cdots, r_k \,\} \in \mathrm{GL}(2n_1, \mathbb{R})$.)

$$\begin{array}{ccc}\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}(2n_1, \mathbb{R})) \times \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{Sp}(2n_0, \mathbb{R})) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(2n, \mathbb{R})) \\ (\pi_1, \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}(2n_1,\mathbb{R}) \times \mathrm{Sp}(2n_0,\mathbb{R})}^{\mathrm{Sp}(2n,\mathbb{R})} \pi_1 \otimes \pi_0\end{array}$$

$$\mathrm{Unip}_{\check{\mathcal{O}}'_b}(\mathrm{GL}(2n_1, \mathbb{R})) = \left\{ \mathrm{Ind}_{\mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_k)}^{\mathrm{GL}(2n_1,\mathbb{R})} \mathrm{sgn}^{\epsilon_1} \otimes \cdots \otimes \mathrm{sgn}^{\epsilon_k} \;\middle|\; \epsilon_1, \cdots, \epsilon_k \in \mathbb{Z}/2\mathbb{Z} \right\}$$

- We assume $\check{\mathcal{O}}$ has good parity from now on.
- Use *theta correspondence* to construct $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.

Construct unipotent representations by theta lifting

- $\mathbf{G} = \mathrm{Sp}(2n, \mathbb{C}), \mathbf{G}^\vee = \mathrm{SO}(2n+1, \mathbb{C})$.
- Good parity orbit $\check{\mathcal{O}} \in \mathbf{Nil}^{gp}(\mathfrak{g}^\vee) \subset \mathbf{Nil}(\mathfrak{g}^\vee)$ is an orbit satisfying
$$\check{\mathcal{O}} = (R_{2a} \geq R_{2a-1} \geq \cdots \geq R_0 > 0) \quad \text{all } R_i \text{ are odd}$$
- Quasi-distinguished: $R_{2i} > R_{2i-1}$ for $i = 1, \cdots, a$.
- $\{$ good parity $\} \supset \{$ qusi-distinguished $\} \supset \{$ distinguished $\}$
- $\rightsquigarrow$ infinitesimal character $\chi_{\check{\mathcal{O}}} : \mathcal{U}(\mathfrak{g})^{\mathbf{G}} \to \mathbb{C}$:
$$\chi_{\check{\mathcal{O}}} := (\rho(R_{2a}), \rho(R_{2a-1}), \cdots, \rho(R_0), 0, \cdots, 0)$$
  where
$$\rho(R) := (1, 2, \cdots, \frac{R-1}{2}).$$
- When $G = \mathrm{Sp}(p,q)$, $\mathrm{Unip}_{\check{\mathcal{O}}}(G) \neq \emptyset \iff \check{\mathcal{O}}$ is qusi-dist.

Descent of nilpotent orbits: $\mathbf{G} = \mathrm{Sp}(2n, \mathbb{C})$

- Take $\check{\mathcal{O}} \in \mathbf{Nil}^{gp}(\mathfrak{g}^\vee)$ (nilpotent orbits with good parity).
- Descent sequence on the dual side:

$$\mathcal{O}^\vee = \mathcal{O}^\vee_{2a} \qquad \mathcal{O}^\vee_{2a-1} \qquad \cdots \qquad \mathcal{O}^\vee_0$$

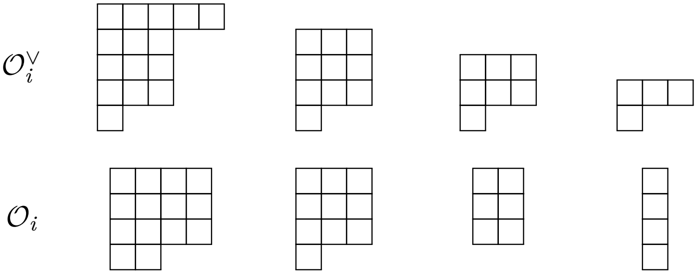
  $\mathcal{O}^\vee_i =$ removing the first rows of $\mathcal{O}^\vee_{i+1}$.
- A descent sequence for $\mathcal{O}$ is a sequence of real classical groups

$$G = G_{2a} \qquad G_{2a-1} \qquad \cdots \qquad G_0$$

  - G_{2k} is a symplectic group
    allow $G_0 = \mathrm{Sp}(0, \mathbb{R}) =$ the trivial group.
  - $G_{2k-1} = \mathrm{O}(p_k, q_k)$
  - $\mathcal{O}^\vee_i$ is nilpotent orbit of $\mathbf{G}^\vee_i$
- (G_i, G_{i-1}) forms a reductive dual pair.
- $\mathcal{O}_i =$ delete the first column of $\mathcal{O}_{i+1}$ and may add one box back to the remaining longest column making the size correct.

Example of descent sequences

$\mathbf{G}_i^\vee$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$



$\mathbf{G}_i$	$\mathrm{Sp}(14,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{Sp}(6,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

- Let $\chi = \chi_{2a-1} \otimes \mathbf{1} \otimes \chi_{2a-3} \otimes \cdots \otimes \chi_1 \otimes \mathbf{1}$ be an 1-dim repn. of $G_{2a-1} \times G_{2a-2} \times \cdots \times G_0$.
- $\chi_i \in \{\mathbf{1}, \mathrm{sgn}^{+,-}, \mathrm{sgn}^{-,+}, \det\}$
- Define a smooth representation of $G = G_{2a}$ (the symplectic group).

$$\pi_\chi := \left(\omega_{G_{2a}, G_{2a-1}} \widehat{\otimes} \omega_{G_{2a-2}, G_{2a-3}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1, G_0} \otimes \chi\right)_{G_{2a-1} \times G_{2a-2} \times \cdots \times G_0}$$

Theorem (Barbasch-M.-Sun-Zhu)

Let $\check{\mathcal{O}}^\vee$ be an orbit with good parity. Then

- $\pi_\chi = 0$ *or*
- π_χ *is in* $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ *and unitarizable.*
- *Moreover,*

$$\mathrm{Unip}_{\mathcal{O}^\vee}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}\,.$$

Example: Coincidences of theta lifting

Lift to $G = \mathrm{Sp}(6,\mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4,\mathbb{C})$.
$\check{\mathcal{O}} = [3,3,1]$ and $\mathcal{O} = (3,3)$.

$$
\begin{tikzcd}
\mathrm{O}(4,0) & & \theta(\mathrm{sgn}^{+,-}) \arrow[dr, equal] & \\
\mathrm{O}(3,1) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) \arrow[dr, equal] & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(2,2) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) \arrow[dr, equal] & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(1,3) & \theta(\mathbf{1}) & \theta(\mathrm{sgn}^{+,-}) \arrow[dr, equal] & \theta(\mathrm{sgn}^{-,+}) \\
\mathrm{O}(0,4) & & & \theta(\mathrm{sgn}^{-,+})
\end{tikzcd}
$$

Some comments

- Many people have studied the problem

  Adams, Barbasch, He, Huang, Li, Loke, Mœglin, Paul, Przebinda, Trapa, ...
- Unitarity:
  - Estimate of matrix coefficients using the explicit realization of the Weil representations.
    Work of **Li**, **He**, and an idea of **Harris-Li-Sun** showing the nonnegativity of a matrix coefficient integral.
- non-vanishing, and compute associated cycle:
  - **Geometry**: moment maps provide the upper bound.
  - **Analysis**: degenerate principal series force the lower bound.
  - **Geometry meets Analysis**: the equality.
- Exhaustion: Combinatorics (very recent breakthrough!)
- Corollary: (using [Gomez-Zhu]) The Whittaker cycle of π_χ equals to its Wavefront cycle.

Counting unipotent representations I

- $\check{\mathcal{O}} \in \mathbf{Nil}^{gp}(\mathbf{G}^\vee)$

  $\rightsquigarrow$ special representation $\tau \leftrightarrow \mathcal{O}$

  Springer/Lusztig left cell $\mathcal{C}_{\mathcal{O}} = \{ \tau_1, \cdots, \tau_{2^l} \}$ containing τ
- $\mathcal{K}_\rho(G)$: the Grothendieck group of finite length Harish-Chandra modules with infinitesimal character ρ.

$$\#\mathrm{Unip}_{\mathcal{O}^\vee}(G) = \sum_{\tau_i} [\tau_i : \mathcal{K}_\rho(G)]$$

- Example $G = \mathrm{Sp}(2n, \mathbb{R})$

$$\mathcal{K}_\rho(G) = \sum_{\substack{p,q,t,s, \\ \sigma \in \widehat{S}_s}} \mathrm{Ind}_{S_t \times W_{2s} \times W_p \times W_q}^{W_n} [\mathrm{sgn} \otimes (\sigma \times \sigma) \otimes \mathbf{1} \otimes \mathbf{1}].$$

- RHS of blue part can be counted by $\mathrm{PBP}(\check{\mathcal{O}})$.
- $LS(\mathcal{O}) \subset \mathcal{K}_{\mathcal{O}}(\mathbf{K})$: all local systems could be obtained by theta lifting.
- When $\mathcal{O}^\vee \in \mathbf{Nil}^{qd}(\mathfrak{g}^\vee)$, $\#\mathrm{PBP}(\mathcal{O}) = \#\mathrm{AOD}(\mathcal{O}) = \#\mathrm{LS}(\mathcal{O})$

Counting unipotent representations II

- $\mathrm{PBP}(\check{\mathcal{O}})$ is complicate.
- $\mathrm{LS}(\check{\mathcal{O}})$ is also complicate.
- Proof of Exhaustion

  Define a bijection (inductively)

$$\begin{tikzcd}
\mathrm{PBP}(\check{\mathcal{O}}) \arrow[r, leftrightarrow] \arrow[d] & \mathrm{Unip}_{\check{\mathcal{O}}}(G) \arrow[d, "\text{associate cycle}"] \\
MYD(\check{\mathcal{O}}) \arrow[r, leftrightarrow] & \mathrm{LS}(\check{\mathcal{O}})
\end{tikzcd}$$

  compatible with the theta lifting.
- The injectivity of theta lifting is crucial!

Unipotent Arthur packet

- Arthur parameter: $\psi\colon W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{G}^\vee \rtimes \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

  Here $W_{\mathbb{R}} = \mathbb{C} \rtimes \langle j \rangle$.
- Arthur's Arthur packet $\Pi^A_\psi(G)$:

  {local components of automorphic representations}

  They are unitary by definition!
- Unipotent Arthur parameter: $\psi|_{\mathbb{C}^\times}$ is trivial.

  Mœglin: $\pi_{\psi,\eta}$ is zero or multiplicity free ($\eta \in \mathrm{Irr}(\pi_1(Z_{\mathbf{G}^\vee}(\psi)))$).

  Warning: $\Pi^A_\psi(G) \cap \Pi^A_{\psi'}(G) \neq \emptyset$ in general.
- "Corollary":

$$\Pi^A_\psi(G) = \Pi^{ABV}_\psi(G)$$

- Question: How to describe $\pi_{\psi,\eta}$ explicitly?

Thank you for your attention!