SPECIAL UNIPOTENT REPRESENTATIONS OF REAL CLASSICAL GROUPS: COUNTING AND REDUCTION TO GOOD PARITY

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ABSTRACT. Let G be a real reductive group in Harish-Chandra's class. We derive some consequences of theory of coherent continuation representations to the counting of irreducible representations of G with a given infinitesimal character and a given bound in the complex associated variety. When G is a real classical group (including the real metaplectic group), we investigate the set of special unipotent representations of G attached to $\check{\mathcal{O}}$, in the sense of Barbasch and Vogan. Here $\check{\mathcal{O}}$ is a nilpotent adjoint orbit in the Langlands dual of G (or the metaplectic dual of G when G is a real metaplectic group). We give a precise count for the number of special unipotent representations of G attached to $\check{\mathcal{O}}$. We also reduce the problem of constructing special unipotent representations attached to $\check{\mathcal{O}}$ to the case when $\check{\mathcal{O}}$ has good parity. The paper is the first in a series of two papers on the classification of special unipotent representations of real classical groups.

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1. INTRODUCTION

1.1. **Background and goals.** In [Art84, Art89], Arthur introduced certain families of representations of a semisimple algebraic group over $\mathbb{R}$ or $\mathbb{C}$, in connection with his conjecture on square-integrable automorphic forms. They are the special unipotent representations in the title of this paper and are attached to a nilpotent adjoint orbit $\check{\mathcal{O}}$ in the Langlands dual of G. For its definition, we will use that of Barbasch-Vogan [BV85], which refines that of Arthur and is in the language of primitive ideals.

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Apart from their clear interest for the theory of automorphic forms, special unipotent representations belong to a fundamental class of unitary representations which are associated to nilpotent coadjoint orbits in the Kirillov philosophy (the orbit method; see [Kir62, Kos70, Vog87b]). These are known informally as unipotent representations, which are expected to play a central role in the unitary dual problem and therefore have been a subject of great interest. See [Vog87, Vog87b, Vog91].

In the context alluded to, a long standing problem in representation theory, known as the Arthur-Barbasch-Vogan conjecture [ABV91, Introduction], is to show that every special unipotent representation is unitary.

In a series of two papers, the authors will construct and classify special unipotent representations of real classical groups, and will prove the Arthur-Barbasch-Vogan conjecture for these groups as a consequence of the classification.

The current paper is the first in the series, which have the following two goals. The first is to give a precise count of special unipotent representations attached to $\check{\mathcal{O}}$. The second is to reduce the problem of constructing special unipotent representations attached to $\check{\mathcal{O}}$ to the case when $\check{\mathcal{O}}$ has good parity.

1.2. **Approach.** We will work in the category of Cassellman-Wallach representations [Wal92, Chapter 11]. The main tool for the counting of irreducible representations in general and special unipotent representations in particular is the coherent continuation representation for this category ([Vog81, Chapter 7]). This is a representation of a certain integral Weyl group, and it can be compared, through a certain embedding, with an analogous coherent continuation representation for the category of highest weight modules. The latter has been intensively studied in Kazhdan-Lusztig theory and is amenable to detailed analysis through the theory of cells and special representations (in the sense of Lusztig), as well as the theory of primitive ideals. In particular one may derive precise information on what representations of the integral Weyl group may contribute to the counting. (Cf. [BV85, Section 5] in the context of complex semisimple groups.) In effectively carrying out the mentioned steps, we build on earlier ideas of several authors including Joseph [Jos80, Jos80b], Vogan [Vog81], King [Kin81], Barbasch-Vogan [BV85], and Casian [Cas86]. For the precise counting of irreducible representations, a key technical issue is a certain relationship (expected by Vogan) between cell representations in the Casselman-Wallach setting (Harish-Chandra cells) and the highest weight module setting (double cells), which we resolve assuming a weak form of Vogan duality. Weaving things together, we arrive at counting results of general nature on irreducible representations of G with a given infinitesimal character and a given bound in the complex associated variety, which are valid for any real reductive group G in Harish-Chandra's class. Together with an explicit formula of the coherent continuation representation (due to Barbasch-Vogan) and explicit branching rules of Weyl group representations, this will finally yield the counting of special unipotent representations of real classical groups, described in terms of combinatorial constructs called painted bipartitions.

We now outline our approach to the second goal. The case of type A groups is relatively easy and so we focus on groups of type B, C, and D. When $\check{\mathcal{O}}$ is not necessarily of good parity, we introduce a $\check{\mathcal{O}}$-relevant parabolic subgroup P, whose Levi component is of the form $G'_{\mathrm{b}} \times G_{\mathrm{g}}$, where G'_{b} is a general linear group, and G_{g} is of the same classical type as G. In their respective Langlands duals, we have nilpotent adjoint orbits $\check{\mathcal{O}}'_{\mathrm{b}}$ and $\check{\mathcal{O}}_{\mathrm{g}}$ (determined by $\check{\mathcal{O}}$). We will show that the normalized smooth parabolic induction from P to G yields a bijective map from a pair of special unipotent representations of G'_{b} (attached to $\check{\mathcal{O}}'_{\mathrm{b}}$) and G_{g}

(attached to $\check{\mathcal{O}}_{\mathrm{g}}$) to special unipotent representations of G (attached to $\check{\mathcal{O}}$). Firstly, there is a classical group G_{b} with the following properties:

- $G_{\mathrm{b}} \times G_{\mathrm{g}}$ is an endoscopy group of G;
- G'_{b} is naturally isomorphic to the Levi component of a $\check{\mathcal{O}}_{\mathrm{b}}$-relevant parabolic subgroup P_{b} of G_{b}.

Here $\check{\mathcal{O}}_{\mathrm{b}}$ is a nilpotent adjoint orbit in the Langlands dual of G_{b} (determined by $\check{\mathcal{O}}$). General considerations as well as detailed information about coherent continuation representations allow us to separate bad parity and good parity, namely we may determine special unipotent representations of G in terms of those of G_{b} and G_{g}. The normalized smooth parabolic induction from P_{b} to G_{b} yields a bijection from special unipotent representations of G'_{b} (attached to $\check{\mathcal{O}}'_{\mathrm{b}}$) to special unipotent representations of G_{b} (attached to $\check{\mathcal{O}}_{\mathrm{b}}$). On the other hand, Vogan's character theory for real reductive groups [Vog82] implies that irreducibility is persevered in the endoscopy setting. This will finally imply the bijectivity of the induction map from P to G.

1.3. **Organization.** In Section 2, we state our main results first on the counting of irreducible representations in our general setup, and then on the counting of special unipotent representations of real classical groups. For the latter, the more involved cases of groups of type B, C and D, the main results are in Theorem 2.12 (reduction to good parity) and Theorem 2.18 (counting in the case of good parity). In Section 3, we recall some generalities on coherent continuation representations for highest weight modules. This includes a new notion of the τ-invariant of a Weyl group representation, and its role in a certain duality notion of double cells (Section 3.8). In Section 4, we discuss some analogous results on coherent continuation representations of Casselman-Wallach representations. As mentioned in the introductory section, we establish a certain relationship between Harish-Chandra cells and double cells assuming a weak form of Vogan duality (Section 4.5). In Section 5, we prove the first part of our main results on the counting of irreducible representations with a given infinitesimal character and a given bound in the complex associated variety. In Section 6, we separate good parity and bad parity for coherent continuation representations, as a preparation for the reduction step in Section 9. Section 7 and Section 8 are devoted to proof of the second part of our main results, which explicitly counts special unipotent representations, for groups of type A, and groups of type B, C and D, respectively. In Section 9, we carry out the reduction step to the case of good parity and prove Theorem 2.12. In Section 10, we develop combinatorics of painted bipartitions which we need for the proof of Theorem 2.18 in Section 8. This includes a notion of descent for painted bipartitions, critical in matching combinatorial parameters of the current paper with special unipotent representations to be constructed in [BMSZ21] by the method of theta lifting (the second paper in the series).

2. The main results

2.1. **Lie algebra notation.** Let $\mathfrak{g}$ be a reductive complex Lie algebra. Its universal enveloping algebra is denoted by $\mathcal{U}(\mathfrak{g})$, and the center of $\mathcal{U}(\mathfrak{g})$ is denoted by $\mathcal{Z}(\mathfrak{g})$. Let ${}^a\mathfrak{h}$ denote the universal Cartan algebra of $\mathfrak{g}$ (also called the abstract Cartan subalgbra in [Vog82]). Let $W \subseteq \mathrm{GL}({}^a\mathfrak{h})$ denote the Weyl group. By the Harish-Chandra isomorphism, there is a 1-1 correspondence between W-orbits of $\nu \in {}^a\mathfrak{h}^*$ (a superscript $*$ indicates the dual space) and algebraic characters $\chi_\nu : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$. We say that an ideal of $\mathcal{U}(\mathfrak{g})$ has infinitesimal character ν if it contains the kernel of χ_ν. By a result of Dixmier [Bor76, Section 3], there is a unique maximal ideal of $\mathcal{U}(\mathfrak{g})$ that has infinitesimal character ν. Write I_ν for this maximal ideal. (We remark that every maximal ideal of $\mathcal{U}(\mathfrak{g})$ is primitive.)

Let $\mathrm{Nil}(\mathfrak{g})$ (resp. $\mathrm{Nil}(\mathfrak{g}^*)$) be the set of nilpotent elements in $[\mathfrak{g},\mathfrak{g}]$ (resp. $[\mathfrak{g},\mathfrak{g}]^*$). Denote $\mathrm{Ad}(\mathfrak{g})$ the inner automorphism group of $\mathfrak{g}$, and put

$$\overline{\mathrm{Nil}}(\mathfrak{g}) := \mathrm{Ad}(\mathfrak{g})\backslash \mathrm{Nil}(\mathfrak{g}), \qquad \overline{\mathrm{Nil}}(\mathfrak{g}^*) := \mathrm{Ad}(\mathfrak{g})\backslash \mathrm{Nil}(\mathfrak{g}^*),$$

the set of $\mathrm{Ad}(\mathfrak{g})$-orbits in $\mathrm{Nil}(\mathfrak{g})$ and $\mathrm{Nil}(\mathfrak{g}^*)$ (which are finite). The Killing form on $[\mathfrak{g},\mathfrak{g}]$ yields an identification $\overline{\mathrm{Nil}}(\mathfrak{g}) = \overline{\mathrm{Nil}}(\mathfrak{g}^*)$. Since $\mathfrak{g}$ is the direct sum of its center with $[\mathfrak{g},\mathfrak{g}]$, $[\mathfrak{g},\mathfrak{g}]^*$ is obviously viewed as a subspace of $\mathfrak{g}^*$.

2.2. **Counting irreducible representations.** Let G be a real reductive group in Harish-Chandra's class whose complexified Lie algebra equals $\mathfrak{g}$. Let $\mathrm{Rep}(G)$ be the category of Casselman-Wallach representations of G, whose Grothendieck group (with $\mathbb{C}$-coefficients) is denoted by $\mathcal{K}(G)$. Let $\mathrm{Irr}(G)$ be the set of isomorphism classes of irreducible Casselman-Wallach representations of G, which forms a basis of $\mathcal{K}(G)$.

Let $\nu \in {}^a\mathfrak{h}^*$ and let S be an $\mathrm{Ad}(\mathfrak{g})$-stable Zariski closed subset of $\mathrm{Nil}(\mathfrak{g}^*)$. Let $\mathrm{Irr}_{\nu,\mathsf{S}}(G)$ denote the subset of $\mathrm{Irr}(G)$ consisting of representations that have infinitesimal character ν and whose complex associated variety is contained in S. Our aim is to count the set $\mathrm{Irr}_{\nu,\mathsf{S}}(G)$.

To do this, we form the subset Λ of ${}^a\mathfrak{h}^*$, which is the ν-translate of the root lattice. Let $W(\Lambda)$ be the integral Weyl group of ν, which equals the stabilizer of Λ in W. We consider the space $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ of $\mathcal{K}(G)$-valued coherent families on Λ, which is a finite-dimensional representation of $W(\Lambda)$. This is called the coherent continuation representation. See Section 3.1.

Attached to S, we will define a certain subset $\mathrm{Irr}_{\mathsf{S}}(W(\Lambda))$ of $\mathrm{Irr}(W(\Lambda))$ by using the Springer correspondence. See Section 3.7.

Let $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ ($n \in \mathbb{N} := \{0,1,2,\dots\}$) denote the metaplectic double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ (it does not split for $n>0$), to be called a real metaplectic group. Recall that G is is said to be linear if it has a faithful finite-dimensional representation.

Theorem 2.1. *We have the inequality*

$$\sharp(\mathrm{Irr}_{\nu,\mathsf{S}}(G)) \le \sum_{\sigma\in \mathrm{Irr}_{\mathsf{S}}(W(\Lambda))} [1_{W_\nu} : \sigma]\cdot[\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

where 1_{W_ν} denotes the trivial representation of the stabilizer W_ν of ν in W. The equality holds if G is linear or isomorphic to a real metaplectic group.

Here and henceforth, $[\,:\,]$ indicates the multiplicity of the first (irreducible) representation in the second one, and $\sharp$ indicates the cardinality of a finite set.

Let $\mathrm{Irr}(G;I_\nu)$ denote the subset of $\mathrm{Irr}(G)$ consisting of representations that are annihilated by I_ν. Note that $\mathrm{Irr}(G;I_\nu) = \mathrm{Irr}_{\nu,\overline{\mathcal{O}_\nu}}(G)$, where $\overline{\mathcal{O}_\nu}$ is the associated variety of the maximal ideal I_ν. In this case, work of Lusztig [Lus84] and Barbasch-Vogan [BV85] imply that

$$[1_{W_\nu} : \sigma] \le 1$$

for all $\sigma \in \mathrm{Irr}_{\overline{\mathcal{O}_\nu}}(W(\Lambda))$, and the set

$$\{\sigma \in \mathrm{Irr}_{\overline{\mathcal{O}_\nu}}(W(\Lambda)) \mid [1_{W_\nu} : \sigma] \neq 0\}$$

can be explicitly described by the Lusztig left cell ${}^L\mathscr{C}_\nu$ (see (3.21) for its definition). Therefore Theorem 2.1 has the following consequence.

Corollary 2.2. *We have the inequality*

$$\sharp(\mathrm{Irr}(G; I_\nu)) \leq \sum_{\sigma \in {}^L\mathscr{C}_\nu} [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

and the equality holds if G is linear or isomorphic to a real metaplectic group.

Remark 2.3. The equalities in Theorem 2.1 and Corollary 2.2 will be proved more generally for any G satisfying a weak form of Vogan duality. For this, following [FJMN21] we will introduce a new technical tool, namely a notion of τ-invariant for a representation of $W(\Lambda)$, and show that a special irreducible representation of $W(\Lambda)$ is determined by its τ-invariant. See Section 3.8 and Section 4.5.

2.3. **Special unipotent representations of real classical groups.** Suppose that $\star$ is one of the 10 labels, G is a classical Lie group of type $\star$, and $\check{\mathfrak{g}}$ is its complex dual Lie algebra, as in the following table ($n, p, q \in \mathbb{N}$).

Label $\star$	Classical Lie Group G	Complex Dual Lie Algebra $\check{\mathfrak{g}}$
$A^{\mathbb{R}}$	$\mathrm{GL}_n(\mathbb{R})$	$\mathfrak{gl}_n(\mathbb{C})$
$A^{\mathbb{H}}$	$\mathrm{GL}_{\frac{n}{2}}(\mathbb{H})$ (n even)	$\mathfrak{gl}_n(\mathbb{C})$
A	$\mathrm{U}(p,q)$	$\mathfrak{gl}_{p+q}(\mathbb{C})$
$\widetilde{A}$	$\widetilde{\mathrm{U}}(p,q)$	$\mathfrak{gl}_{p+q}(\mathbb{C})$
B	$\mathrm{SO}(p,q)$ ($p+q$ odd)	$\mathfrak{sp}_{p+q-1}(\mathbb{C})$
D	$\mathrm{SO}(p,q)$ ($p+q$ even)	$\mathfrak{o}_{p+q}(\mathbb{C})$
C	$\mathrm{Sp}_{2n}(\mathbb{R})$	$\mathfrak{o}_{2n+1}(\mathbb{C})$
$\widetilde{C}$	$\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$	$\mathfrak{sp}_{2n}(\mathbb{C})$
D^*	$\mathrm{O}^*(2n)$	$\mathfrak{o}_{2n}(\mathbb{C})$
C^*	$\mathrm{Sp}(p/2, q/2)$ (p, q even)	$\mathfrak{o}_{p+q+1}(\mathbb{C})$

In this table $\widetilde{\mathrm{U}}(p,q)$ is the double cover of $\mathrm{U}(p,q)$ defined by a square root of the determinant character. In the last column, $\check{\mathfrak{g}}$ is the complex Lie algebra which is the Langlands dual of the complexified Lie algebra $\mathfrak{g}$ of G, except for the case of $\widetilde{C}$. In this exceptional case, $\check{\mathfrak{g}} = \mathfrak{sp}_{2n}(\mathbb{C})$, to be called the "*metaplectic*" Langlands dual of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$.

Let ${}^a\check{\mathfrak{h}}$ denote the abstract Cartan algebra of $\check{\mathfrak{g}}$. In all cases we identify ${}^a\mathfrak{h}^*$ with ${}^a\check{\mathfrak{h}}$ in the standard way, except for $\widetilde{C}$ where the identification is via the half of the trace form on the Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$.

Let $\check{\mathcal{O}} \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}})$ be a nilpotent $\mathrm{Ad}(\check{\mathfrak{g}})$-orbit. Write $\mathbf{d}_{\check{\mathcal{O}}}$ for the Young diagram attached to $\check{\mathcal{O}}$. It determines the nilpotent orbit $\check{\mathcal{O}}$, unless $\check{\mathfrak{g}} = \mathfrak{o}_{4k}(\mathbb{C})$ ($k \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$) and all the row lengths are even. When there is no confusion, we will not distinguish between $\check{\mathcal{O}}$ and $\mathbf{d}_{\check{\mathcal{O}}}$.

Let $\{\check{e}, \check{h}, \check{f}\}$ be an $\mathfrak{sl}_2$-triple in $\check{\mathfrak{g}}$ representing $\check{\mathcal{O}}$, namely $\check{e} \in \check{\mathcal{O}}$. The element $\check{h}/2$ is semisimple, and its $\mathrm{Ad}(\check{\mathfrak{g}})$-orbit is uniquely determined by $\check{\mathcal{O}}$. Using the identification

$$\mathrm{Ad}(\check{\mathfrak{g}}) \backslash \{\text{semisimple element in } \check{\mathfrak{g}}\} = W \backslash {}^a\check{\mathfrak{h}} = W \backslash {}^a\mathfrak{h}^*, \tag{2.1}$$

we pick an element $\lambda_{\check{\mathcal{O}}} \in {}^a\mathfrak{h}^*$ that represents the same element in $W \backslash {}^a\mathfrak{h}^*$ as $\check{h}/2$. As in [BV85, Section 5], $\lambda_{\check{\mathcal{O}}}$ determines an infinitesimal character (namely an algebraic character of $\mathcal{Z}(\mathfrak{g})$), also called the infinitesimal character associated to $\check{\mathcal{O}}$. Write $I_{\check{\mathcal{O}}} := I_{\star,\check{\mathcal{O}}} := I_{\lambda_{\check{\mathcal{O}}}}$ for the maximal ideal of $\mathcal{U}(\mathfrak{g})$, only dependent on $\check{\mathcal{O}}$. (The subscript $\star$ is one of the labels in the table.)

Let $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) \in \overline{\mathrm{Nil}}(\mathfrak{g}^*)$ denote the Barbasch-Vogan dual of $\check{\mathcal{O}}$, namely the unique Zariski open $\mathrm{Ad}(\mathfrak{g})$-orbit in the associated variety of $I_{\check{\mathcal{O}}}$. See [BV85, Appendix] and [BMSZ20, Section 1]. (This duality notion is an analog of a certain duality notion defined earlier by Spaltenstein in [Spa82].) Note that a primitive ideal of $\mathcal{U}(\mathfrak{g})$ equals the maximal ideal $I_{\check{\mathcal{O}}}$ if and only if it has infinitesimal character $\lambda_{\check{\mathcal{O}}}$ and its associated variety equals the Zariski closure of $\mathcal{O}$ in $\mathfrak{g}^*$.

Following Barbasch-Vogan [BV85], define the set of the special unipotent representations of G attached to $\check{\mathcal{O}}$ by

$$\begin{aligned}\mathrm{Unip}_{\check{\mathcal{O}}}(G) :=&\mathrm{Unip}_{\star,\check{\mathcal{O}}}(G)\\ :=&\begin{cases}\{\pi\in\mathrm{Irr}(G)\mid \pi \text{ is genuine and annihilated by } I_{\check{\mathcal{O}}}\}, & \text{if } \star\in\{\widetilde{A},\widetilde{C}\};\\ \{\pi\in\mathrm{Irr}(G)\mid \pi \text{ is annihilated by } I_{\check{\mathcal{O}}}\}, & \textit{otherwise.}\end{cases}\end{aligned}$$

Here "genuine" means that the central subgroup $\{\pm 1\}$ of G, which is the kernel of the covering homomorphism $\widetilde{\mathrm{U}}(p,q)\to \mathrm{U}(p,q)$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})\to\mathrm{Sp}_{2n}(\mathbb{R})$, acts on π through the nontrivial character.

The main goal of this paper and the second paper in the series [BMSZ21] is to count the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ and to construct all the representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$.

We define a notion of good parity and bad parity which will be used throughout the article and will play an important role in the classification of special unipotent representations (for unitary groups and for groups of type B, C and D). Note that if $\star\in\{A^{\mathbb{R}},A^{\mathbb{H}},C,\widetilde{C},D^*\}$, then n equals the rank of $\mathfrak{g}$. In all other cases, we also let n denote the rank of $\mathfrak{g}$. We call an integer to have the good parity (depends on $\star$ and n) if it has the same parity as

$$\begin{cases} n, & \text{if } \star\in\{A^{\mathbb{R}},A^{\mathbb{H}},A\};\\ 1+n, & \text{if } \star=\widetilde{A};\\ 1, & \text{if } \star\in\{\,C,C^*,D,D^*\,\};\\ 0, & \text{if } \star\in\{\,B,\widetilde{C}\,\}.\end{cases}\tag{2.2}$$

Otherwise we call the integer to have bad parity.

2.4. **General linear groups.** All special unipotent representations of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_{\frac{n}{2}}(\mathbb{H})$ are obtained via normalized smooth parabolic induction from quadratic characters (see [Vog86, Page 450]). We will review their classifications in the framework of this article.

For a Young diagram $\imath$, write

$$\mathbf{r}_1(\imath)\geq\mathbf{r}_2(\imath)\geq\mathbf{r}_3(\imath)\geq\cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath)\geq\mathbf{c}_2(\imath)\geq\mathbf{c}_3(\imath)\geq\cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty}\mathbf{r}_i(\imath)$ the total size of $\imath$.

For any Young diagram $\imath$, we introduce the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as follows:

$$\mathrm{Box}(\imath) := \left\{\,(i,j)\in\mathbb{N}^+\times\mathbb{N}^+ \;\middle|\; j\leq\mathbf{r}_i(\imath)\,\right\}.\tag{2.3}$$

A subset of $\mathbb{N}^+\times\mathbb{N}^+$ of the form (2.3) is also said to constitute the Young diagram $\imath$.

We also introduce five symbols $\bullet$, s, r, c and d, and make the following definitions.

Definition 2.4. *A painting on a Young diagram $\imath$ is an assignment*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *the set of boxes of $\imath$ painted with symbols in S constitutes a Young diagram, when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *every row of $\imath$ has at most one box painted with s, and has at most one box painted with r;*
- *every column of $\imath$ has at most one box painted with c, and has at most one box painted with d.*

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram $\imath$ and a painting $\mathcal{P}$ on $\imath$.

Example 2.5. *The following represents a painted Young diagram.*

$\bullet$	$\bullet$	$\bullet$	$\bullet$	r
$\bullet$	r	d		
s	r			
d	d			

Definition 2.6. *Suppose that $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}$. A painting $\mathcal{P}$ on a Young diagram $\imath$ has type $\star$ if*

- *the symbols of $\mathcal{P}$ are in*

$$\begin{cases} \{\bullet, c, d\}, & \textit{if } \star = A^{\mathbb{R}}; \\ \{\bullet\}, & \textit{if } \star = A^{\mathbb{H}}, \end{cases}$$

- *every column of $\imath$ has an even number of boxes painted with $\bullet$.*

Denote by $\mathsf{PAP}_\star(\imath)$ the set of paintings on $\imath^t$ that has type $\star$, where $\imath^t$ is the transpose of $\imath$.

The middle letter A in PAP refers to the common A in $\{A^{\mathbb{R}}, A^{\mathbb{H}}\}$.

It is easy to check that if $\star = A^{\mathbb{R}}$, then

$$\sharp(\mathsf{PAP}_\star(\check{\mathcal{O}})) = \prod_{i \in \mathbb{N}^+} (1 + \text{the number of rows of length } i \text{ in } \check{\mathcal{O}}),$$

and if $\star = A^{\mathbb{H}}$, then

$$\sharp(\mathsf{PAP}_\star(\check{\mathcal{O}})) = \begin{cases} 1, & \text{if all row lengths of } \check{\mathcal{O}} \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Here and henceforth $\sharp$ indicates the cardinality of a finite set.

If $\star = A^{\mathbb{R}}$, for every $\mathcal{P} \in \mathsf{PAP}_\star(\check{\mathcal{O}})$ we attach a representation $\pi_{\mathcal{P}}$ of G as in what follows. Let $P_{\mathcal{P}}$ be the standard parabolic subgroup of G with Levi component

$$\mathrm{GL}_{\mathbf{r}_1(\check{\mathcal{O}})}(\mathbb{R}) \times \mathrm{GL}_{\mathbf{r}_2(\check{\mathcal{O}})}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{\mathbf{r}_k(\check{\mathcal{O}})}(\mathbb{R}),$$

where k is the number of nonempty rows of $\check{\mathcal{O}}$. On each factor $\mathrm{GL}_{\mathbf{r}_j(\check{\mathcal{O}})}(\mathbb{R})$, put the trivial character or the sign character according to whether the j-th column of $\mathcal{P}$ ends in $\bullet, c$ or d. This yields a character of $P_{\mathcal{P}}$ and let $\pi_{\mathcal{P}}$ be the resulting normalized induced representation of G.

Similarly, if $\star = A^{\mathbb{H}}$, for every $\mathcal{P} \in \mathsf{PAP}_\star(\check{\mathcal{O}})$ we attach a representation $\pi_{\mathcal{P}}$ of G as in what follows. Let $P_{\mathcal{P}}$ be the standard parabolic subgroup of G with Levi component

$$\mathrm{GL}_{\mathbf{r}_1(\check{\mathcal{O}})/2}(\mathbb{H}) \times \mathrm{GL}_{\mathbf{r}_2(\check{\mathcal{O}})/2}(\mathbb{H}) \times \cdots \times \mathrm{GL}_{\mathbf{r}_k(\check{\mathcal{O}})/2}(\mathbb{H}),$$

where k is the number of nonempty rows of $\check{\mathcal{O}}$. Put the trivial character on each factor $\mathrm{GL}_{\mathbf{r}_j(\check{\mathcal{O}})/2}(\mathbb{H})$. This yields a character of $P_{\mathcal{P}}$ and let $\pi_{\mathcal{P}}$ be the resulting normalized induced representation of G.

Then in both cases $\pi_{\mathcal{P}}$ is irreducible and belongs to $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ (see [Vog86, Theorem 3.8] and [ABV91, Example 27.5]). We summarize the classifications by the following theorem.

Theorem 2.7. *Suppose that $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}$. Then the map*

$$\begin{array}{rcl} \mathsf{PAP}_\star(\check{\mathcal{O}}) & \rightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ \mathcal{P} & \mapsto & \pi_{\mathcal{P}} \end{array}$$

is bijective. □

2.5. **Unitary groups.** Similar to Definition 2.6, we make the following definition.

Definition 2.8. *Suppose that $\star \in \{A, \widetilde{A}\}$. A painting $\mathcal{P}$ on a Young diagram $\imath$ has type $\star$ if*

- *the symbols of $\mathcal{P}$ are in $\{\bullet, s, r\}$, and*
- *every row of $\imath$ has an even number of boxes painted with $\bullet$.*

Denote by $\mathsf{PAP}_\star(\imath)$ the set of paintings on $\imath^t$ that has type $\star$, where $\imath^t$ is the transpose of $\imath$.

Now suppose that $\imath$ is a Young diagram and $\mathcal{P}$ is a painting on $\imath$ that has type A or $\widetilde{A}$. Define the signature of $\mathcal{P}$ to be the pair

$$(p_{\mathcal{P}}, q_{\mathcal{P}}) := \left(\frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(r)),\ \frac{\sharp(\mathcal{P}^{-1}(\bullet))}{2} + \sharp(\mathcal{P}^{-1}(s)) \right). \tag{2.4}$$

Example 2.9. *Suppose that*

$$\check{\mathcal{O}} = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \\ \cline{1-1} \end{array} \quad \textit{and} \quad \mathcal{P} = \begin{array}{|c|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet & r \\ \hline \bullet & \bullet \\ \cline{1-2} s & r \\ \cline{1-2} s \\ \cline{1-1} r \\ \cline{1-1} \end{array} \quad \in \mathrm{PAP}_A(\check{\mathcal{O}}).$$

Then $(p_{\mathcal{P}}, q_{\mathcal{P}}) = (6, 5)$.

Given two Young diagrams $\imath$ and $\jmath$, write $\imath \overset{r}{\sqcup} \jmath$ for the Young diagram whose multiset of nonzero row lengths equals the union of those of $\imath$ and $\jmath$. Also write $2\imath = \imath \overset{r}{\sqcup} \imath$. Similarly, we write $\imath \overset{c}{\sqcup} \jmath$ for the Young diagram whose multiset of nonzero column lengths equals the union of those of $\imath$ and $\jmath$.

For unitary groups, we have the following counting result.

Theorem 2.10. *Suppose that $\star \in \{A, \widetilde{A}\}$. If there is a Young diagram decomposition*

$$\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}} \overset{r}{\sqcup} 2\check{\mathcal{O}}'_{\mathrm{b}} \tag{2.5}$$

such that all nonzero row lengths of $\check{\mathcal{O}}_{\mathrm{g}}$ *have good parity and all nonzero row lengths of* $\check{\mathcal{O}}'_{\mathrm{b}}$ *have bad parity, then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp\,\{\,\mathcal{P} \in \mathrm{PAP}_{\star}(\check{\mathcal{O}}_{\mathrm{g}}) \mid (p_{\mathcal{P}} + |\check{\mathcal{O}}'_{\mathrm{b}}|\,, q_{\mathcal{P}} + |\check{\mathcal{O}}'_{\mathrm{b}}|) = (p,q)\,\}\,.$$

Otherwise $\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = 0$*.*

In particular, when $\star = \widetilde{A}$ and $p+q$ is odd, the set $\mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{U}}(p,q))$ is empty.

Suppose that $\star \in \{A, \widetilde{A}\}$ so that $G = \mathrm{U}(p,q)$ or $\widetilde{\mathrm{U}}(p,q)$. By Theorem 2.10, the set $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is empty unless there is a decomposition as in (2.5) such that

$$p, q \geq |\check{\mathcal{O}}'_{\mathrm{b}}|\,. \tag{2.6}$$

Assume this holds, and define two groups $G'_{\mathrm{b}} := \mathrm{GL}_{|\check{\mathcal{O}}'_{\mathrm{b}}|}(\mathbb{C})$ and

$$G_{\mathrm{g}} = \begin{cases} \mathrm{U}(p - |\check{\mathcal{O}}'_{\mathrm{b}}|\,, q - |\check{\mathcal{O}}'_{\mathrm{b}}|), & \text{if } \star = A; \\ \widetilde{\mathrm{U}}(p - |\check{\mathcal{O}}'_{\mathrm{b}}|\,, q - |\check{\mathcal{O}}'_{\mathrm{b}}|), & \text{if } \star = \widetilde{A}. \end{cases}$$

Then up to conjugation there is a unique parabolic subgroup P of G whose Levi quotient (namely its quotient by the unipotent radical) is naturally isomorphic to $G'_{\mathrm{b}} \times G_{\mathrm{g}}$. Note that $\check{\mathcal{O}}_{\mathrm{g}}$ has good parity with respect to $\star$ and $|\check{\mathcal{O}}_{\mathrm{g}}|$.

Let $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}}$ denote the unique element in $\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}})$ (see Section 2.9 for a review on complex classical groups). Then for every $\pi_{\mathrm{g}} \in \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$, the normalized smooth parabolic induced representation (from P to G) $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \ltimes \pi_{\mathrm{g}}$ is irreducible by [Mat96, Theorem 3.2.2] and is an element of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ (*cf.* [MR19, Theorem 5.3]).

Theorem 2.11. *With the assumptions in* (2.5) *and* (2.6)*, and notation as above, the parabolic induction map*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ \pi_{\mathrm{g}} & \mapsto & \pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \ltimes \pi_{\mathrm{g}} \end{array} \tag{2.7}$$

is bijective.

See Section 7.3 for additional results on special unipotent representations of unitary groups.

2.6. **Orthogonal and symplectic groups: relevant parabolic subgroups.** In this subsection and the next two subsections, we assume that $\star \in \{\,B, C, \widetilde{C}, C^*, D, D^*\,\}$. Then there are Young diagram decompositions

$$\mathbf{d}_{\check{\mathcal{O}}} = \mathbf{d}_{\mathrm{b}} \overset{r}{\sqcup} \mathbf{d}_{\mathrm{g}} \quad \text{and} \quad \mathbf{d}_{\mathrm{b}} = 2\mathbf{d}'_{\mathrm{b}} \tag{2.8}$$

such that $\mathbf{d}_{\mathrm{b}}$ has bad parity in the sense that all its nonzero row lengths have bad parity, and $\mathbf{d}_{\mathrm{g}}$ has good parity in the sense that all its nonzero row lengths have good parity. Put

$$n_{\mathrm{b}} := |\mathbf{d}'_{\mathrm{b}}|\,. \tag{2.9}$$

Write V for the standard module of $\mathfrak{g}$, which is either a complex symmetric bilinear space or a complex symplectic space. Recall that $\mathcal{O} \subseteq \mathfrak{g}^*$ is the Barbasch-Vogan dual of $\check{\mathcal{O}}$. We say that a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $\check{\mathcal{O}}$-relevant if the following conditions are satisfied:

- it is the stabilizer of a totally isotropic subspace V'_{b} of V of dimension n_{b};
- the orbit $\mathcal{O}$ equals the induction from $\mathfrak{p}$ to $\mathfrak{g}$ of an orbit in $\overline{\mathrm{Nil}}(\mathfrak{l}^*)$, where $\mathfrak{l}$ is the Levi quotient of $\mathfrak{p}$.

Note that the first condition implies the second one, except for the case when $\star \in \{D, D^*\}$ and $\mathbf{d}_{\check{\mathcal{O}}}$ is nonempty and has bad parity. In all cases, up to conjugation by $\mathrm{Ad}(\mathfrak{g})$ there exists a unique parabolic subalgebra of $\mathfrak{g}$ that is $\check{\mathcal{O}}$-relevant.

We say that a parabolic subgroup of G is $\check{\mathcal{O}}$-relevant if its complexified Lie algebra is. If such a parabolic subgroup exists, it is unique up to conjugation by G. In this case, the orbit $\check{\mathcal{O}}$ is said to be G-relevant.

If $\star = D^*$ and $\mathbf{d}_{\check{\mathcal{O}}}$ is nonempty and has bad parity, then there are precisely two orbits in $\overline{\mathrm{Nil}}(\check{\mathfrak{g}})$ that has the same Young diagram as that of $\check{\mathcal{O}}$. Between these two orbits, exactly one is G-relevant. Excluding this special case, $\check{\mathcal{O}}$ is G-relevant if and only if

$$\text{either } \star \in \{B, D, C^*\} \text{ and } p, q \geq n_{\mathrm{b}}, \quad \text{or} \quad \star \in \{C, \widetilde{C}, D^*\}. \tag{2.10}$$

When (2.10) holds, we put

$$G_{\mathrm{g}} := \begin{cases} \mathrm{SO}(p-n_{\mathrm{b}}, q-n_{\mathrm{b}}), & \text{if } \star \in \{\, B, D \,\}; \\ \mathrm{O}^*(2n-2n_{\mathrm{b}}), & \text{if } \star = D^*; \\ \mathrm{Sp}_{2n-2n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star = C; \\ \widetilde{\mathrm{Sp}}_{2n-2n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{Sp}(\frac{p-n_{\mathrm{b}}}{2}, \frac{q-n_{\mathrm{b}}}{2}), & \text{if } \star = C^*. \end{cases} \tag{2.11}$$

Then the Levi quotient of every $\check{\mathcal{O}}$-relevant parabolic subgroup of G is naturally isomorphic to $G'_{\mathrm{b}} \times G_{\mathrm{g}}$ (or $(G'_{\mathrm{b}} \times G_{\mathrm{g}})/\{\pm 1\}$ when $\star = \widetilde{C}$), where

$$G'_{\mathrm{b}} := \begin{cases} \mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star \in \{\, B, C, D \,\}\,; \\ \widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{GL}_{\frac{n_{\mathrm{b}}}{2}}(\mathbb{H}), & \text{if } \star \in \{\, C^*, D^* \,\}\,. \end{cases} \tag{2.12}$$

Here $\widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R})$ is the double cover of $\mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R})$ that fits the following Cartesian diagram of Lie groups:

$$\begin{tikzcd} \widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}) \arrow[d, "g \mapsto \text{ sign of } \det(g)"] \\ \{\pm 1, \pm\sqrt{-1}\} \arrow[r, "x \mapsto x^2"] & \{\pm 1\}. \end{tikzcd} \tag{2.13}$$

Unless otherwise mentioned, we will use the corresponding lower case Gothic letter to denote the complexified Lie algebra of a Lie group. For example, $\mathfrak{g}'_{\mathrm{b}}$ is the complexified Lie algebra of G'_{b}. Let $\check{\mathfrak{g}}'_{\mathrm{b}}$ denote the Langlands dual of $\mathfrak{g}'_{\mathrm{b}}$, and let $\check{\mathcal{O}}'_{\mathrm{b}} \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}}'_{\mathrm{b}})$ denote the nilpotent orbit with Young diagram $\mathbf{d}'_{\mathrm{b}}$. Likewise let $\check{\mathfrak{g}}_{\mathrm{g}}$ denote the Langlands dual (or the metaplectic Langlands dual when $\star = \widetilde{C}$) of $\mathfrak{g}_{\mathrm{g}}$, and let $\check{\mathcal{O}}_{\mathrm{g}} \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}}_{\mathrm{g}})$ denote the (unique) nilpotent orbit with Young diagram $\mathbf{d}_{\mathrm{g}}$.

2.7. **Orthogonal and symplectic groups: reduction to good parity.** Define

$$\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R})) := \{\pi \in \mathrm{Irr}(\widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R})) \mid \pi \text{ is genuine and annihilated by } I_{\check{\mathcal{O}}'_{\mathrm{b}}} := I_{A^{\mathbb{R}}, \check{\mathcal{O}}'_{\mathrm{b}}}\}.$$

Here and as before, "genuine" means that the central subgroup $\{\pm 1\}$ acts through the non-trivial character. Then we have a bijective map

$$\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R})) \to \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R})), \quad \pi \mapsto \pi \otimes \tilde{\chi}_{n_{\mathrm{b}}},$$

where $\tilde{\chi}_{n_{\mathrm{b}}}$ is the character given by the left vertical arrow of (2.13).

Theorem 2.12. *If G has a $\check{\mathcal{O}}$-relevant parabolic subgroup P, then the normalized smooth parabolic induction from P to G yields a bijection*

$$\begin{array}{ccc} \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ (\pi', \pi_{\mathrm{g}}) & \mapsto & \pi' \ltimes \pi_{\mathrm{g}}. \end{array}$$

Otherwise,

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \emptyset.$$

By Theorem 2.12, we have the more specific results on counting as follows.

(a) Assume that $\star \in \{B, D\}$ so that $G = \mathrm{SO}(p,q)$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \begin{cases} \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) \times \sharp(\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}))), & \text{if } p, q \geq n_{\mathrm{b}}; \\ 0, & \text{otherwise.} \end{cases}$$

(b) Assume that $\star = C^*$ so that $G = \mathrm{Sp}(\frac{p}{2}, \frac{q}{2})$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \begin{cases} \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})), & \text{if } p, q \geq n_{\mathrm{b}}; \\ 0, & \text{otherwise.} \end{cases}$$

(c) Assume that $\star \in \{C, \widetilde{C}\}$ so that $G = \mathrm{Sp}_{2n}(\mathbb{R})$ or $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) \times \sharp(\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}))).$$

(d) Assume that $\star = D^*$ so that $G = \mathrm{O}^*(2n)$. Then

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \begin{cases} 0, & \text{if } \check{\mathcal{O}} \text{ is not } G\text{-relevant}; \\ \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})), & \text{otherwise.} \end{cases}$$

2.8. **Orthogonal and symplectic groups: the case of good parity.** We now assume that $\check{\mathcal{O}}$ has good parity, namely $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$. By Theorem 2.12, the counting problem in general is reduced to this case.

Definition 2.13. *A $\star$-pair is a pair $(i, i+1)$ of consecutive positive integers such that*

$$\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be

- *vacant in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*
- *tailed in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ is positive and even.*

Denote $\mathrm{PP}_\star(\check{\mathcal{O}})$ *the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$.*

Remark 2.14. When $\star \neq \widetilde{C}$, the power set of $\mathrm{PP}_\star(\check{\mathcal{O}})$ gives another description of Lusztig's canonical quotient attached to $\check{\mathcal{O}}$. The set $\mathrm{PP}_\star(\check{\mathcal{O}})$ appears implicitly in [Som01, Section 5].

We attach to $\check{\mathcal{O}}$ a pair of Young diagrams

$$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) := (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})), \tag{2.14}$$

as follows.

The case when $\star = B$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star = \widetilde{C}$. In this case, for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = \left(\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}\right).$$

The case when $\star = \{C, C^*\}$. In this case, for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i-1,2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1,2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

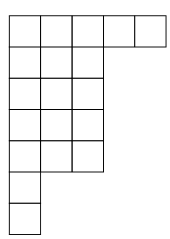
The case when $\star \in \{D, D^*\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0, \end{cases}$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (0,0), & \text{if } (2i,2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0\right), & \text{if } (2i,2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ \left(\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}\right), & \text{otherwise.} \end{cases}$$

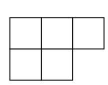
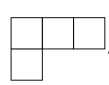
Example 2.15. *Suppose that $\star = C$, and $\check{\mathcal{O}}$ is the following Young diagram which has good parity.*



Then

$$\mathrm{PP}_\star(\check{\mathcal{O}}) = \{(1,2),(5,6)\}$$

and

$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) =$  $\times$ .

Here and henceforth, when no confusion is possible, we write $\alpha \times \beta$ for a pair (α, β). We will also write $\alpha \times \beta \times \gamma$ for a triple (α, β, γ).

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 2.16. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- *$\mathcal{P}$ and $\mathcal{Q}$ have the identical set of boxes painted with $\bullet$;*
- *the symbols of $\mathcal{P}$ are in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, r, c, d\}, & \text{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \text{if } \alpha = D; \\ \{\bullet, s, c\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \text{if } \alpha = C^*; \\ \{\bullet, s\}, & \text{if } \alpha = D^*, \end{cases}$$

- *the symbols of $\mathcal{Q}$ are in*

$$\begin{cases} \{\bullet, s, r, d\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, s\}, & \text{if } \alpha = C; \\ \{\bullet\}, & \text{if } \alpha = D; \\ \{\bullet, r, d\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \text{if } \alpha = C^*; \\ \{\bullet, r\}, & \text{if } \alpha = D^*. \end{cases}$$

For any painted bipartition τ as in Definition 2.16, we write

$$\imath_\tau := \imath,\ \mathcal{P}_\tau := \mathcal{P},\ \jmath_\tau := \jmath,\ \mathcal{Q}_\tau := \mathcal{Q},\ \alpha_\tau := \alpha,\ |\tau| := |\imath| + |\jmath|,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

We further define a pair $\mathrm{Sign}(\tau) := (p_\tau, q_\tau)$ of natural numbers given by the following recipe:

- If $\star_\tau \in \{B, D, C^*\}$, then (p_τ, q_τ) is given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := (\#\bullet) + 2(\#r) + (\#c) + (\#d) + (\#B^+); \\ q_\tau := (\#\bullet) + 2(\#s) + (\#c) + (\#d) + (\#B^-). \end{cases} \tag{2.15}$$

  Here

$$\#\bullet := \#(\mathcal{P}^{-1}(\bullet)) + \#(\mathcal{Q}^{-1}(\bullet))$$

  and the other terms are similarly defined.
- If $\star_\tau \in \{C, \widetilde{C}, D^*\}$, then $p_\tau := q_\tau := |\tau|$.

We also define a classical group

$$G_\uptau := \begin{cases} \mathrm{SO}(p_\uptau, q_\uptau), & \text{if } \star_\uptau = B \text{ or } D; \\ \mathrm{Sp}_{2|\uptau|}(\mathbb{R}), & \text{if } \star_\uptau = C; \\ \widetilde{\mathrm{Sp}}_{2|\uptau|}(\mathbb{R}), & \text{if } \star_\uptau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\uptau}{2}, \frac{q_\uptau}{2}), & \text{if } \star_\uptau = C^*; \\ \mathrm{O}^*(2\,|\uptau|), & \text{if } \star_\uptau = D^*. \end{cases}$$

Define

$$\mathsf{PBP}_\star(\check{\mathcal{O}}) := \{\, \uptau \text{ is a painted bipartition} \mid \star_\uptau = \star, \text{ and } (\imath_\uptau, \jmath_\uptau) = (\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) \,\}\,, \tag{2.16}$$

and

$$\mathsf{PBP}_G(\check{\mathcal{O}}) := \{\, \uptau \in \mathsf{PBP}_\star(\check{\mathcal{O}}) \mid G_\uptau = G \,\}\,.$$

Here "$G_\uptau = G$" amounts to saying that $(p_\uptau, q_\uptau) = (p, q)$ if $\star \in \{B, D, C^*\}$.

Example 2.17. *Suppose that $\star = B$ and*

$$\check{\mathcal{O}} = \begin{array}{|c|c|c|c|c|c|} \hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \end{array}.$$

Then

$$\uptau := \begin{array}{|c|c|} \hline \bullet & \bullet \\ \hline \bullet \\ \cline{1-1} c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & \bullet & d \\ \hline \bullet \\ \cline{1-1} d \\ \cline{1-1} \end{array} \times B^+ \in \mathrm{PBP}_\star(\check{\mathcal{O}}),$$

and

$$G_\uptau = \mathrm{SO}(10, 9).$$

We now state our final result on the counting of special unipotent representations.

Theorem 2.18. *Assume that $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and $\check{\mathcal{O}}$ has good parity. Then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \begin{cases} \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{C^*, D^*\}; \\ 2^{\sharp(\mathrm{PP}_\star(\check{\mathcal{O}}))} \cdot \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{B, C, D, \widetilde{C}\}. \end{cases}$$

In [BMSZ21], the authors will construct all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ by the method of theta lifting, when $\check{\mathcal{O}}$ has good parity. See [BMSZ21, Theorem 4.1].

Example 2.19. *Suppose that $\star = C$, $G = \mathrm{Sp}_4(\mathbb{R})$, and*

$$\check{\mathcal{O}} = \begin{array}{|c|c|c|} \hline \ & \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} \end{array}.$$

Then $\mathrm{PP}_\star(\check{\mathcal{O}}) = \{(1, 2)\}$, and the set $\mathsf{PBP}_G(\check{\mathcal{O}})$ has 4 elements in total:

$$\boxed{\bullet} \times \boxed{\bullet} \times C, \quad \boxed{r} \times \boxed{s} \times C, \quad \boxed{c} \times \boxed{s} \times C, \quad \boxed{d} \times \boxed{s} \times C.$$

Thus there are precisely 8 special unipotent representations of G attached to $\check{\mathcal{O}}$.

2.9. **The case of complex classical groups.** Special unipotent representations of complex classical groups are all well-understood and known to be unitarizable ([BV85], [Bar89]). We briefly review their counting and constructions in what follows. As the methods of this paper and [BMSZ21] work for complex classical groups as well, we will present the results in the complex case parallel to those of this paper and [BMSZ21]. For this subsection, we introduce five more labels $A^{\mathbb{C}}, B^{\mathbb{C}}, D^{\mathbb{C}}, C^{\mathbb{C}}$, and $\widetilde{C}^{\mathbb{C}}$, and let $\star$ be one of them. Let G be a complex classical group of type $\star$, namely

$$G = \mathrm{GL}_n(\mathbb{C}), \quad \mathrm{SO}_{2n+1}(\mathbb{C}), \quad \mathrm{SO}_{2n}(\mathbb{C}), \quad \mathrm{Sp}_{2n}(\mathbb{C}), \quad \text{or} \quad \mathrm{Sp}_{2n}(\mathbb{C}) \qquad (n \in \mathbb{N}),$$

respectively. Let $\mathfrak{g}_0$ denote the Lie algebra of G, which is a complex Lie algebra. The Langlands dual (or metaplectic Langlands dual when $\star = \widetilde{C}^{\mathbb{C}}$) $\check{\mathfrak{g}}_0$ of $\mathfrak{g}_0$ is respectively defined to be

$$\mathfrak{gl}_n(\mathbb{C}), \quad \mathfrak{sp}_{2n}(\mathbb{C}), \quad \mathfrak{o}_{2n}(\mathbb{C}), \quad \mathfrak{o}_{2n+1}(\mathbb{C}), \quad \text{or} \quad \mathfrak{sp}_{2n}(\mathbb{C}).$$

Let $\check{\mathcal{O}} \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}}_0)$. As in the real case we have a maximal ideal $I_{\check{\mathcal{O}}} := I_{\star,\check{\mathcal{O}}}$ of $\mathcal{U}(\mathfrak{g}_0)$.

Write $\overline{\mathfrak{g}}_0$ for the complex Lie algebra equipped with a conjugate linear isomorphism $^- : \mathfrak{g}_0 \to \overline{\mathfrak{g}_0}$. The latter induces a conjugate linear isomorphism $^- : \mathcal{U}(\mathfrak{g}_0) \to \mathcal{U}(\overline{\mathfrak{g}_0})$. Note that $\mathfrak{g}_0 \times \overline{\mathfrak{g}_0}$ equals the complexified Lie algebra $\mathfrak{g}$ of G. Define the set of special unipotent representations of G attached to $\check{\mathcal{O}}$ by

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) := \mathrm{Unip}_{\star,\check{\mathcal{O}}}(G) := \{\pi \in \mathrm{Irr}(G) \mid \pi \text{ is annihilated by } I_{\check{\mathcal{O}}} \otimes \mathcal{U}(\overline{\mathfrak{g}_0}) + \mathcal{U}(\mathfrak{g}_0) \otimes \overline{I_{\check{\mathcal{O}}}}\,\}.$$

If $\star = A^{\mathbb{C}}$ so that $G = \mathrm{GL}_n(\mathbb{C})$, then $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is a singleton whose unique element is given by the normalized smooth parabolic induction $\mathrm{Ind}_P^G 1_P$, where P is the standard parabolic subgroup whose Levi component equals

$$\mathrm{GL}_{\mathbf{r}_1(\check{\mathcal{O}})}(\mathbb{C}) \times \mathrm{GL}_{\mathbf{r}_2(\check{\mathcal{O}})}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{\mathbf{r}_{\mathbf{c}_1(\check{\mathcal{O}})}(\check{\mathcal{O}})}(\mathbb{C}),$$

and 1_P denotes the trivial representation of P.

Now suppose that $\star \in \{B^{\mathbb{C}}, D^{\mathbb{C}}, C^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}}\}$. Write

$$\mathbf{d}_{\check{\mathcal{O}}} = \mathbf{d}_{\mathrm{b}} \overset{r}{\sqcup} \mathbf{d}_{\mathrm{g}} \quad \text{and} \quad \mathbf{d}_{\mathrm{b}} = 2\mathbf{d}'_{\mathrm{b}}$$

as in (2.8), and put $n_{\mathrm{b}} := |\mathbf{d}'_{\mathrm{b}}|$ as before. Let $\mathfrak{p}_0$ be a parabolic subalgebra of $\mathfrak{g}_0$ that is $\check{\mathcal{O}}$-relevant (defined in Section 2.7). Let P be the parabolic subgroup of G with Lie algebra $\mathfrak{p}_0$. Then the Levi quotient of P is naturally isomorphic to $G'_{\mathrm{b}} \times G_{\mathrm{g}}$, where $G'_{\mathrm{b}} = \mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{C})$ and

$$G_{\mathrm{g}} := \begin{cases} \mathrm{SO}_{2n-2n_{\mathrm{b}}+1}(\mathbb{C}), & \text{if } \star = B^{\mathbb{C}}; \\ \mathrm{SO}_{2n-2n_{\mathrm{b}}}(\mathbb{C}), & \text{if } \star = D^{\mathbb{C}}; \\ \mathrm{Sp}_{2n-2n_{\mathrm{b}}}(\mathbb{C}), & \text{if } \star \in \{C^{\mathbb{C}}, \widetilde{C}^{\mathbb{C}}\}. \end{cases}$$

Define the set $\mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}})$ as in the real case. Then by the work of Barbasch-Vogan ([BV85, Corollary 5.29], integral case) and Moeglin-Renard ([MR17, Theorem 6.12 and Theorem 10.1], general case), we have that

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) = 2^{\sharp(\mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}}))}.$$

As in the real case, every representation in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is obtained through irreducible parabolic induction from P to G via those of $\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$ (*cf.* Theorem 2.12), and every representation in $\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$ is obtained via iterated theta lifting (see [Bar17, Theorem 3.5.1], [Mœg17] and [BMSZ21]). The method of [BMSZ21] also shows that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are unitarizable.

3. Generalities on coherent families of highest weight modules

We retain the notation of Section 2.1. Write

$$\Delta^+ \subseteq \Delta \subseteq {}^a\mathfrak{h}^* \quad \text{and} \quad \check{\Delta}^+ \subseteq \check{\Delta} \subseteq {}^a\mathfrak{h}$$

for the positive root system, the root system, the positive coroot system, and the coroot system, respectively, for the reductive complex Lie algebra $\mathfrak{g}$. Write $Q_\mathfrak{g}$ and $Q^\mathfrak{g}$ for the root lattice and weight group of $\mathfrak{g}$, respectively. More precisely, $Q_\mathfrak{g}$ is the subgroup of ${}^a\mathfrak{h}^*$ spanned by Δ, and

$$Q^\mathfrak{g} := \{\nu \in {}^a\mathfrak{h}^* \mid \langle \nu, \check{\alpha}\rangle \in \mathbb{Z} \ \text{ for all } \check{\alpha} \in \check{\Delta}\}.$$

3.1. **Coherent continuation representations.** Throughout this article we take $\mathbb{C}$ as the coefficient ring to define Grothendieck groups. When no confusion is possible, for every object O in an abelian category, we still use the same symbol to indicate the Grothendieck group element represented by the object O.

Let $\Lambda \subseteq {}^a\mathfrak{h}^*$ be a Q-coset, where Q is a W-stable subgroup of $Q^\mathfrak{g}$ containing $Q_\mathfrak{g}$ (which is determined by Λ). The stabilizer of Λ in W is denoted by W_Λ. Let $\mathrm{Rep}(\mathfrak{g}, Q)$ denote the category of all finite-dimensional representations of $\mathfrak{g}$ whose weights (viewed as elements of $Q^\mathfrak{g} \subseteq {}^a\mathfrak{h}^*$) are contained in Q. Write $\mathcal{R}(\mathfrak{g}, Q)$ for the Grothendieck group of this category, which is a commutative $\mathbb{C}$-algebra under the tensor product of representations.

Given a $\mathcal{R}(\mathfrak{g}, Q)$-module $\mathcal{K}$, a Λ-family in $\mathcal{K}$ is a family $\{\mathcal{K}_\nu\}_{\nu\in\Lambda}$ of subspaces of $\mathcal{K}$ such that

- $\mathcal{K}_{w\nu} = \mathcal{K}_\nu$ for all $w \in W_\Lambda$ and $\nu \in \Lambda$;
- for all representations F in $\mathrm{Rep}(\mathfrak{g}, Q)$ and all $\nu \in \Lambda$,

$$F \cdot \mathcal{K}_\nu \subseteq \sum_{\mu \in {}^a\mathfrak{h}^* \text{ is a weight of } F} \mathcal{K}_{\nu+\mu}.$$

Definition 3.1. *Let $\mathcal{K}$ be a $\mathcal{R}(\mathfrak{g}, Q)$-module with a Λ-family $\{\mathcal{K}_\nu\}_{\nu\in\Lambda}$ in it. A $\mathcal{K}$-valued coherent family on Λ is a map*

$$\Psi : \Lambda \to \mathcal{K}$$

satisfying the following two conditions:

- *for all $\nu \in \Lambda$, $\Psi(\nu) \in \mathcal{K}_\nu$;*
- *for all representations F in $\mathrm{Rep}(\mathfrak{g}, Q)$ and all $\nu \in \Lambda$,*

$$F \cdot (\Psi(\nu)) = \sum_\mu \Psi(\nu + \mu),$$

  where μ runs over all weights of F, counted with multiplicities.

In the notation of Definition 3.1, let $\mathrm{Coh}_\Lambda(\mathcal{K})$ denote the vector space of all $\mathcal{K}$-valued coherent families on Λ. It is a representation of W_Λ under the action

$$(w \cdot \Psi)(\nu) = \Psi(w^{-1}\nu), \qquad \text{for all } \ w \in W_\Lambda, \ \Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}), \ \nu \in \Lambda.$$

This is called a coherent continuation representation. When specifying a coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K})$, we will often explicitly describe $\mathcal{K}$ as a Grothendieck group, while the $\mathcal{R}(\mathfrak{g}, Q)$-module structure and the Λ-family $\{\mathcal{K}_\nu\}_{\nu\in\Lambda}$ are the ones which are clear from the context (which are often specified by the infinitesimal characters).

The assignment $\mathrm{Coh}_\Lambda(\mathcal{K})$ is functorial in $\mathcal{K}$ in the following sense: suppose that $\mathcal{K}'$ is another $\mathcal{R}(\mathfrak{g}, Q)$-module with a Λ-family $\{\mathcal{K}'_\nu\}_{\nu\in\Lambda}$ in it, and $\eta : \mathcal{K} \to \mathcal{K}'$ is a $\mathcal{R}(\mathfrak{g}, Q)$-homomorphism such that $\eta(\mathcal{K}_\nu) \subseteq \mathcal{K}'_\nu$ for all $\nu \in \Lambda$, then

$$\eta_* : \mathrm{Coh}_\Lambda(\mathcal{K}) \to \mathrm{Coh}_\Lambda(\mathcal{K}'), \quad \Psi \mapsto \eta \circ \Psi \tag{3.1}$$

is a well-defined W_Λ-equivariant linear map.

3.2. **Coherent continuation representations for highest weight modules.** Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, and let $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$ denote the category of finitely generated $\mathfrak{g}$-modules that are unions of finite-dimensional $\mathfrak{b}$-submodules. For each $\nu \in {}^a\mathfrak{h}^*$, let $\mathrm{Rep}_\nu(\mathfrak{g}, \mathfrak{b})$ denote the full subcategory of $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$ consisting of modules that have generalized infinitesimal character ν (by definition, a $\mathfrak{g}$-module has generalized infinitesimal character ν if every vector in it is annihilated by $(\ker(\chi_\nu))^k$ for some $k \in \mathbb{N}^+$). Write $\mathcal{K}(\mathfrak{g}, \mathfrak{b})$ for the Grothendieck group of $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$, and similarly define the Grothendieck groups $\mathcal{K}_\nu(\mathfrak{g}, \mathfrak{b})$. The Grothendieck group $\mathcal{K}(\mathfrak{g}, \mathfrak{b})$ is a $\mathcal{R}(\mathfrak{g}, Q)$-module under the tensor products. Using the Λ-family $\{\mathcal{K}_\nu(\mathfrak{g}, \mathfrak{b})\}_{\nu\in\Lambda}$, we form the coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$.

Write $\rho \in {}^a\mathfrak{h}^*$ for the half sum of positive roots. For each $\nu \in {}^a\mathfrak{h}^*$, define the Verma module

$$\mathrm{M}(\nu) := \mathrm{M}(\mathfrak{g}, \mathfrak{b}, \nu) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\nu-\rho},$$

where $\mathbb{C}_{\nu-\rho}$ is the one-dimensional ${}^a\mathfrak{h}$-module corresponding to the character $\nu - \rho \in {}^a\mathfrak{h}^*$, and every ${}^a\mathfrak{h}$-module is viewed an a $\mathfrak{b}$-module via the canonical map $\mathfrak{b} \to {}^a\mathfrak{h}$. Write $\mathrm{L}(\lambda) := \mathrm{L}(\mathfrak{g}, \mathfrak{b}, \lambda)$ for the unique irreducible quotient of $\mathrm{M}(\lambda)$.

Recall that an element $\nu \in {}^a\mathfrak{h}^*$ is said to be dominant if

$$\langle \nu, \check\alpha\rangle \notin -\mathbb{N}^+ \qquad \text{for all } \check\alpha \in \check\Delta^+, \tag{3.2}$$

and is said to be regular if

$$\langle \nu, \check\alpha\rangle \neq 0 \qquad \text{for all } \check\alpha \in \check\Delta.$$

For every $w \in W$, define a map

$$\begin{array}{rcl} \Psi_w : \Lambda & \to & \mathcal{K}(\mathfrak{g}, \mathfrak{b}), \\ \nu & \mapsto & \mathrm{M}(w\nu). \end{array}$$

Then $\Psi_w \in \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$, and

$$\{\Psi_w\}_{w\in W} \text{ is a basis of } \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b})). \tag{3.3}$$

There is a unique element $\overline{\Psi}_w \in \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ such that

$$\overline{\Psi}_w(\nu) = \mathrm{L}(w\nu) \quad \text{for all regular dominant element } \nu \in \Lambda. \tag{3.4}$$

Then $\{\overline{\Psi}_w \mid w \in W\}$ is also a basis of the coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$. See [Hum08, Section 7.10].

Let W act on $\mathcal{K}(\mathfrak{g}, \mathfrak{b})$ as $\mathcal{R}(\mathfrak{g}, Q)$-module automorphisms by

$$w \cdot (\mathrm{M}(\nu)) = \mathrm{M}(w\nu) \quad \text{for all } \nu \in {}^a\mathfrak{h}^*.$$

By the functority (3.1), this yields an action of W on $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ as automorphisms of W_Λ-representations. Note that the resulting action of $W \times W_\Lambda$ on $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ is explicitly given by

$$(w_1, w_2) \cdot \Psi_w = \Psi_{w_1 w w_2^{-1}} \quad \text{for all } w_1 \in W,\ w_2 \in W_\Lambda,\ w \in W. \tag{3.5}$$

Let $\Lambda^{\mathfrak{g}} \subseteq {}^a\mathfrak{h}^*$ denote the $Q^{\mathfrak{g}}$-coset containing Λ, and let $\Lambda_{\mathfrak{g}} \subseteq \Lambda$ be a $Q_{\mathfrak{g}}$-coset. Put

$$\Delta(\Lambda) := \{\alpha \in \Delta \mid \langle \check\alpha, \nu\rangle \in \mathbb{Z} \text{ for some (and all) } \nu \in \Lambda\}.$$

Here and henceforth, $\check{\alpha} \in {}^a\mathfrak{h}$ denotes the coroot corresponding to α. This is a root system with the corresponding coroots

$$\check{\Delta}(\Lambda) := \{\check{\alpha} \in \check{\Delta} \mid \langle \check{\alpha}, \nu \rangle \in \mathbb{Z} \text{ for some (and all) } \nu \in \Lambda\}.$$

Let $W(\Lambda) \subseteq W$ denote the Weyl group of the root system $\Delta(\Lambda)$, to be called the integral Weyl group (attached to Λ). Then

$$W(\Lambda_{\mathfrak{g}}) = W(\Lambda) = W(\Lambda^{\mathfrak{g}}) = W_{\Lambda_{\mathfrak{g}}} \subseteq W_\Lambda \subseteq W_{\Lambda^{\mathfrak{g}}}.$$

To understand the coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$, it suffices to consider the case when $\Lambda = \Lambda^{\mathfrak{g}}$ by the following lemma.

Lemma 3.2. *The restriction map*

$$\mathrm{Coh}_{\Lambda^{\mathfrak{g}}}(\mathcal{K}(\mathfrak{g}, \mathfrak{b})) \to \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$$

is a $W \times W_\Lambda$-equivariant linear isomorphism, and the restriction map

$$\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b})) \to \mathrm{Coh}_{\Lambda_{\mathfrak{g}}}(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$$

is a $W \times W(\Lambda)$-equivariant linear isomorphism.

Proof. This follows from (3.3). □

3.3. **Jantzen matrix.** Define the Jantzen matrix $\{a_\Lambda(w_1, w_2)\}_{w_1, w_2 \in W}$, where $a_\Lambda(w_1, w_2) \in \mathbb{Z}$ is specified by the following equation in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$:

$$\overline{\Psi}_{w_1} = \sum_{w_2 \in W} a_\Lambda(w_1, w_2) \Psi_{w_2},$$

for each $w_1 \in W$.

Lemma 3.3 (Bernstein-Gelfand-Gelfand)**.** *Let $w_1, w_2 \in W$. If $w_1 W(\Lambda) \neq w_2 W(\Lambda)$, then $a_\Lambda(w_1, w_2) = 0$.*

Proof. This is a consequence of the theorem of Bernstein-Gelfand-Gelfand on the composition factors of a Verma module. See [Hum08, Corollary 5.2]. □

Put

$$\Delta^+(\Lambda) := \Delta(\Lambda) \cap \Delta^+$$

and

$$D(\Lambda) := \left\{ w \in W \mid w(\Delta^+(\Lambda)) \subseteq \Delta^+ \right\}. \tag{3.6}$$

Then the group multiplication yields a bijective map

$$D(\Lambda) \times W(\Lambda) \to W.$$

Lemma 3.4 (Joseph)**.** *For all $w' \in D(\Lambda)$ and $w_1, w_2 \in W(\Lambda)$,*

$$a_\Lambda(w' w_1, w' w_2) = a_\Lambda(w_1, w_2), \tag{3.7}$$

and

$$a_\Lambda(w_1, w_2) = a_\Lambda(w_1^{-1}, w_2^{-1}). \tag{3.8}$$

Proof. The equations are proved by relating highest weight modules with principal series representations of complex semisimple groups. See [Jos79b, Theorem 4.12 and Theorem 5.4] and [Jos79, Corollary 3.3] for (3.7) and (3.8) respectively. □

Remark 3.5. The equation (3.7) is also a direct consequence of a theorem of Soergel (see [Soe90, Section 2.5, Theorem 11]). The matrix $\{ a_\Lambda(w_1,w_2) \}_{w_1,w_2\in W(\Lambda)}$ only depends on the set of simple reflections in the Weyl group $W(\Lambda)$. More precisely, suppose that $(\mathfrak{g}',\Lambda',W(\Lambda'))$ is a triple as $(\mathfrak{g},\Lambda,W(\Lambda))$, and $\eta : W(\Lambda)\to W(\Lambda')$ is a group isomorphism that restricts to a bijection between the sets of simple reflections, then

$$a_\Lambda(w_1,w_2) = a_{\Lambda'}(\eta(w_1),\eta(w_2))$$

for all $w_1,w_2\in W(\Lambda)$.

Recall that a polynomial function on ${}^a\mathfrak{h}^*$ or ${}^a\mathfrak{h}$ is said to be $W(\Lambda)$-harmonic if it is annihilated by all the $W(\Lambda)$-invariant constant coefficient differential operators without constant term.

For every $w\in W$, define a polynomial function $\tilde{p}_{\Lambda,w}$ on ${}^a\mathfrak{h}\times{}^a\mathfrak{h}^*$ by

$$\tilde{p}_{\Lambda,w}(x,\nu) := \sum_{w_1\in W} a_\Lambda(w,w_1)\cdot\langle x, w_1\nu\rangle^{m_{\Lambda,w}}, \quad \text{for all } \ x\in{}^a\mathfrak{h},\nu\in{}^a\mathfrak{h}^*, \tag{3.9}$$

where $m_{\Lambda,w}$ is the smallest non-negative integer (which always exists) that makes the right-hand side of (3.9) a nonzero polynomial function.

Lemma 3.6 (Joseph [Jos84, § 5.1])**.** *Let $w\in W$. There is a $W(w\Lambda)$-harmonic polynomial function $p'_{\Lambda,w}$ on ${}^a\mathfrak{h}$ and a $W(\Lambda)$-harmonic polynomial function $p_{\Lambda,w}$ on ${}^a\mathfrak{h}^*$ such that*

$$\tilde{p}_{\Lambda,w}(x,\nu) = p'_{\Lambda,w}(x)\cdot p_{\Lambda,w}(\nu), \quad \text{for all } x\in\mathfrak{h},\ \nu\in\mathfrak{h}^*.$$

The two harmonic polynomial functions are nonzero, of degree $m_{\Lambda,w}$ and are uniquely determined up to scalar multiplication.

Proof. This follows from [Jos80b, Lemma 2.3 (i), Lemma 2.5] and [BV83, Theorem 2.6 (b)]. □

3.4. **Left, right, and double cells.** We define a basal vector space to be a complex vector space V equipped with a basis $\mathcal{D}\subseteq V$, and call elements of $\mathcal{D}$ the basal elements in V. A subspace of a basal space V is called a basal subspace if it is spanned by a set of basal elements of V. Every basal subspace is obviously a basal space.

As an example, if $\mathcal{K}$ is the Grothendieck group of an abelian category in which all objects have finite length, then $\mathcal{K}$ is a basal vector space with the irreducible objects as the basal elements.

Definition 3.7. *Let E be a finite group. A basal representation of E is basal vector space carrying a representation of E. A basal subrepresentation of a basal representation V is a subrepresentation of V that is simultaneously a basal subspace.*

Let V be a basal representation of a finite group E, with basal elements $\mathcal{D}\subseteq V$. For each subset $\mathcal{C}\subseteq\mathcal{D}$, write $\langle\mathcal{C}\rangle$ for the smallest basal subrepresentation of V containing $\mathcal{C}$. For each $\phi\in\mathcal{D}$, write $\langle\phi\rangle := \langle\{\phi\}\rangle$ for simplicity. We define an equivalence relation $\approx$ on $\mathcal{D}$ by

$$\phi_1\approx\phi_2 \quad \text{if and only if} \quad \langle\phi_1\rangle = \langle\phi_2\rangle \qquad (\phi_1,\phi_2\in\mathcal{D}).$$

An equivalence class of the relation $\approx$ on the set $\mathcal{D}$ is called a cell in V.

We say that a subset $\mathcal{C}$ of $\mathcal{D}$ is *order closed* if $\langle\mathcal{C}\rangle$ is spanned by $\mathcal{C}$. In general, write $\overline{\mathcal{C}} := \langle\mathcal{C}\rangle\cap\mathcal{D}$, which is the smallest order closed subset of $\mathcal{D}$ containing $\mathcal{C}$.

When $\mathcal{C}$ is a cell in V, define the cell representation attached to $\mathcal{C}$ by

$$V(\mathcal{C}) := \langle\overline{\mathcal{C}}\rangle/\langle\overline{\mathcal{C}} \setminus \mathcal{C}\rangle.$$

Note that the set $\overline{\mathcal{C}} \setminus \mathcal{C}$ is order closed, and $\{\phi + \langle\overline{\mathcal{C}} \setminus \mathcal{C}\rangle\}_{\phi\in\mathcal{C}}$ is a basis of $V(\mathcal{C})$.

We view $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ as a basal representation of $W \times W_\Lambda$ with the basal elements $\{\overline{\Psi}_w \mid w \in W\}$. Write

$$\mathrm{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})) := \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})), \tag{3.10}$$

to be viewed as a basal representation of $W \times W(\Lambda)$. Likewise, write

$$\mathrm{Coh}^{L}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})) := \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})),$$

to be viewed as a basal representation of W, and write

$$\mathrm{Coh}^{R}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})) := \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})),$$

to be viewed as a basal representation of $W(\Lambda)$ (as a subgroup of W_Λ).

Cells in $\mathrm{Coh}^{L}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$, $\mathrm{Coh}^{R}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$, and $\mathrm{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ are respectively called *left, right, and double cells* in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$. It is clear that every left (right) cell is contained in a double cell.

For every set $\mathcal{C}$ of basal elements in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$, write $\langle\mathcal{C}\rangle_L$ for the smallest basal subrepresentation of $\mathrm{Coh}^{L}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ containing $\mathcal{C}$. Similarly define $\langle\mathcal{C}\rangle_R$ and $\langle\mathcal{C}\rangle_{LR}$.

3.5. **Left cells and classification of primitive ideals.** Let $\mathrm{Ann}(M) \subseteq \mathcal{U}(\mathfrak{g})$ denote the annihilator ideal of a $\mathcal{U}(\mathfrak{g})$-molude M. By Duflo [Duf77, Theorem 1], each primitive ideal in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ has the form $\mathrm{Ann}\,\overline{\Psi}_w(\nu)$ for a $w \in W$ and a dominant $\nu \in {}^a\mathfrak{h}^*$.

For each $w \in W$, let $p_{\Lambda,w}$ be the $W(\Lambda)$-harmonic polynomial function as in Lemma 3.6.

Lemma 3.8. *Let $w_1, w_2 \in W$. The following conditions are equivalent.*

(i) The basal elements $\overline{\Psi}_{w_1}$ and $\overline{\Psi}_{w_2}$ lie in a common left cell.
(ii) For some regular dominant $\nu \in \Lambda$, $\mathrm{Ann}\,\overline{\Psi}_{w_1}(\nu) = \mathrm{Ann}\,\overline{\Psi}_{w_1}(\nu)$.
(iii) For every regular dominant $\nu \in \Lambda$, $\mathrm{Ann}\,\overline{\Psi}_{w_1}(\nu) = \mathrm{Ann}\,\overline{\Psi}_{w_1}(\nu)$.
(iv) The equality $\mathbb{C} \cdot p_{\Lambda,w_1} = \mathbb{C} \cdot p_{\Lambda,w_2}$ holds.

Proof. The equivalence of (i) and (ii) is the equivalence of (a) and (b) in [BV83, Proposition 2.9]. The equivalence of (ii) and (iii) follows from the translation principle, see [Vog79b, Lemma 2.7]. The equivalence of (ii) and (iv) is [Jos80b, Theorem 5.1 and 5.5] when $w_1W(\Lambda) = w_2W(\Lambda)$. The general case can be deduced by relating the problem to the setting of complex semisimple groups using [Jos79b, Theorem 4.12] and [Duf77, Proposition 3.7]. □

Using Lemma 3.8, we obviously attach a polynomial function $p_{\mathcal{C}^L}$ on ${}^a\mathfrak{h}^*$ to every left cell $\mathcal{C}^L$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$, which is uniquely determined up to scalar multiplication.

Lemma 3.9. *For every $w \in W$ and every simple reflection $s \in W(\Lambda)$, the following statements are equivalent.*

(i) $s \cdot p_{\Lambda,w} = -p_{\Lambda,w}$.
(ii) $s \cdot \overline{\Psi}_w = -\overline{\Psi}_w$.
(iii) $w(\alpha_s) \in \Delta^+$, where $\alpha_s \in \Delta^+(\Lambda)$ is the simple root corresponding to s.

Proof. The equivalence between (i) and (ii) is due to the fact that $p_{\Lambda,w}$ is essentially the Bernstein degree polynomial, see [Jos80b, Theorem 4.10] and [Vog78, p85]. The equivalence between (ii) and (iii) is due to Jantzen, see [BV83, Proposition 2.20]. $\square$

For every basal element $\Psi = \overline{\Psi}_w$ ($w \in W$) in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$, define the Borho-Jantzen-Duflo τ-invariant of Ψ to be the set of simple reflections satisfying the equivalent conditions in Lemma 3.9:

$$\tau_\Psi := \tau_{\Lambda,w} := \{s \in W(\Lambda) \text{ is a simple reflection } \mid s \cdot p_{\Lambda,w} = -p_{\Lambda,w}\}.$$

This only depends on the left cell containing Ψ, and thus the τ-invariant $\tau_{\mathcal{C}^L}$ is obviously defined for each left cell $\mathcal{C}^L$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$.

For each $\mu \in {}^a\mathfrak{h}^*$, write W_μ for the stabilizer of μ in W. The following is a reformulation of Jantzen's result on the translation to wall, see [Hum08, Theorem 7.9] and also [Vog81, Theorem 7.3.22].

Lemma 3.10 (Jantzen)**.** *Let $w \in W$ and let $\nu \in \Lambda$ be a dominant element. Then*

$$\overline{\Psi}_w(\nu) = \begin{cases} \mathrm{L}(w\nu), & \text{if } \tau_{\Lambda,w} \cap W_\nu = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.10 and the translation principle (see [Vog79b, Lemma 2.7]) imply the following classification of primitive ideals at a general infinitesimal character.

Proposition 3.11 (Joseph)**.** *Let $\nu \in \Lambda$ be a dominant element. Then the map*

$$\begin{array}{rcl} \{\, w \in W \mid \tau_{\Lambda,w} \cap W_\nu = \emptyset \,\} & \to & \{\, \textit{primitive ideal of } \mathcal{U}(\mathfrak{g}) \textit{ of infinitesimal character } \nu \,\} \\ w & \mapsto & \mathrm{Ann}(\mathrm{L}(w\nu)) \end{array}$$

is surjective. Furthermore for all w_1, w_2 in the domain of this map,

$$\mathrm{Ann}(\mathrm{L}(w_1\nu)) = \mathrm{Ann}(\mathrm{L}(w_2\nu)) \quad \textit{if and only if} \quad \mathbb{C} \cdot p_{\Lambda,w_1} = \mathbb{C} \cdot p_{\Lambda,w_2}.$$

For every left cell $\mathcal{C}^L$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ and every dominant element $\nu \in \Lambda$, we define

$$I_{\nu,\mathcal{C}^L} := \mathrm{Ann}(\Psi(\nu)) \tag{3.11}$$

where $\Psi \in \mathcal{C}^L$. Note that this definition is independent of the choice of Ψ, and $I_{\nu,\mathcal{C}^L}$ is a primitive ideal if $\tau_{\mathcal{C}^L} \cap W_\nu = \emptyset$ and is $\mathcal{U}(\mathfrak{g})$ otherwise.

By Proposition 3.11, for every dominant element $\nu \in \Lambda$, we have an obvious bijection

$$\begin{array}{ll} & \{\, \mathcal{C}^L \mid \mathcal{C}^L \text{ is a left cell in } \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b})) \text{ such that } \tau_{\mathcal{C}^L} \cap W_\nu = \emptyset \,\} \\ \xrightarrow{\mathcal{C}^L \mapsto I_{\nu,\mathcal{C}^L}} & \{\text{primitive ideal of } \mathcal{U}(\mathfrak{g}) \text{ of infinitesimal character } \nu\}. \end{array} \tag{3.12}$$

For every primitive ideal I of infinitesimal character ν, write $\mathcal{C}^L_{\nu,I}$ for the left cell that corresponds to it under this bijection. In particular when $\nu \in \Lambda$ is regular dominant, for each basal element Ψ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$, the left cell corresponding to the primitive ideal $\mathrm{Ann}\,\Psi(\nu)$ is just the left cell containing Ψ.

Lemma 3.12. *Let $\nu \in \Lambda$ be a dominant element. Then for all primitive ideals I_1 and I_2 of $\mathcal{U}(\mathfrak{g})$ of infinitesimal character ν,*

$$I_1 \subseteq I_2 \quad \textit{if and only if} \quad \langle \mathcal{C}^L_{\nu,I_1} \rangle_L \supseteq \langle \mathcal{C}^L_{\nu,I_2} \rangle_L.$$

Proof. This follows from Proposition 3.11 and [BV83, Proposition 2.9]. $\square$

3.6. **Double cells and special representations.** For every $\sigma \in \mathrm{Irr}(W)$, its fake degree is defined to be

$$a(\sigma) := \min\{k \in \mathbb{N} \mid \sigma \text{ occurs in the } k\text{-th symmetric power } \mathrm{S}^k({}^a\mathfrak{h})\}. \tag{3.13}$$

This is well-defined since every $\sigma \in \mathrm{Irr}(W)$ occurs in the symmetric algebra $\mathrm{S}({}^a\mathfrak{h})$. The representation σ is said to be *univalent* if it occurs in $\mathrm{S}^{a(\sigma)}({}^a\mathfrak{h}_{\mathrm{s}})$ with multiplicity one, where ${}^a\mathfrak{h}_{\mathrm{s}} := \mathrm{Span}(\check{\Delta})$ denotes the span of the coroots.

Recall Lusztig's notion of a special representation of a Weyl group ([Lus79]). An irreducible representation of W is said to be Springer if it corresponds to the trivial local system on a nilpotent orbit in $\mathfrak{g}^*$ via the Springer correspondence. Note that every special irreducible representation is Springer, and every Springer representation is univalent [BM81].

We have a decomposition

$${}^a\mathfrak{h} = (\Delta(\Lambda))^{\perp} \oplus \mathrm{Span}(\check{\Delta}(\Lambda))$$

where

$$(\Delta(\Lambda))^{\perp} := \{x \in {}^a\mathfrak{h} \mid \langle x, \alpha\rangle = 0 \text{ for all } \alpha \in \Delta(\Lambda)\}.$$

For every univalent irreducible representation σ_0 of $W(\Lambda)$, whenever it is convenient, we view it as a subrepresentation of $\mathrm{S}^{a(\sigma_0)}({}^a\mathfrak{h})$ via the inclusions

$$\sigma_0 = \mathbb{C} \otimes \sigma_0 \subseteq \mathrm{S}^0((\Delta(\Lambda))^{\perp}) \otimes \mathrm{S}^{a(\sigma_0)}(\mathrm{Span}(\check{\Delta}(\Lambda))) \subseteq \mathrm{S}^{a(\sigma_0)}({}^a\mathfrak{h})$$

By the work of Macdonald, Lusztig, and Spaltenstein ([Car93, Chapter 11]), the W-subrepresentation of $\mathrm{S}^{a(\sigma_0)}({}^a\mathfrak{h})$ generated by σ_0 is irreducible and univalent, with the same fake degree as that of σ_0. This irreducible representation of W is called the j-induction of σ_0, to be denoted by $j_{W(\Lambda)}^{W}(\sigma_0)$.

If σ_0 is special, then the j-induction $j_{W(\Lambda)}^{W}\sigma_0$ is Springer. Write $\mathcal{O}_{\sigma_0} \in \overline{\mathrm{Nil}}(\mathfrak{g}^*)$ for the nilpotent orbit corresponding to $j_{W(\Lambda)}^{W}\sigma_0$. Then

$$\dim \mathcal{O}_{\sigma_0} = 2 \cdot (\sharp(\Delta^+) - a(\sigma_0)).$$

See [Hot84, Theorem 3].

Suppose that $\mathcal{C}$ is a double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$. Put $m_{\mathcal{C}} := m_{\Lambda,w}$, where $w \in W$ is an element such that $\overline{\Psi}_w \in \mathcal{C}$. Note that $m_{\mathcal{C}}$ is independent of the choice of w (see [BV83, Corollary 2.15 and 2.16]). For every $\Psi = \sum_{w\in W} a_w \Psi_w \in \langle \mathcal{C}\rangle_{LR}$, define a polynomial function $\tilde{p}_\Psi$ on ${}^a\mathfrak{h} \times {}^a\mathfrak{h}^*$ by

$$\tilde{p}_\Psi(x, \nu) := \sum_{w\in W} a_w \cdot \langle x, w\nu\rangle^{m_{\mathcal{C}}}, \quad \text{for all } \ x \in {}^a\mathfrak{h}, \nu \in {}^a\mathfrak{h}^*. \tag{3.14}$$

Then the linear map

$$\langle \mathcal{C}\rangle_{LR} \to \mathrm{S}({}^a\mathfrak{h}^*) \otimes \mathrm{S}({}^a\mathfrak{h}), \quad \Psi \mapsto \tilde{p}_\Psi \quad (\mathrm{S} \text{ indicates the symmetric algebra}) \tag{3.15}$$

is $W \times W(\Lambda)$-equivariant and descends to a linear map

$$\mathrm{Coh}_\Lambda^{LR}(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C}) \to \mathrm{S}({}^a\mathfrak{h}^*) \otimes \mathrm{S}({}^a\mathfrak{h}). \tag{3.16}$$

See [Jos80b, § 3] and [BV83, Corollary 2.15].

We define a preorder $\underset{LR}{\leq}$ on $\mathrm{Irr}(W(\Lambda))$ by $\sigma_1 \underset{LR}{\leq} \sigma_2$ if and only if there exist double cells $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ such that $\langle \mathcal{C}_1\rangle_{LR} \supseteq \langle \mathcal{C}_2\rangle_{LR}$, and σ_i occurs in $\mathrm{Coh}_\Lambda^{LR}(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C}_i)$ ($i =$

1, 2). This preorder only depends on $W(\Lambda)$ as a Coxeter group (see [BV83, Proposition 2.28]). Define an equivalence relation $\approx$ on $\mathrm{Irr}(W(\Lambda))$ by

$$\sigma_1 \approx \sigma_2 \quad \text{if and only if} \quad \sigma_1 \underset{LR}{\leq} \sigma_2 \text{ and } \sigma_2 \underset{LR}{\leq} \sigma_1.$$

An equivalence class of this equivalence relation is called a double cell in $\mathrm{Irr}(W(\Lambda))$. Note that this definition of double cell in $\mathrm{Irr}(W(\Lambda))$ coincides with that of Lusztig in [Lus82] (also [Car93, Section 13.2]).

Let $\mathrm{Irr}^{\mathrm{sp}}(W(\Lambda))$ denote the subset of $\mathrm{Irr}(W(\Lambda))$ consisting of the special irreducible representations. Recall the definition of $p_{\mathcal{C}^L}$ after Lemma 3.8.

We recollect some results due to Barbasch-Vogan, Lusztig and Joseph in the following proposition. See [BV83, Section 2].

Proposition 3.13. *Let $\mathcal{C}$ be a double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$.*

(a) There is a unique $\sigma_{\mathcal{C}} \in \mathrm{Irr}^{\mathrm{sp}}(W(\Lambda))$ such that the image of (3.16) equals $\left(j^W_{W(\Lambda)}\sigma_{\mathcal{C}}\right) \otimes \sigma_{\mathcal{C}}$. Moreover, $a(\sigma_{\mathcal{C}}) = m_{\mathcal{C}}$ and for every $\sigma' \in \mathrm{Irr}(W(\Lambda))$ that occurs in $\langle\mathcal{C}\rangle_{LR}$, either $\sigma' = \sigma_{\mathcal{C}}$ or $a(\sigma') > m_{\mathcal{C}}$.
(b) The representation $\sigma_{\mathcal{C}}$ is the unique element of $\mathrm{Irr}^{\mathrm{sp}}(W(\Lambda))$ that occurs in $\mathrm{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C})$. Moreover, the map

$$\{\textit{double cell in } \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))\} \to \mathrm{Irr}^{\mathrm{sp}}(W(\Lambda)), \quad \mathcal{C} \mapsto \sigma_{\mathcal{C}}$$

is bijective.
(c) As a representation of $W \times W(\Lambda)$,

$$\mathrm{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C}) \cong \bigoplus_{\substack{\sigma \in \mathrm{Irr}(W(\Lambda)),\\ \sigma \approx \sigma_{\mathcal{C}}}} \left(\mathrm{Ind}^W_{W(\Lambda)} \sigma\right) \otimes \sigma.$$

From now on, we let $\sigma_{\mathcal{C}}$ denote the unique special irreducible representation attached to a double cell $\mathcal{C}$ as in Proposition 3.13. We also define the Goldie rank representation of a primitive ideal I of infinititemal character ν, where $\nu \in \Lambda$ is dominant, to be

$$\sigma_{\nu,I} := \sigma_{\mathcal{C}^{LR}_{\nu,I}} \in \mathrm{Irr}^{\mathrm{sp}}(W(\Lambda)). \tag{3.17}$$

Here $\mathcal{C}^{LR}_{\nu,I}$ is the double cell containing the left cell $\mathcal{C}^L_{\nu,I}$.

Lemma 3.14 ([Jos80b, Theorem 5.5])**.** *Let $\mathcal{C}$ be a double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$. Then the family*

$$\{p_{\mathcal{C}^L}\}_{\mathcal{C}^L \textit{ is a left cell contained in } \mathcal{C}}$$

is linearly independent in $\mathrm{S}^{m_{\mathcal{C}}}({}^a\mathfrak{h})$ and spans a $W(\Lambda)$-subrepresentation that is isomorphic to $\sigma_{\mathcal{C}}$.

Lemma 3.15 ([BV83, Proposition 2.25])**.** *For all $\sigma_1, \sigma_2 \in \mathrm{Irr}(W(\Lambda))$, $\sigma_1 \underset{LR}{\leq} \sigma_2$ if and only if $\sigma_2 \otimes \mathrm{sgn} \underset{LR}{\leq} \sigma_1 \otimes \mathrm{sgn}$.*

3.7. **Associated varieties.** For every primitive ideal I of $\mathcal{U}(\mathfrak{g})$, denote by $\mathrm{AV}(I)$ its associated variety.

Recall the Goldie rank representation $\sigma_{\nu,I}$ attached to a primitive ideal I in (3.17). The following result is due to Joseph.

Proposition 3.16 ([Jos85, Proposition 2.10]). *Let $\nu \in \Lambda$ be a dominant element and let I be a primitive ideal of $\mathcal{U}(\mathfrak{g})$ of infinitesimal character ν. Then the associated variety $\mathrm{AV}(I)$ equals the Zariski closure $\overline{\mathcal{O}_{\sigma_{\nu,I}}}$ of $\mathcal{O}_{\sigma_{\nu,I}}$ in $\mathfrak{g}^*$.*

Let $\mathsf{S} \subseteq \mathrm{Nil}(\mathfrak{g}^*)$ be an $\mathrm{Ad}(\mathfrak{g})$-stable Zariski closed subset. Let $\mathrm{Rep}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b})$ denote the full subcategory of $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b})$ of the modules M such that the associated variety of $\mathrm{Ann}(M)$ is contained in S. We obviously have the coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}))$, where $\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b})$ denote the Grothendieck group of $\mathrm{Rep}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b})$.

For every $w \in W$, put $\mathcal{O}_{\Lambda,w} := \mathcal{O}_{\sigma_{\mathcal{C}}}$, where $\mathcal{C}$ is the double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ containing $\overline{\Psi}_w$.

Lemma 3.17. *Let $w \in W$. The coherent family $\overline{\Psi}_w$ belongs to $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}))$ if and only if $\mathcal{O}_{\Lambda,w} \subseteq \mathsf{S}$.*

Proof. This follows from Lemma 3.10, Proposition 3.11, Proposition 3.16 and [Vog81, Proposition 7.3.11] (for the translation outside the dominant cone). □

Lemma 3.18. *Let $w_1, w_2 \in W$. If $\langle \overline{\Psi}_{w_1} \rangle_L \supsetneq \langle \overline{\Psi}_{w_2} \rangle_L$, then $\overline{\mathcal{O}_{\Lambda,w_1}} \supsetneq \overline{\mathcal{O}_{\Lambda,w_2}}$.*

Proof. Pick a regular dominant element $\nu \in \Lambda$. Put $I_i := \mathrm{Ann}(\overline{\Psi}_{w_i}(\nu))$ $(i = 1, 2)$. Lemma 3.12 implies that $I_1 \subsetneq I_2$ and hence

$$\mathrm{AV}(I_1) \supsetneq \mathrm{AV}(I_2).$$

On the other hand, Proposition 3.16 implies that $\mathrm{AV}(I_i) = \overline{\mathcal{O}_{\Lambda,w_i}}$ $(i = 1, 2)$. Hence the lemma follows. □

Lemma 3.19. *The space $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}))$ is a $W \times W_\Lambda$-submodule of $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ spanned by the basal elements*

$$\{\overline{\Psi}_w \mid w \in W, \mathcal{O}_{\Lambda,w} \subseteq \mathsf{S}\}. \tag{3.18}$$

Proof. Lemma 3.17 implies that $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}))$ is spanned by the set (3.18). The space $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}))$ is clearly W_Λ-stable, and Lemma 3.18 implies that it is also W-stable. This proves the proposition. □

Define

$$\mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W(\Lambda)) := \{\, \sigma_0 \in \mathrm{Irr}^{\mathrm{sp}}(W(\Lambda)) \mid \mathcal{O}_{\sigma_0} \subseteq \mathsf{S} \,\}$$

and

$$\mathrm{Irr}_{\mathsf{S}}(W(\Lambda)) := \{\, \sigma \in \mathrm{Irr}(W(\Lambda)) \mid \text{there is a } \sigma_0 \in \mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W(\Lambda)) \text{ such that } \sigma \approx \sigma_0 \,\}.$$

By Lemma 3.19, $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}))$ is a basal subrepresentation of $\mathrm{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$.

Proposition 3.20. *As a representation of $W \times W(\Lambda)$,*

$$\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b})) \cong \bigoplus_{\sigma \in \mathrm{Irr}_{\mathsf{S}}(W(\Lambda))} \left(\mathrm{Ind}^W_{W(\Lambda)} \sigma\right) \otimes \sigma.$$

Proof. Write $\mathcal{D}_0$ for the set of basal elements in $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b}))$. Then it is elementary to see that we may choose a filtration

$$\mathcal{D}_0 \supset \mathcal{D}_1 \supset \cdots \supset \mathcal{D}_k = \emptyset \qquad (k \in \mathbb{N})$$

such that

- $\langle \mathcal{D}_i \rangle_{LR}$ is spanned by $\mathcal{D}_i$ $(i = 0, 1, 2, \cdots, k)$,

- $\mathcal{C}_i := \mathcal{D}_i \setminus \mathcal{D}_{i+1}$ is a double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ for all $i = 0, 1, \ldots, k-1$, and
- for all $i_1, i_2 \in \{0, 1, \ldots, k-1\}$, $\langle \mathcal{C}_{i_1} \rangle_{LR} \subseteq \langle \mathcal{C}_{i_2} \rangle_{LR}$ implies that $i_1 \geq i_2$.

Then Proposition 3.13 and Lemma 3.17 imply that

$$\{\sigma_{\mathcal{C}_0}, \sigma_{\mathcal{C}_1}, \ldots, \sigma_{\mathcal{C}_{k-1}}\} = \mathrm{Irr}^{\mathrm{sp}}_{\mathsf{S}}(W(\Lambda)).$$

Thus we have that

$$\begin{aligned} & \mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(\mathfrak{g}, \mathfrak{b})) \\ \cong\; & \bigoplus_{i=0}^{k-1} \langle \mathcal{D}_i \rangle_{LR} / \langle \mathcal{D}_{i+1} \rangle_{LR} \\ \cong\; & \bigoplus_{i=0}^{k-1} \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C}_i) \\ \cong\; & \bigoplus_{\sigma \in \mathrm{Irr}_{\mathsf{S}}(W(\Lambda))} \left(\mathrm{Ind}^W_{W(\Lambda)} \sigma\right) \otimes \sigma \qquad \text{(by Proposition 3.13 (c)).} \end{aligned}$$

$\square$

Let $\nu \in \Lambda$ be a dominant element. Recall that for a primitive ideal I of infinitesimal character ν, we have the corresponding left cell $\mathcal{C}^L_{\nu,I}$ (Section 3.5). In the rest of the section, we focus on left cells which come from maximal primitive ideals.

Write $\mathcal{O}_\nu \in \overline{\mathrm{Nil}}(\mathfrak{g}^*)$ for the nilpotent orbit whose Zariski closure $\overline{\mathcal{O}_\nu} \subseteq \mathfrak{g}^*$ equals the associated variety of the maximal ideal I_ν.

Recall the J-induction defined in [Lus82, § 11] (see also [Lus84, (4.1.7)]).

Proposition 3.21 (Barbasch-Vogan). *As W-representations,*

$$\mathrm{Coh}^L_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C}^L_{\nu,I_\nu}) \cong \mathrm{Ind}^W_{W(\Lambda)} \left(\left(J^{W(\Lambda)}_{W_\nu} \mathrm{sgn}\right) \otimes \mathrm{sgn}\right)$$

and as $W(\Lambda)$-representations,

$$\left(\left(J^{W(\Lambda)}_{W_\nu} \mathrm{sgn}\right) \otimes \mathrm{sgn}\right) \cong \bigoplus_{\sigma \in \mathrm{Irr}_{\overline{\mathcal{O}_\nu}}(W(\Lambda)),\, [1_{W_\nu} : \sigma] \neq 0} \sigma. \tag{3.19}$$

Moreover,

$$\sigma_{\nu,I_\nu} \cong \left(j^{W(\Lambda)}_{W_\nu} \mathrm{sgn}\right) \otimes \mathrm{sgn},$$

and σ_{ν,I_ν} is the unique special irreducible representation of $W(\Lambda)$ that occurs in (3.19).

Proof. In view of Proposition 3.13 (b) and Lemma 3.18, this follows by the same proof of [BV85, 5.30]. $\square$

The isomorphism (3.19) and explicit formulas of the J-induction (see [Lus84, §4.4-4.13]) imply that

$$[1_{W_\nu} : \sigma] \leq 1 \tag{3.20}$$

for all $\nu \in \Lambda$ and all $\sigma \in \mathrm{Irr}_{\overline{\mathcal{O}_\nu}}(W(\Lambda))$. We call the set

$$ {}^L\mathscr{C}_\nu := \left\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \left(J^{W(\Lambda)}_{W_\nu} \mathrm{sgn}\right) \otimes \mathrm{sgn}\right\}. \tag{3.21}$$

the *Lusztig left cell* attached to $\nu \in \Lambda$.

Lemma 3.22. *Let $\nu \in \Lambda$ be a dominant element and let I be a primitive ideal of $\mathcal{U}(\mathfrak{g})$ of infinitesimal character ν. Then $I = I_\nu$ if and only if $\sigma_{\nu,I} \cong \left(j_{W_\nu}^{W(\Lambda)} \operatorname{sgn}\right) \otimes \operatorname{sgn}$.*

Proof. The "only if" part follows from Proposition 3.21. For the proof of the "if" part, note that $I \subseteq I_\nu$. Then $\sigma_{\nu,I} \cong \sigma_{\nu,I_\nu}$ implies that I and I_ν have the same associated variety by Proposition 3.16. Thus $I = I_\nu$ by the maximality of I_ν and the proof is complete. □

3.8. **τ-invariants and duals of double cells.** Let $\Pi(\Lambda)$ denote the set of simple reflections in $W(\Lambda)$. For each subset $S \subseteq \Pi(\Lambda)$, define an element

$$e_S := \frac{1}{\sharp(W_{\Pi(\Lambda)\setminus S}) \cdot \sharp(W_S)} \sum_{g \in W_{\Pi(\Lambda)\setminus S}} \sum_{h \in W_S} \operatorname{sgn}(h)\, gh$$

in the group algebra $\mathbb{C}[W(\Lambda)]$, where W_S and $W_{\Pi(\Lambda)\setminus S}$ are the subgroups of $W(\Lambda)$ generated by S and $\Pi(\Lambda) \setminus S$ respectively.

Following [FJMN21], for every finite-dimensional representation σ of $W(\Lambda)$, define its τ-invariant to be the set

$$\tau_\sigma := \{S \subseteq \Pi(\Lambda) \mid \text{the } e_S \text{ action on } \sigma \text{ is nonzero}\}.$$

For every double cell $\mathcal{C}$ in $\operatorname{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$, we define its τ-invariant to be the set

$$\tau_{\mathcal{C}} := \{\tau_\Psi \mid \Psi \in \mathcal{C}\}.$$

Recall that we have a representation $\sigma_{\mathcal{C}} \in \operatorname{Irr}^{\mathrm{sp}}(W(\Lambda))$ attached to $\mathcal{C}$.

Lemma 3.23. *For every double cell $\mathcal{C}$ in $\operatorname{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$, $\tau_{\mathcal{C}} = \tau_{\sigma_{\mathcal{C}}}$.*

Proof. Let M denote the cell representation $\operatorname{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C})$ to be viewed as a $W(\Lambda)$-representation. By [FJMN21, Theorem 2.10], we know that $\tau_{\mathcal{C}} = \tau_M$. Since $\sigma_{\mathcal{C}}$ occurs in M, we have that

$$\tau_{\sigma_{\mathcal{C}}} \subseteq \tau_M = \tau_{\mathcal{C}}.$$

Now it remains to show that $\tau_{\mathcal{C}} \subseteq \tau_{\sigma_{\mathcal{C}}}$. Let $\Psi \in \mathcal{C}$ and put $S := \tau_\Psi$. Write

$$e_S = Q(S') \cdot R(S),$$

where $S' := \Pi(\Lambda) \setminus S$,

$$R(S) := \frac{1}{\sharp(W_S)} \sum_{h \in W_S} \operatorname{sgn}(h)\, h$$

and

$$Q(S') := \frac{1}{\sharp(W_{S'})} \sum_{g \in W_{S'}} g.$$

For each $\Phi \in \mathcal{C}$, write $[\Phi] \in M$ for the class represented by Φ. Then

$$R(S) \cdot [\Psi] = [\Psi],$$

and [FJMN21, Proof of Proposition 2.6] implies that

$$Q(S') \cdot [\Psi] = [\Psi] + \sum_{\Psi' \in \mathcal{C}, \tau_{\Psi'} \not\subseteq S} c_{\Psi'} [\Psi'],$$

where $c_{\Psi'} \in \mathbb{C}$. Thus

$$e_S \cdot [\Psi] = [\Psi] + \sum_{\Psi' \in \mathcal{C}, \tau_{\Psi'} \not\subseteq S} c_{\Psi'} [\Psi']. \tag{3.22}$$

Through Proposition 3.13, we view $\sigma_{\mathcal{C}}$ as a subspace of $\mathrm{S}^{mc}({}^a\mathfrak{h})$. The same proposition also implies that there is a $W(\Lambda)$-homomorphism

$$\phi : M \to \sigma_{\mathcal{C}}$$

such that $\phi([\Phi])$ is a nonzero scalar multiple of $p_{\mathcal{C}^L_\Phi}$ for all $\Phi \in \mathcal{C}$, where $\mathcal{C}^L_\Phi$ is the left cell containing Φ. Applying ϕ to the quality (3.22), we obtain that

$$e_S \cdot p_{\mathcal{C}^L_\Psi} = p_{\mathcal{C}^L_\Psi} + \sum_{\Psi' \in \mathcal{C}, \tau_{\Psi'} \not\supseteq S} c'_{\Psi'} p_{\mathcal{C}^L_{\Psi'}},$$

where $c'_{\Psi'} \in \mathbb{C}$. This is nonzero by Lemma 3.8 and Lemma 3.14, and by noting that $\mathcal{C}^L_\Psi \neq \mathcal{C}^L_{\Psi'}$ for all $\Psi' \in \mathcal{C}$ with $\tau_{\Psi'} \not\supseteq S$. Therefore $S \in \tau_{\sigma_{\mathcal{C}}}$ and the proof is complete. □

Proposition 3.24. *Let $\sigma_1, \sigma_2 \in \mathrm{Irr}^{\mathrm{sp}}(W(\Lambda))$. If $\tau_{\sigma_1} = \tau_{\sigma_2}$, then $\sigma_1 = \sigma_2$.*

Proof. Clearly it suffices to consider the case when the root system $\Delta(\Lambda)$ is irreducible. The lemma follows from Lemma 3.23 and the results of [FJMN21] (Theorem 2.12 for classical groups and the discussion in § 6 for exceptional groups).

□

Let $(\check{\mathfrak{g}}, \check{\mathfrak{b}}, \check{\Lambda})$ be a triple with the same property as the triple $(\mathfrak{g}, \mathfrak{b}, \Lambda)$. Let $\check{W}$ denote the Weyl group of $\check{\mathfrak{g}}$. As before we have the basal space $\mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{\mathfrak{g}}, \check{\mathfrak{b}}))$ carrying a representation of $\check{W} \times \check{W}(\check{\Lambda})$, where $\check{W}(\check{\Lambda}) \subseteq \check{W}$ is the integral Weyl group attached to $\check{\Lambda}$.

Suppose we are given an identification $W(\Lambda) = \check{W}(\check{\Lambda})$ of Coxeter groups, namely a group isomorphism that sends simple reflections to simple reflections. We say that two subsets τ_1 and τ_2 of the power set of $\Pi(\Lambda)$ are dual to each other if

$$\tau_2 = \{\Pi(\Lambda) \setminus S \mid S \in \tau_1\}.$$

Proposition 3.25. *We are given an identification $W(\Lambda) = \check{W}(\check{\Lambda})$ of Coxeter groups. For every double cell $\mathcal{C}$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$, there is a unique double cell $\check{\mathcal{C}}$ in $\mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{\mathfrak{g}}, \check{\mathfrak{b}}))$ whose τ-invariant is dual to that of $\mathcal{C}$. Moreover, the map*

$$\begin{aligned} &\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in } \mathrm{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))(\mathcal{C})\} \\ \xrightarrow{\sigma \mapsto \sigma \otimes \mathrm{sgn}} \; &\{\check{\sigma} \in \mathrm{Irr}(\check{W}(\check{\Lambda})) \mid \check{\sigma} \text{ occurs in } \mathrm{Coh}^{LR}_{\check{\Lambda}}(\mathcal{K}(\check{\mathfrak{g}}, \check{\mathfrak{b}}))(\check{\mathcal{C}})\} \end{aligned}$$

is well-defined and bijective.

Proof. Let $\mathscr{C} := \{\, w \in W(\Lambda) \mid \overline{\Psi}_w \in \mathcal{C} \,\}$. Then $\mathscr{C}$ is a double cell of $W(\Lambda)$ in the sense of Kazhdan-Lusztig ([KL79]). By [BV83, Corollary 2.23], $\mathcal{C}$ has the form

$$\{\, \overline{\Psi}_{w'w} \mid w' \in D(\Lambda), w \in \mathscr{C} \,\} \qquad \text{(see (3.6) for the definition of } D(\Lambda)).$$

Let w_0 be the element in $W(\Lambda)$ such that $w_0(\Delta^+(\Lambda)) \cap \Delta^+(\Lambda) = \emptyset$. Similar to $D(\Lambda)$, we have a subgroup $D(\check{\Lambda})$ of $\check{W}$.

By [BV83, Corollary 2.24],

$$\check{\mathcal{C}} := \{\, \check{\overline{\Psi}}_{w'w_0w} \mid w' \in D(\check{\Lambda}), w \in \mathscr{C} \,\}$$

is a double cell in $\mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{\mathfrak{g}}, \check{\mathfrak{b}}, \check{\Lambda}))$, where $\check{\overline{\Psi}}_{w'w_0w}$ is a basal element defined as in (3.4). By Lemma 3.9, we have that

$$\tau_{\overline{\Psi}_{w'w}} = \tau_{\overline{\Psi}_w} = \Pi(\Lambda) \setminus \tau_{\check{\overline{\Psi}}_{w_0w}} = \Pi(\Lambda) \setminus \tau_{\check{\overline{\Psi}}_{w''w_0w}}$$

for each $w' \in D(\Lambda)$, $w'' \in D(\check{\Lambda})$ and $w \in W(\Lambda)$. This implies that $\tau_{\mathcal{C}}$ is dual to $\tau_{\check{\mathcal{C}}}$.

The uniqueness of $\check{\mathcal{C}}^{LR}$ follows from Proposition 3.24 and the bijection between double cells and special representations in Proposition 3.13 (b). See also [FJMN21, § 6].

The assertion on the operation of tensoring with sgn is in [BV83, Proposition 2.25]. $\square$

The double cell $\check{\mathcal{C}}$ in Proposition 3.25 will be referred as the dual (in $\mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{\mathfrak{g}}, \check{\mathfrak{b}}))$) of $\mathcal{C}$.

4. Generalities on coherent families of Casselman-Wallach representations

Let G be a real reductive group in Harish-Chandra's class (which may be linear or nonlinear). Suppose that $\mathfrak{g}$ is the complexified Lie algebra of G. Let $\mathrm{Rep}(G)$ denote the category of Casselman-Wallach representations of G.

The complex associated variety of a representation π in $\mathrm{Rep}(G)$, denoted by $\mathrm{AV}_{\mathbb{C}}(\pi)$, is defined to be the associated variety of $\mathrm{Ann}(\pi) \subseteq \mathcal{U}(\mathfrak{g})$. Let S be an $\mathrm{Ad}(\mathfrak{g})$-stable Zariski closed subset of $\mathrm{Nil}(\mathfrak{g}^*)$. Let $\mathrm{Rep}_{\mathsf{S}}(G)$ denote the category of Casselman-Wallach representations of G whose complex associated variety is contained in S.

For every $\mu \in {}^a\mathfrak{h}^*$, respectively let $\mathrm{Rep}_\mu(G)$ and $\mathrm{Rep}_{\mu,\mathsf{S}}$ denote the full subcategories of $\mathrm{Rep}(G)$ and $\mathrm{Rep}_{\mathsf{S}}(G)$ consisting of the Casselman-Wallach representations of generalized infinitesimal character μ. Denote by $\mathcal{K}(G)$, $\mathcal{K}_\mu(G)$ and $\mathcal{K}_{\mu,\mathsf{S}}(G)$ for the Grothedieck groups of $\mathrm{Rep}(G)$, $\mathrm{Rep}_\mu(G)$ and $\mathrm{Rep}_{\mu,\mathsf{S}}(G)$ respectively. The set of irreducible objects in $\mathrm{Rep}_\mu(G)$ and $\mathrm{Rep}_{\mu,\mathsf{S}}(G)$ will be denoted by $\mathrm{Irr}_\mu(G)$ and $\mathrm{Irr}_{\mu,\mathsf{S}}(G)$, respectively.

4.1. **The parameter set for coherent continuation representations.** Suppose that we are given a connected reductive complex Lie group $G_{\mathbb{C}}$ together with a Lie group homomorphism $\iota : G \to G_{\mathbb{C}}$ such that its differential $\mathrm{d}\iota : \mathrm{Lie}(G) \to \mathrm{Lie}(G_{\mathbb{C}})$ has the following two properties:

- the kernel of $\mathrm{d}\iota$ is contained in the center of the Lie algebra $\mathrm{Lie}(G)$ of G;
- the image of $\mathrm{d}\iota$ is a real form of the Lie algebra $\mathrm{Lie}(G_{\mathbb{C}})$.

The analytic weight lattice of $G_{\mathbb{C}}$ is identified with a subgroup of ${}^a\mathfrak{h}^*$ via $\mathrm{d}\iota$. We write $Q_\iota \subseteq {}^a\mathfrak{h}^*$ for this subgroup.

When $G_{\mathbb{C}} = \mathrm{Ad}(\mathfrak{g})$ and ι is the adjoint representation, Q_ι equals the root lattice $Q_{\mathfrak{g}}$. In general Q_ι is W-stable and $Q_{\mathfrak{g}} \subseteq Q_\iota \subseteq Q^{\mathfrak{g}}$. In the rest of this section we assume that $Q = Q_\iota$. Recall that $\Lambda \subseteq {}^a\mathfrak{h}^*$ is a Q-coset.

The algebra $\mathcal{R}(\mathfrak{g}, Q)$ is obviously identified with the Grothendieck group of the category of holomorphic finite-dimensional representations of $G_{\mathbb{C}}$. Thus $\mathcal{K}(G)$ is naturally a $\mathcal{R}(\mathfrak{g}, Q)$-module by using tensor product and the homomorphism ι. As before, we form the coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ of W_Λ.

Suppose that H is a Cartan subgroup of G. Write $\mathfrak{t}$ for the complexified Lie algebra of the unique maximal compact subgroup of H. As before, denote by $\Delta_{\mathfrak{h}} \subseteq \mathfrak{h}^*$ the root system of $\mathfrak{g}$. A root $\alpha \in \Delta_{\mathfrak{h}}$ is called imaginary if $\check{\alpha} \in \mathfrak{t}$. An imaginary root $\alpha \in \Delta_{\mathfrak{h}}$ is said to be compact if the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are contained in the complexified Lie algebra of a common compact subgroup of G.

Note that every Casselman-Wallach representation of H is finite dimensional. Since G is in the Harish-Chandra's class, for every $\Gamma \in \mathrm{Irr}(H)$, there is a unique element $\mathrm{d}\Gamma \in \mathfrak{h}^*$ such that the differential of Γ is isomorphic to a direct sum of one-dimensional representations of $\mathfrak{h}$ attached to $\mathrm{d}\Gamma$.

For every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, write

$$\xi_{\mathfrak{b}} : {}^a\mathfrak{h} \to \mathfrak{h} \tag{4.1}$$

for the linear isomorphism attached to $\mathfrak{b}$, whose transpose inverse is still denoted by $\xi_{\mathfrak{b}} : {}^a\mathfrak{h}^* \to \mathfrak{h}^*$. Write

$$W({}^a\mathfrak{h}^*, \mathfrak{h}^*) := \{\xi_{\mathfrak{b}} : {}^a\mathfrak{h}^* \to \mathfrak{h}^* \mid \mathfrak{b} \text{ is a Borel subalgebra of } \mathfrak{g} \text{ containing } \mathfrak{h}\}.$$

Recall that $\Delta^+ \subseteq {}^a\mathfrak{h}^*$ denotes the set of positive roots. For every element $\xi \in W({}^a\mathfrak{h}^*, \mathfrak{h}^*)$, put

$$\delta(\xi) := \frac{1}{2} \cdot \sum_{\alpha \text{ is an imaginary root in } \xi\Delta^+} \alpha - \sum_{\beta \text{ is a compact imaginary root in } \xi\Delta^+} \beta \in \mathfrak{h}^*. \tag{4.2}$$

Write $\mathscr{P}_\Lambda(G)$ for the set of all triples $\gamma = (H, \xi, \Gamma)$ where H is a Cartan subgroup of G, $\xi \in W({}^a\mathfrak{h}^*, \mathfrak{h}^*)$, and

$$\Gamma : \Lambda \to \mathrm{Irr}(H), \quad \nu \mapsto \Gamma_\nu$$

is a map with the following properties:

- $\Gamma_{\nu+\beta} = \Gamma_\nu \otimes \xi(\beta)$ for all $\beta \in Q$ and $\nu \in \Lambda$;
- $\mathrm{d}\Gamma_\nu = \xi(\nu) + \delta(\xi)$ for all $\nu \in \Lambda$.

Here $\xi(\beta)$ is naturally viewed as a character of H by using the homomorphism $\iota : H \to H_{\mathbb{C}}$, and $H_{\mathbb{C}}$ is the Cartan subgroup of $G_{\mathbb{C}}$ containing $\iota(H)$.

The group G obviously acts on $\mathscr{P}_\Lambda(G)$, and we define the set of parameters for $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ to be

$$\mathcal{P}_\Lambda(G) := G\backslash\mathscr{P}_\Lambda(G). \tag{4.3}$$

For each $\gamma \in \mathcal{P}_\Lambda(G)$ that is represented by $\gamma = (H, \xi, \Gamma)$, by [Vog81, Theorem 8.2.1], we have two $\mathcal{K}(G)$-valued coherent families Ψ_γ and $\overline{\Psi}_\gamma$ on Λ such that

$$\Psi_\gamma(\nu) = \mathrm{X}((\Gamma_\nu, \xi(\nu)) \quad \text{and} \quad \overline{\Psi}_\gamma(\nu) = \overline{\mathrm{X}}(\Gamma_\nu, \xi(\nu)) \tag{4.4}$$

for all regular dominant element $\nu \in \Lambda$. Here $\mathrm{X}((\Gamma_\nu, \xi(\nu))$ is the standard representation defined in [Vog81, Notational Convention 6.6.3] and $\overline{\mathrm{X}}(\Gamma_\nu, \xi(\nu)))$ is its unique irreducible subrepresentation (see [Vog81, Theorem 6.5.12]).

By Langlands classification, $\{\overline{\Psi}_\gamma\}_{\gamma\in\mathcal{P}_\Lambda(G)}$ is a basis of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ (see [Vog81, Theorem 6.6.14]), and we view $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ as a basal representation of W_Λ with this basis. The family $\{\Psi_\gamma\}_{\gamma\in\mathcal{P}_\Lambda(G)}$ is also a basis of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ ([Vog81, Proposition 6.6.7]).

The coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ is independent of Q in the sense of the following lemma.

Lemma 4.1. *Let $\Lambda_{\mathfrak{g}}$ be a $Q_{\mathfrak{g}}$ coset in Λ. Form the coherent continuation representation $\mathrm{Coh}_{\Lambda_{\mathfrak{g}}}(\mathcal{K}(G))$ by using the adjoint representation $G \to \mathrm{Ad}(\mathfrak{g})$. Then the restriction yields a linear isomorphism*

$$\mathrm{Coh}_\Lambda(\mathcal{K}(G)) \xrightarrow{\sim} \mathrm{Coh}_{\Lambda_{\mathfrak{g}}}(\mathcal{K}(G)).$$

Proof. Note that the restriction yields a bijective map

$$\mathcal{P}_\Lambda(G) \to \mathcal{P}_{\Lambda_{\mathfrak{g}}}(G).$$

This implies the lemma. □

We have the cross action of W_Λ on the set $\mathscr{P}_\Lambda(G)$ (see [Vog82, Definition 4.2]):

$$w \times (H, \xi, \Gamma) = \left(H, \xi w^{-1}, (\nu \mapsto \Gamma_\nu \otimes (\xi w^{-1}\nu + \delta(\xi w^{-1}) - \xi\nu - \delta(\xi)))\right).$$

This commutes with the action of G and thus descends to an action

$$W_\Lambda \times \mathcal{P}_\Lambda(G) \to \mathcal{P}_\Lambda(G), \quad (w, \gamma) \mapsto w \times \gamma.$$

Since G is in Harish-Chandra's class, the real Weyl group

$$W_H := (\text{the normalize of } H \text{ in } G)/H$$

is identified with a subgroup of the Weyl group $W_{\mathfrak{h}}$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Write $\mathfrak{t}_{\mathrm{im}}$ for the subspace of $\mathfrak{t}$ spanned by the set $\{\check{\alpha} \mid \alpha \text{ is an imaginary root in } \Delta_{\mathfrak{h}}\}$. Then W_H stabilizers $\mathfrak{t}_{\mathrm{im}}$, and we define a character

$$\mathrm{sgn}_{\mathrm{im}} : W_H \to \mathbb{C}^\times, \quad w \mapsto (\text{the determinant of the map } w : \mathfrak{t}_{\mathrm{im}} \to \mathfrak{t}_{\mathrm{im}}).$$

This is a quadratic character.

Let $\gamma \in \mathcal{P}_\Lambda(G)$. Write W_γ for the stabilizer of γ in W_Λ under the cross action. Pick an element $\boldsymbol{\gamma} = (H, \xi, \Gamma) \in \mathscr{P}_\Lambda(G)$ that represents γ. It is clear that $\xi w \xi^{-1} \in W_H$ for all $w \in W_\gamma$. We define a quadratic character

$$\mathrm{sgn}_\gamma : W_\gamma \to \mathbb{C}^\times, \qquad w \mapsto \mathrm{sgn}_{\mathrm{im}}(\xi w \xi^{-1}).$$

This is independent of the choice of $\boldsymbol{\gamma}$.

The coherent continuation representation may be computed by using the basis $\{\, \Psi_\gamma \,\}_{\gamma \in \mathcal{P}_\Lambda(G)}$ (see [Vog82, Section 14]). The following result is due to Barbasch-Vogan, in a suitably modified form from [BV83b, Proposition 2.4]. As its proof follows the same line as that of [BV83b, Proposition 2.4], we will be contented to state the precise result.

Theorem 4.2 (*cf.* [BV83b, Proposition 2.4])**.** *As a representation of W_Λ,*

$$\mathrm{Coh}_\Lambda(\mathcal{K}(G)) \cong \bigoplus_\gamma \mathrm{Ind}_{W_\gamma}^{W_\Lambda} \mathrm{sgn}_\gamma,$$

where γ runs over a representative set of the W_Λ-orbits in $\mathcal{P}_\Lambda(G)$ under the cross action.

4.2. **Some properties of the coherent continuation representation** $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$**.** For every $\nu \in \Lambda$, by evaluating at ν we get a linear map

$$\mathrm{ev}_\nu \colon \mathrm{Coh}_\Lambda(\mathcal{K}(G)) \to \mathcal{K}_\nu(G).$$

We first present a number of lemmas concerning ev_ν, which may be found in (or easily deduced from) [Vog81, Vog82].

Lemma 4.3. *Let $\nu \in \Lambda$. The map ev_ν is surjective, and it is bijective when ν is regular.*

Proof. The surjectivity is due to Schmid and Zuckerman, see [Vog81, Theorem 7.2.7]. The injectivity (for ν regular) is due to Schmid, see [Vog81, Proposition 7.2.23]. ☐

Lemma 4.4. *Let Ψ be a basal element in $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$. Then there is a unique left cell $\mathcal{C}^L_\Psi$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ such that $\mathrm{Ann}(\Psi(\nu)) = I_{\nu, \mathcal{C}^L_\Psi}$ for every dominant element $\nu \in \Lambda$.*

Proof. Since a coherent family restricted to the dominant cone can be constructed via the translation functor, the proposition follows from [Vog79b, Lemma 2.7 and Lemma 2.8]. ☐

For every basal element $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, define its τ-invariant to be $\tau_\Psi := \tau_{\mathcal{C}^L_\Psi}$, where $\mathcal{C}^L_\Psi$ is as in Lemma 4.4. For every simple reflection s in $W(\Lambda)$, $s \in \tau_\Psi$ if and only if $s \cdot \Psi = -\Psi$ (see [Vog79, Theorem 2.4] and [Vog81, Proposition 7.3.7]).

Lemma 4.5. *([Vog82, Corollary 7.3.23]) Let $\nu \in \Lambda$ be a dominant element. Then for every basal element $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, $\Psi(\nu)$ is*

$$\begin{cases} \textit{irreducible}, & \textit{if } \tau_\Psi \cap W_\nu = \emptyset; \\ 0, & \textit{otherwise}. \end{cases}$$

Moreover, for every basal element $\Phi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, if $\Phi(\nu) = \Psi(\nu) \neq 0$, then $\Phi = \Psi$.

Let $\mathcal{C}^{LR}$ be a double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$. Define $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(G))$ to be the basal subspace of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ spanned by all the basal elements Ψ such that $\mathcal{C}^L_\Psi \subseteq \langle \mathcal{C}^{LR} \rangle_{LR}$. Recall that $\langle \mathcal{C}^{LR} \rangle_{LR}$ denotes the smallest basal subrepresentation of $\mathrm{Coh}^{LR}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ containing $\mathcal{C}^{LR}$.

Lemma 4.6. *For every double cell $\mathcal{C}^{LR}$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$, the space $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(G))$ is a basal $W(\Lambda)$-subrepresentation of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$.*

Proof. By passing to a suitable covering group, we assume without loss of generallity that the derived group of $G_\mathbb{C}$ is simply connected.

Let Ψ be a basal element in $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(G))$, and let α be a simple root in $\Delta(\Lambda)$. We need to show that $s_\alpha \cdot \Psi \in \mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(G))$, where $s_\alpha \in W(\Lambda)$ is the simple reflection associated to α.

Our assumption of simply connectedness implies that there is a regular dominant element $\mu \in \Lambda$ such that $\langle \mu, \check{\alpha} \rangle \in 2\mathbb{Z}$. Put

$$\nu := \mu - \frac{\langle \mu, \check{\alpha} \rangle}{2} \alpha \in \Lambda,$$

which is a dominant element such that $W_\nu = \{1, s_\alpha\}$.

If $\Psi(\nu) = 0$, then $s_\alpha \cdot \Psi = -\Psi \in \mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(G))$. Now we assume that $\Psi(\nu) \neq 0$. Then $\Psi(\nu)$ is irreducible.

Let F denote the irreducible representation of G with extremal weight $\frac{1}{2} \langle \mu, \check{\alpha} \rangle \alpha$ that occurs in the symmetric power $\mathrm{S}(\mathfrak{g})$. Then there are two nonzero G-homomorphisms

$$\Psi(\mu) \xrightarrow{\iota_1} \Psi(\nu) \otimes F \xrightarrow{\iota_2} \Psi(\mu)$$

such that $\iota_2 \circ \iota_1 = 0$. Moreover, $\Psi(\mu)$ occurs in the composition series of $\Psi(\nu) \otimes F$ with multiplicity two, and

$$(s_\alpha \cdot \Psi)(\mu) = \mathrm{Pr}_\mu(\Psi(\nu) \otimes F) - \Psi(\mu) \in \mathcal{K}(G), \tag{4.5}$$

where Pr_μ denotes the projection map

$$\mathrm{Pr}_\mu : \mathcal{K}(G) = \bigoplus_{\lambda \in W \backslash {}^a\mathfrak{h}^*} \mathcal{K}_\lambda(G) \to \mathcal{K}_\mu(G). \tag{4.6}$$

See [Vog81, Theorem 7.3.16 and Corollary 7.3.18].

By (4.5), we know that $s_\alpha \cdot \Psi$ is a positive integral linear combination of basal elements of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$. Let Ψ' be a basal element that occurs in the positive integral combination. Then

$$\mathrm{Ann}(\Psi'(\mu)) \supseteq \mathrm{Ann}((\Psi(\nu) \otimes F)).$$

Pick an element $w \in W$ such that $\mathrm{Ann}(\Psi(\mu)) = \mathrm{Ann}(\overline{\Psi}_w(\mu))$, where $\overline{\Psi}_w$ is defined as in (3.4). The translation principle [Vog79b, Lemma 2.7] implies that $\mathrm{Ann}(\Psi(\nu)) = \mathrm{Ann}(\overline{\Psi}_w(\nu))$. In particular, $\overline{\Psi}_w(\nu)$ is nonzero and hence irreducible.

As in the group case, $\overline{\Psi}_w(\mu)$ occurs as a subrepresentation of $\Psi(\nu) \otimes F$ (and also as a quotient representation), and occurs in the composition series of $\overline{\Psi}_w(\nu) \otimes F$ with multiplicity two. Moreover,

$$(s_\alpha \cdot \overline{\Psi}_w)(\mu) = \mathrm{Pr}_\mu(\overline{\Psi}_w(\nu) \otimes F) - \overline{\Psi}_w(\mu) \in \mathcal{K}(\mathfrak{g}, \mathfrak{b}), \tag{4.7}$$

where Pr_μ denotes the projection map analogous to (4.6).

We have an ideal $I \subseteq \mathcal{U}(\mathfrak{g})$ that is a finite product of primitive ideals of $\mathcal{U}(\mathfrak{g})$ with infinitesimal character distinct from μ, and a sequence $w_1, w_2, \cdots, w_{k-1}, w_k = w$ in W $(k \in \mathbb{N}^+)$ such that

$$\mathrm{Ann}(\overline{\Psi}_w(\nu) \otimes F) \supseteq \mathrm{Ann}(\overline{\Psi}_w(\mu)) \cdot \mathrm{Ann}(\overline{\Psi}_{w_1}(\mu)) \cdot \mathrm{Ann}(\overline{\Psi}_{w_2})(\mu) \cdot \cdots \cdot \mathrm{Ann}(\overline{\Psi}_{w_k}(\mu)) \cdot I,$$

and

$$(s_\alpha \cdot \overline{\Psi}_w) = \overline{\Psi}_{w_1} + \overline{\Psi}_{w_1} \cdots + \overline{\Psi}_{w_k} \in \mathcal{K}(\mathfrak{g}, \mathfrak{b}).$$

Then

$$\mathrm{Ann}(\Psi'(\mu)) \supseteq \mathrm{Ann}(\overline{\Psi}_w(\mu)) \cdot \mathrm{Ann}(\overline{\Psi}_{w_1}(\mu)) \cdot \mathrm{Ann}(\overline{\Psi}_{w_2}(\mu)) \cdot \cdots \cdot \mathrm{Ann}(\overline{\Psi}_{w_k}(\mu)) \cdot I.$$

Since the primitive ideal $\mathrm{Ann}(\Psi'(\mu))$ is prime, this implies that $\mathrm{Ann}(\Psi'(\mu)) \supseteq \mathrm{Ann}(\overline{\Psi}_{w_i}(\mu))$ for some $i \in \{1, 2, \ldots, k\}$. Hence $\mathcal{C}^L_{\Psi'} \subseteq \langle \overline{\Psi}_{w_i} \rangle_L$ by Lemma 3.12. On the other hand, $\overline{\Psi}_{w_i}$ is in $\langle \overline{\Psi}_w \rangle_R$. We conclude that $\Psi' \in \mathrm{Coh}_{\Lambda, \mathcal{C}^{LR}}(\mathcal{K}(G))$. The proof is complete. □

4.3. **The coherent continuation representation** $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$**.** Let H be a Cartan subgroup of G. Let $\mathfrak{b}'$ be a Borel subalgebra of $\mathfrak{g}$ containing the complexified Lie algebra $\mathfrak{h}$ of H. Recall that a $(\mathfrak{g}, H)$-module is defined to be a $\mathfrak{g}$-module V together with a locally-finite representation of H on it such that

- $h \cdot (X \cdot (h^{-1} \cdot u)) = (\mathrm{Ad}_h(X)) \cdot u$, for all $h \in H, X \in \mathcal{U}(\mathfrak{g}), u \in V$ (Ad stands for the Adjoint representation);
- the differential of the representation of H and the restriction of the representation of $\mathfrak{g}$ yields the same representation of $\mathfrak{h}$ on V.

Let $\mathrm{Rep}(\mathfrak{g}, H, \mathfrak{b}')$ denote the category of finitely generated $(\mathfrak{g}, H)$-modules that are unions of finite-dimensional $\mathfrak{b}'$-submodules. As before, we form the Grothendieck group $\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}')$, and the coherent continuation representation $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$ of $W(\Lambda)$.

Write $\mathscr{P}_\Lambda(\mathfrak{g}, H, \mathfrak{b}')$ for the set of all pairs (w, Γ) where $w \in W$, and

$$\Gamma : \Lambda \to \mathrm{Irr}(H), \quad \nu \mapsto \Gamma_\nu$$

is a map such that

- $\Gamma_{\nu+\beta} = \Gamma_\nu \otimes \xi_{\mathfrak{b}'}(\beta)$ for all $\beta \in Q$ and $\nu \in \Lambda$;
- $\mathrm{d}\Gamma_\nu = \xi_{\mathfrak{b}'}(w\nu - \rho)$ for all $\nu \in \Lambda$.

Here $\xi_{\mathfrak{b}'}$ is defined in (4.1), $\xi_{\mathfrak{b}'}(\beta)$ is viewed as a character of H by pulling-back through the homomorphism $\iota : H \to H_\mathbb{C}$, and $H_\mathbb{C}$ is the Cartan subgroup of $G_\mathbb{C}$ containing $\iota(H)$.

For each $\sigma \in \mathrm{Irr}(H)$, put

$$\mathrm{M}(\sigma) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}')} \sigma,$$

which is a module in $\mathrm{Rep}(\mathfrak{g}, H, \mathfrak{b}')$, where the $\mathcal{U}(\mathfrak{g})$-action is given by the left multiplication, and the H-action is given by

$$h \cdot (X \otimes u) := \mathrm{Ad}_h(X) \otimes h \cdot u, \qquad h \in H,\ X \in \mathcal{U}(\mathfrak{g}),\ u \in \sigma.$$

As before, the $(\mathfrak{g}, H)$-module $\mathrm{M}(\sigma)$ has a unique irreducible quotient, to be denoted by $\mathrm{L}(\sigma)$.

For each $(w,\Gamma) \in \mathscr{P}_\Lambda(\mathfrak{g}, H, \mathfrak{b}')$, we have two $\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}')$-valued coherent families $\Psi_{w,\Gamma}$ and $\overline{\Psi}_{w,\Gamma}$ on Λ such that

$$\Psi_{w,\Gamma}(\nu) = \mathrm{M}(\Gamma_\nu) \quad \text{for all } \nu \in \Lambda, \tag{4.8}$$

and

$$\overline{\Psi}_{w,\Gamma}(\nu) = \mathrm{L}(\Gamma_\nu) \quad \text{for all regular dominant element } \nu \in \Lambda. \tag{4.9}$$

As before, both $\{\Psi_{w,\Gamma}\}_{(w,\Gamma)\in\mathscr{P}_\Lambda(\mathfrak{g},H,\mathfrak{b}')}$ and $\{\overline{\Psi}_{w,\Gamma}\}_{(w,\Gamma)\in\mathscr{P}_\Lambda(\mathfrak{g},H,\mathfrak{b}')}$ are bases of $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$. We view $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$ as a basal representation of $W(\Lambda)$ by using the second basis.

The analog of Lemma 4.4 for $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$ still holds (with the same proof), and we define similarly the left cell $\mathcal{C}^L_\Psi$ in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ for every basal element $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$.

Let $\mathcal{C}^{LR}$ be a double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}))$ as before. Define $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$ to be the basal subspace of $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$ spanned by all the basal elements Ψ such that $\mathcal{C}^L_\Psi \subseteq \langle \mathcal{C}^{LR}\rangle_{LR}$. By the same argument as that of Lemma 4.6, the space $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$ is a basal $W(\Lambda)$-subrepresentation of $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$.

Lemma 4.7. *The representation $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}'))$ of $W(\Lambda)$ is isomorphic to a subrepresentation of $(\langle \mathcal{C}^{LR}\rangle_{LR})^k$, for some $k \in \mathbb{N}$.*

Proof. Let Q act on the set $\mathrm{Irr}(H)$ by

$$\beta \cdot \sigma = \sigma \otimes \xi_{\mathfrak{b}'}(\beta), \qquad \beta \in Q,\ \sigma \in \mathrm{Irr}(H),$$

where $\xi_{\mathfrak{b}'}(\beta)$ is as in (4.1). For each Q-orbit $\Omega \subseteq \mathrm{Irr}(H)$, let $\mathrm{Rep}_\Omega(\mathfrak{g}, H, \mathfrak{b}')$ be the full subcategory of $\mathrm{Rep}(\mathfrak{g}, H, \mathfrak{b}')$ whose objects are modules V such that every irreducible subquotient of $V|_H$ (viewed as a representation of H) belongs to Ω. Write $\mathcal{K}_\Omega(\mathfrak{g}, H, \mathfrak{b}')$ for the Grothendieck group of the category $\mathrm{Rep}_\Omega(\mathfrak{g}, H, \mathfrak{b}')$. Then $\mathrm{Coh}_\Lambda(\mathcal{K}_\Omega(\mathfrak{g}, H, \mathfrak{b}'))$ is a basal $W(\Lambda)$-representation. As $W(\Lambda)$-representations,

$$\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}')) = \bigoplus_{i=1}^{k} \mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}_{\Omega_i}(\mathfrak{g}, H, \mathfrak{b}')),$$

where $\Omega_1, \Omega_2, \cdots, \Omega_k$ $(k \in \mathbb{N})$ are certain Q-orbits in $\mathrm{Irr}(H)$, and

$$\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}_{\Omega_i}(\mathfrak{g}, H, \mathfrak{b}')) := \mathrm{Coh}_\Lambda(\mathcal{K}_{\Omega_i}(\mathfrak{g}, H, \mathfrak{b}')) \cap \mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(\mathfrak{g}, H, \mathfrak{b}')).$$

Define a linear isomorphism

$$T_{\mathfrak{b},\mathfrak{b}'} : \mathcal{K}(\mathfrak{g}, \mathfrak{b}) \to \mathcal{K}(\mathfrak{g}, \mathfrak{b}')$$

such that $T_{\mathfrak{b},\mathfrak{b}'}(\mathrm{M}(\mathfrak{g}, \mathfrak{b}, \nu)) = \mathrm{M}(\mathfrak{g}, \mathfrak{b}', \nu)$ for all $\nu \in {}^a\mathfrak{h}^*$. The map $T_{\mathfrak{b},\mathfrak{b}'}$ is W-equivariant and induces an isomorphism

$$\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b})) \to \mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g}, \mathfrak{b}')).$$

By using this isomorphism we assume without loss of generality that $\mathfrak{b} = \mathfrak{b}'$.

Denote by $\mathcal{F}$ the forgetful functor from $\mathrm{Rep}(\mathfrak{g}, H, \mathfrak{b}')$ to $\mathrm{Rep}(\mathfrak{g}, \mathfrak{b}')$. Since $\Psi \mapsto \mathcal{F} \circ \Psi$ gives an injective $W(\Lambda)$-homomorphism from $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}_{\Omega_i}(\mathfrak{g}, H, \mathfrak{b}'))$ to $\langle \mathcal{C}^{LR}\rangle_{LR}$, the lemma follows. $\square$

4.4. **An embedding of coherent continuation representations.** As before $\mathcal{C}^{LR}$ is a double cell in $\mathrm{Coh}_\Lambda\mathcal{K}(\mathfrak{g},\mathfrak{b})$. The purpose of this subsection is to prove the following proposition.

Proposition 4.8. *The representation* $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(G))$ *of* $W(\Lambda)$ *is isomorphic to a subrepresentation of* $(\langle\mathcal{C}^{LR}\rangle_{LR})^k$*, for some* $k\in\mathbb{N}$.

Let $\{H_1,H_2,\cdots,H_r\}$ ($r\in\mathbb{N}^+$) be a set of representatives of the conjugacy classes of Cartan subgroups of G. For each $i=1,2,\cdots,r$, fix a Borel subalgebra $\mathfrak{b}_i$ of $\mathfrak{g}$ that contains the complexified Lie algebra of H_i.

Lemma 4.9. *There is an injective* $\mathcal{R}(\mathfrak{g},Q)$*-module homomorphism*

$$\gamma_G:\mathcal{K}(G)\to\bigoplus_{i=1}^r\mathcal{K}(\mathfrak{g},H_i,\mathfrak{b}_i)$$

such that

$$\gamma_G(\mathcal{K}_I(G))\subseteq\bigoplus_{i=1}^r\mathcal{K}_I(\mathfrak{g},H_i,\mathfrak{b}_i)\tag{4.10}$$

for every ideal I *of* $\mathcal{U}(\mathfrak{g})$*. Here* $\mathcal{K}_I(G)$ *is the Grothedieck group of* $\mathrm{Rep}_I(G)$*,* $\mathrm{Rep}_I(G)$ *is the full subcategory of* $\mathrm{Rep}(G)$ *consisting of the representations that are annihilated by* I*, and* $\mathcal{K}_I(\mathfrak{g},H_i,\mathfrak{b}_i)$ *is the similarly defined subspace of* $\mathcal{K}(\mathfrak{g},H_i,\mathfrak{b}_i)$.

Proof. This follows from the work of Casian ([Cas86]). Since the proposition is not explicitly stated in [Cas86], we briefly recall the argument of Casian for the convenience of the reader.

Let $\mathfrak{n}_i$ denote the nilpotent radical of $[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{b}_i$ ($i=1,2,\cdots,r$). For every $q\in\mathbb{Z}$, let $\gamma^q_{\mathfrak{n}_i}$ denote the q-th right derived functor of the following left exact functor from the category of $\mathfrak{g}$-modules to itself:

$$V\to\{u\in V\mid\mathfrak{n}_i^k\cdot u=0\text{ for some }k\in\mathbb{N}^+\}.$$

Fix a Cartan involution θ of G and write K for its fixed point group (which is a maximal compact subgroup of G). Without loss of generality we assume that all H_i's are θ-stable.

For every Casselman-Wallach representation V of G, write $V_{[K]}$ for the space of K-finite vectors in V, which is a $(\mathfrak{g},K)$-module of finite length. Then $\gamma^q_{\mathfrak{n}_i}(V_{[K]})$ is naturally a representation in $\mathrm{Rep}(\mathfrak{g},H_i,\mathfrak{b}_i)$ ([Cas86, Corollary 4.9]).

We define a linear map

$$\gamma_G:\mathcal{K}(G)\to\bigoplus_{i=1}^r\mathcal{K}(\mathfrak{g},H_i,\mathfrak{b}_i)$$

given by

$$\gamma_G(V)=\left\{\sum_{q\in\mathbb{Z}}(-1)^q\gamma^q_{\mathfrak{n}_i}(V_{[K]})\right\}_{i=1,2,\cdots,r}$$

for every Casselman-Wallach representation V of G. A form of the Osborne conjecture (see [Cas86, Theorem 3.1]) and [Cas86, Corollary 4.9]) implies that the map γ_G is well-defined and injective.

Proposition 4.11 of [Cas86] implies that the functor $\gamma^q_{\mathfrak{n}_i}$ commutes with tensor product with finite-dimensional representations. Thus γ_G is a $\mathcal{R}(\mathfrak{g},Q)$-homomorphism. Finally, [Cas86, Corollary 4.15] implies that γ_G satisfies the property in (4.10). $\square$

Proof of Proposition 4.8. Lemma 4.9 implies that the representation $\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(G))$ of $W(\Lambda)$ is isomorphic to a subrepresentation of $\bigoplus_{i=1}^{r}\mathrm{Coh}_{\Lambda,\mathcal{C}^{LR}}(\mathcal{K}(\mathfrak{g},H_i,\mathfrak{b}_i))$. Together with Lemma 4.7, this implies Proposition 4.8. ☐

4.5. **Harish-Chandra cells and double cells.** We now discuss the consequences of Proposition 4.8. View $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ as a basal representation of $W(\Lambda)$ and let $\mathcal{C}$ be a cell in it, to be called a Harish-Chandra cell.

Lemma 4.10. *Suppose $\mathcal{C}$ is a Harisch-Chandra cell. Then there is a unique double cell $\mathcal{C}^{LR}$ such that $\mathcal{C}^L_\Psi \subseteq \mathcal{C}^{LR}$ for every basal element Ψ in $\mathcal{C}$.*

Proof. pass ☐

Define the τ-invariant of $\mathcal{C}$ to be the set

$$\tau_{\mathcal{C}} := \{\tau_\Psi \mid \Psi \in \mathcal{C}\}.$$

Clearly $\tau_{\mathcal{C}}$ is a prior a subset of $\tau_{\mathcal{C}^{LR}}$ where $\mathcal{C}^{LR}$ is the double cell in).

For every basal element $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, put $\sigma_\Psi := \sigma_{\mathcal{C}^{LR}_\Psi}$, where $\mathcal{C}^{LR}_\Psi$ is the double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ containing $\mathcal{C}^L_\Psi$, and $\mathcal{C}^L_\Psi$ is the left cell in Proposition 4.4.

Working Hypothesis:

Proposition 4.11. *Among all irreducible representations of $W(\Lambda)$ that occur in $\langle\mathcal{C}\rangle$, there is a unique one, to be denote by $\sigma_{\mathcal{C}}$, that has minimal fake degree. Moreover, $\tau_{\mathcal{C}} = \tau_{\sigma_{\mathcal{C}}}$, and $\sigma_{\mathcal{C}} = \sigma_\Psi$ for all $\Psi \in \mathcal{C}$.*

Proof. By [Vog82, Theorem 14.10] (see also [Kin81] and [Cas86, Theorem 3.4]), there is a $W(\Lambda)$-equivariant linear map

$$\mathrm{ch} : \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}) \to \mathrm{S}({}^a\mathfrak{h}) \tag{4.11}$$

with the following properties:

- for every $\Psi \in \mathcal{C}$, the image of the class of Ψ under the map (4.11) equals $p_{\mathcal{C}^L_\Psi}$ up to a nonzero scalar multiplication;
- the image of (4.11) is a special irreducible representation of $W(\Lambda)$, to be denoted by $\sigma_{\mathcal{C}}$.

Let $\mathcal{C}^{LR}$ be the double cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(\mathfrak{g},\mathfrak{b}))$ corresponding to $\sigma_{\mathcal{C}}$ as in Proposition 3.13. Then Lemma 3.14 implies that

$$\mathcal{C}^{LR} = \bigcup_{\Psi\in\mathcal{C}} \mathcal{C}^L_\Psi.$$

Consequently, $\sigma_{\mathcal{C}} = \sigma_\Psi$ for every basal element $\Psi \in \mathcal{C}$.

By Lemma 3.23, we have that

$$\tau_{\mathcal{C}} = \{\tau_{\mathcal{C}^L_\Psi} \mid \Psi \in \mathcal{C}\} = \tau_{\mathcal{C}^{LR}} = \tau_{\sigma_{\mathcal{C}}}.$$

Finally, Proposition 4.8 implies that $\langle\mathcal{C}\rangle$ is isomorphic to a subrepresentation of $(\langle\mathcal{C}^{LR}\rangle_{LR})^k$ ($k \in \mathbb{N}^+$), and Proposition 3.13 implies that $\sigma_{\mathcal{C}}$ is the unique irreducible representation of $W(\Lambda)$ with minimal fake degree that occurs in $(\langle\mathcal{C}^{LR}\rangle_{LR})^k$. Therefore $\sigma_{\mathcal{C}}$ is also the unique irreducible representation of $W(\Lambda)$ with minimal fake degree that occurs in $\langle\mathcal{C}\rangle$. This finishes the proof of the proposition. ☐

Let $\sigma_{\mathcal{C}}$ be as in Proposition 4.11, to be called the special irreducible representation attached to $\mathcal{C}$. The following conjecture is widely anticipated, although to our knowledge no proof has appeared in the literature. See [Vog82, page 1055].

Conjecture 4.12. *The set*

$$\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in the cell representation } \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})\}$$

is contained in the double cell in $\mathrm{Irr}(W(\Lambda))$ *containing* $\sigma_{\mathcal{C}}$*.*

The conjecture holds for unitary group by [Bož02, Theorem 5]. We note McGovern's observation ([McG98, Page 213]) which amounts to the assertion in Conjecture 4.12 for linear groups. It appears to us that the argument is inadequate as presented. In what follows we will give a proof assuming a weak form of Vogan duality, which therefore works for linear groups and the real metaplectic group.

We consider a 5-tuple $(G, G_{\mathbb{C}}, \iota : G \to G_{\mathbb{C}}, \Lambda, \mathcal{C})$, where the conditions required of the 4-tuple $(G, G_{\mathbb{C}}, \iota : G \to G_{\mathbb{C}}, \Lambda)$ are specified in Section 4.1, and $\mathcal{C}$ is a Harish-Chandra cell in $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$.

Let $(\check{G}, \check{G}_{\mathbb{C}}, \check{\iota} : \check{G} \to \check{G}_{\mathbb{C}}, \check{\Lambda}, \check{\mathcal{C}})$ be another 5-tuple that has the same properties as the 5-tuple $(G, G_{\mathbb{C}}, \iota : G \to G_{\mathbb{C}}, \Lambda, \mathcal{C})$. Thus $\check{\mathcal{C}}$ is a Harish-Chandra cell in $\mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{G}))$, considered as a basal representation of $\check{W}(\check{\Lambda})$. Here $\check{W}$ is the Weyl group of the complexified Lie algebra of $\check{G}$, and $\check{W}(\check{\Lambda}) \subseteq \check{W}$ is the integral Weyl group attached to $\check{\Lambda}$.

We say that $(\check{G}, \check{G}_{\mathbb{C}}, \check{\iota} : \check{G} \to \check{G}_{\mathbb{C}}, \check{\Lambda}, \check{\mathcal{C}})$ is a weak Vogan dual of $(G, G_{\mathbb{C}}, \iota : G \to G_{\mathbb{C}}, \Lambda, \mathcal{C})$ if there is given an identification $W(\Lambda) = \check{W}(\check{\Lambda})$ of Coxeter groups such that

$$\mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{G}))(\check{\mathcal{C}}) \cong (\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})) \otimes \mathrm{sgn}$$

as representations of $W(\Lambda) = \check{W}(\check{\Lambda})$.

Lemma 4.13. *Suppose that* $(\check{G}, \check{G}_{\mathbb{C}}, \check{\iota} : \check{G} \to \check{G}_{\mathbb{C}}, \check{\Lambda}, \check{\mathcal{C}})$ *is a weak Vogan dual of* $(G, G_{\mathbb{C}}, \iota : G \to G_{\mathbb{C}}, \Lambda, \mathcal{C})$*. Then the* τ*-invariants* $\tau_{\check{\mathcal{C}}}$ *and* $\tau_{\mathcal{C}}$ *are dual to each other.*

Proof. Write $M := \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})$, to be viewed as a representation of $W(\Lambda)$. Likewise write $\check{M} := \mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{G}))(\check{\mathcal{C}})$, to be viewed as a representation of $\check{W}(\check{\Lambda})$. Then by [FJMN21, Theorem 2.10],

$$\tau_M = \tau_{\mathcal{C}}, \qquad \tau_{\check{M}} = \tau_{\check{\mathcal{C}}}.$$

and τ_M is dual to $\tau_{\check{M}}$. Therefore the lemma follows. □

Proposition 4.14. *If the* 5*-tuple* $(G, G_{\mathbb{C}}, \iota : G \to G_{\mathbb{C}}, \Lambda, \mathcal{C})$ *has a weak Vogan dual, then*

$$\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \text{ occurs in the cell representation } \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})\}$$

is contained in the double cell in $\mathrm{Irr}(W(\Lambda))$ *containing* $\sigma_{\mathcal{C}}$*.*

Proof. Let $(\check{G}, \check{G}_{\mathbb{C}}, \check{\iota} : \check{G} \to \check{G}_{\mathbb{C}}, \check{\Lambda}, \check{\mathcal{C}})$ be a weak Vogan dual of $(G, G_{\mathbb{C}}, \iota : G \to G_{\mathbb{C}}, \Lambda, \mathcal{C})$. Similar to $\sigma_{\mathcal{C}}$, we have a special representation $\sigma_{\check{\mathcal{C}}} \in \mathrm{Irr}^{\mathrm{sp}}(\check{W}(\check{\Lambda}))$ associated to the Harish-Chandra cell $\check{\mathcal{C}}$.

Suppose that $\sigma \in \mathrm{Irr}(W(\Lambda))$ occurs in the cell representation $\mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})$. Then Proposition 4.8 implies that

$$\sigma_{\mathcal{C}} \underset{LR}{\leq} \sigma.$$

Since $\sigma \otimes \mathrm{sgn}$ occurs in the cell representation $\mathrm{Coh}_{\check{\Lambda}}(\mathcal{K}(\check{G}))(\check{\mathcal{C}})$, Proposition 4.8 also implies that

$$\sigma_{\check{\mathcal{C}}} \underset{LR}{\leq} \sigma \otimes \mathrm{sgn}\,.$$

Then Lemma 3.15 implies that

$$\sigma \underset{LR}{\leq} \sigma_{\check{\mathcal{C}}} \otimes \mathrm{sgn}\,.$$

Since $\tau_{\check{\mathcal{C}}}$ is dual to $\tau_{\mathcal{C}}$ by Lemma 4.13, Proposition 4.11 implies that $\tau_{\sigma_{\check{\mathcal{C}}}}$ is dual to $\tau_{\sigma_{\mathcal{C}}}$. Then Lemma 3.23 and Proposition 3.25 imply that

$$\sigma_{\check{\mathcal{C}}} \otimes \mathrm{sgn} \approx \sigma_{\mathcal{C}}.$$

Finally we conclude that $\sigma_{\mathcal{C}} \approx \sigma$ and the proposition is proved. □

Since weak Vogan duals exist for linear groups ([Vog82, Corollary 14.9]) and the real metaplectic group ([RT03, Theorem 5.2]), we obtain the following corollary.

Corollary 4.15. *Suppose that G has a faithful finite-dimensional representation, or G is isomorphic to the metaplectic group $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ ($n \in \mathbb{N}$), then*

$$\{\sigma \in \mathrm{Irr}(W(\Lambda)) \mid \sigma \textit{ occurs in the cell representation } \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C})\}$$

is contained in the double cell in $\mathrm{Irr}(W(\Lambda))$ containing $\sigma_{\mathcal{C}}$.

5. Counting irreducible representations

The purpose of this section is to prove two counting results on irreducible representations, valid for any G satisfying a weak form of Vogan duality. When G is linear or isomorphic to a real metaplectic group, the results appear as Theorem 2.1 and Corollary 2.2 in Section 2.2.

For every basal element $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}(G))$, let $\mathcal{O}_\Psi := \mathcal{O}_{\sigma_\Psi} \in \overline{\mathrm{Nil}}(\mathfrak{g}^*)$ be the nilpotent orbit corresponding to $j^W_{W(\Lambda)}\sigma_\Psi$. By the translation principle [Vog79b, Lemma 2.7], we have that

$$\mathrm{Ann}(\Psi(\nu)) = \overline{\mathcal{O}_\Psi} \qquad \text{(the Zariski closure of } \mathcal{O}_\Psi)$$

for all dominant $\nu \in \Lambda$ such that $\Psi(\nu)) \neq 0$. By the translation outside the dominant cone as in [Vog81, Proposition 7.3.11], we have that

$$\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}_{\overline{\mathcal{O}_\Psi}}(G)).$$

Recall that S is an $\mathrm{Ad}(\mathfrak{g})$-stable Zariski closed subset of $\mathrm{Nil}(\mathfrak{g}^*)$.

Lemma 5.1. *For all $\nu \in \Lambda$, the map*

$$\mathrm{ev}_\nu : \mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(G)) \to \mathcal{K}_{\nu,\mathsf{S}}(G), \quad \Psi \mapsto \Psi(\nu)$$

descends to a linear isomorphism

$$\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(G))_{W_\nu} \xrightarrow{\sim} \mathcal{K}_{\nu,\mathsf{S}}(G).$$

Here the subscript group indicates the coinvariant space.

Proof. Without loss of generality we assume that ν is dominant. Put

$$\mathcal{B} := \{\, \Psi \mid \Psi \text{ is basal and } \mathcal{O}_\Psi \subseteq \mathsf{S} \,\},$$

which is a basis of $\mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(G))$ by Lemma 4.4.

We have that

$$\begin{aligned} \ker(\mathrm{ev}_\nu) &= \mathrm{Span}\{\, \Psi \in \mathcal{B} \mid \Psi(\nu) = 0 \,\} \\ &= \mathrm{Span}\{\, \Psi - s\cdot\Psi \mid \Psi \in \mathcal{B}, s \text{ is a simple reflection in } W_\nu \,\} \\ &\quad\ \text{(by Lemma 4.5)} \\ &\subseteq \mathrm{Span}\{\, \Psi - w\cdot\Psi \mid \Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}),\ w \in W_\nu \,\} \\ &\subseteq \ker(\mathrm{ev}_\nu). \end{aligned}$$

Therefore

$$\ker(\mathrm{ev}_\nu) = \mathrm{Span}\,\{\, \Psi - w \cdot \Psi \mid \Phi \in \mathrm{Coh}_\Lambda(\mathcal{K}),\, w \in W_\nu \,\}\,.$$

Together with Lemma 4.3, this implies the lemma. $\square$

For every $\nu \in {}^a\mathfrak{h}^*$, let $\mathrm{Irr}_{\nu,\mathsf{S}}(G)$ denote the set of isomorphism classes of irreducible representations in $\mathrm{Rep}_{\nu,\mathsf{S}}(G)$.

Proposition 5.2 (Vogan). *For every $\nu \in \Lambda$,*

$$\sharp(\mathrm{Irr}_{\nu,\mathsf{S}}(G)) = [1_{W_\nu} : \mathrm{Coh}_\Lambda(\mathcal{K}_\mathsf{S}(G))].$$

Proof. Since $\sharp(\mathrm{Irr}_{\nu,\mathsf{S}}(G)) = \dim \mathcal{K}_{\nu,\mathsf{S}}(G)$, this is implied by Lemma 5.1. $\square$

Proposition 5.3. *Let Ψ be a basal element of $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$, and let $\nu \in \Lambda$ be a dominant element. Then $\Psi(\nu) \in \mathrm{Irr}_{\nu,\overline{\mathcal{O}_\nu}}(G)$ if and only if $\sigma_\Psi \cong \left(j_{W_\nu}^{W(\Lambda)} \mathrm{sgn}\right) \otimes \mathrm{sgn}$ and there is no element of τ_Ψ that fixes ν.*

Proof. First suppose that $\sigma_\mathcal{C} \cong \left(j_{W_\nu}^{W(\Lambda)} \mathrm{sgn}\right) \otimes \mathrm{sgn}$ and there is no element of τ_Ψ that fixes ν. Then Lemma 4.5 implies that $\Psi(\nu) \in \mathrm{Irr}(G)$ and Proposition 4.11 implies that

$$\sigma_{\nu,\mathrm{Ann}(\Psi(\nu))} \cong \left(j_{W_\nu}^{W(\Lambda)} \mathrm{sgn}\right) \otimes \mathrm{sgn}\,.$$

Thus $\mathrm{Ann}(\Psi(\nu)) = I_\nu$ by Lemma 3.22, and hence $\Psi(\nu) \in \mathrm{Irr}_{\nu,\overline{\mathcal{O}_\nu}}(G)$.

Now we suppose that $\Psi(\nu) \in \mathrm{Irr}_{\nu,\overline{\mathcal{O}_\nu}}(G)$. Lemma 4.5 implies that there is no element of τ_Ψ that fixes ν. As $\mathrm{Ann}(\Psi(\nu)) = I_\nu$, Lemma 3.22 and Proposition 4.11 imply that $\sigma_\mathcal{C} \cong \left(j_{W_\nu}^{W(\Lambda)} \mathrm{sgn}\right) \otimes \mathrm{sgn}$. This completes the proof of the proposition. $\square$

Proposition 5.4. *For all $\sigma \in \mathrm{Irr}(W(\Lambda)) \setminus \mathrm{Irr}_\mathsf{S}(W(\Lambda))$,*

$$[\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}_\mathsf{S}(G))] = 0.$$

Proof. This is implied by Proposition 4.8 and Proposition 3.20. $\square$

Proposition 5.2 and Proposition 5.4 imply that for every $\nu \in \Lambda$,

$$\sharp(\mathrm{Irr}_{\nu,\mathsf{S}}(G)) = \sum_{\sigma \in \mathrm{Irr}_\mathsf{S}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}_\mathsf{S}(G))]. \tag{5.1}$$

Theorem 5.5. *For all $\nu \in \Lambda$,*

$$\sharp(\mathrm{Irr}_{\nu,\mathsf{S}}(G)) \leq \sum_{\sigma \in \mathrm{Irr}_\mathsf{S}(W(\Lambda))} [1_{W_\nu} : \sigma] \cdot [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

and the equality holds if Conjecture 4.12 holds for G.

Proof. The first assertion is immediate from (5.1). As a representation of $W(\Lambda)$,

$$\mathrm{Coh}_\Lambda(\mathcal{K}(G)) \cong \bigoplus_{\mathcal{C} \text{ is a cell in } \mathrm{Coh}_\Lambda(\mathcal{K}(G))} \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}).$$

Proposition 4.11 implies that

$$\mathrm{Coh}_\Lambda(\mathcal{K}_\mathsf{S}(G)) \cong \bigoplus_{\mathcal{C} \text{ is a cell in } \mathrm{Coh}_\Lambda(\mathcal{K}(G)),\ \sigma_\mathcal{C} \in \mathrm{Irr}_\mathsf{S}^{\mathrm{sp}}(W(\Lambda))} \mathrm{Coh}_\Lambda(\mathcal{K}(G))(\mathcal{C}),$$

where $\sigma_{\mathcal{C}}$ is as in Proposition 4.11. Under the assumption that Conjecture 4.12 holds for G,

$$[\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}_{\mathsf{S}}(G))] = [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))] \quad \text{for all } \sigma \in \mathrm{Irr}_{\mathsf{S}}(W(\Lambda)).$$

Together with (5.1), this implies the second assertion. $\square$

Recall that $\mathcal{O}_\nu$ is the nilpotent orbit whose Zariski closure $\overline{\mathcal{O}_\nu} \subseteq \mathfrak{g}^*$ equals the associated variety of the maximal ideal I_ν. Recall also the Lusztig left cell ${}^L\mathscr{C}_\nu$ from (3.21).

Corollary 5.6. *For all $\nu \in \Lambda$,*

$$\sharp(\mathrm{Irr}_{\nu,\overline{\mathcal{O}_\nu}}(G)) \leq \sum_{\sigma \in {}^L\mathscr{C}_\nu} [\sigma : \mathrm{Coh}_\Lambda(\mathcal{K}(G))],$$

and the equality holds if Conjecture 4.12 holds for G.

Proof. In view of (3.20), Proposition 3.21 and Proposition 5.3, this follows from Theorem 5.5. $\square$

6. Separating good parity and bad parity for coherent continuation representations

We continue the notation of the last section. We further assume that G is a classical Lie group as in Sections 2.3-2.8,

$$G_{\mathbb{C}} := \begin{cases} \mathrm{GL}_n(\mathbb{C}), & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}; \\ \mathrm{GL}_{p+q}(\mathbb{C}), & \text{if } \star \in \{A, \widetilde{A}\}; \\ \mathrm{SO}_{p+q}(\mathbb{C}), & \text{if } \star \in \{\, B, D \,\}; \\ \mathrm{SO}_{2n}(\mathbb{C}), & \text{if } \star = D^*; \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } \star \in \{C, \widetilde{C}\}; \\ \mathrm{Sp}_{p+q}(\mathbb{C}), & \text{if } \star = C^*, \end{cases} \tag{6.1}$$

and $\iota : G \to G_{\mathbb{C}}$ is the usual complexification homomorphism.

In the rest of the paper we will freely use the notation of Sections 2.3-2.8.

If $\star \in \{\widetilde{A}, \widetilde{C}\}$, we let $\mathcal{K}'(G)$ denote the Grothendieck group of the category of genuine Casselman-Wallach representations of G. Otherwise, put $\mathcal{K}'(G) := \mathcal{K}(G)$. Similar notations such as $\mathcal{P}'_\Lambda(G)$, $\mathcal{K}'_\nu(G)$ and $\mathrm{Irr}'_\nu(G)$ will be used without further explanation.

6.1. Standard representations of classical groups.

Define a partition

$$\{\{A^{\mathbb{R}}\}, \{A^{\mathbb{H}}\}, \{A, \widetilde{A}\}, \{B, D\}, \{C, \widetilde{C}\}, \{C^*\}, \{D^*\}\}$$

of the set of the symbols. Let $[\star]$ denote the equivalence class containing $\star$ of the corresponding equivalence relation.

We define a $[\star]$-space to be a finite dimensional complex vector space V equipped with a $[\star]$-structure that is defined as in what follows.

The case when $\star \in \{\, A^{\mathbb{R}}, A^{\mathbb{H}} \,\}$. In this case, a $[\star]$-structure on V is a conjugate linear automorphism $\mathbf{j} : V \to V$ such that

$$\mathbf{j}^2 = \begin{cases} 1, & \text{if } \star = A^{\mathbb{R}}; \\ -1, & \text{if } \star = A^{\mathbb{H}}. \end{cases}$$

The case when $\star \in \{ A, \widetilde{A} \}$. In this case, a $[\star]$-structure on V is a non-degenerate Hermitian form $\langle\,,\,\rangle : V \times V \to \mathbb{C}$ (which is by convention linear on the first variable and conjugate linear on the second variable).

The case when $\star \in \{ C, \widetilde{C}, C^* \}$. In this case, a $[\star]$-structure on V is a symplectic form $\langle\,,\,\rangle : V \times V \to \mathbb{C}$ together will a conjugate linear automorphism $\mathbf{j} : V \to V$ such that

$$\mathbf{j}^2 = \begin{cases} 1, & \text{if } \star \in \{C, \widetilde{C}\}; \\ -1, & \text{if } \star = C^*, \end{cases}$$

and for all $u, v \in V$,

$$\langle \mathbf{j}(u), \mathbf{j}(v) \rangle = \text{the complex conjugate of } \langle u, v \rangle.$$

The case when $\star \in \{ B, D, D^* \}$. In this case, a $[\star]$-structure on V is a triple $(\langle\,,\,\rangle, \mathbf{j}, \omega)$ where $\langle\,,\,\rangle : V \times V \to \mathbb{C}$ is a non-degenerate symmetric bilinear form, $\mathbf{j} : V \to V$ is a conjugate linear automorphism, and $\omega \in \wedge^{\dim V} V$ (to be called the orientation), subject to the following conditions:

- $$\mathbf{j}^2 = \begin{cases} 1, & \text{if } \star \in \{B, D\}; \\ -1, & \text{if } \star = D^*; \end{cases}$$
- for all $u, v \in V$,
$$\langle \mathbf{j}(u), \mathbf{j}(v) \rangle = \text{the complex conjugate of } \langle u, v \rangle;$$
- $\langle \omega, \omega \rangle = 1$, where $\langle\,,\,\rangle : \wedge^{\dim V} V \times \wedge^{\dim V} V \to \mathbb{C}$ is the symmetric bilinear form given by

$$\langle u_1 \wedge u_2 \wedge \cdots \wedge u_{\dim V}, v_1 \wedge v_2 \wedge \cdots \wedge v_{\dim V} \rangle = \det\left([\langle u_i, v_j \rangle]_{i,j=1,2,\ldots,\dim V} \right); \tag{6.2}$$

- if $\star = D^*$, then

$$\omega = \sqrt{-1} u_1 \wedge \mathbf{j}(u_1) \wedge \sqrt{-1} u_2 \wedge \mathbf{j}(u_2) \wedge \cdots \wedge \sqrt{-1} u_{\frac{\dim V}{2}} \wedge \mathbf{j}(u_{\frac{\dim V}{2}}), \tag{6.3}$$

  for some vectors $u_1, u_2, \ldots, u_{\frac{\dim V}{2}} \in V$;
- if $\star \in \{B, D\}$ and $\dim V = 0$, then $\omega = 1$ (as an element of $\wedge^{\dim V} V = \mathbb{C}$).

We remark that when $\star = D^*$, the existence of $\mathbf{j}$ implies that $\dim V$ is even, and up to a positive scalar multiplication the vector

$$\sqrt{-1} u_1 \wedge \mathbf{j}(u_1) \wedge \sqrt{-1} u_2 \wedge \mathbf{j}(u_2) \wedge \cdots \wedge \sqrt{-1} u_{\frac{\dim V}{2}} \wedge \mathbf{j}(u_{\frac{\dim V}{2}}) \tag{6.4}$$

is independent of the choice of $u_1, u_2, \ldots, u_{\frac{\dim V}{2}} \in V$ that makes (6.4) a nonzero vector.

Suppose that V is a $[\star]$-space, with a $[\star]$-structure as above. Whenever it is defined, write $\mathbf{j}_V := \mathbf{j}$, $\langle\,,\,\rangle_V := \langle\,,\,\rangle$, and $\omega_V := \omega$.

Isomorphisms of $[\star]$-spaces are obviously defined. Given two $[\star]$-spaces V_1 and V_2, the product $V_1 \times V_2$ is obviously a $[\star]$-space. We remark that when $[\star] = \{B, D\}$ and both $\dim V_1$ and $\dim V_2$ are odd, the switching map $V_1 \times V_2 \to V_2 \times V_1$ does not preserve the orientation.

We define $\mathsf{G}(V)$ to be the subgroup of $\mathrm{GL}(V)$ fixing the $[\star]$-structure, which is a real classical group. Let $\mathsf{G}_{\mathbb{C}}(V)$ denote the Zariski closure of $\mathsf{G}(V)$ in $\mathrm{GL}(V)$. Put

$$\mathsf{G}_\star(V) := \begin{cases} \text{the } \det^{\frac{1}{2}}\text{-double cover of } \mathsf{G}(V), & \text{if } \star = \widetilde{A}; \\ \text{the metaplectic double cover of } \mathsf{G}(V), & \text{if } \star = \widetilde{C}; \\ \mathsf{G}(V), & \text{otherwise.} \end{cases}$$

Note that $\mathsf{G}_\star(V)$ may be a special orthogonal group of type D even when $\star = B$, and vice versa.

Now we assume that G is identified with $\mathsf{G}_\star(V)$. We call V the standard representation of G. Recall than n is the rank of $\mathfrak{g}$ in all cases. Take a flag

$$\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_n \tag{6.5}$$

in V such that

- $\dim V_i = i$ for all $i = 1, 2, \ldots, n$.
- if $\star \notin \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}$, then V_n is totaly isotropic (with respect to the bilinear form $\langle\,,\,\rangle_V : V \times V \to \mathbb{C}$);
- if $\star \in \{D, D^*\}$, then V_n is ω_V-compatible in the sense that
$$\omega_V = u_1 \wedge u_2 \wedge \cdots \wedge u_{2n}$$
for some elements $u_1, u_2, \ldots, u_{2n} \in V_{2n}$ such that $u_1, u_2, \cdots, u_n \in V_n$ and
$$\langle u_i, u_j \rangle = \begin{cases} 1, & \text{if } i + j = 2n + 1; \\ 0, & \text{if } i + j \neq 2n + 1, \end{cases}$$
for all $i, j = 1, 2, \cdots, 2n$.

The flag (6.5) is unique up to the conjugation by $G_{\mathbb{C}}$. We remark that in the case when $\star = D^*$ and n is even, the condition (6.3) ensures that there exists a $\mathbf{j}_V$-stable totally isotropic subspace of V of dimension n that is ω_V-compatible.

The stabilizer of the flag (6.5) in $\mathfrak{g}$ is a Borel subalgebra of $\mathfrak{g}$. Using this Borel subgroup, we get an identification

$$^a\mathfrak{h} = \prod_{i=1}^{n} \mathfrak{gl}(V_i/V_{i-1}) = \mathbb{C}^n. \tag{6.6}$$

This identification is independent of the choice of the flag (6.5). As in Section 4.1, we assume that $Q = Q_\iota$ (the analytic weight lattice). Note that

$$Q = \mathbb{Z}^n \subseteq \mathbb{C}^n = (\mathbb{C}^n)^* = {}^a\mathfrak{h}^*, \tag{6.7}$$

and the positive roots are

$$\Delta^+ = \begin{cases} \{e_i - e_j \mid 1 \leq i < j \leq n\}, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}; \\ \{e_i \pm e_j \mid 1 \leq i < j \leq n\}, & \text{if } \star \in \{D, D^*\}; \\ \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}, & \text{if } \star = B; \\ \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{2e_i \mid 1 \leq i \leq n\}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}. \end{cases}$$

Here $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. The Weyl group

$$W = \begin{cases} \mathsf{S}_n, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}; \\ \mathsf{W}_n, & \text{if } \star \in \{B, C, \widetilde{C}, C^*\}; \\ \mathsf{W}'_n, & \text{if } \star \in \{D, D^*\}, \end{cases} \tag{6.8}$$

where $\mathsf{S}_n \subseteq \mathrm{GL}_n(\mathbb{Z})$ is the group of the permutation matrices, $\mathsf{W}_n \subseteq \mathrm{GL}_n(\mathbb{Z})$ is the subgroup generated by S_n and all the diagonal matrices with diagonal entries ± 1, and $\mathsf{W}'_n \subseteq \mathrm{GL}_n(\mathbb{Z})$ is the subgroup generated by S_n and all the diagonal matrices with determinant 1 and diagonal entries ± 1.

6.2. **Good parity and bad parity in the Langlands dual.** Recall that an integer has good parity (depends on $\star$ and $n = \operatorname{rank} \check{\mathfrak{g}}$) if it has the same parity as

$$\begin{cases} n, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A\}; \\ 1+n, & \text{if } \star = \widetilde{A}; \\ 1, & \text{if } \star \in \{\, C, C^*, D, D^* \,\}; \\ 0, & \text{if } \star \in \{\, B, \widetilde{C} \,\}. \end{cases}$$

Otherwise it has bad parity.

Let $\check{V}$ denote the standard representation of $\check{\mathfrak{g}}$ so that $\check{\mathfrak{g}}$ is identified with a Lie subalgebra of $\mathfrak{gl}(\check{V})$. When $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}$, $\mathfrak{g} = \mathfrak{gl}(\check{V})$. When $\star \in \{B, \widetilde{C}\}$, the space $\check{V}$ is equipped with a $\check{\mathfrak{g}}$-invariant symplectic form $\langle\, ,\, \rangle_{\check{V}} : \check{V} \times \check{V} \to \mathbb{C}$. When $\star \in \{C, C^*, D, D^*\}$, the space $\check{V}$ is equipped with a $\check{\mathfrak{g}}$-invariant non-degenerate symmetric bilinear form $\langle\, ,\, \rangle_{\check{V}} : \check{V} \times \check{V} \to \mathbb{C}$ together with an element $\omega_{\check{V}} \in \wedge^{\dim \check{V}} \check{V}$ with $\langle \omega_{\check{V}}, \omega_{\check{V}} \rangle_{\check{V}} = 1$ (see (6.2)).

As in (6.6) we also have that ${}^a\check{\mathfrak{h}} = \mathbb{C}^n$. Thus we have an identification ${}^a\check{\mathfrak{h}} = {}^a\mathfrak{h}^*$ that identifies $\check{\mathfrak{g}}$ as the Langlands dual (or the metaplectic Langlands dual) of $\mathfrak{g}$.

Recall the semisimple element $\lambda^\circ_{\check{\mathcal{O}}} \in \check{\mathfrak{g}}$ which is uniquely determined by $\check{\mathcal{O}}$ up to $\mathrm{Ad}(\check{\mathfrak{g}})$-conjugation. We have a Lie algebra homomorphism

$$\eta : \mathfrak{sl}_2(\mathbb{C}) \to \check{\mathfrak{g}}$$

such that

$$\eta\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \in \check{\mathcal{O}} \quad \text{and} \quad \eta\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 2\lambda^\circ_{\check{\mathcal{O}}}. \tag{6.9}$$

We view $\check{V}$ as an $\mathfrak{sl}_2(\mathbb{C})$-module via η. Put

$$\check{V}_{\mathrm{b}} := \text{sum of irreducible } \mathfrak{sl}_2(\mathbb{C})\text{-submodules of } \check{V} \text{ with bad parity dimension.}$$

Define $\check{V}_{\mathrm{g}}$ similarly by using the good parity so that

$$\check{V} = \check{V}_{\mathrm{b}} \times \check{V}_{\mathrm{g}}. \tag{6.10}$$

Write $\check{\mathfrak{g}}_{\mathrm{b}}$ for the Lie subalgebra of $\check{\mathfrak{g}}$ consisting the elements that annihilate $\check{V}_{\mathrm{g}}$ and stabilize $\check{V}_{\mathrm{b}}$. Define $\check{\mathfrak{g}}_{\mathrm{g}}$ similarly so that $\check{\mathfrak{g}}_{\mathrm{b}} \times \check{\mathfrak{g}}_{\mathrm{g}}$ is the stabilizer of the decomposition (6.10) in $\check{\mathfrak{g}}$. We have natural isomorphisms

$$(\check{\mathfrak{g}}_{\mathrm{b}}, \check{\mathfrak{g}}_{\mathrm{g}}) \cong \begin{cases} (\mathfrak{gl}_{n_{\mathrm{b}}}(\mathbb{C}), \mathfrak{gl}_{n_{\mathrm{g}}}(\mathbb{C})), & \text{if } \star \in \{\, A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A} \,\}; \\ (\mathfrak{sp}_{2n_{\mathrm{b}}}(\mathbb{C}), \mathfrak{sp}_{2n_{\mathrm{g}}}(\mathbb{C})), & \text{if } \star \in \{\, B, \widetilde{C} \,\}; \\ (\mathfrak{o}_{2n_{\mathrm{b}}}(\mathbb{C}), \mathfrak{o}_{2n_{\mathrm{g}}+1}(\mathbb{C})), & \text{if } \star \in \{\, C, C^* \,\}; \\ (\mathfrak{o}_{2n_{\mathrm{b}}}(\mathbb{C}), \mathfrak{o}_{2n_{\mathrm{g}}}(\mathbb{C})), & \text{if } \star \in \{\, D, D^* \,\}, \end{cases}$$

where n_{b} and n_{g} are ranks of $\check{\mathfrak{g}}_{\mathrm{b}}$ and $\check{\mathfrak{g}}_{\mathrm{g}}$ respectively so that $n_{\mathrm{b}} + n_{\mathrm{g}} = n$. Note that when $\star \in \{B, C, \widetilde{C}, C^*, D, D^*\}$, n_{b} agrees with the one defined in (2.9).

When $\star \in \{B, C, \widetilde{C}, C^*, D, D^*\}$, (6.10) is an orthogonal decomposition, and the form $\langle\, ,\, \rangle_{\check{V}}$ on $\check{V}$ restricts to bilinear forms on $\check{V}_{\mathrm{b}}$ and $\check{V}_{\mathrm{g}}$, to be respectively denoted by $\langle\, ,\, \rangle_{\check{V}_{\mathrm{b}}}$ and $\langle\, ,\, \rangle_{\check{V}_{\mathrm{b}}}$. When $\star \in \{C, C^*, D, D^*\}$, we equip on $\check{V}_{\mathrm{b}}$ an element $\omega_{\check{V}_{\mathrm{b}}} \in \wedge^{\dim \check{V}_{\mathrm{b}}} \check{V}_{\mathrm{b}}$ and equip on $\check{V}_{\mathrm{g}}$ an element $\omega_{\check{V}_{\mathrm{g}}} \in \wedge^{\dim \check{V}_{\mathrm{g}}} \check{V}_{\mathrm{g}}$ such that

$$\langle \omega_{\check{V}_{\mathrm{b}}}, \omega_{\check{V}_{\mathrm{b}}} \rangle_{\check{V}_{\mathrm{b}}} = \langle \omega_{\check{V}_{\mathrm{g}}}, \omega_{\check{V}_{\mathrm{g}}} \rangle_{\check{V}_{\mathrm{g}}} = 1 \quad \text{and} \quad \omega_{\check{V}_{\mathrm{b}}} \wedge \omega_{\check{V}_{\mathrm{g}}} = \omega_{\check{V}}. \tag{6.11}$$

Then as in (6.6), in all cases we have identifications

$$ {}^a\check{\mathfrak{h}}_{\mathrm{b}} = \mathbb{C}^{n_{\mathrm{b}}}, \quad {}^a\check{\mathfrak{h}}_{\mathrm{g}} = \mathbb{C}^{n_{\mathrm{g}}} \quad \text{and} \quad {}^a\check{\mathfrak{h}} = {}^a\check{\mathfrak{h}}_{\mathrm{b}} \times {}^a\check{\mathfrak{h}}_{\mathrm{g}} = \mathbb{C}^n, $$

where ${}^a\check{\mathfrak{h}}_{\mathrm{b}}$ and ${}^a\check{\mathfrak{h}}_{\mathrm{g}}$ are the abstract Cartan algebras of $\check{\mathfrak{g}}_{\mathrm{b}}$ and $\check{\mathfrak{g}}_{\mathrm{g}}$, respectively.

The homomorphism η obviously induces Lie algebra homomorphisms

$$ \eta_{\mathrm{g}} : \mathfrak{sl}_2(\mathbb{C}) \to \check{\mathfrak{g}}_{\mathrm{g}} \quad \text{and} \quad \eta_{\mathrm{b}} : \mathfrak{sl}_2(\mathbb{C}) \to \check{\mathfrak{g}}_{\mathrm{b}}. $$

By using these two homomorphisms, we obtain $\check{\mathcal{O}}_{\mathrm{g}} \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}}_{\mathrm{g}})$, $\lambda^{\circ}_{\check{\mathcal{O}}_{\mathrm{g}}} \in \check{\mathfrak{g}}_{\mathrm{g}}$, $\check{\mathcal{O}}_{\mathrm{b}} \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}}_{\mathrm{b}})$, and $\lambda^{\circ}_{\check{\mathcal{O}}_{\mathrm{b}}} \in \check{\mathfrak{g}}_{\mathrm{b}}$ as in (6.9). Then

$$ \mathbf{d}_{\check{\mathcal{O}}} = \mathbf{d}_{\check{\mathcal{O}}_{\mathrm{g}}} \overset{r}{\sqcup} \mathbf{d}_{\check{\mathcal{O}}_{\mathrm{b}}}. $$

Put

$$ \star_{\mathrm{g}} := \begin{cases} A, & \text{if } \star = \widetilde{A} \text{ and } p+q \text{ is odd;} \\ \star, & \text{otherwise,} \end{cases} $$

and

$$ \star_{\mathrm{b}} := \begin{cases} \widetilde{A}, & \text{if } \star = A \text{ and } p+q \text{ is odd;} \\ D, & \text{if } \star = C; \\ D^*, & \text{if } \star = C^*. \\ \star, & \text{otherwise.} \end{cases} $$

The orbit $\check{\mathcal{O}}_{\mathrm{g}}$ has good parity in the sense that all its nonzero row lengths have good parity with respect to $\star_{\mathrm{g}}$ and $\check{\mathfrak{g}}_{\mathrm{g}}$, and likewise the orbit $\check{\mathcal{O}}_{\mathrm{b}}$ has bad parity.

Put

$$ \Lambda_{\mathrm{b}} := \begin{cases} \mathbb{Z}^{n_{\mathrm{b}}}, & \text{if } \star \in \{A^{\mathbb{R}}, A\} \text{ and } n \text{ is even, or } \star \in \{A^{\mathbb{H}}, B, \widetilde{C}\}; \\ (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) + \mathbb{Z}^{n_{\mathrm{b}}}, & \text{otherwise,} \end{cases} $$

which is the $\mathbb{Z}^{n_{\mathrm{b}}}$-coset in $\mathbb{C}^{n_{\mathrm{b}}} = {}^a\check{\mathfrak{h}}_{\mathrm{b}}$ containing all the elements in ${}^a\check{\mathfrak{h}}_{\mathrm{b}}$ that represent the $\mathrm{Ad}(\check{\mathfrak{g}}_{\mathrm{b}})$-conjugacy class of $\lambda^{\circ}_{\check{\mathcal{O}}_{\mathrm{b}}}$. Likewise put

$$ \Lambda_{\mathrm{g}} := \begin{cases} (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) + \mathbb{Z}^{n_{\mathrm{g}}}, & \text{if } \star \in \{A^{\mathbb{R}}, A\} \text{ and } n \text{ is even, or } \star \in \{A^{\mathbb{H}}, B, \widetilde{C}\}; \\ \mathbb{Z}^{n_{\mathrm{g}}}, & \text{otherwise,} \end{cases} $$

which is the $\mathbb{Z}^{n_{\mathrm{g}}}$-coset in $\mathbb{C}^{n_{\mathrm{g}}} = {}^a\check{\mathfrak{h}}_{\mathrm{g}}$ containing all the elements in ${}^a\check{\mathfrak{h}}_{\mathrm{g}}$ that represent the $\mathrm{Ad}(\check{\mathfrak{g}}_{\mathrm{g}})$-conjugacy class of $\lambda^{\circ}_{\check{\mathcal{O}}_{\mathrm{g}}}$. We also define

$$ \Lambda := \Lambda_{\mathrm{b}} \times \Lambda_{\mathrm{g}} \subseteq {}^a\mathfrak{h}^*_{\mathrm{b}} \times {}^a\mathfrak{h}^*_{\mathrm{g}} = \mathbb{C}^{n_{\mathrm{b}}} \times \mathbb{C}^{n_{\mathrm{g}}} = \mathbb{C}^n = {}^a\check{\mathfrak{h}} = {}^a\mathfrak{h}^*. \tag{6.12} $$

6.3. **Separating the abstract Cartan algebra.** When $\star \in \{ B, C, \widetilde{C}, C^*, D, D^* \}$, we have defined in Section 2.7 the notion of $\check{\mathcal{O}}$-relevant parabolic subgroups of G. We extend this notion to the other cases: If $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}$, then a parabolic subgroup of G is said to be $\check{\mathcal{O}}$-relevant if it is the stabilizer group of $\mathbf{j}$-stable subspace of V of dimension n_{b}. If $\star \in \{A, \widetilde{A}\}$, then a parabolic subgroup of G is said to be $\check{\mathcal{O}}$-relevant if it is the stabilizer group of totally isotropic subspace of V of dimension $\frac{n_{\mathrm{b}}}{2}$ (in particular, the existence of such a parabolic subgroup implies that n_{b} is even).

Through a case-by-case verification, we record the following lemma without proof.

Lemma 6.1. *For every $\gamma = (H, \xi, \Gamma) \in \mathscr{P}'_{\Lambda}(G)$, H is contained in a parabolic subgroup of G that is $\check{\mathcal{O}}'$-relevant for some $\check{\mathcal{O}}' \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}})$ with $\mathbf{d}_{\check{\mathcal{O}}'} = \mathbf{d}_{\check{\mathcal{O}}}$.*

In the rest of this paper we assume that G has a parabolic subgroups that is $\check{\mathcal{O}}'$-relevant for some $\check{\mathcal{O}}' \in \overline{\mathrm{Nil}}(\check{\mathfrak{g}})$ with $\mathbf{d}_{\check{\mathcal{O}}'} = \mathbf{d}_{\check{\mathcal{O}}}$. Otherwise Lemma 6.1 and Lemma 4.3 imply that $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is empty. The assumption is equivalent to saying that

$$n_{\mathrm{b}} \text{ is even if } \star = \widetilde{A}, \tag{6.13}$$

and

$$p, q \geq \begin{cases} \frac{n_{\mathrm{b}}}{2}, & \text{if } \star \in \{A, \widetilde{A}\}; \\ n_{\mathrm{b}}, & \text{if } \star \in \{B, C^*, D\}. \end{cases} \tag{6.14}$$

In particular, $\star_{\mathrm{g}} = \star$ in all cases.

Define two classical groups

$$(G_{\mathrm{b}}, G_{\mathrm{g}}) = \begin{cases} (\mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}), \mathrm{GL}_{n_{\mathrm{g}}}(\mathbb{R})), & \text{if } \star = A^{\mathbb{R}}; \\ (\mathrm{GL}_{\frac{n_{\mathrm{b}}}{2}}(\mathbb{H}), \mathrm{GL}_{\frac{n_{\mathrm{g}}}{2}}(\mathbb{H})), & \text{if } \star = A^{\mathbb{H}}; \\ (\mathrm{U}(\frac{n_{\mathrm{b}}}{2}, \frac{n_{\mathrm{b}}}{2}), \mathrm{U}(p_{\mathrm{g}}, q_{\mathrm{g}})), & \text{if } \star = A \text{ and } p+q \text{ is even}; \\ (\widetilde{\mathrm{U}}(\frac{n_{\mathrm{b}}}{2}, \frac{n_{\mathrm{b}}}{2}), \mathrm{U}(p_{\mathrm{g}}, q_{\mathrm{g}})), & \text{if } \star = A \text{ and } p+q \text{ is odd}; \\ (\widetilde{\mathrm{U}}(\frac{n_{\mathrm{b}}}{2}, \frac{n_{\mathrm{b}}}{2}), \widetilde{\mathrm{U}}(p_{\mathrm{g}}, q_{\mathrm{g}})), & \text{if } \star = \widetilde{A}; \\ (\mathrm{SO}(n_{\mathrm{b}}, n_{\mathrm{b}}+1), \mathrm{SO}(p_{\mathrm{g}}, q_{\mathrm{g}})), & \text{if } \star = B; \\ (\mathrm{SO}(n_{\mathrm{b}}, n_{\mathrm{b}}), \mathrm{Sp}_{2n_{\mathrm{g}}}(\mathbb{R})), & \text{if } \star = C; \\ (\mathrm{O}^*(2n_{\mathrm{b}}), \mathrm{Sp}(\frac{p_{\mathrm{g}}}{2}, \frac{q_{\mathrm{g}}}{2})), & \text{if } \star = C^*; \\ (\widetilde{\mathrm{Sp}}_{2n_{\mathrm{b}}}(\mathbb{R}), \widetilde{\mathrm{Sp}}_{2n_{\mathrm{g}}}(\mathbb{R})), & \text{if } \star = \widetilde{C}; \\ (\mathrm{SO}(n_{\mathrm{b}}, n_{\mathrm{b}}), \mathrm{SO}(p_{\mathrm{g}}, q_{\mathrm{g}})), & \text{if } \star = D; \\ (\mathrm{O}^*(2n_{\mathrm{b}}), \mathrm{O}^*(2n_{\mathrm{g}})), & \text{if } \star = D^*, \end{cases}$$

where

$$(p_{\mathrm{g}}, q_{\mathrm{g}}) = \begin{cases} (p - \frac{n_{\mathrm{b}}}{2}, q - \frac{n_{\mathrm{b}}}{2}), & \text{if } \star \in \{A, \widetilde{A}\}; \\ (p - n_{\mathrm{b}}, q - n_{\mathrm{b}}), & \text{if } \star \in \{B, C^*, D\}. \end{cases}$$

Then G_{b} has type $\star_{\mathrm{b}}$ and G_{g} has type $\star$.

As in (6.1), we have the complex groups $G_{\mathrm{b},\mathbb{C}}$ and $G_{\mathrm{g},\mathbb{C}}$, and the complexification homomorphisms $\iota_{\mathrm{b}} : G_{\mathrm{b}} \to G_{\mathrm{b},\mathbb{C}}$ and the complexification homomorphisms $\iota_{\mathrm{g}} : G_{\mathrm{g}} \to G_{\mathrm{g},\mathbb{C}}$.

Suppose that G_{b} is identified with $\mathsf{G}_{\star_{\mathrm{b}}}(V_{\mathrm{b}})$ so that $G_{\mathrm{b},\mathbb{C}}$ is identified with $\mathsf{G}_{\mathbb{C}}(V_{\mathrm{b}})$, where V_{b} is a $[\star_{\mathrm{b}}]$-space. Similarly, suppose that G_{g} is identified with $\mathsf{G}_{\star_{\mathrm{g}}}(V_{\mathrm{g}})$ so that $G_{\mathrm{g},\mathbb{C}}$ is identified with $\mathsf{G}_{\mathbb{C}}(V_{\mathrm{g}})$, where V_{g} is a $[\star]$-space.

Respectively write ${}^a\mathfrak{h}_{\mathrm{b}}$ and ${}^a\mathfrak{h}_{\mathrm{g}}$ for the abstract Cartan algebras for $\mathfrak{g}_{\mathrm{b}}$ and $\mathfrak{g}_{\mathrm{g}}$. Then we have identifications

$${}^a\mathfrak{h}_{\mathrm{b}}^* = \mathbb{C}^{n_{\mathrm{b}}} = {}^a\check{\mathfrak{h}}_{\mathrm{b}} \qquad \text{and} \qquad {}^a\mathfrak{h}_{\mathrm{g}}^* = \mathbb{C}^{n_{\mathrm{g}}} = {}^a\check{\mathfrak{h}}_{\mathrm{g}}.$$

Using these identifications, we view $\check{\mathfrak{g}}_{\mathrm{b}}$ as the Langlands dual (or the metaplectic Langlands dual when $\star = \widetilde{C}$) of $\mathfrak{g}_{\mathrm{b}}$, and view $\check{\mathfrak{g}}_{\mathrm{g}}$ as the Langlands dual (or the metaplectic Langlands dual when $\star = \widetilde{C}$) of $\mathfrak{g}_{\mathrm{g}}$.

We also have identifications

$${}^a\mathfrak{h}_{\mathrm{b}}^* \times {}^a\mathfrak{h}_{\mathrm{g}}^* = \mathbb{C}^{n_{\mathrm{b}}} \times \mathbb{C}^{n_{\mathrm{g}}} = \mathbb{C}^n = {}^a\mathfrak{h}^*.$$

Under this identification, we have that

$$\Delta_{\mathrm{b}}^+ = \Delta_{\mathrm{b}} \cap \Delta^+ \qquad \text{and} \qquad \Delta_{\mathrm{g}}^+ = \Delta_{\mathrm{g}} \cap \Delta^+,$$

where $\Delta_{\mathrm{b}}^+ \subseteq \Delta_{\mathrm{b}} \subseteq {}^a\mathfrak{h}_{\mathrm{b}}^*$ are respectively the positive root system and the root system of $\mathfrak{g}_{\mathrm{b}}$, and likewise $\Delta_{\mathrm{g}}^+ \subseteq \Delta_{\mathrm{g}} \subseteq {}^a\mathfrak{h}_{\mathrm{g}}^*$ are respectively the positive root system and the root system of $\mathfrak{g}_{\mathrm{g}}$.

Similar to (6.7), we have the analytic weight lattices

$$Q_{\iota_{\mathrm{b}}} = \mathbb{Z}^{n_{\mathrm{b}}} \subseteq \mathbb{C}^{n_{\mathrm{b}}} = (\mathbb{C}^{n_{\mathrm{b}}})^* = {}^a\mathfrak{h}_{\mathrm{b}}^*,$$

and

$$Q_{\iota_{\mathrm{g}}} = \mathbb{Z}^{n_{\mathrm{g}}} \subseteq \mathbb{C}^{n_{\mathrm{g}}} = (\mathbb{C}^{n_{\mathrm{g}}})^* = {}^a\mathfrak{h}_{\mathrm{g}}^*.$$

Then Λ_{b} is the unique $Q_{\iota_{\mathrm{b}}}$-coset in ${}^a\mathfrak{h}_{\mathrm{b}}^*$ that contains some (and all) elements in ${}^a\mathfrak{h}_{\mathrm{b}}^*$ that represents the $\mathrm{Ad}(\check{\mathfrak{g}}_{\mathrm{b}})$-conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}_{\mathrm{b}}}$. Likewise, Λ_{g} is the unique $Q_{\iota_{\mathrm{g}}}$-coset in ${}^a\mathfrak{h}_{\mathrm{g}}^*$ that contains some (and all) elements in ${}^a\mathfrak{h}_{\mathrm{g}}^*$ that represents the $\mathrm{Ad}(\check{\mathfrak{g}}_{\mathrm{g}})$-conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}_{\mathrm{g}}}$.

Write $W_{\mathrm{b}} \subseteq \mathrm{GL}({}^a\mathfrak{h}_{\mathrm{b}})$ for the Weyl group for $\mathfrak{g}_{\mathrm{b}}$ so that

$$W_{\mathrm{b}} = \begin{cases} \mathsf{S}_{n_{\mathrm{b}}}, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}; \\ \mathsf{W}_{n_{\mathrm{b}}}, & \text{if } \star \in \{B, \widetilde{C}\}; \\ \mathsf{W}'_{n_{\mathrm{b}}}, & \text{if } \star \in \{C, C^*, D, D^*\}, \end{cases} \tag{6.15}$$

and write $W_{\mathrm{g}} \subseteq \mathrm{GL}({}^a\mathfrak{h}_{\mathrm{g}})$ for the Weyl group for $\mathfrak{g}_{\mathrm{g}}$ so that

$$W_{\mathrm{g}} = \begin{cases} \mathsf{S}_{n_{\mathrm{g}}}, & \text{if } \star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}; \\ \mathsf{W}_{n_{\mathrm{g}}}, & \text{if } \star \in \{B, C, \widetilde{C}, C^*\}; \\ \mathsf{W}'_{n_{\mathrm{g}}}, & \text{if } \star \in \{D, D^*\}. \end{cases} \tag{6.16}$$

Let $W'_{\mathrm{g}} \subseteq \mathrm{GL}({}^a\mathfrak{h}_{\mathrm{g}})$ denote the integral Weyl group of Λ_{g}, namely the Weyl group of the root system

$$\{\alpha \in \Delta_{\mathrm{g}} \mid \langle \check{\alpha}, \nu \rangle \in \mathbb{Z} \text{ for some (and all) } \nu \in \Lambda_{\mathrm{g}}\}.$$

Then

$$W'_{\mathrm{g}} = \begin{cases} \mathsf{W}'_{n_{\mathrm{g}}}, & \text{if } \star = \widetilde{C}; \\ W_{\mathrm{g}}, & \text{otherwise.} \end{cases}$$

Note that

$$W(\Lambda) = W_{\mathrm{b}} \times W'_{\mathrm{g}} \subseteq W_{\mathrm{b}} \times W_{\mathrm{g}} \subseteq W_\Lambda. \tag{6.17}$$

Let $\lambda_{\check{\mathcal{O}}_{\mathrm{b}}} \in \Lambda_{\mathrm{b}}$ be the unique dominant element that represents the $\mathrm{Ad}(\check{\mathfrak{g}}_{\mathrm{b}})$-conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}_{\mathrm{b}}}$. Also take a dominant element $\lambda_{\check{\mathcal{O}}_{\mathrm{g}}} \in \Lambda_{\mathrm{g}}$ that represents the $\mathrm{Ad}(\check{\mathfrak{g}}_{\mathrm{g}})$-conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}_{\mathrm{g}}}$, which is unique unless $\star = \widetilde{C}$. Put $\lambda_{\check{\mathcal{O}}} := (\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}, \lambda_{\check{\mathcal{O}}_{\mathrm{g}}}) \in \Lambda$.

Lemma 6.2. *The element $\lambda_{\check{\mathcal{O}}} \in {}^a\mathfrak{h}^*$ is dominant and represents the conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}}$.*

Proof. Note that for all $\alpha \in \Delta$,

$$\langle \lambda_{\check{\mathcal{O}}}, \check{\alpha} \rangle \in \mathbb{Z} \qquad \text{implies} \qquad \alpha \in \Delta_{\mathrm{b}} \sqcup \Delta_{\mathrm{g}}.$$

This implies that $\lambda_{\check{\mathcal{O}}}$ is dominant.

It is easy to see that there exists an element $\lambda'_{\check{\mathcal{O}}_{\mathrm{b}}} \in {}^a\mathfrak{h}_{\mathrm{b}}^*$ that represents the conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}_{\mathrm{b}}}$ and an element $\lambda_{\check{\mathcal{O}}_{\mathrm{g}}} \in {}^a\mathfrak{h}_{\mathrm{g}}^*$ that represents the conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}_{\mathrm{g}}}$ such that $\lambda'_{\check{\mathcal{O}}} := (\lambda'_{\check{\mathcal{O}}_{\mathrm{b}}}, \lambda'_{\check{\mathcal{O}}_{\mathrm{g}}})$ represents the conjugacy class of $\lambda_{\check{\mathcal{O}}}$. Since $\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}$ is W_{b}-conjugate to $\lambda'_{\check{\mathcal{O}}_{\mathrm{b}}}$, and $\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}$ is W_{g}-conjugate to $\lambda'_{\check{\mathcal{O}}_{\mathrm{b}}}$, $\lambda_{\check{\mathcal{O}}}$ is W-conjugate to $\lambda'_{\check{\mathcal{O}}}$. Thus $\lambda_{\check{\mathcal{O}}} \in {}^a\mathfrak{h}^*$ also represents the conjugacy class of $\lambda^\circ_{\check{\mathcal{O}}}$. $\square$

We say that a $[\star]$-space V_0 is split if

- $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, C, \widetilde{C}\}$, or
- $\star \in \{A, \widetilde{A}\}$ and there exists a totally isotropic subspace of V_0 of dimension $\frac{\dim V_0}{2}$, or
- $\star \in \{B, D, C^*, D^*\}$ and there exists a $\mathbf{j}$-stable totally isotropic subspace of V of dimension $\frac{\dim V}{2}$.

Up to isomorphism a split $[\star]$-spaces is determined by its dimension.

Recall that $\star_{\mathrm{g}} = \star$ but $\star_{\mathrm{b}}$ may not equal $\star$ in general. As a modification of V_{b}, we define a split $[\star]$-space $V_{\mathrm{b}}^{\mathrm{g}}$ such that

$$\dim V_{\mathrm{b}}^{\mathrm{g}} = \begin{cases} \dim V_{\mathrm{b}} - 1, & \text{if } \star = B; \\ \dim V_{\mathrm{b}}, & \text{otherwise.} \end{cases}$$

For simplicity, write $G_{\mathrm{b}}^{\mathrm{g}} := \mathsf{G}_{\star_{\mathrm{g}}}(V_{\mathrm{b}}^{\mathrm{g}})$. Note that

$$V \cong V_{\mathrm{b}}^{\mathrm{g}} \times V_{\mathrm{g}} \tag{6.18}$$

as $[\star]$-representations.

6.4. **Matching relevant Cartan subgroups.** By a $\star$-representation of a Lie group E, we mean a $[\star]$-space V_1 together with a Lie group homomorphism $E \to \mathsf{G}_\star(V_1)$. An isomorphism of $\star$-representation is defined to be a $[\star]$-sapce isomorphism $f : V_1 \to V_2$ of $\star$-representations that makes the diagram

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ \mathsf{G}_\star(V_1) \arrow[rr, "\text{the isomorphism induced by } f"] & & \mathsf{G}_\star(V_2). \end{tikzcd}$$

commutes. The product $V_1 \times V_2$ is obviously a $[\star]$-representation of E whenever V_1 and V_2 is are $[\star]$-representation of E. Note that a B-representation is the same as a D-representation.

We say that a Cartan subgroup H_{b} of G_{b} is relevant if

- $\star_{\mathrm{b}} \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}$, or
- $\star \in \{A, \widetilde{A}\}$ and H_{b} stabilizes a totally isotropic subspace of V of dimension $\frac{n_{\mathrm{b}}}{2}$, or
- $\star \in \{B, C, \widetilde{C}, D, C^*, D^*\}$ and H_{b} stabilizes a $\mathbf{j}_{V_{\mathrm{b}}}$-stable totally isotropic subspace of V of dimension n_{b}.

Suppose that H_{b} be a relevant Cartan subgroup of G_{b}. Then V_{b} is naturally a $\star_{\mathrm{b}}$-representation of H_{b}.

The following lemma can be verified directly.

Lemma 6.3. *There exits a unique (up to isomorphism) $\star$-representation of H_{b} on the $[\star]$-space $V_{\mathrm{b}}^{\mathrm{g}}$ with the following property.*

(a) When $\star = B$, $V_{\mathrm{b}} \cong V_1 \times V_{\mathrm{b}}^{\mathrm{g}}$ as $\star$-representations of H_{b} for a one-dimensional $\star$-representation V_1 of H_{b} with trivial action.

(b) When $\star \in \{C, C^*\}$, $V_{\mathrm{b}} \cong V_{\mathrm{b}}^{\mathrm{g}}$ as representations of H_{b}.

(c) When $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}, \widetilde{C}, D, D^*\}$, $V_{\mathrm{b}} \cong V_{\mathrm{b}}^{\mathrm{g}}$ as $\star$-representations of H_{b}.

We view $V_{\mathrm{b}}^{\mathrm{g}}$ as a $\star$-representation of H_{b} as in Lemma 6.3.

Let H_{g} be a Cartan subgroup of G_{g} so that V_{g} is naturally an H_{g}-representation. Then $V_{\mathrm{b}}^{\mathrm{g}} \times V_{\mathrm{g}}$ is naturally a $\star$-representation of $H_{\mathrm{b}} \times H_{\mathrm{g}}$. The isomorphism (6.18) yields a Lie group homomorphism

$$H_{\mathrm{b}} \times H_{\mathrm{g}} \to G. \tag{6.19}$$

Write H for the image of the homomorphism (6.19), and let $\zeta : H_{\mathrm{b}} \times H_{\mathrm{g}} \to H$ denote the homomorphism induced by (6.19). Note that H is a Cartan subgroup of G. We call the pair (H, ζ) a matching of $H_{\mathrm{b}} \times H_{\mathrm{g}}$, which is uniquely defined up to conjugation by G.

6.5. **Matching the parameters.** Given a parameter $\gamma_{\mathrm{b}} \in \mathcal{P}'_{\Lambda_{\mathrm{b}}}(G_{\mathrm{b}})$ that is represented by $(H_{\mathrm{b}}, \xi_{\mathrm{b}}, \Gamma_{\mathrm{b}})$, and a parameter $\gamma_{\mathrm{g}} \in \mathcal{P}'_{\Lambda_{\mathrm{g}}}(G_{\mathrm{g}})$ that is represented by $(H_{\mathrm{g}}, \xi_{\mathrm{g}}, \Gamma_{\mathrm{g}})$, we define a parameter $\varphi(\gamma_{\mathrm{b}}, \gamma_{\mathrm{g}}) \in \mathcal{P}'_{\Lambda}(G)$ as in what follows. Note that Lemma 6.1 implies that H_{b} is relevant. Take a matching (H, ζ) of $H_{\mathrm{b}} \times H_{\mathrm{g}}$. Let ξ be the composition of

$$^{a}\mathfrak{h}^* = {}^{a}\mathfrak{h}_{\mathrm{b}}^* \times {}^{a}\mathfrak{h}_{\mathrm{g}}^* \xrightarrow{\xi_{\mathrm{b}} \times \xi_{\mathrm{g}}} \mathfrak{h}_{\mathrm{b}}^* \times \mathfrak{h}_{\mathrm{g}}^* \xrightarrow{\text{the transpose inverse of the complexified differential of } \zeta} \mathfrak{h}^*.$$

Let $\Gamma : \Lambda \to \mathrm{Irr}'(G)$ to be the map

$$\nu = (\nu_{\mathrm{b}}, \nu_{\mathrm{g}}) \mapsto \Gamma_{\mathrm{b}}(\nu_{\mathrm{b}}) \otimes \Gamma_{\mathrm{g}}(\nu_{\mathrm{g}}),$$

where $\Gamma_{\mathrm{b}}(\nu_{\mathrm{b}}) \otimes \Gamma_{\mathrm{g}}(\nu_{\mathrm{g}})$ (which is originally defined as an irreducible representation of $H_{\mathrm{b}} \times H_{\mathrm{g}}$) is viewed as a representation of H via the descent through the homomorphism $\xi : H_{\mathrm{b}} \times H_{\mathrm{g}} \to H$.

Lemma 6.4. *The triple* (H, ξ, Γ) *defined above is an element of* $\mathscr{P}'_{\Lambda}(G)$.

Proof. Note that every imaginary root of G with respect to H is either an imaginary root of G_{b} with respect to H_{b} or an imaginary root of G_{g} with respect to H_{g} (here $\mathfrak{h}$ is identified with $\mathfrak{h}_{\mathrm{b}} \times \mathfrak{h}_{\mathrm{g}}$ via ξ). This implies that $\delta(\xi) = \delta(\xi_{\mathrm{b}}) + \delta(\xi_{\mathrm{g}})$, and the lemme then easily follows. □

Now we define $\varphi(\gamma_{\mathrm{b}}, \gamma_{\mathrm{g}}) \in \mathcal{P}'_{\Lambda}(G)$ to be the G-orbit of the triple (H, ξ, Γ) defined above. This is independent of the choices of the representatives $(H_{\mathrm{b}}, \xi_{\mathrm{b}}, \Gamma_{\mathrm{b}})$, $(H_{\mathrm{g}}, \xi_{\mathrm{g}}, \Gamma_{\mathrm{g}})$, and the matching (H, ζ). It is easy to see that the map

$$\varphi : \mathcal{P}'_{\Lambda_{\mathrm{b}}}(G_{\mathrm{b}}) \times \mathcal{P}'_{\Lambda_{\mathrm{g}}}(G_{\mathrm{g}}) \to \mathcal{P}'_{\Lambda}(G) \tag{6.20}$$

is $W_{\mathrm{b}} \times W_{\mathrm{g}}$-equivariant under the cross actions.

Proposition 6.5. *If either* $\star = C^*$ *and* $n_{\mathrm{b}} > 0$, *or* $\star = D^*$ *and* $n_{\mathrm{b}}, n_{\mathrm{g}} > 0$, *then the map* (6.20) *is injective and*

$$\mathcal{P}'_{\Lambda}(G) = \varphi(\mathcal{P}'_{\Lambda_{\mathrm{b}}}(G_{\mathrm{b}}) \times \mathcal{P}'_{\Lambda_{\mathrm{g}}}(G_{\mathrm{g}})) \sqcup w \cdot \Big(\varphi(\mathcal{P}'_{\Lambda_{\mathrm{b}}}(G_{\mathrm{b}}) \times \mathcal{P}'_{\Lambda_{\mathrm{g}}}(G_{\mathrm{g}}))\Big)$$

for every $w \in W_{\Lambda} \setminus (W_{\mathrm{b}} \times W_{\mathrm{g}})$. *The map* (6.20) *is bijective in all other cases.*

Proof. In view of Lemma 6.1, this follows from a case-by-case verification. □

Define a linear map

$$\varphi : \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \otimes \mathrm{Coh}'_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \to \mathrm{Coh}_{\Lambda}(\mathcal{K}'(G)) \tag{6.21}$$

such that

$$\varphi(\Psi_{\gamma_{\mathrm{b}}} \otimes \Psi_{\gamma_{\mathrm{g}}}) = \Psi_{\varphi(\gamma_{\mathrm{b}}, \gamma_{\mathrm{g}})}$$

for all $\gamma_{\mathrm{b}} \in \mathcal{P}'_{\Lambda_{\mathrm{b}}}(G_{\mathrm{b}})$ and $\gamma_{\mathrm{g}} \in \mathcal{P}'_{\Lambda_{\mathrm{g}}}(G_{\mathrm{g}})$.

Proposition 6.6. *The linear map* (6.21) *is* $W_{\mathrm{b}} \times W_{\mathrm{g}}$*-equivariant. Moreover*

$$\varphi(\overline{\Psi}_{\bar{\gamma}_{\mathrm{b}}} \otimes \overline{\Psi}_{\bar{\gamma}_{\mathrm{g}}}) = \overline{\Psi}_{\varphi(\bar{\gamma}_{\mathrm{b}}, \bar{\gamma}_{\mathrm{g}})}$$

for all $\gamma_{\mathrm{b}} \in \mathcal{P}'_{\Lambda_{\mathrm{b}}}(G_{\mathrm{b}})$ *and* $\gamma_{\mathrm{g}} \in \mathcal{P}'_{\Lambda_{\mathrm{g}}}(G_{\mathrm{g}})$.

Proof. This follows from the Kazhdan-Lusztig-Vogan algorithm (see [Vog82, Theorem 12.17]). The case of the odd orthogonal group is treated in [GI19, Section 3] and the case of the real metaplectic group is treated in [RT03, Theorem 5.2]. □

By Proposition 6.5 and Proposition 6.6, we have the following corollary.

Corollary 6.7. *If* $\star \in \{C^*, D^*\}$, *then*

$$\mathrm{Coh}_{\Lambda}(\mathcal{K}'(G)) \cong \mathrm{Ind}_{W_{\mathrm{b}} \times W_{\mathrm{g}}}^{W_{\Lambda}}(\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \otimes \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})))$$

as representations of W_{Λ}. *In all the other cases,*

$$\mathrm{Coh}_{\Lambda}(\mathcal{K}'(G)) \cong \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \otimes \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}}))$$

as representations of $W_{\mathrm{b}} \times W_{\mathrm{g}}$.

The following proposition is a reformulation of results in [Mat96, Lemma 4.1.3] and [GI19, Lemma 3.3].

Proposition 6.8. *Let* $\nu = (\nu_{\mathrm{b}}, \nu_{\mathrm{g}}) \in \Lambda_{\mathrm{b}} \times \Lambda_{\mathrm{g}} = \Lambda$. *Then there is a unique linear map* $\varphi_\nu : \mathcal{K}'_{\nu_{\mathrm{b}}}(G_{\mathrm{b}}) \otimes \mathcal{K}'_{\nu_{\mathrm{g}}}(G_{\mathrm{g}}) \to \mathcal{K}'_{\nu}(G)$ *that makes the diagram*

$$\begin{array}{ccc}
\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \otimes \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) & \xrightarrow{\varphi} & \mathrm{Coh}_{\Lambda}(\mathcal{K}'(G)) \\
{\scriptstyle \mathrm{ev}_{\nu_{\mathrm{b}}} \otimes \mathrm{ev}_{\nu_{\mathrm{g}}}} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_{\nu}} \\
\mathcal{K}'_{\nu_{\mathrm{b}}}(G_{\mathrm{b}}) \otimes \mathcal{K}'_{\nu_{\mathrm{g}}}(G_{\mathrm{g}}) & \xrightarrow{\varphi_\nu} & \mathcal{K}'_{\nu}(G)
\end{array}$$

commutes, where ev *indicates the evaluation maps. Moreover,* φ_ν *is injective, and*

$$\varphi_\nu(\mathrm{Irr}'_{\nu_{\mathrm{b}}}(G_{\mathrm{b}}) \times \mathrm{Irr}'_{\nu_{\mathrm{g}}}(G_{\mathrm{g}})) \subseteq \mathrm{Irr}'_{\nu}(G).$$

Proof. The first assertion follows directly from Lemma 5.1, since the map φ is $W(\lambda) = W_{\mathrm{b}} \times W'_{\mathrm{g}}$-equivariant. Since φ is injective, Lemma 5.1 also implies that φ_ν is injective.

For the proof of the second assertion, we assume without loss of generality that ν is dominant. Let $\pi_{\mathrm{b}} \in \mathrm{Irr}'_{\nu_{\mathrm{b}}}(G_{\mathrm{b}})$ and $\pi_{\mathrm{g}} \in \mathrm{Irr}'_{\nu_{\mathrm{g}}}(G_{\mathrm{g}})$. Pick basal elements

$$\Psi_{\mathrm{b}} \in \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \quad \text{and} \quad \Psi_{\mathrm{g}} \in \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}}))$$

such that $\Psi_{\mathrm{b}}(\nu_{\mathrm{b}}) = \pi_{\mathrm{b}}$ and $\Psi_{\mathrm{g}}(\nu_{\mathrm{g}}) = \pi_{\mathrm{g}}$. Then

$$\varphi_\nu(\pi_{\mathrm{b}} \otimes \pi_{\mathrm{g}}) = \mathrm{ev}_\nu(\varphi(\Psi_{\mathrm{b}} \otimes \Psi_{\mathrm{g}})).$$

Proposition 6.6 implies that $\varphi(\Psi_{\mathrm{b}} \otimes \Psi_{\mathrm{g}})$ is a basel element of $\mathrm{Coh}_{\Lambda}(\mathcal{K}'(G))$. Thus by Lemma 4.5, $\mathrm{ev}_\nu(\varphi(\Psi_{\mathrm{b}} \otimes \Psi_{\mathrm{g}}))$ is either irreducible or zero. It is clearly nonzero since φ_ν is injective. This proves the last assertion of the proposition. □

7. Special unipotent representations in type A

We begin the explicit counting of special unipotent representations for a real classical group G.

Recall that $\mathcal{K}'(G)$ is the Grothendieck group of the category of genuine Casselman-Wallach representations of G if $\star \in \{\widetilde{A}, \widetilde{C}\}$, and $\mathcal{K}'(G) := \mathcal{K}(G)$ otherwise. As an obvious variant of Corollary 5.6 for $\nu = \lambda_{\check{\mathcal{O}}}$, we have that

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sum_{\sigma \in {}^{L}\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}} [\sigma : \mathrm{Coh}_{\Lambda}(\mathcal{K}'(G))]. \tag{7.1}$$

(The equality holds because of Corollary 4.15.) Here $\Lambda \subseteq {}^{a}\mathfrak{h}^{*}$ is defined in (6.12), and the Lusztig left cell ${}^{L}\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}$ is defined in (3.21).

7.1. **Some Weyl group representations.** The group S_n is identified with the permutation group of the set $\{1, 2, \dots, n\}$, and W_n is identified with $\mathsf{S}_n \ltimes \{\pm 1\}^n$.

Define a quadratic character

$$\varepsilon : \mathsf{W}_n \to \{\pm 1\}, \quad (s, (x_1, x_2, \cdots, x_n)) \mapsto x_1 x_2 \cdots x_n. \tag{7.2}$$

Then W'_n is the kernel of this character. As always, sgn denotes the sign character (of an appropriate Weyl group). Since S_n is a quotient of W_n, we may inflate the sign character of S_n to obtain a character of W_n, to be denoted by $\overline{\mathrm{sgn}}$. Then we have that

$$\varepsilon = \mathrm{sgn} \otimes \overline{\mathrm{sgn}}.$$

The group W_n is naturally embedded in S_{2n} via the homomorphism determined by

$$(i, i+1) \in \mathsf{S}_n \mapsto (2i-1, 2i+1)(2i, 2i+2), \qquad (1 \leq i \leq n-1), \tag{7.3}$$

and

$$(1, \cdots, 1, \underbrace{-1}_{j\text{-th term}}, 1, \cdots, 1) \in \{\pm 1\}^n \mapsto (2j-1, 2j), \qquad (1 \leq j \leq n). \tag{7.4}$$

Here $(i, i+1)$, $(2i-1, 2i+1)$, etc., indicate the involutions in the permutation groups. Note that ε is also the restriction of sgn of S_{2n} to W_n.

Let YD_n be the set of Young diagrams of total size n. We identify YD_n with the set $\overline{\mathrm{Nil}}(\mathfrak{g}) := \mathrm{GL}_n(\mathbb{C}) \backslash \mathrm{Nil}(\mathfrak{gl}_n(\mathbb{C}))$ of complex nilpotent orbits and also with the set $\mathrm{Irr}(\mathsf{S}_n)$ via the Springer correspondence (see [Car93, 11.4]). More specifically, for $\mathcal{O}' \in \overline{\mathrm{Nil}}(\mathfrak{g}) = \mathsf{YD}_n$, the Springer correspondence is given by Macdonald's construction for S_n via j-induction:

$$\mathrm{Springer}(\mathcal{O}') = j_{\prod_j \mathsf{S}_{\mathbf{c}_j(\mathcal{O}')}}^{\mathsf{S}_n} \mathrm{sgn}\,.$$

Suppose that $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}$ in the rest of this section. Recall that $W(\Lambda) = \mathsf{S}_{n_{\mathrm{b}}} \times \mathsf{S}_{n_{\mathrm{g}}}$. It is easy to verify that

$$\begin{aligned} W_{\lambda_{\check{\mathcal{O}}}} &= \prod_{i \in \mathbb{N}^+} \mathsf{S}_{\mathbf{c}_i(\check{\mathcal{O}}_{\mathrm{b}})} \times \prod_{i \in \mathbb{N}^+} \mathsf{S}_{\mathbf{c}_i(\check{\mathcal{O}}_{\mathrm{g}})}, \\ {}^{L}\mathscr{C}_{\lambda_{\check{\mathcal{O}}}} &= \{\tau_{\lambda_{\check{\mathcal{O}}}}\}, \quad \text{where } \tau_{\lambda_{\check{\mathcal{O}}}} := (j_{W_{\lambda_{\check{\mathcal{O}}}}}^{W(\Lambda)} \mathrm{sgn}) \otimes \mathrm{sgn} = \check{\mathcal{O}}_{\mathrm{b}}^{t} \otimes \check{\mathcal{O}}_{\mathrm{g}}^{t}. \end{aligned} \tag{7.5}$$

Here and as before, a superscript "t" indicates the transpose of a Young diagram.

The proof of Propositions 7.1-7.3 follows from Theorem 4.2 by direct computation. We omit the details.

Proposition 7.1. *Suppose that* $\star = A^{\mathbb{R}}$. *For all* $l \in \mathbb{N}$, *put*

$$\mathcal{C}_l := \bigoplus_{\substack{t,c,d\in\mathbb{N}\\ 2t+c+d=l}} \mathrm{Ind}_{\mathsf{W}_t\times\mathsf{S}_c\times\mathsf{S}_d}^{\mathsf{S}_l} \varepsilon\otimes 1\otimes 1.$$

Then

$$\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \cong \mathcal{C}_{n_{\mathrm{b}}} \quad \text{and} \quad \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \cong \mathcal{C}_{n_{\mathrm{g}}}.$$

Proposition 7.2. *Suppose that* $\star = A^{\mathbb{H}}$. *For all even number* $l \in \mathbb{N}$, *put*

$$\mathcal{C}_l := \mathrm{Ind}_{\mathsf{W}_{\frac{l}{2}}}^{\mathsf{S}_l} \varepsilon.$$

Then

$$\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \cong \mathcal{C}_{n_{\mathrm{b}}} \quad \text{and} \quad \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \cong \mathcal{C}_{n_{\mathrm{g}}}.$$

Proposition 7.3. *Suppose that* $\star \in \{A, \widetilde{A}\}$. *Then*

$$\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \cong \mathrm{Ind}_{\mathsf{W}_{\frac{n_{\mathrm{b}}}{2}}}^{\mathsf{S}_{n_{\mathrm{b}}}} 1$$

and

$$\mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \cong \bigoplus_{\substack{t,s,r\in\mathbb{N}\\ t+r=p_{\mathrm{g}},t+s=q_{\mathrm{g}}}} \mathrm{Ind}_{\mathsf{W}_t\times\mathsf{S}_s\times\mathsf{S}_r}^{\mathsf{S}_{n_{\mathrm{g}}}} 1\otimes\mathrm{sgn}\otimes\mathrm{sgn}\,.$$

7.2. **Special unipotent representations of** $\mathrm{GL}_n(\mathbb{R})$ **and** $\mathrm{GL}_n(\mathbb{H})$**.** Recall from Definition 2.6 and Definition 2.8 the set $\mathsf{PAP}_\star(\check{\mathcal{O}})$, which is the set of paintings on $\check{\mathcal{O}}^t$ that has type $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}, A, \widetilde{A}\}$. Recall that if $\star = A^{\mathbb{R}}$, then

$$\#(\mathsf{PAP}_\star(\check{\mathcal{O}})) = \prod_{r\in\mathbb{N}^+} (\#\{\, i\in\mathbb{N}^+ \mid \mathbf{r}_i(\check{\mathcal{O}}) = r\,\}+1). \tag{7.6}$$

Also note that if $\star = A^{\mathbb{H}}$, then

$$\#(\mathsf{PAP}_\star(\check{\mathcal{O}})) = \begin{cases} 1, & \text{if } \check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}};\\ 0, & \text{otherwise.}\end{cases} \tag{7.7}$$

Proposition 7.4. *Suppose that* $\star \in \{A^{\mathbb{R}}, A^{\mathbb{H}}\}$. *Then*

$$[\tau_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Coh}_{[\Lambda]}(\mathcal{K}'(G))] = \#(\mathsf{PAP}_\star(\check{\mathcal{O}}_{\mathrm{g}})) \times \#(\mathsf{PAP}_\star(\check{\mathcal{O}}_{\mathrm{b}})) = \#(\mathsf{PAP}_\star(\check{\mathcal{O}})).$$

Proof. In view of Proposition 7.1 and the description of the left cell representation $\tau_{\lambda_{\check{\mathcal{O}}}}$ in (7.5), the first equality follows from Pieri's rule ([GW09, Corollary 9.2.4]) and the following branching formula (see [BV83b, Lemma 4.1 (b)]):

$$\mathrm{Ind}_{\mathsf{W}_t}^{\mathsf{S}_{2t}} \varepsilon = \bigoplus_{\substack{\sigma\in\mathsf{YD}_{2t}\\ \mathbf{c}_i(\sigma) \text{ is even for all } i\in\mathbb{N}^+}} \sigma \qquad (t\in\mathbb{N}). \tag{7.8}$$

The last equality follows from (7.6) and (7.7). □

We are in the setting of Theorem 2.7. Recall the map

$$\begin{array}{rcl} \mathsf{PAP}_\star(\check{\mathcal{O}}) & \to & \mathrm{Unip}_{\check{\mathcal{O}}}(G),\\ \mathcal{P} & \mapsto & \pi_{\mathcal{P}}, \end{array}$$

where $\pi_{\mathcal{P}}$ is defined in Section 2.4. It is proved in [Vog86, Theorem 3.8] that the above map is injective. Then the map is bijective by (7.1) and Proposition 7.4. This proves Theorem 2.7.

7.3. **Special unipotent representations of unitary groups.** In this subsection, we suppose $\star \in \{ A, \widetilde{A} \}$ so that

$$G = \begin{cases} \mathrm{U}(p,q), & \text{if } \star = A; \\ \widetilde{\mathrm{U}}(p,q), & \text{if } \star = \widetilde{A}. \end{cases}$$

For $\mathcal{P} \in \mathsf{PAP}_\star(\check{\mathcal{O}})$, we have defined its signature $(p_{\mathcal{P}}, q_{\mathcal{P}})$ in (2.4). Recall that $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}}) = \check{\mathcal{O}}^t \subseteq \mathrm{Nil}(\mathfrak{g}^*)$. Let $\overline{\mathrm{Nil}}_G(\mathcal{O})$ denote the set of G-orbits in $(\sqrt{-1}\mathfrak{g}_0^*) \cap \mathcal{O}$, where $\mathfrak{g}_0$ denotes the Lie algebra of G which equals $\mathfrak{u}(p,q)$.

We first consider the case when $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$. In this good parity setting, we will state a counting result on $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$. The elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ can be constructed by cohomological induction explicitly and they are irreducible and unitary due to [Mat96, Tra01], see also [Tra04, Section 2] and [MR19, Section 4]. We refer the reader to [BMSZ21] for the construction of all elements of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ by the method of theta lifting.

Theorem 7.5 (*cf.* [BV83b, Theorem 4.2] and [Tra04, Theorem 2.1])**.** *Suppose that $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$. Then*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp\{ \mathcal{P} \in \mathrm{PAP}_{A^*}(\check{\mathcal{O}}) \mid (p_{\mathcal{P}}, q_{\mathcal{P}}) = (p,q) \} = \sharp(\overline{\mathrm{Nil}}_G(\mathcal{O})). \tag{7.9}$$

Moreover, for every $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, its wavefront set $\mathrm{WF}(\pi)$ is the closure in $\sqrt{-1}\mathfrak{g}_0^*$ of a unique orbit $\mathscr{O}_\pi \in \overline{\mathrm{Nil}}_G(\mathcal{O})$, and the map

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}}(G) & \longrightarrow & \overline{\mathrm{Nil}}_G(\mathcal{O}), \\ \pi & \mapsto & \mathscr{O}_\pi \end{array} \tag{7.10}$$

is bijective.

Proof. In view of the counting formula in (7.1), the first equality in (7.9) follows from Proposition 7.3, (7.5), (7.8), and Pieri's rule ([GW09, Corollary 9.2.4]). The second equality in (7.9) will follow directly from the bijectivity of (7.10).

The assignment of wavefront set yields a bijection

$$\{\text{cell in the basal representation } \mathrm{Coh}_{[\Lambda]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G))/\mathrm{Coh}_{[\Lambda]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))\} \to \overline{\mathrm{Nil}}_G(\mathcal{O}),$$

and every cell representation in $\mathrm{Coh}_{[\Lambda]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G))/\mathrm{Coh}_{[\Lambda]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))$ is irreducible and isomorphic to $\tau_{\lambda_{\check{\mathcal{O}}}}$. See [BV83b, Theorem 4.2] and [Bož02, Theorem 5] (we have used [SV00, Theorem 1.4] to rephrase the result in terms of real nilpotent orbits).

Note that (3.20) implies that

$$[1_{W_{\lambda_{\check{\mathcal{O}}}}} : \tau_{\lambda_{\check{\mathcal{O}}}}] = 1.$$

Recall from Lemma 6.2 that $\lambda_{\check{\mathcal{O}}} \in {}^a\mathfrak{h}^*$ is dominant. As in the proof of Theorem 5.4, for each cell $\mathcal{C}$ in $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G))$ that is not a cell in $\mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))$, there is a unique element $\Psi_{\mathcal{C}} \in \mathcal{C}$ such that $\Psi_{\mathcal{C}}(\lambda_{\check{\mathcal{O}}}) \neq 0$. Then

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{\Psi_{\mathcal{C}}(\lambda_{\check{\mathcal{O}}})\}_{\mathcal{C} \text{ is a cell in } \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}}(G)) \text{ that is not a cell in } \mathrm{Coh}_{[\lambda_{\check{\mathcal{O}}}]}(\mathcal{K}'_{\overline{\mathcal{O}}\setminus\mathcal{O}}(G))}.$$

This implies the bijectivity assertion of the theorem. □

Lemma 7.6. *If $\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) \neq 0$, then $\check{\mathcal{O}}_{\mathrm{b}} = 2\check{\mathcal{O}}'_{\mathrm{b}}$ for some Young diagram $\check{\mathcal{O}}'_{\mathrm{b}}$.*

Proof. Suppose that $\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) \neq 0$. Then

$$[\tau_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}} : \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}}))] \neq 0. \tag{7.11}$$

Recall from Proposition 7.3 that

$$\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \cong \mathrm{Ind}_{\mathsf{W}_{\frac{n_{\mathrm{b}}}{2}}}^{\mathsf{S}_{n_{\mathrm{b}}}} 1.$$

Thus (7.11) is the same as saying that

$$[\check{\mathcal{O}}_{\mathrm{b}}^t : \mathrm{Ind}_{\mathsf{W}_t}^{\mathsf{S}_{2t}} 1] \neq 0.$$

Similar to (7.8), we have that

$$\mathrm{Ind}_{\mathsf{W}_t}^{\mathsf{S}_{2t}} 1 = \bigoplus_{\substack{\sigma \in \mathsf{YD}_{2t} \\ \mathbf{r}_i(\sigma) \text{ is even for all } i \in \mathbb{N}^+}} \sigma \qquad (t \in \mathbb{N}). \tag{7.12}$$

This implies the lemma. □

Now we assume that $\check{\mathcal{O}}_{\mathrm{b}} = 2\check{\mathcal{O}}'_{\mathrm{b}}$ for some Young diagram $\check{\mathcal{O}}'_{\mathrm{b}}$.

The group G has an $\check{\mathcal{O}}$-relevant parabolic subgroup P whose Levi quotient is naturally isomorphic to $G'_{\mathrm{b}} \times G_{\mathrm{g}}$, where $G'_{\mathrm{b}} := \mathrm{GL}_{\frac{n_{\mathrm{b}}}{2}}(\mathbb{C})$. Let $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}}$ denote the unique element in $\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(\mathrm{GL}_{\frac{n_{\mathrm{b}}}{2}}(\mathbb{C}))$. Then for every $\pi_{\mathrm{g}} \in \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$, the normalized smooth parabolic induction $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \rtimes \pi_{\mathrm{g}}$ is irreducible by [Mat96, Theorem 3.2.2] and is an element of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ (*cf.* [MR19, Theorem 5.3]).

Theorem 7.7. *The equality*

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) \tag{7.13}$$

holds, and the map

$$\begin{array}{ccc} \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) & \longrightarrow & \mathrm{Unip}_{\check{\mathcal{O}}}(G), \\ \pi_0 & \mapsto & \pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \ltimes \pi_0 \end{array} \tag{7.14}$$

is bijective.

Proof. By the counting formula in (7.1),

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G)) = [\tau_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Coh}_{\Lambda}(\mathcal{K}'(G))].$$

Similarly,

$$\sharp(\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})) = [\tau_{\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}} : \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}}))].$$

The proof of Lemma 7.6 shows that

$$[\tau_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}} : \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}}))] = 1.$$

The above three equalities clearly imply (7.13).

By the calculation of the wavefront set of the induced representations ([Bar00, Corollary 5.0.10]), Theorem 7.5 implies that all the representations $\pi_{\check{\mathcal{O}}'_{\mathrm{b}}} \rtimes \pi_{\mathrm{g}}$, where π_{g} varies in $\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$, have pairwise distinct wavefront sets. Thus the map (7.14) is injective. Hence it is bijection by the counting assertion (7.13). □

Theorems 2.10 and 2.11 then follow from Theorems 7.5 and 7.7.

Remark 7.8. When $\check{\mathcal{O}} \neq \check{\mathcal{O}}_{\mathrm{g}}$, the wavefront set $\mathrm{WF}(\pi)$, where $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, may not be the closure of a single orbit in $\overline{\mathrm{Nil}}_G(\mathcal{O})$.

8. Special unipotent representations in type BCD : counting

In this section, we assume that $\star \in \{ B, C, \widetilde{C}, C^*, D, D^* \}$. If $\star \in \{C, C^*, D, D^*\}$, we say that $\check{\mathcal{O}}_{\mathrm{b}}$ has type I if

$$\text{the number of negative entries of } \lambda_{\check{\mathcal{O}}_{\mathrm{b}}} \text{ has the same parity as } \frac{n_{\mathrm{b}}}{2}; \tag{8.1}$$

otherwise we say that $\check{\mathcal{O}}_{\mathrm{b}}$ has type II.

8.1. **The coherent continuation representations.** Let $\mathsf{H}_t := \mathsf{W}_t \ltimes \{ \pm 1 \}^t$ $(t \in \mathbb{N})$, to be viewed as a subgroup in W_{2t} such that

- the first factor W_t sits in $\mathsf{S}_{2t} \subseteq \mathsf{W}_{2t}$ as in (7.3) and (7.4),
- the element $(1, \cdots, 1, \underbrace{-1}_{i\text{-th term}}, 1, \cdots, 1) \in \{ \pm 1 \}^t$ acts on $\mathbb{C}^{2t}$ by

$$(x_1, x_2, \cdots, x_{2t}) \mapsto (x_1, \cdots, x_{2i-2}, -x_{2i}, -x_{2i-1}, x_{2i+1}, \cdots, x_{2t}).$$

Note that H_t is also a subgroup of W'_{2t}. Define a quadratic character

$$\begin{array}{rcl} \tilde{\varepsilon} : \mathsf{H}_t = \mathsf{W}_t \ltimes \{ \pm 1 \}^t & \to & \{ \pm 1 \}, \\ (g, (a_1, a_2, \cdots, a_t)) & \mapsto & a_1 a_2 \cdots a_t. \end{array}$$

The proof of the following Propositions 8.1 and 8.2 also follows from Theorem 4.2 by direct computation. We omit the details.

Proposition 8.1. *As representations of W_{b},*

$$\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \cong \begin{cases} \bigoplus_{2t+c+d=n_{\mathrm{b}}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{W}_c \times \mathsf{W}_d}^{\mathsf{W}_{n_{\mathrm{b}}}} \tilde{\varepsilon} \otimes 1 \otimes 1, & \text{if } \star \in \{B, \widetilde{C}\}; \\ \bigoplus_{2t+a=n_{\mathrm{b}}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{S}_a}^{\mathsf{W}_{n_{\mathrm{b}}}} \tilde{\varepsilon} \otimes 1, & \text{if } \star \in \{C, D\}; \\ \mathrm{Ind}_{\mathsf{H}_{\frac{n_{\mathrm{b}}}{2}}}^{\mathsf{W}'_{n_{\mathrm{b}}}} \tilde{\varepsilon}, & \text{if } \star \in \{C^*, D^*\}, \end{cases}$$

where in the case when $\star \in \{C, D\}$, the right-hand side space is viewed as a representation of $W_{\mathrm{b}} = \mathsf{W}'_{n_{\mathrm{b}}}$ by restriction.

Proposition 8.2. *As representations of W_{g},*

$$\mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \cong \begin{cases} \bigoplus_{\substack{0 \leq p_{\mathrm{g}} - (2t+a+2r) \leq 1, \\ 0 \leq q_{\mathrm{g}} - (2t+a+2s) \leq 1}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{S}_a \times \mathsf{W}_s \times \mathsf{W}_r}^{\mathsf{W}_{n_{\mathrm{g}}}} \tilde{\varepsilon} \otimes 1 \otimes \mathrm{sgn} \otimes \mathrm{sgn}, & \text{if } \star = B; \\ \bigoplus_{2t+a+c+d=n_{\mathrm{g}}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{S}_a \times \mathsf{W}_c \times \mathsf{W}_d}^{W_{n_{\mathrm{g}}}} \tilde{\varepsilon} \otimes \mathrm{sgn} \otimes 1 \otimes 1, & \text{if } \star = C; \\ \bigoplus_{2t+a+a'=n_{\mathrm{g}}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{S}_a \times \mathsf{S}_{a'}}^{\mathsf{W}_{n_{\mathrm{g}}}} \tilde{\varepsilon} \otimes \mathrm{sgn} \otimes 1, & \text{if } \star = \widetilde{C}; \\ \bigoplus_{(t+s, t+r) = (\frac{p_{\mathrm{g}}}{2}, \frac{q_{\mathrm{g}}}{2})} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{W}_s \times \mathsf{W}_t}^{\mathsf{W}_{n_{\mathrm{g}}}} \tilde{\varepsilon} \otimes \mathrm{sgn} \otimes \mathrm{sgn}, & \text{if } \star = C^*; \\ \bigoplus_{\substack{2t+c+d+2r=p_{\mathrm{g}} \\ 2t+c+d+2s=q_{\mathrm{g}}}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{W}_s \times \mathsf{W}_r \times \mathsf{W}'_c \times \mathsf{W}_d}^{\mathsf{W}_{n_{\mathrm{g}}}} \tilde{\varepsilon} \otimes \overline{\mathrm{sgn}} \otimes \overline{\mathrm{sgn}} \otimes 1 \otimes 1, & \text{if } \star = D; \\ \bigoplus_{2t+a=n_{\mathrm{g}}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{S}_a}^{\mathsf{W}'_{n_{\mathrm{g}}}} \tilde{\varepsilon} \otimes \mathrm{sgn}, & \text{if } \star = D^*, \end{cases}$$

where in the case when $\star = D$, the right-hand side space is viewed as a representation of $W_{\mathrm{g}} = \mathsf{W}'_{n_{\mathrm{g}}}$ by restriction.

8.2. **The Lusztig left cells.** To ease the notation, for every sequence $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ ($k \geq 0$) of integers, we let $[a_1, a_2, \cdots, a_k]_{\mathrm{col}}$ denote the Young diagram whose i-th column has length a_i if $1 \leq i \leq k$ and length 0 otherwise. Likewise, we let $[a_1, a_2, \cdots, a_k]_{\mathrm{row}}$ denote the Young diagram whose i-th row has length a_i if $1 \leq i \leq k$ and length 0 otherwise. All Weyl group notation such as W_b, W'_g and $W(\Lambda)$ are in Section 6.3.

As usual, we identify $\mathrm{Irr}(\mathsf{W}_t)$ ($t \in \mathbb{N}$) with the set of bipartitions $\tau = (\tau_L, \tau_R)$ of total size t ([Car93, Section 11.4]). Here the total size refers to $|\tau_L| + |\tau_R|$. We also let $(\tau_L, \tau_R)_I \in \mathrm{Irr}(\mathsf{W}'_t)$ denote the irreducible representation given by

- the restriction of $(\tau_L, \tau_R) \in \mathrm{Irr}(\mathsf{W}_t)$ if $\tau_L \neq \tau_R$, and
- the induced representation $\mathrm{Ind}_{\mathsf{S}_t}^{\mathsf{W}'_t} \tau_L$ if $\tau_L = \tau_R$.

Take an element $w \in \mathsf{W}_t$ such that $w\mathsf{W}'_t$ generate the group $\mathsf{W}_t/\mathsf{W}'_t$. Define $(\tau_L, \tau_R)_{II} \in \mathrm{Irr}(\mathsf{W}'_t)$ to be the twist of $(\tau_L, \tau_R)_I$ by the conjugation by w, namely there is a linear isomorphism $\kappa : (\tau_L, \tau_R)_I \to (\tau_L, \tau_R)_{II}$ such that

$$\kappa(g \cdot u) = (wgw^{-1}) \cdot (\kappa(u)), \quad \text{for all } g \in \mathsf{W}'_t,\ u \in (\tau_L, \tau_R)_I.$$

Note that

$$(\tau_L, \tau_R)_I = (\tau_R, \tau_L)_I \quad \text{and} \quad (\tau_L, \tau_R)_{II} = (\tau_R, \tau_L)_{II}$$

in all cases, and $(\tau_L, \tau_R)_I = (\tau_R, \tau_L)_{II}$ when $\tau_L \neq \tau_R$.

In the rest of this subsection, we describe the Lusztig left cell ${}^L\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}$ attached to $\lambda_{\check{\mathcal{O}}}$. Define two Young diagrams

(8.2)
$$\tau_{L,\mathrm{b}} := \begin{cases} \left[\frac{1}{2}(\mathbf{r}_1(\check{\mathcal{O}}'_{\mathrm{b}}) + 1), \frac{1}{2}(\mathbf{r}_2(\check{\mathcal{O}}'_{\mathrm{b}}) + 1), \cdots, \frac{1}{2}(\mathbf{r}_c(\check{\mathcal{O}}'_{\mathrm{b}}) + 1)\right]_{\mathrm{col}}, & \text{if } \star \in \{ B, \widetilde{C} \}; \\ \left[\frac{1}{2}\mathbf{r}_1(\check{\mathcal{O}}'_{\mathrm{b}}), \frac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}'_{\mathrm{b}}), \cdots, \frac{1}{2}\mathbf{r}_c(\check{\mathcal{O}}'_{\mathrm{b}})\right]_{\mathrm{col}}, & \text{if } \star \in \{ C, C^*, D, D^* \}, \end{cases}$$

and

(8.3)
$$\tau_{R,\mathrm{b}} := \begin{cases} \left(\frac{1}{2}(\mathbf{r}_1(\check{\mathcal{O}}'_{\mathrm{b}}) - 1), \frac{1}{2}(\mathbf{r}_2(\check{\mathcal{O}}'_{\mathrm{b}}) - 1), \cdots, \frac{1}{2}(\mathbf{r}_c(\check{\mathcal{O}}'_{\mathrm{b}}) - 1)\right)_{\mathrm{col}}, & \text{if } \star \in \{ B, \widetilde{C} \}; \\ \left(\frac{1}{2}\mathbf{r}_1(\check{\mathcal{O}}'_{\mathrm{b}}), \frac{1}{2}\mathbf{r}_2(\check{\mathcal{O}}'_{\mathrm{b}}), \cdots, \frac{1}{2}\mathbf{r}_c(\check{\mathcal{O}}'_{\mathrm{b}})\right)_{\mathrm{col}}, & \text{if } \star \in \{ C, C^*, D, D^* \}, \end{cases}$$

where $c := \mathbf{c}_1(\check{\mathcal{O}}'_{\mathrm{b}})$.

Define an irreducible representation $\tau_{\mathrm{b}} \in \mathrm{Irr}(W_{\mathrm{b}})$ attached to $\check{\mathcal{O}}_{\mathrm{b}}$ by

$$\tau_{\mathrm{b}} := \begin{cases} (\tau_{L,\mathrm{b}}, \tau_{R,\mathrm{b}}), & \text{if } \star \in \{ B, \widetilde{C} \}; \\ (\tau_{L,\mathrm{b}}, \tau_{R,\mathrm{b}})_I, & \text{if } \star \in \{ C, C^*, D, D^* \} \text{ and } \check{\mathcal{O}}_{\mathrm{b}} \text{ has type I}; \\ (\tau_{L,\mathrm{b}}, \tau_{R,\mathrm{b}})_{II}, & \text{if } \star \in \{ C, C^*, D, D^* \} \text{ and } \check{\mathcal{O}}_{\mathrm{b}} \text{ has type II}. \end{cases} \tag{8.4}$$

Recall the set $\mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}})$ from Definition 2.13. Put

$$\mathrm{A}(\check{\mathcal{O}}) := \mathrm{A}(\check{\mathcal{O}}_{\mathrm{g}}) := \text{the power set of } \mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}}),$$

which is identified with the free $\mathbb{F}_2$-vector space with free basis $\mathrm{PP}(\check{\mathcal{O}}_{\mathrm{g}})$. Here $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ is the field with two elements only. Note that $\{\emptyset, \mathrm{PP}(\check{\mathcal{O}}_{\mathrm{g}})\}$ is a subgroup of $\mathrm{A}(\check{\mathcal{O}})$. Define

$$\bar{\mathrm{A}}(\check{\mathcal{O}}) := \bar{\mathrm{A}}(\check{\mathcal{O}}_{\mathrm{g}}) := \begin{cases} \mathrm{A}(\check{\mathcal{O}})/\{\emptyset, \mathrm{PP}(\check{\mathcal{O}}_{\mathrm{g}})\}, & \text{if } \star = \widetilde{C}; \\ \mathrm{A}(\check{\mathcal{O}}), & \text{otherwise.} \end{cases}$$

Generalizing (2.14), for each $\wp \in \mathrm{A}(\check{\mathcal{O}})$, we define a pair

$$(\imath_\wp, \jmath_\wp) := (\imath_\star(\check{\mathcal{O}}, \wp), \jmath_\star(\check{\mathcal{O}}, \wp))$$

of Young diagrams as in what follows.

If $\star = B$, then

$$\mathbf{c}_1(\jmath_\wp) = \frac{\mathbf{r}_1(\check{\mathcal{O}}_{\mathrm{g}})}{2},$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_{i+1}(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_{\mathrm{g}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})}{2}), & \text{if } (2i, 2i+1) \in \wp; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_{\mathrm{g}})}{2}), & \text{otherwise.} \end{cases}$$

If $\star = \widetilde{C}$, then for all $i \geq 1$,

$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_i(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})}{2}), & \text{if } (2i-1, 2i) \in \wp; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})}{2}), & \text{otherwise.} \end{cases}$$

If $\star \in \{D, D^*\}$, then

$$\mathbf{c}_1(\imath_\wp) = \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}_{\mathrm{g}}) > 0; \\ 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}_{\mathrm{g}}) = 0, \end{cases}$$

and for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_{i+1}(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{if } (2i, 2i+1) \in \wp; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (0, 0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{otherwise.} \end{cases}$$

If $\star \in \{C, C^*\}$, then for all $i \geq 1$,

$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_i(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{if } (2i-1, 2i) \in \wp; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (0, 0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}_{\mathrm{g}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}}_{\mathrm{g}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}}_{\mathrm{g}})+1}{2}), & \text{otherwise.} \end{cases}$$

We define an element $\tau_\wp \in \mathrm{Irr}(W'_{\mathrm{g}})$ by

$$\tau_\wp := \begin{cases} (\imath_\wp, \jmath_\wp), & \text{if } \star \in \{ B, C, C^* \}; \\ (\imath_\wp, \jmath_\wp)_I = (\imath_\wp, \jmath_\wp)_{II}, & \text{if } \star \in \{ D, D^* \}; \\ (\imath_\wp, \jmath_\wp)_I, & \text{if } \star = \widetilde{C} \text{ and } \frac{n_{\mathrm{g}}}{2} \text{ is even;} \\ (\imath_\wp, \jmath_\wp)_{II}, & \text{if } \star = \widetilde{C} \text{ and } \frac{n_{\mathrm{g}}}{2} \text{ is odd.} \end{cases} \tag{8.5}$$

Note that if $\star = \widetilde{C}$, then $\tau_\wp = \tau_{\wp^c}$, where $\wp^c$ is the complement of $\wp$ in $\mathrm{PP}_\star(\check{\mathcal{O}}_{\mathrm{g}})$. Therefore in all cases, $\tau_{\bar{\wp}} \in \mathrm{Irr}(W'_{\mathrm{g}})$ is obviously defined for every $\bar{\wp} \in \bar{\mathrm{A}}(\check{\mathcal{O}})$.

To simplify the notation, we write ${}^L\mathscr{C}_{\check{\mathcal{O}}} := {}^L\mathscr{C}_{\lambda_{\check{\mathcal{O}}}}$. Recall that ${}^L\mathscr{C}_{\check{\mathcal{O}}}$ is the set of all $\sigma \in \mathrm{Irr}(W(\Lambda))$ that occurs in the multiplicity free representation

$$\left(J_{W_{\lambda_{\check{\mathcal{O}}}}}^{W(\Lambda)} \mathrm{sgn} \right) \otimes \mathrm{sgn}\, .$$

Proposition 8.3 (*cf.* Barbasch-Vogan [BV85, Proposition 5.28])**.** *The map*

$$\begin{array}{rcl} \bar{\mathrm{A}}(\check{\mathcal{O}}) & \to & {}^L\mathscr{C}_{\check{\mathcal{O}}}, \\ \bar{\wp} & \mapsto & \tau_{\mathrm{b}} \otimes \tau_{\bar{\wp}} \end{array}$$

is well-defined and bijective, and

$$\tau_{\check{\mathcal{O}}} := \tau_{\mathrm{b}} \otimes \tau_{\emptyset}$$

is the unique special representation in ${}^L\mathscr{C}_{\check{\mathcal{O}}}$. *Moreover,*

$$\mathrm{Springer}^{-1}(j_{W(\Lambda)}^{W}(\tau_{\check{\mathcal{O}}})) = d_{\mathrm{BV}}(\check{\mathcal{O}}), \tag{8.6}$$

as $\mathrm{Ad}(\mathfrak{g})$*-orbits, and*

$$d_{\mathrm{BV}}(\check{\mathcal{O}}) = (\check{\mathcal{O}}_{\mathrm{b}}')^t \stackrel{c}{\sqcup} (\check{\mathcal{O}}_{\mathrm{b}}')^t \stackrel{c}{\sqcup} d_{\mathrm{BV}}(\check{\mathcal{O}}_{\mathrm{g}}) \tag{8.7}$$

as Young diagrams.

Proof. When $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$ and $\star \neq \widetilde{C}$, the proposition is proved in [BV85, Proposition 5.28]. When $\star \neq \widetilde{C}$, the equalities (8.6) and (8.7) are proved in [BV85, Proposition A2]. In general, the proposition follows from Lusztig's formula of J-induction in [Lus84, §4.4-4.6]. □

Recall that $W_{\mathrm{g}}' = \mathsf{W}_{n_{\mathrm{g}}}'$ when $\star \in \{\widetilde{C}, D, D^*\}$. Since the representation theory of W_n is more elementary than that of W_n', we prefer to work with W_n instead of W_n' in some situations. For this reason, we also define for all cases

$$\widetilde{\tau}_{\wp} = (\imath_{\wp}, \jmath_{\wp}) \in \mathrm{Irr}(\mathsf{W}_{n_{\mathrm{g}}}), \qquad \wp \in \mathrm{A}(\check{\mathcal{O}}). \tag{8.8}$$

See (8.5) for the description of $\imath_{\wp}$ and $\jmath_{\wp}$.

For later use, we record the following lemma, which follows immediately from our explicit descriptions of τ_{b} and $\tau_{\wp}$.

Lemma 8.4. *Let* $\wp \in \mathrm{A}(\check{\mathcal{O}})$. *If* $\star = \widetilde{C}$, *then*

$$\mathrm{Ind}_{\mathsf{W}_{n_{\mathrm{b}}} \times \mathsf{W}_{n_{\mathrm{g}}}'}^{\mathsf{W}_{n_{\mathrm{b}}} \times \mathsf{W}_{n_{\mathrm{g}}}} \tau_{\mathrm{b}} \otimes \tau_{\wp} = \begin{cases} \tau_{\mathrm{b}} \otimes \widetilde{\tau}_{\emptyset}, & \text{if } n_{\mathrm{g}} = 0; \\ (\tau_{\mathrm{b}} \otimes \widetilde{\tau}_{\wp}) \oplus (\tau_{\mathrm{b}} \otimes \widetilde{\tau}_{\wp^c}), & \text{otherwise.} \end{cases}$$

If $\star \in \{D, D^*\}$, *then*

$$\mathrm{Ind}_{\mathsf{W}_{n_{\mathrm{b}}}' \times \mathsf{W}_{n_{\mathrm{g}}}'}^{\mathsf{W}_{n_{\mathrm{b}}} \times \mathsf{W}_{n_{\mathrm{g}}}} \tau_{\mathrm{b}} \otimes \tau_{\wp} \cong \begin{cases} \widetilde{\tau}_{\mathrm{b}}, & \text{if } n_{\mathrm{g}} = 0; \\ (\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_{\wp}) \oplus (\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_{\wp}^s), & \text{otherwise.} \end{cases}$$

Here $\widetilde{\tau}_{\mathrm{b}} = \mathrm{Ind}_{\mathsf{W}_{n_{\mathrm{b}}}'}^{\mathsf{W}_{n_{\mathrm{b}}}} \tau_{\mathrm{b}}$ *and* $\widetilde{\tau}_{\wp}^s := \widetilde{\tau}_{\wp} \otimes \varepsilon$ *(recall the quadratic character* ε *from* (7.2)*).*

8.3. **From coherent continuation representation to counting.** We have defined in (2.16) the set $\mathsf{PBP}_{\star}(\check{\mathcal{O}})$ when $\check{\mathcal{O}}$ has good parity. Similarly, we make the following definition in the bad parity case.

Definition 8.5. *Let* $\mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}})$ *be the set of all triples* $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \star$ *where* $(\imath, \mathcal{P})$ *and* $(\jmath, \mathcal{Q})$ *are painted partitions such that*

- $(\imath, \jmath) = (\tau_{L,\mathrm{b}}, \tau_{R,\mathrm{b}})$ *(see* (8.4)*);*
- *the symbols of* $\mathcal{P}$ *are in*

$$\begin{cases} \{\bullet, c, d\}, & \text{if } \star \in \{B, \widetilde{C}\}; \\ \{\bullet, d\}, & \text{if } \star \in \{C, D\}; \\ \{\bullet\}, & \text{if } \star \in \{C^*, D^*\}; \end{cases}$$

- *the symbols of $\mathcal{Q}$ are in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \star \in \{C, D\}; \\ \{\bullet\}, & \text{if } \star \in \{B, \widetilde{C}, C^*, D^*\}. \end{cases}$$

We introduce some additional notation. For each bipartition τ, let

$$\mathsf{PBP}_\star(\tau) := \{\,\uptau \text{ is a painted bipartition } \mid \star_\uptau = \star, (\imath_\uptau, \jmath_\uptau) = \tau\,\}$$

and

$$\mathsf{PBP}_G(\tau) := \{\,\uptau \text{ is a painted bipartition } \mid G_\uptau = G, (\imath_\uptau, \jmath_\uptau) = \tau\,\}.$$

Put

$$\widetilde{\mathsf{PBP}}_\star(\check{\mathcal{O}}) := \bigsqcup_{\wp \subseteq \mathrm{PP}(\check{\mathcal{O}})} \mathsf{PBP}_\star(\widetilde{\tau}_\wp)$$

and

$$\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}}) := \bigsqcup_{\wp \subseteq \mathrm{PP}(\check{\mathcal{O}})} \mathsf{PBP}_G(\widetilde{\tau}_\wp),$$

where $\widetilde{\tau}_\wp := (\imath_\wp, \jmath_\wp)$ (see (8.8)).

Recall the notion of $\check{\mathcal{O}}$ being G-relevant (Section 2.7). Note that if $\star = D^*$, then $\check{\mathcal{O}}$ is not G-relevant if and only if $\check{\mathcal{O}}$ has bad parity and type II (see (8.1)).

Proposition 8.6. *If $\star = D^*$ and $\check{\mathcal{O}}$ is not G-relevant, then*

$$\sum_{\bar{\wp} \in \bar{\mathrm{A}}(\check{\mathcal{O}})} [\tau_{\mathrm{b}} \otimes \tau_{\bar{\wp}} : \mathrm{Coh}_\Lambda(\mathcal{K}'(G))] = 0.$$

In all other cases,

$$\sum_{\bar{\wp} \in \bar{\mathrm{A}}(\check{\mathcal{O}})} [\tau_{\mathrm{b}} \otimes \tau_{\bar{\wp}} : \mathrm{Coh}_\Lambda(\mathcal{K}'(G))] = \sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}})) \cdot \sharp(\widetilde{\mathsf{PBP}}_{G_{\mathrm{g}}}(\check{\mathcal{O}}_{\mathrm{g}})).$$

Proof. We use the following formulas ([McG98, p220 (6)]) to compute the multiplicities:

$$\begin{aligned} \mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}_{2t}} \tilde{\varepsilon} &\cong \bigoplus_{\sigma \in \mathrm{Irr}(\mathsf{S}_t)} (\sigma, \sigma), \\ \mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}'_{2t}} \tilde{\varepsilon} &\cong \bigoplus_{\sigma \in \mathrm{Irr}(\mathsf{S}_t)} (\sigma, \sigma)_I, \\ \mathrm{Ind}_{\mathsf{S}_t}^{\mathsf{W}_t} \mathrm{sgn} &\cong \bigoplus_{a+b=t} \mathrm{Ind}_{\mathsf{W}_a \times \mathsf{W}_b}^{\mathsf{W}_t} \overline{\mathrm{sgn}} \otimes \mathrm{sgn} \cong \bigoplus_{a+b=t} ([a]_{\mathrm{col}}, [b]_{\mathrm{col}}), \\ \mathrm{Ind}_{\mathsf{S}_t}^{\mathsf{W}_t} 1 &\cong \bigoplus_{a+b=t} \mathrm{Ind}_{\mathsf{W}_a \times \mathsf{W}_b}^{\mathsf{W}_t} 1 \otimes \epsilon \cong \bigoplus_{a+b=t} ([a]_{\mathrm{row}}, [b]_{\mathrm{row}}), \end{aligned} \tag{8.9}$$

where $t \in \mathbb{N}$.

We skip the details when $\star \in \{B, \widetilde{C}, C, D, C^*\}$, and present the computation for $\star = D^*$, which is the most complicated case. Suppose that $\star = D^*$ so that $G = \mathrm{O}^*(2n)$. If $\check{\mathcal{O}}$ has bad parity, then

$$\begin{aligned}
&\sum_{\bar\wp \in \bar{\mathrm{A}}(\check{\mathcal{O}})} [\tau_{\mathrm{b}} \otimes \tau_{\bar\wp} : \mathrm{Coh}_\Lambda(\mathcal{K}'(G))] \\
= \ & [\tau_{\mathrm{b}} : \mathrm{Coh}_\Lambda(\mathcal{K}'(G))] \\
= \ & [\tau_{\mathrm{b}} : \mathrm{Ind}_{\mathsf{H}_t}^{\mathsf{W}'_{2t}} \tilde\varepsilon] \\
= \ & \begin{cases} 1 = \sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}})) \cdot \sharp(\widetilde{\mathsf{PBP}}_{G_{\mathrm{g}}}(\check{\mathcal{O}}_{\mathrm{g}})), & \text{if } \check{\mathcal{O}} \text{ is } G\text{-relevant;} \\ 0, & \text{if } \check{\mathcal{O}} \text{ is not } G\text{-relevant.} \end{cases}
\end{aligned}$$

Now we assume that $\check{\mathcal{O}}$ does not have bad parity so that $n_{\mathrm{g}} > 0$. Put

$$\mathsf{W}'_{n_{\mathrm{b}},n_{\mathrm{g}}} := \mathsf{W}'_n \cap (\mathsf{W}_{n_{\mathrm{b}}} \times \mathsf{W}_{n_{\mathrm{g}}}) \supseteq \mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}} = W_{\mathrm{b}} \times W_{\mathrm{g}}.$$

Recall that

$$\widetilde{\tau}_{\mathrm{b}} := \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}}^{\mathsf{W}_{n_{\mathrm{b}}}} \tau_{\mathrm{b}} = (\mathcal{O}'_{\mathrm{b}}, \mathcal{O}'_{\mathrm{b}}) \in \mathrm{Irr}(\mathsf{W}_{n_{\mathrm{b}}})$$

and

$$\widetilde{\tau}_{\wp} := (\imath_\wp, \jmath_\wp) \in \mathrm{Irr}(\mathsf{W}_{n_{\mathrm{g}}}) \quad \text{ for all } \wp \subseteq \mathrm{PP}(\check{\mathcal{O}}_{\mathrm{g}}).$$

Note that $\imath_\wp \neq \jmath_\wp$ since $\mathbf{c}_1(\imath_\wp) > \mathbf{c}_1(\jmath_\wp)$, which implies that

$$\mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}'_{n_{\mathrm{b}},n_{\mathrm{g}}}} \tau_{\mathrm{b}} \otimes \tau_\wp \cong (\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_\wp)|_{\mathsf{W}'_{n_{\mathrm{b}},n_{\mathrm{g}}}}.$$

For ease of notation, write $\mathsf{W}'' := \mathsf{W}'_{n_{\mathrm{b}},n_{\mathrm{g}}}$. Put

$$\mathcal{C}_{\mathrm{b}} := \mathrm{Ind}_{\mathsf{H}_{\frac{n_{\mathrm{b}}}{2}}}^{\mathsf{W}_{n_{\mathrm{b}}}} \tilde\varepsilon \quad \text{and} \quad \mathcal{C}_{\mathrm{g}} := \bigoplus_{2t+a=n_{\mathrm{g}}} \mathrm{Ind}_{\mathsf{H}_t \times \mathsf{S}_a}^{\mathsf{W}'_{n_{\mathrm{g}}}} \tilde\varepsilon \otimes \mathrm{sgn},$$

where $\mathcal{C}_{\mathrm{b}}$ is viewed as a representation of $\mathsf{W}'_{n_{\mathrm{b}}}$ by restriction. For every finite group E and any two finite-dimensional representations V_1 and V_2 of E, put

$$[V_1, V_2]_E := \dim \mathrm{Hom}_E(V_1, V_2).$$

For each $\wp \in \mathrm{A}(\check{\mathcal{O}}_{\mathrm{g}})$, we have that

$$\begin{aligned}
[\tau_{\mathrm{b}} \otimes \tau_\wp : \mathrm{Coh}_\Lambda(\mathcal{K}'(G))] =& [\tau_{\mathrm{b}} \otimes \tau_\wp : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \mathcal{C}_{\mathrm{b}} \otimes \mathcal{C}_{\mathrm{g}}]_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}} \\
=& [\mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \tau_{\mathrm{b}} \otimes \tau_\wp : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \mathcal{C}_{\mathrm{b}} \otimes \mathcal{C}_{\mathrm{g}}]_{\mathsf{W}''} \\
=& [(\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_\wp)|_{\mathsf{W}''} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}''} \mathcal{C}_{\mathrm{b}} \otimes \mathcal{C}_{\mathrm{g}}]_{\mathsf{W}''} \\
=& [\widetilde{\tau}_{\mathrm{b}} \otimes \widetilde{\tau}_\wp : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}} \times \mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}_{n_{\mathrm{b}}} \times \mathsf{W}_{n_{\mathrm{g}}}} \mathcal{C}_{\mathrm{b}} \otimes \mathcal{C}_{\mathrm{g}}]_{\mathsf{W}_{n_{\mathrm{b}}} \times \mathsf{W}_{n_{\mathrm{g}}}} \\
=& [\widetilde{\tau}_{\mathrm{b}} : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{b}}}}^{\mathsf{W}_{n_{\mathrm{b}}}} \mathcal{C}_{\mathrm{b}}]_{\mathsf{W}_{n_{\mathrm{b}}}} \cdot [\widetilde{\tau}_\wp : \mathrm{Ind}_{\mathsf{W}'_{n_{\mathrm{g}}}}^{\mathsf{W}_{n_{\mathrm{g}}}} \mathcal{C}_{\mathrm{g}}]_{\mathsf{W}_{n_{\mathrm{g}}}} \\
=& \sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}}))) \cdot \sharp(\mathsf{PBP}_{G_{\mathrm{g}}}(\widetilde{\tau}_\wp)).
\end{aligned}$$

The last equality follows from the branching rules (8.9) and Pieri's rule ([GW09, Corollary 9.2.4]). $\square$

Now Proposition 8.6, Proposition 8.3 and (7.1) imply the following corollary.

Corollary 8.7. *The following equality holds:*

$$\begin{aligned}&\sharp(\mathrm{Unip}_{\check{\mathcal{O}}}(G))\\ = &\begin{cases}0, & \text{if } \star = D^* \text{ and } \check{\mathcal{O}} \text{ is not } G\text{-relevant;}\\ \sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}}))\cdot\sharp(\widetilde{\mathsf{PBP}}_{G_{\mathrm{g}}}(\check{\mathcal{O}}_{\mathrm{g}})), & \text{otherwise.}\end{cases}\end{aligned}$$

When $\check{\mathcal{O}}$ has good parity, we will see from Proposition 10.1 and Proposition 10.2 that

$$\sharp(\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}})) = \begin{cases}\sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{C^*, D^*\};\\ 2^{\sharp(\mathrm{PP}_\star(\check{\mathcal{O}}))}\cdot\sharp(\mathsf{PBP}_G(\check{\mathcal{O}})), & \text{if } \star \in \{B, C, D, \widetilde{C}\}.\end{cases}$$

Corollary 8.7 will thus imply Theorem 2.18.

9. Special unipotent representations in type BCD : reduction to good parity

The goal of this section is to prove Theorem 2.12. Corollary 8.7 implies that $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is empty if $\check{\mathcal{O}}$ is not G-relevant (this notion is defined in Section 2.7). Thus we further assume that $\check{\mathcal{O}}$ is G-relevant. With our earlier assumptions (6.13) and (6.14), this is equivalent to saying that $\check{\mathcal{O}}$ has type I when $\star = D^*$ and $\check{\mathcal{O}}$ has bad parity.

If $\star = C^*$, or $\star = D^*$ and $\check{\mathcal{O}}$ does not have bad parity, by possibly changing $\omega_{\check{V}_{\mathrm{b}}}$ and $\omega_{\check{V}_{\mathrm{g}}}$ defined in (6.11) to their negatives, we assume without loss of generality that $\check{\mathcal{O}}_{\mathrm{b}}$ has type I.

9.1. Separating bad parity and good parity for special unipotent representations.

By Proposition 6.8, we have an injective linear map

$$\varphi_{\lambda_{\check{\mathcal{O}}}} : \mathcal{K}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}(G_{\mathrm{b}}) \otimes \mathcal{K}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}}(G_{\mathrm{g}}) \to \mathcal{K}'_{\lambda_{\check{\mathcal{O}}}}(G)$$

and an injective map

$$\varphi_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Irr}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}(G_{\mathrm{b}}) \times \mathrm{Irr}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}}(G_{\mathrm{g}}) \to \mathrm{Irr}'_{\lambda_{\check{\mathcal{O}}}}(G). \tag{9.1}$$

Lemma 9.1. *The set* $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ *is contained in the image of the map* (9.1).

Proof. We assume that $\star \in \{C^*, D^*\}$, and $n_{\mathrm{b}}, n_{\mathrm{g}} > 0$. Otherwise the map (9.1) is bijective and the lemma is obviously true. Let $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$. Let $\Psi \in \mathrm{Coh}_\Lambda(\mathcal{K}'(G))$ be the unique basal element such that $\Psi(\lambda_{\check{\mathcal{O}}}) = \pi$ (see Lemma 4.5). Denote by $\mathcal{C}$ the cell containing Ψ in the basal representation $\mathrm{Coh}_\Lambda(\mathcal{K}'(G))$ of $W(\Lambda)$. Then Proposition 5.3 implies that

$$\sigma_{\mathcal{C}} \cong \left(j_{W_{\lambda_{\check{\mathcal{O}}}}}^{W(\Lambda)}\,\mathrm{sgn}\right) \otimes \mathrm{sgn} \cong \tau_{\mathrm{b}} \otimes \tau_{\emptyset}.$$

In particular, τ_{b} occurs in the cell representation $\langle\mathcal{C}\rangle/\langle\overline{\mathcal{C}}\setminus\mathcal{C}\rangle$ (in the notation of Section 3.4).

Write $\mathrm{Coh} := \mathrm{Coh}_\Lambda(\mathcal{K}'(G))$ and write $\mathrm{Coh}_{\mathrm{I}}$ for the image of the map

$$\varphi : \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \otimes \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \to \mathrm{Coh}_\Lambda(\mathcal{K}'(G))$$

which is defined in (6.3). Assume by contradiction that π is not contained in the image of the map (9.1). Then $\mathcal{C}$ has no intersection with $\mathrm{Coh}_{\mathrm{I}}$, and the natural homomorphism

$$\langle\mathcal{C}\rangle/\langle\overline{\mathcal{C}}\setminus\mathcal{C}\rangle \to \mathrm{Coh}/(\mathrm{Coh}_{\mathrm{I}} + \langle\overline{\mathcal{C}}\setminus\mathcal{C}\rangle)$$

is injective. Note that τ_{b} does not occur in $\mathrm{Coh}/(\mathrm{Coh}_{\mathrm{I}} + \langle\overline{\mathcal{C}}\setminus\mathcal{C}\rangle)$ since it does not occur in $\mathrm{Coh}/\mathrm{Coh}_{\mathrm{I}}$ by Corollary 6.7 and Proposition 8.1. This is a contradiction and the lemma is now proved. □

Proposition 9.2. *The map* (9.1) *restricts to a bijective map*

$$\varphi_{\lambda_{\check{\mathcal{O}}}} : \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) \to \mathrm{Unip}_{\check{\mathcal{O}}}(G).$$

Proof. Let $\pi_{\mathrm{b}} \in \mathrm{Irr}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}(G_{\mathrm{b}})$ and $\pi_{\mathrm{g}} \in \mathrm{Irr}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}}(G_{\mathrm{g}})$. Pick a basal element Ψ_{b} of $\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}}))$ such that $\Psi_{\mathrm{b}}(\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}) = \pi_{\mathrm{b}}$. Likewise pick a basal element Ψ_{g} of $\mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{g}}))$ such that $\Psi_{\mathrm{g}}(\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}) = \pi_{\mathrm{g}}$. Write $\Psi := \varphi(\Psi_{\mathrm{b}} \otimes \Psi_{\mathrm{g}})$. Denote by $\mathcal{C}$ the cell in $\mathrm{Coh}_{\Lambda}(\mathcal{K}'(G))$ containing Ψ, and similarly define the cells $\mathcal{C}_{\mathrm{g}} \ni \Psi_{\mathrm{g}}$ and $\mathcal{C}_{\mathrm{b}} \ni \Psi_{\mathrm{b}}$.

Put $\pi := \varphi_{\lambda_{\check{\mathcal{O}}}}(\pi_{\mathrm{b}}, \pi_{\mathrm{g}}) = \Psi(\lambda_{\check{\mathcal{O}}})$. Write

$$G_2 := G_{\mathrm{b}} \times G_{\mathrm{g}}, \qquad \pi_2 := \pi_{\mathrm{b}} \widehat{\otimes} \pi_{\mathrm{g}} \in \mathrm{Irr}(G_2),$$

and

$$I_{2,\lambda_{\check{\mathcal{O}}}} := \text{the maximal ideal of } \mathcal{U}(\mathfrak{g}_2) \text{ with infinitesimal character } \lambda_{\check{\mathcal{O}}}.$$

Then

$$\begin{aligned} & \pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G) \\ \Longleftrightarrow\ & \sigma_{\mathcal{C}} \cong \left(j_{W_{\lambda_{\check{\mathcal{O}}}}}^{W(\Lambda)} \mathrm{sgn} \right) \otimes \mathrm{sgn} \quad \text{(by Proposition 5.3)} \\ \Longleftrightarrow\ & \sigma_{\mathcal{C}_{\mathrm{b}}} \cong \left(j_{W_{\mathrm{b},\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}}^{W_{\mathrm{b}}} \mathrm{sgn} \right) \otimes \mathrm{sgn} \ \text{ and } \ \sigma_{\mathcal{C}_{\mathrm{g}}} \cong \left(j_{W'_{\mathrm{g},\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}}}^{W'_{\mathrm{g}}} \otimes \mathrm{sgn} \right) \otimes \mathrm{sgn} \\ \Longleftrightarrow\ & \pi_{\mathrm{b}} \in \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}}) \ \text{ and } \ \pi_{\mathrm{g}} \in \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) \quad \text{(by Proposition 5.3)}. \end{aligned}$$

Here $W_{\mathrm{b},\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}$ denote the stabilizer of $\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}$ in $W_{\mathrm{b},\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}$, and likewise $W'_{\mathrm{g},\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}}$ denote the stabilizer of $\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}$ in W_{g}.

Thus we have proved that the inverse image of $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ under the map (9.1) equals $\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$. Together with Lemma 9.1, this proves the proposition. □

9.2. **The case of bad parity.** Recall from (2.12) the group

$$G'_{\mathrm{b}} := \begin{cases} \mathrm{GL}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star \in \{ B, C, D \}; \\ \widetilde{\mathrm{GL}}_{n_{\mathrm{b}}}(\mathbb{R}), & \text{if } \star = \widetilde{C}; \\ \mathrm{GL}_{\frac{n_{\mathrm{b}}}{2}}(\mathbb{H}), & \text{if } \star \in \{ C^*, D^* \}. \end{cases}$$

Lemma 9.3. *The equalities*

$$\sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}})) = \sharp(\mathsf{PAP}_{\star'}(\check{\mathcal{O}}'_{\mathrm{b}})) = \sharp(\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}))$$

hold, where

$$\star' := \begin{cases} A^{\mathbb{R}}, & \text{if } \star \in \{ B, C, \widetilde{C}, D \}; \\ A^{\mathbb{H}}, & \text{if } \star \in \{ C^*, D^* \}. \end{cases} \tag{9.2}$$

Proof. Suppose that $\star \in \{ C^*, D^* \}$. Then

$$\sharp(\mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}}; \tau_{\mathrm{b}})) = \sharp(\mathsf{PAP}_{A^{\mathbb{H}}}(\check{\mathcal{O}}'_{\mathrm{b}})) = 1.$$

Suppose that $\star \in \left\{ B, C, \widetilde{C}, D \right\}$. It is easy to see that we have a bijection

$$\begin{array}{ccc} \mathsf{PBP}_\star(\check{\mathcal{O}}_{\mathrm{b}}) & \to & \mathsf{PAP}_{A^{\mathbb{R}}}(\check{\mathcal{O}}'_{\mathrm{b}}), \\ (\tau_{L,b}, \mathcal{P}) \times (\tau_{R,b}, \mathcal{Q}) \times \star & \mapsto & ((\check{\mathcal{O}}'_{\mathrm{b}})^t, \mathcal{P}'), \end{array}$$

where $\mathcal{P}'$ is defined by the condition that

$$\mathcal{P}(\mathbf{c}_j(\tau_{L,b}), j) = d \Longleftrightarrow \mathcal{P}'(\mathbf{r}_j(\check{\mathcal{O}}'_{\mathrm{b}}), j) = d, \quad \text{for all } j = 1, 2, \cdots, \mathbf{c}_1(\check{\mathcal{O}}'_{\mathrm{b}}).$$

The last equality is in Theorem 2.7. $\square$

Let P_{b} be a parabolic subgroup of G_{b} that is $\check{\mathcal{O}}_{\mathrm{b}}$-relevant as defined in Section 2.7. Then G'_{b} is naturally isomorphic to the Levi quotient of P_{b}.

Proposition 9.4. *For every $\pi' \in \mathrm{Unip}_{G'_{\mathrm{b}}}(\check{\mathcal{O}}'_{\mathrm{b}})$, the normalized induced representation $\mathrm{Ind}_{P_{\mathrm{b}}}^{G_{\mathrm{b}}} \pi'$ is irreducible and belongs to $\mathrm{Unip}_{G'_{\mathrm{b}}}(\check{\mathcal{O}}'_{\mathrm{b}})$. Moreover, the map*

$$\begin{array}{rcl} \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}) & \to & \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}}), \\ \pi' & \mapsto & \mathrm{Ind}_{P_{\mathrm{b}}}^{G_{\mathrm{b}}} \pi' \end{array} \tag{9.3}$$

is bijective.

Proof. By the construction of π' (see Section 2.4) and a result of Barbasch [Bar00, Corollary 5.0.10], the wavefront cycle of $\mathrm{Ind}_{P_{\mathrm{b}}}^{G_{\mathrm{b}}} \pi'$ is a single orbit $\mathscr{O}$ with multiplicity one and its complexification is $d_{\mathrm{BV}}(\check{\mathcal{O}}_{\mathrm{b}})$. Note that every irreducible summand of $\mathrm{Ind}_{P_{\mathrm{b}}}^{G_{\mathrm{b}}} \pi'$ belongs to $\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}})$. Hence $\mathrm{Ind}_{P_{\mathrm{b}}}^{G_{\mathrm{b}}} \pi'$ has to be irreducible.

We now suppose that $\star = D$ so that $G_{\mathrm{b}} = \mathrm{SO}(n_{\mathrm{b}}, n_{\mathrm{b}})$. We will prove the injectivity of the map (9.3). Fix a split Cartan subgroup

$$H = (\mathbb{R}^\times)^{n_{\mathrm{b}}} \subseteq G'_{\mathrm{b}} \subseteq G_{\mathrm{b}}$$

and write $H = MA$ where $M = \{\pm 1\}^{n_{\mathrm{b}}}$ is the compact part of H and $A = (\mathbb{R}_+^\times)^{n_{\mathrm{b}}}$ is the split part of H. We identify a character of A with a vector in $\mathbb{C}^{n_{\mathrm{b}}}$ as usual. Let

$$K := \{(g_1, g_2) \in \mathrm{O}(n_{\mathrm{b}}) \times \mathrm{O}(n_{\mathrm{b}}) \mid \det(g_1) \cdot \det(g_2) = 1\}$$

be a maximal compact subgroup of G_{b}, and let B be a Borel subgroup of G_{b} containing H and the unipotent radical of P_{b}.

For each integer l such that $0 \leq l \leq n_{\mathrm{b}}$, put

$$\delta_l = \underbrace{1_1 \otimes \cdots \otimes 1_1}_{l} \otimes \underbrace{\mathrm{sgn}_1 \otimes \cdots \otimes \mathrm{sgn}_1}_{n_{\mathrm{b}} - l} \in \mathrm{Irr}(M),$$

where 1_1 denotes the trivial character of $\{\pm 1\}$ and sgn_1 denotes the non-trivial character of $\{\pm 1\}$. It is a fine M-type ([Vog81, Definition 4.3.8]). Let λ_l be the restriction to K of the irreducible $\mathrm{O}(n_{\mathrm{b}}) \times \mathrm{O}(n_{\mathrm{b}})$-representation $\wedge^l \mathbb{C}^{n_{\mathrm{b}}} \otimes \wedge^0 \mathbb{C}^{n_{\mathrm{b}}}$. Then λ_l is a fine K-type, see [BGG75, §6].

Vogan proves that there is a well-defined injective map

$$\begin{array}{rrcl} \mathfrak{X}\colon & W_H \backslash \mathrm{Irr}(H) & \to & \mathrm{Irr}(G_{\mathrm{b}}), \\ & \text{the } W_H\text{-orbit of } \chi \in \mathrm{Irr}(H) & \mapsto & (\mathrm{Ind}_B^{G_{\mathrm{b}}} \chi)(\lambda_{l_\chi}), \end{array}$$

where W_H denotes the real Weyl group of G_{b} with respect to H, $l_\chi \in \{0, 1, \ldots, n_{\mathrm{b}}\}$ is the integer such that $\chi|_M$ is conjugate to δ_{l_χ} by W_H, and $(\mathrm{Ind}_B^{G_{\mathrm{b}}} \chi)(\lambda_{l_\chi})$ denotes the unique irreducible subquotient in the normalized induced representation $\mathrm{Ind}_B^{G_{\mathrm{b}}} \chi$ containing the K-type λ_l. See [Vog81, Theorem 4.4.8].

Now take

$$\pi' = 1_{n_1} \times \cdots \times 1_{n_r} \times \mathrm{sgn}_{n_{r+1}} \times \cdots \times \mathrm{sgn}_{n_k} \in \mathrm{Unip}_{G'_{\mathrm{b}}}(\check{\mathcal{O}}'_{\mathrm{b}}),$$

where $k \geq r \geq 0$, and $n_1, n_2, \ldots, n_k$ are positive integers with $n_1 + n_2 + \cdots + n_k = n_{\mathrm{b}}$.

Write

$$\begin{aligned}\delta_{\pi'} &:= \delta_{n_1+n_2+\cdots+n_r} \in \mathrm{Irr}(M),\\ \nu_{\pi'} &:= \left(\tfrac{n_1-1}{2}, \tfrac{n_1-3}{2}, \ldots, \tfrac{1-n_1}{2}, \ldots, \tfrac{n_k-1}{2}, \tfrac{n_k-3}{2}, \ldots, \tfrac{1-n_k}{2}\right) \in \mathrm{Irr}(A).\end{aligned}$$

Clearly the map

$$\begin{aligned}\mathfrak{P} : \mathrm{Unip}_{G'_{\mathrm{b}}}(\check{\mathcal{O}}'_{\mathrm{b}}) &\to W_H\backslash \mathrm{Irr}(H),\\ \pi' &\mapsto \text{the } W_H\text{-orbit of } \delta_{\pi'} \otimes \nu_{\pi'}\end{aligned}$$

is well-defined and injective.

Note that the map (9.3) equals the composition $\mathfrak{X} \circ \mathfrak{P}$, and hence it is also injective. This proves the injectivity of (9.3) in the case when $\star = D$. The same proof works in the case when $\star \in \{B, C, \widetilde{C}\}$ and we omit the details. When $\star \in \{C^*, D^*\}$, the map (9.3) has to be injective since its domain is a singleton. Thus (9.3) is injective in all cases.

The bijection of (9.3) follows from the injectivity and the counting inequalities below:

$$\left|\mathsf{PAP}_{\star'}(\check{\mathcal{O}}'_{\mathrm{b}})\right| = \left|\mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}})\right| \leq \left|\mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{b}}}(G_{\mathrm{b}})\right| \leq \left|\mathsf{PBP}_{\star}(\check{\mathcal{O}}_{\mathrm{b}})\right| = \left|\mathsf{PAP}_{\star'}(\check{\mathcal{O}}'_{\mathrm{b}})\right|.$$

Here $\star' \in \{ A^{\mathbb{R}}, A^{\mathbb{H}} \}$ is as in (9.2), the first inequality follows from the injectivity of (9.3), the second inequality follows from Corollary 8.7, and the last equality is in Lemma 9.3. $\square$

9.3. **Coherent continuation representations and parabolic induction.** The normalized smooth parabolic induction from P_{b} to G_{b} yields a linear map

$$\mathrm{Ind} : \mathcal{K}'(G'_{\mathrm{b}}) \to \mathcal{K}'(G_{\mathrm{b}}).$$

This induces a linear map

$$\begin{aligned}\mathrm{Ind} : \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G'_{\mathrm{b}})) &\to \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})),\\ \Psi'_{\mathrm{b}} &\mapsto (\nu \mapsto \mathrm{Ind}(\Psi'_{\mathrm{b}}(\nu))).\end{aligned}$$

Let P be a $\check{\mathcal{O}}$-relevant parabolic subgroup of G as in Theorem 2.12. Then $G'_{\mathrm{b}} \times G_{\mathrm{g}}$ (or its quotient by $\{\pm 1\}$ when $\star = \widetilde{C}$) is naturally isomorphic to the Levi quotient of P. The normalized smooth parabolic induction using P yields a linear map

$$\mathrm{Ind} : \mathcal{K}'(G'_{\mathrm{b}}) \otimes \mathcal{K}'(G_{\mathrm{g}}) \to \mathcal{K}'(G),$$

which further induces a linear map (see [Vog81, Corollary 7.2.10])

$$\begin{aligned}\mathrm{Ind} : \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G'_{\mathrm{b}})) \otimes \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) &\to \mathrm{Coh}_{\Lambda}(\mathcal{K}'(G)),\\ \Psi'_{\mathrm{b}} \otimes \Psi_{\mathrm{g}} &\mapsto ((\nu_{\mathrm{b}}, \nu_{\mathrm{g}}) \mapsto \mathrm{Ind}(\Psi'_{\mathrm{b}}(\nu_{\mathrm{b}})) \otimes \Psi_{\mathrm{g}}(\nu_{\mathrm{g}}))).\end{aligned}$$

The following result follows from Langlands' construction of standard modules. See [Vog81, Lemma 6.6.12 and Theorem 6.6.15].

Lemma 9.5. *The diagram*

$$\begin{tikzcd}[column sep=tiny] & \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G'_{\mathrm{b}})) \otimes \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \arrow[dl, "\mathrm{Ind} \otimes \mathrm{Id}"'] \arrow[dr, "\mathrm{Ind}"] & \\ \mathrm{Coh}_{\Lambda_{\mathrm{b}}}(\mathcal{K}'(G_{\mathrm{b}})) \otimes \mathrm{Coh}_{\Lambda_{\mathrm{g}}}(\mathcal{K}'(G_{\mathrm{g}})) \arrow[rr, "\varphi"] & & \mathrm{Coh}_{\Lambda}(\mathcal{K}'(G))\end{tikzcd} \tag{9.4}$$

commutes.

Proposition 9.6. *For all $\pi' \in \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}})$ and $\pi_{\mathrm{g}} \in \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}})$, the normalized smooth induction $\mathrm{Ind}_P^G(\pi_{\mathrm{g}}\widehat{\otimes}\pi')$ is irreducible and belongs to $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$.*

Proof. By using the evaluation maps, the commutative diagram (9.4) descends to a commutative diagram

$$\begin{tikzcd} & \mathcal{K}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}(G'_{\mathrm{b}}) \otimes \mathcal{K}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}}(G_{\mathrm{g}}) \arrow[dl, "\mathrm{Ind}\otimes\mathrm{Id}"'] \arrow[dr, "\mathrm{Ind}"] & \\ \mathcal{K}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{b}}}}(G_{\mathrm{b}}) \otimes \mathcal{K}'_{\lambda_{\check{\mathcal{O}}_{\mathrm{g}}}}(G_{\mathrm{g}}) \arrow[rr, "\varphi_{\lambda_{\check{\mathcal{O}}}}"] & & \mathcal{K}'_{\lambda_{\check{\mathcal{O}}}}(G). \end{tikzcd} \tag{9.5}$$

Here $\varphi_{\lambda_{\check{\mathcal{O}}}}$ is as in Proposition 6.8. Thus $\mathrm{Ind}_P^G(\pi_{\mathrm{g}}\widehat{\otimes}\pi')$ is irreducible by Propositions 9.4 and 6.8. As before, [Bar00, Corollary 5.0.10] implies that $\mathrm{Ind}_P^G(\pi_{\mathrm{g}}\widehat{\otimes}\pi') \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$. □

By Proposition 9.6, we have a well-defined map

$$\mathrm{Ind} : \mathrm{Unip}_{\check{\mathcal{O}}'_{\mathrm{b}}}(G'_{\mathrm{b}}) \times \mathrm{Unip}_{\check{\mathcal{O}}_{\mathrm{g}}}(G_{\mathrm{g}}) \longrightarrow \mathrm{Unip}_{\check{\mathcal{O}}}(G). \tag{9.6}$$

In view of the commutative diagram (9.5), Propositions 6.8 and 9.4 implies that the map (9.6) is injective. On the other hands, Propositions 9.2 and 9.4 imply that the domain and codomain of the map (9.6) have the same cardinality. Thus the map (9.6) is bijective. This finishes the proof of Theorem 2.12.

10. Combinatorics of painted bipartitions

Throughout this section, we assume that $\star \in \{B, C, \widetilde{C}, C^*, D, D^*\}$, and $\check{\mathcal{O}} = \check{\mathcal{O}}_{\mathrm{g}}$, namely $\check{\mathcal{O}}$ has $\star$-good parity.

Recall the set $\mathrm{PP}_\star(\check{\mathcal{O}})$ of primitive $\star$-pairs in $\check{\mathcal{O}}$ (Definition 2.13). For each subset $\wp$ of $\mathrm{PP}_\star(\check{\mathcal{O}})$, we have defined a pair $(\imath_\wp, \jmath_\wp)$ of Young diagrams in Section 8.2. Our main object of interest is the disjoint union:

$$\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}}) := \bigsqcup_{\wp \subseteq \mathrm{PP}(\check{\mathcal{O}})} \mathsf{PBP}_G(\check{\mathcal{O}}, \wp),$$

where

$$\mathsf{PBP}_G(\check{\mathcal{O}}, \wp) := \{\, \boldsymbol{\tau} \text{ is a painted bipartition} \mid (\imath_{\boldsymbol{\tau}}, \jmath_{\boldsymbol{\tau}}) = (\imath_\wp, \jmath_\wp), G_{\boldsymbol{\tau}} = G \,\}.$$

10.1. **Main combinatorial results.** We say that the orbit $\check{\mathcal{O}}$ is quasi-distinguished if there is no $\star$-pair that is balanced in $\check{\mathcal{O}}$ (see Definition 2.13).

For the quaternionic groups, we have the following non-existence result on painted bipartitions.

Proposition 10.1. *Suppose that $\star \in \{\, C^*, D^* \,\}$. If $\mathsf{PBP}_G(\check{\mathcal{O}}, \wp)$ is nonempty for some $\wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}})$, then $\check{\mathcal{O}}$ is quasi-distinguished and $\wp = \emptyset$. Consequently,*

$$\sharp(\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}})) = \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})).$$

Proof. Suppose that $\boldsymbol{\tau} = (\imath_\wp, \mathcal{P}) \times (\jmath_\wp, \mathcal{Q}) \times \star \in \mathsf{PBP}_G(\check{\mathcal{O}}, \wp)$. Assume by contradiction that $\wp \neq \emptyset$.

First assume that $\star = C^*$. Pick an element $(2i-1, 2i) \in \wp$. Then we have that

$$\mathbf{c}_i(\imath_\wp) = \tfrac{1}{2}(\mathbf{r}_{2i-1}(\check{\mathcal{O}}) + 1) > \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) - 1) = \mathbf{c}_i(\jmath_\wp). \tag{10.1}$$

By the requirements of a painted bipartition, we also have that

$$\begin{array}{rcl} \mathbf{c}_i(\imath_\wp) & = & \sharp\{\, j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\imath_\wp),\ \mathcal{P}(i,j) = \bullet \,\} \\ & = & \sharp\{\, j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\jmath_\wp),\ \mathcal{Q}(i,j) = \bullet \,\} \\ & \leq & \mathbf{c}_i(\jmath_\wp). \end{array}$$

This contradicts (10.1) and therefore $\wp = \emptyset$.

Now assume that $\star = D^*$. Pick an element $(2i, 2i+1) \in \wp$. Then we have that

$$\mathbf{c}_{i+1}(\imath_\wp) = \tfrac{1}{2}(\mathbf{r}_{2i}(\check{\mathcal{O}}) + 1) > \tfrac{1}{2}(\mathbf{r}_{2i+1}(\check{\mathcal{O}}) - 1) = \mathbf{c}_i(\jmath_\wp). \tag{10.2}$$

By the requirements of a painted bipartition, we also have that

$$\begin{array}{rcl} \mathbf{c}_{i+1}(\imath_\wp) & \leq & \sharp\{\, j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\imath_\wp),\ \mathcal{P}(i,j) = \bullet \,\} \\ & = & \sharp\{\, j \in \mathbb{N}^+ \mid (i,j) \in \mathrm{Box}(\jmath_\wp),\ \mathcal{Q}(i,j) = \bullet \,\} \\ & \leq & \mathbf{c}_i(\jmath_\wp). \end{array}$$

This contradicts (10.2) and therefore $\wp = \emptyset$.

It is also straightforward to show that $\check{\mathcal{O}}$ is quasi-distinguished from the requirements of a painted bipartition. We omit the details. □

In view of Proposition 10.1, we assume in the rest of this section that $\check{\mathcal{O}}$ is quasi-distinguished when $\star \in \{C^*, D^*\}$.

In the rest of the section, we will define a notion of descent for painted bipartitions which will be used as an induction mechanism and we will deduce certain counting formulas by a generating function approach. The following proposition will then be an immediate consequence of a certain (independence) property of the generating functions in Proposition 10.10 and Proposition 10.11.

Proposition 10.2. *Suppose that $\star \in \{\, B, C, \widetilde{C}, D \,\}$. Then*

$$\sharp(\mathsf{PBP}_G(\check{\mathcal{O}}, \wp)) = \sharp(\mathsf{PBP}_G(\check{\mathcal{O}}, \emptyset)), \quad \textit{for all } \wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}}).$$

Consequently,

$$\sharp(\widetilde{\mathsf{PBP}}_G(\check{\mathcal{O}})) = 2^{\sharp(\mathrm{PP}_\star(\check{\mathcal{O}}))} \cdot \sharp(\mathsf{PBP}_G(\check{\mathcal{O}})).$$

10.2. **Type C and $\widetilde{C}$: shape shifting.** In this subsection, we assume that $\star \in \{\, C, \widetilde{C} \,\}$. We will define a shape shifting operation on painted bipartitions which we will be used in defining the notion of descent.

Recall that $\check{\mathcal{O}}$ is an orbit with good parity. In this subsection we assume that $(1,2) \in \mathrm{PP}_\star(\check{\mathcal{O}})$, namely the pair $(1,2)$ is primitive in $\check{\mathcal{O}}$. Let $\wp$ be a subset of $\mathrm{PP}_\star(\check{\mathcal{O}})$ such that $(1,2) \notin \wp$. Put $\wp_\uparrow := \wp \cup \{\, (1,2) \,\} \subseteq \mathrm{PP}_\star(\check{\mathcal{O}})$.

Note that

$$(\mathbf{c}_1(\imath_{\wp_\uparrow}), \mathbf{c}_1(\jmath_{\wp_\uparrow})) = \begin{cases} (\mathbf{c}_1(\jmath_\wp) + 1, \mathbf{c}_1(\imath_\wp) - 1), & \text{when } \star = C; \\ (\mathbf{c}_1(\jmath_\wp), \mathbf{c}_1(\imath_\wp)), & \text{when } \star = \widetilde{C}, \end{cases}$$

and

$$(\mathbf{c}_i(\imath_{\wp_\uparrow}), \mathbf{c}_i(\jmath_{\wp_\uparrow})) = (\mathbf{c}_i(\imath_\wp), \mathbf{c}_i(\jmath_\wp)), \qquad \text{for } i = 2, 3, 4, \ldots.$$

For $\tau := (\imath_\wp, \mathcal{P}_\tau) \times (\jmath_\wp, \mathcal{Q}_\tau) \times \alpha \in \mathsf{PBP}_\star(\check{\mathcal{O}}, \wp)$, we will define an element

$$\tau_\uparrow := (\imath_{\wp_\uparrow}, \mathcal{P}_{\tau_\uparrow}) \times (\jmath_{\wp_\uparrow}, \mathcal{Q}_{\tau_\uparrow}) \times \alpha \in \mathsf{PBP}_\star(\check{\mathcal{O}}, \wp_\uparrow)$$

by the following recipe.

The case when $\star = C$. Note that $\mathbf{c}_1(\imath_{\wp_\uparrow}) > \mathbf{c}_1(\imath_\wp) \geq 1$ since $(1,2) \in \mathrm{PP}_\star(\check{\mathcal{O}})$. For all $(i,j) \in \mathrm{Box}(\imath_{\wp_\uparrow})$, we define $\mathcal{P}_{\tau_\uparrow}(i,j)$ case by case as in what follows.

(a) Suppose that $\mathcal{P}_\tau(\mathbf{c}_1(\imath_\wp),1) \neq \bullet$.

- If $\mathbf{c}_1(\imath_\wp) \geq 2$ and $\mathcal{P}_\tau(\mathbf{c}_1(\imath_\wp)-1,1) = c$, we define

$$\mathcal{P}_{\tau_\uparrow}(i,j) := \begin{cases} r, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp)-1 \leq i \leq \mathbf{c}_1(\imath_{\wp_\uparrow})-2; \\ c, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp_\uparrow})-1,1); \\ d, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp_\uparrow}),1); \\ \mathcal{P}_\tau(i,j), & \text{otherwise.} \end{cases}$$

- Otherwise, we define

$$\mathcal{P}_{\tau_\uparrow}(i,j) := \begin{cases} r, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp) \leq i \leq \mathbf{c}_1(\imath_{\wp_\uparrow})-1; \\ \mathcal{P}_\tau(\mathbf{c}_1(\imath_\wp),1), & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp_\uparrow}),1); \\ \mathcal{P}_\tau(i,j), & \text{otherwise,} \end{cases}$$

(b) Suppose that $\mathcal{P}_\tau(\mathbf{c}_1(\imath_\wp),1) = \bullet$.

- If $\mathbf{c}_2(\imath_\wp) = \mathbf{c}_1(\imath_\wp)$ and $\mathcal{P}_\tau(\mathbf{c}_1(\imath_\wp),2) = r$, we define

$$\mathcal{P}_{\tau_\uparrow}(i,j) := \begin{cases} r, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp) \leq i \leq \mathbf{c}_1(\imath_{\wp_\uparrow})-1; \\ c, & \text{if } (i,j) = (\mathbf{c}_2(\imath_\wp),2); \\ d, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp_\uparrow}),1); \\ \mathcal{P}_\tau(i,j), & \text{otherwise.} \end{cases}$$

- Otherwise, we define

$$\mathcal{P}_{\tau_\uparrow}(i,j) := \begin{cases} r, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp) \leq i \leq \mathbf{c}_1(\imath_{\wp_\uparrow})-2; \\ c, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp_\uparrow})-1,1); \\ d, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp_\uparrow}),1); \\ \mathcal{P}_\tau(i,j), & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Box}(\jmath_{\wp_\uparrow}) \subseteq \mathrm{Box}(\jmath_\wp)$. For all $(i,j) \in \mathrm{Box}(\jmath_{\wp_\uparrow})$, we define

$$\mathcal{Q}_{\tau_\uparrow}(i,j) := \mathcal{Q}_\tau(i,j).$$

The case when $\star = \widetilde{C}$. For all $(i,j) \in \mathrm{Box}(\imath_{\wp_\uparrow})$, define

$$\mathcal{P}_{\tau_\uparrow}(i,j) := \begin{cases} s, & \text{if } j=1 \text{ and } \mathcal{Q}_\tau(i,j) = r; \\ c, & \text{if } j=1 \text{ and } \mathcal{Q}_\tau(i,j) = d; \\ \mathcal{P}_\tau(i,j), & \text{otherwise.} \end{cases} \tag{10.3}$$

For all $(i,j) \in \mathrm{Box}(\jmath_{\wp_\uparrow})$,

$$\mathcal{Q}_{\tau_\uparrow}(i,j) := \begin{cases} r, & \text{if } j=1 \text{ and } \mathcal{P}_\tau(i,j) = s; \\ d, & \text{if } j=1 \text{ and } \mathcal{P}_\tau(i,j) = c; \\ \mathcal{Q}_\tau(i,j), & \text{otherwise.} \end{cases} \tag{10.4}$$

In both cases, it is routine to check that $\tau_\uparrow$ is a painted bipartition and is an element of $\mathsf{PBP}_\star(\check{\mathcal{O}}, \wp_\uparrow)$.

Lemma 10.3. *The map*

$$T_{\wp,\wp_\uparrow}\colon \mathsf{PBP}_\star(\check{\mathcal{O}},\wp)\rightarrow \mathsf{PBP}_\star(\check{\mathcal{O}},\wp_\uparrow),\quad \tau\mapsto\tau_\uparrow$$

defined above is bijective.

Proof. Let $\tau_\uparrow=(\imath_{\wp_\uparrow},\mathcal{P}_{\tau_\uparrow})\times(\jmath_{\wp_\uparrow},\mathcal{Q}_{\tau_\uparrow})\times\alpha\in\mathsf{PBP}_\star(\check{\mathcal{O}},\wp_\uparrow)$. We will define a painted bipartition $\tau\in\mathsf{PBP}_\star(\check{\mathcal{O}},\wp)$ by the following recipe.

The case when $\star=C$. Note that

$$\mathbf{c}_1(\imath_{\wp_\uparrow})=\mathbf{c}_1(\jmath_\wp)+1>\mathbf{c}_1(\imath_\wp)=\mathbf{c}_1(\jmath_{\wp_\uparrow})+1.$$

We define $\mathcal{P}_\tau$ and $\mathcal{Q}_\tau$ case by case as in what follows.

(a) Suppose that $\mathcal{P}_{\tau_\uparrow}(\mathbf{c}_1(\imath_{\wp_\uparrow})-1,1)=c$.

- If $\mathbf{c}_1(\jmath_{\wp_\uparrow})\geq 1$ and $\mathcal{P}_{\tau_\uparrow}(\mathbf{c}_1(\jmath_{\wp_\uparrow}),1)=r$, for all $(i,j)\in\mathrm{Box}(\imath_\wp)$ we define

$$\mathcal{P}_\tau(i,j):=\begin{cases}c, & \text{if } (i,j)=(\mathbf{c}_1(\imath_\wp)-1,1);\\ d, & \text{if } (i,j)=(\mathbf{c}_1(\imath_\wp),1);\\ \mathcal{P}_{\tau_\uparrow}(i,j), & \text{otherwise,}\end{cases}$$

  and for all $(i,j)\in\mathrm{Box}(\jmath_\wp)$ we define

$$\mathcal{Q}_\tau(i,j):=\begin{cases}s, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp)\leq i\leq \mathbf{c}_1(\jmath_\wp);\\ \mathcal{Q}_{\tau_\uparrow}(i,j), & \text{otherwise.}\end{cases}$$

- Otherwise, for all $(i,j)\in\mathrm{Box}(\imath_\wp)$ we define

$$\mathcal{P}_\tau(i,j):=\begin{cases}\bullet, & \text{if } (i,j)=(\mathbf{c}_1(\imath_\wp),1);\\ \mathcal{P}_{\tau_\uparrow}(i,j), & \text{otherwise,}\end{cases}$$

  and for all $(i,j)\in\mathrm{Box}(\jmath_\wp)$ we define

$$\mathcal{Q}_\tau(i,j):=\begin{cases}\bullet, & \text{if } (i,j)=(\mathbf{c}_1(\imath_\wp),1);\\ s, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp)+1\leq i\leq \mathbf{c}_1(\jmath_\wp);\\ \mathcal{Q}_{\tau_\uparrow}(i,j), & \text{otherwise.}\end{cases}$$

(b) Suppose that $\mathcal{P}_{\tau_\uparrow}(\mathbf{c}_1(\imath_{\wp_\uparrow})-1,1)=r$.

- If $\mathbf{c}_2(\imath_{\wp_\uparrow})=\mathbf{c}_1(\jmath_{\wp_\uparrow})+1$ and $(\mathcal{P}_{\tau_\uparrow}(\mathbf{c}_1(\imath_{\wp_\uparrow}),1),\mathcal{P}_{\tau_\uparrow}(\mathbf{c}_2(\imath_{\wp_\uparrow}),2))=(d,c)$, for all $(i,j)\in\mathrm{Box}(\imath_\wp)$ we define

$$\mathcal{P}_\tau(i,j):=\begin{cases}\bullet, & \text{if } (i,j)=(\mathbf{c}_1(\imath_\wp),1);\\ r, & \text{if } (i,j)=(\mathbf{c}_2(\imath_\wp),2);\\ \mathcal{P}_{\tau_\uparrow}(i,j), & \text{otherwise,}\end{cases}$$

  and for all $(i,j)\in\mathrm{Box}(\jmath_\wp)$ we define

$$\mathcal{Q}_\tau(i,j):=\begin{cases}\bullet, & \text{if } (i,j)=(\mathbf{c}_1(\imath_\wp),1);\\ s, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp)+1\leq i\leq \mathbf{c}_1(\jmath_\wp);\\ \mathcal{Q}_{\tau_\uparrow}(i,j), & \text{otherwise.}\end{cases}$$

▪ Otherwise, for all $(i,j) \in \mathrm{Box}(\imath_\wp)$ we define

$$\mathcal{P}_\tau(i,j) := \begin{cases} \mathcal{P}_{\tau_\uparrow}(\mathbf{c}_1(\imath_{\wp_\uparrow}),1), & \text{if } (i,j) = (\mathbf{c}_1(\imath_\wp),1); \\ \mathcal{P}_{\tau_\uparrow}(i,j), & \text{otherwise,} \end{cases}$$

and for all $(i,j) \in \mathrm{Box}(\jmath_\wp)$ we define

$$\mathcal{Q}_\tau(i,j) := \begin{cases} s, & \text{if } j=1 \text{ and } \mathbf{c}_1(\imath_\wp) \le i \le \mathbf{c}_1(\jmath_\wp); \\ \mathcal{Q}_{\tau_\uparrow}(i,j), & \text{otherwise.} \end{cases}$$

The case when $\star = \widetilde{C}$. The definition of $T_{\wp_\uparrow,\wp}$ is given by the formulas (10.3) and (10.4) with the role of $\wp$ and $\wp_\uparrow$ switched.

In both cases, it is routine to check that

$$T_{\wp_\uparrow,\wp} \colon \mathsf{PBP}_\star(\check{\mathcal{O}},\wp_\uparrow) \to \mathsf{PBP}_\star(\check{\mathcal{O}},\wp), \qquad \tau_\uparrow \mapsto \tau$$

is well-defined and is the inverse of $T_{\wp,\wp_\uparrow}$. □

10.3. **Naive descent of a painted bipartition.** As before, let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.

Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or }\ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or }\ D^*.$$

We call $\star'$ the Howe dual of $\star$.

For a Young diagram $\imath$, its naive descent, which is denoted by $\nabla_{\mathrm{naive}}(\imath)$, is defined to be the Young diagram obtained from $\imath$ by removing the first column. By convention, $\nabla_{\mathrm{naive}}(\emptyset) = \emptyset$.

We start with the notion of the naive descent of a painted bipartition. Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ be a painted bipartition such that $\star_\tau = \star$. Put

$$\alpha'_{\mathrm{naive}} = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ does not occur in the first column of } (\imath,\mathcal{P}); \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ occurs in the first column of } (\imath,\mathcal{P}); \\ \star', & \text{if } \alpha \ne \widetilde{C}. \end{cases} \tag{10.5}$$

Lemma 10.4. *If $\star \in \{\, B, C, C^* \,\}$, then there is a unique painted bipartition of the form $\tau'_{\mathrm{naive}} = (\imath'_{\mathrm{naive}}, \mathcal{P}'_{\mathrm{naive}}) \times (\jmath'_{\mathrm{naive}}, \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'_{\mathrm{naive}}$ with the following properties:*

- $(\imath'_{\mathrm{naive}}, \jmath'_{\mathrm{naive}}) = (\imath, \nabla_{\mathrm{naive}}(\jmath))$;
- *for all $(i,j) \in \mathrm{Box}(\imath'_{\mathrm{naive}})$,*

$$\mathcal{P}'_{\mathrm{naive}}(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \ \mathcal{P}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}(i,j), & \text{if } \ \mathcal{P}(i,j) \notin \{\bullet, s\}; \end{cases}$$

- *for all $(i,j) \in \mathrm{Box}(\jmath'_{\mathrm{naive}})$,*

$$\mathcal{Q}'_{\mathrm{naive}}(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \ \mathcal{Q}(i,j+1) \in \{\bullet, s\}; \\ \mathcal{Q}(i,j+1), & \text{if } \ \mathcal{Q}(i,j+1) \notin \{\bullet, s\}. \end{cases}$$

Proof. First assume that the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$. Then the image of $\mathcal{P}$ is in fact contained in $\{\bullet\}$, and $(\imath, \jmath)$ is right interlaced in the sense that

$$\mathbf{c}_1(\jmath) \geq \mathbf{c}_1(\imath) \geq \mathbf{c}_2(\jmath) \geq \mathbf{c}_2(\imath) \geq \mathbf{c}_3(\jmath) \geq \mathbf{c}_3(\imath) \geq \cdots .$$

Hence $(\imath', \jmath') := (\imath, \nabla(\jmath))$ is left interlaced in the sense that

$$\mathbf{c}_1(\imath') \geq \mathbf{c}_1(\jmath') \geq \mathbf{c}_2(\imath') \geq \mathbf{c}_2(\jmath') \geq \mathbf{c}_3(\imath') \geq \mathbf{c}_3(\jmath') \geq \cdots .$$

Then it is clear that there is a unique painted bipartition of the form $\tau'_{\mathrm{naive}} = (\imath'_{\mathrm{naive}}, \mathcal{P}'_{\mathrm{naive}}) \times (\jmath'_{\mathrm{naive}}, \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'_{\mathrm{naive}}$ such that symbols of $\mathcal{P}'_{\mathrm{naive}}$ and $\mathcal{Q}'_{\mathrm{naive}}$ are both in $\{\bullet, s\}$. This proves the lemma in the special case when the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$.

The proof of the lemma in the general case is easily reduced to this special case. $\square$

Lemma 10.5. *If $\star \in \{\widetilde{C}, D, D^*\}$, then there is a unique painted bipartition of the form $\tau'_{\mathrm{naive}} = (\imath'_{\mathrm{naive}}, \mathcal{P}'_{\mathrm{naive}}) \times (\jmath'_{\mathrm{naive}}, \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'_{\mathrm{naive}}$ with the following properties:*

- $(\imath'_{\mathrm{naive}}, \jmath'_{\mathrm{naive}}) = (\nabla_{\mathrm{naive}}(\imath), \jmath)$;
- *for all* $(i, j) \in \mathrm{Box}(\imath'_{\mathrm{naive}})$,
$$\mathcal{P}'_{\mathrm{naive}}(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i, j+1) \in \{\bullet, s\}; \\ \mathcal{P}(i, j+1), & \text{if } \mathcal{P}(i, j+1) \notin \{\bullet, s\}; \end{cases}$$
- *for all* $(i, j) \in \mathrm{Box}(\jmath'_{\mathrm{naive}})$,
$$\mathcal{Q}'_{\mathrm{naive}}(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i, j) \in \{\bullet, s\}; \\ \mathcal{Q}(i, j), & \text{if } \mathcal{Q}(i, j) \notin \{\bullet, s\}. \end{cases}$$

Proof. The proof is similar to that of Lemma 10.4. $\square$

In the notation of Lemma 10.4 and Lemma 10.5, we call τ'_{naive} the naive descent of τ, to be denoted by $\nabla_{\mathrm{naive}}(\tau)$.

Example 10.6. *If*

$$\tau = \begin{array}{llll} \bullet & \bullet & \bullet & c \\ \bullet & s & c & \\ s & & & \\ c & & & \end{array} \times \begin{array}{lll} \bullet & \bullet & \bullet \\ \bullet & r & d \\ d & d & \end{array} \times \widetilde{C},$$

then

$$\nabla_{\mathrm{naive}}(\tau) = \begin{array}{lll} \bullet & \bullet & c \\ \bullet & c & \end{array} \times \begin{array}{lll} \bullet & \bullet & s \\ \bullet & r & d \\ d & d & \end{array} \times B^-.$$

10.4. **Descent of a painted bipartition.** The dual descent of a Young diagram $\check{\mathcal{O}}$ is defined to be

$$\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}}) := \check{\nabla}_\star(\check{\mathcal{O}}) := \begin{cases} \square, & \text{if } \star \in \{D, D^*\} \text{ and } |\check{\mathcal{O}}| = 0; \\ \check{\nabla}_{\mathrm{naive}}(\check{\mathcal{O}}), & \text{otherwise,} \end{cases}$$

where $\square$ denotes the Young diagram that has total size 1, and $\check{\nabla}_{\mathrm{naive}}(\check{\mathcal{O}})$ denotes the Young diagram obtained from $\check{\mathcal{O}}$ by removing the first row. We also view $\check{\mathcal{O}}'$ as a nilpotent orbit in the Langlands dual (or metaplectic Langlands dual) of the complexified Lie algebra of a classical real Lie group of type $\star'$.

The dual descent of a subset $\wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}})$ is defined to be the subset

$$\wp' := \check{\nabla}(\wp) := \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \wp\} \subseteq \mathrm{PP}_{\star'}(\check{\mathcal{O}}').$$

Note that

$$(\imath_{\wp'}, \jmath_{\wp'}) = \begin{cases} (\imath_\wp, \nabla_{\mathrm{naive}}(\jmath_\wp)), & \text{if } \star = B, \text{ or } \star \in \{C, C^*\} \text{ and } (1,2) \notin \wp; \\ (\imath_{\wp_\downarrow}, \nabla_{\mathrm{naive}}(\jmath_{\wp_\downarrow})), & \text{if } \star \in \{C, C^*\} \text{ and } (1,2) \in \wp; \\ (\nabla_{\mathrm{naive}}(\imath_\wp), \jmath_\wp), & \text{if } \star \in \{D, D^*\}, \text{ or } \star = \widetilde{C} \text{ and } (1,2) \notin \wp; \\ (\nabla_{\mathrm{naive}}(\imath_{\wp_\downarrow}), \jmath_{\wp_\downarrow}), & \text{if } \star = \widetilde{C} \text{ and } (1,2) \in \wp, \end{cases}$$

where $\imath_{\wp'} := \imath_{\star'}(\check{\mathcal{O}}', \wp')$, $\jmath_{\wp'} := \jmath_{\star'}(\check{\mathcal{O}}', \wp')$, and $\wp_\downarrow = \wp \setminus \{(1,2)\}$. Suppose that

$$\boldsymbol{\tau} = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathsf{PBP}_\star(\check{\mathcal{O}}, \wp).$$

We will define an element

$$\boldsymbol{\tau}' := (\imath_{\wp'}, \mathcal{P}') \times (\jmath_{\wp'}, \mathcal{Q}') \times \alpha'_{\mathrm{naive}} \in \mathsf{PBP}_{\star}(\check{\mathcal{O}}', \wp'),$$

to be called the descent of $\boldsymbol{\tau}$.

The case when $\star = B$.

(a) If

$$\begin{cases} \alpha = B^+; \\ (2,3) \notin \wp; \\ \mathbf{r}_2(\check{\mathcal{O}}) > 0; \\ \mathcal{Q}(\mathbf{c}_1(\imath_\wp), 1) \in \{\, r, d \,\}, \end{cases}$$

we define

$$\mathcal{P}'(i,j) := \begin{cases} s, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp'}), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise} \end{cases}$$

for all $(i,j) \in \mathrm{Box}(\imath_{\wp'})$, and $\mathcal{Q}' := \mathcal{Q}'_{\mathrm{naive}}$.

(b) If

$$\begin{cases} \alpha = B^+; \\ (2,3) \in \wp; \\ \mathcal{Q}(\mathbf{c}_2(\jmath_\wp), 1) \in \{\, r, d \,\}, \end{cases}$$

we define $\mathcal{P}' := \mathcal{P}'_{\mathrm{naive}}$, and

$$\mathcal{Q}'(i,j) := \begin{cases} r, & \text{if } (i,j) = (\mathbf{c}_1(\jmath_{\wp'}), 1); \\ \mathcal{Q}'_{\mathrm{naive}}(i,j), & \text{otherwise} \end{cases}$$

for all $(i,j) \in \mathrm{Box}(\jmath_{\wp'})$.

(c) Otherwise, we define $\mathcal{P}' := \mathcal{P}'_{\mathrm{naive}}$ and $\mathcal{Q}' := \mathcal{Q}'_{\mathrm{naive}}$.

The case when $\star = D$.

(a) If

$$\begin{cases} \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0; \\ \mathcal{P}(\mathbf{c}_2(\imath_\wp), 2) = c; \\ \mathcal{P}(i, 1) \in \{\, r, d \,\}, \qquad \text{for all } \mathbf{c}_2(\imath_\wp) \leq i \leq \mathbf{c}_1(\imath_\wp), \end{cases}$$

we define

$$\mathcal{P}'(i,j) := \begin{cases} r, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp'}), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise} \end{cases}$$

for all $(i,j) \in \mathrm{Box}(\imath_{\wp'})$, and $\mathcal{Q}' := \mathcal{Q}'_{\mathrm{naive}}$.
(b) If

$$\begin{cases} (2,3) \in \wp; \\ \mathcal{P}(\mathbf{c}_2(\imath_\wp) - 1, 1) \in \{ r, c \}, \end{cases}$$

we define

$$\mathcal{P}'(i,j) := \begin{cases} r, & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp'}) - 1, 1); \\ \mathcal{P}(\mathbf{c}_2(\imath_\wp) - 1, 1), & \text{if } (i,j) = (\mathbf{c}_1(\imath_{\wp'}), 1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise} \end{cases}$$

for all $(i,j) \in \mathrm{Box}(\imath_{\wp'})$, and $\mathcal{Q}' := \mathcal{Q}'_{\mathrm{naive}}$.
(c) Otherwise, we define $\mathcal{P}' := \mathcal{P}'_{\mathrm{naive}}$ and $\mathcal{Q}' := \mathcal{Q}'_{\mathrm{naive}}$.

The Case when $\star \in \{ C, \widetilde{C}, C^*, D^* \}$.
(a) If $(1,2) \notin \wp$, we define

$$\tau' := \nabla_{\mathrm{naive}}(\tau).$$

(b) If $(1,2) \in \wp$, then $\star \in \{ C, \widetilde{C} \}$ (by Proposition 10.1) and we define

$$\tau' := \nabla_{\mathrm{naive}}(\tau_{\wp_\downarrow}),$$

where $\wp_\downarrow := \wp \setminus \{ (1,2) \}$, $\tau_{\wp_\downarrow} := T^{-1}_{\wp_\downarrow,\wp}(\tau)$, and $T_{\wp_\downarrow,\wp}$ is as in Lemma 10.3.

In all cases, it is routine to check that τ' thus constructed is an element in $\mathsf{PBP}_{\star'}(\check{\mathcal{O}}', \wp')$, which is called the descent of τ. To summarize, we have a map

$$\nabla\colon \mathsf{PBP}_\star(\check{\mathcal{O}}, \wp) \to \mathsf{PBP}_{\star'}(\check{\mathcal{O}}', \wp'). \tag{10.6}$$

We call ∇ the descent map of painted bipartitions.

10.5. **Properties of the descent map.** We first define the notion of the tail of a painted bipartition, when $\star \in \{ B, D, C^* \}$.

Put

$$\star_{\mathrm{t}} := \begin{cases} D, & \text{if } \star \in \{ B, D \}; \\ C^*, & \text{if } \star = C^*, \end{cases} \quad \text{and} \quad k := \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} + 1, & \text{if } \star \in \{B, D\}; \\ \lfloor \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}}) - 1}{2} \rfloor, & \text{if } \star = C^*. \end{cases}$$

Note that k is a non-negative integer, and is positive when $\star \in \{ B, D \}$.

From the pair $(\star, \check{\mathcal{O}})$, we define a Young diagram $\check{\mathcal{O}}_{\mathbf{t}}$ as follows.

- If $\star \in \{ B, D \}$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of two rows with lengths $2k-1$ and 1.
- If $\star = C^*$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of one row with length $2k+1$.

Note that in all the three cases $\check{\mathcal{O}}_{\mathbf{t}}$ has good parity (with respect to $\star_{\mathbf{t}}$), $\mathrm{PP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}) = \emptyset$, and every element in $\mathsf{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ has the form

$$
\begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \emptyset \times D, \qquad
\begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \emptyset \times D, \qquad \text{or} \qquad
\emptyset \times \begin{array}{|c|}\hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times C^*, \tag{10.7}
$$

according to $\star = B, D$, or C^*, respectively. When $k = 0$ (and thus $\star = C^*$), the element in $\mathsf{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ is understood to be $\emptyset \times \emptyset \times C^*$.

When $\star \in \{ B, D, C^* \}$, the tail $\boldsymbol{\tau}_{\mathbf{t}} \in \mathsf{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ of an element $\boldsymbol{\tau} = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_{\star}(\check{\mathcal{O}}, \wp)$ will be the painted bipartition in (10.7), specified by the multiset $\{x_1, x_2, \cdots, x_k\}$ case by case as follows.

The case when $\star = B$. In this case, $\mathbf{c}_1(\imath) \leq \mathbf{c}_1(\jmath)$. The multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{Q}(j, 1) \mid \mathbf{c}_1(\imath) + 1 \leq j \leq \mathbf{c}_1(\jmath) \}$$

with the set

$$\begin{cases} \{ c \}, & \text{if } \alpha = B^+, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ s \}, & \text{if } \alpha = B^-, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ \mathcal{Q}(\mathbf{c}_1(\imath), 1) \}, & \text{otherwise.} \end{cases}$$

The case when $\star = D$. In this case, $\mathbf{c}_1(\imath) > \mathbf{c}_1(\jmath)$ when $|\check{\mathcal{O}}| > 0$. The multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{P}(j, 1) \mid \mathbf{c}_1(\jmath) + 2 \leq j \leq \mathbf{c}_1(\imath) \}$$

with the set

$$\begin{cases} \{ d \}, & \text{if } |\check{\mathcal{O}}|{=}0; \\ \{ c \}, & \text{if } \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0, \ \mathcal{P}(\mathbf{c}_1(\imath), 1) \in \{ r, d \}, \text{ and} \\ & \qquad (\mathcal{P}(\mathbf{c}_1(\jmath) + 1, 1), \mathcal{P}(\mathbf{c}_1(\jmath) + 1, 2)) = (r, c); \\ \{ \mathcal{P}(\mathbf{c}_1(\jmath) + 1, 1) \}, & \text{otherwise.} \end{cases}$$

The case $\star = C^*$. In this case, $\check{\mathcal{O}}$ is quasi-distinguished, $\wp = \emptyset$, and $\mathbf{c}_1(\imath) \leq \mathbf{c}_1(\jmath)$. The multiset $\{x_1, x_2, \cdots, x_k\}$ equals the multiset

$$\{ \mathcal{Q}(j, 1) \mid \mathbf{c}_1(\imath) + 1 \leq j \leq \mathbf{c}_1(\jmath) \}.$$

We introduce one final notation when $\star \in \{ B, D \}$:

$$x_{\boldsymbol{\tau}} := \mathcal{P}_{\boldsymbol{\tau}_{\mathbf{t}}}(k, 1),$$

which is the symbol in the last box of the tail $\boldsymbol{\tau}_{\mathbf{t}}$.

The following two propositions summarize key properties of the descent map. Both readily follow from the detailed description of the descent algorithm.

Proposition 10.7. *Suppose that $\star \in \{ C, \widetilde{C}, D^* \}$.*
(a) When $\star = D^*$ or $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$, the descent map (10.6) *is bijective.*

(b) When $\star \in \{C, \widetilde{C}\}$ and $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$, the descent map (10.6) *is injective and its image equals*

$$\left\{ \tau' \in \mathsf{PBP}_{\star'}(\check{\mathcal{O}}', \wp') \,\middle|\, x_{\tau'} \neq s \right\}.$$

Recall that $\mathcal{O} = d_{\mathrm{BV}}(\check{\mathcal{O}})$ is the Barbasch-Vogan dual of $\check{\mathcal{O}}$. Write

$$\check{\mathcal{O}}'' := \check{\nabla}^2(\check{\mathcal{O}}) := \check{\nabla}(\check{\mathcal{O}}') \quad \text{and} \quad \wp'' := \check{\nabla}^2(\wp) := \check{\nabla}(\wp').$$

Then we have the double descent map

$$\nabla^2 := \nabla \circ \nabla : \ \mathsf{PBP}_\star(\check{\mathcal{O}}, \wp) \longrightarrow \mathsf{PBP}_\star(\check{\mathcal{O}}'', \wp'').$$

Proposition 10.8. *Suppose that $\star \in \{ B, D, C^* \}$. Consider the map*

$$\mathsf{PBP}_\star(\check{\mathcal{O}}, \wp) \longrightarrow \mathsf{PBP}_\star(\check{\mathcal{O}}'', \wp'') \times \mathsf{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}), \qquad \tau \mapsto (\nabla^2(\tau), \tau_{\mathbf{t}}). \tag{10.8}$$

(a) Suppose that $\star = C^*$ or $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then the map (10.8) *is bijective, and for every $\tau \in \mathsf{PBP}_\star(\check{\mathcal{O}})$,*

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}), \mathbf{c}_2(\mathcal{O})) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}).$$

(b) Suppose that $\star \in \{ B, D \}$ and $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$. Then the map (10.8) *is injective with image*

$$\left\{ (\tau'', \tau_0) \in \mathsf{PBP}_\star(\check{\mathcal{O}}'', \wp'') \times \mathsf{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}}) \;\middle|\; \begin{array}{l} \text{either } x_{\tau''} = d, \text{ or} \\ x_{\tau''} \in \{ r, c \} \text{ and } \mathcal{P}_{\tau_0}^{-1}(\{ s, c \}) \neq \emptyset, \end{array} \right\}$$

and for every $\tau \in \mathsf{PBP}_\star(\check{\mathcal{O}}, \wp)$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}).$$

The following corollary will be used in [BMSZ21].

Corollary 10.9. *Suppose that $\star \in \{ B, D, C^* \}$. Denote*

$$\varepsilon_\tau := \begin{cases} 0, & \text{if } \star \in \{ B, D \} \text{ and } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

Then the map

$$\begin{array}{rcl} \mathsf{PBP}_\star(\check{\mathcal{O}}, \wp) & \to & \mathsf{PBP}_{\star'}(\check{\mathcal{O}}', \wp') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array}$$

is injective.

Proof. This is easy to verify in the special case where $\mathbf{r}_3(\check{\mathcal{O}}) = 0$ and

$$(\mathbf{r}_1(\check{\mathcal{O}}), \mathbf{r}_2(\check{\mathcal{O}})) = \begin{cases} (2k-2, 0), & \text{if } \star = B; \\ (2k-1, 1), & \text{if } \star = D; \\ (2k-1, 0), & \text{if } \star = C^*. \end{cases}$$

In general, it follows from the injectivity of ∇ in Proposition 10.7 and the signature formula in Proposition 10.8. ∎

10.6. **Generating functions.** We introduce some generating functions to count painted bipartitions. When $\star \in \{ B, D, C^* \}$, we define

$$f_{\star,\check{\mathcal{O}},\wp}(\mathbf{p},\mathbf{q}) := \sum_{\tau \in \mathsf{PBP}_\star(\check{\mathcal{O}},\wp)} \mathbf{p}^{p_\tau}\mathbf{q}^{q_\tau},$$

where (p_τ, q_τ) is as in (2.15). This is an element in the polynomial ring $\mathbb{Z}[\mathbf{p},\mathbf{q}]$, where $\mathbf{p}$ and $\mathbf{q}$ are indeterminants. The coefficient of $\mathbf{p}^p\mathbf{q}^q$ in $f_{\star,\check{\mathcal{O}},\wp}(\mathbf{p},\mathbf{q})$ equals the cardinality of $\mathsf{PBP}_G(\check{\mathcal{O}},\wp)$.

When $\star \in \{B, D\}$, for each subset $S \subseteq \{ c, d, r, s \}$, we also define

$$\mathsf{PBP}^S_\star(\check{\mathcal{O}},\wp) := \{ \tau \in \mathsf{PBP}_\star(\check{\mathcal{O}},\wp) \mid x_\tau \in S \}$$

and the corresponding generating function

$$f^S_{\star,\check{\mathcal{O}},\wp}(\mathbf{p},\mathbf{q}) := \sum_{\tau \in \mathsf{PBP}^S_\star(\check{\mathcal{O}},\wp)} \mathbf{p}^{p_\tau}\mathbf{q}^{q_\tau}.$$

Note that

$$f^{\{d\}}_{D,[0]_{\mathrm{row}},\emptyset} := 1, \quad f^{\{c,r\}}_{D,[0]_{\mathrm{row}},\emptyset} := 0, \quad \text{and} \quad f^{\{s\}}_{D,[0]_{\mathrm{row}},\emptyset} := 0.$$

In the following we will compute $f^{\{d\}}_{\star,\check{\mathcal{O}},\wp}$, $f^{\{c,r\}}_{\star,\check{\mathcal{O}},\wp}$ and $f^{\{s\}}_{\star,\check{\mathcal{O}},\wp}$ whose sum is the desired function $f_{\star,\check{\mathcal{O}},\wp}$.

For every integer k, we define

$$\nu_k := \begin{cases} \sum_{i=0}^k \mathbf{p}^{2i}\mathbf{q}^{2(k-i)}, & \text{if } k \geq 0; \\ 0, & \text{if } k < 0. \end{cases}$$

It is straightforward to check the following identities: For $k \in \mathbb{N}$

$$f^S_{B,[2k]_{\mathrm{row}},\emptyset} = \begin{cases} (\mathbf{p}^2\mathbf{q} + \mathbf{p}\mathbf{q}^2)\nu_{k-1}, & \text{if } S = \{ d \}; \\ \mathbf{p}\nu_k + \mathbf{p}^2\mathbf{q}\nu_{k-1}, & \text{if } S = \{ c, r \}; \\ \mathbf{q}^{2k+1}, & \text{if } S = \{ s \}, \end{cases}$$

and for $k \in \mathbb{N}^+$

$$f^S_{D,[2k-1,1]_{\mathrm{row}},\emptyset} = \begin{cases} \mathbf{p}\mathbf{q}\nu_{k-1} + \mathbf{p}^2\mathbf{q}^2\nu_{k-2}, & \text{if } S = \{ d \}; \\ (\mathbf{p}^2 + \mathbf{p}\mathbf{q})\nu_{k-1}, & \text{if } S = \{ c, r \}; \\ \mathbf{q}^{2k}, & \text{if } S = \{ s \}. \end{cases}$$

For every $k \in \mathbb{N}$, we also define

$$\begin{aligned} h^{\{d\}}_k &:= (\mathbf{p}^2\mathbf{q}^2 + \mathbf{p}\mathbf{q}^3)\nu_{k-2}, \\ h^{\{c,r\}}_k &:= \mathbf{p}\mathbf{q}\nu_{k-1} + \mathbf{p}^2\mathbf{q}^2\nu_{k-2}, \\ h^{\{s\}}_k &:= \mathbf{q}^{2k}. \end{aligned}$$

Proposition 10.10. *Suppose that $\star \in \{ B, D \}$ and $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Let $k := \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} + 1$ and let $S = \{ d \}, \{ c, r \}$, or $\{ s \}$.*

(a) If $(2,3) \in \mathrm{PP}_\star(\check{\mathcal{O}})$, then

$$f^S_{\star,\check{\mathcal{O}},\wp} = (\mathbf{p}\mathbf{q})^{\mathbf{c}_1(\check{\mathcal{O}})} f^S_{D,[2k-1,1]_{\mathrm{row}},\emptyset} \cdot f_{\star,\check{\nabla}^2(\check{\mathcal{O}}),\check{\nabla}^2(\wp)}.$$

(b) If $(2,3) \notin \mathrm{PP}_\star(\check{\mathcal{O}})$*, then*

$$f^S_{\star,\check{\mathcal{O}},\wp} := (\mathbf{pq})^{\mathbf{c}_1(\check{\mathcal{O}})-1}(f^S_{D,[2k-1,1]_{\mathrm{row}},\emptyset} \cdot f^{\{d\}}_{\star,\check{\nabla}^2(\check{\mathcal{O}}),\check{\nabla}^2(\wp)} + h^S_k \cdot f^{\{c,r\}}_{\star,\check{\nabla}^2(\check{\mathcal{O}}),\check{\nabla}^2(\wp)}).$$

Proof. This follows from Proposition 10.8, after a routine calculation. □

Proposition 10.11. *Suppose that* $\star \in \{C, \widetilde{C}\}$ *and* $\mathbf{r}_1(\check{\mathcal{O}}) > 0$.
(a) If $(1,2) \in \mathrm{PP}_\star(\check{\mathcal{O}})$*, then*

$$\#\mathsf{PBP}_\star(\check{\mathcal{O}},\wp) = f_{\star',\check{\nabla}(\check{\mathcal{O}}),\check{\nabla}(\wp)}(1,1).$$

(b) If $(1,2) \notin \mathrm{PP}_\star(\check{\mathcal{O}})$*, then*

$$\#\mathsf{PBP}_\star(\check{\mathcal{O}},\wp) = f^{\{c,r\}}_{\star',\check{\nabla}(\check{\mathcal{O}}),\check{\nabla}(\wp)}(1,1) + f^{\{d\}}_{\star',\check{\nabla}(\check{\mathcal{O}}),\check{\nabla}(\wp)}(1,1).$$

Proof. This follows from Proposition 10.7, after a routine calculation. □

As a consequence of Proposition 10.10 and Proposition 10.11, when $\star \in \{B, C, \widetilde{C}, D\}$, the generating functions $f_{\star,\check{\mathcal{O}},\wp}$ are independent of $\wp$ and so

$$f_{\star,\check{\mathcal{O}},\wp} = f_{\star,\check{\mathcal{O}},\emptyset}, \quad \text{for all } \wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}}).$$

Consequently,

$$\#\mathsf{PBP}_\star(\check{\mathcal{O}},\wp) = \#\mathsf{PBP}_\star(\check{\mathcal{O}},\emptyset), \quad \text{for all } \wp \subseteq \mathrm{PP}_\star(\check{\mathcal{O}}).$$

This proves Proposition 10.2.

The following two results are easy to check using Proposition 10.7 and Proposition 10.8.

Proposition 10.12. *Suppose that* $\star = C^*$*. Then*

$$f_{\star,\check{\mathcal{O}},\emptyset} = \begin{cases} \nu_{\frac{\mathbf{r}_1(\check{\mathcal{O}})-1}{2}}, & \text{if } \mathbf{r}_2(\check{\mathcal{O}}) = 0; \\ (\mathbf{pq})^{\mathbf{r}_2(\check{\mathcal{O}})+1}\nu_k \cdot f_{\star,\check{\nabla}^2(\check{\mathcal{O}}),\emptyset}, & \text{if } \mathbf{r}_2(\check{\mathcal{O}}) > 0. \end{cases}$$

where $k := \frac{\mathbf{r}_1(\check{\mathcal{O}})-\mathbf{r}_2(\check{\mathcal{O}})}{2} - 1$.

Proposition 10.13. *Suppose that* $\star = D^*$*. Then*

$$\#\mathsf{PBP}_G(\check{\mathcal{O}},\emptyset) = f_{\star',\check{\nabla}(\check{\mathcal{O}}),\emptyset}(1,1).$$

The following result is clear by Proposition 10.12 and Proposition 10.13.

Proposition 10.14. *Suppose that* $\star \in \{C^*, D^*\}$*. Then the number of* $\mathsf{PBP}_G(\check{\mathcal{O}})$ *equals the number of* G*-orbits in* $(\sqrt{-1}\mathfrak{g}_0^*) \cap \mathcal{O}$*, where* $\mathfrak{g}_0$ *denotes the Lie algebra of* G *and* $\mathcal{O} := d_{\mathrm{BV}}(\check{\mathcal{O}})$.

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