SPECIAL UNIPOTENT REPRESENTATIONS : ORTHOGONAL AND SYMPLECTIC GROUPS

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Contents

1. Introduction and the main results

1.1. **Unitary representations and the orbit method.** A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [40, 41]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [6, 40]. For more general Lie groups, work of Mackey and Duflo [25, 52] suggest that one should focus attention on reductive Lie groups. As expounded by Vogan in his writings (see for example [78, 80, 81]), the problem finally is to quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We mention two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [4, 5], Adams, Barbasch and Vogan established some important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [2]. The problem of finding (integral) special unipotent representations for complex semisimple groups (as well as their distribution characters) was solved earlier by Barbasch and Vogan [15] and the unitarity of these representations was established by Barbasch for complex classical groups [7]. In a similar vein, Barbasch outlined his proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [8]. The second major development is Vogan's theory of associated varieties [79] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a

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representation attached to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [36], the description of discrete spectrum by Adams [1] and the preservation of unitarity in stable range theta lifting by Li [47]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [12, 16, 31, 33, 34, 64, 67, 70, 76]. Particularly worth mentioning were the work of Przebinda [67] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [31] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [26, 49, 58, 60], which among other things demonstrate the tight link between the orbit method and Howe's theory.

In the present article we will demonstrate that the orbit method and Howe's theory in fact have perfect synergy when it comes to unipotent representations. (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [12, 31, 55, 76].) We will restrict our attention to a real classical group G of orthogonal or symplectic type and we will construct all unipotent representations of $\widetilde{G}$ attached to $\mathcal{O}$ (in the sense of Barbasch and Vogan) via the method of theta lifting. Here $\widetilde{G}$ is either G, or the metaplectic cover of G if G is a real symplectic group, and $\mathcal{O}$ is a member of our preferred set of complex nilpotent orbits, large enough to include all those which are rigid special. Here we wish to emphasize that there have been extensive investigations of unipotent representations for real reductive groups by Vogan and his collaborators (see e.g. [2, 78, 79]), only for unitary groups complete results are known (*cf.* [14, 76]), in which case all such representations may also be described in terms of cohomological induction.

1.2. **Special unipotent representations of classical groups of type B, C or D.** In this article, we are aimed to understand special unipotent representations of classical groups of type B, C or D. As the cases of complex orthogonal groups and complex symplectic groups are well-understood (see [15] and [12]), we will focus on the following groups:

$$\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p,q),\ \mathrm{O}^*(2n), \tag{1.1}$$

where $p, q, n \in \mathbb{N} := \{0, 1, 2, \cdots\}$. Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the real metaplectic group, namely the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n = 0$.

Let G be one of the groups in (1.1). As usual, we view it as a real form of $G_{\mathbb{C}}$, or a double cover of a real form of $G_{\mathbb{C}}$ in the metaplecitic case, where

$$G_{\mathbb{C}} := \begin{cases} \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \text{ or } \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2p+2q}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Respectively write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ for the Lie algebras of G and $G_{\mathbb{C}}$. Then $\mathfrak{g}$ is viewed as a complexification of $\mathfrak{g}_{\mathbb{R}}$.

Let $r_{\mathfrak{g}}$ denote the rank of $\mathfrak{g}$. Let $W_{r_{\mathfrak{g}}}$ denote the subgroup of $\mathrm{GL}_{r_{\mathfrak{g}}}(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. Then as usual, Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} = (\mathrm{S}(\mathbb{C}^{r_{\mathfrak{g}}}))^{W_{r_{\mathfrak{g}}}}. \tag{1.2}$$

Here and henceforth, "U" indicates the universal enveloping algebra of a Lie algebra, a superscript group indicate the space of invariant vectors under the group action, and "S" indicates the symmetric algebra. Unless $G_{\mathbb{C}}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. By (1.2), we have the following parameterization of characters of $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$:

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}, \mathbb{C}) = W_{r_{\mathfrak{g}}}\backslash(\mathbb{C}^{r_{\mathfrak{g}}})^* = W_{r_{\mathfrak{g}}}\backslash\mathbb{C}^{r_{\mathfrak{g}}}$$

Here "$\mathrm{Hom}_{\mathrm{alg}}$" indicates the set of $\mathbb{C}$-algebra homomorphisms, and a superscript " $*$ " over a vector space indicates the dual space.

We define the Langlands dual of G to be the complex group

$$\check{G} := \begin{cases} \mathrm{Sp}_{p+q-1}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is odd;} \\ \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is even;} \\ \mathrm{O}_{2n+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{O}_{2p+2q+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$.

Denote by $\mathrm{Nil}(\check{\mathfrak{g}})$ the set of nilpotent $\check{G}$-orbits in $\check{\mathfrak{g}}$, namely the set of all orbits of nilpotent matrices under the adjoint action of $\check{G}$ in $\check{\mathfrak{g}}$. When no confusion is possible, we will not distinguish a nilpotent orbit in $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ with its corresponding Young diagram. In particular, the zero orbit is also regarded as the Young with one nonempty column at most.

Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. It determines a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$ as in what follows. For every integer $a \geqslant 0$, write

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd;} \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even;} \end{cases}$$

By convention, $\rho(1)$ and $\rho(0)$ are the empty sequence. Write $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ ($s \geqslant 0$) for the row lengths of $\check{\mathcal{O}}$. Define

$$\chi(\check{\mathcal{O}}) := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0), \tag{1.3}$$

to be viewed as a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} \to \mathbb{C}$. Here the number of 0's is

$$\left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor.$$

Recall the following well-known result of Dixmier ([17, Section 3]): for every algebraic character χ of $\mathrm{Z}(\mathfrak{g})$, there exists a unique maximal ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of χ. As an easy consequence, we know that there is a unique maximal $G_{\mathbb{C}}$-stable ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.

Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation ([19,83]) if its Harish-Chandra module has finite length. When $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a metaplectic group, write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then a representation of G is said to be genuine if ε_G acts on it through the scalar multiplication by -1. Following Barbasch and Vogan ([2, 15]), we make the following definition.

Definition 1.1. *Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. An irreducible Casselman-Wallach representation π of G is attached to $\check{\mathcal{O}}$ if*

- *$I_{\check{\mathcal{O}}}$ annihilates π; and*
- *V is genuine if G is a metaplectic group.*

Write $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ for the set of all isomorphism classes of irreducible Casselman-Wallach representations of G that are attached to $\check{\mathcal{O}}$. We say that an irreducible Casselman-Wallach representation of G is special unipotent if it is attached to $\check{\mathcal{O}}$, for some $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. We will construct all the special unipotent representations, and show that they are all unitarizable as predicted by the Arthur-Barbasch-Vogan conjecture [2, Introduction].

1.3. **Painted bipartitions.** We introduce six symbols B, C, D, $\widetilde{C}$, C^* and D^* to indicate the types of the groups that we are considering as in (1.1), namely odd real orthogonal groups, real symplectic groups, even real orthogonal groups, real metaplectic groups, quaternionic symplectic groups and quaternionic orthogonal groups, respectively. Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$.

For every Young diagram $\imath$, write

$$\mathbf{r}_1(\imath) \geqslant \mathbf{r}_2(\imath) \geqslant \mathbf{r}_3(\imath) \geqslant \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots$$

for its column lengths. Denote by $|\imath| := \sum_{i=1}^{\infty} \mathbf{r}_i(\imath)$ the total size of $\imath$.

For every Young diagram $\imath$, we view the set $\mathrm{Box}(\imath)$ of boxes of $\imath$ as the following subset of $\mathbb{N}^+ \times \mathbb{N}^+$ ($\mathbb{N}^+$ denotes the set of positive integers):

$$\mathrm{Box}(\imath) := \left\{ (i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ \;\middle|\; j \leqslant \mathbf{r}_i(\imath) \right\}. \tag{1.4}$$

We introduce five symbols $\bullet$, s, r, c and d.

Definition 1.2. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is the set of boxes of a Young diagram when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$ when $S = \{s\}$ or $\{r\}$;*
- *every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$ when $S = \{c\}$ or $\{d\}$.*

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram and a painting on it.

Example. Suppose that $\imath = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$, then there are $25 + 12 + 6 + 2 = 45$ paintings on $\imath$ in total as listed below.

$\begin{array}{|c|c|}\hline \bullet & \alpha \\ \hline \beta \\ \cline{1-1}\end{array}$ $\alpha, \beta \in \{\bullet, s, r, c, d\}$

$\begin{array}{|c|c|}\hline s & \alpha \\ \hline \beta \\ \cline{1-1}\end{array}$ $\alpha \in \{r, c, d\}, \beta \in \{s, r, c, d\}$

$\begin{array}{|c|c|}\hline r & \alpha \\ \hline \beta \\ \cline{1-1}\end{array}$ $\alpha \in \{c, d\}, \beta \in \{r, c, d\}$

$\begin{array}{|c|c|}\hline c & \alpha \\ \hline d \\ \cline{1-1}\end{array}$ $\alpha \in \{c, d\}$

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.3. *A painted bipartition is a triple* $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$*, where* $(\imath, \mathcal{P})$ *and* $(\jmath, \mathcal{Q})$ *are painted Young diagrams, and* $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$*, subject to the following conditions:*

- $\mathcal{P}^{-1}(\bullet) = \mathcal{Q}^{-1}(\bullet)$*;*
- *the image of* $\mathcal{P}$ *is contained in*

$$\begin{cases} \{\bullet, c\}, & \textit{if } \alpha = B^+ \textit{ or } B^-; \\ \{\bullet, r, c, d\}, & \textit{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \textit{if } \alpha = D; \\ \{\bullet, s, c\}, & \textit{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \textit{if } \alpha = C^*; \\ \{\bullet, s\}, & \textit{if } \alpha = D^*, \end{cases}$$

- *the image of* $\mathcal{Q}$ *is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \textit{if } \alpha = B^+ \textit{ or } B^-; \\ \{\bullet, s\}, & \textit{if } \alpha = C; \\ \{\bullet\}, & \textit{if } \alpha = D; \\ \{\bullet, r, d\}, & \textit{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \textit{if } \alpha = C^*; \\ \{\bullet, r\}, & \textit{if } \alpha = D^*. \end{cases}$$

For every painted bipartition τ as in Definition 1.3, we write

$$\imath_\tau := \imath, \ \mathcal{P}_\tau := \mathcal{P}, \ \jmath_\tau := \jmath, \ \mathcal{Q}_\tau := \mathcal{Q}, \ \alpha_\tau := \alpha,$$

and

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

If $\star_\tau \in \{B, C, C^*\}$, we call the first column of the painted Young diagram $(\jmath, \mathcal{Q})$ the leading column of τ; if $\star_\tau \in \{\widetilde{C}, D, D^*\}$, we call the first column of $(\imath, \mathcal{P})$ the leading column of τ. We further define some objects attached to τ in what follows:

$$|\tau|, \ p_\tau, \ q_\tau, \ G_\tau, \ \dim \tau, \ \varepsilon_\tau.$$

$|\tau|$: This is the natural number

$$|\tau| := |\imath| + |\jmath|.$$

p_τ and q_τ: This is a pair of natural numbers given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := \# \bullet + 2\#r + \#c + \#d + \#B^+; \\ q_\tau := \# \bullet + 2\#s + \#c + \#d + \#B^-. \end{cases}$$

G_τ: This is a classical group given by

$$G_\tau := \begin{cases} \mathrm{O}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

$\dim\tau$: This is the dimension of the standard representation of the complexification of G_τ, equvalently,

$$\dim\tau := \begin{cases} 2\,|\tau|+1, & \text{if } \star_\tau = B; \\ 2\,|\tau|\,, & \text{otherwise.} \end{cases}$$

ε_τ: This is the element in $\mathbb{Z}/2\mathbb{Z}$ such that

$$\varepsilon_\tau = 0 \Leftrightarrow \text{the symbol } d \text{ occurs in the leading column of } \tau.$$

Example. Suppose that

$$\tau = \begin{array}{|c|c|} \hline \bullet & c \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} c & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & s & r \\ \hline \bullet & \multicolumn{2}{c}{} \\ \cline{1-1} r & \multicolumn{2}{c}{} \\ \cline{1-1} r & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \times B^+.$$

Then

$$\begin{cases} |\tau| = 10; \\ p_\tau = 4+6+2+0+1 = 13; \\ q_\tau = 4+2+2+0+0 = 8; \\ G_\tau = \mathrm{O}(13,8); \\ \dim\tau = 21; \\ \varepsilon_\tau = 1. \end{cases}$$

1.4. **Painted bipartitions attached to $\check{\mathcal{O}}$.** We suppose that the classical group G has type $\star$. Following [56, Definition 4.1], We say that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity if

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \check{G} \text{ is a complex symplectic group; and} \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \check{G} \text{ is a complex orthogonal group.} \end{cases}$$

The study of the special unipotent representations in general case reduced to the case when $\check{\mathcal{O}}$ has $\star$-good parity. See Appendix **??**. Now we suppose that $\check{\mathcal{O}}$ has $\star$-good parity. Equivalently but in a completely combinatorial setting, we consider $\check{\mathcal{O}}$ as a Young diagram that has $\star$-good parity in the following sense:

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \star \in \{B, \widetilde{C}\}; \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \star \in \{C, D, C^*, D^*\}; \text{ and} \\ \text{the total size } \left|\check{\mathcal{O}}\right| \text{ is odd if and only if } \star \in \{C, C^*\}. \end{cases}$$

Definition 1.4. *A $\star$-pair is a pair $(i, i+1)$ of successive positive integers such that*

$$\begin{cases} i \textit{ is odd}, & \textit{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \textit{ is even}, & \textit{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be

- *vacant in $\check{\mathcal{O}}$, if* $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$*;*
- *balanced in $\check{\mathcal{O}}$, if* $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$*;*
- *tailed in $\check{\mathcal{O}}$, if* $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ *is positive and odd;*
- *primitive in $\check{\mathcal{O}}$, if* $\mathbf{r}_i(\check{\mathcal{O}}) - \mathbf{r}_{i+1}(\check{\mathcal{O}})$ *is positive and even.*

Let $\mathrm{PP}_\star(\check{\mathcal{O}})$ denote the set of all $\star$-pairs that are primitive in $\check{\mathcal{O}}$. We define a pair

$$(\imath_{\check{\mathcal{O}}}, \jmath_{\check{\mathcal{O}}}) := (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}}))$$

of Young diagrams as in what follows.

The case when $\star = B$. In this case,

$$\mathbf{c}_1(\jmath_{\check{\mathcal{O}}}) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}).$$

The case when $\star = \widetilde{C}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_{\check{\mathcal{O}}}), \mathbf{c}_i(\jmath_{\check{\mathcal{O}}})) = (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}).$$

The case when $\star \in \{C, C^*\}$. In this case, for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_i(\imath_{\check{\mathcal{O}}})) = \begin{cases} (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (0,0), & \text{if } (2i-1, 2i) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

The case when $\star \in \{D, D^*\}$. In this case,

$$\mathbf{c}_1(\imath_{\check{\mathcal{O}}}) = \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0; \\ 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0, \end{cases}$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_{\check{\mathcal{O}}}), \mathbf{c}_{i+1}(\imath_{\check{\mathcal{O}}})) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ (0,0), & \text{if } (2i, 2i+1) \text{ is vacant in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

Define

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) := \left\{ \tau \text{ is a painted bipartition} \mid \star_\tau = \star \text{ and } (\imath_\tau, \jmath_\tau) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}})) \right\}.$$

We also define the following extended parameter set:

$$\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) := \begin{cases} \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\wp \subset \mathrm{PP}_\star(\check{\mathcal{O}})\}, & \text{if } \star \in \{B, C, D, \widetilde{C}\}; \\ \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \{\varnothing\}, & \text{if } \star \in \{C^*, D^*\}. \end{cases}$$

We use $\boldsymbol{\tau} = (\tau, \wp)$ to denote an element in $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$.

Put

$$\mathrm{Unip}_\star(\check{\mathcal{O}}) := \begin{cases} \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|+1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = B; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{|\check{\mathcal{O}}|-1}(\mathbb{R})), & \text{if } \star = C; \\ \bigsqcup_{p,q\in\mathbb{N}, p+q=|\check{\mathcal{O}}|} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = D; \\ \mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{Sp}}_{|\check{\mathcal{O}}|}(\mathbb{R})), & \text{if } \star = \widetilde{C}; \\ \bigsqcup_{p,q\in\mathbb{N}, 2p+2q=|\check{\mathcal{O}}|-1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q)), & \text{if } \star = C^*; \\ \mathrm{Unip}_{\check{\mathcal{O}}}\, \mathrm{O}^*(|\check{\mathcal{O}}|), & \text{if } \star = D^*. \end{cases}$$

Now we can state the main result in [13]:

Theorem 1.5. *Suppose $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity and $|\check{\mathcal{O}}| > 0$. Then*

$$|\mathrm{Unip}_\star \check{\mathcal{O}}| = \begin{cases} |\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})| & \textit{when } \star \in \{ C, \widetilde{C}, C^*, D^* \}; \\ 2\,|\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})| & \textit{when } \star \in \{ B, D \}. \end{cases}$$

We will use the set $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$ to count the set $\mathrm{Unip}_\star(\check{\mathcal{O}})$.

1.5. **Descents of painted bipartitions and theta lifts.** Let $\star$ and $\check{\mathcal{O}}$ be as before. For every $\boldsymbol{\tau} = (\tau, \wp) \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, in what follows we will construct a representation $\pi_{\tau,\wp}$ of G_τ by theta lift. For the starting case when $\check{\mathcal{O}}$ is the empty Young diagram, define

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \text{the one dimensional genuine representation,} & \text{if } \star = \widetilde{C} \text{ so that } G_\tau = \widetilde{\mathrm{Sp}}_0(\mathbb{R}); \\ \text{the one dimensional trivial representation,} & \text{if } \star \neq \widetilde{C}. \end{cases}$$

Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or } \ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or } \ D^*.$$

We call $\star'$ the Howe dual of $\star$. Now we assume that $\check{\mathcal{O}}$ is nonempty, and define its dual decent to be the Young diagram

$$\check{\mathcal{O}}' := \check{\nabla}(\check{\mathcal{O}}) := \text{the one obtained from } \check{\mathcal{O}} \text{ by removing the first row.}$$

Note that $\check{\mathcal{O}}'$ has $\star'$-good parity, and

$$\mathrm{PP}_{\star'}(\check{\mathcal{O}}') = \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \mathrm{PP}_\star(\check{\mathcal{O}})\}.$$

Define the dual descent of $\wp$ to be

$$\wp' := \check{\nabla}(\wp) := \{(i, i+1) \mid i \in \mathbb{N}^+,\ (i+1, i+2) \in \wp\} \subset \mathrm{PP}_{\star'}(\check{\mathcal{O}}'). \tag{1.5}$$

In section 2.2, we will define the descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

We the descent map naturally extends to the parameter space

$$\nabla \colon \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}') \qquad (\tau, \wp) \mapsto (\nabla(\tau), \check{\nabla}(\wp))$$

Put $\boldsymbol{\tau}' := (\tau', \wp') := \nabla(\boldsymbol{\tau})$. Let $(W_{\tau,\tau'}, \langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\tau'})$ be a real symplectic space of dimension $\dim \tau \cdot \dim \tau'$. As usual, there are continuous homomorphisms $G_\tau \to \mathrm{Sp}(W_{\tau,\tau'})$ and $G_{\tau'} \to \mathrm{Sp}(W_{\tau,\tau'})$ whose images form a reductive dual pair in $\mathrm{Sp}(W_{\tau,\tau'})$. We form the semidirect prduct

$$J_{\tau,\tau'} := (G_\tau \times G_{\tau'}) \ltimes \mathrm{H}(W_{\tau,\tau'}),$$

where

$$\mathrm{H}(W_{\tau,\tau'}) := W_{\tau,\tau'} \times \mathbb{R}$$

is the Heisenberg group with group multiplication

$$(w,t)(w',t') := (w + w', t + t' + \langle w, w'\rangle_{\tau,\tau'}), \quad w, w' \in W_{\tau,\tau'},\ t, t' \in \mathbb{R}.$$

Let $\omega_{\tau,\tau'}$ be a suitably normalized smooth oscillator representation of $J_{\tau,\tau'}$ such that every $t \in \mathbb{R} \subset J_{\tau,\tau'}$ acts on it through the scalar multiplication by $e^{2\pi\sqrt{-1}\,t}$ (the letter π often denotes a representation, but here it stands for the circumference ratio). See Section **??** for details.

For every Casselman-Wallach representation π' of $G_{\tau'}$, write

$$\check{\Theta}^\tau_{\tau'}(\pi') := (\omega_{\tau,\tau'} \widehat{\otimes} \pi')_{G_{\tau'}} \qquad \text{(the Hausdorff coinvariant space),}$$

where $\widehat{\otimes}$ indicates the complete projective tensor product. This is a Casselman-Wallach representation of G_τ. Now we define the representation $\pi_{\tau,\wp}$ of G_τ by induction on $\mathbf{c}_1(\check{\mathcal{O}})$:

$$\pi_{\boldsymbol{\tau}} := \begin{cases} \check{\Theta}^\tau_{\tau'}(\pi_{\boldsymbol{\tau}'}) \otimes (1^{+,-}_{p_\tau,q_\tau})^{\varepsilon_\tau}, & \text{if } \star_\tau = B \text{ or } D; \\ \check{\Theta}^\tau_{\tau'}(\pi_{\boldsymbol{\tau}'} \otimes \det{}^{\varepsilon_\wp}), & \text{if } \star_\tau = C \text{ or } \widetilde{C}; \\ \check{\Theta}^\tau_{\tau'}(\pi_{\boldsymbol{\tau}'}), & \text{if } \star_\tau = C^* \text{ or } D^*. \end{cases}$$

Here $1^{+,-}_{p_\tau,q_\tau}$ denotes the character of $\mathrm{O}(p_\tau,q_\tau)$ whose restriction to $\mathrm{O}(p_\tau)\times\mathrm{O}(q_\tau)$ equals $1\otimes\det$ (1 stands for the trivial character), and

$$\varepsilon_\wp := \begin{cases} 1 & \text{if } (1,2)\in\wp \\ 0 & \text{if } (1,2)\notin\wp \end{cases}$$

$$\varepsilon_\tau := \begin{cases} 0 & \text{if } (\star, \mathcal{P}_\tau(\mathbf{c}_1(\imath_\tau),1)) = (D,d) \\ 0 & \text{if } (\star, \mathcal{Q}_\tau(\mathbf{c}_1(\jmath_\tau),1)) = (B,d) \\ 1 & \text{otherwise} \end{cases}$$

We are now ready to formulate our first main theorem.

Theorem 1.6. *Let $\star\in\{B,C,D,\widetilde{C},C^*,D^*\}$, and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.*
(a) For every $\boldsymbol{\tau}=(\tau,\wp)\in\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, the representation $\pi_{\boldsymbol{\tau}}$ of G_τ is irreducible and attached to $\check{\mathcal{O}}$.
(b) If $\star\in\{B,D\}$ and $\left|\check{\mathcal{O}}\right|>0$, then the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})\times\mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ (\boldsymbol{\tau},\epsilon) & \mapsto & \pi_{\boldsymbol{\tau}}\otimes\det^\epsilon \end{array}$$

is bijective.
(c) In all other cases, the map

$$\begin{array}{rcl} \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}), \\ \boldsymbol{\tau} & \mapsto & \pi_{\boldsymbol{\tau}} \end{array}$$

is bijective.

By the above theorem (and the reduction in **??**), we have explicitly constructed all special unipotent representations of the classical groups in (1.1). By this construction and the method of matrix coefficient integrals, we are able to prove the unitarity of these representations.

Theorem 1.7. *All special unipotent representations of the classical groups in* (1.1) *are unitarizable.*

1.6. **Associated cycles.** Let $G, G_\mathbb{C}, \mathfrak{g}_\mathbb{R}, \mathfrak{g}, \check{\mathfrak{g}}$ and $\check{\mathcal{O}}\in\mathrm{Nil}(\check{\mathfrak{g}})$ be as in Section 1.2. Fix a maximal compact subgroup K of G whose Lie algebra is denoted by $\mathfrak{k}_\mathbb{R}$. Then we have an orthogonal decomposition

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p},$$

where $\mathfrak{k}$ is the complexification of $\mathfrak{k}_\mathbb{R}$, and $\mathfrak{g}$ is equipped with the trace form. By taking the dual spaces, we have a decomposition

$$\mathfrak{g}^*=\mathfrak{k}^*\oplus\mathfrak{p}^*,$$

Write $K_\mathbb{C}$ for the complexification of the complex group K. It is a complex algebraic group with an obvious algebraic action on $\mathfrak{p}^*$.

As before, suppose that G has type $\star$ and $\check{\mathcal{O}}$ has $\star$-good parity. Write $\mathcal{O}\in\mathrm{Nil}(\mathfrak{g}^*)$ for the Barbarsch-Vogan dual of $\check{\mathcal{O}}$ so that its Zariski closure in $\mathfrak{g}^*$ equals the associated variety of the ideal $I_{\check{\mathcal{O}}}\subset\mathrm{U}(\mathfrak{g})$. The algebraic variety $\mathcal{O}\cap\mathfrak{p}^*$ is a finite union of $K_\mathbb{C}$-orbits. Given such an orbit $\mathscr{O}$, write $\mathcal{K}_{\mathscr{O}}(K_\mathbb{C})$ for the Grothendieck group of the category of $K_\mathbb{C}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{\mathcal{O}}(K_\mathbb{C}) := \bigoplus_{\mathscr{O}\text{ is a }K_\mathbb{C}\text{-orbit in }\mathcal{O}\cap\mathfrak{p}^*}\mathcal{K}_{\mathscr{O}}(K_\mathbb{C}).$$

We say that a Casselman-Wallach representation of G is $\mathcal{O}$-bounded if the associated variety of its annihilator ideal is contained in the Zariski closure of $\mathcal{O}$. Note that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are $\mathcal{O}$-bounded. Write $\mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}}$ for the Grothendieck group of the category of all such representations. From [79, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{AC}_{\mathcal{O}}\colon \mathcal{K}(G)_{\mathcal{O}-\mathrm{bounded}} \longrightarrow \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}}).$$

We call $\mathrm{AC}_{\mathcal{O}}(\pi)$ the associated cycle of π, where π is an $\mathcal{O}$-bounded Casselman-Wallach representation of G. This is a very important invariant attached to π.

Following Vogan [79, Section 8], we make the following definition.

Definition 1.8. *Let $\mathscr{O}$ be a $K_{\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}^*$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $K_{\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_X$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_X$;*
- *if $\star = \widetilde{C}$, then ε_G acts on $\mathcal{E}$ by the scalar multiplication by -1.*

Here $X \in \mathscr{O}$, $\mathcal{E}_X$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_X$ denotes the Lie algebra of the stabilizer of X in $K_{\mathbb{C}}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_X$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$.

Note that in the situation of the classical groups we consider here, all admissible orbit data are line bundles. Denote by $\mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$. Put

$$\mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}}) \subset \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}}).$$

Recall that a nilpotent orbit in $\check{\mathfrak{g}}$ is said to be distinguished if it is has no intersection with every proper Levi subalgebra of $\check{\mathfrak{g}}$. Combinatorially, this is equivalent to saying that no pair of rows of the Young diagram have equal nonzero length. Note that all distinguished nilpotent orbits in $\check{\mathfrak{g}}$ has $\star$-good parity.

Definition 1.9. *(a) The orbit $\check{\mathcal{O}}$ (which has $\star$-good parity) is said to be quasi-distinguished if there is no $\star$-pair that is balanced in $\check{\mathcal{O}}$.*
(b) If $\star \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be be pseudo-distinguished if there is no positive even integer i such that $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = \mathbf{r}_{i+2}(\check{\mathcal{O}}) = \mathbf{r}_{i+3}(\check{\mathcal{O}}) > 0$.
(c) If $\star \in \{C, \widetilde{C}, C^*\}$ so that its Howe dual $\star' \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be be pseudo-distinguished if either it is the empty Young diagram or it is nonempty and its dual descent $\check{\mathcal{O}}'$ (which is a Young diagram that has $\star'$-good parity) is pseudo-distinguished.

We will calculate the associated cycle of π_τ for every painted bipartition τ that has good parity. The calculation is a key ingredient in the proof of Theorem 1.6. Especially, we have the following theorem concerning the associated characters of the special unipotent representations.

Theorem 1.10. *Let G be a group in* (1.1) *that has type $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For every $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the associated cycle $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$ is a nonzero sum of pairwise distinct elements of $\mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}})$.
(b) If $\check{\mathcal{O}}$ is pseudo-distinguished, then the map

$$\mathrm{AC}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$$

is injective.
(c) If $\check{\mathcal{O}}$ is quasi-distinguished, then the map $\mathrm{AC}_{\check{\mathcal{O}}}$ induces a bijection

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}}).$$

Remark. Suppose that $\star \in \{C^*, D^*\}$ so that G is quaternionic. Then there is precisely one admissible orbit datum over $\mathscr{O}$ for each $K_{\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}^*$. Thus

$$\mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}}) = K_{\mathbb{C}} \backslash (\mathcal{O} \cap \mathfrak{p}^*).$$

If $\check{\mathcal{O}}$ is not quasi-distinghuished, then $\mathcal{O} \cap \mathfrak{p}^*$ is empty (see [21, Theorems 9.3.4 and 9.3.5]), and hence $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is also empty.

2. The descents of painted bipartitions

As before, let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$ and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity. Put

$$l := l_{\star,\check{\mathcal{O}}} := \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2}; & \text{if } \star \in \{B, \widetilde{C}\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})-1}{2}, & \text{if } \star \in \{C, C^*\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \star \in \{D, D^*\}. \end{cases} \tag{2.1}$$

This is the length of the leading column of every element of $\mathrm{PBP}_\star(\check{\mathcal{O}})$.

In various context, we use $\varnothing$ to denote the empty set, the empty Young diagram or the painted Young diagram whose underlying Young diagram is empty. For every Young diagram $\imath$, its descent, which is denoted by $\nabla(\jmath)$, is defined to be the Young diagram obtained from $\jmath$ by removing the first column. By convention, $\nabla(\varnothing) = \varnothing$.

In the rest of this section, we assume that $\check{\mathcal{O}} \neq \varnothing$, and write $\check{\mathcal{O}}'$ for its dual descent. Write $\star'$ for the Howe dual of $\star$ so that $\check{\mathcal{O}}'$ has $\star'$-good parity. Put

$$l' := l_{\star',\check{\mathcal{O}}'}$$

2.1. Naive descents of painted bipartitions.

In this subsection, let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ be a painted bipartition such that $\star_\tau = \star$. Write $\star'$ for the Howe dual of $\star$ and put

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } \mathcal{P}_\tau(l_{\star,\check{\mathcal{O}}}, 1), 1) \neq c; \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } \mathcal{P}_\tau(l_{\star,\check{\mathcal{O}}}, 1), 1) = c; \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{2.2}$$

[

$$\alpha' = \begin{cases} B^+, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ does not occur in the leading column of } \tau; \\ B^-, & \text{if } \alpha = \widetilde{C} \text{ and } c \text{ occurs in the leading column of } \tau; \\ \star', & \text{if } \alpha \neq \widetilde{C}. \end{cases} \tag{2.3}$$

]

Lemma 2.1. *If $\star \in \{B, C, C^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\imath, \nabla(\jmath))$;
- *for all $(i, j) \in \mathrm{Box}(\imath')$,*

$$\mathcal{P}'(i, j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i, j) \in \{\bullet, s\}; \\ \mathcal{P}(i, j), & \text{if } \mathcal{P}(i, j) \notin \{\bullet, s\}; \end{cases}$$

- *for all $(i, j) \in \mathrm{Box}(\jmath')$,*

$$\mathcal{Q}'(i, j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}(i, j+1) \in \{\bullet, s\}; \\ \mathcal{Q}(i, j+1), & \text{if } \mathcal{Q}(i, j+1) \notin \{\bullet, s\}. \end{cases}$$

Proof. First assume that the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$. Then the image of $\mathcal{P}$ is in fact contained in $\{\bullet\}$, and $(\imath, \jmath)$ is right interlaced in the sense that

$$\mathbf{c}_1(\jmath) \geqslant \mathbf{c}_1(\imath) \geqslant \mathbf{c}_2(\jmath) \geqslant \mathbf{c}_2(\imath) \geqslant \mathbf{c}_3(\jmath) \geqslant \mathbf{c}_3(\imath) \geqslant \cdots .$$

Hence $(\imath', \jmath') := (\imath, \nabla(\jmath))$ is left interlaced in the sense that

$$\mathbf{c}_1(\imath') \geqslant \mathbf{c}_1(\jmath') \geqslant \mathbf{c}_2(\imath') \geqslant \mathbf{c}_2(\jmath') \geqslant \mathbf{c}_3(\imath') \geqslant \mathbf{c}_3(\jmath') \geqslant \cdots .$$

Then it is clear that there is unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ such that images of $\mathcal{P}'$ and $\mathcal{Q}'$ are both contained in $\{\bullet, s\}$. This proves the lemma in the special case when the images of $\mathcal{P}$ and $\mathcal{Q}$ are both contained in $\{\bullet, s\}$.

The proof of the lemma in the general case is easily reduced to this special case. $\square$

Lemma 2.2. *If $\star \in \{\widetilde{C}, D, D^*\}$, then there is a unique painted bipartition of the form $\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$ with the following properties:*

- $(\imath', \jmath') = (\nabla(\imath), \jmath)$;
- *for all* $(i, j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i, j+1) \in \{\bullet, s\}; \\ \mathcal{P}(i, j+1), & \text{if } \mathcal{P}(i, j+1) \notin \{\bullet, s\}; \end{cases}$$

- *for all* $(i, j) \in \mathrm{Box}(\jmath')$,

$$\mathcal{Q}'(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}(i, j) \in \{\bullet, s\}; \\ \mathcal{Q}(i, j), & \text{if } \mathcal{Q}(i, j) \notin \{\bullet, s\}. \end{cases}$$

Proof. The proof is similar to that of Lemma 2.1.

$\square$

Definition 2.3. *In the notation of Lemma 2.1 and 2.2, we call τ' the naive descent of τ, to be denoted by $\nabla_{\mathrm{naive}}(\tau)$.*

Example. If

$$\tau = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & \bullet & c \\ \hline \bullet & s & c & \\ \cline{1-3} s & & & \\ \cline{1-1} c & & & \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & \bullet & \bullet \\ \hline \bullet & r & d \\ \hline d & d & \\ \cline{1-2} \end{array} \times \widetilde{C},$$

then

$$\nabla_{\mathrm{naive}}(\tau) = \begin{array}{|c|c|c|} \hline \bullet & \bullet & c \\ \hline \bullet & c & \\ \cline{1-2} \end{array} \times \begin{array}{|c|c|c|} \hline \bullet & \bullet & s \\ \hline \bullet & r & d \\ \hline d & d & \\ \cline{1-2} \end{array} \times B^-.$$

2.2. **Descents of painted bipartitions.** Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ and write

$$\tau'_{\mathrm{naive}} = (\imath', \mathcal{P}'_{\mathrm{naive}}) \times (\jmath', \mathcal{Q}'_{\mathrm{naive}}) \times \alpha'$$

for the naive descent of τ. This is clearly an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$.

The following two lemmas are easily verified and we omit the proofs. We will give an example for each of them.

Lemma 2.4. *Suppose that*

$$\begin{cases} \alpha = B^+; \\ \mathbf{r}_2(\check{\mathcal{O}}) > 0; \\ \mathcal{Q}(l, 1) \in \{r, d\}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i,j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} s, & \text{if } (i,j) = (l',1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|}\hline \bullet & r \\ \hline r & d \\ \hline \end{array} \times B^+,$$

then

$$\tau'_{\mathrm{naive}} = \begin{array}{|c|c|}\hline s & c \\ \hline c & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C} \qquad \text{and} \qquad \tau' = \begin{array}{|c|c|}\hline s & c \\ \hline s & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times \begin{array}{|c|}\hline r \\ \hline d \\ \hline \end{array} \times \widetilde{C}.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $4,4,4,2$, and $l' = 2$.

Lemma 2.5. *Suppose that*

$$\begin{cases} \alpha = D; \\ \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0; \\ \mathcal{P}(l'+1,1) = r; \\ \mathcal{P}(l'+1,2) = c; \\ \mathcal{P}(l,1) \in \{r,d\}. \end{cases}$$

Then there is a unique element in $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$ *of the form*

$$\tau' = (\imath', \mathcal{P}') \times (\jmath', \mathcal{Q}') \times \alpha'$$

such that $\mathcal{Q}' = \mathcal{Q}'_{\mathrm{naive}}$ *and for all* $(i,j) \in \mathrm{Box}(\imath')$,

$$\mathcal{P}'(i,j) = \begin{cases} r, & \text{if } (i,j) = (l'+1,1); \\ \mathcal{P}'_{\mathrm{naive}}(i,j), & \text{otherwise.} \end{cases}$$

Example. If

$$\tau = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & s \\ \hline \bullet & s \\ \hline r & c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times D,$$

then

$$\tau'_{\mathrm{naive}} = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline c \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times C, \qquad \text{and} \qquad \tau' = \begin{array}{|c|}\hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline r \\ \hline \end{array} \times \begin{array}{|c|c|}\hline \bullet & s \\ \hline \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \bullet & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \times C.$$

Note that in this case, the nonzero row lengths of $\check{\mathcal{O}}$ are $7,7,7,3$, and $l' = 3$.

Definition 2.6. *We define the descent of* τ *to be*

$$\nabla(\tau) := \begin{cases} \tau', & \text{if the condition of Lemma 2.4 or 2.5 holds;} \\ \nabla_{\mathrm{naive}}(\tau), & \text{otherwise,} \end{cases}$$

which is an element of $\mathrm{PBP}_{\star'}(\check{\mathcal{O}}')$. *Here* τ' *is as in Lemmas 2.4 and 2.5.*

In conclusion, we have defined the descent map

$$\nabla : \mathrm{PBP}_{\star}(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

3. Painted bipartitions

3.1. **Vanishing propotition.** Because of the following proposition, we assume in the rest of this paper that $\check{\mathcal{O}}$ is quasi-distinghuished when $\star \in \{C^*, D^*\}$.

Proposition 3.1. *Suppose that $\star \in \{C^*, D^*\}$. If the set $\mathrm{PBP}_\star(\check{\mathcal{O}})$ is nonempty, then $\check{\mathcal{O}}$ is quasi-distinguished.*

Proof. Suppose that $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. If $\star = C^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_i(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots.$$

This forces that $\check{\mathcal{O}}$ is quasi-distinguished.

If $\star = D^*$, then the definition of painted bipartitions implies that

$$\mathbf{c}_{i+1}(\imath) \leqslant \mathbf{c}_i(\jmath) \qquad \text{for all } i = 1, 2, 3, \cdots.$$

This also forces that $\check{\mathcal{O}}$ is quasi-distinguished. □

3.2. **Tails of painted bipartitions.** In the rest of this subsection, we assume that $\star \in \{B, D, C^*\}$. Let $(\imath, \jmath) = (\imath_\star(\check{\mathcal{O}}), \jmath_\star(\check{\mathcal{O}}))$. Put

$$\star_{\mathbf{t}} := \begin{cases} D, & \text{if } \star \in \{B, D\}; \\ C^*, & \text{if } \star = C^*. \end{cases} \quad \text{and} \quad k := \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} + 1 & \text{if } \star \in \{B, D\}; \\ \frac{\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})}{2} - 1 & \text{if } \star = C^*. \end{cases}$$

Here $k = \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, $\mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath)$, and $\mathbf{c}_1(\imath) - \mathbf{c}_1(\jmath)$ when $\star = B, C^*, D^*$, respectively.

Let $\check{\mathcal{O}}_{\mathbf{t}}$ be the following Young diagram that is determined by the pair $(\star, \check{\mathcal{O}})$.

- If $\star \in \{ B, D \}$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of two rows with lengths $2k - 1$ and 1.
- If $\star = C^*$, then $\check{\mathcal{O}}_{\mathbf{t}}$ consists of one row with length $2k + 1$.

Note that in all these three cases $\check{\mathcal{O}}_{\mathbf{t}}$ has $\star_{\mathbf{t}}$-good parity and every element in $\mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ has the form

$$\begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \varnothing \times D, \qquad \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times \varnothing \times D \qquad \text{or} \qquad \varnothing \times \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \vdots \\ \hline x_k \\ \hline \end{array} \times C^*, \tag{3.1}$$

respectively if $\star = B, D$ or C^*.

Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ be as before.

The case when $\star = B$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the first painted bipartition in (3.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{Q}(j, 1) \mid \mathbf{c}_1(\imath) + 1 \leqslant j \leqslant \mathbf{c}_1(\jmath) \}$$

with the set

$$\begin{cases} \{ c \}, & \text{if } \alpha = B^+, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ s \}, & \text{if } \alpha = B^-, \text{ and either } \mathbf{c}_1(\imath) = 0 \text{ or } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{ \bullet, s \}; \\ \{ \mathcal{Q}(\mathbf{c}_1(\imath), 1) \}, & \text{if } \mathbf{c}_1(\imath) > 0 \text{ and } \mathcal{Q}(\mathbf{c}_1(\imath), 1) \in \{r, d\}. \end{cases}$$

The case when $\star = D$. In this case, we define the tail $\tau_{\mathbf{t}}$ of τ to be the second painted bipartition in (3.1) such that the multiset $\{x_1, x_2, \cdots, x_k\}$ is the union of the multiset

$$\{ \mathcal{P}(j, 1) \mid \mathbf{c}_1(\jmath) + 2 \leqslant j \leqslant \mathbf{c}_1(\imath) \}$$

with the set

$$\begin{cases} \{ c \}, & \begin{array}{l} \text{if } \mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}), \\ \quad (\mathcal{P}(\mathbf{c}_1(\jmath)+1,1), \mathcal{P}(\mathbf{c}_1(\jmath)+1,2)) = (r,c) \\ \quad \text{and } \mathcal{P}(l,1) \in \{ r, d \}; \end{array} \\ \{ \mathcal{P}(\mathbf{c}_1(\jmath)+1,1) \}, & \text{otherwise.} \end{cases}$$

The case $\star = C^*$. When $k = 0$, we define the tail $\tau_{\mathbf{t}}$ of τ to be $\emptyset \times \emptyset \times C^*$. When $k > 0$, we define the tail $\tau_{\mathbf{t}}$ of τ to be the third painted bipartition in (3.1) such that

$$(x_1, x_2, \cdots, x_k) = (\mathcal{Q}(\mathbf{c}_1(\imath)+1,1), \mathcal{Q}(\mathbf{c}_1(\imath)+2,1), \cdots, \mathcal{Q}(\mathbf{c}_1(\jmath),1)).$$

When $\star \in \{ B, D \}$, the symbol in the last box of the tail $\tau_{\mathbf{t}} \in \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}})$ will be impotent for us. We write x_τ for it, namely

$$x_\tau := \mathcal{P}_{\tau_{\mathbf{t}}}(k,1).$$

The following lemma is easy to check.

Lemma 3.2. *If $\star = B$, then*

$$x_\tau = s \Longleftrightarrow \begin{cases} \alpha = B^-; \\ \mathcal{Q}(\mathbf{c}_1(\jmath),1) \in \{\bullet, s\}, \end{cases}$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{Q}(\mathbf{c}_1(\jmath),1) = d.$$

If $\star = D$, then

$$x_\tau = s \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath),1) = s,$$

and

$$x_\tau = d \Longleftrightarrow \mathcal{P}(\mathbf{c}_1(\imath),1) = d.$$

□

3.3. **Some properties of the descent maps.** The key properties of the descent map when $\star \in \{ C, \widetilde{C}, D^* \}$ are summarized in the following proposition.

Proposition 3.3. *Suppose that $\star \in \{ C, \widetilde{C}, D^* \}$ and cosider the descent map*

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}'). \tag{3.2}$$

(a) If $\star = D^*$ or $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$, then the map (3.2) is bijective.

(b) If $\star \in \{C, \widetilde{C}\}$ and $\mathbf{r}_1(\check{\mathcal{O}}) = \mathbf{r}_2(\check{\mathcal{O}})$, then the map (3.2) is injective and its image equals

$$\left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, x_{\tau'} \neq s \right\}.$$

Proof. We give the detailed proof of part (b) when $\star = \widetilde{C}$. The proofs in the other cases are similar and are left to the reader.

By the definition of descent map (see (2.3)), we have a well defined map

$$\nabla : \left\{ \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \,\middle|\, \mathcal{P}_\tau(\mathbf{c}_1(\imath),1) \neq c \right\} \to \left\{ \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \,\middle|\, \alpha_{\tau'} = B^+ \right\}. \tag{3.3}$$

Suppose that τ' is an element in the codomain of the map (3.3). Similar to the proof of Lemma 2.1, there is a unique element in $\tau := \nabla^{-1}(\tau') \in \mathrm{PBP}_\star(\check{\mathcal{O}})$ such that for all $i = 1, 2, \cdots, \mathbf{c}_1(\imath)$,

$$\mathcal{P}_\tau(i,1) \in \{\bullet, s\},$$

and for all $(i,j) \in \mathrm{Box}(\tau')$,

$$\mathcal{P}_\tau(i,j+1) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{P}_{\tau'}(i,j) \in \{\bullet, s\}; \\ \mathcal{P}_{\tau'}(i,j), & \text{if } \mathcal{P}_{\tau'}(i,j) \notin \{\bullet, s\}, \end{cases}$$

and

$$\mathcal{Q}_\tau(i,j) = \begin{cases} \bullet \text{ or } s, & \text{if } \mathcal{Q}_{\tau'}(i,j) \in \{\bullet, s\}; \\ \mathcal{Q}_{\tau'}(i,j), & \text{if } \mathcal{Q}_{\tau'}(i,j) \notin \{\bullet, s\}. \end{cases}$$

Note that τ is in the domain of (3.3). It is then routine to check that the map

$$\nabla^{-1} : \{\, \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \mid \alpha_{\tau'} = B^+ \,\} \to \{\, \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \mid \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq c \,\}$$

and the map (3.3) are inverse to each other. Hence the map (3.3) is bijective.

Similarly, the map

$$\nabla : \{\, \tau \in \mathrm{PBP}_\star(\check{\mathcal{O}}) \mid \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) = c \,\} \to \{\, \tau' \in \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \mid \alpha_{\tau'} = B^-, \mathcal{Q}_{\tau'}(\mathbf{c}_1(\imath), 1) \in \{r, d\} \,\}$$

is well-defined, and we show that it is bijective by explicitly constructing its inverse. In view of Lemma 3.2, this proves the proposition in the case we are considering.

[Suppose $\star = \widetilde{C}$ and $\mathbf{r}_1(\check{\mathcal{O}}) > \mathbf{r}_2(\check{\mathcal{O}})$. Then $\mathbf{c}_1(\imath) > \mathbf{c}_1(\jmath)$ and the "inverse" of descent is constructed in an obvious way such that $\mathcal{P}(\mathbf{c}_1(\imath), 1) = c \Leftrightarrow \alpha = B^-$.

Suppose $\star = C$ and $\mathbf{r}_1(\check{\mathcal{O}}) - \mathbf{r}_2(\check{\mathcal{O}})$. Then $\mathbf{c}_1(\imath) = \mathbf{c}_1(\jmath) + 1$. So $\mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq s$. Hence $\mathcal{P}_{\tau'}(\mathbf{c}_1(\imath), 1) = \mathcal{P}_\tau(\mathbf{c}_1(\imath), 1) \neq s$. The "inverse" of descent is constructed in an obvious way.

Suppose $\star = D^*$. Then $\mathbf{c}_1(\imath) \geqslant \mathbf{c}_1(\jmath) + 1$. The "inverse" of descent is constructed in an obvious way.] □

The key properties of the descent map when $\star \in \{\, D, B, C^* \,\}$ are summarized in the following two propositions.

The following proposition is easy to verify:

Proposition 3.4. *(a) Suppose $k \in \mathbb{N}^+$, $\star = D$ and $\check{\mathcal{O}}$ consists of two rows with lengths $2k - 1$ and 1. The following map is bijective:*

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \left\{ (p, q, \varepsilon) \;\middle|\; \begin{array}{l} p, q \in \mathbb{N} \text{ such that } p + q = 2k, \\ \varepsilon = 1 \text{ if } pq = 0, \text{ and} \\ \varepsilon \in \{\, 0, 1 \,\} \text{ otherwise.} \end{array} \right\} \qquad \tau \mapsto (p_\tau, q_\tau, \varepsilon_\tau).$$

(b) Suppose $k \in \mathbb{N}$, $\star = C^*$ and $\check{\mathcal{O}}$ consists of one row with length $2k + 1$. The following map is bijective:

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \{\, (2p, 2q) \mid p, q \in \mathbb{N} \text{ such that } p + q = k \,\} \qquad \tau \mapsto (p_\tau, q_\tau).$$

□

For every painted bipartition τ, write

$$\mathrm{Sign}(\tau) := (p_\tau, q_\tau).$$

When $\mathbf{r}_2(\check{\mathcal{O}}) > 0$, the double descent $\nabla^2(\tau) := \nabla(\nabla(\tau))$ is well-defined whenever $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. As in the Introduction, $\mathcal{O}$ denotes the Barbasch-Vogan dual of $\check{\mathcal{O}}$. We also consider it as a Young diagram.

Proposition 3.5. *Assume that $\star \in \{\, D, B, C^* \,\}$ and $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Write $\check{\mathcal{O}}'' := \check{\nabla}(\check{\mathcal{O}}')$ and consider the map*

$$\delta \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}), \qquad \tau \mapsto (\nabla^2(\tau), \tau_{\mathbf{t}}). \tag{3.4}$$

(a) Suppose that $\star = C^*$ or $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then the map (3.4) is a bijective, and for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}), \mathbf{c}_2(\mathcal{O})) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{3.5}$$

(b) *Suppose that* $\star \in \{B, D\}$ *and* $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}})$. *Then the map* (3.4) *is injection and its image equals*

$$\left\{ (\tau'', \tau_0) \in \mathrm{PBP}_\star(\check{\mathcal{O}}'') \times \mathrm{PBP}_D(\check{\mathcal{O}}_{\mathbf{t}}) \;\middle|\; \begin{array}{l} x_{\tau''} = d \text{ or} \\ x_{\tau''} \in \{r, c\} \text{ and } \mathcal{P}_{\tau_0}^{-1}(\{s, c\}) \neq \varnothing \end{array} \right\}. \tag{3.6}$$

Moreover, for every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$,

$$\mathrm{Sign}(\tau) = (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\nabla^2(\tau)) + \mathrm{Sign}(\tau_{\mathbf{t}}). \tag{3.7}$$

Proof. We give the detailed proof of part (b) when $\star = B$. The proofs in the other cases are similar and are left to the reader. Let $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$ and $\tau'' = \nabla^2(\tau)$.

According to the descent algorithm, we have

$$\begin{aligned} \mathcal{P}(i,j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{P}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 1; \end{cases} \\ \mathcal{Q}(i,j) &= \begin{cases} \bullet & \text{if } i < \mathbf{c}_1(\imath), j = 1; \\ \mathcal{Q}_{\tau''}(i, j-1) & \text{if } i < \mathbf{c}_1(\imath) \text{ or } j > 2; \end{cases} \end{aligned} \tag{3.8}$$

We label the boxes avoided in (3.8) as the following:

$$\tau: \ \begin{array}{|c|}\hline x_0 \\ \hline \end{array} \times \begin{array}{|c|c|}\hline x_1 & x'' \\ \hline x_2 & \\ \cline{1-1} \vdots & \\ \vdots & \\ \cline{1-1} x_k & \\ \cline{1-1} \end{array} \times \alpha \quad \mapsto \quad \tau'': \ \varnothing \times \begin{array}{|c|}\hline y'' \\ \hline \end{array} \times \alpha'' \tag{3.9}$$

where $k := \mathbf{c}_1(\jmath) - \mathbf{c}_1(\imath) + 1$, x_0, x_1 and y'' has coordinate $(\mathbf{c}_1(\imath), 1)$ in the corresponding painted Young diagram. By the descent algorithm, $x'' = y''$. Also note that $\mathbf{c}_2(\mathcal{O}) = 2\mathbf{c}_1(\imath)$.

We now discuss case by case according to the paint of x_1. When $x_1 = s$, we have $(x_0, \alpha'') = (c, B^-)$. Now

$$\begin{aligned} &\mathrm{Sign}(\tau) - \mathrm{Sign}(\tau'') \\ &= (2\mathbf{c}_1(\imath) - 2, 2\mathbf{c}_1(\imath) - 2) + (1,1) + (0,2) - (0,1) + \mathrm{Sign}(\alpha) + \mathrm{Sign}(x_2 \cdots x_k) \\ &= (\mathbf{c}_2(\mathcal{O}) - 1, \mathbf{c}_2(\mathcal{O}) - 1) + \mathrm{Sign}(\tau_{\mathbf{t}}). \end{aligned} \tag{3.10}$$

In second line of the above formula, the terms count the signatures of extra bullets, x_0, x_1, α', α and $x_1 \cdots x_k$ respectively.

When $x_1 = \bullet$, we have $(x_0, \alpha'') = (\bullet, B^+)$. When $x_1 \in \{r, d\}$, we have $x'' = y'' = d$ $(x_0, \alpha'') = (c, \alpha)$. In these two cases, the signature formula can be checked similarly.It also clear that the image of δ is in (3.6).

It is not hard to construct the map from (3.6) to $\mathrm{PBP}_\star(\check{\mathcal{O}})$, and then the proposition follows. To illustrate the idea, we give the detail construction of $\delta^{-1}(\tau'', \tau_0)$ when $\mathcal{P}_{\tau_0}(k, 1) = c$: most of the entries in $\mathcal{P}$ and $\mathcal{Q}$ are defined by (3.8); we refer to (3.9) for the labeling of the rest of the entries, and set $\alpha := B^-$, $x'' := \mathcal{Q}_{\tau''}(\mathbf{c}_1(\imath), 1)$,

$$(x_0, x_1) := \begin{cases} (\bullet, \bullet) & \text{if } \alpha'' = B^+, \\ (c, s) & \text{if } \alpha'' = B^-, \end{cases} \quad \text{and} \quad x_i := \mathcal{P}_{\tau_0}(i-1, 1) \quad \text{for } i = 2, 3, \cdots, k.$$

[In fact, it suffice to check the proposition for the orbit $\check{\mathcal{O}}$ consists of three rows with lengths $2k$, 2, and 2.]

[Suppose $\star = C^*$. Then τ is obtained by attaching a column of $\mathbf{c}_1(\imath)$ bullets on the left of $\mathcal{P}$, and a column of $\mathbf{c}_1(\imath)$ bullets concatenated with $\tau_{\mathbf{t}}$ on the left of $\mathcal{Q}$. The bijectivity is clear. The signature formula follows from $2\mathbf{c}_1(\imath) = \mathbf{c}_2(\mathcal{O})$.

Suppose $\star = D$ and $\mathbf{r}_2(\check{\mathcal{O}}) > \mathbf{r}_3(\check{\mathcal{O}})$. Then τ is obtained by attaching a column of $\mathbf{c}_1(\jmath)$ bullets concatenated with $\tau_{\mathbf{t}}$ on the left of $\mathcal{P}$, and a column of $\mathbf{c}_1(\jmath)$ bullets on the left of $\mathcal{Q}$. The bijectivity is clear. The signature formula follows from $2\mathbf{c}_1(\jmath) = \mathbf{c}_2(\mathcal{O})$.
] □

Proposition 3.6. *Suppose that $\star \in \{ D, B, C^* \}$ and $\mathbf{r}_2(\check{\mathcal{O}}) > 0$. Then the map*

$$\delta' \colon \mathrm{PBP}_\star(\check{\mathcal{O}}) \longrightarrow \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathrm{PBP}_{\star_{\mathbf{t}}}(\check{\mathcal{O}}_{\mathbf{t}}) \qquad \tau \mapsto (\nabla(\tau), \tau_{\mathbf{t}}) \tag{3.11}$$

is injective. Moreover, (3.11) *is bijective unless $\star \in \{ B, D \}$ and $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$.*

Proof. It follows from the injectivity results in Proposition 3.5 and Proposition 3.3. □

Combining the above propotition with Proposition 3.4, we get the following.

Corollary 3.7. *If $\star \in \{ B, D, C^* \}$, then the map*

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \to & \mathrm{PBP}_{\star'}(\check{\mathcal{O}}') \times \mathbb{N} \times \mathbb{N} \times \mathbb{Z}/2\mathbb{Z}, \\ \tau & \mapsto & (\nabla(\tau), p_\tau, q_\tau, \varepsilon_\tau) \end{array} \tag{3.12}$$

is injective. □

4. Proof of theorem on counting

We do induction on the number of rows of the nilpotent orbits $\check{\mathcal{O}}$.

4.1. Properties of $\mathrm{AC}(\tau)$.

Proposition 4.1. *(a) For each $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}')$, $\mathrm{AC}(\tau') \neq 0$.*

In this section, we let $\star \in \{ B, D, C^* \}$. Now $\star' = \widetilde{C}, C, D^*$ respectively.
We assume all the claim had been proven when $\mathcal{O}$ has at most two columns, that is

- When $\star = D$, $\mathbf{r}_3(\check{\mathcal{O}}) = 0$;
- When $\star = B$, $\mathbf{r}_2(\check{\mathcal{O}}) = 0$;
- When $\star = C^*$, $\mathbf{r}_2(\check{\mathcal{O}}) = 0$.

Proposition 4.2. *We have:*

i) For each $\tau' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$, $\mathrm{AC}(\tau') \neq 0$.
ii) For each $\tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$, $\mathrm{AC}(\tau) \neq 0$.
iii) Let τ'_1, τ'_2 be two distinct elements in $\mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$. Then either
- *$\mathrm{AC}(\tau'_1) \neq \mathrm{AC}(\tau'_2)$; or*
- *$(\tau''_1, \varepsilon'_1) \neq (\tau''_2, \varepsilon'_2)$ and $\mathrm{Sign}(\tau''_1) = \mathrm{Sign}(\tau''_2)$ where $(\tau''_i, \varepsilon'_i) := \nabla(\tau'_i)$ for $i = 1, 2$.*

iv) Let τ_1, τ_2 be two distinct elements in $\mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}})$. Then either
- *$\mathrm{AC}(\tau_1) \neq \mathrm{AC}(\tau_2)$; or*
- *$\tau'_1 \neq \tau'_2$ and $\varepsilon_1 = \varepsilon_2$ where $(\tau'_i, \varepsilon_i) := \nabla(\tau_i)$ for $i = 1, 2$.*

v) When $\star \in \{ B, D \}$ The local system $\mathrm{AC}(\tau)$ is disjoint with their determinant twist:

$$\{ \mathrm{AC}(\tau) \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \} \cap \{ \mathrm{AC}(\tau) \otimes \det \mid \tau \in \mathrm{PBP}^{\mathrm{ext}}_\star(\check{\mathcal{O}}) \} = \varnothing. \tag{4.1}$$

vi) We have
(i) If $x_\tau = s$, then $\Lambda_+(\tau) = \Lambda_-(\tau) = 0$.
(ii) If $x_\tau \in \{ r, c \}$, then $\Lambda_+(\tau) \neq 0$ and $\Lambda_-(\tau) = 0$.
(iii) If $x_\tau = d$, then $\Lambda_+(\tau) \neq 0$ and $\Lambda_-(\tau) \neq 0$.

vii) When $\check{\mathcal{O}}'$ is weakly-distinguished, the map $\mathrm{AC} \colon \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\mathcal{O}') \longrightarrow K_{\star'}(\mathcal{O}')$ is injective.

viii) *When $\check{\mathcal{O}}$ is weakly-distinguished, the map* $\mathrm{AC}\colon \mathrm{PBP}^{\mathrm{ext}}_{\star}(\mathcal{O}) \longrightarrow K_{\star}(\mathcal{O})$ *is a injective.*

ix) *When $\check{\mathcal{O}}$ is +weakly-distinguished, the maps*

$$\begin{aligned} \Upsilon^{+}_{\mathcal{O}}\colon \{\, \mathrm{AC}(\tau) \mid \Lambda_{+}(\tau) \neq 0 \,\} &\longrightarrow \{\, \Lambda_{+}(\tau) \,\} \\ \mathrm{AC}(\tau) &\longmapsto \Lambda_{+}(\tau) \qquad \textit{and} \\ \Upsilon^{-}_{\mathcal{O}}\colon \{\, \mathrm{AC}(\tau) \mid \Lambda_{-}(\tau) \neq 0 \,\} &\longrightarrow \{\, \Lambda_{-}(\tau) \,\} \\ \mathrm{AC}(\tau) &\longmapsto \Lambda_{-}(\tau) \end{aligned}$$

are injective.

x) *When $\mathcal{O}$ is noticed, the map*

$$\begin{aligned} \Upsilon_{\mathcal{O}}\colon \{\, \mathrm{AC}(\tau) \mid \Lambda_{+}(\tau) \neq 0 \,\} &\longrightarrow \{\, (\Lambda_{+}(\tau), \Lambda_{-}(\tau)) \mid \Lambda_{+}(\tau) \neq 0 \,\} \\ \mathrm{AC}(\tau) &\longmapsto (\Lambda_{+}(\tau), \Lambda_{-}(\tau)) \end{aligned} \tag{4.2}$$

is injective.[1]

4.2. **The initial case:** When $k = 0$, $\mathcal{O} = (C_1 = 2c_1, C_0 = 2c_0)$ has at most two columns. Now $\pi_{\tau'}$ is the trivial representation of $\mathrm{Sp}(2c_0, \mathbb{R})$. The lift $\pi_{\tau'} \mapsto \bar{\Theta}(\pi_{\tau'})$ is in fact a stable range theta lift.For $\tau \in \mathrm{DRC}(\mathcal{O})$, let $(p_1, q_1) = \mathrm{Sign}(\mathbf{x}_\tau)$ and x_τ be the foot of τ. It is easy to see that (using associated character formula)

$$\mathrm{AC}(\pi_\tau) = \mathcal{T}_\tau \cdot \mathcal{P}_\tau, \text{ where } \mathcal{T}_\tau := \dagger\dagger_{c_0, c_0}, \text{ and } \mathcal{P}_\tau := \begin{cases} \ddagger_{p_1, q_1} & x_\tau \neq d, \\ \dagger_{p_1, q_1} & x_\tau = d. \end{cases}$$

Note that $\mathrm{AC}(\pi_\tau)$ is irreducible.

Let $\mathcal{O}_1 = (2(c_1 - c_0)) \in \mathrm{Nil}^{\mathrm{dpe}}(D)$. Using the above formula, one can check that the following maps are bijections

$$\begin{array}{ccccc} \mathrm{DRC}(\mathcal{O}_1) & \longleftarrow & \mathrm{DRC}(\mathcal{O}) & \longrightarrow & {}^{\ell}\mathrm{LS}(\mathcal{O}) \\ \mathbf{x}_\tau & \longleftarrow\!\shortmid & \tau & \longmapsto & \mathrm{AC}(\tau). \end{array}$$

To see that $\mathrm{DRC}(\mathcal{O}) \ni \tau \mapsto \mathrm{AC}(\tau)$ is an injection, one check the following lemma holds:

Lemma 4.3. *The map* $\mathrm{DRC}(\mathcal{O}_1) \longrightarrow \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$ *given by* $\tau \mapsto (\mathrm{Sign}(\tau), \epsilon_\tau)$ *is injective (see* **??** *for the definition of ϵ_τ). Moreover, $\epsilon_\tau = 0$ only if* $\mathrm{Sign}(\tau) \geqslant (0, 1)$.

Moreover, $\mathrm{AC}(\tau) \otimes \det \notin {}^{\ell}\mathrm{LS}(\mathcal{O})$ for any $\tau \in \mathrm{DRC}(\mathcal{O})$. In fact, if $\mathrm{Sign}(\mathbf{x}_\tau) \geq (1, 0)$, $\mathrm{AC}(\tau)$ on the 1-rows with +-signs is trivial and $\mathrm{AC}(\tau) \otimes \det$ has non-trivial restriction. When $\mathrm{Sign}(\mathbf{x}_\tau) \ngeq (1, 0)$, $\mathbf{x}_\tau = s \cdots s$, all 1-rows of $\mathrm{AC}(\tau)$ are marked by "=" and all 1-rows of $\mathrm{AC}(\tau) \otimes \det$ are marked by "$-$".

Therefore our main **??** and main proposition **??** holds for $\mathcal{O}$.

Let $C_i = \mathbf{c}_i(\check{\mathcal{O}})$ for $i = 1, 2, \cdots$.

4.3. **The descent case.** Now we assume $(2, 3) \in \mathrm{PP}_{\star}(\check{\mathcal{O}})$.

4.3.1. We first calculate the local system attached to $\check{\mathcal{O}}'$.

Let $\tau' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\check{\mathcal{O}}')$.

For $\mathcal{L}'' \in \mathrm{AOD}(\mathcal{O}'')$, let $(p_1, q_1) = {}^{l}\mathrm{Sign}(\mathcal{L}'')$ and $(p_2, q_2) = (C_2 - q_1, C_2 - p_1)$. Then

$$\vartheta(\mathcal{L}'') = (\maltese^{y} \dagger \mathcal{L}'') \cdot \dagger_{(n_0, n_0)}$$

where y is an integer determined by $\mathrm{Sign}(\tau'')$ and $n_0 = (C_2 - C_3)/2$.

---

[1] In fact, we only need to know whether $\Lambda_{-}(\tau)$ is non-zero or not.

Therefore, $\mathrm{LS}(\mathcal{O}'') \longrightarrow \mathrm{LS}(\mathcal{O}')$ given by $\mathcal{L} \mapsto \vartheta(\mathcal{L})$ is an injection between abelian groups.

By (4.1), we have a injection

$$\begin{aligned} {}^{\ell}\mathrm{LS}(\mathcal{O}'') \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow {}^{\ell}\mathrm{LS}(\mathcal{O}') \\ (\mathcal{L}'', \epsilon') &\longmapsto \vartheta(\mathcal{L}'' \otimes \det{}^{\epsilon'}). \end{aligned} \tag{4.3}$$

In particular, we have $\mathrm{AC}(\tau') \neq 0$ for every $\tau' \in \mathrm{DRC}(\mathcal{O}')$.

Now suppose $\tau'_1 \neq \tau'_2$. Suppose $\mathrm{AC}(\tau'_1) = \mathrm{AC}(\tau'_2)$. By (4.3),

$$\varepsilon_{\tau'_1} = \varepsilon_{\tau'_2} \tag{4.4}$$

and $\mathrm{AC}(\tau''_1) = \mathrm{AC}(\tau''_2)$. In particular, we have $\mathrm{Sign}(\tau''_1) = \mathrm{Sign}(\mathrm{AC}(\tau''_1)) = \mathrm{Sign}(\mathrm{AC}(\tau''_2)) = \mathrm{Sign}(\tau''_1)$. We now claim that $\tau''_1 \neq \tau''_2$. It suffice to consider the case where $\wp_{\tau''_1} = \wp_{\tau''_2}$. Then $\wp_{\tau'_1} = \wp_{\tau'_2}$ by (4.4) and so $\tau'_1 \neq \tau'_2$. By Proposition 3.3, $\tau''_1 \neq \tau''_2$ and this proves the claim.

[Now suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$ and $\mathrm{AC}(\tau'_1) = \mathrm{AC}(\tau'_2)$. By (4.3), $\epsilon'_1 = \epsilon'_2$ and $\mathrm{AC}(\tau''_1) = \mathrm{AC}(\tau''_2)$. In particular, τ''_1 and τ''_2 have the same signature. By ?? and the induction hypothesis, $\tau''_1 \neq \tau''_2$ and so $\pi_{\tau''_1} \otimes \det^{\epsilon'_1} \neq \pi_{\tau''_2} \otimes \det^{\epsilon'_2}$. Now the injectivity of theta lifting yields

$$\pi_{\tau'_2} = \bar{\Theta}(\pi_{\tau''_1} \otimes \det{}^{\epsilon'_1}) \neq \bar{\Theta}(\pi_{\tau''_2} \otimes \det{}^{\epsilon'_2}) = \pi_{\tau'_2}$$

Suppose $\mathcal{O}'$ is noticed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is noticed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is an injection into $\mathrm{LS}(\mathcal{O}'')$. Now (4.3) implies $\tau' \mapsto \mathrm{AC}(\tau')$ is also an injection.

Suppose $\mathcal{O}'$ is quasi-distingushed. Then $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ is quasi-distingushed by definition and $(\tau'', \epsilon') \mapsto \mathrm{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$ is a bijection with $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}'')$. Since the component group of each K-nilpotent orbit in $\mathcal{O}'$ is naturally isomorphic to the component group of its descent. So we deduce that $\tau' \mapsto \mathrm{AC}(\tau')$ is a bijection onto $\mathrm{LS}^{\mathrm{aod}}(\mathcal{O}')$.

This proves the main proposition for $\mathcal{O}'$.]

4.3.2. Now we consider the local system attached to $\check{\mathcal{O}}$. Let $\tau \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$. By the signature formula (3.5)

$$\begin{aligned} \mathrm{AC}(\tau) &= \left(\left(\maltese^{y'}(\mathrm{AC}(\tau'') \otimes (\det)^{\varepsilon_{\wp'}}) \cdot 1^{(n_0,n_0)}\right) \cdot 1^{\mathrm{Sign}(\tau_{\mathbf{t}})}\right) \otimes (1^{+,-})^{\varepsilon_\tau} \\ &= \left(\maltese^{y'}(\mathrm{AC}(\tau'') \otimes (\det)^{\varepsilon_{\wp'}}) \otimes (1^{+,-})^{\varepsilon_\tau}\right) \cdot \mathrm{AC}(\tau_{\mathbf{t}}) \\ &= \left(\maltese^{y}\mathrm{AC}(\tau') \otimes (1^{+,-})^{\varepsilon_\tau}\right) \cdot \mathrm{AC}(\tau_{\mathbf{t}}) \end{aligned} \tag{4.5}$$

where $y' = (p_\tau - q_\tau - p_{\tau''} + q_{\tau''})/2$ and y is determined by $\mathrm{Sign}(\tau)$ when $\star \in \{B, D\}$. Whne $\star \in \{C^*\}$, $y' = y'' = 0$.

Now suppose $\tau_1 \neq \tau_2 \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\check{\mathcal{O}})$ and $\mathrm{AC}(\tau_1) = \mathrm{AC}(\tau_2)$. By (4.5),

By (4.5), we have $\mathrm{AC}(\tau_{1\mathbf{t}}) = \mathrm{AC}(\tau_{2\mathbf{t}})$ and . Therefore $\tau_{1\mathbf{t}} = \tau_{2\mathbf{t}}$, $\varepsilon_{\tau_1} = \varepsilon_{\tau_2}$ and $\mathrm{AC}(\tau'_1) = \mathrm{AC}(\tau'_2)$. Thanks to Proposition 3.6, $\tau'_1 \neq \tau'_2$.

The claims in item vi) follows from (4.5) and the initial case for $\tau_{\mathbf{t}}$.

4.3.3. Note that $\mathrm{AC}(\tau)$ is obtained from $\mathrm{AC}(\tau')$ by attaching the peduncle determined by $\mathbf{x}_\tau$. The claims about noticed and quasi-distinguished orbit are easy to verify. We leave them to the reader.

4.4. **The general descent case.** We assume $\mathbf{r}_2(\check{\mathcal{O}}) = \mathbf{r}_3(\check{\mathcal{O}}) > 0$. Note that in this case, $\star \in \{B, D\}$.

4.4.1. We now describe the local systems $\mathrm{AC}(\tau')$.

$$\mathrm{AC}(\tau') = \vartheta^{\mathsf{s}}_{\mathsf{s}'}(\mathrm{AC}(\tau'')) = \maltese^{y'}(\Lambda_+(\mathrm{AC}(\tau'')) + \Lambda_-(\mathrm{AC}(\tau''))) \cdot 1^{(0,0)} \neq 0 \tag{4.6}$$

Here y' is dentermined by $\mathrm{Sign}(\tau'')$.

According to Proposition 3.3, $x_{\tau''} \neq s$. By induction hypothesis, $\Lambda_+(\mathrm{AC}(\tau'')) \neq 0$. Hence $\mathrm{AC}(\tau') \neq 0$.

Let $(p_1'', q_1'') := {}^l\mathrm{Sign}(\mathrm{AC}(\tau''))$. Then

$$ {}^l\mathrm{Sign}(\Lambda_+(\tau'')) = (p_0 - 1, q_0), \text{ and } {}^l\mathrm{Sign}(\Lambda_-(\tau'')) = (p_0, q_0 - 1) \text{ if } \Lambda_-(\tau'') \neq \varnothing. \tag{4.7}$$

$$\mathrm{AC}(\tau') = \vartheta(\mathrm{AC}(\tau'')) = \maltese^{\frac{|\mathrm{Sign}(\tau'')|}{2}}(\dagger\Lambda_+(\tau'') + \dagger\Lambda_-(\tau'')) \neq 0 \tag{4.8}$$

where ${}^l\mathrm{Sign}(\dagger\Lambda_+(\tau'')) = (q_1'', p_1'' - 1)$ and ${}^l\mathrm{Sign}(\dagger\Lambda_-(\tau'')) = (q_1'' - 1, p_1'')$ (if $\dagger\Lambda_-(\tau'') \neq 0$).

Let $(p_1, q_1) := {}^l\mathrm{Sign}(\mathrm{AC}(\tau))$ and $(e, f) := (p_1 - p_1'' + 1, q_1 - q_1'' + 1)$. Using **??** one can show that

$$(e, f) = \mathrm{Sign}(\tau_{\mathbf{t}}).$$

[It suffice to consider the most left three columns of the peduncle part: this part has signature ${}^l\mathrm{Sign}(\mathcal{T}_\tau) + (1,1) = \mathrm{Sign}(\tau_{\mathbf{t}} x_{\tau''}) = \mathrm{Sign}(\tau_{\mathbf{t}}) + {}^l\mathrm{Sign}(\mathcal{D}_{\tau''})$. Therfore,

$$\mathrm{Sign}(\tau_{\mathbf{t}}) = {}^l\mathrm{Sign}(\mathcal{T}_\tau) - {}^l\mathrm{Sign}(\mathcal{D}_{\tau''}) + (1,1) = {}^l\mathrm{Sign}(\mathrm{AC}(\tau)) - {}^l\mathrm{Sign}(\mathrm{AC}(\tau'')) + (1,1).$$

]

Now

(4.9)

$$\mathrm{AC}(\tau) = \begin{cases} (\mathbf{1}^{+,-} \otimes \maltese^t(\Lambda_+\mathrm{AC}(\tau'') \cdot 1^{(0,0)}) \cdot \mathcal{P}^+_\tau + (\mathbf{1}^{+,-} \otimes (\Lambda_-\mathrm{AC}(\tau'') \cdot 1^{(0,0)}) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau \neq d \\ (\dagger\maltese^t\dagger\Lambda_+(\tau'')) \cdot \mathcal{P}^+_\tau + (\dagger\maltese^t\dagger\Lambda_-(\tau'')) \cdot \mathcal{P}^-_\tau & \text{if } x_\tau = d \end{cases}$$

where

$$t = \frac{|\mathrm{Sign}(\tau) - \mathrm{Sign}(\tau'')|}{2} = \frac{|\mathrm{Sign}(\tau_{\mathbf{t}})|}{2}$$

$$\mathcal{P}^+_\tau = \begin{cases} 1^{(e,(-1)^{\varepsilon_\tau}(f-1))} & \text{when } f \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e,f-1} & \text{when } f \geqslant 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{P}^-_\tau = \begin{cases} \ddagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau \neq d \\ \dagger_{e-1,f} & \text{when } e \geqslant 1 \text{ and } x_\tau = d\,. \\ 0 & \text{otherwise} \end{cases}$$

Lemma 4.4. *The local system* $\mathrm{AC}(\tau) \neq 0$.

Proof. Since $x_{\tau''} \neq s$, $\mathrm{AC}(\tau'') \neq 0$. When $\mathrm{Sign}(\tau_{\mathbf{t}}) \succeq (0,1)$, the first term in (4.9) dose not vanish. Now suppose $\mathrm{Sign}(\tau_{\mathbf{t}}) \nsucc (0,1)$. By Proposition 3.5, $x_{\tau''} = d$, and so $\Lambda_-\mathrm{AC}(\tau'') \neq 0$ by induction hypothesis. Since $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (2,0)$, the second term in (4.9) dose not vanish. Hence $\mathrm{AC}(\tau) \neq 0$ in all cases. □

Lemma 4.5. *The local system* $\mathrm{AC}(\tau)$ *satisfies the following properties:*

(i) When $x_\tau = s$, *then* $\Lambda_+(\mathrm{AC}(\tau)) = \Lambda_-(\mathrm{AC}(\tau)) = 0$.
(ii) When $x_\tau = r/c$, *then* $\Lambda_+(\mathrm{AC}(\tau)) \neq 0$ *and* $\Lambda_-(\mathrm{AC}(\tau)) = 0$.
(iii) When $x_\tau = d$, *then* $\Lambda_+(\mathrm{AC}(\tau)) \neq 0$ *and* $\Lambda_-(\mathrm{AC}(\tau)) \neq 0$.

Proof. If $x_\tau = s/r/c$, $\mathrm{AC}(\tau) \not\ni 1^{(0,1)}$ by (4.9). So $\Lambda_-(\mathrm{AC}(\tau)) = 0$ in these cases.

(i) Suppose $x_\tau = s$. We have $\mathrm{Sign}(\tau_{\mathbf{t}}) = (0, 2n_0)$ where $n_0 = |\tau_{\mathbf{t}}|$. Therefore, only the first term in (4.9) is non-zero and $\mathcal{P}^+_\tau = 1^{(0,-(2n_0-1))}$. Hence $\Lambda_+(\mathrm{AC}(\tau)) = 0$.

(ii) Suppose $x_\tau \in \{r, c\}$. If $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (1,1)$, the first term in (4.9) is non-zero and $\Lambda_+(\mathrm{AC}(\tau)) \succ 1^{(1,0)}$. So $\Lambda_+(\mathrm{AC}(\tau)) \neq 0$. Now suppose $\mathrm{Sign}(\tau_{\mathbf{t}}) \not\succ (1,1)$. Then $\tau_{\mathbf{t}} = r \cdots r$ and $x_{\tau''} = d$ by Proposition 3.3. Now the second term of (4.9) is non-zero and $\mathcal{P}^-_\tau \succ 1^{(1,0)}$. So $\Lambda_+ \mathrm{AC}(\tau) \neq 0$.

(iii) Suppose $x_\tau = d$. Suppose $\mathrm{Sign}(\tau_{\mathbf{t}}) \succ (1,2)$. Then the first term in (4.9) is non-zero and $\Lambda_+(\mathrm{AC}(\tau)) \succ 1^{(1,1)}$. So $\Lambda_+(\mathrm{AC}(\tau)) \neq 0$ and $\Lambda_-(\mathrm{AC}(\tau)) \neq 0$. Now suppose $\mathrm{Sign}(\tau_{\mathbf{t}}) \not\succ (1,2)$. Then $\tau_{\mathbf{t}} = r \cdots rd$ [2] and $x_{\tau''} = d$ by Proposition 3.3. So the both terms in (4.9) are non-zero. Since $\mathcal{P}^+_\tau \succ 1^{(1,0)}$ and $\mathcal{P}^-_\tau \succ 1^{(0,1)}$, we get the conclusion. □

4.4.2. *Unipotent representations attached to $\mathcal{O}'$.* In this section, we let $\tau' \in \mathrm{PBP}^{\mathrm{ext}}_{\star'}(\mathcal{O}')$ and $\tau'' = \nabla(\tau') \in \mathrm{PBP}^{\mathrm{ext}}_{\star}(\mathcal{O}'')$. First note that $x_{\tau''} \neq s$ and so $\Lambda_+(\mathrm{AC}(\tau''))$ always non-zero by induction hypothesis.

Recall (4.8). We claim that we can recover $\Lambda_+(\mathrm{AC}\tau''$ and $\Lambda_-(\tau'')$ from $\mathrm{AC}(\tau')$:

Lemma 4.6. *The map* $\Omega_{\mathcal{O}'} \colon \mathrm{AC}(\tau') \mapsto (\Lambda_+(\mathrm{AC}(\tau'')), \Lambda_-(\mathrm{AC}(\tau'')))$ *is a well defined map.*

Proof. When ${}^l\mathrm{Sign}(\mathrm{AC}(\tau'))$ has two elements, $x_{\tau''} = d$ and ${}^l\mathrm{Sign}(\mathrm{AC}(\tau')) = \{\, (q''_1, p''_1 - 1), (q''_1 - 1, p''_1) \,\}$. In (4.8), $\dagger\Lambda_+(\mathrm{AC}\tau''$ (resp. $\dagger\Lambda_-(\tau'')$) consists of components whose first column has siginature $(q''_1, p''_1 - 1)$ (resp. $(q''_1 - 1, p''_1)$). When ${}^l\mathrm{Sign}(\mathrm{AC}(\tau'))$ has only one elements, $x_\tau = r/c$, ${}^l\mathrm{Sign}(\mathrm{AC}(\tau')) = \{\, (q''_1, p''_1 - 1) \,\}$, $\mathrm{AC}(\tau') = \dagger\Lambda_+(\mathrm{AC}\tau''$ and $\Lambda_-(\tau'') = 0$

In any case, we get

$$\mathrm{Sign}(\tau'') = (n_0, n_0) + (p''_1, q''_1) \quad \text{where } 2n_0 = |\nabla(\mathcal{O}'')| . \tag{4.10}$$

Using the signature $\mathrm{Sign}(\tau'')$, we could recover the twisting characters in the theta lifting of local system. Therefore $\Lambda_+(\mathrm{AC}\tau''$ and $\Lambda_-(\tau'')$ can be recovered from $\mathrm{AC}(\tau')$. □

Lemma 4.7. *Suppose* $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$. *Then* $\pi_{\tau'_1} \neq \pi_{\tau'_2}$.

Proof. It suffice to consider the case when $\mathrm{AC}(\tau'_1) = \mathrm{AC}(\tau'_2)$. By (4.10) in the proof of Lemma 4.6, $\mathrm{Sign}(\tau''_1) = \mathrm{Sign}(\tau''_2)$. On the other hand, $\epsilon_1 = \epsilon_2 = 0$ and $\tau''_1 \neq \tau''_2$ by **??**. So

$$\pi_{\tau'_1} = \bar{\Theta}(\pi_{\tau''_1}) \neq \bar{\Theta}(\pi_{\tau''_2}) = \pi_{\tau'_2}$$

by the injectivity of theta lift. □

4.4.3. *Unipotent representations attached to $\mathcal{O}$.*

Lemma 4.8. *Suppose* $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$. *Then* $\pi_{\tau_1} \neq \pi_{\tau_2}$.

Proof. It suffice to consider the case that $\mathrm{AC}(\tau_1) = \mathrm{AC}(\tau_2)$. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. On the other hand, the twisting $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ since it is determined by the local system: $\epsilon_{\tau_i} = 0$ if and only if $\mathrm{AC}(\tau_i) \supset \boxed{-}$.

We conclude that $\tau''_1 \neq \tau''_2$ and $\tau'_1 \neq \tau'_2$ by **??** and **??**. By Lemma 4.7 and the injectivity of theta lift,

$$\pi_{\tau_1} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_1}} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (\mathbf{1}^{+,-})^{\epsilon_{\tau_2}} = \pi_{\tau_2}$$

□

[2] $\tau_{\mathbf{t}} = d$ if it has length 1.

4.4.4. *Noticed orbits.* In this section, we prove the claims about noticed and +noticed orbits.

Lemma 4.9. *Suppose $\mathcal{O}'$ is noticed. Then*

(i) The map $\{\, \mathrm{AC}(\tau'') \mid x_{\tau''} \neq s \,\} \longrightarrow \{\, \mathrm{AC}(\tau') \,\} = {}^{\ell}\mathrm{LS}(\mathcal{O}')$ *given by* $\mathrm{AC}(\tau'') \mapsto \mathrm{AC}(\tau') = \vartheta(\mathrm{AC}(\tau''))$ *is a bijection.*

(ii) The map $\mathrm{DRC}(\mathcal{O}') \to {}^{\ell}\mathrm{LS}(\mathcal{O}')$ *given by* $\tau' \mapsto \mathrm{AC}(\tau')$ *is a bijection.*

Proof. (i) Note that $\mathcal{O}''$ is noticed by definition. Hence $\Upsilon_{\mathcal{O}''}$ is invertible. Recall Lemma 4.6, we see that

$$(\Upsilon_{\mathcal{O}''})^{-1} \circ \Omega_{\mathcal{O}'} \colon \mathrm{AC}(\tau') \mapsto (\Lambda_{+}(\mathrm{AC}\tau'', \Lambda_{-}(\tau'')) \mapsto \mathrm{AC}(\tau'')$$

gives the inverse of the map in the claim.

(ii) Suppose $\tau'_1 \neq \tau'_2 \in \mathrm{DRC}(\mathcal{O}')$. By **??**, $\tau''_1 \neq \tau''_2$. Hence $\mathrm{AC}(\tau''_1) \neq \mathrm{AC}(\tau''_2)$ by the induction hypothesis. Therefore, $\mathrm{AC}(\tau'_1) \neq \mathrm{AC}(\tau'_2)$ by (i) of the claim.

□

Lemma 4.10. *Suppose $\mathcal{O}$ is noticed,* $\mathcal{L} \colon \mathrm{DRC}(\mathcal{O}) \mapsto {}^{\ell}\mathrm{LS}(\mathcal{O})$ *given by* $\tau \mapsto \mathrm{AC}(\tau)$ *is bijective.*

Proof. We prove the claim by contradiction. We assume $\tau_1 \neq \tau_2$ such that $\mathrm{AC}(\tau_1) = \mathrm{AC}(\tau_2)$.

Recall (4.9): there is an pair of non-negative integers $(e_i, f_i) = \mathrm{Sign}(\mathbf{u}_{\tau_i})$ such that

$$\mathrm{AC}(\tau_i) = \dagger\dagger\Lambda_{+}(\mathrm{AC}\tau''_i \cdot \mathcal{P}^{+}_{\tau_i} + \dagger\dagger\Lambda_{-}(\tau'') \cdot \mathcal{P}^{-}_{\tau_i}.$$

(i) Suppose that $\mathrm{AC}(\tau_i)$ contains a 1-row marked by $-$ or $=$. Then we have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$, which can be read from the mark $-/=$. Moreover, the term $\dagger\dagger\Lambda_{+}(\mathrm{AC}\tau''_i \cdot \mathcal{P}^{+}_{\tau_i} \neq 0$ and we can recover it from $\mathrm{AC}(\tau_i)$. So $\Lambda_{+}(\mathrm{AC}\tau''_1 = \Lambda_{+}(\mathrm{AC}\tau''_2$. Since $\mathcal{O}''$ is +noticed, $\tau''_1 = \tau''_2$ by (4.12) and the induction hypothesis. Note that $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$, we get $\tau_1 = \tau_2$ by **??** which is contradict to our assumption.

(ii) Now we assume that $\mathrm{AC}(\tau_i)$ does not contains a 1-row marked by $-/=$. By (4.9), we conclude that $x_{\tau} \neq d$ and $\tau_{\mathbf{t}} = r \cdots r$ or $r \cdots rc$. Therefore $\mathbf{p}_{\tau_i}$ must be one of the following form by **??**:

$$\mathbf{p}_1 = \begin{array}{|c|c|} \hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} r & \\ \cline{1-1} \end{array}, \quad \mathbf{p}_2 = \begin{array}{|c|c|} \hline r & c \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} c & \\ \cline{1-1} \end{array}, \quad \text{and} \quad \mathbf{p}_3 = \begin{array}{|c|c|} \hline r & d \\ \hline \vdots & \\ \cline{1-1} r & \\ \cline{1-1} r & \\ \cline{1-1} \end{array}$$

We consider case by case according to the set $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\}$:

(a) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_1 \,\}$ or $\{\, \mathbf{p}_2 \,\}$. In these cases, $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by **??**. On the other hand, $\Lambda_{+}(\mathrm{AC}\tau''_1 = \Lambda_{+}(\mathrm{AC}\tau''_2$ by (4.9). Since $\mathcal{O}''$ is +noticed, we get $\tau''_1 = \tau''_2$ a contradiction.

(b) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_3 \,\}$. We still have $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ and $\tau''_1 \neq \tau''_2$ by **??**. On the other hand, $\Lambda_{-}(\tau''_1) = \Lambda_{-}(\tau''_2)$ by (4.9). This again contradict to the injectivity of $\mathrm{AC}(\tau'') \mapsto \Lambda_{-}(\tau'')$.

(c) Suppose $\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \mathbf{p}_1, \mathbf{p}_2 \,\}$. By the argument above, we have $\Lambda_{+}(\mathrm{AC}\tau''_1 = \Lambda_{+}(\mathrm{AC}\tau''_2$ Now $\tau''_1 \neq \tau''_2$ since $\{\, x_{\tau''_1}, x_{\tau''_2} \,\} = \{\, r, c \,\}$. This contradict to that $\mathcal{O}''$ is +noticed again.

(d) Suppose $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ or $\mathbf{p}_2, \mathbf{p}_{\tau_2} = \mathbf{p}_3$. Then

$$\mathrm{AC}(\tau_1) = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_1}|}{2}}\dagger\Lambda_{+}(\mathrm{AC}\tau''_1 \cdot \mathcal{P}^{+}_{\tau''_1} \quad \text{and} \quad \mathrm{AC}(\tau_2) = \dagger\maltese^{\frac{|\mathbf{u}_{\tau_2}|}{2}}\dagger\Lambda_{-}(\tau''_2) \cdot \mathcal{P}^{-}_{\tau''_2}.$$

This implies $|\mathrm{Sign}(\tau''_1)| \equiv |\mathrm{Sign}(\tau''_2)| + 2 \pmod 4$ and $\maltese\dagger\Lambda_{+}(\mathrm{AC}\tau''_1 = \dagger\Lambda_{-}(\tau''_2)$.

Claim 4.11. *We have* $\Lambda_{-}(\tau''_2) \supset \boxed{+}$.

Proof. If $\mathcal{O}''$ is obtained by the usual descent, we have $\mathrm{AC}(\tau) \geq \begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array}$ and we are done.
Now assume $\mathcal{O}''$ is obtained by generalized descent and $\mathcal{O}''$ is $+$noticed, i.e. $\mathbf{u}_{\tau''}$ has at least length 2. Suppose $\mathrm{Sign}(\mathbf{u}_{\tau''}) \succ (1,2)$. Applying (4.9) to τ'', we see that the first term is non-zero and $\mathrm{AC}(\tau'') \supseteq \dagger_{(1,1)}$.
Otherwise, $\mathrm{Sign}(\mathbf{u}_{\tau''}) = (2n_0-1,1) \geq (3,1)$ where n_0 is the length of $\mathbf{u}_{\tau''}$. By (4.9), the second term is non-zero and $\mathrm{AC}(\tau'') \succ \dagger_{(2n_0-1,1)} \succ +$. ☐
The claim leads to a contradiction: Thanks to the character twist in the theta lifting formula of the local system, we see that the the associated character restricted on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$ of ✠$\dagger\Lambda_+(\mathrm{AC}\tau_1''$ and $\dagger\Lambda_-(\tau_2'')$ must be are different, a contradiction.

We finished the proof of the lemma. ☐

4.4.5. *The maps $\Upsilon^+_{\mathcal{O}}$, $\Upsilon^-_{\mathcal{O}}$ and $\Upsilon_{\mathcal{O}}$.*

Lemma 4.12. *Suppose $\mathcal{O}$ is $+$noticed. The map $\Upsilon^+_{\mathcal{O}}\colon \{\,\mathrm{AC}(\tau) \mid \Lambda_+(\mathrm{AC}(\tau)) \neq 0\,\} \to \{\,\Lambda_+(\mathrm{AC}(\tau)) \neq 0\,\}$ is injective.*

Proof. We have two cases:

(i) $\mathrm{AC}(\tau)$ contain an irreducible component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$. This is equivalent to $\Lambda_+(\mathrm{AC}(\tau))$ has an irreducible component $\succ \begin{array}{|c|}\hline + \\ \hline\end{array}$. Now all irreducible components of $\mathrm{AC}(\tau) \succ \begin{array}{|c|}\hline + \\ \hline\end{array}$ and $\mathrm{AC}(\tau) \mapsto \Lambda_+(\mathrm{AC}(\tau))$ will not kill any irreducible components. Hence we could recover $\mathrm{AC}(\tau)$ from $\Lambda_+(\mathrm{AC}(\tau))$.

(ii) Suppose that $\mathrm{AC}(\tau_1) \neq \mathrm{AC}(\tau_2)$ do not contain a component $\succ \begin{array}{|c|}\hline + \\ \hline + \\ \hline\end{array}$.

By Lemma 4.10 $\tau_1 \neq \tau_2$.[3] We now show that $\Lambda_+(\mathrm{AC}\tau_1 = \Lambda_+(\mathrm{AC}\tau_2 \neq 0$ leads to a contradiction. Clearly, $\mathrm{Sign}(\tau_1) = \mathrm{Sign}(\tau_2)$. By (4.9) and checking the properties in **??**, we have

$$\mathbf{p}_{\tau_i} = \begin{array}{|c|c|}\hline s & x_{\tau''} \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} s & \multicolumn{1}{c}{} \\ \cline{1-1} x_{\tau_i} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \text{where } x_{\tau_i''} = c/d, x_{\tau_i} = c/d.$$

On the other hand, when $\mathbf{p}_{\tau_i}$ have the above form, the first term of (4.9) is non-zero. So we conclude that $\Lambda_+(\mathrm{AC}\tau_1 = \Lambda_+(\mathrm{AC}\tau_2$ implies

$$\Lambda_+(\mathrm{AC}\tau_1'' = \Lambda_+(\mathrm{AC}\tau_2''. \tag{4.11}$$

Moreover, we see that every components of $\mathrm{AC}(\tau_i)$ has a 1-row of mark "$-/=$" and we can determine ϵ_{τ_i} by the mark $-/=$ of the 1-row appeared in $\Lambda_+(\mathrm{AC}\tau_i$. In particular, we have $\epsilon_1 = \epsilon_2$. Now **??** implies $\tau_1'' \neq \tau_2''$. This is contradict to (4.11) and our induction hypothesis since $\mathcal{O}''$ is also $+$noticed.

☐

Lemma 4.13. *Suppose $\mathcal{O}$ is noticed. Then the map $\Upsilon_{\mathcal{O}}\colon \mathrm{AC}(\tau) \mapsto (\Lambda_+(\mathrm{AC}(\tau)), \Lambda_-(\tau))$ is injective.*

Proof. It suffice to consider the case where $\mathcal{O}$ is noticed but not $+$noticed, i.e. $C_{2k+1} = C_{2k}+1$. In this case, $n = 1$. The peduncle $\mathbf{p}_\tau$ have four possible cases:

$$\begin{array}{|c|c|}\hline r & c \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline c & c \\ \hline\end{array}, \quad \begin{array}{|c|c|}\hline c & d \\ \hline\end{array}, \quad \text{or} \quad \begin{array}{|c|c|}\hline d & d \\ \hline\end{array}.$$

By (4.9), $\Lambda_+(\tau) = \dagger\dagger\Lambda_+(\tau'')$.

Now suppose $\tau_1 \neq \tau_2$ such that $\Lambda_+(\tau_1) = \Lambda_+(\tau_2)$. Since $\mathcal{O}''$ is $+$noticed, we have $\tau_1'' = \tau_2''$ by the induction hypothesis (see Lemma 4.12). By **??**, this only happens when

[3] $\mathrm{AC}(\tau_1) \neq \mathrm{AC}(\tau_2)$ clearly implies $\tau_1 \neq \tau_2$.

$\{\, \mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2} \,\} = \{\, \begin{array}{|c|c|}\hline c & d \\ \hline\end{array}, \begin{array}{|c|c|}\hline d & d \\ \hline\end{array} \,\}$. Since one of $\Lambda_-(\tau_i)$ is zero and the other is non-zero, we conclude that $\Upsilon_{\mathcal{O}}(\mathrm{AC}(\tau_1)) \neq \Upsilon_{\mathcal{O}}(\mathrm{AC}(\tau_2))$. □

Lemma 4.14. *Suppose $\mathcal{O}$ is +noticed. The map $\Upsilon_{\mathcal{O}}^-\colon \{\, \mathrm{AC}(\tau) \mid \Lambda_-(\tau) \neq 0 \,\} \to \{\, \Lambda_-(\tau) \neq 0 \,\}$ is injective.*

Proof. The proof is similar to that of Lemma 4.12. We have two cases:

(i) $\mathrm{AC}(\tau)$ contain an irreducible component $\succ \begin{array}{|c|}\hline - \\ \hline - \\ \hline\end{array}$. This is equivalent to $\Lambda_-(\tau)$ has an irreducible component $\succ \boxed{-}$ and $\mathrm{AC}(\tau) \mapsto \Lambda_-(\tau)$ will not kill any irreducible components. Hence we could recover $\mathrm{AC}(\tau)$ from $\Lambda_-(\tau)$.

(ii) Now we make a weaker assumption that $\tau_1 \neq \tau_2 \in \mathrm{DRC}(\mathcal{O})$ such that $\mathrm{AC}(\tau_1)$ and $\mathrm{AC}(\tau_2)$ do not contain a component $\succ \begin{array}{|c|}\hline - \\ \hline - \\ \hline\end{array}$.

By (4.9) and the properties in **??**, we see that $\mathbf{p}_{\tau_i}$ must be one of the following

$$\mathbf{p}_1 = \begin{array}{|c|c|}\hline r & c \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \text{or} \qquad \mathbf{p}_2 = \begin{array}{|c|c|}\hline r & d \\ \hline \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} r & \multicolumn{1}{c}{} \\ \cline{1-1} d & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Without of loss of generality, we have the following possibilities:

(a) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_1$. We have $\Lambda_+(\tau_1'') = \Lambda_+(\tau_2'')$ and obtain the contradiction.

(b) $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_2$. We have $\Lambda_-(\tau_1'') = \Lambda_-(\tau_2'')$ and obtain the contradiction.

(c) $\mathbf{p}_{\tau_1} = \mathbf{p}_1$ and $\mathbf{p}_{\tau_2} = \mathbf{p}_2$. Now we apply the same argument in the case (d) of the proof of Lemma 4.10. We get $\maltese\dagger\Lambda_+(\tau_1'') = \dagger\Lambda_-(\tau_2'')$, $\Lambda_-(\tau_2'') \supset +$ and a contradiction by looking at the associated character on the 2-row $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$.

This finished the proof. □

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