

# COMBINATORICS FOR UNIPOTENT REPRESENTATIONS

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## CONTENTS

1. Statement of the main result	1
2. Matching doc-r-c diagrams, local systems, and unipotent representations	5
3. The definition of $\bar{\nabla}$	6
4. Convention on the local system	14
5. Proof of main theorem in the type C/D case	19
6. Type B/M case	26
References	33
Index	39

## 1. STATEMENT OF THE MAIN RESULT

**1.1. Nilpotent Orbits.** Let  $\mathbf{G} = \mathrm{Sp}, \mathrm{O}$ , and  $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$  is a complex nilpotent orbit. “Type M” means that  $\tilde{\mathbf{G}}$  is a metaplectic group. its dual group  $\check{\mathbf{G}}$  is a symplectic group with the same rank, and its duality means the metaplectic dual. Let  $\mathrm{DRC}(\mathcal{O})$  be the set of dot-r-c diagrams parameterizing unipotent representations of the real forms of  $\mathbf{G}$ .

We say an orbit  $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathbf{G}})$  is purely even (good parity in Moeglin-Renard’s notation) if

all row lengths of  $\check{\mathcal{O}}$  are even when  $\check{\mathbf{G}}$  is a symplectic group.  
all row lengths of  $\check{\mathcal{O}}$  are odd when  $\check{\mathbf{G}}$  is an orthogonal group.

**Definition 1.1.** Now we describe the nilpotent orbit  $\mathcal{O} \in \mathrm{Nil}(\mathbf{G})$  who is the Barbasch-Vogan dual of a purely even orbit  $\check{\mathcal{O}}$ . If  $G$  is

**Type C** Then  $\mathcal{O}$  is special and it has columns

$(C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0)$  such that  $C_{2i+1} > C_{2i}$  if  $C_{2i}$  is even for  $i = 0, \dots, k-1$ .

Note that, by the definition of special orbit, we have  $C_{2i} \equiv C_{2i-1} \pmod{2}$ ,  $C_{2i} = C_{2i-1}$  if  $C_{2i}$  is odd for  $i = 1, \dots, k$ .

**Type M** Then  $\mathcal{O}$  is metaplectic special and it has columns

$(C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0)$  such that  $C_{2i+1} > C_{2i}$  if  $C_{2i}$  is odd for  $i = 0, \dots, k-1$ .

Note that, by the definition of metaplectic special, we have  $C_{2i} \equiv C_{2i-1} \pmod{2}$ ,  $C_{2i} = C_{2i-1}$  if  $C_{2i}$  is even for  $i = 1, \dots, k$ .

**Type D** and  $\mathcal{O}$  is special and it has columns

$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0)$  such that  $C_{2i+1} > C_{2i}$  if  $C_{2i}$  is even for  $i = 0, \dots, k$ .

Note that, the condition is equivalent to  $\nabla(\mathcal{O})$  is purely even of type C and  $C_{2k+1}$  is always even.

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**Type B** and  $\mathcal{O}$  is special and it has columns

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ if } C_{2i} \text{ is odd for } i = 0, \dots, k.$$

Note that, the condition is equivalent to  $\nabla(\mathcal{O})$  is purely even of type  $M$  and  $C_{2k+1}$  is always odd.

Let  $\text{Nil}^{\text{dpe}}(\star)$  the set of duals of purely even orbits for type  $\star = B, C, D, M$  respectively.

**1.2. Descent of orbits in the purely even case.** For  $\{\star, \star'\} = \{C, D\}, \{B, M\}$ , we define the notion of descent of orbits. The descent map is defined in the following way (under the above notation):

$$\overline{\nabla}: \quad \text{Nil}^{\text{dpe}}(\star) \longrightarrow \text{Nil}^{\text{dpe}}(\star')$$

$$\mathcal{O} \longmapsto \mathcal{O}'$$

such that

- when  $\star$  is B or D,  $\mathcal{O}' := \nabla(\mathcal{O})$ ;
- when  $\star$  is C or M,

$$\mathcal{O}' := \begin{cases} \nabla(\mathcal{O}) & \text{if } C_{2k} \text{ is even and } \star = B \\ & \text{or } C_{2k} \text{ is odd and } \star = M; \\ (C_{2k-1} + 1, C_{2k-2}, C_{2k-3}, \dots, C_0) & \text{otherwise.} \end{cases}$$

In the second case,  $C_{2k} = C_{2k-1}$  and  $\mathcal{O} \mapsto \overline{\nabla}(\mathcal{O})$  is a good generalized descent.

The following set of orbit are essential to us:

**Definition 1.2.** A nilpotent orbit  $\mathcal{O} \in \text{Nil}(\mathbf{G})$  is called noticed if  $G$  is

**Type D** and  $\mathcal{O}$  is special and it has columns

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \dots, k.$$

**Type B** and  $\mathcal{O}$  is special and it has columns

$$(C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0) \quad \text{such that } C_{2i+1} > C_{2i} \text{ for } i = 0, \dots, k.$$

**Type C** and  $\overline{\nabla}(\mathcal{O})$  is noticed of type D.

**Type M** and  $\overline{\nabla}(\mathcal{O})$  is notice of type B.

Furthermore, We say a noticed orbit  $\mathcal{O}$  of type D or B is +noticed if  $C_{2k+1} - C_{2k} \geq 2$ .

The following inductive definition of noticed orbit is easy to verify.

**Lemma 1.3.** An orbit  $\mathcal{O}$  of type D (resp. type B) is noticed if and only if

- i)  $\overline{\nabla}^2(\mathcal{O})$  is noticed, when  $C_{2k}$  is even (resp. odd), or
- ii)  $\overline{\nabla}^2(\mathcal{O})$  is +noticed, when  $C_{2k}$  is odd (resp. even).

**1.3. Relevant Weyl group representations and dot-r-c diagrams.** Retain the notation in Definition 1.2. Now we describe all relevant representations of the Weyl group appear in the counting algorithm.

[ We use “s” and “d” to denote “r” and “c” in Dan’s notes respectively. ]

**Type D** We define  $c_i$  as the following

$$c_{2i} := \lfloor \frac{C_{2i}}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1} + 1}{2} \rfloor \quad \forall i = 0, 1, \dots, k,$$

$$c_{2k+1} := \frac{C_{2k+1}}{2}$$

The (special) Weyl group representation associated to the special orbit  $\mathcal{O}$  has columns

$$\tau = \tau_L \times \tau_R = (c_{2k+1}, c_{2k-1}, \dots, c_1) \times (c_{2k}, \dots, c_0).$$

The other representations is obtained from the special one by interchanging following pairs when  $c_{2i-1} \neq c_{2i} + 1$  and  $i = 1, \dots, k$  (these are pairs coming from even length columns)

$$(c_{2i-1}, c_{2i}) \longleftrightarrow (c_{2i} + 1, c_{2i-1} - 1).$$

[ Note that we always have  $c_{2i-1} - 1 \geq c_{2i-2}$  by our assumptions of the shape of the orbit. ]

The dot-r-c diagrams of type D: For each representation  $\tau = \tau_L \times \tau_R$ , dots are filled in both sides forming the same shape of Young diagram, a column of “s” is added next to dots on the left side diagram, a column of “r” is added next to dots on the left side diagram, add a row of “c”s and finally add a row of “d”s on the left side of the diagram; Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step. [ At most one r/s is added in each row of the existing dots diagram and at most one c/d is added to each column. ]

In summary, s/r/c/d are all added on the left diagram. Fixing the representation  $\tau$ , the left diagram  $\tau_L$  determin  $\tau_R$  completely.

**Type C** Set  $C_{2k+1} = 0$ . The Weyl group representations are obtained by the same formula of Type D.

The dot-r-c diagrams of type C are obtained in almost the same way as that of type D, except that s are added on the right side of the diagram.

**Type B** We define  $c_i$  as the following

$$c_{2i} := \lfloor \frac{C_{2i} + 1}{2} \rfloor \quad \text{and} \quad c_{2i-1} = \lfloor \frac{C_{2i-1}}{2} \rfloor \quad \forall i = 1, 2, \dots, k,$$

$$c_{2k+1} := \frac{C_{2k+1} - 1}{2}$$

Note that  $C_{2k+1}$  is always odd and  $C_0 = 0$ . The (special) Weyl group representation associated to the special orbit  $\mathcal{O}$  has columns

$$\tau = \tau_L \times \tau_R = (c_{2k}, \dots, c_2) \times (c_{2k+1}, c_{2k-1}, \dots, c_1).$$

The other representations is obtained from the special one by interchanging following pairs when  $c_{2i-1} \neq c_{2i} + 1$  and  $i = 1, \dots, k$  (these are pairs coming from odd length columns)

$$(c_{2i}, c_{2i-1}) \longleftrightarrow (c_{2i-1}, c_{2i}).$$

[ Note that we always have  $c_{2i-1} \geq c_{2i-2}$  by our assumptions of the shape of the orbit. ]

The dot-r-c diagrams of type B: For each representation  $\tau = \tau_L \times \tau_R$ , dots are filled in both sides forming the same shape of Young diagram, a column of “s” is added next to dots on the right side of the diagram, a column of “r” is added next to dots on the right side diagram, add a row of “d”s on the right side of the diagram, and finally add a row of “c”s on the left side of the diagram. Through this procedure, one must make sure that the shape of the diagram is a valid Young diagram after each step. [ At most one r/s is added in each row of the existing dots diagram and at most one c/d is added to each column. ]

In summary, s/r/d are added on the right and c is added on the left.

Each dot-r-c diagram of type B is extended into two diagrams by adding an extra label “a” or “b”. This extension helps us to determine the signature of special orthogonal groups.

Fixing the representation  $\tau$ , the right diagram  $\tau_R$  determin  $\tau_L$  completely.

**Type M** Set  $C_{2k+1} = 0$ . The Weyl group representations are obtained by the same formula of Type B.

The dot-r-c diagrams of type M are obtained in almost the same way as the type B except that  $s$  are added on the left side of the diagram and we do not add labels a/b.

We let  $\text{DRC}(\mathcal{O})$  denote the set of dot-r-c diagrams defined above. Note that for type B, there is an additional mark  $a/b$  attached to the diagram.

In addition, for  $\star = B, C, D, M$ , let

$$\text{DRC}(\star) := \bigsqcup_{\mathcal{O} \in \text{Nil}^{\text{dpe}}(\star)} \text{DRC}(\mathcal{O}).$$

**1.4. Main theorem.** Let  $\text{LS}(\mathcal{O})$  denote the Grothendieck group of  $\tilde{\mathbf{K}}$ -equivariant local systems on  $\mathcal{O}$  for various real forms of  $\mathbf{G}$ . The following is our main theorem on the counting of unipotent representations.

Let  $\tilde{\mathcal{O}}$  be a purely even orbit and  $\mathcal{O}$  be its dual orbit.

**Theorem 1.4.** *We can define an injective map using convergence range theta lifting:*

$$\begin{aligned} \pi : \text{DRC}(\mathcal{O}) &\longrightarrow \text{Unip}(\mathcal{O}) \\ \tau &\longmapsto \pi_\tau \end{aligned}$$

- i) When  $\mathcal{O}$  is of type C or type M,  $\pi$  is a bijection.
- ii) When  $\mathcal{O}$  is of type D or type B, the following map is a bijection

$$\begin{aligned} \text{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z} &\longrightarrow \text{Unip}(\mathcal{O}) \\ (\tau, \varepsilon) &\longmapsto \pi_\tau \otimes (\det)^\varepsilon \end{aligned}$$

- iii) All unipotent representations attached to  $\mathcal{O}$  are unitarizable.
- iv) Suppose  $\mathcal{O}$  is noticed. The associated character map is an injective map:

$$\text{Ch}_{\mathcal{O}} : \text{Unip}(\mathcal{O}) \hookrightarrow \text{LS}(\mathcal{O})$$

- v) Suppose  $\mathcal{O}$  is quasi-distinguished. The image of associated character map is exactly the set of admissible orbit data. In other words, we have a bijection

$$\text{Ch}_{\mathcal{O}} : \text{Unip}(\mathcal{O}) \hookrightarrow \text{LS}^{\text{aod}}(\mathcal{O})$$

According to Moeglin-Renard’s work, Arthur’s unipotent Arthur packet (defined by endoscopic transfer) is constructed by iterated theta lifting. The theorem implies that ABV’s unipotent Arthur packet coincide with Arthur’s.

Two unipotent representation  $\pi_1$  and  $\pi_2$  are called in the same LS packet if  $\mathcal{L}_{\pi_1} = \mathcal{L}_{\pi_2}$ . For a local system  $\mathcal{L} \in \text{LS}(\mathcal{O})$ , let

$$[\mathcal{L}] = \{ \pi \mid \text{Ch}_{\mathcal{O}}(\pi) = \mathcal{L} \}$$

denote the set of all unipotent representations attached to  $\mathcal{L}$ . We call  $[\mathcal{L}]$  a LS packet.

*Remark.* 1. We use convergence range theta lift to construct  $\pi$  inductively with respect to the number of columns in  $\mathcal{O}$ .

2. When  $\mathcal{O}$  is not noticed, a LS packet may not be a singleton and we use the injectivity of theta lifting for a fixed dual pair in an essential way.

3. The size of LS packet  $[\mathcal{L}]$  could be arbitrary large when the size of  $\mathcal{O}$  groups. For example,  $\mathcal{O}$  is the regular orbit of  $\mathrm{Sp}(2n, \mathbb{C})$ , there is only 5 possible local systems when  $n \geq 3$ . But there are  $2n + 1$  unipotent representations.
4. One will see easily from the proof that the associated character  $\mathrm{Ch}_{\mathcal{O}}(\pi)$  for every  $\pi \in \mathrm{Unip}(\mathcal{O})$  is always multiplicity free. Even when  $\mathrm{Ch}_{\mathcal{O}}(\pi)$  is not irreducible, the character of the local system attached the same row length in different irreducible components are always the same.

## 2. MATCHING DOC-R-C DIAGRAMS, LOCAL SYSTEMS, AND UNIPOTENT REPRESENTATIONS

**2.1. Descent maps of dot-r-c diagram in the algorithm.** Suppose  $\mathbf{G}$  is of type B or D, we define a map

$$\begin{aligned} \text{Sign: } \quad \mathrm{DRC}(B) \sqcup \mathrm{DRC}(D) &\longrightarrow \mathbb{N} \times \mathbb{N} \\ \tau &\longmapsto (p, q) \end{aligned}$$

Here  $(p, q)$  is the signature of the orthogonal group  $\mathrm{O}(p, q)$  corresponding to the dot-r-c diagram  $\tau$ , which is calculated by the following formula

$$\begin{aligned} p &= \# \bullet + 2\#r + \#c + \#d + \#A \\ q &= \# \bullet + 2\#s + \#c + \#d + \#B \end{aligned}$$

Suppose  $\mathcal{O}$  is the trivial orbit of  $\mathrm{Sp}(2c_0, \mathbb{R})$  ( $c_0 \geq 0$ ) or  $\mathrm{Mp}(0, \mathbb{R})$  ( $c_0 = 0$ ). Let

$$(2.1) \quad \tau_{\mathcal{O}} = \begin{array}{ccc} & \emptyset & s \\ & \times & \vdots \\ & & s \end{array}$$

such that there are  $c_0$ -entries marked by “ $s$ ” in the right diagram. Then  $\mathrm{DRC}(\mathcal{O}) = \{\tau_{\mathcal{O}}\}$  is a singleton. Define  $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$  the trivial representation of  $\mathrm{Sp}(2c_0, \mathbb{R})$  or  $\mathrm{Mp}(0, \mathbb{R}) = \mathbf{1}$ . That by convention,  $\mathrm{Sp}(0, \mathbb{R}) = \mathrm{Mp}(0, \mathbb{R})$  is the trivial group. Let

$$\mathrm{DRC}_0(C) = \{\tau_{\mathcal{O}} \mid \mathcal{O} = (2c_0), c_0 = 0, 1, 2, \dots\} \text{ and } \mathrm{DRC}_0(M) = \{\tau_{(0)} = \emptyset \times \emptyset\}.$$

The algorithm of matching dot-r-c diagrams is encoded in the descent map  $\overline{\nabla}$  of dot-r-c diagrams defined below. For  $(\star, \star') = (D, C), (B, M)$ , we define maps

$$\begin{aligned} \overline{\nabla}: \quad \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) &\longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star)) \times \mathbb{Z}/2\mathbb{Z} \\ \tau &\longmapsto (\tau', \epsilon). \end{aligned}$$

Here

- i)  $\tau$  is in  $\mathrm{DRC}(\mathcal{O})$  where  $\mathcal{O} \in \mathrm{Nil}^{\mathrm{dpe}}(\star) \sqcup \mathrm{Nil}^{\mathrm{dpe}}(\star')$ ;
- ii)  $\tau' \in \mathrm{DRC}(\mathcal{O}')$  where  $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$ ;
- iii)  $i\epsilon = 0$  when  $\mathcal{O} \mapsto \mathcal{O}'$  is a generalized descent from type C or M to type B or D.

To ease the notations, we define

$$\overline{\nabla}_1: \mathrm{DRC}(\star) \sqcup (\mathrm{DRC}(\star') - \mathrm{DRC}_0(\star')) \longrightarrow (\mathrm{DRC}(\star') \sqcup \mathrm{DRC}(\star))$$

such that  $\overline{\nabla}_1(\tau)$  is the first component of  $\overline{\nabla}(\tau)$ .

We define  $\mathcal{L}$  and  $\pi$  inductively with respect to the number of columns of  $\mathcal{O}$ .

- i) Our reduction terminates when  $\mathcal{O}$  is the trivial orbit of  $\mathrm{Sp}(2c_0, \mathbb{R})$  or  $\mathrm{Mp}(0, \mathbb{R})$ . In this case,  $\mathrm{DRC}(\mathcal{O}) = \{\tau_{\mathcal{O}}\}$  which is attached to the trivial representation. We remark that the theta lift of the trivial representation  $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$  of  $\mathrm{Sp}(0, \mathbb{R})$  or  $\mathrm{Mp}(0, \mathbb{R})$  to  $\mathrm{O}(p, q)$  for any signature  $p, q$  is the trivial representation  $\mathbf{1}_{p,q}^{+,+}$  of  $\mathrm{O}(p, q)$ .<sup>1</sup>
- ii) Suppose  $\mathcal{O} \in \mathrm{Nil}(\star) \sqcup \mathrm{Nil}(\star')$  is a nontrivial orbit. Let  $\tau \in \mathrm{DRC}(\mathcal{O})$  and

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \in \mathrm{DRC}(\mathcal{O}') \times \mathbb{Z}/2\mathbb{Z} \text{ where } \mathcal{O}' = \overline{\nabla}(\mathcal{O}).$$

- a) Suppose  $\tau \in \mathrm{DRC}(C)$  or  $\mathrm{DRC}(M)$ . Now  $\tilde{G} = \mathrm{Sp}(2n, \mathbb{R})$  or  $\mathrm{Mp}(2n, \mathbb{R})$  with  $n = |\tau|$ . By induction,  $\tau'$  is attached to a unipotent representation  $\pi_{\tau'}$  of  $\tilde{G}' = \mathrm{O}(p, q)$  where  $(p, q) = \mathrm{Sign}(\tau')$ . We define

$$\pi_{\tau} := \bar{\Theta}_{\tilde{G}', \tilde{G}}(\pi_{\tau'} \otimes (\det)^{\epsilon}).$$

Here  $\det$  is the determinant character of  $\mathrm{O}(p, q)$ .

- b) Suppose  $\tau \in \mathrm{DRC}(D)$  or  $\mathrm{DRC}(B)$ . Now  $\tilde{G} = \mathrm{O}(p, q)$  with  $(p, q) = \mathrm{Sign}(\tau)$ . By induction,  $\tau'$  is attached to a unipotent representation of  $\tilde{G}'$  where  $\tilde{G}' = \mathrm{Sp}(2n, \mathbb{R})$  or  $\mathrm{Mp}(2n, \mathbb{R})$  and  $n = |\tau'|$ . We define

$$\pi_{\tau} := \bar{\Theta}_{\tilde{G}', \tilde{G}}(\pi_{\tau'} \otimes (\mathbf{1}_{p,q}^{+, -})^{\epsilon}).$$

We let

$$\begin{aligned} \mathcal{L}_{\tau} &:= \mathrm{Ch}_{\mathcal{O}}(\pi_{\tau}), \text{ and} \\ {}^{\ell}\mathrm{LS}(\mathcal{O}) &:= \{ \mathcal{L}_{\tau} \mid \tau \in \mathrm{DRC}(\mathcal{O}) \} \end{aligned}$$

to be the set of local systems obtained from our algorithm.

We summarize our definitions in the following diagram with  $\star = \mathcal{O}/C/D/B/M$ .

$$\begin{array}{ccccc} \mathrm{DRC}(\star) & \xrightarrow{\pi} & \mathrm{Unip}(\star) & & \\ \downarrow \mathcal{L} & \begin{array}{ccc} \tau & \xrightarrow{\quad} & \pi_{\tau} \\ \downarrow & & \downarrow \\ \mathcal{L}_{\tau} & \xleftarrow{\quad} & \mathcal{L}_{\tau} \end{array} & \downarrow \mathrm{Ch} & \\ \mathcal{K}_{\mathrm{M}}(\star) & \xrightarrow{\quad} & \mathrm{LS}(\star) & & \end{array}$$

Here  $\mathcal{K}_{\mathrm{M}}(\star)$  is a combinatorially defined monoid parameterizing the local systems on the relevant nilpotent orbits, see.

### 3. THE DEFINITION OF $\overline{\nabla}$

From now on, we assume

$$\tau = (\tau_{L,0}, \tau_{L,1}, \dots) \times (\tau_{R,0}, \tau_{R,1}, \dots).$$

For any  $\tau$  let  $\tau^{\bullet}$  be the subdiagram consisting of  $\bullet$  and  $s$  entries only.

<sup>1</sup>When  $c_0 = 0$ , this is an assignment by convention.

**3.1. dot-s switching algorithm.** We say a bipartition  $\tau = \tau_L \times \tau_R$  is interlaced with larger left part if  $\tau$  has columns  $(\tau_{L,0} \geq \tau_{L,1}, \dots) \times (\tau_{R,0}, \tau_{R,1}, \dots)$  such that

$$\tau_{L,0} \geq \tau_{R,0} \geq \tau_{L,1} \geq \tau_{R,1} \geq \tau_{L,2} \geq \dots$$

$\text{bi}\mathcal{P}_L$  denote the set of all interlaced bipartitions with larger left parts. Similarly, define  $\text{bi}\mathcal{P}_R$  be the set of bipartitions with larger right parts:

$$\text{bi}\mathcal{P}_R = \{ \tau = (\tau_{L,0} \geq \tau_{L,1}, \dots) \times (\tau_{R,0}, \tau_{R,1}, \dots) \mid \tau_{L,0} \geq \tau_{R,0} \geq \tau_{L,1} \geq \tau_{R,1} \geq \tau_{L,2} \geq \dots \}$$

Let  $\text{DS}_L$  (resp.  $\text{DS}_R$ ) be the set of filled diagrams with only “s” entries on the left (resp. right) diagram and the rest are all “•”.

Note that for each  $\tau \in \text{DS}_L$ , its shape  $\tau$  is in  $\text{bi}\mathcal{P}_L$ . On the other hand, for each  $\tau \in \text{bi}\mathcal{P}_L$ , we can fill “•” in all entries on the right diagram and corresponding entries on the left, and then fill “s” in the rest entries on the left. This procedure yields a valid dot-s diagram. In summary,  $\text{bi}\mathcal{P}_L$  and  $\text{DS}_L$  are naturally bijective to each other.

Let  $\bar{\nabla}_L: \text{bi}\mathcal{P}_L \rightarrow \text{bi}\mathcal{P}_R$  (resp.  $\bar{\nabla}_R: \text{bi}\mathcal{P}_R \rightarrow \text{bi}\mathcal{P}_L$ ) be the operation of deleting the longest column on the left (resp. on the right). Then the operation gives a well defined maps between  $\text{DS}_L$  and  $\text{DS}_R$  making the following diagrams commute (the vertical maps are the natural identification discussed above):

$$\begin{array}{ccc} \text{bi}\mathcal{P}_L & \xrightarrow{\bar{\nabla}_L} & \text{bi}\mathcal{P}_R \\ \parallel & & \parallel \\ \text{DS}_L & \xrightarrow{\bar{\nabla}_L} & \text{DS}_R \end{array} \quad \begin{array}{ccc} \text{bi}\mathcal{P}_R & \xrightarrow{\bar{\nabla}_R} & \text{bi}\mathcal{P}_L \\ \parallel & & \parallel \\ \text{DS}_R & \xrightarrow{\bar{\nabla}_R} & \text{DS}_L \end{array}$$

By abuse of notation, we also call the dashed arrow above  $\bar{\nabla}_L$  and  $\bar{\nabla}_R$ .

**3.2. Bijection between “special” and “non-special” diagrams for type C.** We define a bijection between “special” diagram  $\tau^s$  and “non-special” diagrams  $\tau$ .

**Definition 3.1.** Let  $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_1, C_0, C_{-1} = 0) \in \text{Nil}^{\text{dpe}}(C)$  with  $k \geq 0$ . Let

$$\tau = (\tau_{2k-1}, \dots, \tau_1, \tau_{-1} = 0) \times (\tau_{2k}, \dots, \tau_0).$$

be a Weyl group representation attached to  $\mathcal{O} \in \text{Nil}^{\text{dpe}}(C)$ . We say  $\tau$  has

- special shape if  $\tau_{2k-1} \leq \tau_{2k} + 1$ ;
- non-special shape if  $\tau_{2k-1} > \tau_{2k} + 1$ .

Clearly, this gives a partition of the set of Weyl group representations attached to  $\mathcal{O}$ .

Suppose  $k \geq 1$ ,  $C_{2k} = 2c_{2k}$  and  $C_{2k-1} = 2c_{2k-1}$ . The special shape representation  $\tau^s$

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k-1}, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \dots, \tau_0)$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k} + 1, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \dots, \tau_0)$$

In the pairing  $\tau^s \leftrightarrow \tau^{ns}$ ,  $\tau_j$  is unchanged for  $j \leq 2k - 2$ . We call  $\tau^s$  the special shape and  $\tau^{ns}$  the corresponding non-special shape

**Definition 3.2.** Suppose  $C_{2k}$  is even. We retain the notation in Definition 3.1 where the shapes  $\tau^s$  and  $\tau^{ns}$  correspond with each other. We define a bijection between  $\text{DRC}(\tau^s)$  and  $\text{DRC}(\tau^{ns})$ :

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \longleftrightarrow & \text{DRC}(\tau^{ns}) \\ \tau^s & \longleftrightarrow & \tau^{ns} \end{array}$$

Without loss of generality, we can assume that  $\tau^s \in \text{DRC}(\tau^s)$  and  $\tau^{ns} \in \text{DRC}(\tau^{ns})$  have the following shapes:

$$(3.1) \quad \tau^s : \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ x_0 * \cdots & v_0 \cdots & & \\ x_1 x_2 \cdots & x_3 & & \\ & \begin{array}{c} s \\ \vdots \\ s \end{array} & & \end{array} \longleftrightarrow \tau^{ns} : \begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ y_0 * \cdots & w_0 \cdots & & \\ \begin{array}{c} r \\ \vdots \\ r \end{array} y_2 \cdots & & & \\ & \begin{array}{c} y_1 \\ y_3 \end{array} & & \end{array}$$

(Here the grey parts have length  $(C_{2k} - C_{2k-1})/2$  (could be zero length), the row contains  $x_0$  and  $y_0$  may be empty, and  $*/\cdots$  are arbitrary entries. We remark that the value of  $v_0$  and  $w_0$  are completely determined by  $x_0$  and  $y_0$ . In particular,  $v_0$  and  $w_0$  are non-empty if and only if  $x_0/y_0 \neq \emptyset$ , i.e.  $C_{2k-1} \geq 4$ .)

The correspondence between  $(x_0, x_1, x_2, x_3)$ , with  $(y_0, y_1, y_2, y_3)$  is given by Table 1. The rest part of the diagrams are unchanged.

We define

$$\text{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \text{DRC}(\tau^s) \quad \text{and} \quad \text{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \text{DRC}(\tau^{ns})$$

where  $\tau^s$  (resp.  $\tau^{ns}$ ) runs over all special (resp. non-special) shape representations attached to  $\mathcal{O}$ . Clearly  $\text{DRC}(\mathcal{O}) = \text{DRC}^s(\mathcal{O}) \sqcup \text{DRC}^{ns}(\mathcal{O})$ .

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

**3.3. Special and non-special shapes of type D.** Now we define the notion of special and non-special shape of type D.

**Definition 3.3.** Let

$$\tau = (\tau_{2k+1}, \tau_{2k-1}, \dots, \tau_1) \times (\tau_{2k}, \dots, \tau_0)$$

be a Weyl group representation attached to  $\mathcal{O} \in \text{Nil}^{\text{dpe}}(D)$ . We say  $\tau$  has

- special shape if  $\tau_{2k-1} \leq \tau_{2k} + 1$ ;
- non-special shape if  $\tau_{2k-1} > \tau_{2k} + 1$ .

Suppose  $\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \dots, C_0 \geq 0)$  with  $k \geq 1$ .<sup>2</sup> A representation  $\tau^s$  attached to  $\mathcal{O}$  has the shape

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k}, \tau_{2k-2}, \dots, \tau_0)$$

is paired with

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k+1}, c_{2k} + 1, \tau_{2k-3}, \dots, \tau_1) \times (c_{2k-1} - 1, \tau_{2k-2}, \dots, \tau_0).$$

In the pairing  $\tau^s \leftrightarrow \tau^{ns}$ ,  $\tau_j$  is unchanged for  $j \leq 2k - 2$ .

**3.4. Definition of  $\bar{\nabla}$  of Type D and C.** The induction starts with type C: For the trivial orbit  $\mathcal{O}$  of  $\tilde{G} = \text{Sp}(2c_0, \mathbb{R})$ ,  $\text{DRC}(\mathcal{O}) = \{\tau_{\mathcal{O}}\}$  and  $\pi_{\tau_{\mathcal{O}}} = \mathbf{1}$  is the trivial representation, see (2.1).

<sup>2</sup>Note that  $C_{2k}$  and  $C_{2k-1}$  are non-zero even integers.



### 3.4.1. The definition of $\epsilon$ .

- (1). When  $\tau \in \text{DRC}(D)$ ,  $\epsilon$  is determined by the “basal disk” of  $\tau$  (see (6.3) for the definition of  $x_\tau$ ):

$$\epsilon_\tau := \begin{cases} 0, & \text{if } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

- (2). When  $\tau \in \text{DRC}(C)$ , the  $\epsilon$  is determined by the lengths of  $\tau_{L,0}$  and  $\tau_{R,0}$ :

$$\epsilon_\tau := \begin{cases} 0, & \text{if } \tau \text{ has special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| \leq 1; \\ 1, & \text{if } \tau \text{ has non-special shape, i.e. } |\tau_{L,0}| - |\tau_{R,0}| > 1. \end{cases}$$

### 3.4.2. Initial cases. Let $\mathcal{O}$ be a nilpotent orbit of type D with at most 2 columns.

- (3). Suppose  $\mathcal{O} = (2c_1, 2c_0)$ . Then  $\mathcal{O}' := \overline{\nabla}(\mathcal{O})$  is the trivial orbit of  $\text{Sp}(2c_0, \mathbb{R})$ .  $\text{DRC}(\mathcal{O})$  consists of diagrams of shape  $\tau_L \times \tau_R = (c_1, ) \times (c_0, )$ . The set  $\text{DRC}(\mathcal{O}')$  is a singleton  $\{\tau_{\mathcal{O}'}\}$ , and every element  $\tau \in \text{DRC}(\mathcal{O})$  maps to  $\tau_{\mathcal{O}'}$  (see (2.1)):

$$\text{DRC}(\mathcal{O}) \ni \tau : \begin{array}{c} \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \end{array} \times \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \mapsto \tau_{\mathcal{O}'} : \begin{array}{c} \emptyset \\ \vdots \\ s \end{array} \times \begin{array}{c} s \\ \vdots \\ s \end{array}$$

	$\tau'$	$\tau^s$	$\tau^{ns}$	
	$\begin{array}{ccc} x'_0 & * & * \\ x'_1 & x'_2 & \\ & \times & \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ x_1 & x_2 & x_3 \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} y_0 & * & * * \\ & \begin{array}{c} r \\ \vdots \\ r \end{array} & y_2 \\ & \vdots & \times \\ & y_1 & \\ & y_3 & \end{array}$	
$\begin{array}{l} x'_1 = s \\ \text{then} \\ z_0 = \emptyset / s \\ x_0 = \emptyset / \bullet \\ x_1 = \bullet \\ x_2 = x'_2 \end{array}$	$\begin{array}{ccc} x'_0 & * & * \\ s & x_2 & \\ & \times & \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ \bullet & x_2 & \bullet \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ & \begin{array}{c} r \\ \vdots \\ r \end{array} & x_2 \\ & \vdots & \times \\ & c & \\ & d & \end{array}$	$x_2 \neq r$
		$\begin{array}{ccc} x_0 & * & * * \\ \bullet & r & \bullet \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ & \begin{array}{c} r \\ \vdots \\ r \end{array} & c \\ & \vdots & \times \\ & r & \\ & d & \end{array}$	$x_2 = r$
$\begin{array}{l} x'_1 \neq s \\ \text{then} \\ x_1 = x'_1 \\ x_2 = x'_2 \end{array}$	$\begin{array}{ccc} x'_0 & * & * \\ x'_1 & x_2 & \\ & \times & \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ x_1 & x_2 & s \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} x_0 & * & * * \\ & \begin{array}{c} r \\ \vdots \\ r \end{array} & x_2 \\ & \vdots & \times \\ & r & \\ & x_1 & \end{array}$	$\begin{array}{l} x'_0 \neq c \\ \text{then} \\ x'_0 = \emptyset / s / r \\ x_0 = \emptyset / \bullet / r \end{array}$
		$\begin{array}{ccc} c & * & * * \\ d & x_2 & s \\ & \times & \begin{array}{c} s \\ \vdots \\ s \end{array} \end{array}$	$\begin{array}{ccc} r & * & * * \\ & \begin{array}{c} r \\ \vdots \\ r \end{array} & x_2 \\ & \vdots & \times \\ & c & \\ & d & \end{array}$	$\begin{array}{l} x'_0 = c \\ \text{then} \\ x_0 = x'_0 = c \\ x_1 = x'_1 = d \\ x_2 = x'_2 = \emptyset / d \end{array}$

TABLE 1. “special-non-special” switch

We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \text{ and } x_\tau := x_n$$

We call  $\mathbf{p}_\tau = \mathbf{x}_\tau$  the “peduncle” part of  $\tau$  and  $x_\tau$  the “basal disk” of  $\tau$ . Note that  $\mathbf{x}_\tau$  consists of entries marked by  $s/r/c/d$ .

3.4.3. *The descent from C to D.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

- (4). Suppose  $|\tau_{L,0}| - |\tau_{R,0}| < 2$ , i.e.  $\tau$  has special shape. Keep  $r, c, d$  unchanged, delete  $\tau_{R,0}$  and fill the remaining part with “•” and “s” by  $\bar{\nabla}_R(\tau^\bullet)$  using the dot-s switching algorithm.
- (5). Suppose  $|\tau_{L,0}| - |\tau_{R,0}| \geq 2$ . We define

$$\tau' = \bar{\nabla}_1(\tau) := \bar{\nabla}_1(\tau^s)$$

where  $\tau^s$  is the special diagram corresponding to  $\tau$  defined in Definition 3.2.

**Lemma 3.4.** *Suppose  $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$  with  $k \geq 1$  and  $C_{2k}$  even. Let  $\mathcal{O}' := \bar{\nabla}(\mathcal{O}) = (C_{2k-1}, \dots, C_0)$ . Then*

$$\begin{array}{ccc} \text{DRC}^s(\mathcal{O}) & \xrightarrow{\bar{\nabla}_1} & \text{DRC}(\mathcal{O}') \\ \text{DRC}^{ns}(\mathcal{O}) & \xrightarrow{\bar{\nabla}_1} & \text{DRC}(\mathcal{O}') \end{array}$$

are a bijections.

*Proof.* The claim for  $\text{DRC}^s(\mathcal{O})$  is clear by the definition of descent. The claim for  $\text{DRC}^{ns}(\mathcal{O})$  reduces to that of  $\text{DRC}^s(\mathcal{O})$  using Definition 3.2.  $\square$

**Lemma 3.5.** *Suppose  $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$  with  $k \geq 1$  and  $C_{2k}$  odd. Let  $\mathcal{O}' := \bar{\nabla}(\mathcal{O}) = (C_{2k-1} + 1, \dots, C_0)$ . Then the following map is a bijection*

$$\text{DRC}(\mathcal{O}) \xrightarrow{\bar{\nabla}_1} \{ \tau' \in \text{DRC}(\mathcal{O}') \mid x_\tau \neq s \}.$$

*Proof.* This is clear by the descent algorithm.  $\square$

3.4.4. *Descent from D to C.* Now we consider the general case of the descent from type  $D$  to type  $C$ . So we assume  $\mathcal{O}$  has at least 3 columns. First note that the shape of  $\tau' = \bar{\nabla}(\tau)$  is the shape of  $\tau$  deleting the longest column on the left.

The definition splits in cases below. In all these cases we define

$$(3.2) \quad \mathbf{x}_\tau := x_1 \cdots x_n \text{ and } x_\tau := x_n$$

which is marked by  $s/r/c/d$ . For the part marked by  $*/\dots$ ,  $\bar{\nabla}$  keeps  $r, c, d$  and maps the rest part consisting of • and  $s$  by dot-s switching algorithm.

- (6). When  $C_{2k} = C_{2k-1}$  is odd, we could assume  $\tau$  and  $\tau'$  have the following forms with  $n = (C_{2k+1} - C_{2k} + 1)/2$ .

$$\tau : \begin{array}{ccccccc} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & w_0 & * & \dots & \dots \\ x_1 & y_1 & \dots & \times & & & \\ \vdots & & & & & & \\ & & & & & & x_n \end{array} \mapsto \tau' : \begin{array}{ccccccc} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & \dots & & w_0 & * & \dots & \dots \\ z_1 & \dots & \times & & & & \end{array}$$

Suppose  $y_1 = c$  and  $(x_1, \dots, x_n) = (r, \dots, r)$  or  $(r, \dots, r, d)$ , then let  $z_1 = r$ ; otherwise, set  $z_1 := y_1$ . We define

$$\mathbf{p}_\tau := \begin{matrix} x_1 & y_1 \\ \vdots & \\ x_n & \end{matrix}.$$

- (7). When  $C_{2k}$  is even and  $\tau$  has special shape, the descent is given by the following diagram

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots \\ \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots \\ \bullet \\ \vdots \\ \bullet \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \ \dots \end{array} \times \begin{array}{c} \dots\dots\dots \\ s \\ \vdots \\ s \end{array}$$

- (8). When  $C_{2k}$  is even and  $\tau$  has non-special shape, then

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots\dots\dots \\ s \ r \ \dots\dots\dots \\ \vdots \\ s \ r \\ x_0 \ y_1 \\ x_1 \ y_2 \\ \vdots \\ x_n \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \ \dots\dots\dots \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \ * \ \dots\dots\dots \\ r \ \vdots \ \dots\dots\dots \\ r \\ z_1 \\ z_2 \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \ \dots\dots\dots \end{array}$$

where  $y_1, y_2$  are non empty. In most of the case, we just delete the longest column on the left and set  $(z_1, z_2) = (y_1, y_2)$ . The exceptional cases are listed below:

- When  $x_0 = r$ , we have  $(y_1, y_2) = (c, d)$ . We let

$$\begin{array}{c} x_0 \ y_1 \quad r \ c \quad z_1 \quad r \\ x_1 \ y_2 = x_1 \ d \mapsto z_2 := r \\ \vdots \quad \vdots \\ x_n \quad x_n \end{array}$$

- When  $x_0 = c$ , we have  $n = 1$  and  $(x_0, x_1) = (y_1, y_2) = (c, d)$ . We let

$$\begin{array}{c} x_0 \ y_1 = c \ c \mapsto z_1 := r \\ x_1 \ y_2 = d \ d \quad z_2 := c \end{array}$$

- We remark that  $x_0$  never equals to  $d$ .

3.4.5. *A key property in the descent case.* We retain the notation in Definition 3.3, where

$$\mathcal{O} = (C_{2k+1} = 2c_{2k+1}, C_{2k} = 2c_{2k}, C_{2k-1} = 2c_{2k-1}, \dots, C_0) \text{ such that } k \geq 1.$$

Let  $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$  and  $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ . The following lemma is the key property satisfied by our definition

**Lemma 3.6.** *Let  $\tau^s$  and  $\tau^{ns}$  are two representations attached to  $\mathcal{O}$  as in Definition 3.3. Then*

- i) *For every  $\tau \in \text{DRC}(\tau^s) \sqcup \text{DRC}(\tau^{ns})$ , the shape of  $\overline{\nabla}^2(\tau)$  is*

$$\tau'' = (c_{2k-1}, \tau_{2k-3}, \tau_1) \times (\tau_{2k-2}, \dots, \tau_0)$$

- ii) *Let  $\mathcal{O}_1 = (2(c_{2k+1} - c_{2k}),) \in \text{Nil}(D)$  be the trivial orbit of  $\text{O}(2(c_{2k-1} - c_{2k}), \mathbb{C})$ . We define  $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$  by  $\delta(\tau) = (\overline{\nabla}^2(\tau), \mathbf{x}_\tau)$ . The following maps are bijections*

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \xrightarrow{\delta} & \text{DRC}(\tau'') \times \text{DRC}(\mathcal{O}_1) \xleftarrow{\delta} \text{DRC}(\tau^{ns}) \\ \tau^s & \longmapsto & (\overline{\nabla}^2(\tau^s), \mathbf{x}_{\tau^s}) \\ & & (\overline{\nabla}^2(\tau^{ns}), \mathbf{x}_{\tau^{ns}}) \longleftarrow \tau^{ns} \end{array}$$

*In particular, we obtain an one-one correspondence  $\tau^s \leftrightarrow \tau^{ns}$  such that  $\delta(\tau^s) = \delta(\tau^{ns})$ .*

iii) Suppose  $\tau^s$  and  $\tau^{ns}$  correspond as the above such that  $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$ . Then

$$(3.3) \quad \text{Sign}(\tau^s) = \text{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \text{Sign}(\tau'') + \text{Sign}(\mathbf{x}).$$

*Proof.* Suppose  $\tau^s$  has special shape. the the behavior of  $\tau^s$  under the descent map  $\bar{\nabla}$  is illustrated as the following:

$$\tau^s : \begin{array}{c} \begin{array}{|c|} \hline \dots \\ * \dots \\ \bullet \\ \vdots \\ \bullet \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ * \dots \\ \bullet \\ \vdots \\ \bullet \\ \hline \end{array} \mapsto \tau'^s \times \begin{array}{|c|} \hline \dots \\ * \dots \\ s \\ \vdots \\ s \\ \hline \end{array} \mapsto \tau'' : \begin{array}{|c|} \hline \dots \\ * \dots \\ \vdots \\ * \dots \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ * \dots \\ \vdots \\ * \dots \\ \hline \end{array} \\ x_1 \\ \vdots \\ x_n \end{array}$$

Note that  $\tau^s$  is obtained from  $\tau''$  by attaching the grey part consisting totally  $2c_{2k}$  dots and  $\mathbf{x}$ . Now the claims for  $\tau^s$  is clear.

Now consider the descent of a non-special diagram  $\tau^{ns}$ :

$$\tau^{ns} : \begin{array}{c} \begin{array}{|c|} \hline \dots \\ * y_0 * \dots \\ s \ r \ y_2 \dots \\ \vdots \\ s \ r \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ u_0 \dots \\ \hline \end{array} \mapsto \tau'^{ns} : \begin{array}{c} \begin{array}{|c|} \hline \dots \\ y'_0 * \dots \\ r \ y'_2 \dots \\ \vdots \\ r \\ \hline \end{array} \times \begin{array}{|c|} \hline \dots \\ w_0 \dots \\ \hline \end{array} \mapsto \tau'' : \begin{array}{|c|} \hline \dots \\ x'_0 * \dots \\ x'_1 x'_2 \dots \\ \hline \end{array} \\ x_0 \ y_1 \\ x_1 \ y_3 \\ \vdots \\ x_n \end{array}$$

The bijection follows from the observation that 1.  $\tau^{ns}$  is obtained by attaching the grey part  $x_0, \dots, x_n$  and  $y_0, y_1, y_2, y_3$  to the  $*/\dots$  part of  $\tau''$ ; 2. the value of  $(y_0, y_1, y_2, y_3)$  is completely determined by  $(x'_0, x'_1, x'_2)$  and  $\mathbf{x} = x_1 \cdots x_n$ .

We leave it to the reader for checking case by case that the grey part of  $\tau^{ns}$  has signature  $(C_{2k} - 2, C_{2k} - 2)$  and

$$\text{Sign}(x_0 y_0 y_1 y_2 y_3) - \text{Sign}(x'_0 x'_1 x'_2) = (2, 2).$$

Therefore (6.4) follows.

[ First assume that  $C_{2k} = 2$ . Note that we can not/(or now?) assume  $k = 1$ , consider the orbit  $\mathcal{O}'' = (2, 1, 1, 1, 1)$ .

- (i)  $x'_1 = s$ , now  $y'_0 = \emptyset/\bullet$  when  $x'_0 = \emptyset/s$ 
  - (a) Suppose  $x'_2 \neq r$ . Then  $(y'_2, y'_1, y'_3) = (x'_2, r, d)$ ,  $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, x'_2, c, d)$ . Therefore, the sign difference is  $\text{Sign}(s, c, d) - \text{Sign}(s) = (2, 2)$ .
  - (b) Suppose  $x'_2 = r$ . Then  $(y'_2, y'_1, y'_3) = (c, r, d)$ .  $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, c, r, d)$ . Therefore, the sign difference is  $\text{Sign}(s, c, r, d) - \text{Sign}(s, r) = (2, 2)$ .
- (ii)  $x'_1 \neq s$ .
  - (a) Suppose  $x'_0 \neq c$ . Then  $x'_0 = \emptyset/s/r$ ,  $x'_1 = r/c/d$ ,  $(y'_0, y'_2, y'_1, y'_3) = (\emptyset/\bullet/r, x'_2, r, x'_1)$ .
    - 1).  $x'_1 = r$ . Then  $(x_0, y_0, y_2, y_1, y_3) = (r, x'_0, x'_2, c, d)$ . Therefore, the sign difference is  $\text{Sign}(r, c, d) - \text{Sign}(r) = (2, 2)$ .
    - 2).  $x'_1 = c$ . Then  $x_1 = d$ .  $(x_0, y_0, y_2, y_1, y_3) = (c, x'_0, x'_2, c, d)$ . Therefore, the sign difference is  $\text{Sign}(c, c, d) - \text{Sign}(c) = (2, 2)$ .
    - 3).  $x'_1 = d$ . Then  $(x_0, y_0, y_2, y_1, y_3) = (s, x'_0, y'_2, y'_1, y'_3) = (s, x'_0, x'_2, r, d)$ . Therefore, the sign difference is  $\text{Sign}(s, r, d) - \text{Sign}(d) = (2, 2)$ .
  - (b)  $(x_0, y_0, y_2, y_1, y_3) = (s, r, x'_2, c, d)$ . Therefore, the sign difference is  $\text{Sign}(s, r, x'_2, c, d) - \text{Sign}(c, d, x'_2) = (2, 2)$ .

- (b) Suppose  $x'_0 = c$ . Then  $x'_1 = d$ ,  $(y'_0, y'_2, y'_1, y'_3) = (r, x'_2, c, d)$ .  $(x_0, y_0, y_2, y_1, y_3) = (s, r, x'_2, c, d)$ . Therefore, the sign difference is  $\text{Sign}(s, r, x'_2, c, d) - \text{Sign}(c, d, x'_2) = (2, 2)$ .

]

□

3.4.6. *A key proposition in the generalized descent case.* We now assume that  $k \geq 1$  and  $C_{2k}$  is odd, i.e.  $\mathcal{O}' \rightsquigarrow \mathcal{O}''$  is a generalized descent.

Without loss of generality, we can assume the dot-r-c diagrams have the following shapes with  $n = (C_{2k+1} - C_{2k} + 1)/2$ :

$$(3.4) \quad \tau : \begin{array}{c} \begin{array}{cccc} * & * & * & * \\ \vdots & \vdots & \vdots & \vdots \end{array} \times \begin{array}{c} \begin{array}{cc} * & * \\ \vdots & \vdots \end{array} \end{array} \xrightarrow{\bar{\nabla}_1} \tau' : \begin{array}{c} \begin{array}{ccc} * & * & * \\ \vdots & \vdots & \vdots \end{array} \times \begin{array}{c} \begin{array}{cc} * & * \\ \vdots & \vdots \end{array} \end{array} \xrightarrow{\bar{\nabla}_1} \tau'' : \begin{array}{c} \begin{array}{ccc} * & * & * \\ \vdots & \vdots & \vdots \end{array} \times \begin{array}{c} \begin{array}{cc} * & * \\ \vdots & \vdots \end{array} \end{array}$$

The diagram  $\tau$  is obtained from the diagram of  $\tau$  by adding  $\bullet$  at the grey parts, changing  $x_{\tau''}$  to  $x_0$  and attaching  $x_1 \cdots x_n$ .

We define

$$\mathbf{p}_\tau := \begin{array}{c} x_1 \ x_0 \\ \vdots \\ x_n \end{array}$$

and call  $\mathbf{p}_\tau$  the peduncle part of  $\tau$ . Let  $\mathcal{O}_1 = (C_{2k+1} - C_{2k} + 1, ) \in \text{Nil}^{\text{dpe}}(D)$ . We define  $\mathbf{u}_\tau \in \text{DRC}(\mathcal{O}_1)$  by the following formula:

$$(3.5) \quad \mathbf{u}_\tau := u_1 \cdots u_n = \begin{cases} r \cdots rc, & \text{when } \mathbf{p}_\tau = \begin{array}{c} r \\ \vdots \\ r \end{array} \begin{array}{c} c \\ \\ \end{array} \\ r \cdots cd, & \text{when } \mathbf{p}_\tau = \begin{array}{c} r \\ \vdots \\ d \end{array} \begin{array}{c} c \\ \\ \end{array} \\ x_1 \cdots, x_n & \text{otherwise.} \end{cases}$$

Let  $\tilde{\mathcal{O}} = (C_{2k+1} - C_{2k} + 1, 1, 1, ) \in \text{Nil}^{\text{dpe}}(D)$  and  $\tilde{\mathcal{O}}'' = \bar{\nabla}^2(\tilde{\mathcal{O}}) = (2) \in \text{Nil}^{\text{dpe}}(D)$ . Then  $\mathbf{p}_\tau \in \text{DRC}(\tilde{\mathcal{O}})$  and  $x_\tau \in \text{Nil}^{\text{dpe}}(D)$ . There is some restriction on the possibilities of  $\mathbf{u}_\tau$ .

**Lemma 3.7.** *Let*

$$\tau = \begin{array}{c} x_1 \ x_0 \\ \vdots \\ x_n \end{array} \in \text{DRC}(\tilde{\mathcal{O}}) \quad \text{and} \quad \tau'' := \bar{\nabla}_1^2(\tau).$$

*Then we have the following properties which is easy to verify.*

- i) If  $\tau'' = r$ , then  $s \in \mathbf{u}_\tau$  or  $c \in \mathbf{u}_\tau$ . In particular,  $\text{Sign}(\mathbf{u}_\tau) > (0, 1)$ . If  $\text{Sign}(\mathbf{u}_\tau) \nprec (0, 2)$ , we have

$$\tau = \begin{array}{c} r \\ \vdots \\ r \end{array} \begin{array}{c} c \\ \\ \end{array} \quad \text{and} \quad \mathbf{u}_\tau = \begin{array}{c} r \\ \vdots \\ c \end{array}.$$

- ii) If  $\tau'' = c$ , then  $s \in \mathbf{u}_\tau$  or  $c \in \mathbf{u}_\tau$ . In particular,  $\text{Sign}(\mathbf{u}_\tau) \geq (0, 1)$ .

- iii) If  $\tau'' = d$ , there is no restriction on  $\mathbf{u}_\tau$ .

- iv) We have  $x_n = d$  if and only  $d \in \mathbf{u}_\tau$ .

- v) If  $x_n = d$  and  $\tau'' = r/c$ , we have  $n \geq 2$  and  $\text{Sign}(\mathbf{u}_\tau) \geq (0, 2)$ .

- vi) If  $\mathbf{u}_\tau = r \cdots r$ , we have  $\tau'' = d$ .

**Lemma 3.8.** *The map  $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$  given by  $\tau \mapsto (\bar{\nabla}_1^2(\tau), \mathbf{u}_\tau)$  is injective. Moreover,*

$$\text{Sign}(\tau) = \text{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \text{Sign}(\mathbf{u}_\tau).$$

*The map  $\tilde{\delta}: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$  given by  $\tau \mapsto (\bar{\nabla}_1^2(\tau), \text{Sign}(\tau), \epsilon_\tau)$  is injective.*

*Proof.* By our algorithm,

$$(x_{\tau''}, \mathbf{u}_{\tau}) = (x_{\nabla_1^2(\mathbf{p}_{\tau})}, \mathbf{u}_{\mathbf{p}_{\tau}}).$$

The injectivity of  $\delta$  and the signature formula follows directly from the definition of  $\mathbf{u}_{\tau}$ . Now the injectivity of  $\tilde{\delta}$  follows from the injectivity of  $\mathbf{u}_{\tau} \mapsto (\text{Sign}(\mathbf{u}_{\tau}), \epsilon)$  by Lemma 5.2.  $\square$

#### 4. CONVENTION ON THE LOCAL SYSTEM

An irreducible local system is represented by a *marked Young diagram*. A general local system is understood as the union of irreducible ones. By our computation later, it would be clear that the local systems appeared in our cases are always *multiplicity one*.

**4.1. Commutative free monoid.** Given a set  $\mathbf{A}$ , the commutative free monoid  $\mathbf{C}(\mathbf{A})$  on  $\mathbf{A}$  consists all finite multisets with elements drawn from  $\mathbf{A}$ . It is equipped with the binary operation, say  $\bullet$  such that

$$\{a_1, \dots, a_{k_1}\} \bullet \{b_1, \dots, b_{k_2}\} := \{a_1, \dots, a_{k_1}, b_1, \dots, b_{k_2}\}.$$

In the following, we always identify  $\mathbf{A}$  with the subset of  $\mathbf{C}(\mathbf{A})$  consists of singletons.

Suppose  $\mathbf{M}$  is a commutative monoid, a map  $f: \mathbf{A} \rightarrow \mathbf{M}$  is uniquely extends to a morphism  $\mathbf{C}(\mathbf{A}) \rightarrow \mathbf{M}$  which is still called  $f$  by abuse of notations.

In particular a map  $f: \mathbf{A} \rightarrow \mathbf{B}$  between sets uniquely extends to an morphism  $f: \mathbf{C}(\mathbf{A}) \rightarrow \mathbf{C}(\mathbf{B})$  by

$$f(\{a_1, \dots, a_k\}) = \{f(a_1), \dots, f(a_k)\}.$$

**4.2. Marked Young diagram.** We now explain the combinatorial objects parameterizing local systems. Basically, we use  $+/-$  (resp.  $*/=$ ) to denote the associated character on the corresponding factor of the isotropic group. Let  $\star = B, C, D, M$ .

- **Young diagram** Let  $\mathbf{YD}$  be the commutative free monoid  $\mathbf{C}(\mathbb{N}^+)$  on the set of positive integers  $\mathbb{Z}_{\geq 1}$ . An element  $\mathcal{O} = \{R_1, \dots, R_k\}$  is identified with the Young diagram whose set of row lengths is  $\mathcal{O}$ . Therefore, we identify  $\mathbf{YD}$  with the set of Young diagrams. Let

$$|\cdot| : \mathbf{YD} \longrightarrow \mathbb{N}$$

be the map sending  $\{R_1, \dots, R_k\}$  to  $\sum_{i=1}^k R_i$ .

Let  $\mathbf{YD}(\star)$  to denote the Young diagrams corresponding to partitions of type  $\star$ . The set  $\mathbf{YD}(\star)$  parameterizes the set of nilpotent orbits of type  $\star$ . Note that  $\mathbf{YD}(C) = \mathbf{YD}(M)$ .

- **Signed Young diagram** Let  $\mathbf{S}$  be the set of non-empty finite length strings of alternating signs of the form  $+ - + \dots$  or  $- + - \dots$ . Let  $\mathbf{SYD} := \mathbf{C}(\mathbf{S})$  be the monoid on  $\mathbf{S}$ .

The monoid  $\mathbf{SYD}$  is identified with the set of all signed Young diagrams:  $\mathcal{O} = \{\mathbf{r}_1, \dots, \mathbf{r}_k\} \in \mathbf{SYD}$  is identified with the signed Young diagram whose rows are filled by  $\mathbf{r}_j$ . The map  $\mathbf{S} \rightarrow \mathbb{Z}_{\geq 1}$  sending a string to its length extends to the morphism

$$\begin{aligned} \mathcal{Y}: \mathbf{SYD} &\longrightarrow \mathbf{YD} \\ \{\mathbf{r}_1, \dots, \mathbf{r}_k\} &\longmapsto \{|\mathbf{r}_1|, \dots, |\mathbf{r}_k|\} \end{aligned}$$

where  $|\mathbf{r}_j|$  denote the length of the string  $\mathbf{r}_j$ . By abused of notation, we also denote  $|\cdot| \circ \mathcal{Y}$  by  $|\cdot|$ .

We define  $\text{SYD}(\star)$  to be a subset of the preimage of  $\text{YD}(\star)$  under the above map: When  $\star = C, M$  (resp.  $\star = D, B$ ),

$$\text{SYD}(\star) = \left\{ \mathcal{O} \in \text{SYD} \left| \begin{array}{l} \mathcal{Y}(\mathcal{O}) \in \text{YD}(\star) \\ \text{For each positive odd number (resp.} \\ \text{even number) } r, \text{ the strings } + - \cdots \\ \text{and } - + \cdots \text{ of the length } r \text{ ap-} \\ \text{pears in } \mathcal{O} \text{ with the same multiplic-} \\ \text{ity (could be 0).} \end{array} \right. \right\}$$

The set  $\text{SYD}(\star)$  parameterizes the real nilpotent orbits of type  $\star$  and  $\mathbf{K}$ -orbits in  $\mathfrak{p}$ .

We define two signature maps  $\mathbf{S} \rightarrow \mathbb{N} \times \mathbb{N}$  as the following. For  $\mathbf{r} \in \mathbf{S}$ , define

$$\text{Sign}(\mathbf{r}) := (\# + (\mathbf{r}), \# - (\mathbf{r})) \quad \text{and}$$

$${}^l\text{Sign}(\mathbf{r}) := \begin{cases} (1, 0) & \text{if } \mathbf{r} = + \cdots \\ (0, 1) & \text{if } \mathbf{r} = - \cdots \end{cases}$$

The above defined signature maps naturally extends to signature maps  $\text{Sign}: \text{SYD} \rightarrow \mathbb{N} \times \mathbb{N}$  and  $\text{Sign}: \text{SYD} \rightarrow \mathbb{N} \times \mathbb{N}$ .

- **Marked Young diagram** Let  $\mathbf{M}$  be the set of finite length strings of alternating marks of the form  $+ - + \cdots, - + - \cdots, * = * \cdots$ , or  $= * = \cdots$ .

Let  $\text{MYD} := \mathbf{C}(\mathbf{M})$  which is identified with the set of Young diagrams marked with alternating marks  $+/-, */=$ . Define  $\mathcal{F}: \mathbf{M} \rightarrow \mathbf{S}$  by the formula

$$\mathcal{F}(\mathbf{r}) := \begin{cases} + - + \cdots & \text{if } \mathbf{r} = + - + \cdots \text{ or } * = * \cdots \\ - + - \cdots & \text{if } \mathbf{r} = - + - \cdots \text{ or } = * = \cdots \end{cases} \quad \forall \mathbf{r} \in \mathbf{M}$$

such that  $\mathcal{F}(\mathbf{r})$  and  $\mathbf{r}$  have the same length.  $\mathcal{F}$  extends to the map  $\mathcal{F}: \text{MYD} \rightarrow \text{SYD}$ .

By abused of notation, we also denote  $| \circ \mathcal{Y}$  by  $| \cdot |$ .

For  $\mathcal{L} \in \text{MYD}$ , we define

$$\text{Sign}(\mathcal{L}) = \text{Sign}(\mathcal{F}(\mathcal{L})) \quad \text{and} \quad {}^l\text{Sign}(\mathcal{L}) = {}^l\text{Sign}(\mathcal{F}(\mathcal{L})).$$

We define  $\text{MYD}(\star)$  to be a subset of the preimage of the above map:

$$\text{MYD}(\star) = \left\{ \mathcal{L} \in \text{MYD}(\star) \left| \begin{array}{l} \mathcal{F}(\mathcal{L}) \in \text{SYD}(\star); \\ \mathbf{r}_1 = \mathbf{r}_2 \text{ if } \mathbf{r}_1, \mathbf{r}_2 \in \mathcal{L} \text{ and } \mathcal{F}(\mathbf{r}_1) = \mathcal{F}(\mathbf{r}_2); \\ \mathbf{r} = + - + \cdots \text{ or } - + - \cdots \text{ if the length} \\ \text{of } \mathbf{r} \text{ is odd (resp. even) when } \star \in \{C, M\} \\ \text{(resp. } \star \in \{B, D\}). \end{array} \right. \right\}$$

We will use  $\text{MYD}(\star)$  to parameterize irreducible local systems attached to the nilpotent orbits of type  $\star$ .

For monoid  $\text{YD}$ ,  $\text{SYD}$  or  $\text{MYD}$ , the identity element is the empty set  $\emptyset$ , and the monoid binary operator is denoted by “.”. The  $\emptyset$  is always in  $\text{MYD}(\star)$  by convention. According to the definition of  $\text{MYD}(\star)$ , it could happen that  $\mathcal{L}_1 \cdot \mathcal{L}_2 \notin \text{MYD}(\star)$  for  $\mathcal{L}_1, \mathcal{L}_2 \in \text{MYD}(\star)$ .

We define an involution

$$(4.1) \quad \overline{\phantom{x}}: \mathbf{M} \rightarrow \mathbf{M}$$

on  $\mathbf{M}$  by switching the symbols  $+$  with  $*$  and  $-$  with  $=$ . For example  $\overline{+ - +} = * = *$  and  $\overline{=*} = - +$ .

We define the map

$$(4.2) \quad \dagger: \mathbf{M} \rightarrow \mathbf{M}$$

by extending the length of the strings to the left by one. For example  $\dagger(+ - +) = - + - +$  and  $\dagger(=*) = *=*$ .

For  $\mathcal{L}, \mathcal{C} \in \text{MYD}$ , the expression  $\mathcal{L} \succ \mathcal{C}$  means that there is  $\mathcal{B} \in \text{MYD}$  such that the factorization  $\mathcal{L} = \mathcal{B} \cdot \mathcal{C}$  holds.

In the following, we will represent an element  $\mathcal{L} \in \text{MYD}$  by the Young diagram whose rows are filled with the strings in  $\mathcal{L}$ . The terminology “ $l$ -row” means a row of length  $l$ . If we do not care whether the mark is  $*$  or  $+$  (resp.  $=$  or  $=$ ), we will use  $*$  (resp.  $\div$ ) to mark the corresponding position.

**Example 4.1.** *The following two diagrams  $\dagger_{p,q}$  and  $\dagger_{p,q}$  are in  $\text{MYD}(B) \cup \text{MYD}(D)$ .*

$$\dagger_{p,q} := \begin{array}{c} \dagger \\ \vdots \\ + \\ = \\ \vdots \\ = \end{array}$$

*represents the local system on the trivial orbit of  $O(p, q)$  which has the trivial character on  $O(p)$  and  $\det$  on  $O(q)$ . Similarly,*

$$\dagger_{p,q} := \begin{array}{c} \dagger \\ \vdots \\ + \\ - \\ \vdots \\ - \end{array}$$

*represents the trivial local system on the trivial orbit.*

**4.2.1. Local systems.** We define  $\mathcal{K}_M$  to be the free commutative monoid generated by  $\text{MYD}$ . The binary operation in  $\mathcal{K}_M$  is denoted by “ $+$ ”.

The operation “ $\cdot$ ” naturally extends to  $\mathcal{K}_M$ :

$$\left( \sum_i m_i \mathcal{L}_i \right) \cdot \left( \sum_j m_j \mathcal{L}'_j \right) := \sum_{i,j} m_i m_j (\mathcal{L}_i \cdot \mathcal{L}'_j).$$

where  $\mathcal{L}_i, \mathcal{L}'_j \in \text{MYD}$ ,  $m_i, m_j \in \mathbb{Z}_{\geq 0}$ .

Now  $(\mathcal{K}_M, +, \cdot)$  is in fact a commutative semiring. We let  $0$  be the additive identity element in  $\mathcal{K}_M$  and  $\emptyset$  is the multiplicative identity. Note that  $\mathcal{L} \cdot 0 = 0$  for any  $\mathcal{L} \in \mathcal{K}_M$  by definition.

For  $\mathcal{L}, \mathcal{D} \in \mathcal{K}_M$ ,  $\mathcal{L} \succ \mathcal{D}$  means that there exists  $\mathcal{B} \in \mathcal{K}_M$  such that  $\mathcal{L} = \mathcal{B} \cdot \mathcal{D}$ . In addition, we write  $\mathcal{L} \supset \mathcal{D}$  if there is  $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{K}_M$  such that  $\mathcal{L}_1 \succ \mathcal{D}$  and  $\mathcal{L}_1 + \mathcal{L}_2 = \mathcal{L}$ .

Let  $\mathcal{K}_M(\star)$  be the sub-monoid of  $\mathcal{K}_M$  generated by  $\text{MYD}(\star)$ . Note that  $\mathcal{K}_M(\star)$  is not a sub semi-ring of  $\mathcal{K}_M$ .

For  $\mathcal{L} = \sum_{i=1}^k \mathcal{L}_i \in \mathcal{K}_M$  with  $\mathcal{L}_i \in \text{MYD}$ , we let

$$\text{Sign}(\mathcal{L}) = \{ \text{Sign}(\mathcal{L}_i) \mid i = 1, \dots, k \} \text{ and } {}^l\text{Sign}(\mathcal{L}) = \{ {}^l\text{Sign}(\mathcal{L}_i) \mid i = 1, \dots, k \}$$

**4.3. Operations on  $\mathcal{K}_M(\star)$ .** The operation  $\dagger$  defined in (4.2) extends to an operation

$$\dagger: \mathcal{K}_M \longrightarrow \mathcal{K}_M$$

on  $\mathcal{K}_M$  which adding a column on the left of the original marked Young diagram.

**4.3.1. Character twists on  $\mathcal{K}_M(D)$  and  $\mathcal{K}_M(B)$ .** Recall that  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  parameterizes the characters of a indefinite orthogonal group  $O(p, q)$ :

$$\chi = (\chi^+, \chi^-) \longleftrightarrow (\mathbf{1}^{-,+})^{\chi^+} \otimes (\mathbf{1}^{+,-})^{\chi^-}.$$



We define the action of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  action on  $\mathbf{M}$  by:

$$\mathbf{r} \otimes \chi := \begin{cases} \bar{\mathbf{r}} & \text{if } p + q \equiv 1 \text{ and } \chi^+ p + \chi^- q \equiv 1 \pmod{2} \\ \mathbf{r} & \text{otherwise} \end{cases}$$

where  $\mathbf{r} \in \mathbf{M}$ ,  $\chi = (\chi^+, \chi^-) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  and  $(p, q) = \text{Sign}(\mathbf{r})$ .

The  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  action extends to actions on  $\mathcal{K}_{\mathbf{M}}(D)$  and  $\mathcal{K}_{\mathbf{M}}(B)$ .

4.3.2. *Twists on  $\mathcal{K}_{\mathbf{M}}(C)$  and  $\mathcal{K}_{\mathbf{M}}(M)$ .* We define an involution  $\mathfrak{X}_C: \mathbf{M} \rightarrow \mathbf{M}$  on  $\mathbf{M}$  by

$$\mathfrak{X}(\mathbf{r}) := \begin{cases} \bar{\mathbf{r}} & \text{if } |\mathbf{r}| \equiv 2 \pmod{4} \\ \mathbf{r} & \text{otherwise.} \end{cases}$$

The involution  $\mathfrak{X}$  extends to an involution on  $\mathcal{K}_{\mathbf{M}}$ . We will only apply  $\mathfrak{X}$  on  $\mathcal{L} \in \mathcal{K}_{\mathbf{M}}(C)$  or  $\mathcal{K}_{\mathbf{M}}(M)$ .

4.3.3. *Truncations.* We define the Truncation maps

$$\mathsf{T}^+: \mathcal{K}_{\mathbf{M}} \longrightarrow \mathcal{K}_{\mathbf{M}}$$

such that for  $\mathcal{L} \in \text{MYD}$

$$\mathsf{T}^+(\mathcal{L}) = \begin{cases} \mathcal{L}_1 & \text{if there is } \mathcal{L}_1 \in \text{MYD} \text{ such that } \mathcal{L}_1 \cdot + = \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$

We define  $\mathsf{T}^-: \mathcal{K}_{\mathbf{M}} \longrightarrow \mathcal{K}_{\mathbf{M}}$  similarly such that  $\mathcal{L} = \mathsf{T}^-(\mathcal{L}) \cdot -$  if  $\mathcal{L} > -$  and  $\mathsf{T}^-(\mathcal{L}) = 0$  otherwise for  $\mathcal{L} \in \text{MYD}$ .

To symplify the notation, we write  $\mathcal{L}^\pm := \mathsf{T}^\pm(\mathcal{L})$  for  $\mathcal{L} \in \mathcal{K}_{\mathbf{M}}$ .

#### 4.4. Theta lifts of local systems.

4.4.1. *Lift from type C to D.* For each signature  $(p, q) \in \mathbb{N} \times \mathbb{N}$  such that  $p + q$  is even, we define a map

$$\vartheta_{CD, (p, q)}: \mathcal{K}_{\mathbf{M}}(C) \rightarrow \mathcal{K}_{\mathbf{M}}(D)$$

as the following: Let  $\mathcal{L} \in \text{MYD}(C)$ . Let  $(p_1, q_1) = {}^l\text{Sign}(\mathcal{L})$  and

$$(p_0, q_0) = (p, q) - \text{Sign}(\mathcal{L}) - (q_1, p_1).$$

We define

$$\vartheta_{CD, (p, q)}(\mathcal{L}) = \begin{cases} (\dagger \mathfrak{X}^{\frac{p-q}{2}}(\mathcal{L})) \cdot \dagger_{p_0, q_0} & \text{if } p_0 \geq 0 \text{ and } q_0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

4.4.2. *Lift from type D to type C.* For each non-zero integer  $n \in \mathbb{N}$ , we define a map

$$\vartheta_{DC, n}: \mathcal{K}_{\mathbf{M}}(D) \rightarrow \mathcal{K}_{\mathbf{M}}(C)$$

as the following: Let  $\mathcal{L} \in \text{MYD}(D)$ . Let  $(p_1, q_1) = {}^l\text{Sign}(\mathcal{L})$  and  $(p, q) = \text{Sign}(\mathcal{L})$ . Let

$$n_0 = n - (p + q)/2 - (p_1 + q_1)/2$$

We define

$$\vartheta_{DC, n}(\mathcal{L}) = \begin{cases} \mathfrak{X}^{\frac{p-q}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geq 0} \\ \mathfrak{X}^{\frac{p-q}{2}}(\dagger \mathcal{L}^+ + \dagger \mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that  $\mathfrak{X}^{\frac{p-q}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) = (\mathfrak{X}^{\frac{p-q}{2}}(\dagger \mathcal{L})) \cdot \dagger_{n_0, n_0}$ .

[ We take a splitting  $\text{O}(p, q) \times \text{Sp}(n, \mathbb{R})$  in to the big metaplectic group  $\text{Mp}$  such that  $\text{O}(p, \mathbb{C}) \times \text{O}(q, \mathbb{C})$  acts on the Fock model linearly. The maximal compact  $K_{\text{Sp}(n, \mathbb{R})} = \text{U}(2n)$  acts on the Fock model by the character  $\zeta = \det^{(p-q)/2}$ . For the component of the rational

nilpotent orbit  $\mathrm{Sp}(2n, \mathbb{R})$ , the factor of the component group is  $\mathrm{Sp}$  if the corresponding row has odd length. Otherwise the component group is  $\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})$  where  $(p_{2k}, q_{2k})$  is the signature corresponding to the  $2k$ -rows.  $\zeta|_{\mathrm{O}(p_{2k})} = \det_{\mathrm{O}(p_{2k})}^{(p-q)k/2}$  which is nontrivial if and only if  $p - q \equiv 2 \pmod{4}$  and  $2k \equiv 2 \pmod{4}$ . This gives the formula above. ]

In the proof later, we don't care above the marks for rows with length longer than 1 in the most of the case. So we will omit  $\boxtimes_{(p,q)}$ . When there is no confusion, we will simply write  $\vartheta$  for  $\vartheta_{CD,(p,q)}$  and  $\vartheta_{CD,(p,q)}$ .

4.4.3. *Lift from type M to B.* For each signature  $(p, q) \in \mathbb{N} \times \mathbb{N}$  such that  $p + q$  is odd, we define a map

$$\vartheta_{MB,(p,q)}: \mathcal{K}_M(M) \rightarrow \mathcal{K}_M(B)$$

as the following: Let  $\mathcal{L} \in \mathrm{MYD}(M)$ . Let  $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$  and

$$(p_0, q_0) = (p, q) - \mathrm{Sign}(\mathcal{L}) - (q_1, p_1).$$

We define

$$\vartheta_{MB,(p,q)}(\mathcal{L}) = \begin{cases} (\dagger \boxtimes^{\frac{p-q+1}{2}}(\mathcal{L})) \cdot \dagger_{p_0, q_0} & \text{if } p_0 \geq 0 \text{ and } q_0 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

4.4.4. *Lift from type B to type M.* For each non-zero integer  $n \in \mathbb{N}$ , we define a map

$$\vartheta_{BM,n}: \mathcal{K}_M(B) \rightarrow \mathcal{K}_M(M)$$

as the following: Let  $\mathcal{L} \in \mathrm{MYD}(D)$ . Let  $(p_1, q_1) = {}^l\mathrm{Sign}(\mathcal{L})$  and  $(p, q) = \mathrm{Sign}(\mathcal{L})$ . Let

$$n_0 = n - (p + q)/2 - (p_1 + q_1)/2$$

We define

$$\vartheta_{BM,n}(\mathcal{L}) = \begin{cases} \boxtimes^{\frac{p-q-1}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) & \text{if } n_0 \in \mathbb{Z}_{\geq 0} \\ \boxtimes^{\frac{p-q-1}{2}}(\dagger \mathcal{L}^+ + \dagger \mathcal{L}^-) & \text{if } n_0 = -1 \\ 0 & \text{otherwise} \end{cases}$$

We remark that  $\boxtimes^{\frac{p-q-1}{2}}((\dagger \mathcal{L}) \cdot \dagger_{n_0, n_0}) = (\boxtimes^{\frac{p-q-1}{2}}(\dagger \mathcal{L})) \cdot \dagger_{n_0, n_0}$ .

[ For odd orthogonal-metaplectic group case, we still take the splitting such that  $\mathrm{O}(p, q)$  acts linearly.

The maximal compact  $\tilde{K}_{\mathrm{Sp}(n, \mathbb{R})} = \widetilde{\mathrm{U}(2n)}$  acts on the Fock model by the character  $\zeta = \det^{(p-q)/2}$ .

Then  $\det^{(p-q)/2}|_{\mathrm{O}(p_{2k}) \times \mathrm{O}(q_{2k})} = \det_{\mathrm{O}(p_{2k})}^{\frac{(p-q)k}{2}} \boxtimes \det_{\mathrm{O}(q_{2k})}^{\frac{(p-q)k}{2}}$ . Note that  $p - q$  is an odd number. When  $k$  is even the character on  $\widetilde{\mathrm{O}(p_{2k})/\mathrm{O}(q_{2k})}$  are  $\mathbf{1}$  or  $\det$ . When  $k$  is odd the character would be  $\det^{\frac{1}{2}}$  or  $\det^{\frac{1}{2}+1}$ .

We assume the default character on  $2k$ -rows is  $\det^{\frac{k}{2}}$ , and we use  $+/-$  to mark the row. Otherwise, we use  $*/=$  to mark the rows. ]

## 4.5. Examples and remarks.

### 4.5.1. Example.

**Example 4.2.** Let  $\mathcal{T} := \dagger \dagger (\dagger_{2,1}) + \dagger \dagger (\dagger_{1,2})$  and  $\mathcal{P} = \dagger_{1,3}$ . Then

$$\mathcal{T} \cdot \mathcal{P} = \begin{array}{cc} * \div * & * \div * \\ * \div * & \div * \div \\ \div * \div & \div * \div \\ + & \cup + \\ = & = \\ = & = \\ = & = \end{array}.$$

4.5.2. *Signs of the local systems obtained by iterated lifting.* Suppose  $\mathcal{L} \in \mathcal{K}_M$  is obtained by iterated theta lifting and character twisting in Section 4.3.1. Then the set  ${}^l\text{Sign}(\mathcal{L})$  has restricted possibilities. First,  $\text{Sign}(\mathcal{L})$  is always a singleton.

When  $\mathcal{O}$  is of type B/D,  ${}^l\text{Sign}(\mathcal{L})$  is a singleton  $\{(p_1, q_1)\}$  where

$$(p_1, q_1) = \text{Sign}(\mathcal{L}) - (n_0, n_0) \quad \text{and} \quad 2n_0 = |\overline{\nabla}(\mathcal{O})|.$$

When  $\mathcal{O}$  is of type C/M,  ${}^l\text{Sign}(\mathcal{L}) = \{(p_1, q_1)\}$  is a singleton in the most of the case. The  ${}^l\text{Sign}(\mathcal{L})$  may have two elements if  $\mathcal{L}$  is obtained by good generalized lifting, and it has the form  ${}^l\text{Sign}(\mathcal{L}) = \{(p_1 + 1, q_1), (p_1, q_1 + 1)\}$ .

## 5. PROOF OF MAIN THEOREM IN THE TYPE C/D CASE

We do induction on the number of columns of the nilpotent orbits.

5.1. **Notation.** In this section,  $\mathcal{O}$  is always an nilpotent orbit of type D in  $\text{Nil}^{\text{dpe}}(D)$ . We always let

$$\begin{aligned} \mathcal{O} &= (C_{2k+1}, C_{2k}, C_{2k-1}, \dots, C_1, C_0), \\ \mathcal{O}' &= \overline{\nabla}(\mathcal{O}) = (C_{2k}, C_{2k-1}, \dots, C_1, C_0). \end{aligned}$$

When  $k \geq 1$ , we let

$$\mathcal{O}'' = \overline{\nabla}^2(\mathcal{O}) = \begin{cases} (C_{2k-1}, \dots, C_1, C_0) & \text{when } C_{2k} \text{ is even,} \\ (C_{2k-1} + 1, \dots, C_1, C_0) & \text{when } C_{2k} \text{ is odd.} \end{cases}$$

For a  $\tau \in \text{DRC}(\mathcal{O})$ , we let

$$(\tau', \epsilon) = \overline{\nabla}(\tau) \quad \text{and} \quad (\tau'', \epsilon') = \overline{\nabla}(\tau') \quad (\text{when } k \geq 1).$$

5.1.1. *Main proposition.* The aim of this section is to prove the following main proposition holds for  $\mathcal{O} \in \text{Nil}^{\text{dpe}}(D)$ .

**Proposition 5.1.** *We have:*

- i)  $\mathcal{L}_\tau$  is non-zero. In particular,  $\pi_\tau \neq 0$ .
- ii) Suppose  $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$ , then  $\pi_{\tau'_1} \neq \pi_{\tau'_2}$ .
- iii) Suppose  $\tau_1 \neq \tau_2 \in \text{DRC}(\mathcal{O})$ , then  $\pi_{\tau_1} \neq \pi_{\tau_2}$ .
- iv) The local system  $\mathcal{L}_\tau$  is disjoint with their determinant twist:

$$(5.1) \quad \{\mathcal{L}_\tau \mid \tau \in \text{DRC}(\mathcal{O})\} \cap \{\mathcal{L}_\tau \otimes \det \mid \tau \in \text{DRC}(\mathcal{O})\} = \emptyset.$$

v) We have

- (i)  $x_\tau = s$ , then  $\mathcal{L}_\tau^+ = \mathcal{L}_\tau^- = 0$ .
- (ii)  $x_\tau = r/c$ , then  $\mathcal{L}_\tau^+ \neq 0$  and  $\mathcal{L}_\tau^- = 0$ .
- (iii)  $x_\tau = d$ , then  $\mathcal{L}_\tau^+ \neq 0$  and  $\mathcal{L}_\tau^- \neq 0$ .

vi) If  $\mathcal{O}' \rightsquigarrow \mathcal{O}''$  is a usual descent, we have a simpler factorization:

$$(5.2) \quad \mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_{x_\tau}.$$

- vii) When  $\mathcal{O}'$  is noticed, the map  $\mathcal{L}: \text{DRC}(\mathcal{O}') \rightarrow {}^\ell\text{LS}(\mathcal{O}')$  is a bijection.
- viii) When  $\mathcal{O}$  is noticed, the map  $\mathcal{L}: \text{DRC}(\mathcal{O}) \rightarrow {}^\ell\text{LS}(\mathcal{O})$  is a bijection.
- ix) When  $\mathcal{O}$  is +noticed, the maps

$$\begin{aligned} \Upsilon_{\mathcal{O}}^+ : \{\mathcal{L}_\tau \mid \mathcal{L}_\tau^+ \neq 0\} &\longrightarrow \{\mathcal{L}_\tau^+\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^+ & \text{and} \\ \Upsilon_{\mathcal{O}}^- : \{\mathcal{L}_\tau \mid \mathcal{L}_\tau^- \neq 0\} &\longrightarrow \{\mathcal{L}_\tau^-\} \\ \mathcal{L}_\tau &\longmapsto \mathcal{L}_\tau^- \end{aligned}$$

are injective.  
 x) When  $\mathcal{O}$  is noticed, the map

$$(5.3) \quad \begin{array}{ccc} \Upsilon_{\mathcal{O}}: \{ \mathcal{L}_{\tau} \mid \mathcal{L}_{\tau}^{+} \neq 0 \} & \longrightarrow & \{ (\mathcal{L}_{\tau}^{+}, \mathcal{L}_{\tau}^{-}) \mid \mathcal{L}_{\tau}^{+} \neq 0 \} \\ \mathcal{L}_{\tau} & \longmapsto & (\mathcal{L}_{\tau}^{+}, \mathcal{L}_{\tau}^{-}) \end{array}$$

is injective.<sup>3</sup>

Note that,  $\text{DRC}(\mathcal{O})$  counts the unipotent representations of special orthogonal groups a priori. But Proposition 5.1 (iv)), implies that the representation  $\pi_{\tau}$  and  $\pi_{\tau} \otimes \det$  are two different representations of the orthogonal groups. Therefore  $\text{DRC}(\mathcal{O}) \times \mathbb{Z}/2\mathbb{Z}$  counts the set of unipotent representations of orthogonal groups.

The main theorem would be a consequence of Proposition 5.1.

5.1.2. *How to think about the factorization of local systems.* When  $\mathcal{O}' \rightsquigarrow \mathcal{O}''$  is the usual descent, the local system  $\mathcal{L}_{\tau}$  could be compared to a water hydra, see in Figure 1.

The tentacles part  $\mathcal{T}_{\tau} = \dagger \mathcal{L}_{\tau'}$  could be reducible. The peduncle part  $\mathcal{P}_{\tau} := \mathcal{L}_{\mathbf{x}_{\tau}}$  and  $\mathcal{L}_{\tau} = \mathcal{T}_{\tau} \cdot \mathcal{P}_{\tau}$ .

When  $\mathcal{O}' \rightsquigarrow \mathcal{O}''$  is a generalized descent, the situation is more complicated as the local system  $\mathcal{L}_{\tau}$  may consists of two hydras, see (5.8).

**5.2. The initial case:** When  $k = 0$ ,  $\mathcal{O} = (C_1 = 2c_1, C_0 = 2c_0)$  has at most two columns. Now  $\pi_{\tau'}$  is the trivial representation of  $\text{Sp}(2c_0, \mathbb{R})$ . The lift  $\pi_{\tau'} \mapsto \bar{\Theta}(\pi_{\tau'})$  is in fact a stable range theta lift. For  $\tau \in \text{DRC}(\mathcal{O})$ , let  $(p_1, q_1) = \text{Sign}(\mathbf{x}_{\tau})$  and  $x_{\tau}$  be the foot of  $\tau$ . It is easy to see that (using associated character formula)

$$\mathcal{L}_{\pi_{\tau}} = \mathcal{T}_{\tau} \cdot \mathcal{P}_{\tau}, \text{ where } \mathcal{T}_{\tau} := \dagger \dagger_{c_0, c_0}, \text{ and } \mathcal{P}_{\tau} := \begin{cases} \dagger_{p_1, q_1} & x_{\tau} \neq d, \\ \dagger_{p_1, q_1} & x_{\tau} = d. \end{cases}$$

Note that  $\mathcal{L}_{\pi_{\tau}}$  is irreducible.

<sup>3</sup>In fact, we only need to know whether  $\mathcal{L}_{\tau}^{-}$  is non-zero or not.

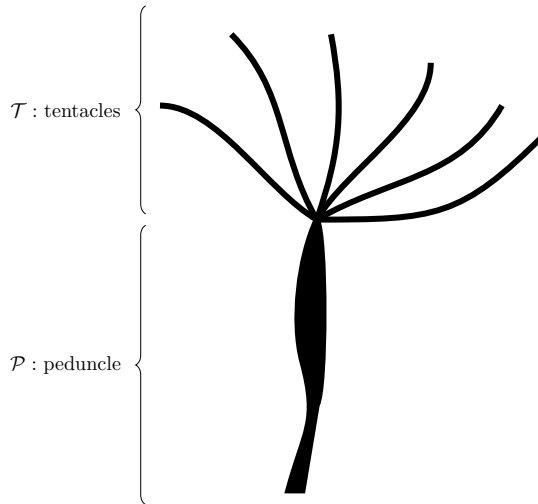


FIGURE 1. Freshwater hydra

Let  $\mathcal{O}_1 = (2(c_1 - c_0)) \in \text{Nil}^{\text{dpe}}(D)$ . Using the above formula, one can check that the following maps are bijections

$$\begin{array}{ccc} \text{DRC}(\mathcal{O}_1) & \longleftarrow & \text{DRC}(\mathcal{O}) \longrightarrow {}^\ell\text{LS}(\mathcal{O}) \\ \mathbf{x}_\tau & \longleftarrow & \tau \longrightarrow \mathcal{L}_\tau. \end{array}$$

To see that  $\text{DRC}(\mathcal{O}) \ni \tau \mapsto \mathcal{L}_\tau$  is an injection, one check the following lemma holds:

**Lemma 5.2.** *The map  $\text{DRC}(\mathcal{O}_1) \longrightarrow \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$  given by  $\tau \mapsto (\text{Sign}(\tau), \epsilon_\tau)$  is injective (see Section 6.2.1 for the definition of  $\epsilon_\tau$ ). Moreover,  $\epsilon_\tau = 0$  only if  $\text{Sign}(\tau) \geq (0, 1)$ .*

Moreover,  $\mathcal{L}_\tau \otimes \det \notin {}^\ell\text{LS}(\mathcal{O})$  for any  $\tau \in \text{DRC}(\mathcal{O})$ . In fact, if  $\text{Sign}(\mathbf{x}_\tau) \geq (1, 0)$ ,  $\mathcal{L}_\tau$  on the 1-rows with +-signs is trivial and  $\mathcal{L}_\tau \otimes \det$  has non-trivial restriction. When  $\text{Sign}(\mathbf{x}_\tau) \not\geq (1, 0)$ ,  $\mathbf{x}_\tau = s \cdots s$ , all 1-rows of  $\mathcal{L}_\tau$  are marked by “=” and all 1-rows of  $\mathcal{L}_\tau \otimes \det$  are marked by “-”.

Therefore our main Theorem 1.4 and main proposition Proposition 5.1 holds for  $\mathcal{O}$ .

**5.3. The descent case.** Now we assume  $k \geq 1$  and  $C_{2k}$  is even. In this case  $\mathcal{O}' \rightsquigarrow \mathcal{O}''$  is a usual descent.

**5.3.1. Unipotent representations attached to  $\mathcal{O}'$ .** Therefore,  $\text{LS}(\mathcal{O}'') \longrightarrow \text{LS}(\mathcal{O}')$  given by  $\mathcal{L} \mapsto \vartheta(\mathcal{L})$  is an injection between abelian groups.

For  $\mathcal{L}'' \in {}^\ell\text{LS}(\mathcal{O}'') \sqcup {}^\ell\text{LS}(\mathcal{O}'') \otimes \det$ , let  $(p_1, q_1) = {}^l\text{Sign}(\mathcal{L})$  and  $(p_2, q_2) = (C_{2k} - q_1, C_{2k} - p_1)$ . Then

$$\vartheta(\mathcal{L}'') = \dagger \mathcal{L}'' \cdot \dagger_{p_2, q_2}.$$

By (5.1), we have a injection

$$(5.4) \quad \begin{array}{ccc} {}^\ell\text{LS}(\mathcal{O}'') \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & {}^\ell\text{LS}(\mathcal{O}') \\ (\mathcal{L}'', \epsilon') & \longmapsto & \vartheta(\mathcal{L}'' \otimes \det^{\epsilon'}). \end{array}$$

In particular, we have  $\mathcal{L}_{\tau'} \neq 0$  and  $\pi_{\tau'} \neq 0$  for every  $\tau' \in \text{DRC}(\mathcal{O}')$ .

**5.3.2. Unipotent representations attached to  $\mathcal{O}$ .** Now suppose  $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$  and  $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$ . By (5.4),  $\epsilon'_1 = \epsilon'_2$  and  $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$ . In particular,  $\tau'_1$  and  $\tau'_2$  have the same signature. By Lemma 3.4 and the induction hypothesis,  $\tau'_1 \neq \tau'_2$  and so  $\pi_{\tau'_1} \otimes \det^{\epsilon'_1} \neq \pi_{\tau'_2} \otimes \det^{\epsilon'_2}$ . Now the injectivity of theta lifting yields

$$\pi_{\tau'_2} = \bar{\Theta}(\pi_{\tau'_1} \otimes \det^{\epsilon'_1}) \neq \bar{\Theta}(\pi_{\tau'_2} \otimes \det^{\epsilon'_2}) = \pi_{\tau'_2}$$

Suppose  $\mathcal{O}'$  is noticed. Then  $\mathcal{O}'' = \bar{\nabla}(\mathcal{O}')$  is noticed by definition and  $(\tau'', \epsilon') \mapsto \text{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$  is an injection into  $\text{LS}(\mathcal{O}'')$ . Now (5.4) implies  $\tau' \mapsto \mathcal{L}_{\tau'}$  is also an injection.

Suppose  $\mathcal{O}'$  is quasi-distinguished. Then  $\mathcal{O}'' = \bar{\nabla}(\mathcal{O}')$  is quasi-distinguished by definition and  $(\tau'', \epsilon') \mapsto \text{Ch}(\pi_{\tau''} \otimes \det^{\epsilon'})$  is a bijection with  $\text{LS}^{\text{aod}}(\mathcal{O}'')$ . Since the component group of each K-nilpotent orbit in  $\mathcal{O}'$  is naturally isomorphic to the component group of its descent. So we deduce that  $\tau' \mapsto \mathcal{L}_{\tau'}$  is a bijection onto  $\text{LS}^{\text{aod}}(\mathcal{O}')$ .

This proves the main proposition for  $\mathcal{O}'$ .

Now let  $\tau \in \text{DRC}(\mathcal{O})$ . By (6.4), we have

$$(5.5) \quad \mathcal{L}_\tau = \mathcal{T}_\tau \cdot \mathcal{P}_\tau \neq 0, \text{ where } \mathcal{T}_\tau := \dagger \mathcal{L}_{\tau'}, \text{ and } \mathcal{P}_\tau := \mathcal{L}_{\mathbf{x}_\tau}.$$

In particular,  $\pi_\tau \neq 0$  and we could determine  $\mathbf{x}_\tau$  from the marks on the 1-rows of  $\mathcal{L}_\tau$ .

Now suppose  $\tau_1 \neq \tau_2$  and  $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ . By (5.5), the result in Section 5.2 and Lemma 3.6 (ii), we have  $\mathbf{x}_{\tau_1} = \mathbf{x}_{\tau_2}$ ,  $\epsilon_1 = \epsilon_2$  and  $\tau'_1 \neq \tau'_2$ . By (5.5), we have  $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$ . Applying the main

proposition of  $\mathcal{O}'$ , we have  $\epsilon'_1 = \epsilon'_2$  and  $\tau'_1 \neq \tau'_2$ . Now by the injectivity of theta lifting, we conclude that

$$\pi_{\tau_2} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (1^{+, -})^{\epsilon_1} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (1^{+, -})^{\epsilon_2} = \pi_{\tau'_2}$$

5.3.3. Note that  $\mathcal{L}_\tau$  is obtained from  $\mathcal{L}_{\tau'}$  by attaching the peduncle determined by  $\mathbf{x}_\tau$ . The claims about noticed and quasi-distinguished orbit are easy to verify. We leave them to the reader.

5.4. **The general descent case.** We assume  $k \geq 1$  and all the properties are satisfied by  $\tau'' \in \text{DRC}(\mathcal{O}'')$ . We retain the notation in Section 3.4.6.

5.4.1. *Local systems.* We now describe the local systems  $\mathcal{L}_{\tau'}$  and  $\mathcal{L}_\tau$  more precisely using our formula of the associated character and the induction hypothesis. Note that these local systems are always non-zero which implies that  $\pi_{\tau'}$  and  $\pi_\tau$  are unipotent representations attached to  $\mathcal{O}'$  and  $\mathcal{O}$  respectively.

First note that in  $\tau'$  and  $\tau''$ ,  $x_{\tau''} \neq s$  by the definition of dot-r-c diagram.<sup>4</sup> By the induction hypothesis,  $\mathcal{L}_{\tau''}^+ \neq 0$ .

Let  $(p_1'', q_1'') := {}^l\text{Sign}(\mathcal{L}_{\tau''})$ . Then

$$(5.6) \quad {}^l\text{Sign}(\mathcal{L}_{\tau''}^+) = (p_0 - 1, q_0), \text{ and } {}^l\text{Sign}(\mathcal{L}_{\tau''}^-) = (p_0, q_0 - 1) \text{ if } \mathcal{L}_{\tau''}^- \neq \emptyset.$$

$$(5.7) \quad \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''}) = \boxtimes^{\frac{|\text{Sign}(\tau'')|}{2}} (\dagger \mathcal{L}_{\tau''}^+ + \dagger \mathcal{L}_{\tau''}^-) \neq 0$$

where  ${}^l\text{Sign}(\dagger \mathcal{L}_{\tau''}^+) = (q_1'', p_1'' - 1)$  and  ${}^l\text{Sign}(\dagger \mathcal{L}_{\tau''}^-) = (q_1'' - 1, p_1'')$  (if  $\dagger \mathcal{L}_{\tau''}^- \neq 0$ ).

Let  $(p_1, q_1) := {}^l\text{Sign}(\mathcal{L}_\tau)$  and  $(e, f) := (p_1 - p_1'' + 1, q_1 - q_1'' + 1)$ . Using Lemma 3.8 one can show that

$$(e, f) = \text{Sign}(\mathbf{u}_\tau).$$

[ It suffice to consider the most left three columns of the peduncle part: this part has signature  ${}^l\text{Sign}(\mathcal{P}_\tau) + (1, 1) = \text{Sign}(\mathbf{u}_\tau x_{\tau''}) = \text{Sign}(\mathbf{u}_\tau) + {}^l\text{Sign}(\mathcal{D}_{\tau''})$ . Therefore,

$$\text{Sign}(\mathbf{u}_\tau) = {}^l\text{Sign}(\mathcal{P}_\tau) - {}^l\text{Sign}(\mathcal{D}_{\tau''}) + (1, 1) = {}^l\text{Sign}(\mathcal{L}_\tau) - {}^l\text{Sign}(\mathcal{L}_{\tau''}) + (1, 1).$$

]

Now

$$(5.8) \quad \mathcal{L}_\tau = \begin{cases} (1^{+, -} \otimes \dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^+) \cdot \mathcal{P}_\tau^+ + (1^{+, -} \otimes \dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^-) \cdot \mathcal{P}_\tau^- & \text{if } x_\tau \neq d \\ (\dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^+) \cdot \mathcal{P}_\tau^+ + (\dagger \boxtimes^t \dagger \mathcal{L}_{\tau''}^-) \cdot \mathcal{P}_\tau^- & \text{if } x_\tau = d \end{cases}$$

where

$$t = \frac{|\text{Sign}(\tau) - \text{Sign}(\tau'')|}{2} = \frac{|\text{Sign}(\mathbf{u}_\tau)|}{2}$$

$$\mathcal{P}_\tau^+ = \begin{cases} \dagger_{e, f-1} & \text{when } f \geq 1 \text{ and } x_\tau \neq d \\ \dagger_{e, f-1} & \text{when } f \geq 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{P}_\tau^- = \begin{cases} \dagger_{e-1, f} & \text{when } e \geq 1 \text{ and } x_\tau \neq d \\ \dagger_{e-1, f} & \text{when } e \geq 1 \text{ and } x_\tau = d \\ 0 & \text{otherwise} \end{cases}.$$

**Lemma 5.3.** *The local system  $\mathcal{L}_\tau \neq 0$ .*

<sup>4</sup>Suppose  $\tau'' \in \text{DRC}(\mathcal{O}'')$  has the shape in (3.4) and  $x_{\tau''} = s$ . By the induction hypothesis,  $\mathcal{L}_{\tau''}^+ = \mathcal{L}_{\tau''}^- = 0$  and so the theta lift of  $\pi_{\tau''}$  vanishes.

*Proof.* Since  $x_{\tau''} \neq s$ ,  $\mathcal{L}_{\tau''}^+ \neq 0$ . When  $\text{Sign}(\mathbf{u}_{\tau}) \geq (0, 1)$ , the first term in (5.8) dose not vanish. Now suppose  $\text{Sign}(\mathbf{u}_{\tau}) \not\geq (0, 1)$ . We have  $x_{\tau''} = d$  by Lemma 3.7 and  $\text{Sign}(\mathbf{u}_{\tau}) > (1, 0)$ . Therefore,  $\mathcal{L}_{\tau''}^- \neq 0$  by the induction hypothesis and the second term in (5.8) dose not vanish. Hence  $\mathcal{L}_{\tau} \neq 0$  in all cases.  $\square$

5.4.2. *Unipotent representations attached to  $\mathcal{O}'$ .* In this section, we let  $\tau' \in \text{DRC}(\mathcal{O}')$  and  $\tau'' = \bar{\nabla}_1(\tau') \in \text{DRC}(\mathcal{O}'')$ . First note that  $x_{\tau''} \neq s$  and so  $\mathcal{L}_{\tau''}^+$  always non-zero.

Recall (5.7). We claim that we can recover  $\mathcal{L}_{\tau''}^+$  and  $\mathcal{L}_{\tau''}^-$  from  $\mathcal{L}_{\tau'}$ :

**Lemma 5.4.** *The map  $\Omega_{\mathcal{O}'}: \mathcal{L}_{\tau'} \mapsto (\mathcal{L}_{\tau''}^+, \mathcal{L}_{\tau''}^-)$  is a well defined map.*

*Proof.* When  ${}^l\text{Sign}(\mathcal{L}_{\tau'})$  has two elements,  $x_{\tau''} = d$  and  ${}^l\text{Sign}(\mathcal{L}_{\tau'}) = \{(q_1'', p_1'' - 1), (q_1'' - 1, p_1'')\}$ . In (5.7),  $\dagger\mathcal{L}_{\tau''}^+$  (resp.  $\dagger\mathcal{L}_{\tau''}^-$ ) consists of components whose first column has signature  $(q_1'', p_1'' - 1)$  (resp.  $(q_1'' - 1, p_1'')$ ). When  ${}^l\text{Sign}(\mathcal{L}_{\tau'})$  has only one elements,  $x_{\tau} = r/c$ ,  ${}^l\text{Sign}(\mathcal{L}_{\tau'}) = \{(q_1'', p_1'' - 1)\}$ ,  $\mathcal{L}_{\tau'} = \dagger\mathcal{L}_{\tau''}^+$  and  $\mathcal{L}_{\tau''}^- = 0$

In any case, we get

$$(5.9) \quad \text{Sign}(\tau'') = (n_0, n_0) + (p_1'', q_1'') \quad \text{where } 2n_0 = |\nabla(\mathcal{O}'')|.$$

Using the signature  $\text{Sign}(\tau'')$ , we could recover the twisting characters in the theta lifting of local system. Therefore  $\mathcal{L}_{\tau''}^+$  and  $\mathcal{L}_{\tau''}^-$  can be recovered from  $\mathcal{L}_{\tau'}$ .  $\square$

**Lemma 5.5.** *Suppose  $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$ . Then  $\pi_{\tau'_1} \neq \pi_{\tau'_2}$ .*

*Proof.* It suffice to consider the case when  $\mathcal{L}_{\tau'_1} = \mathcal{L}_{\tau'_2}$ . By (5.9) in the proof of Lemma 5.4,  $\text{Sign}(\tau'_1) = \text{Sign}(\tau'_2)$ . On the other hand,  $\epsilon_1 = \epsilon_2 = 0$  and  $\tau'_1 \neq \tau'_2$  by Lemma 3.5. So

$$\pi_{\tau'_1} = \bar{\Theta}(\pi_{\tau'_1}) \neq \bar{\Theta}(\pi_{\tau'_2}) = \pi_{\tau'_2}$$

by the injectivity of theta lift.  $\square$

5.4.3. *Unipotent representations attached to  $\mathcal{O}$ .*

**Lemma 5.6.** *Suppose  $\tau_1 \neq \tau_2 \in \text{DRC}(\mathcal{O})$ . Then  $\pi_{\tau_1} \neq \pi_{\tau_2}$ .*

*Proof.* It suffice to consider the case that  $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ . Clearly,  $\text{Sign}(\tau_1) = \text{Sign}(\tau_2)$ . On the other hand, the twisting  $\epsilon_{\tau_1} = \epsilon_{\tau_2}$  since it is determined by the local system:  $\epsilon_{\tau_i} = 0$  if and only if  $\mathcal{L}_{\tau_i} \supset -$ .

We conclude that  $\tau'_1 \neq \tau'_2$  and  $\tau'_1 \neq \tau'_2$  by Lemma 3.8 and Lemma 3.5. By Lemma 5.5 and the injectivity of theta lift,

$$\pi_{\tau_1} = \bar{\Theta}(\pi_{\tau'_1}) \otimes (\mathbf{1}^{+, -})^{\epsilon_{\tau_1}} \neq \bar{\Theta}(\pi_{\tau'_2}) \otimes (\mathbf{1}^{+, -})^{\epsilon_{\tau_2}} = \pi_{\tau_2}$$

$\square$

5.4.4. *Noticed orbits.* In this section, we prove the claims about noticed and +noticed orbits.

**Lemma 5.7.** *Suppose  $\mathcal{O}'$  is noticed. Then*

(i) *The map  $\{\mathcal{L}_{\tau''} \mid x_{\tau''} \neq s\} \longrightarrow \{\mathcal{L}_{\tau'}\} = {}^{\ell}\text{LS}(\mathcal{O}')$  given by  $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau'} = \vartheta(\mathcal{L}_{\tau''})$  is a bijection.*

(ii) *The map  $\text{DRC}(\mathcal{O}') \rightarrow {}^{\ell}\text{LS}(\mathcal{O}')$  given by  $\tau' \mapsto \mathcal{L}_{\tau'}$  is a bijection.*

*Proof.* (i) Note that  $\mathcal{O}''$  is noticed by definition. Hence  $\Upsilon_{\mathcal{O}''}$  is invertible. Recall Lemma 5.4, we see that

$$(\Upsilon_{\mathcal{O}''})^{-1} \circ \Omega_{\mathcal{O}'}: \mathcal{L}_{\tau'} \mapsto (\mathcal{L}_{\tau''}^+, \mathcal{L}_{\tau''}^-) \mapsto \mathcal{L}_{\tau''}$$

gives the inverse of the map in the claim.

(ii) Suppose  $\tau'_1 \neq \tau'_2 \in \text{DRC}(\mathcal{O}')$ . By Lemma 3.5,  $\tau''_1 \neq \tau''_2$ . Hence  $\mathcal{L}_{\tau'_1} \neq \mathcal{L}_{\tau'_2}$  by the induction hypothesis. Therefore,  $\mathcal{L}_{\tau'_1} \neq \mathcal{L}_{\tau'_2}$  by (i) of the claim.  $\square$

**Lemma 5.8.** *Suppose  $\mathcal{O}$  is noticed,  $\mathcal{L}: \text{DRC}(\mathcal{O}) \mapsto {}^\ell\text{LS}(\mathcal{O})$  given by  $\tau \mapsto \mathcal{L}_\tau$  is bijective.*

*Proof.* We prove the claim by contradiction. We assume  $\tau_1 \neq \tau_2$  such that  $\mathcal{L}_{\tau_1} = \mathcal{L}_{\tau_2}$ .

Recall (5.8): there is an pair of non-negative integers  $(e_i, f_i) = \text{Sign}(\mathbf{u}_{\tau_i})$  such that

$$\mathcal{L}_{\tau_i} = \dagger \dagger \mathcal{L}_{\tau''_i}^+ \cdot \mathcal{P}_{\tau_i}^+ + \dagger \dagger \mathcal{L}_{\tau''_i}^- \cdot \mathcal{P}_{\tau_i}^-.$$

(i) Suppose that  $\mathcal{L}_{\tau_i}$  contains a 1-row marked by  $-$  or  $=$ . Then we have  $\epsilon_{\tau_1} = \epsilon_{\tau_2}$ , which can be read from the mark  $-/ =$ . Moreover, the term  $\dagger \dagger \mathcal{L}_{\tau''_i}^+ \cdot \mathcal{P}_{\tau_i}^+ \neq 0$  and we can recover it from  $\mathcal{L}_{\tau_i}$ . So  $\mathcal{L}_{\tau''_1}^+ = \mathcal{L}_{\tau''_2}^+$ . Since  $\mathcal{O}''$  is +noticed,  $\tau''_1 = \tau''_2$  by (5.10) and the induction hypothesis. Note that  $\text{Sign}(\tau_1) = \text{Sign}(\tau_2)$ , we get  $\tau_1 = \tau_2$  by Lemma 3.8 which is contradict to our assumption.

(ii) Now we assume that  $\mathcal{L}_{\tau_i}$  does not contains a 1-row marked by  $-/ =$ . By (5.8), we conclude that  $x_{\tau''} \neq d$  and  $\mathbf{u}_\tau = r \cdots r$  or  $r \cdots rc$ . Therefore  $\mathbf{p}_{\tau_i}$  must be one of the following form by Lemma 3.7:

$$\mathbf{p}_1 = \begin{array}{c} r \ c \\ \vdots \\ r \\ r \end{array}, \quad \mathbf{p}_2 = \begin{array}{c} r \ c \\ \vdots \\ r \\ c \end{array}, \quad \text{and} \quad \mathbf{p}_3 = \begin{array}{c} r \ d \\ \vdots \\ r \\ r \end{array}$$

We consider case by case according to the set  $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\}$ :

- (a) Suppose  $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1\}$  or  $\{\mathbf{p}_2\}$ . In these cases,  $\epsilon_{\tau_1} = \epsilon_{\tau_2}$  and  $\tau''_1 \neq \tau''_2$  by Lemma 3.8. On the other hand,  $\mathcal{L}_{\tau''_1}^+ = \mathcal{L}_{\tau''_2}^+$  by (5.8). Since  $\mathcal{O}''$  is +noticed, we get  $\tau''_1 = \tau''_2$  a contradiction.
- (b) Suppose  $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_3\}$ . We still have  $\epsilon_{\tau_1} = \epsilon_{\tau_2}$  and  $\tau''_1 \neq \tau''_2$  by Lemma 3.8. On the other hand,  $\mathcal{L}_{\tau''_1}^- = \mathcal{L}_{\tau''_2}^-$  by (5.8). This again contradict to the injectivity of  $\mathcal{L}_{\tau''} \mapsto \mathcal{L}_{\tau''}^-$ .
- (c) Suppose  $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{\mathbf{p}_1, \mathbf{p}_2\}$ . By the argument above, we have  $\mathcal{L}_{\tau''_1}^+ = \mathcal{L}_{\tau''_2}^+$ . Now  $\tau''_1 \neq \tau''_2$  since  $\{x_{\tau''_1}, x_{\tau''_2}\} = \{r, c\}$ . This contradict to that  $\mathcal{O}''$  is +noticed again.
- (d) Suppose  $\mathbf{p}_{\tau_1} = \mathbf{p}_1$  or  $\mathbf{p}_2, \mathbf{p}_{\tau_2} = \mathbf{p}_3$ . Then

$$\mathcal{L}_{\tau_1} = \dagger \boxtimes^{\frac{|\mathbf{u}_{\tau_1}|}{2}} \dagger \mathcal{L}_{\tau''_1}^+ \cdot \mathcal{P}_{\tau_1}^+ \quad \text{and} \quad \mathcal{L}_{\tau_2} = \dagger \boxtimes^{\frac{|\mathbf{u}_{\tau_2}|}{2}} \dagger \mathcal{L}_{\tau''_2}^- \cdot \mathcal{P}_{\tau_2}^-.$$

This implies  $|\text{Sign}(\tau''_1)| \equiv |\text{Sign}(\tau''_2)| + 2 \pmod{4}$  and  $\boxtimes \dagger \mathcal{L}_{\tau''_1}^+ = \dagger \mathcal{L}_{\tau''_2}^-$ .

**Claim 5.9.** *We have  $\mathcal{L}_{\tau''_2}^- \supset +$ .*

*Proof.* If  $\mathcal{O}''$  is obtained by the usual descent, we have  $\mathcal{L}_\tau \geq \begin{smallmatrix} - \\ + \end{smallmatrix}$  and we are done.

Now assume  $\mathcal{O}''$  is obtained by generalized descent and  $\mathcal{O}''$  is +noticed, i.e.  $\mathbf{u}_{\tau''}$  has at least length 2. Suppose  $\text{Sign} \mathbf{u}_{\tau''} > (1, 2)$ . Applying (5.8) to  $\tau''$ , we see that the first term is non-vanishing and  $\mathcal{L}_{\tau''} \supseteq \dagger_{(1,1)}$ .

Otherwise,  $\text{Sign}(\mathbf{u}_{\tau''}) = (2n_0 - 1, 1) \geq (3, 1)$  where  $n_0$  is the length of  $\mathbf{u}_{\tau''}$ . By (5.8), the second term is non-vanishing and  $\mathcal{L}_{\tau''} > \dagger_{(2n_0-1,1)} > +$ .  $\square$

The claim leads to a contradiction: Thanks to the character twist in the theta lifting formula of the local system, we see that the associated character restricted on the 2-row  $- +$  of  $\boxtimes \dagger \mathcal{L}_{\tau''_1}^+$  and  $\dagger \mathcal{L}_{\tau''_2}^-$  must be are different, a contradiction.

We finished the proof of the lemma.  $\square$



5.4.5. The maps  $\Upsilon_{\mathcal{O}}^+$ ,  $\Upsilon_{\mathcal{O}}^-$  and  $\Upsilon_{\mathcal{O}}$ .

**Lemma 5.10.** *Suppose  $\mathcal{O}$  is +noticed. The map  $\Upsilon_{\mathcal{O}}^+ : \{\mathcal{L}_{\tau} \mid \mathcal{L}_{\tau}^+ \neq 0\} \rightarrow \{\mathcal{L}_{\tau}^+ \neq 0\}$  is injective.*

*Proof.* We have two cases:

(i)  $\mathcal{L}_{\tau}$  contain an irreducible component  $> \begin{smallmatrix} + \\ + \end{smallmatrix}$ . This is equivalent to  $\mathcal{L}_{\tau}^+$  has an irreducible component  $> \begin{smallmatrix} + \\ + \end{smallmatrix}$ . Now all irreducible components of  $\mathcal{L}_{\tau} > \begin{smallmatrix} + \\ + \end{smallmatrix}$  and  $\mathcal{L}_{\tau} \mapsto \mathcal{L}_{\tau}^+$  will not kill any irreducible components. Hence we could recover  $\mathcal{L}_{\tau}$  from  $\mathcal{L}_{\tau}^+$ .

(ii) Suppose that  $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$  do not contain a component  $> \begin{smallmatrix} + \\ + \end{smallmatrix}$ .

By Lemma 5.8  $\tau_1 \neq \tau_2$ .<sup>5</sup> We now show that  $\mathcal{L}_{\tau_1}^+ = \mathcal{L}_{\tau_2}^+ \neq 0$  leads to a contradiction. Clearly,  $\text{Sign}(\tau_1) = \text{Sign}(\tau_2)$ . By (5.8) and checking the properties in Lemma 3.7, we have

$$\mathbf{p}_{\tau_i} = \begin{matrix} s & x_{\tau''} \\ \vdots & \\ s & \end{matrix} \quad \text{where } x_{\tau''} = c/d, x_{\tau_i} = c/d.$$

On the other hand, when  $\mathbf{p}_{\tau_i}$  have the above form, the first term of (5.8) is non-vanishing. So we conclude that  $\mathcal{L}_{\tau_1}^+ = \mathcal{L}_{\tau_2}^+$  implies

$$(5.10) \quad \mathcal{L}_{\tau_1''}^+ = \mathcal{L}_{\tau_2''}^+.$$

Moreover, we see that every components of  $\mathcal{L}_{\tau_i}$  has a 1-row of mark “- / =” and we can determine  $\epsilon_{\tau_i}$  by the mark - / = of the 1-row appeared in  $\mathcal{L}_{\tau_i}^+$ . In particular, we have  $\epsilon_1 = \epsilon_2$ . Now Lemma 3.8 implies  $\tau_1'' \neq \tau_2''$ . This is contradict to (5.10) and our induction hypothesis since  $\mathcal{O}''$  is also +noticed. □

**Lemma 5.11.** *Suppose  $\mathcal{O}$  is noticed. Then the map  $\Upsilon_{\mathcal{O}} : \mathcal{L}_{\tau} \mapsto (\mathcal{L}_{\tau}^+, \mathcal{L}_{\tau}^-)$  is injective.*

*Proof.* It suffice to consider the case where  $\mathcal{O}$  is noticed but not +noticed, i.e.  $C_{2k+1} = C_{2k} + 1$ . In this case,  $n = 1$ . The peduncle  $\mathbf{p}_{\tau}$  have four possible cases:

$$\begin{matrix} r & c, & c & c, & c & d, & \text{or} & d & d. \end{matrix}$$

By (5.8),  $\mathcal{L}_{\tau}^+ = \dagger \dagger \mathcal{L}_{\tau''}^+$ .

Now suppose  $\tau_1 \neq \tau_2$  such that  $\mathcal{L}_{\tau_1}^+ = \mathcal{L}_{\tau_2}^+$ . Since  $\mathcal{O}''$  is +noticed, we have  $\tau_1'' = \tau_2''$  by the induction hypothesis (see Lemma 5.10). By Lemma 3.8, this only happens when  $\{\mathbf{p}_{\tau_1}, \mathbf{p}_{\tau_2}\} = \{ \begin{smallmatrix} c & d \\ d & d \end{smallmatrix} \}$ . Since one of  $\mathcal{L}_{\tau_i}^-$  is zero and the other is non-zero, we conclude that  $\Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_1}) \neq \Upsilon_{\mathcal{O}}(\mathcal{L}_{\tau_2})$ . □

**Lemma 5.12.** *Suppose  $\mathcal{O}$  is +noticed. The map  $\Upsilon_{\mathcal{O}}^- : \{\mathcal{L}_{\tau} \mid \mathcal{L}_{\tau}^- \neq 0\} \rightarrow \{\mathcal{L}_{\tau}^- \neq 0\}$  is injective.*

*Proof.* The proof is similar to that of Lemma 5.10. We have two cases:

(i)  $\mathcal{L}_{\tau}$  contain an irreducible component  $> \begin{smallmatrix} - \\ - \end{smallmatrix}$ . This is equivalent to  $\mathcal{L}_{\tau}^-$  has an irreducible component  $> \begin{smallmatrix} - \\ - \end{smallmatrix}$  and  $\mathcal{L}_{\tau} \mapsto \mathcal{L}_{\tau}^-$  will not kill any irreducible components. Hence we could recover  $\mathcal{L}_{\tau}$  from  $\mathcal{L}_{\tau}^-$ .

(ii) Now we make a weaker assumption that  $\tau_1 \neq \tau_2 \in \text{DRC}(\mathcal{O})$  such that  $\mathcal{L}_{\tau_1}$  and  $\mathcal{L}_{\tau_2}$  do not contain a component  $> \begin{smallmatrix} - \\ - \end{smallmatrix}$ .

---

<sup>5</sup> $\mathcal{L}_{\tau_1} \neq \mathcal{L}_{\tau_2}$  clearly implies  $\tau_1 \neq \tau_2$ .

By (5.8) and the properties in Lemma 3.7, we see that  $\mathbf{p}_{\tau_i}$  must be one of the following

$$\mathbf{p}_1 = \begin{array}{c} r \\ \vdots \\ r \\ d \end{array} \quad \text{or} \quad \mathbf{p}_2 = \begin{array}{c} r \\ \vdots \\ r \\ d \end{array}.$$

Without loss of generality, we have the following possibilities:

- (a)  $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_1$ . We have  $\mathcal{L}_{\tau_1''}^+ = \mathcal{L}_{\tau_2''}^+$  and obtain the contradiction.
- (b)  $\mathbf{p}_{\tau_1} = \mathbf{p}_{\tau_2} = \mathbf{p}_2$ . We have  $\mathcal{L}_{\tau_1''}^- = \mathcal{L}_{\tau_2''}^-$  and obtain the contradiction.
- (c)  $\mathbf{p}_{\tau_1} = \mathbf{p}_1$  and  $\mathbf{p}_{\tau_2} = \mathbf{p}_2$ . Now we apply the same argument in the case (d) of the proof of Lemma 5.8. We get  $\boxtimes \dagger \mathcal{L}_{\tau_1''}^+ = \dagger \mathcal{L}_{\tau_2''}^-$ ,  $\mathcal{L}_{\tau_2''}^- \supset +$  and a contradiction by looking at the associated character on the 2-row  $- +$ .

This finished the proof.  $\square$

## 6. TYPE B/M CASE

### 6.1. “Special” and “non-special” diagrams.

#### 6.1.1. type M.

**Definition 6.1.** Let  $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_1, C_0 = 0) \in \text{Nil}^{\text{dpe}}(M)$  with  $k \geq 1$ . Let

$$\tau = (\tau_{2k}, \dots, \tau_2) \times (\tau_{2k-1}, \dots, \tau_1).$$

be a Weyl group representation attached to  $\mathcal{O} \in \text{Nil}^{\text{dpe}}(M)$ . We say  $\tau$  has

- special shape if  $\tau_{2k} \geq \tau_{2k-1}$ ;
- non-special shape if  $\tau_{2k} < \tau_{2k-1}$ .

Clearly, this gives a partition of the set of Weyl group representations attached to  $\mathcal{O}$ .

Suppose  $k \geq 1$ ,  $C_{2k} = 2c_{2k} - 1$  and  $C_{2k-1} = 2c_{2k-1} + 1$ . The special shape representation  $\tau^s$

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k-1}, \tau_{2k-3}, \dots, \tau_1).$$

is paired with the non-special shape representation

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k-1}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k}, \tau_{2k-3}, \dots, \tau_1)$$

In the pairing  $\tau^s \leftrightarrow \tau^{ns}$ ,  $\tau_j$  is unchanged for  $j \leq 2k - 2$ . We call  $\tau^s$  the special shape and  $\tau^{ns}$  the corresponding non-special shape

**Definition 6.2.** Suppose  $C_{2k}$  is odd. We retain the notation in Definition 6.1 where the shapes  $\tau^s$  and  $\tau^{ns}$  correspond with each other. We define a bijection between  $\text{DRC}(\tau^s)$  and  $\text{DRC}(\tau^{ns})$ :

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \longleftrightarrow & \text{DRC}(\tau^{ns}) \\ \tau^s & \longleftrightarrow & \tau^{ns} \end{array}$$

Without loss of generality, we can assume that  $\tau^s \in \text{DRC}(\tau^s)$  and  $\tau^{ns} \in \text{DRC}(\tau^{ns})$  have the following shapes:

$$(6.1) \quad \tau^s : \begin{array}{c} \dots\dots\dots \\ y_1 \dots\dots x_0 \dots\dots \\ \begin{array}{|c|} \hline s \\ \hline \vdots \\ \hline s \\ \hline \end{array} \\ y_2 \end{array} \times \longleftrightarrow \tau^{ns} : \begin{array}{c} \dots\dots\dots \\ y_0 \dots\dots x_1 \dots\dots \\ \begin{array}{|c|} \hline r \\ \hline \vdots \\ \hline r \\ \hline \end{array} \\ x_2 \end{array}$$

Here the grey parts have length  $(C_{2k} - C_{2k-1})/2$  (could be zero length). The entries  $x_0, x_1, y_0$  and  $y_1$  are either all empty or all non-empty. The correspondence between  $(x_0, x_1, x_2)$ , with  $(y_0, y_1, y_2)$  is given by switching  $r$  and  $s$ ,  $d$  and  $c$ . i.e.

$$\begin{cases} y_i = s \Leftrightarrow x_i = r \\ y_i = c \Leftrightarrow x_i = d \end{cases} \quad \text{for } i = 0, 1, 2.$$

We define

$$\text{DRC}^s(\mathcal{O}) := \bigsqcup_{\tau^s} \text{DRC}(\tau^s) \quad \text{and} \quad \text{DRC}^{ns}(\mathcal{O}) := \bigsqcup_{\tau^{ns}} \text{DRC}(\tau^{ns})$$

where  $\tau^s$  (resp.  $\tau^{ns}$ ) runs over all special (resp. non-special) shape representations attached to  $\mathcal{O}$ . Clearly  $\text{DRC}(\mathcal{O}) = \text{DRC}^s(\mathcal{O}) \sqcup \text{DRC}^{ns}(\mathcal{O})$ .

It is easy to check the bijectivity in the above definition, which we leave it to the reader.

6.1.2. *Special and non-special shapes of type B.* Now we define the notion of special and non-special shape of type B.

**Definition 6.3.** *Let*

$$\tau = (\tau_{2k}, \dots, \tau_2) \times (\tau_{2k+1}, \tau_{2k-1}, \dots, \tau_1)$$

be a Weyl group representation attached to  $\mathcal{O} \in \text{Nil}^{\text{dpe}}(B)$ . We say  $\tau$  has

- special shape if  $\tau_{2k} \geq \tau_{2k-1}$ ;
- non-special shape if  $\tau_{2k} < \tau_{2k-1}$ .

Suppose  $k \geq 1$  and

$$(6.2) \quad \mathcal{O} = (C_{2k+1} = 2c_{2k+1} + 1, C_{2k} = 2c_{2k} - 1, C_{2k-1} = 2c_{2k-1} + 1, \dots, C_1 > 0, C_0 = 0).$$

A representation  $\tau^s$  attached to  $\mathcal{O}$  has the shape

$$\tau^s = \tau_L^s \times \tau_R^s = (c_{2k}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k+1}, c_{2k-1}, \tau_{2k-3}, \dots, \tau_1)$$

is paired with

$$\tau^{ns} = \tau_L^{ns} \times \tau_R^{ns} = (c_{2k-1}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k+1}, c_{2k}, \tau_{2k-3}, \dots, \tau_1).$$

In the pairing  $\tau^s \leftrightarrow \tau^{ns}$  described as the above,  $\tau_j$  are unchanged for  $j \leq 2k - 2$ .

## 6.2. Definition of $\overline{\nabla}$ for type B and M.

6.2.1. *The definition of  $\epsilon$ .*

- (1). When  $\tau \in \text{DRC}(B)$ ,  $\epsilon$  is determined by the “basal disk” of  $\tau$  (see (6.3) for the definition of  $x_\tau$ ):

$$\epsilon_\tau := \begin{cases} 0, & \text{if } x_\tau = d; \\ 1, & \text{otherwise.} \end{cases}$$

- (2). When  $\tau \in \text{DRC}(M)$ , the  $\epsilon$  is determined by the lengths of  $\tau_{L,0}$  and  $\tau_{R,0}$ :

$$\epsilon_\tau := \begin{cases} 0, & \text{if } |\tau_{L,0}| - |\tau_{R,0}| \geq 0; \\ 1, & \text{if } |\tau_{L,0}| - |\tau_{R,0}| < 0. \end{cases}$$

6.2.2. *Initial cases.*

- (3). Suppose  $\mathcal{O} = (2c_1 + 1)$  has only one column. Then  $\mathcal{O}'_0 := \overline{\nabla}(\mathcal{O})$  is the trivial orbit of  $\text{Mp}(0, \mathbb{R})$ .  $\text{DRC}(\mathcal{O})$  consists of diagrams of shape  $\tau_L \times \tau_R = \emptyset \times (c_1, )$ . The set  $\text{DRC}(\mathcal{O}'_0)$  is a singleton  $\{\tau'_0 = \emptyset \times \emptyset\}$ , and every element  $\tau \in \text{DRC}(\mathcal{O})$  maps to  $\tau_0$  (see (2.1)):

$$\text{DRC}(\mathcal{O}) \ni \tau := \begin{array}{c} \emptyset \\ \times \\ x_1 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau'_0 := \emptyset \times \emptyset$$

Here  $m_\tau = a$  or  $b$  is the mark of  $\tau$ . We define

$$\mathbf{p}_\tau := \mathbf{x}_\tau := x_1 \cdots x_n m_\tau.$$

We call  $\mathbf{p}_\tau = \mathbf{x}_\tau$  the “peduncle” part of  $\tau$ .

6.2.3. *The descent from M to B.* The descent of a special shape diagram is simple, the descent of a non-special shape reduces to the corresponding special one:

- (4). Suppose  $|\tau_{L,0}| \geq |\tau_{R,0}|$ , i.e.  $\tau$  has special shape. Keep  $r, d$  unchanged, delete  $\tau_{L,0}$  and fill the remaining part with “ $\bullet$ ” and “ $s$ ” by  $\overline{\nabla}_R(\tau^\bullet)$  using the dot-s switching algorithm. The mark  $m_{\tau'}$  of  $\tau' := \overline{\nabla}_1(\tau)$  is given by the following formula:

$$m_{\tau'} := \begin{cases} a & \text{if } \tau_{L,0} \text{ ends with } s \text{ or } \bullet, \\ b & \text{if } \tau_{L,0} \text{ ends with } c. \end{cases}$$

- (5). Suppose  $|\tau_{L,0}| < |\tau_{R,0}|$ , i.e.  $\tau$  has non-special shape. We define

$$\overline{\nabla}_1(\tau) := \overline{\nabla}_1(\tau^s)$$

where  $\tau^s$  is the special diagram corresponding to  $\tau$  defined in Definition 6.2.

**Lemma 6.4.** *Suppose  $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$  with  $k \geq 1$  and  $C_{2k}$  is odd. Let  $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1}, \dots, C_0)$ . Then*

$$\begin{array}{ccc} \text{DRC}^s(\mathcal{O}) & \xrightarrow{\overline{\nabla}_1} & \text{DRC}(\mathcal{O}') \\ \text{DRC}^{ns}(\mathcal{O}) & \xrightarrow{\overline{\nabla}_1} & \text{DRC}(\mathcal{O}') \end{array}$$

are a bijections.

*Proof.* The claim for  $\text{DRC}^s(\mathcal{O})$  is clear by the definition of descent. The claim for  $\text{DRC}^{ns}(\mathcal{O})$  reduces to that of  $\text{DRC}^s(\mathcal{O})$  using Definition 6.2.  $\square$

**Lemma 6.5.** *Suppose  $\mathcal{O} = (C_{2k}, C_{2k-1}, \dots, C_0)$  with  $k \geq 1$  and  $C_{2k}$  even. Let  $\mathcal{O}' := \overline{\nabla}(\mathcal{O}) = (C_{2k-1} + 1, \dots, C_0)$ . Then the following map is a bijection*

$$\text{DRC}(\mathcal{O}) \xrightarrow{\overline{\nabla}_1} \left\{ \tau' \in \text{DRC}(\mathcal{O}') \mid \begin{array}{l} m_{\tau'} \neq b \text{ and} \\ \tau'_{R,0} \text{ dose not ends with } s. \end{array} \right\}.$$

*Proof.* This is clear by the descent algorithm.  $\square$

6.2.4. *Descent from B to M.* Now we define the general case of the descent from type B to type M. We assume that  $k \geq 1$  i.e.  $\mathcal{O}$  has at least 3 columns. First note that the shape of  $\tau' = \overline{\nabla}(\tau)$  is the shape of  $\tau$  deleting the most left column on the right diagram.

The definition splits in cases below. In all these cases we define

$$(6.3) \quad \tilde{\mathbf{x}}_\tau := x_1 \cdots x_n m_\tau \text{ and } \tilde{x}_\tau := x_n$$

which is marked by  $s/r/c/d$ . For the part marked by  $*/\cdots$ ,  $\overline{\nabla}$  keeps  $r, c, d$  and maps the rest part consisting of  $\bullet$  and  $s$  by dot-s switching algorithm.

- (6). When  $C_{2k}$  is odd and  $\tau$  has special shape, the descent is given by the following diagram

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ \bullet \\ \vdots \\ \bullet \\ y_0 \end{array} \times \begin{array}{c} \dots\dots\dots \\ * * \dots\dots \\ \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ s \\ \vdots \\ s \\ y'_0 \end{array} \times$$

Here the grey columns has length  $(C_{2k} - C_{2k-1})/2$  and  $n = (C_{2k+1} - C_{2k})/2$ . The “\*/...” part of  $\tau'$  is given by keeping the corresponding entries marked by  $r/d/c$  in  $\tau$  unchange and filling  $s/\bullet$  accordingly in the rest of the entries. The entry  $y'_0$  of  $\tau'$  is given by the following formula:

$$y'_0 := \begin{cases} s & \text{if } x_1 = \bullet \text{ or } (x_1, m_\tau) = (r/d, a) \\ c & \text{if } x_1 = s \text{ or } (x_1, m_\tau) = (r/d, b). \end{cases}$$

- (7). When  $C_{2k}$  is odd and  $\tau$  has non-special shape, then

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ s \\ \vdots \\ s \\ x_1 \end{array} \times \begin{array}{c} \dots\dots\dots \\ * * \dots\dots \\ s \ r \\ \vdots \\ s \ r \\ x_1 \ x_0 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ r \\ \vdots \\ r \\ x'_0 \end{array} \times$$

Here the grey columns has length  $(C_{2k} - C_{2k-1})/2$  and  $n = (C_{2k+1} - C_{2k})/2$ . The “\*/...” part of  $\tau'$  is given by keeping the corresponding entries marked by  $r/d/c$  in  $\tau$  unchange and filling  $s/\bullet$  accordingly in the rest of the entries. The entry  $x'_0$  of  $\tau'$  is given by the following formula:

$$x'_0 := \begin{cases} r & \text{if } (x_1, m_\tau) = (r/d, a), \\ d & \text{if } (x_1, m_\tau) = (r/d, b), \\ x_0 & \text{if } x_1 = s. \end{cases}$$

Note that

- (8). When  $C_{2k} = C_{2k-1}$  is even, we could assume  $\tau$  and  $\tau'$  have the following forms with  $n = (C_{2k+1} - C_{2k} + 1)/2$ .

$$\tau : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ y_0 \dots \end{array} \times \begin{array}{c} \dots\dots\dots \\ * * \dots\dots \\ x_1 \ x_0 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ y'_0 \dots \end{array} \times \begin{array}{c} \dots\dots\dots \\ * \dots\dots \\ x'_0 \end{array}$$

In the most of the case,  $\tau'$  is obtained from  $\tau$  by deleting the first column of  $\tau_R$ , keeping  $c/r/d$  and filling  $s/\bullet$  accordingly. The exceptional cases are when  $x_1 = r/d$ . In the exceptional case we always have  $x_0 = d$ . We define

$$x'_0 := d, \\ y'_0 := \begin{cases} s & \text{if } m_\tau = a, \\ c & \text{if } m_\tau = b. \end{cases}$$

Note that this definition is similar to that of the special shape diagrams in the descent case. We define the “peduncle” of  $\tau$  to be

$$\mathbf{p}_\tau = \begin{matrix} x_1 & x_0 \\ \vdots & \\ x_n & \\ m_\tau & \end{matrix}.$$

In all the cases, let  $\mathcal{O}_1 = (2n, )$  is the trivial nilpotent of type D. For each  $\tau \in \text{DRC}(\mathcal{O})$ , we define  $\mathbf{x}_\tau \in \text{DRC}(\mathcal{O}_1)$  by require the following identity of multisets:

$$\{\text{entry of } \mathbf{x}_\tau\} = \begin{cases} \{c, x_2, \dots, x_n\} & \text{if } (x_1, m_\tau) = (\bullet/s, a), \\ \{s, x_2, \dots, x_n\} & \text{if } (x_1, m_\tau) = (\bullet/s, b), \\ \{x_1, x_2, \dots, x_n\} & \text{otherwise } (x_1 = r/d). \end{cases}$$

In the above definition we include the initial case where  $\mathcal{O}$  is a trivial orbit of type B. We define  $\mathbf{x}_\tau = \emptyset$  when  $\mathcal{O}$  is the trivial orbit of  $\text{O}(1, \mathbb{C})$ .

We define the “basal disk” of  $\tau$  as the following:

$$x_\tau := \begin{cases} \mathbf{x}_\tau & \text{if } C_{2k+1} = C_{2k} + 1. \\ c & \text{if } x_n m_\tau = sa \text{ or } \bullet a \\ x_n & \text{otherwise.} \end{cases}$$

6.2.5. *A key proposition for descent case.* Now we assume  $\mathcal{O}$  is given by (6.2) Let  $\mathcal{O}' = \overline{\nabla}(\mathcal{O})$  and  $\mathcal{O}'' = \overline{\nabla}(\mathcal{O}')$ .

The following lemma is the key property satisfied by our definition

**Lemma 6.6.** *Suppose  $k \geq 1$  and  $\tau^s$  and  $\tau^{ns}$  are two representations attached to  $\mathcal{O}$  as in Definition 6.3. Then*

i) *For every  $\tau \in \text{DRC}(\tau^s) \sqcup \text{DRC}(\tau^{ns})$ , the shape of  $\overline{\nabla}^2(\tau)$  is*

$$\tau'' = (\tau_{2k-2}, \dots, \tau_2) \times (c_{2k-1}, \tau_{2k-3}, \tau_1)$$

ii) *We define  $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$  by  $\delta(\tau) = (\overline{\nabla}^2(\tau), \mathbf{x}_\tau)$ . The following maps are bijective*

$$\begin{array}{ccc} \text{DRC}(\tau^s) & \xrightarrow{\delta} & \text{DRC}(\tau'') \times \text{DRC}(\mathcal{O}_1) \xleftarrow{\delta} \text{DRC}(\tau^{ns}) \\ \tau^s & \longmapsto & (\overline{\nabla}^2(\tau^s), \mathbf{x}_{\tau^s}) \\ & & (\overline{\nabla}^2(\tau^{ns}), \mathbf{x}_{\tau^{ns}}) \longleftarrow \tau^{ns} \end{array}$$

*In particular, we obtain an one-one correspondence  $\tau^s \leftrightarrow \tau^{ns}$  such that  $\delta(\tau^s) = \delta(\tau^{ns})$ .*

iii) *Suppose  $\tau^s$  and  $\tau^{ns}$  correspond as the above such that  $\delta(\tau^s) = \delta(\tau^{ns}) = (\tau'', \mathbf{x})$ . Then*

$$(6.4) \quad \text{Sign}(\tau^s) = \text{Sign}(\tau^{ns}) = (C_{2k}, C_{2k}) + \text{Sign}(\tau'') + \text{Sign}(\mathbf{x}).$$

*Note that  $\text{Sign}(\mathbf{x}) \in \mathbb{N} \times \mathbb{N}$ .*

*Proof.* By our assumption, we have  $c_{2k+1} \geq c_{2k} > c_{2k-1}$  and

$$\tau^s = (c_{2k}, \tau_{2k-2}, \dots, \tau_2) \times (c_{2k+1}, c_{2k-1}).$$

The behavior of  $\tau^s$  under the descent map  $\overline{\nabla}$  is illustrated as the following:

$$(6.5) \quad \tau^s : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ \bullet \\ \vdots \\ \bullet \\ x_0 \end{array} \times \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ \bullet \\ \vdots \\ \bullet \\ x_1 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau'^s : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ s \\ \vdots \\ s \\ x'_0 \end{array} \times \cdots \mapsto \tau'' : \begin{array}{c} \text{.....} \\ * \text{ } \cdots \\ \cdots \\ * \text{ } \cdots \\ \cdots \\ * \text{ } \cdots \\ m_{\tau''} \end{array}$$

The grey part consists of totally  $2(c_{2k} - 1) = C_{2k} - 1$  dots. Hence the total signature of the grey part is  $(C_{2k} - 1, C_{2k} - 1)$ . Note that  $\tau^s$  is obtained from the “ $* \cdots$ ” part of  $\tau''$  by attaching the grey part and  $x_0, \dots, x_n, m_\tau$ .

The descent of a non-special diagram  $\tau^{ns}$ :

$$(6.6) \quad \tau^{ns} : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ s \\ \vdots \\ s \\ x_1 \end{array} \times \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ s \text{ } r \\ \vdots \\ s \text{ } r \\ x_1 \text{ } x_0 \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau'^{ns} : \begin{array}{c} \text{.....} \\ * \text{ } * \text{ } \cdots \\ s \\ \vdots \\ s \\ x'_0 \end{array} \times \begin{array}{c} \text{.....} \\ * \text{ } \cdots \\ r \\ \vdots \\ r \\ x'_0 \end{array} \mapsto \tau'' : \begin{array}{c} \text{.....} \\ * \text{ } \cdots \\ \cdots \\ * \text{ } \cdots \\ \cdots \\ * \text{ } \cdots \\ m_{\tau''} \end{array}$$

Here the grey parts in  $\tau$  consists of  $C_{2k} - 1$  marks of  $\bullet$ ,  $s$ , or  $r$  and  $s$  and  $r$  occur with the same multiplicity. Hence the total signature of the grey part is  $(C_{2k} - 1, C_{2k} - 1)$ .

To prove the lemma, it suffice to verify the following claims which we leave to the reader:

- In both the special shape and non-special shape cases, the following map is bijective:

$$x_0 x_1 \cdots x_n m_{\tau^{s/ns}} \mapsto (\mathbf{x}_{\tau^{s/ns}}, m_{\tau''}).$$

- The following equation of signatures holds

$$\text{Sign}(x_0 x_1 \cdots x_n m_\tau) - \text{Sign}(m_{\tau''}) = \text{Sign}(\mathbf{x}_\tau).$$

In the verification, one can use the fact that  $m_\tau = m_{\tau''}$  when  $x_1 = r/d$ .

[ Suppose  $x_1 = r/d$ . Then  $x_0 = c/d$  and

$$\text{Sign}(x_0 \cdots x_n m_\tau) - \text{Sign}(m_{\tau''}) = (1, 1) + \text{Sign}(x_1 \cdots x_n).$$

Now we consider the generic cases, i.e.  $x_1 = \bullet/s$ : In this case, we claim that  $\text{Sign}(x_0 x_1) - \text{Sign}(m_{\tau''}) = (1, 2)$ . Now

$$\begin{aligned} & \text{Sign}(x_0 x_1 \cdots x_n m_\tau) - \text{Sign}(m_{\tau''}) \\ &= (1, 1) + \text{Sign}(x_2 \cdots x_n m_\tau) + (0, 1) \\ &= \begin{cases} (1, 1) + \text{Sign}(x_2 \cdots x_n) + \text{Sign}(c) & \text{if } m_\tau = a \\ (1, 1) + \text{Sign}(x_2 \cdots x_n) + \text{Sign}(s) & \text{if } m_\tau = b \end{cases} \end{aligned}$$

First consider the special shape diagram  $\tau = \tau^s$ .

- Suppose  $m_{\tau''} = a$ . Then  $x_0 \times x_1 = \bullet \times \bullet$ .
- Suppose  $m_{\tau''} = b$ . Then  $x_0 \times x_1 = c \times s$ .

Now consider the non-special shape diagram  $\tau = \tau^{ns}$ .

- Suppose  $m_{\tau''} = a$ . Then  $\emptyset \times x_1 x_0 = \emptyset \times sr$ .
- Suppose  $m_{\tau''} = b$ . Then  $\emptyset \times x_1 x_0 = \emptyset \times sd$ .

The matching between  $\tau^s$  and  $\tau^{ns}$  is also clear by the above listing of cases. ]

□

6.2.6. *A key proposition for the generalized descent case.* Now assume  $C_{2k}$  is even. By our assumption, we have  $c_{2k+1} \geq c_{2k} = c_{2k-1}$ .

$$(6.7) \quad \tau : \begin{array}{c} \text{.....} \\ y_0 \text{.....} \end{array} \times \begin{array}{c} \text{.....} \\ x_1 x_0 \cdots \\ \vdots \\ x_n \\ m_\tau \end{array} \mapsto \tau' : \begin{array}{c} \text{.....} \\ y'_0 \text{.....} \end{array} \times \begin{array}{c} \text{.....} \\ x'_0 \cdots \end{array} \mapsto \tau'' : \begin{array}{c} \text{.....} \\ \text{.....} \\ x'' \cdots \\ m_{\tau''} \end{array}$$

Here the gray parts in  $\tau$  are two columns of  $\bullet$  of length  $C_{2k}/2 - 1$ .

**Lemma 6.7.** *In (6.7),  $x'' = x_0$ . The map  $\delta: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \text{DRC}(\mathcal{O}_1)$  given by  $\tau \mapsto (\bar{\nabla}_1^2(\tau), \mathbf{x}_\tau)$  is injective. Moreover,*

$$\text{Sign}(\tau) = \text{Sign}(\tau'') + (C_{2k} - 1, C_{2k} - 1) + \text{Sign}(\mathbf{x}_\tau).$$

The map  $\tilde{\delta}: \text{DRC}(\mathcal{O}) \rightarrow \text{DRC}(\mathcal{O}'') \times \mathbb{N}^2 \times \mathbb{Z}/2\mathbb{Z}$  given by  $\tau \mapsto (\bar{\nabla}_1^2(\tau), \text{Sign}(\tau), \epsilon_\tau)$  is injective.

*Proof.* The claim  $x'' = x_0$ , the injectivity of  $\delta$  and the signature formula follows directly from our algorithm.

Now the injectivity of  $\tilde{\delta}$  follows from  $\mathbf{x}_\tau \mapsto (\text{Sign}(\mathbf{x}_\tau), \epsilon)$  is injective by Lemma 5.2. [ We can fully recover  $\tau$  from  $\mathbf{x}_\tau$  and  $\tau''$ . ]

Suppose  $\mathbf{x}_\tau$  dose not contains  $s$  or  $c$ . Then  $x_1 = r/d$ ,  $m_\tau = m_{\tau''}$ ,  $y_0 = c$ . Hence the claim holds.

$$\text{Sign}(\tau) = \text{Sign}(\tau'') + (C_{2k} - 2, C_{2k} - 2) + \text{Sign}(c) + \text{Sign}(\mathbf{x}_\tau)$$

Now assume  $\mathbf{x}_\tau$  contain at least one  $s$  or  $c$ . Now

$$m_\tau = \begin{cases} a & \text{if } \mathbf{x}_\tau \text{ contains } c \\ b & \text{otherwise} \end{cases}$$

Therefore,  $\text{Sign}(x_2 \cdots x_n m_\tau) = \text{Sign}(\mathbf{x}_\tau) - (0, 1)$ .

Clearly, we can recover  $x_2 \cdots x_n$  from  $\mathbf{x}_\tau$  by deleting the  $c$  (if it exists) or a  $s$ . On the other hand,  $(y_0, x_1)$  is completely determined by  $m_{\tau''}$ :

$$(y_0, x_1) = \begin{cases} (\bullet, \bullet) & \text{if } m_{\tau''} = a \\ (c, s) & \text{if } m_{\tau''} = b \end{cases}$$

Hence  $\text{Sign}(y_0 x_1) = \text{Sign}(m_{\tau''}) + (1, 2)$ .

Now  $\text{Sign}(y_0 x_0 x_1 \cdots x_n m_\tau) = \text{Sign}(\mathbf{x}_\tau) + \text{Sign}(x'' m_{\tau''}) + (1, 2) - (0, 1)$ . This yields the signature identity. ]  $\square$

**6.3. Proof of the type BM case.** We will reduce the proof to the type CD case.

In this section,  $k \geq 1$  and  $\mathcal{O} = (C_{2k+1}, C_{2k}, \cdots, C_1, C_0 = 0) \in \text{Nil}^{\text{dpe}}(B)$ .  $\mathcal{O}' := \bar{\nabla}(\mathcal{O})$  and  $\mathcal{O}'' := \bar{\nabla}(\mathcal{O}')$ .

Take  $\tau \in \text{DRC}(\mathcal{O})$ , we marks the entries in  $\tau$  as in (6.5) to (6.7).

6.3.1. *Usual decent case.* Thanks to Lemma 6.6, the proof in the descent case is exactly the same as that in Section 5.3.



6.3.2. *Reduction to type CD case in the general descent case.* Suppose  $C_{2k}$  is even. We define  $\tilde{\mathcal{O}} = (C_{2k+1} - C_{2k} + 1, 1, 1) \in \text{Nil}^{\text{dpe}}(D)$  and  $\tilde{\mathcal{O}}'' := \bar{\nabla}_1(\tilde{\mathcal{O}}) = (2)$ .

The key proposition of reduction.

**Proposition 6.8.** *We have the following properties:*

- (a) When  $\tau'' = \bar{\nabla}_1^2(\tau)$  for certain  $\tau \in \text{DRC}(\mathcal{O})$ , we have  $x_{\tau''} \neq s$ ;
- (b) For each  $\tau''$  such that  $x_{\tau''} \neq s$ , we have a bijection

$$\begin{aligned} \bar{\delta}: \{ \tau \in \text{DRC}(\mathcal{O}) \mid \bar{\nabla}_1^2(\tau) = \tau'' \} &\longrightarrow \{ (x_{\tilde{\tau}}, \mathbf{u}_{\tilde{\tau}}) \mid \tilde{\tau} \in \text{DRC}(\tilde{\mathcal{O}}) \text{ s.t. } \bar{\nabla}_1^2(\tilde{\tau}) = x_{\tau''} \} \\ \tau &\longmapsto (x_{\tau''}, \mathbf{x}_{\tau}) \end{aligned}$$

where  $\mathbf{u}_{\tilde{\tau}}$  is defined in (3.5).

The situation could be summarized in the following commutative diagram:

$$\begin{array}{ccccccc} & & \tau & \xrightarrow{\delta} & (\tau'', \mathbf{x}_{\tau}) & \longmapsto & \tau'' \\ & & \downarrow & & \downarrow & & \downarrow \\ x_{\tau} & \longleftarrow & \mathbf{p}_{\tau} & \longrightarrow & (x_{\tau''}, \mathbf{x}_{\tau}) & \longrightarrow & x_{\tau''} \\ & \uparrow & \downarrow & & \parallel & & \parallel \\ & & \tilde{\tau} & \xrightarrow{\delta} & (\tilde{\tau}'', \mathbf{u}_{\tilde{\tau}}) & \longrightarrow & \tilde{\tau}'' \\ & & \uparrow & & \uparrow & & \uparrow \\ & & x_{\tilde{\tau}} & \longleftarrow & \tilde{\tau} & \longrightarrow & \tilde{\tau}'' \end{array}$$

We remark that  $x_{\tau}$  and  $x_{\tilde{\tau}}$  are equal in the most of the case, especially when  $n = 1$ . But there are exceptional cases.

*Proof.* This follows from a case by case verification according to our algorithm. We list all possible cases below:

When  $n = 1$ , see Table 2.

Now assume  $n \geq 1$ . We let  $\mathbf{x} = x_1 x_2 \cdots x_n$  and define

$$\mathbf{x}' = \begin{cases} \mathbf{x} \text{ deleting "c"} & \text{if } c \in \mathbf{x} \\ \mathbf{x} \text{ deleting "s"} & \text{if } c \notin \mathbf{x}, s \in \mathbf{x} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Now all the cases are listed in Table 3. □

Thanks to  $\bar{\delta}$  defined in the above lemma, we can use (5.8). The rest of the proof is similar to that in Section 5.4. We leave the details to the reader.

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$x_\tau$	$\tilde{x}_{\tau''}$ $m_{\tau''}$	$\mathbf{p}_\tau$	$(x_{\tau''}, \mathbf{x}_\tau)$	$\tilde{\tau}_L$
$r$	$r$ $a$	$\bullet \begin{smallmatrix} r \\ b \end{smallmatrix}$	$(r, s)$	$s \ r$
		$\bullet \begin{smallmatrix} r \\ a \end{smallmatrix}$	$(r, c)$	$r \ c$
$r$	$r$ $b$	$s \begin{smallmatrix} r \\ b \end{smallmatrix}$	$(r, s)$	$s \ r$
		$s \begin{smallmatrix} r \\ a \end{smallmatrix}$	$(r, c)$	$r \ c$
$c$	$s$ $a$	$\bullet \begin{smallmatrix} s \\ b \end{smallmatrix}$	$(c, s)$	$s \ c$
		$\bullet \begin{smallmatrix} s \\ a \end{smallmatrix}$	$(c, c)$	$c \ c$
$d$	$d$ $a$	$\bullet \begin{smallmatrix} d \\ b \end{smallmatrix}$	$(d, s)$	$s \ d$
		$\bullet \begin{smallmatrix} d \\ a \end{smallmatrix}$	$(d, c)$	$c \ d$
		$r \begin{smallmatrix} d \\ a \end{smallmatrix}$	$(d, r)$	$r \ d$
		$d \begin{smallmatrix} d \\ a \end{smallmatrix}$	$(d, d)$	$d \ d$
	$d$ $b$	$s \begin{smallmatrix} d \\ b \end{smallmatrix}$	$(d, s)$	$s \ d$
		$s \begin{smallmatrix} d \\ a \end{smallmatrix}$	$(d, c)$	$c \ d$
		$r \begin{smallmatrix} d \\ b \end{smallmatrix}$	$(d, r)$	$r \ d$
		$d \begin{smallmatrix} d \\ b \end{smallmatrix}$	$(d, d)$	$d \ d$

TABLE 2. Reduction when  $n = 1$

$x_\tau$	$\tilde{x}_{\tau''}$ $m_{\tau''}$	$\mathbf{p}_\tau$	$x_{\tau''}, \mathbf{x}_\tau$	$\tilde{\tau}_L$	conditions	
$r$	$r$ $a$	$\begin{smallmatrix} \bullet & r \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$r, \mathbf{x}$	$\mathbf{x} \ r$	$x_1 \neq r$	$c \in \mathbf{X}$
		$\begin{smallmatrix} r & c \\ \mathbf{x}' & \end{smallmatrix}$		$x_1 = r$		
	$\begin{smallmatrix} \bullet & r \\ \mathbf{x}' & \\ b & \end{smallmatrix}$	$r, \mathbf{x}$	$\mathbf{x} \ r$	$c \notin \mathbf{X}, s \in \mathbf{X}$		
$r$	$r$ $b$	$\begin{smallmatrix} s & r \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$r, \mathbf{x}$	$\mathbf{x} \ r$	$x_1 \neq r$	$c \in \mathbf{X}$
		$\begin{smallmatrix} r & c \\ \mathbf{x}' & \end{smallmatrix}$		$x_1 = r$		
	$\begin{smallmatrix} s & r \\ \mathbf{x}' & \\ b & \end{smallmatrix}$	$r, \mathbf{x}$	$\mathbf{x} \ r$	$c \notin \mathbf{X}, s \in \mathbf{X}$		
$c$	$s$ $a$	$\begin{smallmatrix} \bullet & s \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$c, \mathbf{x}$	$\mathbf{x} \ c$	$c \in \mathbf{X}$	$s \in \mathbf{X}$ is automatic
		$\begin{smallmatrix} \bullet & s \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$c \notin \mathbf{X}$	
$d$	$d$ $a$	$\begin{smallmatrix} \bullet & d \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$d, \mathbf{x}$	$\mathbf{x} \ d$	$c \in \mathbf{X}$	
		$\begin{smallmatrix} \bullet & d \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$x_1 = s$	$c \notin \mathbf{X}$
		$\begin{smallmatrix} \mathbf{x} & d \\ a & \end{smallmatrix}$			$x_1 \neq s$	$c \notin \mathbf{X}$
$d$	$d$ $b$	$\begin{smallmatrix} s & d \\ \mathbf{x}' & \\ a & \end{smallmatrix}$	$d, \mathbf{x}$	$\mathbf{x} \ d$	$c \in \mathbf{X}$	
		$\begin{smallmatrix} s & d \\ \mathbf{x}' & \\ b & \end{smallmatrix}$			$x_1 = s$	$c \notin \mathbf{X}$
		$\begin{smallmatrix} \mathbf{x} & d \\ b & \end{smallmatrix}$			$x_1 \neq s$	$c \notin \mathbf{X}$

TABLE 3. Reduction when  $n \geq 1$

## INDEX

non-special shape, 7, 8, 26, 27

Young Diagram, 14

special shape, 7, 8, 26, 27

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