COUNTING UNIPOTENT REPRESENTATIONS OF REAL REUDCTIVE GROUPS

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1. Counting unipotent representations

In this section, let $G_{\mathbb{C}}$ be a connected complex reductive group and $\mathfrak{g}$ is its Lie algebra. Fix a antiholomorphic involution σ on $G_{\mathbb{C}}$ and a corresponding Cartan involution θ of $G_{\mathbb{C}}$. Let G be a finite central extension of a open subgroup os $G_{\mathbb{C}}^{\sigma}$ and

$$\mathrm{pr}\colon G \to G_{\mathbb{C}}^{\sigma}$$

be the canonical projection. Let $K = \mathrm{pr}^{-1}(G_{\mathbb{C}}^{\sigma})$.

Let ${}^a\mathfrak{h}$ be the abstract Cartan subalgebra of $\mathfrak{g}$ and ${}^aX$ be the lattice of abstarct weight spaces. Let ${}^aR \subseteq {}^aX$, ${}^aR^+$ and ${}^aQ$ be the abstract root system, the set of positive roots and the root lattice. Let

$$\mathsf{C} = \left\{ \mu \in {}^a\mathfrak{h}^* \;\middle|\; \begin{array}{l} \text{either } \langle \mathrm{Re}(\mu), \check{\alpha} \rangle > 0 \text{ or} \\ \langle \mathrm{Re}(\mu), \check{\alpha} \rangle = 0 \text{ and } \sqrt{-1}\,\langle \mathrm{Im}(\mu), \check{\alpha} \rangle > 0 \end{array} \right\}$$

and $\overline{\mathsf{C}}$ be the closure of C in ${}^a\mathfrak{h}$.

1.1. **Coherent family.**

For each finite dimensional $\mathfrak{g}$-module or $G_{\mathbb{C}}$-module F, let F^* be its contragredient representation and let $\Delta(F) \subseteq {}^aX$ denote the multi-set of weights in F.

Let $\Pi_{\Lambda_0}(G_{\mathbb{C}})$ be the set of irreducible finite dimensional representations of $G_{\mathbb{C}}$ with extermal weight in Λ_0 and $\mathcal{G}_{\Lambda}(G_{\mathbb{C}})$ be the subgroup generated by $\Pi_{\Lambda_0}(G_{\mathbb{C}})$. Let

$${}^aP := \{\, \mu \in {}^aX \mid \mu \text{ is a } {}^a\mathfrak{h}\text{-weight of an } F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}}) \,\}.$$

Via the highest weight theory, every $W(G_{\mathbb{C}})$-orbit $W \cdot \mu$ in ${}^aP$ corresponds with the irreducible finite dimensional representation $F \in \Pi_{\mathrm{fin}}(G_{\mathbb{C}})$ with extermal weight μ.

Now the Grothendieck group $\mathcal{G}(G_{\mathbb{C}})$ of finite dimensional representation of $G_{\mathbb{C}}$ is identified with $\mathbb{Z}[{}^aP/W]$. In fact $\mathcal{G}(G_{\mathbb{C}})$ is a $\mathbb{Z}$-algebra under the tensor product and equiped with the involution $F \mapsto F^*$.

Fix a W-invariant sub-lattice $\Lambda_0 \subset {}^aX$ containing ${}^aQ$.

Take a lattice $\Lambda = \lambda + \Lambda_0 \in \mathfrak{h}^*/\Lambda$ with $\lambda \in \overline{\mathsf{C}}$. Let

$$R(\lambda) := \{\, \alpha \in {}^aR \mid \langle \lambda, \check{\alpha} \rangle = 0 \,\} \quad \text{and} \quad W(\lambda) := \langle s_\alpha | \alpha \in R(\lambda) \rangle.$$

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Definition 1.1. *Suppose $\mathcal{M}$ is an abelian group with $\mathcal{G}_\Lambda(G_\mathbb{C})$-action*

$$\mathcal{G}_\Lambda(G_\mathbb{C}) \times \mathcal{M} \ni (F, m) \mapsto F \otimes m.$$

In addition, we fix a subgroup $\mathcal{M}_\mu$ of $\mathcal{M}$ for each for each $\mu \in \Lambda$.

A function $f\colon \Lambda \to \mathcal{M}$ is called a coherent family based on Λ if it satisfies $f(\mu) \in \mathcal{M}_\mu$ and

$$F \otimes f(\mu) = \sum_{\nu \in \Delta(F)} f(\mu + \nu) \qquad \forall \mu \in \Lambda, F \in \Pi_{\Lambda_0}(G_\mathbb{C}).$$

Let $\mathrm{Coh}_\Lambda(\mathcal{M})$ *be the abelian group of all coherent families based on Λ and value in $\mathcal{M}$.*

In this paper, we will consider the following cases.

Suppose $\mathcal{M} = \mathbb{Q}$ and $F \otimes m = \dim(F) \cdot m$ for $F \in \Pi_{\Lambda_0}(G_\mathbb{C})$ and $m \in \mathcal{M}$. We let $\mathcal{M}_\mu = \mathcal{M}$ for every $\mu \in \Lambda$. When $\Lambda = \Lambda_0$, the set of $W(G_\mathbb{C})$-harmonic polynomials on ${}^a\mathfrak{h}^*$ is natrually identified with $\mathrm{Coh}_\Lambda(\mathcal{M})$ via restriction (Vogan's result)

Let $\mathcal{G}(\mathfrak{g}, K)$ be the Grothendieck group of finite length $(\mathfrak{g}, K)$-modules and $\mathcal{G}_\mu(\mathfrak{g}, K)$ be the subgroup of $\mathcal{G}(\mathfrak{g}, K)$ generated by the set of irreducible $(\mathfrak{g}, K)$-modules with infinitesimal character μ.

Then $\mathrm{Coh}_\Lambda(\mathcal{G}(\mathfrak{g}, K))$ is the group of coherent families of Harish-Chandra modules.

2. Parameterize of Unipotent representations

We fix an abstract complex Cartan subgroup $\mathbf{H}_a$ and $\mathfrak{h}_a$ in $\mathbf{G}$ and a set of simple roots Π_a. Let $\mathcal{P}(\mathbf{G})$ be the set of all Langlands parameters of G-modules with character ρ (i.e. the infinitesimal character of the trivial representation). For $\gamma \in \mathcal{P}(\mathbf{G})$, let $\mathcal{L}(\gamma)$, $\mathcal{S}(\gamma)$ and Φ_γ be the corresponding Langlands quotient, standard module and coherent family such that $\Phi_\gamma(\rho) = \mathcal{L}(\gamma)$. Let $\mathcal{M}(\mathbf{G})$ be the span of $\mathcal{L}(\gamma)$. Let $\{ \mathbb{B} \}$ be the set of all blocks. Then $\mathcal{P}(\mathbf{G}) = \bigsqcup_{\mathcal{B}} \mathcal{B}$. The Weyl group $W = W(G)$ acts on $\mathcal{M}(\mathbf{G})$ by coherent continuation. Let $\mathcal{M}_\mathcal{B}$ be the submodule of $\mathcal{M}(\mathbf{G})$ spand by $\gamma \in \mathcal{B}$, then

$$\mathcal{M}(\mathbf{G}) = \bigoplus_{\mathcal{B}} \mathcal{M}_\mathcal{B}$$

Let $\tau(\gamma) \subset \Pi_a$ be the τ-invariant of γ.

Let $\check{\mathcal{O}}$ be even orbit. $\lambda = \frac{1}{2}\check{h}$. Define

$$S(\lambda) = \{ \alpha \in \Pi_a \mid \langle \alpha, \lambda \rangle = 0 \}.$$

Let $\mathcal{P}_\lambda(\mathbf{G})$ be the set of all Langlands parameters with infinitesimal character λ. Let $T_{\lambda,\rho}$ be the translation functor. Let

$$\mathcal{B}(S) = \{ \gamma \in \mathcal{B} \mid S \cap \tau(\gamma) = \varnothing \}$$

and

$$\mathcal{P}(\mathbf{G}, S) = \bigsqcup_{\mathcal{B}} \mathcal{B}(S)$$

Then

$$\begin{aligned} \mathcal{P}(\mathbf{G}, S) &\longrightarrow \mathcal{P}_\lambda(\mathbf{G}) \\ \gamma &\longmapsto T_{\lambda,\rho}(\gamma) \end{aligned}$$

Let $\mathcal{O}$ be a complex nilpotent orbit in $\mathfrak{g}$. Let

$$\mathcal{B}(S, \mathcal{O}) = \{ \gamma \in \mathcal{B}(S) \mid \mathrm{AV}_\mathbb{C}(\mathcal{L}(\gamma)) \subset \overline{\mathcal{O}} \}$$

Let

$$\begin{aligned} m_S(\sigma) &= [\sigma : \mathrm{Ind}_{W(S)}^W \mathbf{1}] \\ m_\mathcal{B}(\sigma) &= [\sigma : \mathcal{M}_\mathcal{B}] \end{aligned}$$

Barbasch [10, Theorem 9.1] established the following theorem.

Theorem 2.1.
$$|\mathcal{B}(S,\mathcal{O})| = \sum_{\sigma} m_{\mathcal{B}}(\sigma) m_S(\sigma)$$
Here $\sigma\times\sigma$ running over the $W\times W$ appears in the double cell $\mathcal{C}(\mathcal{O})$.

Proof. We need to take the graded module of $\mathcal{M}(\mathbf{G})$ with respect to the $\overset{LR}{\leqslant}$. By abuse of notation, we identify the basis $\mathcal{P}(\mathbf{G})$ with its image in the graded module. Note that $S\cap\tau(\lambda)=\varnothing$ if and only if $W(S)$ acts on γ trivially by [70, Lemma 14.7]. On the other hand, by [70, Theorem 14.10, and page 58], $\mathrm{AV}_{\mathbb{C}}(\mathcal{L}(\gamma))\subset\overline{\mathcal{O}}$ only if γ generate a W-module in the double cell of $\mathcal{O}$. $\square$

Now assume $S=S(\lambda)$. By [12, Cor 5.30 b) and c)], $[\sigma:\mathrm{Ind}_{W(S)}^{W}\mathbf{1}]=[\mathbf{1}|_{W(S)}:\sigma]\leqslant 1$.

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