1. Introduction and the main results

1.1. **Unitary representations and the orbit method.** A fundamental problem in representation theory is to determine the unitary dual of a given Lie group G, namely the set of equivalent classes of irreducible unitary representations of G. A principal idea, due to Kirillov and Kostant, is that there is a close connection between irreducible unitary representations of G and the orbits of G on the dual of its Lie algebra [?Ki62, ?Ko70]. This is known as orbit method (or the method of coadjoint orbits). Due to its resemblance with the process of attaching a quantum mechanical system to a classical mechanical system, the process of attaching a unitary representation to a coadjoint orbit is also referred to as quantization in the representation theory literature.

As it is well-known, the orbit method has achieved tremendous success in the context of nilpotent and solvable Lie groups [?Ki62, ?AK]. For more general Lie groups, work of Mackey and Duflo [?Ma, ?Du82] suggest that one should focus attention on reductive Lie groups. As expounded by Vogan in his writings (see for example [?VoBook, ?Vo98, ?Vo00]), the problem finally is to quantize nilpotent coadjoint orbits in reductive Lie groups. The "corresponding" unitary representations are called unipotent representations.

Significant developments on the problem of unipotent representations occurred in the 1980's. We mention two. Motivated by Arthur's conjectures on unipotent representations in the context of automorphic forms [?ArPro, ?ArUni], Adams, Barbasch and Vogan established some important local consequences for the unitary representation theory of the group G of real points of a connected reductive algebraic group defined over $\mathbb{R}$. See [?ABV]. The problem of finding (integral) special unipotent representations for complex semisimple groups (as well as their distribution characters) was solved earlier by Barbasch and Vogan [?BVUni] and the unitarity of these representations was established by Barbasch for complex classical groups [?B.Class]. In a similar vein, Barbasch outlined his proof of the unitarity of special unipotent representations for real classical groups in his 1990 ICM talk [?B.Uni]. The second major development is Vogan's theory of associated varieties [?Vo89] in which Vogan pursues the method of coadjoint orbits by investigating the relationship between a Harish-Chandra module and its associate variety. Roughly speaking, the Harish-Chandra module of a representation attached to a nilpotent coadjoint orbit should have a simple structure after taking the "classical limit", and it should have a specified support dictated by the nilpotent coadjoint orbit via the Kostant-Sekiguchi correspondence.

Simultaneously but in an entirely different direction, there were significant developments in Howe's theory of (local) theta lifting and it was clear by the end of 1980's that the theory has much relevance for unitary representations of classical groups. The relevant works include the notion of rank by Howe [?HoweRank], the description of discrete spectrum by Adams [?Ad83] and the preservation of unitarity in stable range theta lifting by Li [?Li89]. Therefore it was natural, and there were many attempts, to link the orbit method with Howe's theory, and in particular to construct unipotent representations in this formalism. See for example [?Sa, ?Pz, ?HZ, ?HL, ?Br, ?He, ?Tr, ?PT, ?B17]. Particularly worth mentioning were the work of Przebinda [?Pz] in which a double fiberation of moment maps made its appearance in the context of theta lifting, and the work of He [?He] in which an innovative technique called quantum induction was devised to show the non-vanishing of the lifted representations. More recently the double fiberation of moment maps was successfully used by a number of authors to understand refined (nilpotent) invariants of representations such as associated cycles and generalized Whittaker models [?NOTYK, ?NZ, ?GZ, ?LM], which among other things demonstrate the tight link between the orbit method and Howe's theory.

In the present article we will demonstrate that the orbit method and Howe's theory in fact have perfect synergy when it comes to unipotent representations. (Barbasch, Mœglin, He and Trapa pursued a similar theme. See [?B17, ?Mo17, ?He, ?Tr].) We will restrict our attention to a real classical group G of orthogonal or symplectic type and we will construct all unipotent representations of $\widetilde{G}$ attached to $\mathcal{O}$ (in the sense of Barbasch and Vogan) via the method of theta lifting. Here $\widetilde{G}$ is either G, or the metaplectic cover of G if G is a real symplectic group, and $\mathcal{O}$ is a member of our preferred set of complex nilpotent orbits, large enough to include all those which are rigid special. Here we wish to emphasize that there have been extensive investigations of unipotent representations for real reductive groups by Vogan and his collaborators (see e.g. [?VoBook, ?Vo89, ?ABV]), only for unitary groups complete results are known (*cf.* [?BV83, ?Tr]), in which case all such representations may also be described in terms of cohomological induction.

1.2. **Special unipotent representations of classical groups of type B, C or D.** In this article, we are aimed to understand special unipotent representations of classical groups of type B, C or D. As the cases of complex orthogonal groups and complex symplectic groups are well-understood (see [?BVUni] and [?B17]), we will focus on the following groups:

$$\mathrm{O}(p,q), \mathrm{Sp}_{2n}(\mathbb{R}),\ \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}),\ \mathrm{Sp}(p,q),\ \mathrm{O}^*(2n), \tag{1.1}$$

where $p,q,n \in \mathbb{N} := \{0,1,2,\cdots\}$. Here $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ denotes the real metaplectic group, namely the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ that does not split unless $n = 0$.

Let G be one the groups in (**??**). As usual, we view it as a real form of $G_{\mathbb{C}}$, or a double cover of a real form of $G_{\mathbb{C}}$ in the metaplecitic case, where

$$G_{\mathbb{C}} := \begin{cases} \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}) \text{ or } \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2p+2q}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Respectively write $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{g}$ for the Lie algebras of G and $G_{\mathbb{C}}$. Then $\mathfrak{g}$ is viewed as a complexification of $\mathfrak{g}_{\mathbb{R}}$.

Let $r_{\mathfrak{g}}$ denote the rank of $\mathfrak{g}$. Let $W_{r_{\mathfrak{g}}}$ denote the subgroup of $\mathrm{GL}_{r_{\mathfrak{g}}}(\mathbb{C})$ generated by the permutation matrices and the diagonal matrices with diagonal entries ± 1. Then as usual, Harish-Chandra isomorphism yields an identification

$$\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}} = (\mathrm{S}(\mathbb{C}^{r_{\mathfrak{g}}}))^{W_{r_{\mathfrak{g}}}}. \tag{1.2}$$

Here and henceforth, "U" indicates the universal enveloping algebra of a Lie algebra, a superscript group indicate the space of invariant vectors under the group action, and "S" indicates the symmetric algebra. Unless $G_{\mathbb{C}}$ is an even orthogonal group, $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$ equals the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$. By (**??**), we have the following parameterization of characters of $\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}$:

$$\mathrm{Hom}_{\mathrm{alg}}(\mathrm{U}(\mathfrak{g})^{G_{\mathbb{C}}}, \mathbb{C}) = W_{r_{\mathfrak{g}}} \backslash (\mathbb{C}^{r_{\mathfrak{g}}})^* = W_{r_{\mathfrak{g}}} \backslash \mathbb{C}^{r_{\mathfrak{g}}}$$

Here "$\mathrm{Hom}_{\mathrm{alg}}$" indicates the set of $\mathbb{C}$-algebra homomorphisms, and a superscript "$*$" over a vector space indicates the dual space.

We define the Langlands dual of G to be the complex group

$$\check{G} := \begin{cases} \mathrm{Sp}_{p+q-1}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is odd;} \\ \mathrm{O}_{p+q}(\mathbb{C}), & \text{if } G = \mathrm{O}(p,q) \text{ and } p+q \text{ is even;} \\ \mathrm{O}_{2n+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}_{2n}(\mathbb{R}); \\ \mathrm{Sp}_{2n}(\mathbb{C}), & \text{if } G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}); \\ \mathrm{O}_{2p+2q+1}(\mathbb{C}), & \text{if } G = \mathrm{Sp}(p,q); \\ \mathrm{O}_{2n}(\mathbb{C}), & \text{if } G = \mathrm{O}^*(2n). \end{cases}$$

Write $\check{\mathfrak{g}}$ for the Lie algebra of $\check{G}$.

Denote by $\mathrm{Nil}(\check{\mathfrak{g}})$ the set of nilpotent $\check{G}$-orbits in $\check{\mathfrak{g}}$, namely the set of all orbits of nilpotent matrices under the adjoint action of $\check{G}$ in $\check{\mathfrak{g}}$. When no confusion is possible, we will not distinguish a nilpotent orbit in $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$ with its corresponding Young diagram. In particular, the zero orbit is also regarded as the Young with one nonempty column at most.

Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. It determines a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_\mathbb{C}} \to \mathbb{C}$ as in what follows. For every integer $a \geqslant 0$, write

$$\rho(a) := \begin{cases} (1, 2, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is odd;} \\ (\frac{1}{2}, \frac{3}{2}, \cdots, \frac{a-1}{2}), & \text{if } a \text{ is even;} \end{cases}$$

By convention, $\rho(1)$ and $\rho(0)$ are the empty sequence. Write $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_s > 0$ ($s \geqslant 0$) for the row lengths of $\check{\mathcal{O}}$. Define

$$\chi(\check{\mathcal{O}}) := (\rho(a_1), \rho(a_2), \cdots, \rho(a_s), 0, 0, \cdots, 0), \tag{1.3}$$

to be viewed as a character $\chi(\check{\mathcal{O}}) : \mathrm{U}(\mathfrak{g})^{G_\mathbb{C}} \to \mathbb{C}$. Here the number of 0's is

$$\left\lfloor \frac{\text{the number of odd rows of the Young diagram of } \check{\mathcal{O}}}{2} \right\rfloor.$$

Recall the following well-known result of Dixmier ([?Bor, Section 3]): for every algebraic character χ of $\mathrm{Z}(\mathfrak{g})$, there exists a unique maximal ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of χ. As an easy consequence, we know that there is a unique maximal $G_\mathbb{C}$-stable ideal of $\mathrm{U}(\mathfrak{g})$ that contains the kernel of $\chi(\check{\mathcal{O}})$. Write $I_{\check{\mathcal{O}}}$ for this ideal.

Recall that a smooth Fréchet representation of moderate growth of a real reductive group is called a Casselman-Wallach representation ([?Ca89, ?Wa2]) if its Harish-Chandra module has finite length. When $G = \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ is a metaplectic group, write ε_G for the non-trivial element in the kernel of the covering map $G \to \mathrm{Sp}_{2n}(\mathbb{R})$. Then a representation of G is said to be genuine if ε_G acts on it through the scalar multiplication by -1. Following Barbasch and Vogan ([?ABV, ?BVUni]), we make the following definition.

Definition 1.1. *Let $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. An irreducible Casselman-Wallach representation π of G is attached to $\check{\mathcal{O}}$ if*

- *$I_{\check{\mathcal{O}}}$ annihilates π; and*
- *V is genuine if G is a metaplectic group.*

Write $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ for the set of all isomorphism classes of irreducible Casselman-Wallach representations of G that are attached to $\check{\mathcal{O}}$. We say that an irreducible Casselman-Wallach representation of G is special unipotent if it is attached to $\check{\mathcal{O}}$, for some $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$. We will construct all the special unipotent representations, and show that they are all unitarizable as predicted by the Arthur-Barbasch-Vogan conjecture [?ABV, Introduction].

1.3. **Combinatorics : bipartitions.** For every Young diagram $\mathcal{O}$, write

$$\mathbf{r}_1(\mathcal{O}) \geqslant \mathbf{r}_2(\mathcal{O}) \geqslant \mathbf{r}_3(\mathcal{O}) \geqslant \cdots$$

for its row lengths, and similarly, write

$$\mathbf{c}_1(\mathcal{O}) \geqslant \mathbf{c}_2(\mathcal{O}) \geqslant \mathbf{c}_3(\mathcal{O}) \geqslant \cdots$$

for its column lengths. Denote by $|\mathcal{O}| := \sum_{i=1}^{\infty} \mathbf{r}_i(\mathcal{O})$ the total size of $\mathcal{O}$.

We introduce six symbols B, C, D, $\widetilde{C}$, C^* and D^* to indicate the types of the groups that we are considering as in (**??**), namely odd real orthogonal groups, real symplectic groups, even real orthogonal groups, real metaplectic groups, quaternionic symplectic groups and quaternionic orthogonal groups, respectively. Let $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and suppose that G has type $\star$.

Following [?MR, Definition 4.1], We say that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity if

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \check{G} \text{ is a complex symplectic group; and} \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \check{G} \text{ is a complex orthogonal group.} \end{cases}$$

The study of the special unipotent representations in general case reduced to the case when $\check{\mathcal{O}}$ has $\star$-good parity. See Appendix **??**. Now we suppose that $\check{\mathcal{O}}$ has $\star$-good parity. Equivalently but in a completely combinatorial setting, we consider $\check{\mathcal{O}}$ as a Young diagram that has $\star$-good parity in the following sense:

$$\begin{cases} \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are even if } \star \in \{B, \widetilde{C}\}; \\ \text{all nonzero row lengths of } \check{\mathcal{O}} \text{ are odd if } \star \in \{C, D, C^*, D^*\}; \text{ and} \\ \text{the total size } \left|\check{\mathcal{O}}\right| \text{ is odd if and only if } \star \in \{C, C^*\}. \end{cases}$$

Definition 1.2. *A $\star$-pair is a pair $(i, i+1)$ of successive positive integers such that*

$$\begin{cases} i \text{ is odd}, & \text{if } \star \in \{C, \widetilde{C}, C^*\}; \\ i \text{ is even}, & \text{if } \star \in \{B, D, D^*\}. \end{cases}$$

A $\star$-pair $(i, i+1)$ is said to be

- *empty in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *balanced in $\check{\mathcal{O}}$, if $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) > 0$;*
- *tailed in $\check{\mathcal{O}}$, if $\star \in \{C, D, C^*, D^*\}$ and $\mathbf{r}_i(\check{\mathcal{O}}) > \mathbf{r}_{i+1}(\check{\mathcal{O}}) = 0$;*
- *distinguished in $\check{\mathcal{O}}$, if it is not empty, balanced or tailed in $\check{\mathcal{O}}$.*

Let $\mathrm{DP}_\star(\check{\mathcal{O}})$ denote the set of all $\star$-pairs that are distinghuished in $\check{\mathcal{O}}$. For every subset $\wp \subset \mathrm{DP}_\star(\check{\mathcal{O}})$, we define a pair

$$(\imath_\wp, \jmath_\wp) := (\imath_\star(\check{\mathcal{O}}, \wp), \jmath_\star(\check{\mathcal{O}}, \wp))$$

of Young diagrams as in what follows.

If $\star = B$, then

$$\mathbf{c}_1(\jmath_\wp) = \frac{\mathbf{r}_1(\check{\mathcal{O}})}{2},$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_{i+1}(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}), & \text{if } (2i, 2i+1) \in \wp; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})}{2}), & \text{otherwise.} \end{cases}$$

If $\star = \widetilde{C}$, then for all $i \geqslant 1$,

$$(\mathbf{c}_i(\imath_\wp), \mathbf{c}_i(\jmath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}), & \text{if } (2i-1, 2i) \in \wp; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})}{2}), & \text{otherwise.} \end{cases}$$

If $\star = D$ or D^*, then

$$\mathbf{c}_1(\imath_\wp) = \begin{cases} \frac{\mathbf{r}_1(\check{\mathcal{O}})+1}{2}, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) > 0; \\ 0, & \text{if } \mathbf{r}_1(\check{\mathcal{O}}) = 0, \end{cases}$$

and for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_{i+1}(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{if } (2i, 2i+1) \in \wp; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i, 2i+1) \text{ is tailed in } \check{\mathcal{O}}; \\ (0,0), & \text{if } (2i, 2i+1) \text{ is empty in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i+1}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

If $\star = C$ or C^*, then for all $i \geqslant 1$,

$$(\mathbf{c}_i(\jmath_\wp), \mathbf{c}_i(\imath_\wp)) = \begin{cases} (\frac{\mathbf{r}_{2i}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})+1}{2}), & \text{if } (2i-1, 2i) \in \wp; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, 0), & \text{if } (2i-1, 2i) \text{ is tailed in } \check{\mathcal{O}}; \\ (0,0), & \text{if } (2i-1, 2i) \text{ is empty in } \check{\mathcal{O}}; \\ (\frac{\mathbf{r}_{2i-1}(\check{\mathcal{O}})-1}{2}, \frac{\mathbf{r}_{2i}(\check{\mathcal{O}})+1}{2}), & \text{otherwise.} \end{cases}$$

In all cases, it is known that (see [?Carter, Section13.2])

$$(\imath_\star(\check{\mathcal{O}}, \wp), \jmath_\star(\check{\mathcal{O}}, \wp)) = (\imath_\star(\check{\mathcal{O}}', \wp'), \jmath_\star(\check{\mathcal{O}}', \wp')) \quad \text{if and only if} \quad \begin{cases} \check{\mathcal{O}} = \check{\mathcal{O}}'; \\ \wp = \wp', \end{cases} \tag{1.4}$$

where $\check{\mathcal{O}}'$ is another Young diagram that has $\star$-good parity, and $\wp' \subset \mathrm{DP}_\star(\check{\mathcal{O}}')$. Put

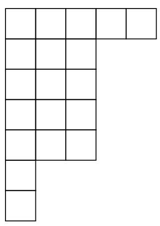
$$\mathrm{BP}_\star(\check{\mathcal{O}}) := \{(\imath_\wp, \jmath_\wp) \mid \wp \subset \mathrm{DP}_\star(\check{\mathcal{O}})\}$$

and

$$\mathrm{BP}_\star := \bigcup_{\check{\mathcal{O}} \text{ is Young diagram that has } \star\text{-good parity}} \mathrm{BP}_\star(\check{\mathcal{O}}).$$

The latter is a disjoint union by (**??**), and its elements are called $\star$-bipartitions. Recall that the set $\mathrm{BP}_\star(\check{\mathcal{O}})$ parametrizes certain Weyl group representations ([?Carter, Section 11.4]).

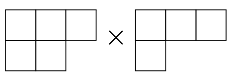
Example. Suppose that $\star = C$, and $\check{\mathcal{O}}$ is the following Young diagram which has $\star$-good parity.

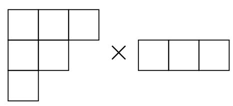


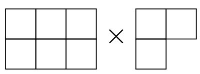
Then

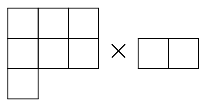
$$\mathrm{DP}_\star(\check{\mathcal{O}}) = \{(1,2), (5,6)\}.$$

and $(\imath_\wp, \jmath_\wp)$ has the following form.

$\wp = \varnothing$: 

$\wp = \{(1,2)\}$: 

$\wp = \{(5,6)\}$: 

$\wp = \{(1,2), (5,6)\}$: 

Here and henceforth, when no confusion is possible, we write $\alpha \times \beta$ for a pair (α, β). We will also write $\alpha \times \beta \times \gamma$ for a triple (α, β, γ).

1.4. **Combinatorics : painted bipartitions.** Let $\imath$ be a Young diagram. Write $\mathrm{Box}(\imath)$ for the set of all boxes in $\imath$, which has $|\imath|$ elements in total. We say that a Young diagram $\imath'$ is contained in $\imath$ (and write $\imath' \subset \imath$) if

$$\mathbf{r}_i(\imath') \leqslant \mathbf{r}_i(\imath) \qquad \text{for all } i = 1, 2, 3, \cdots .$$

When this is the case, $\mathrm{Box}(\imath')$ is viewed as a subset of $\mathrm{Box}(\imath)$ concentrating on the upper-left corner. We say that a subset of $\mathrm{Box}(\imath)$ is a Young subdiagram if it equals $\mathrm{Box}(\imath')$ for a Young diagram $\imath' \subset \imath$. In this case, we call $\imath'$ the Young diagram corresponding to this Young subdiagram.

We introduce five symbols $\bullet$, s, r, c and d.

Definition 1.3. *A painting on a Young diagram $\imath$ is a map*

$$\mathcal{P} : \mathrm{Box}(\imath) \to \{\bullet, s, r, c, d\}$$

with the following properties:

- *$\mathcal{P}^{-1}(S)$ is a Young subdiagram of $\mathrm{Box}(\imath)$ when $S = \{\bullet\}, \{\bullet, s\}, \{\bullet, s, r\}$ or $\{\bullet, s, r, c\}$;*
- *every row of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$ when $S = \{s\}$ or $\{r\}$;*
- *every column of $\imath$ has at most one box in $\mathcal{P}^{-1}(S)$ when $S = \{c\}$ or $\{d\}$.*

A painted Young diagram is a pair $(\imath, \mathcal{P})$ consisting of a Young diagram and a painting on it.

Example. Suppose that $\tau = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$, then there are $25 + 12 + 6 + 2 = 45$ paintings on τ in total as listed below.

$\begin{array}{|c|c|}\hline \bullet & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ $\quad \alpha, \beta \in \{\bullet, s, r, c, d\}$ $\qquad$ $\begin{array}{|c|c|}\hline s & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ $\quad \alpha \in \{r, c, d\}, \beta \in \{s, r, c, d\}$

$\begin{array}{|c|c|}\hline r & \alpha \\ \hline \beta & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ $\quad \alpha \in \{c, d\}, \beta \in \{r, c, d\}$ $\qquad$ $\begin{array}{|c|c|}\hline c & \alpha \\ \hline d & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ $\quad \alpha \in \{c, d\}$

We introduce two more symbols B^+ and B^-, and make the following definition.

Definition 1.4. *A painted bipartition is a triple $\tau = (\imath, \mathcal{P}) \times (\jmath, \mathcal{Q}) \times \alpha$, where $(\imath, \mathcal{P})$ and $(\jmath, \mathcal{Q})$ are painted Young diagrams, and $\alpha \in \{B^+, B^-, C, D, \widetilde{C}, C^*, D^*\}$, subject to the following conditions:*

- *$(\imath, \jmath) \in \mathrm{BP}_\alpha$ if $\alpha \notin \{B^+, B^-\}$, and $(\imath, \jmath) \in \mathrm{BP}_B$ if $\alpha \in \{B^+, B^-\}$;*
- *$\mathcal{P}^{-1}(\bullet)$ and $\mathcal{Q}^{-1}(\bullet)$ corresponding to the same Young diagram;*
- *the image of $\mathcal{P}$ is contained in*

$$\begin{cases} \{\bullet, c\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, r, c, d\}, & \text{if } \alpha = C; \\ \{\bullet, s, r, c, d\}, & \text{if } \alpha = D; \\ \{\bullet, s, c\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet\}, & \text{if } \alpha = C^*; \\ \{\bullet, s\}, & \text{if } \alpha = D^*, \end{cases}$$

- *the image of $\mathcal{Q}$ is contained in*

$$\begin{cases} \{\bullet, s, r, d\}, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \{\bullet, s\}, & \text{if } \alpha = C; \\ \{\bullet\}, & \text{if } \alpha = D; \\ \{\bullet, r, d\}, & \text{if } \alpha = \widetilde{C}; \\ \{\bullet, s, r\}, & \text{if } \alpha = C^*; \\ \{\bullet, r\}, & \text{if } \alpha = D^*. \end{cases}$$

With the notation of Definition **??**, we define in what follows some objects attached to the painted bipartition τ:

$$\star_\tau,\ \check{\mathcal{O}}_\tau,\ \wp_\tau,\ |\tau|,\ \varepsilon_\tau,\ p_\tau,\ q_\tau,\ G_\tau\ \ \dim\tau.$$

$\star_\tau$: This is the symbol given by

$$\star_\tau := \begin{cases} B, & \text{if } \alpha = B^+ \text{ or } B^-; \\ \alpha, & \text{otherwise.} \end{cases}$$

$\check{\mathcal{O}}_\tau$ and $\wp_\tau$: This is the unique pair such that

- $\check{\mathcal{O}}_\tau$ is a Young diagram that has $\star_\tau$-good parity;
- $\wp_\tau \subset \mathrm{DP}_{\star_\tau}(\check{\mathcal{O}}_\tau)$; and
- $(\imath, \jmath) = (\imath_{\star_\tau}(\check{\mathcal{O}}_\tau, \wp_\tau), \jmath_{\star_\tau}(\check{\mathcal{O}}_\tau, \wp_\tau))$.

$|\tau|$: This is the natural number

$$|\tau| := |\imath| + |\jmath|\,.$$

Note that

$$|\tau| := \begin{cases} \frac{|\check{\mathcal{O}}_\tau|-1}{2}, & \text{if } \star_\tau = C \text{ or } C^*; \\ \frac{|\check{\mathcal{O}}_\tau|}{2}, & \text{otherwise.} \end{cases}$$

ε_τ: This is an element of $\mathbb{Z}/2\mathbb{Z}$. It equals 0 in the following cases:

(a) $\star_\tau = B$, $|\tau| > 0$ and the symbol d occurs in the first column of the painted Young diagram $(\jmath, \mathcal{Q})$;
(b) $\star_\tau = D$, $|\tau| > 0$ and the symbol d occurs in the first column of the painted Young diagram $(\imath, \mathcal{P})$;
(c) $\star_\tau \in \{C, \widetilde{C}\}$ and $(1, 2) \notin \wp_\tau$.

In all other case, $\varepsilon_\tau = 1$.

p_τ and q_τ: This is a pair of natural numbers given by counting the various symbols appearing in $(\imath, \mathcal{P})$, $(\jmath, \mathcal{Q})$ and $\{\alpha\}$:

$$\begin{cases} p_\tau := \#\bullet + 2\#r + \#c + \#d + \#B^+; \\ q_\tau := \#\bullet + 2\#s + \#c + \#d + \#B^-. \end{cases}$$

G_τ: This is a classical group given by

$$G_\tau := \begin{cases} \mathrm{O}(p_\tau, q_\tau), & \text{if } \star_\tau = B \text{ or } D; \\ \mathrm{Sp}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = C; \\ \widetilde{\mathrm{Sp}}_{2|\tau|}(\mathbb{R}), & \text{if } \star_\tau = \widetilde{C}; \\ \mathrm{Sp}(\frac{p_\tau}{2}, \frac{q_\tau}{2}), & \text{if } \star_\tau = C^*; \\ \mathrm{O}^*(2\,|\tau|), & \text{if } \star_\tau = D^*. \end{cases}$$

$\dim \tau$: This is the dimension of the standard representation of the complexification of G_τ, equvalently,

$$\dim \tau := \begin{cases} 2\,|\tau| + 1, & \text{if } \star_\tau = B; \\ 2\,|\tau|\,, & \text{otherwise.} \end{cases}$$

Example. Suppose that

$$\tau = \begin{array}{|c|c|}\hline \bullet & \bullet \\ \hline \bullet \\ \cline{1-1} c \\ \cline{1-1} \end{array} \times \begin{array}{|c|c|c|c|}\hline \bullet & \bullet & r & d \\ \hline \bullet \\ \cline{1-1} d \\ \cline{1-1} \end{array} \times B^+.$$

Then

$$\star_\tau = B, \qquad \check{\mathcal{O}}_\tau = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \ & \ \\ \cline{1-2} \end{array}\,, \qquad \wp_\tau = \{(6,7)\}, \quad |\tau| = 4 + 6 = 10, \qquad \varepsilon_\tau = 0,$$

and

$$p_\tau = 6{+}2{+}1{+}2{+}1 = 12, \qquad q_\tau = 6{+}0{+}1{+}2{+}0 = 9, \qquad G_\tau = \mathrm{O}(12,9), \qquad \dim \tau = 21.$$

Write PBP for the set of all painted bipartitions. Then

$$\mathrm{PBP} = \bigsqcup_{(\star, \check{\mathcal{O}})} \mathrm{PBP}_\star(\check{\mathcal{O}}),$$

where $\star$ runs over the set $\{B, \mathrm{C}, D, \widetilde{C}, C^*, D^*\}$, $\check{\mathcal{O}}$ runs over all Young diagrams that has $\star$-good parity, and

$$\mathrm{PBP}_\star(\check{\mathcal{O}}) := \{\tau \in \mathrm{PBP} \mid \star_\tau = \star,\ \check{\mathcal{O}}_\tau = \check{\mathcal{O}}\}.$$

1.5. **Descents of painted bipartitions and theta lifts.** Let $\star$ and $\check{\mathcal{O}}$ be as before. For every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, in what follows we will construct a representation π_τ of G_τ by theta lift. For the starting case when $\check{\mathcal{O}}$ is the empty Young diagram, define

$$\pi_\tau := \begin{cases} \text{the one dimensional genuine representation,} & \text{if } \star = \widetilde{C} \text{ so that } G_\tau = \widetilde{\mathrm{Sp}}_0(\mathbb{R}); \\ \text{the one dimensional trivial representation,} & \text{if } \star \neq \widetilde{C}. \end{cases}$$

Define a symbol

$$\star' := \widetilde{C},\ D,\ C,\ B,\ D^* \ \text{ or } \ C^*$$

respectively if

$$\star = B,\ C,\ D,\ \widetilde{C},\ C^* \ \text{ or } \ D^*.$$

We call $\star'$ the Howe dual of $\star$. Now we assume that $\check{\mathcal{O}}$ is nonempty, and define its decent to be the Young diagram

$$\check{\mathcal{O}}' := \text{the one obtained from } \check{\mathcal{O}} \text{ by removing the first row.}$$

Note that $\check{\mathcal{O}}'$ has $\star'$-good parity. In section **??**, we will define the descent map

$$\nabla : \mathrm{PBP}_\star(\check{\mathcal{O}}) \to \mathrm{PBP}_{\star'}(\check{\mathcal{O}}').$$

Let $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$. Put $\tau' := \nabla(\tau)$. Let $(W_{\tau,\tau'}, \langle\,\cdot\,,\,\cdot\,\rangle_{\tau,\tau'})$ be a real symplectic space of dimension $\dim \tau \cdot \dim \tau'$. As usual, there are continuous homomorphisms $G_\tau \to \mathrm{Sp}(W_{\tau,\tau'})$ and $G_{\tau'} \to \mathrm{Sp}(W_{\tau,\tau'})$ whose images form a reductive dual pair in $\mathrm{Sp}(W_{\tau,\tau'})$. We form the semidirect prduct

$$J_{\tau,\tau'} := (G_\tau \times G_{\tau'}) \ltimes \mathrm{H}(W_{\tau,\tau'}),$$

where

$$\mathrm{H}(W_{\tau,\tau'}) := W_{\tau,\tau'} \times \mathbb{R}$$

is the Heisenberg group with group multiplication

$$(w,t)(w',t') := (w+w', t+t'+\langle w,w'\rangle_{\tau,\tau'}), \quad w,w' \in W_{\tau,\tau'},\ t,t' \in \mathbb{R}.$$

Let $\omega_{\tau,\tau'}$ be a suitably normalized smooth oscillator representation of $J_{\tau,\tau'}$ such that every $t \in \mathbb{R} \subset J_{\tau,\tau'}$ acts on it through the scalar multiplication by $e^{2\pi\sqrt{-1}\,t}$ (the letter π often denotes a representation, but here it stands for the circumference ratio). See Section **??** for details.

For every Casselman-Wallach representation π' of $G_{\tau'}$, write

$$\check{\Theta}_{\tau'}^{\tau}(\pi') := (\omega_{\tau,\tau'}\widehat{\otimes}\pi')_{G_{\tau'}} \qquad \text{(the Hausdorff coinvariant space)},$$

where $\widehat{\otimes}$ indicates the complete projective tensor product. This is a Casselman-Wallach representation of G_τ. Now we define the representation π_τ of G_τ by induction on $\mathbf{c}_1(\check{\mathcal{O}})$:

$$\pi_\tau := \begin{cases} \check{\Theta}_{\tau'}^{\tau}(\pi_{\tau'}) \otimes (1_{p_\tau,q_\tau}^{+,-})^{\varepsilon_\tau}, & \text{if } \star_\tau = B \text{ or } D;\\ \check{\Theta}_{\tau'}^{\tau}(\pi_{\tau'} \otimes \det^{\varepsilon_\tau}), & \text{if } \star_\tau = C \text{ or } \widetilde{C};\\ \check{\Theta}_{\tau'}^{\tau}(\pi_{\tau'}), & \text{if } \star_\tau = C^* \text{ or } D^*. \end{cases}$$

Here $1_{p_\tau,q_\tau}^{+,-}$ denotes the character of $\mathrm{O}(p_\tau,q_\tau)$ whose restriction to $\mathrm{O}(p_\tau)\times\mathrm{O}(q_\tau)$ equals $1\otimes\det$ (1 stands for the trivial character).

Put

$$\mathrm{Unip}_\star(\check{\mathcal{O}}) := \begin{cases} \bigsqcup_{p,q\in\mathbb{N},\,p+q=|\check{\mathcal{O}}|+1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = B;\\ \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}_{|\check{\mathcal{O}}|-1}(\mathbb{R})), & \text{if } \star = C;\\ \bigsqcup_{p,q\in\mathbb{N},\,p+q=|\check{\mathcal{O}}|} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{O}(p,q)), & \text{if } \star = D;\\ \mathrm{Unip}_{\check{\mathcal{O}}}(\widetilde{\mathrm{Sp}}_{|\check{\mathcal{O}}|}(\mathbb{R})), & \text{if } \star = \widetilde{C};\\ \bigsqcup_{p,q\in\mathbb{N},\,2p+2q=|\check{\mathcal{O}}|-1} \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}(p,q)), & \text{if } \star = C^*;\\ \mathrm{Unip}_{\check{\mathcal{O}}}\, \mathrm{O}^*(|\check{\mathcal{O}}|), & \text{if } \star = D^*. \end{cases}$$

We are now ready to formulate our first main theorem.

Theorem 1.5. *Let $\star \in \{B,C,D,\widetilde{C},C^*,D^*\}$, and let $\check{\mathcal{O}}$ be a Young diagram that has $\star$-good parity.*
(a) For every $\tau \in \mathrm{PBP}_\star(\check{\mathcal{O}})$, the representation π_τ of G_τ is irreducible and attached to $\check{\mathcal{O}}$.
(b) If $\star \in \{B,D\}$ and $|\check{\mathcal{O}}| > 0$, then the map

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) \times \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}),\\ (\tau,\epsilon) & \mapsto & \pi_\tau \otimes \det^\epsilon \end{array}$$

is bijective.
(c) In all other cases, the map

$$\begin{array}{rcl} \mathrm{PBP}_\star(\check{\mathcal{O}}) & \to & \mathrm{Unip}_\star(\check{\mathcal{O}}),\\ \tau & \mapsto & \pi_\tau \end{array}$$

is bijective.

By the above theorem (and the reduction in Appendix **??**), we have explicitly constructed all special unipotent representations of the classical groups in (**??**). By this construction and the method of matrix coefficient integrals, we are able to prove the unitarity of these representations.

Theorem 1.6. *All special unipotent representations of the classical groups in* (**??**) *are unitarizable.*

1.6. **Associated cycles.** Let $G, G_{\mathbb{C}}, \mathfrak{g}_{\mathbb{R}}, \mathfrak{g}, \check{\mathfrak{g}}$ and $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ be as in Section **??**. Fix a maximal compact subgroup K of G whose Lie algebra is denoted by $\mathfrak{k}_{\mathbb{R}}$. Then we have an orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the complexification of $\mathfrak{k}_{\mathbb{R}}$, and $\mathfrak{g}$ is equipped with the trace form. By taking the dual spaces, we have a decomposition

$$\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*,$$

Write $K_{\mathbb{C}}$ for the complexification of the complex group K. It is a complex algebraic group with an obvious algebraic action on $\mathfrak{p}^*$.

As before, suppose that G has type $\star$ and $\check{\mathcal{O}}$ has $\star$-good parity. Write $\mathcal{O} \in \mathrm{Nil}(\mathfrak{g}^*)$ for the Barbarsch-Vogan dual of $\check{\mathcal{O}}$ so that its Zariski closure in $\mathfrak{g}^*$ equals the associated variety of the ideal $I_{\check{\mathcal{O}}} \subset \mathrm{U}(\mathfrak{g})$. The algebraic variety $\mathcal{O} \cap \mathfrak{p}^*$ is a finite union of $K_{\mathbb{C}}$-orbits. Given such an orbit $\mathscr{O}$, write $\mathcal{K}_{\mathscr{O}}(K_{\mathbb{C}})$ for the Grothendieck group of the category of $K_{\mathbb{C}}$-equivariant algebraic vector bundles on $\mathscr{O}$. Put

$$\mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}}) := \bigoplus_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathcal{K}_{\mathscr{O}}(K_{\mathbb{C}}).$$

We say that a Casselman-Wallach representation of G is $\mathcal{O}$-bounded if the associated variety of its annihilator ideal is contained in the Zariski closure of $\mathcal{O}$. Note that all representations in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ are $\mathcal{O}$-bounded. Write $\mathcal{K}(G)_{\mathcal{O}-\text{bounded}}$ for the Grothendieck group of the category of all such representations. From [?Vo89, Theorem 2.13], we have a canonical homomorphism

$$\mathrm{AC}_{\mathcal{O}} \colon \mathcal{K}(G)_{\mathcal{O}-\text{bounded}} \longrightarrow \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}}).$$

We call $\mathrm{AC}_{\mathcal{O}}(\pi)$ the associated cycle of π, where π is an $\mathcal{O}$-bounded Casselman-Wallach representation of G. This is a very important invariant attached to π.

Following Vogan [?Vo89, Section 8], we make the following definition.

Definition 1.7. *Let $\mathscr{O}$ be a $K_{\mathbb{C}}$-orbit in $\mathcal{O} \cap \mathfrak{p}^*$. An admissible orbit datum over $\mathscr{O}$ is an irreducible $K_{\mathbb{C}}$-equivariant algebraic vector bundle $\mathcal{E}$ on $\mathscr{O}$ such that*

- *$\mathcal{E}_X$ is isomorphic to a multiple of $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ as a representation of $\mathfrak{k}_X$;*
- *if $\star = \widetilde{C}$, then ε_G acts on $\mathcal{E}$ by the scalar multiplication by -1.*

Here $X \in \mathscr{O}$, $\mathcal{E}_X$ is the fibre of $\mathcal{E}$ at X, $\mathfrak{k}_X$ denotes the Lie algebra of the stabilizer of X in $K_{\mathbb{C}}$, and $(\bigwedge^{\mathrm{top}} \mathfrak{k}_X)^{\frac{1}{2}}$ is a one-dimensional representation of $\mathfrak{k}_X$ whose tensor square is the top degree wedge product $\bigwedge^{\mathrm{top}} \mathfrak{k}_X$.

Note that in the situation of the classical groups we consider here, all admissible orbit data are line bundles. Denote by $\mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}})$ the set of isomorphism classes of admissible orbit data over $\mathscr{O}$, to be viewed as a subset of $\mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$. Put

$$\mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}}) := \bigsqcup_{\mathscr{O} \text{ is a } K_{\mathbb{C}}\text{-orbit in } \mathcal{O} \cap \mathfrak{p}^*} \mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}}) \subset \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}}).$$

Recall that a nilpotent orbit in $\check{\mathfrak{g}}$ is said to be distinguished if it is has no intersection with every proper Levi subalgebra of $\check{\mathfrak{g}}$. Combinatorially, this is equivalent to saying that no pair of rows of the Young diagram has equal nonzero length. Note that all distinguished nilpotent orbits in $\check{\mathfrak{g}}$ has $\star$-good parity.

Definition 1.8. *(a) The orbit $\check{\mathcal{O}}$ (which has $\star$-good parity) is said to be quasi-distinguished if there is no $\star$-pair that is balanced in $\check{\mathcal{O}}$.*
(b) If $\star \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be be pseudo-distinguished if there is no positive even integer i such that $\mathbf{r}_i(\check{\mathcal{O}}) = \mathbf{r}_{i+1}(\check{\mathcal{O}}) = \mathbf{r}_{i+2}(\check{\mathcal{O}}) = \mathbf{r}_{i+3}(\check{\mathcal{O}}) > 0$.

(c) If $\star \in \{C, \widetilde{C}, C^*\}$ so that its Howe dual $\star' \in \{B, D, D^*\}$, then $\check{\mathcal{O}}$ is said to be be pseudo-distinguished if either it is the empty Young diagram or it is nonempty and its descent $\check{\mathcal{O}}'$ (which is a Young diagram that has $\star'$-good parity) is pseudo-distinguished.

We will calculate the associated character of π_τ for every bipartition τ. The calculation is a key ingredient in the proof of Theorem **??**. Especially, we have the following theorem concerning the associated characters of the special unipotent representations.

Theorem 1.9. *Let G be a group in (**??**) that has type $\star \in \{B, C, D, \widetilde{C}, C^*, D^*\}$, and suppose that $\check{\mathcal{O}} \in \mathrm{Nil}(\check{\mathfrak{g}})$ has $\star$-good parity.*
(a) For every $\pi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$, the associated cycle $\mathrm{AC}_{\mathcal{O}}(\pi) \in \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$ is a nonzero sum of pairwise distinct elements of $\mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}})$.
(b) If $\check{\mathcal{O}}$ is pseudo-distinguished, then the map

$$\mathrm{AC}_{\mathcal{O}} : \mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathcal{K}_{\mathcal{O}}(K_{\mathbb{C}})$$

is injective.
(c) If $\check{\mathcal{O}}$ is quasi-distinguished, then the map $\mathrm{AC}_{\check{\mathcal{O}}}$ induces a bijection

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) \to \mathrm{AOD}_{\mathcal{O}}(K_{\mathbb{C}}).$$

Remark. Suppose that $\star \in \{C^*, D^*\}$ so that G is quaternionic. Then there is precisely one admissible orbit datum over $\mathscr{O}$ for each $K_{\mathbb{C}}$-orbit $\mathscr{O} \subset \mathcal{O} \cap \mathfrak{p}^*$. Thus

$$\mathrm{AOD}_{\mathscr{O}}(K_{\mathbb{C}}) = K_{\mathbb{C}} \backslash (\mathcal{O} \cap \mathfrak{p}^*).$$

If $\check{\mathcal{O}}$ is not quasi-distinghuished, then $\mathcal{O} \cap \mathfrak{p}^*$ is empty (see [?CM, Theorems 9.3.4 and 9.3.5]), and hence $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$ is also empty.

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