CSCI 4470/6470 Algorithms, Spring 2017

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Department of Computer Science, UGA

Syllabus: http://cobweb.cs.uga.edu/~cai/courses/algo/2017Spring/

January 17, 2017

An Introduction to the Introduction

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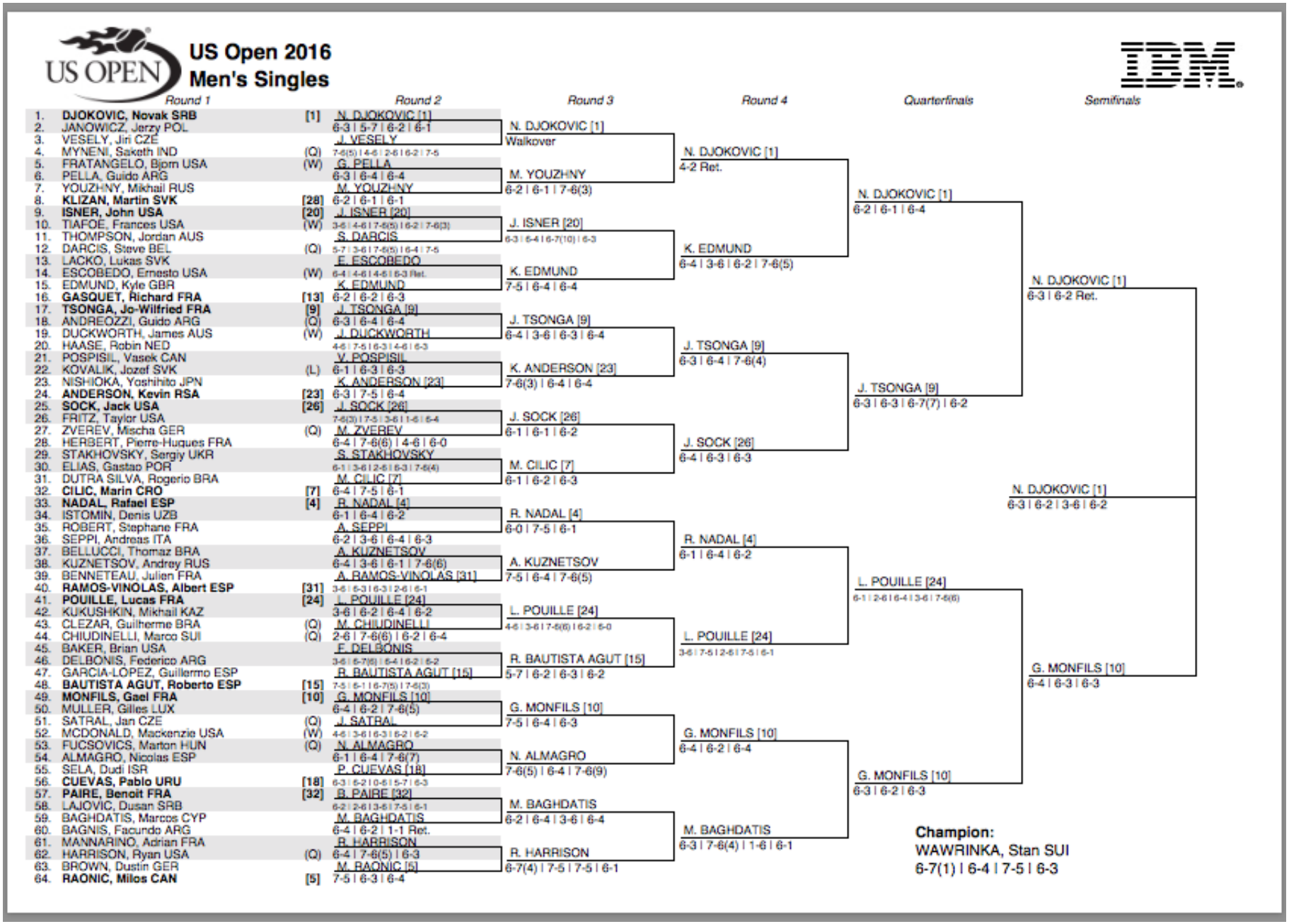

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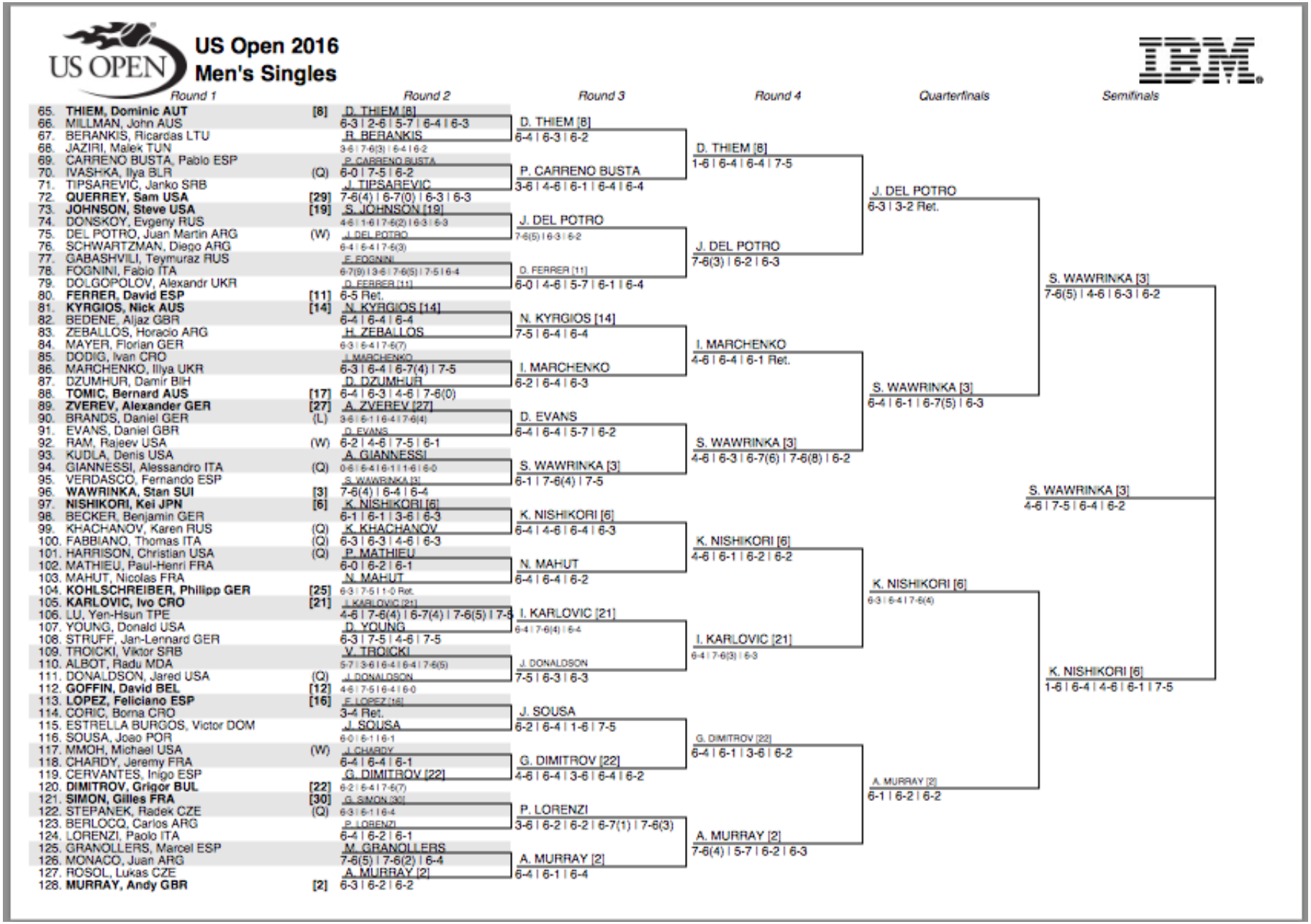

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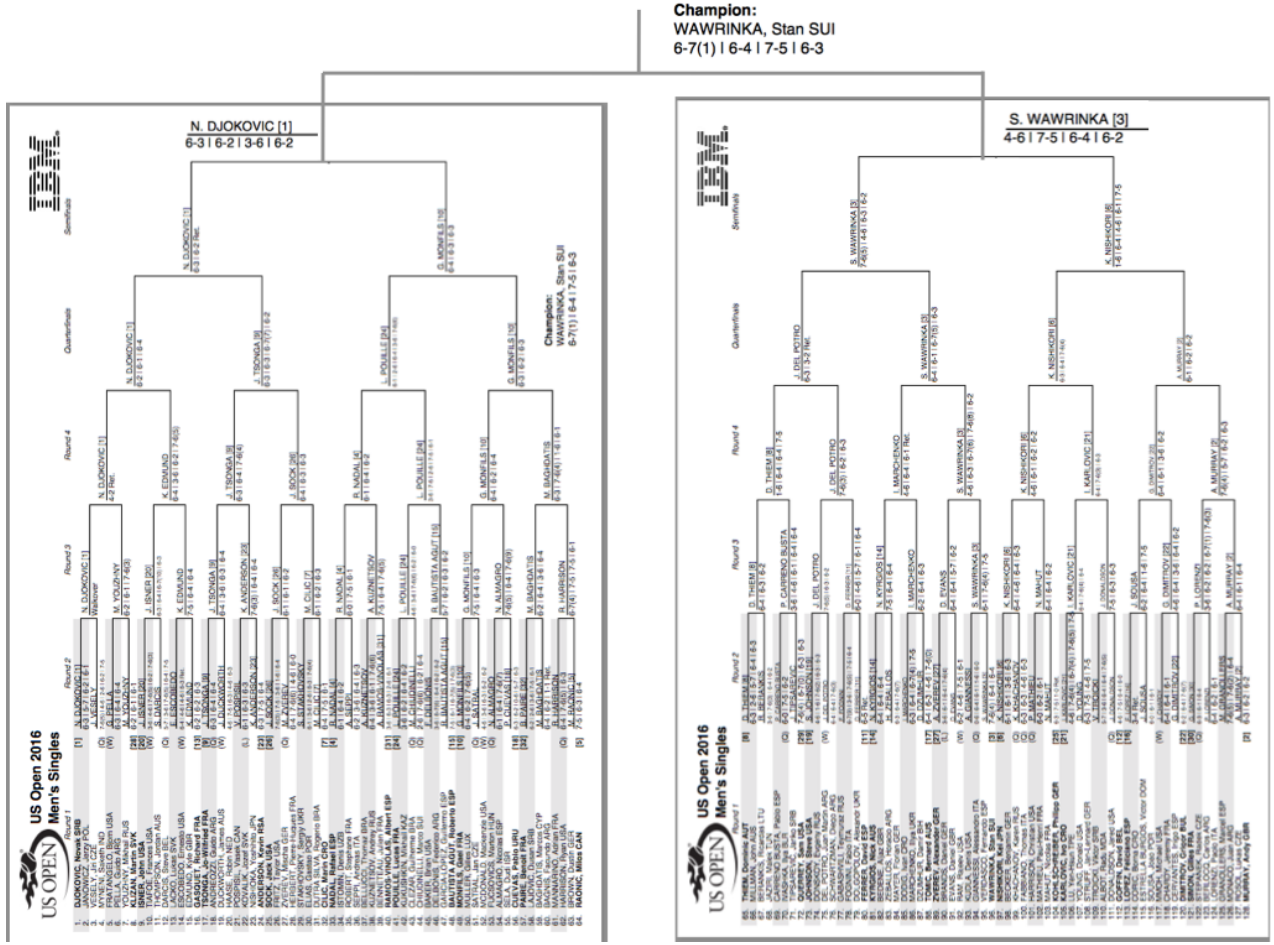

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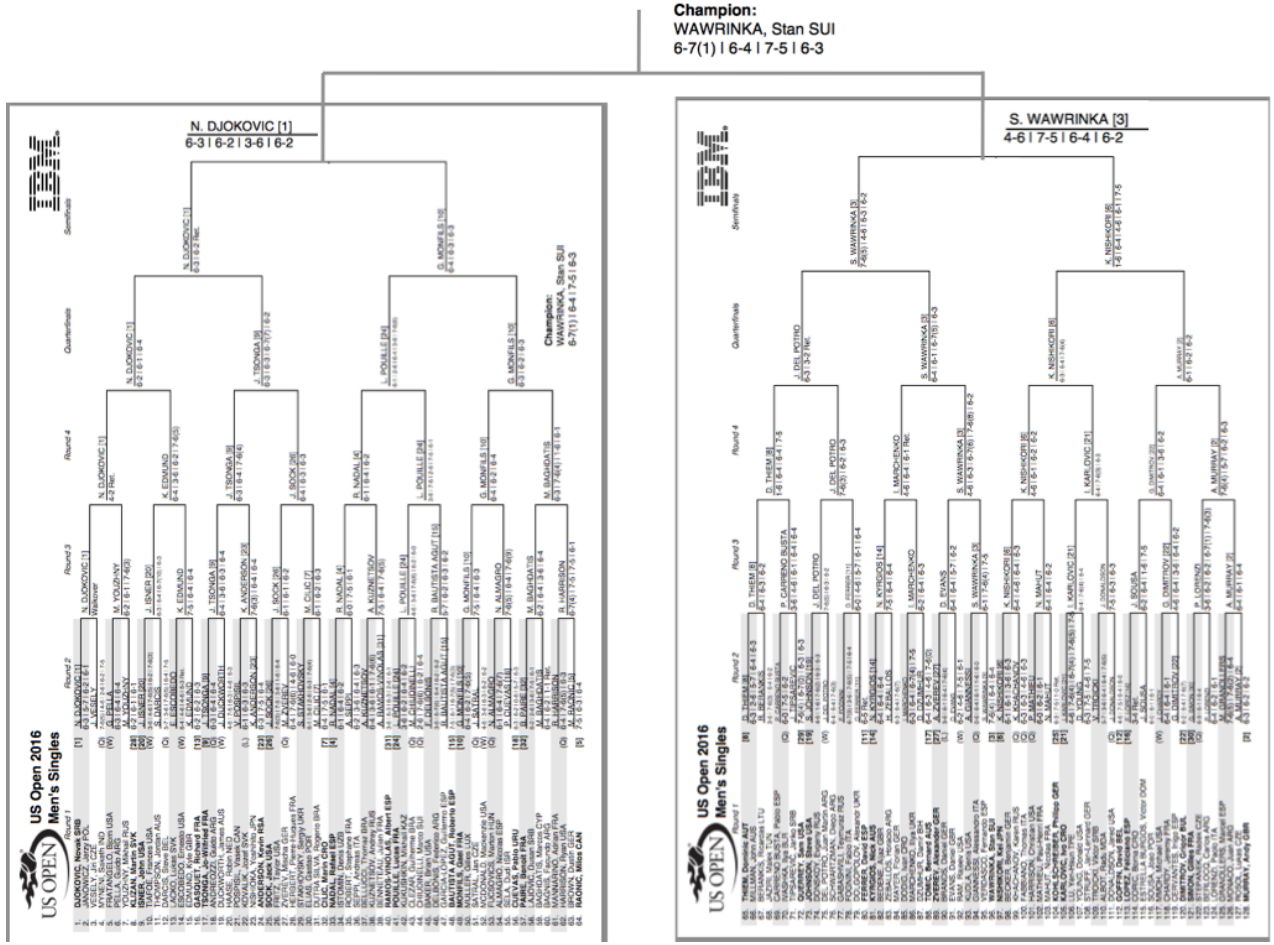

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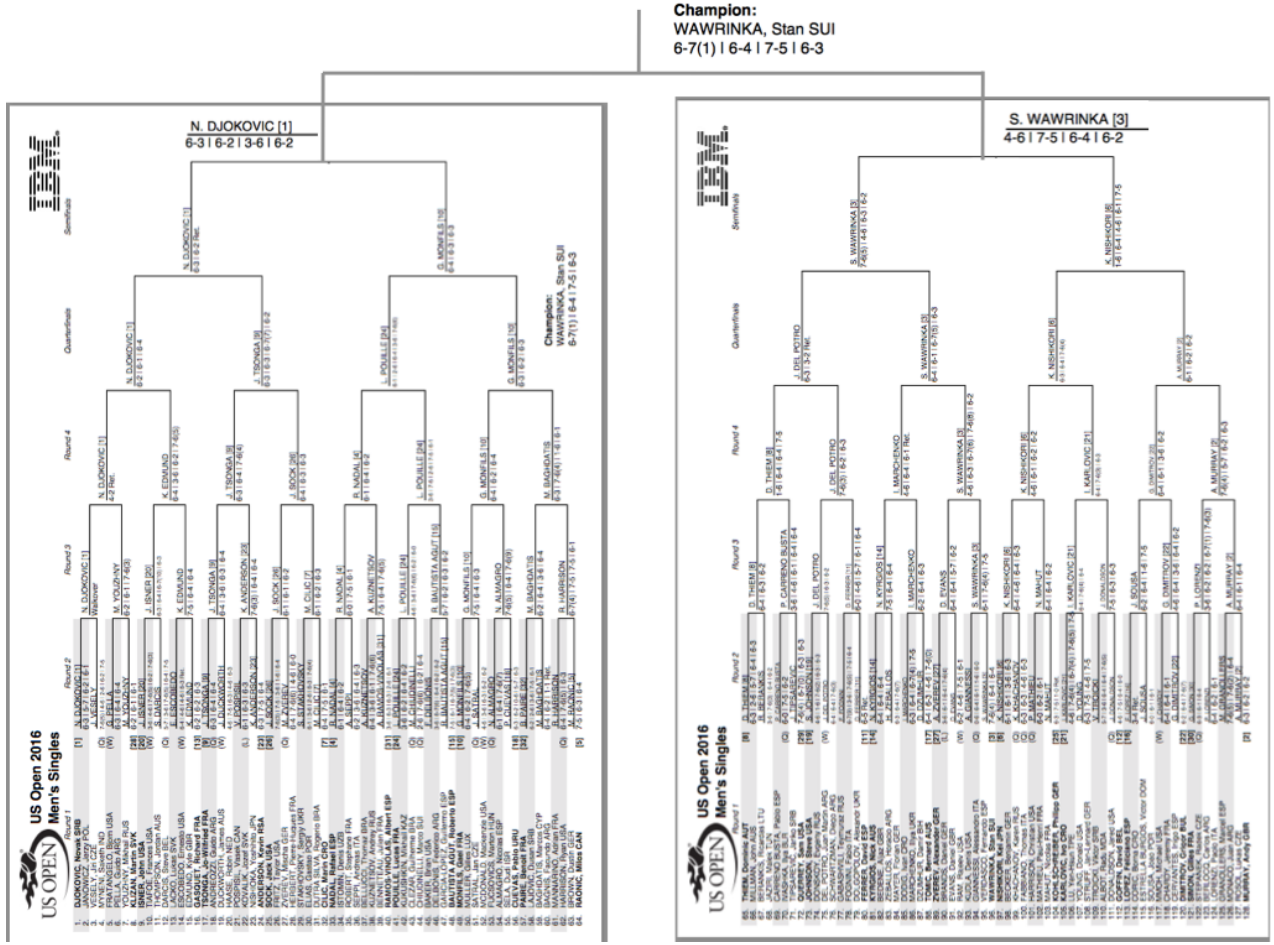

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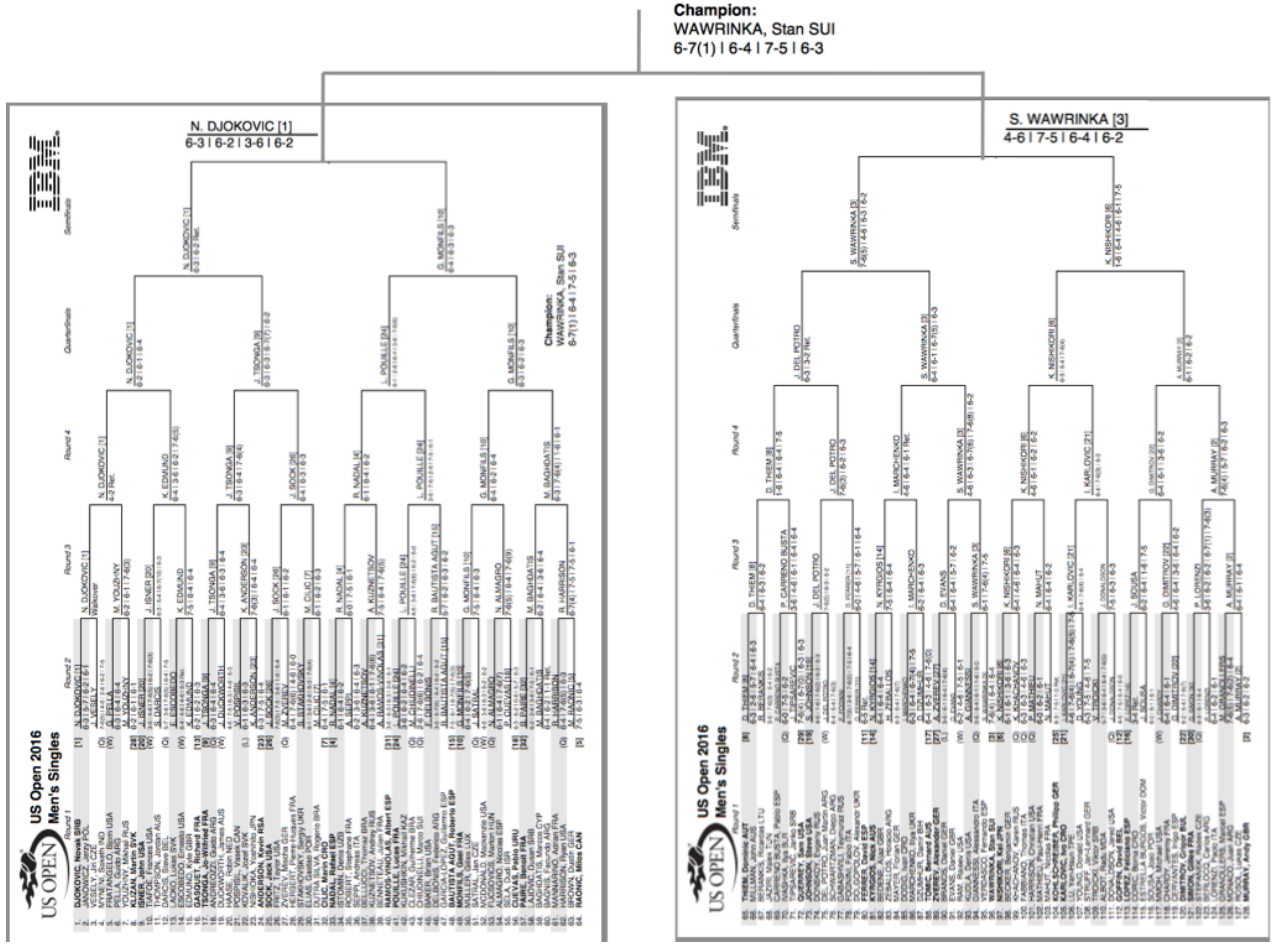

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- 127 matches were played to determine who was the champion.
- **Is it possible to just play fewer matches?**

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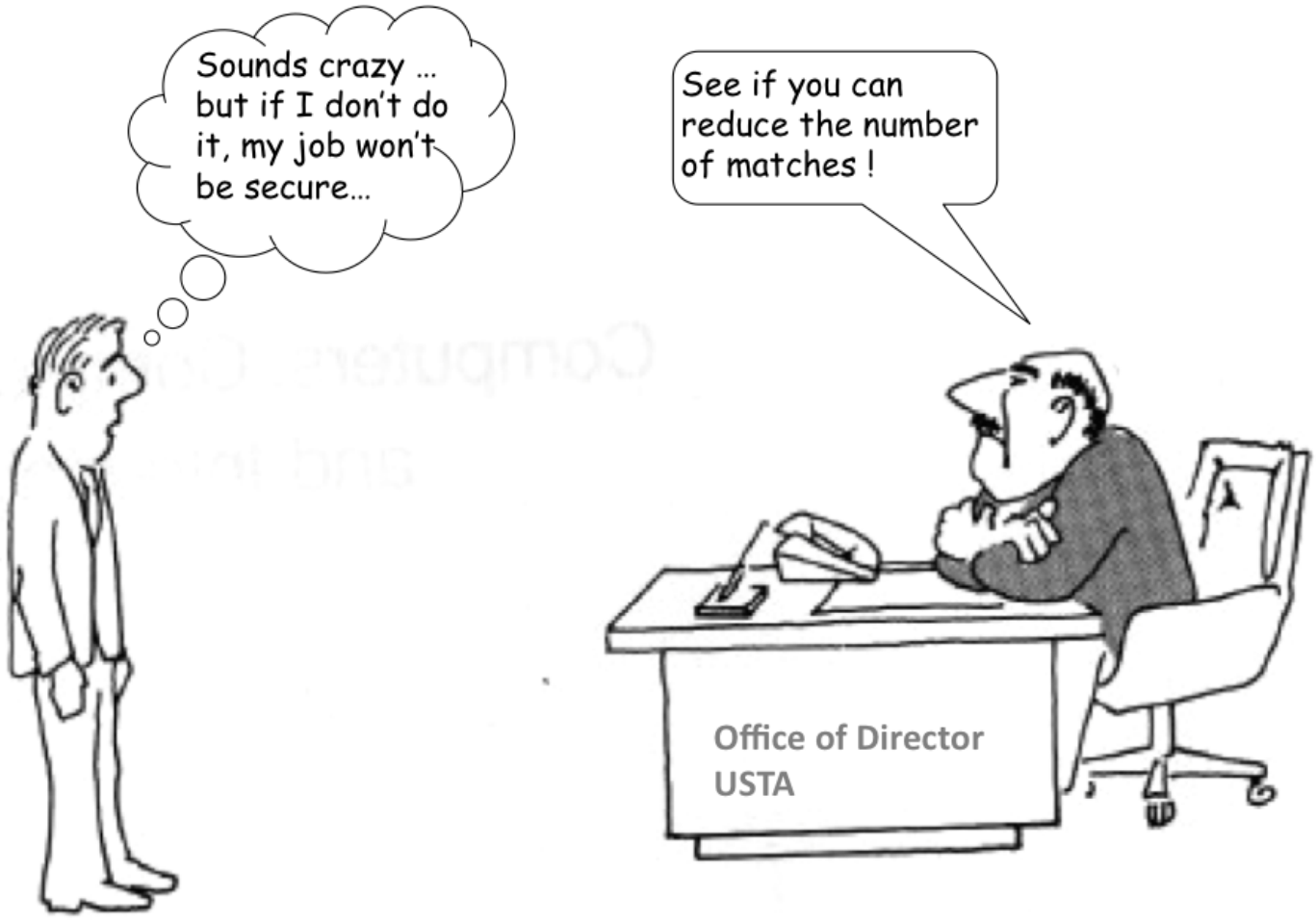

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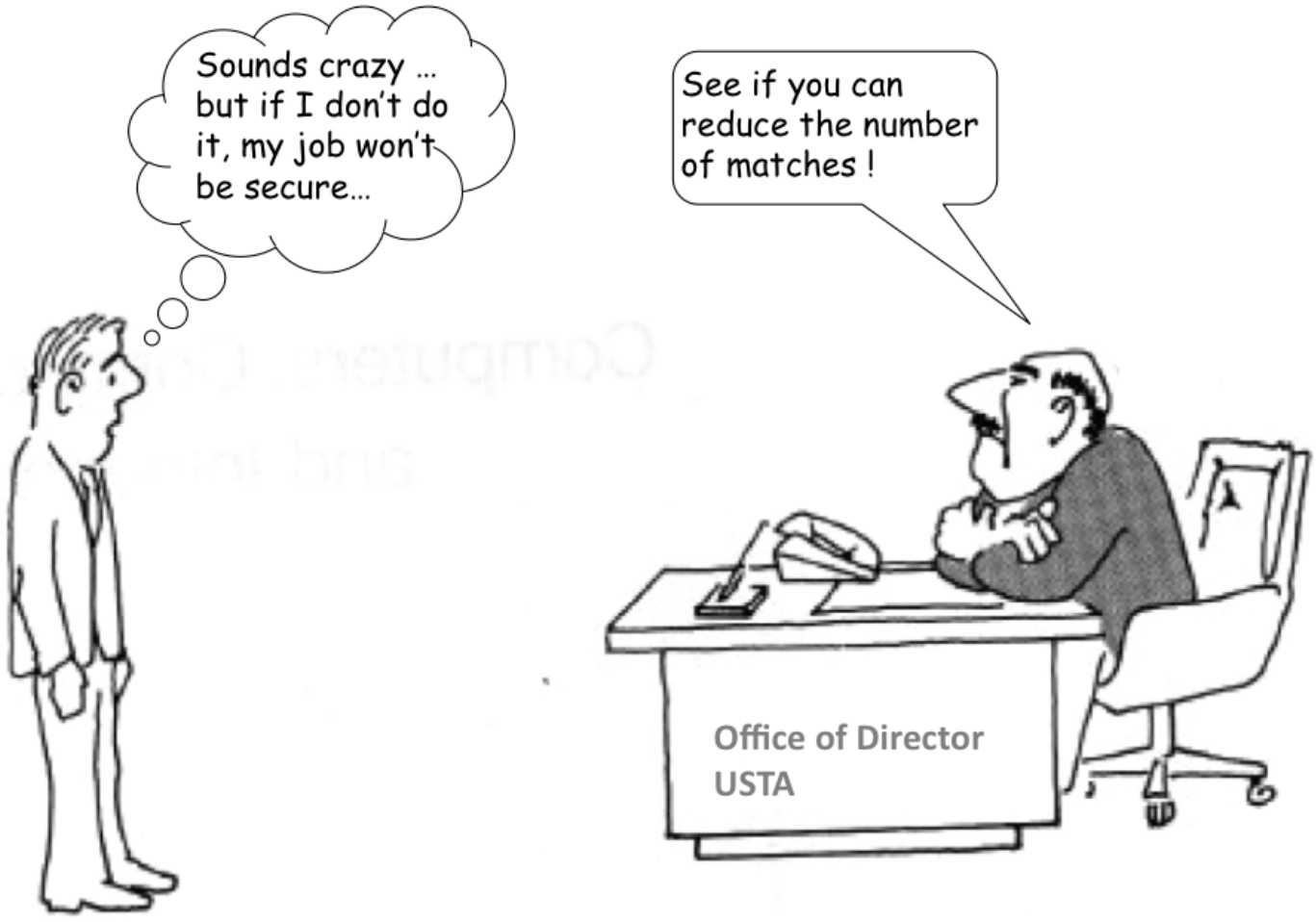

You were called to answer this question;

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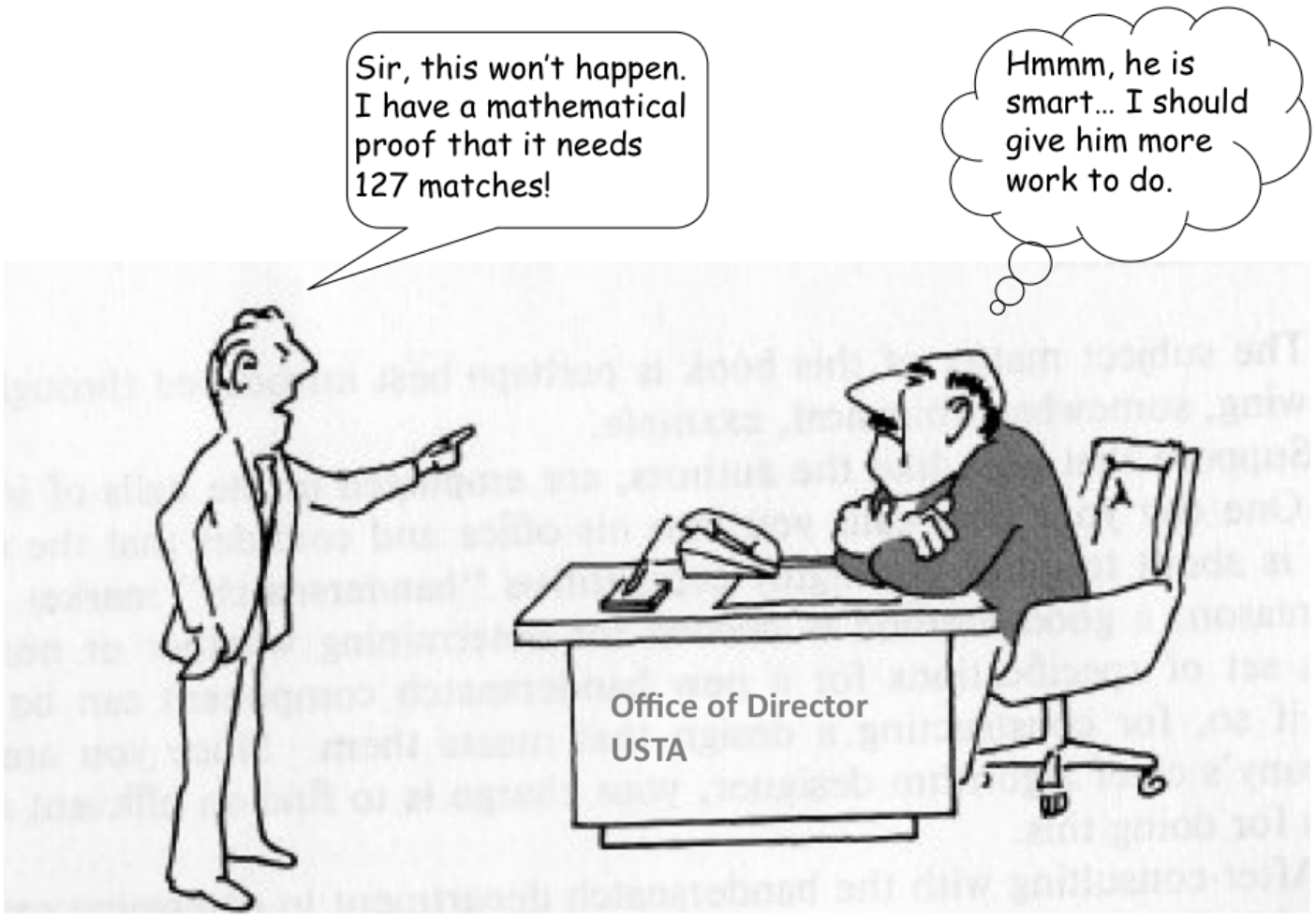

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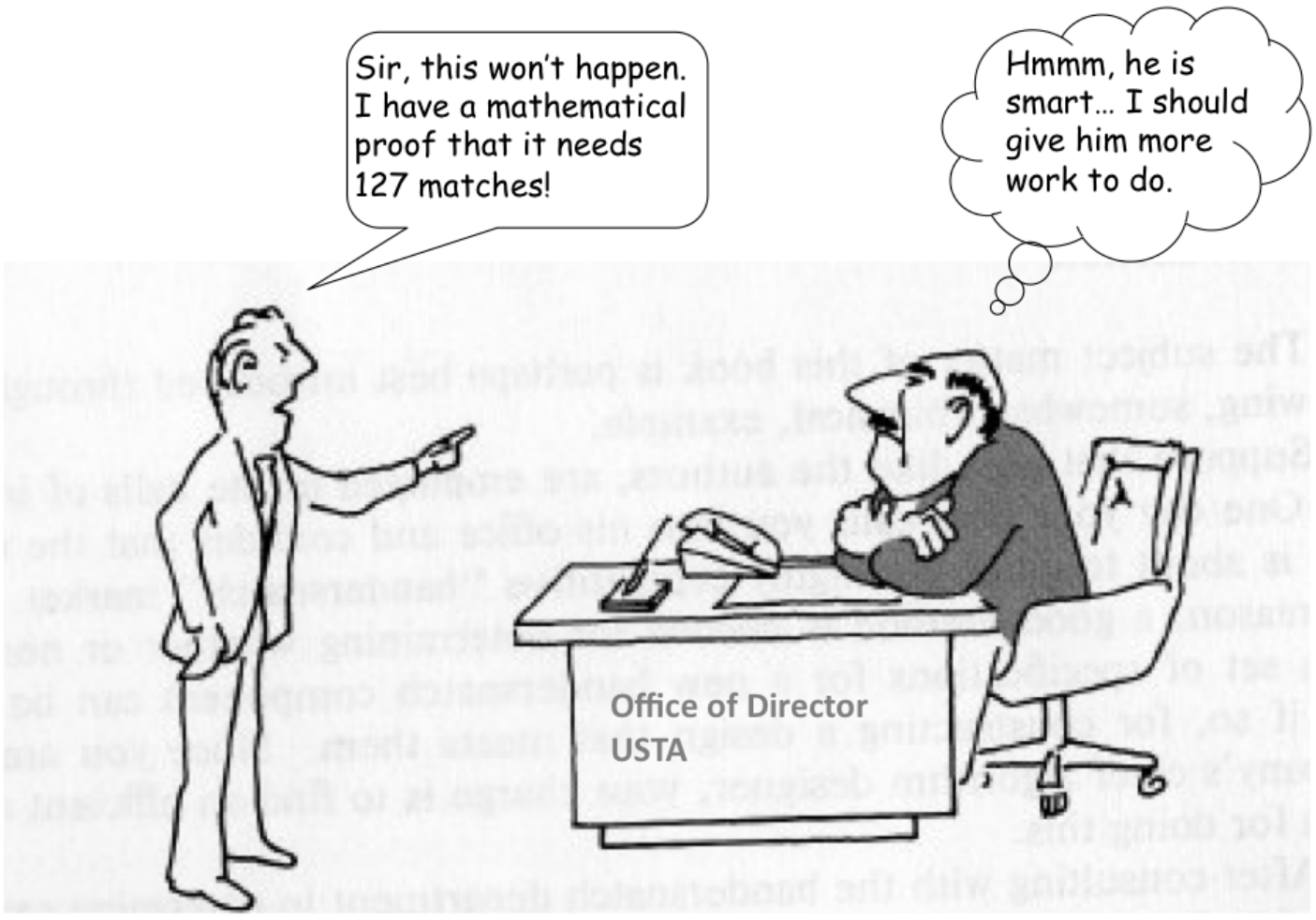

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Same problems	Tennis Tournament	Finding Maximum
Input:	128 players	n numbers
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- actually you proved the "USTA algorithm" was already the optimal.

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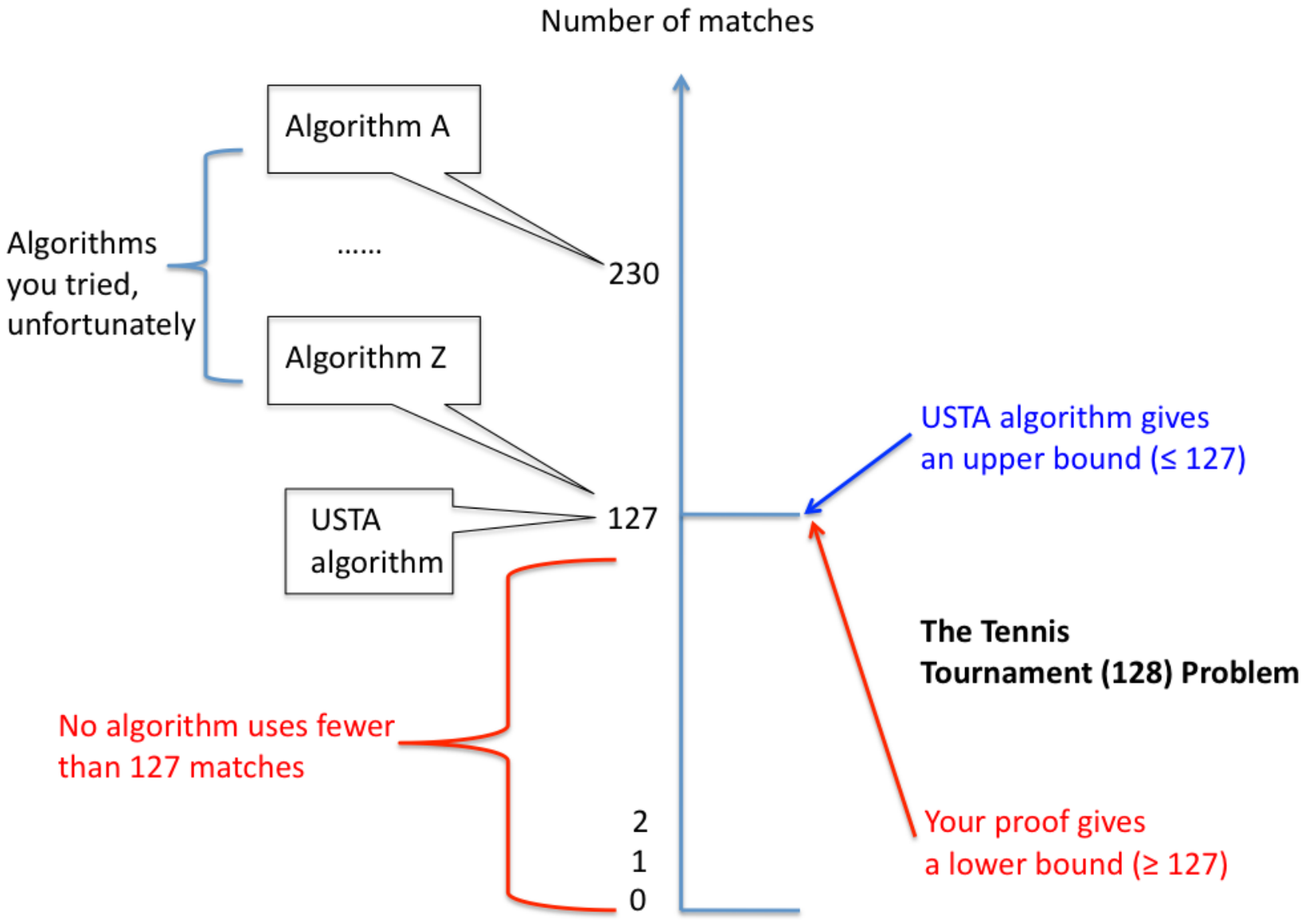

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  Also the corresponding algorithms (that offer the upper bound) are called **optimal** for Π.

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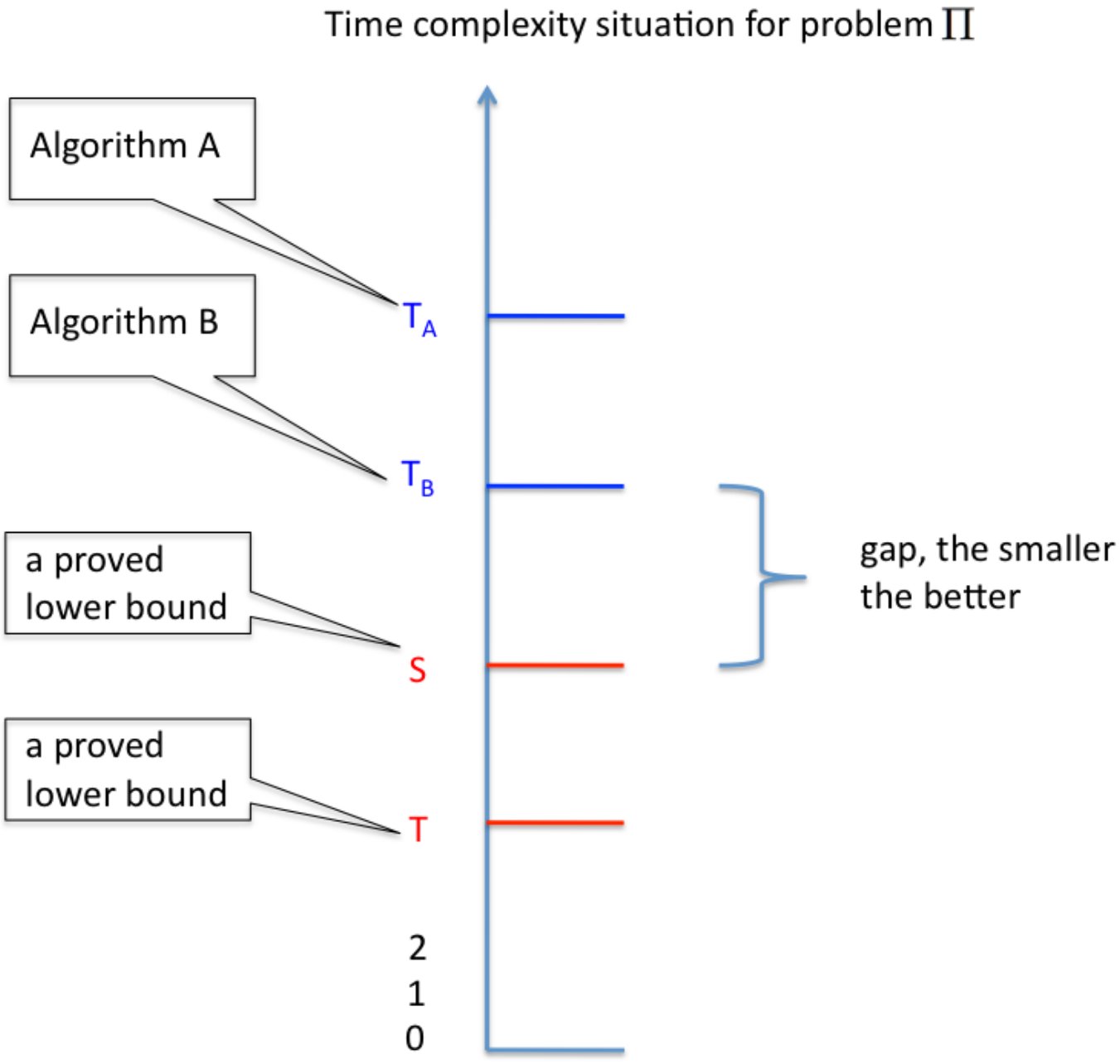

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1. to achieve tighter upper bounds
   - we need be familiar with techniques for algorithm complexity analysis;
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2. and to achieve tighter lower bounds,
   - we need to know the methods for lower bound proofs.

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  - to master some fundamental algorithmic techniques
  - to study advanced algorithmic skills
  - to understand computational intractability

Part I. Foundations

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- Chapter 1. The role of algorithms in computing
- Chapter 2. Getting started
- Chapter 3. Growth of functions
- Chapter 4. Solving recurrences
- Chapter 5. Probabilistic analysis and randomized algorithms

Part I. Foundations

The theme of the course

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Design and analysis of algorithms are closely related.

Part I. Foundations

Example: the Fibonacci sequence.

$$f(n) = \begin{cases} f(n-1) + f(n-2) & \text{if } n \geq 3 \\ 1, & \text{otherwise} \end{cases}$$

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Example: the Fibonacci sequence.

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That is:

n	:	1	2	3	4	5	6	7	8	9	...
$f(n)$	:	1	1	2	3	5	8	13	21	34	...

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Two different types of algorithms: *recursive* and *iterative*

- recursive: task decomposition, top-down, recursive calls;
- iterative: more tightly coupled tasks, bottom-up approaches;

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- suitable for execution of subroutines
- but oblivious, cannot remember any completed subroutine.

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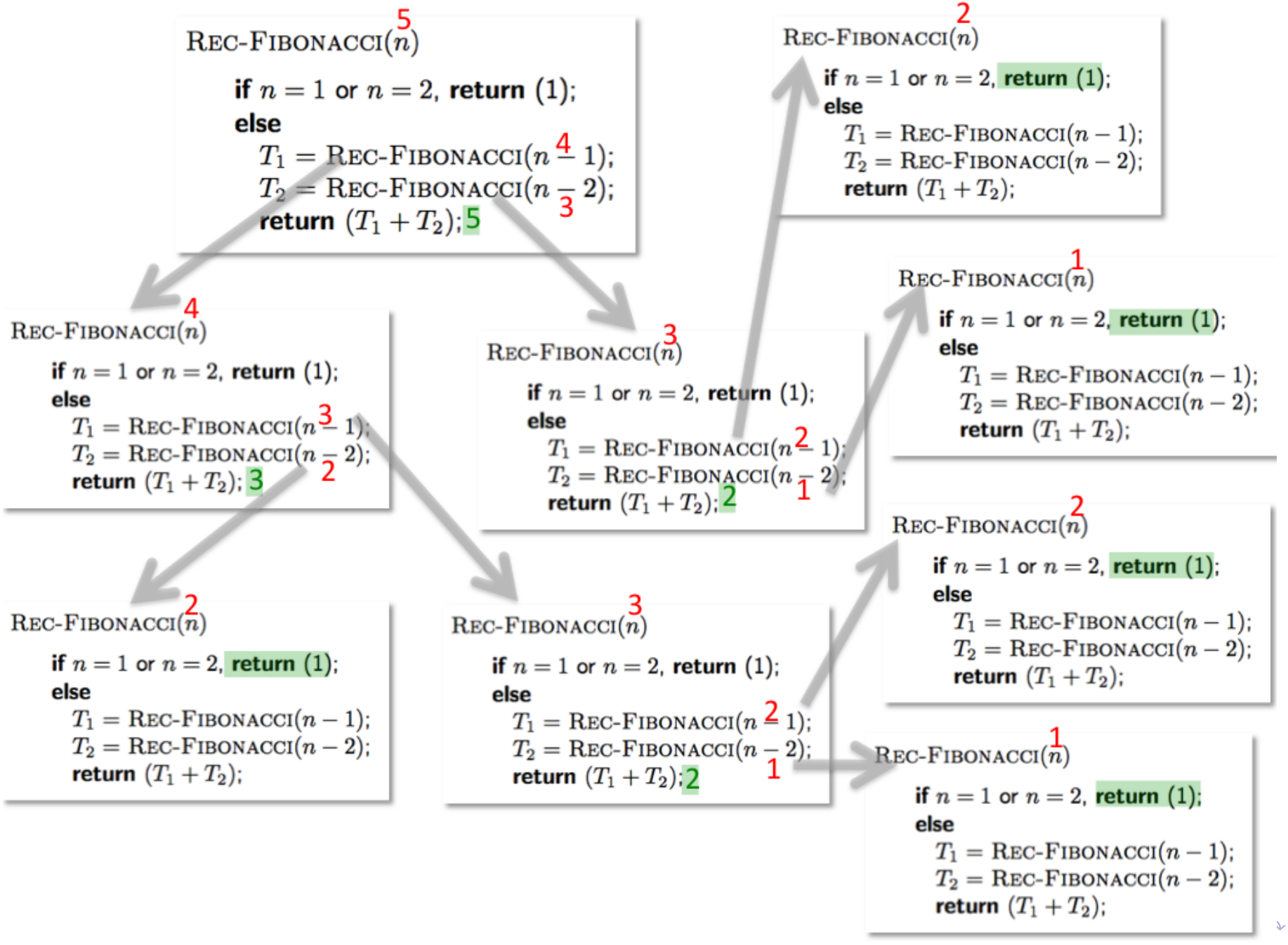

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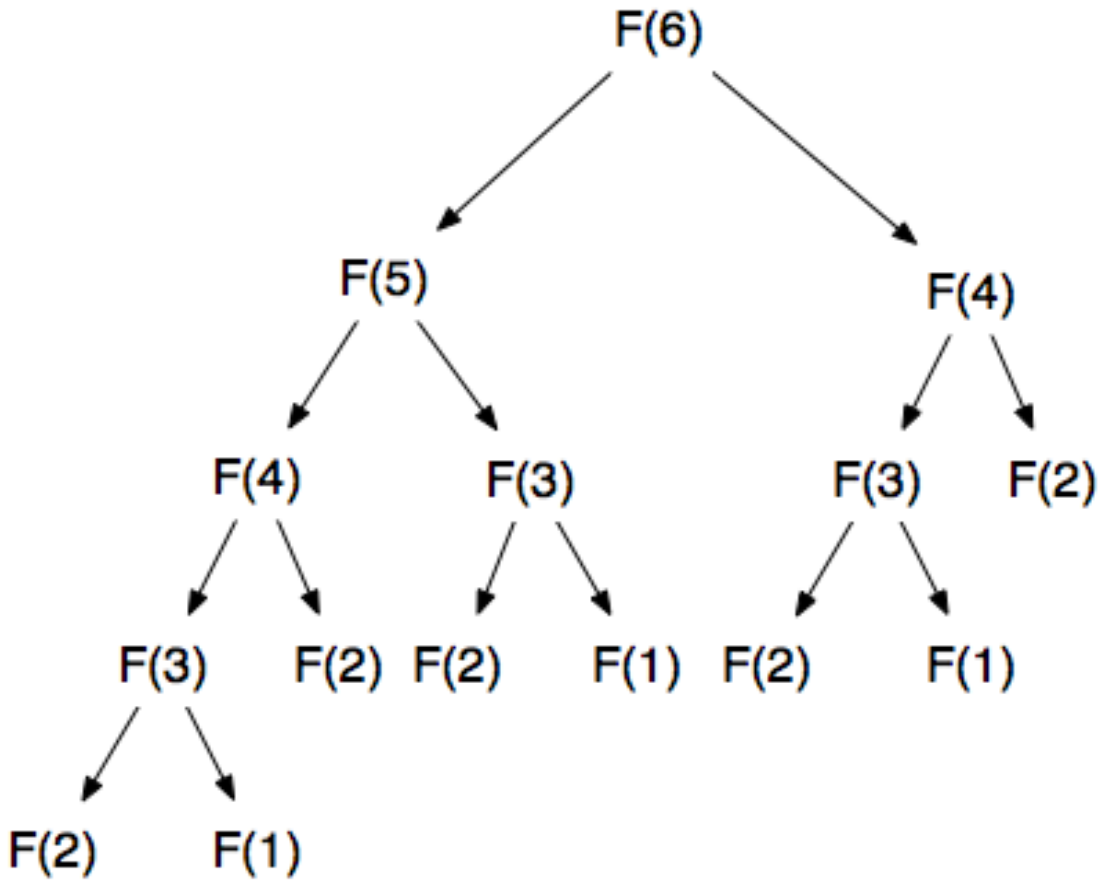

Repeated computations everywhere!

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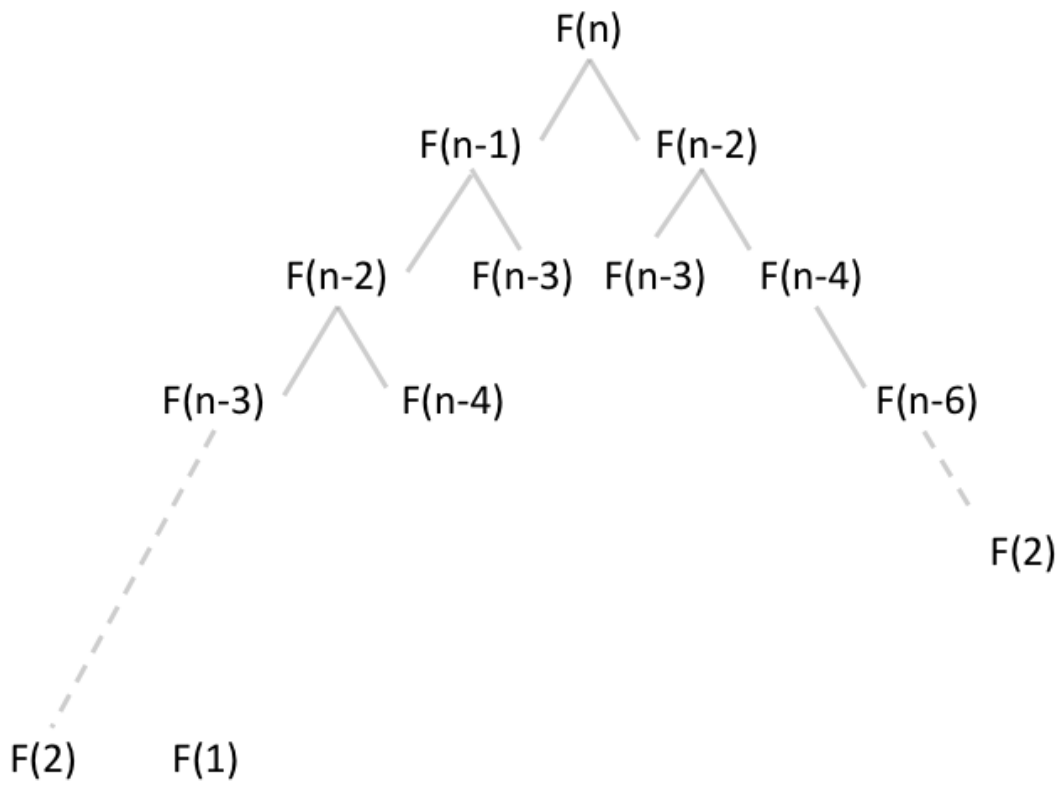

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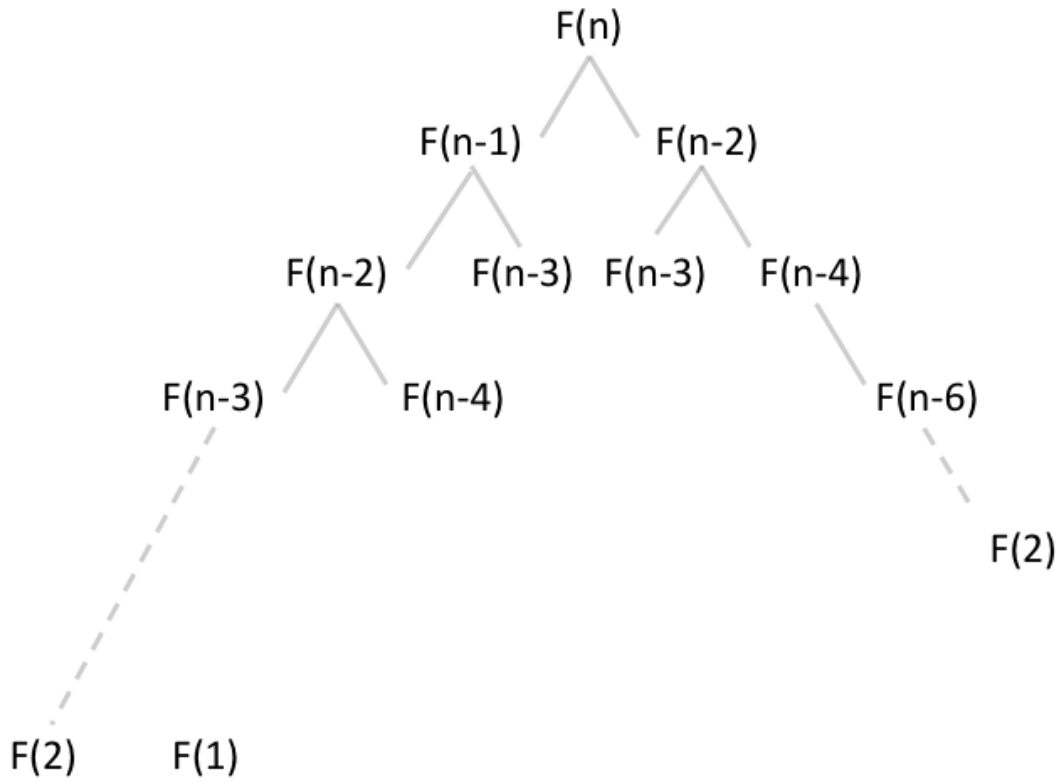

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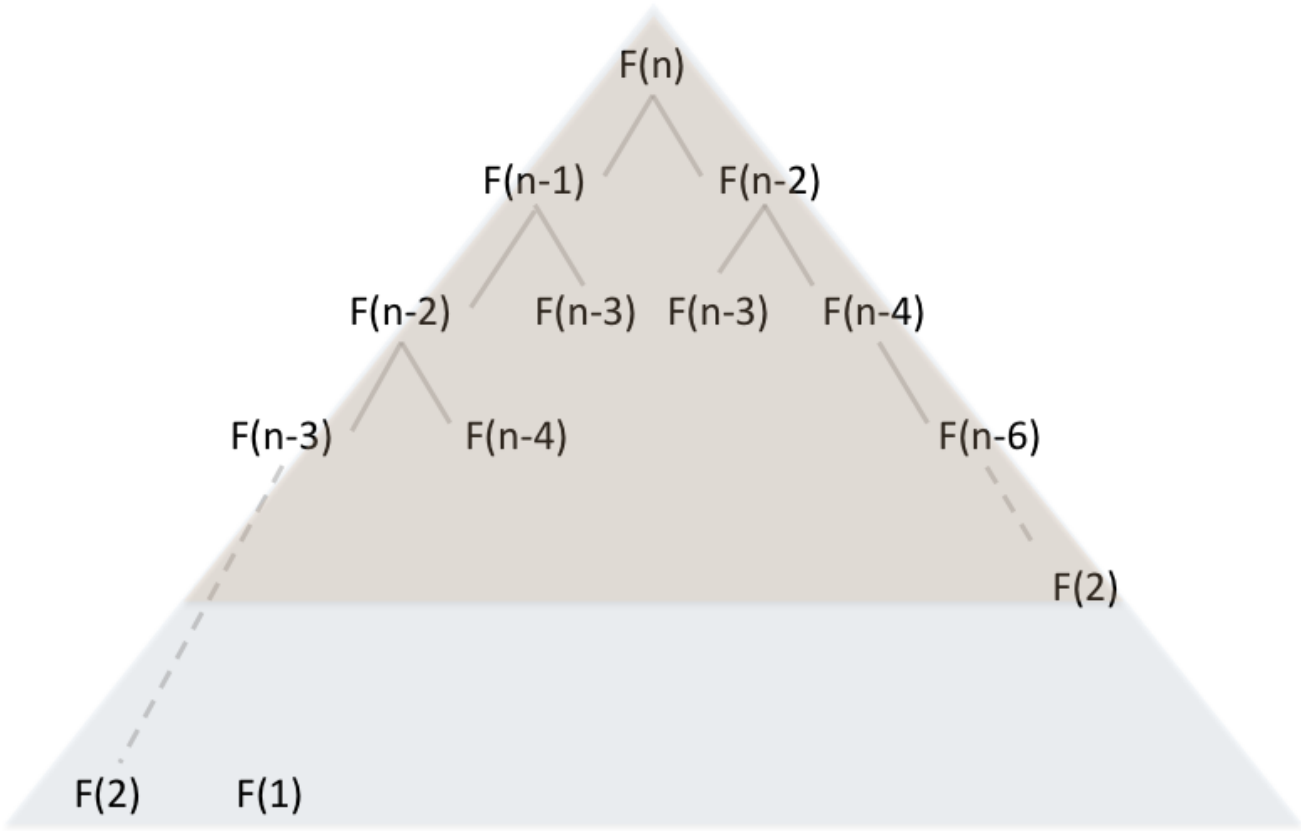

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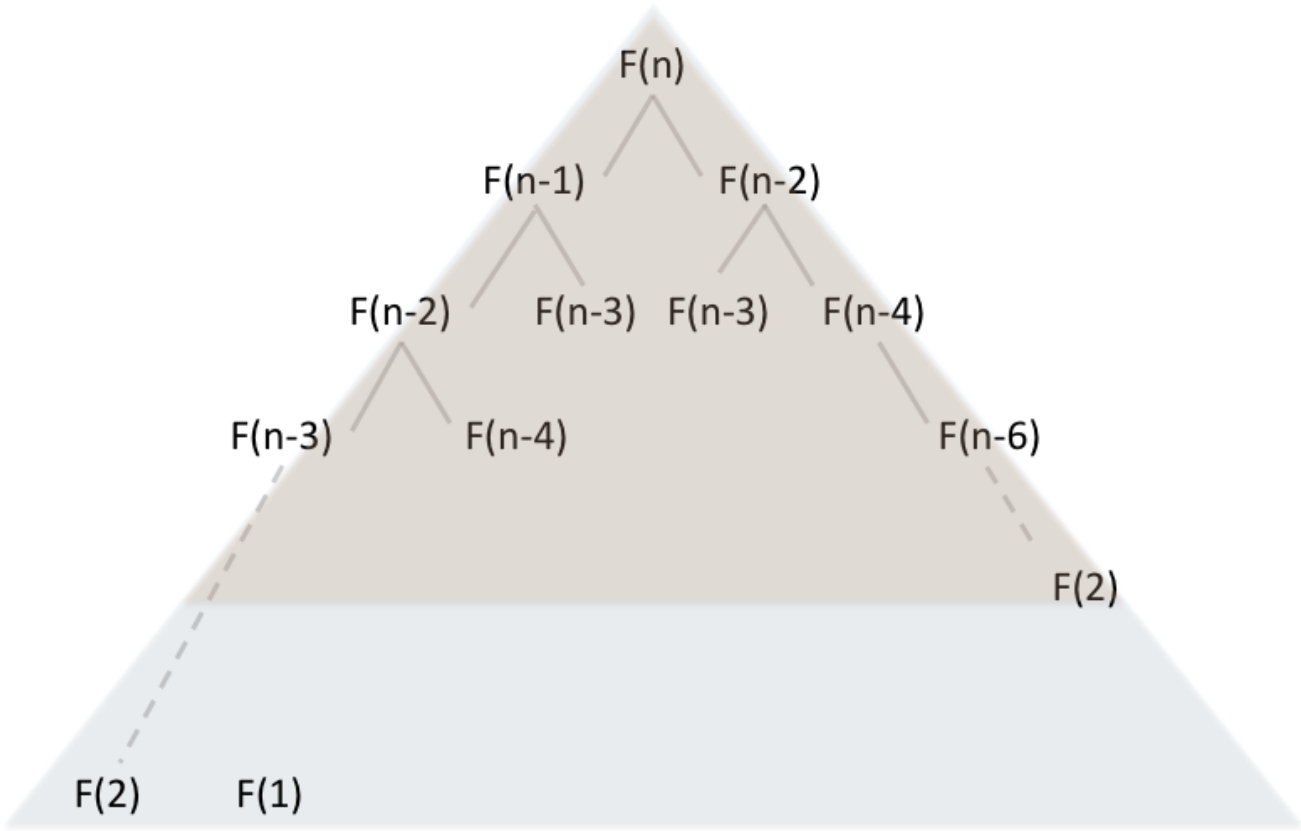

small triangle $\leq$ size of tree $\leq$ large triangle

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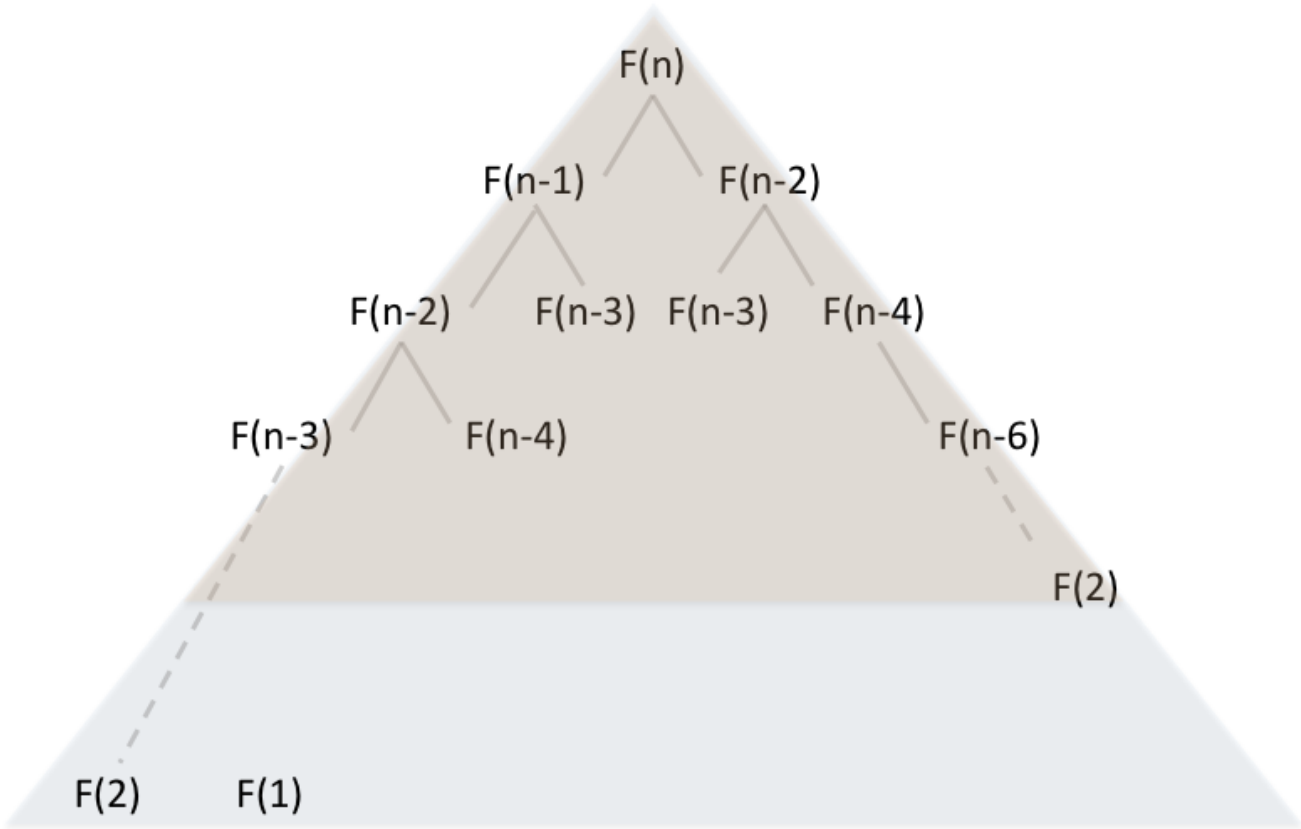

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roughly: $2^{\frac{n}{2}} \leq$ size of tree $\leq 2^n$

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ITERATIVE-FIBONACCI(n) is a simple dynamic programming algorithm.

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$$T(n) = c \times n$$

Chapter 1. The role of algorithms in computing

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What is an Algorithm: a well-defined, finite procedure that takes an input and produces an output.

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Example 2: An algorithm skeleton;

Algorithm MAXIMUM;

INPUT: list $X = \{a_1, \cdots, a_n\}$;
Body that is a series of instructions;
OUTPUT: y, the maximum of $a_1, \cdots, a_n$.

Chapter 1. The Role of Algorithms in Computing

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For some problems, the functions computed are predicates, i.e., output $y \in \{\texttt{TRUE}, \texttt{FALSE}\}$

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Chapter 2. Getting started

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As the process goes, the sorted sublist gets longer, the unsorted sublist gets shorter, until the unsorted becomes empty.

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- (correctness proof): to show that the algorithm is as desired;
- (efficiency proof): to show a guaranteed efficiency of the algorithm

Chapter 2. Getting Started

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Correctness proof: this is to prove

the pre-condition (condition for the input)

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If the algorithm consists of sequential blocks of instructions,
the task is to prove the correct transformation by each block.

This means we need to prove that every sequential statement
in the algorithm transforms the given pre-condition to
the given post-condition.

Chapter 2. Getting Started

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In INSERTION-SORT, the loop invariant is

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However, finding loop invariants is difficult!

Chapter 2. Getting Started

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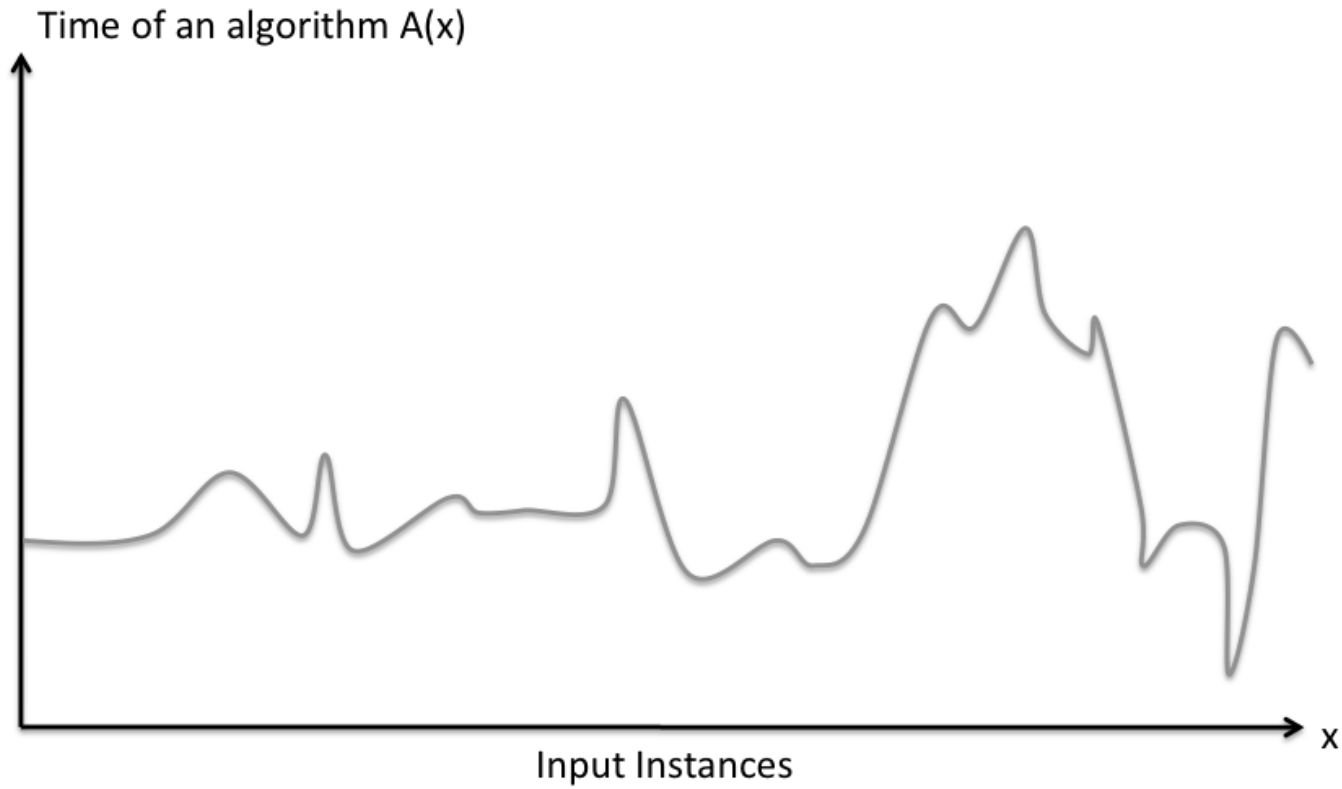
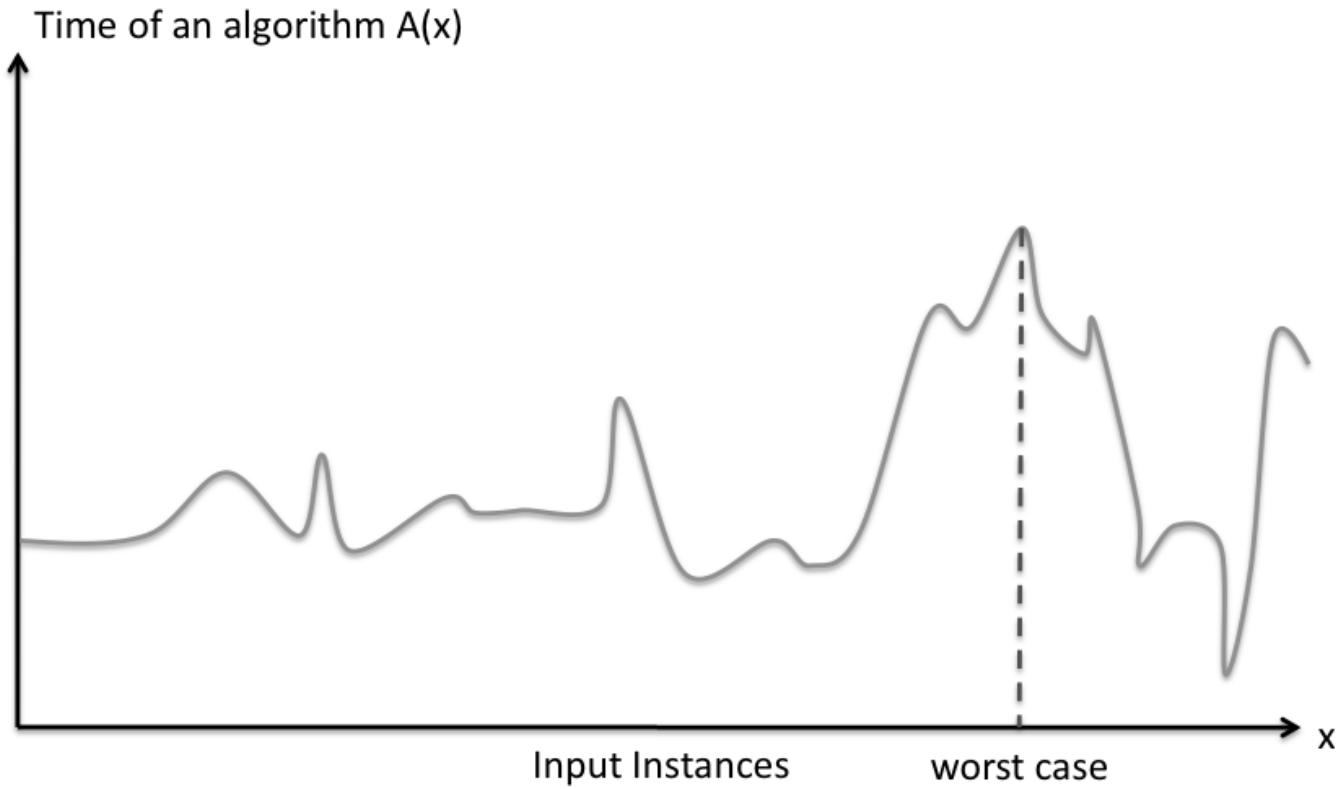
- however, the unit measured is not real time or memory unit.

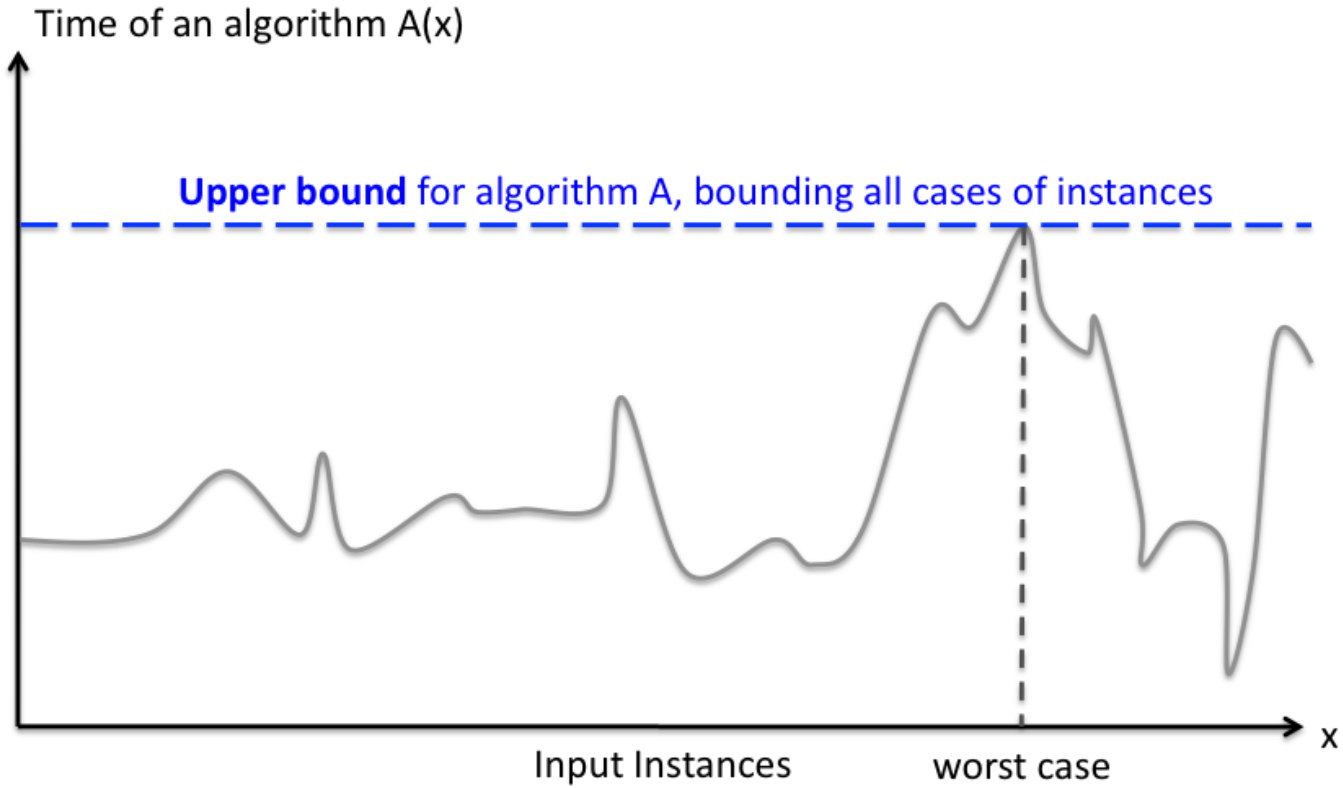
Chapter 2. Getting Started



Chapter 2. Getting Started



Chapter 2. Getting Started



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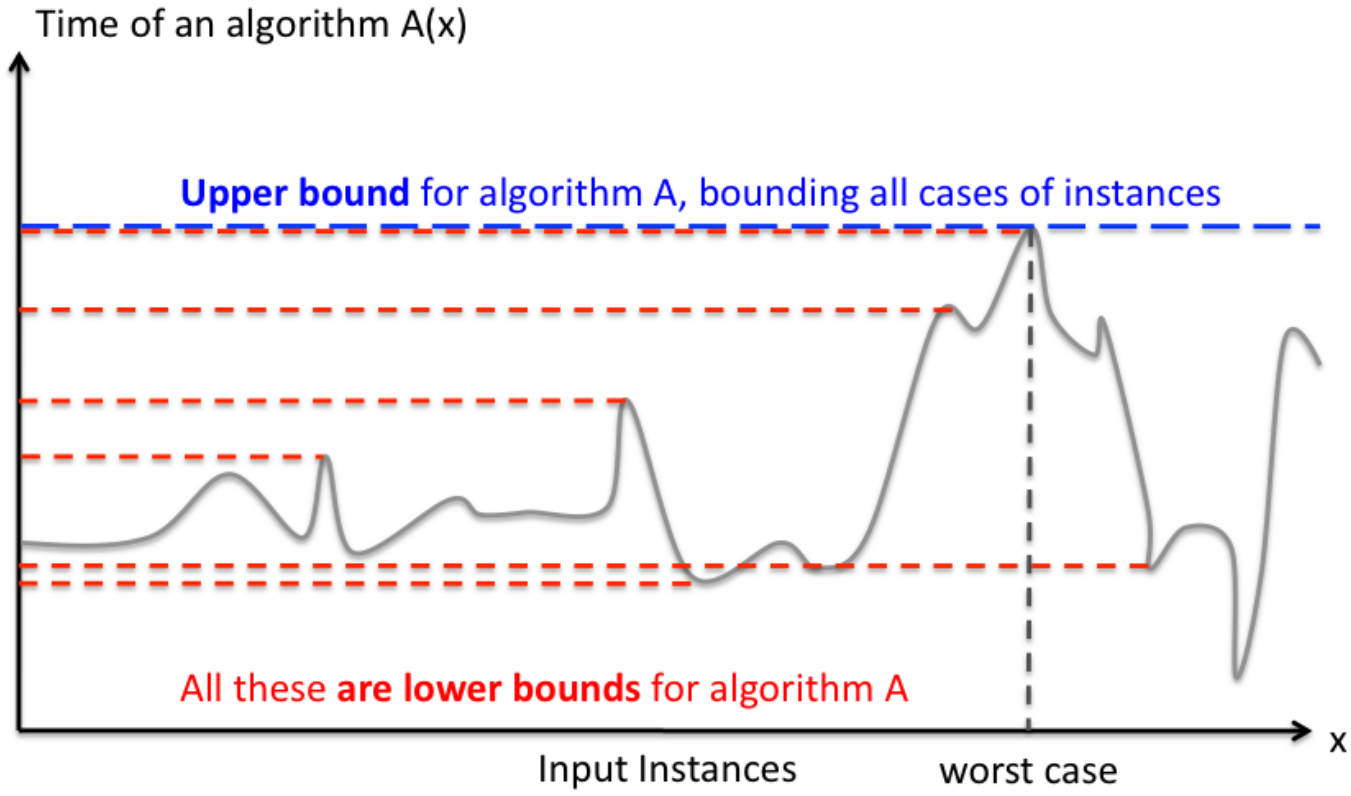

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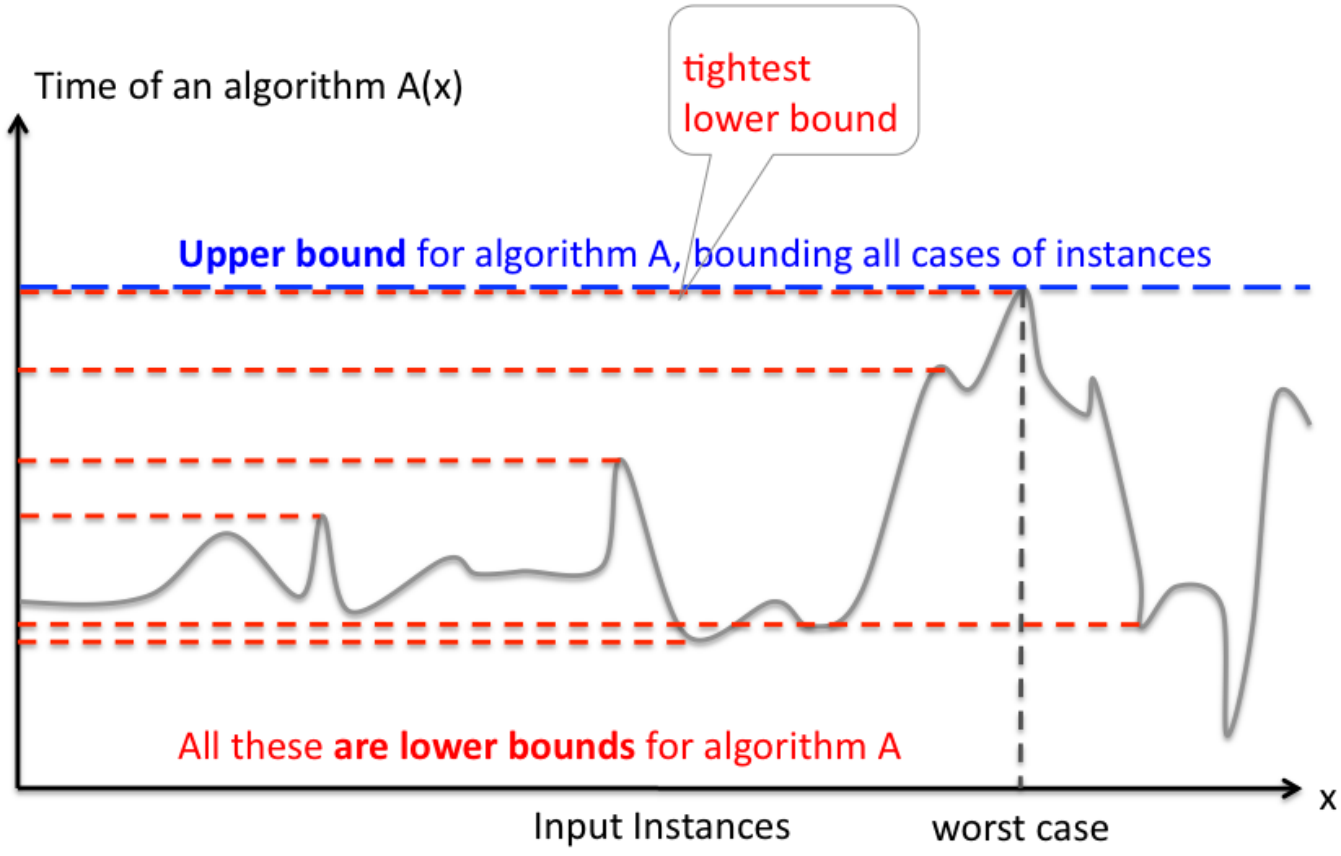

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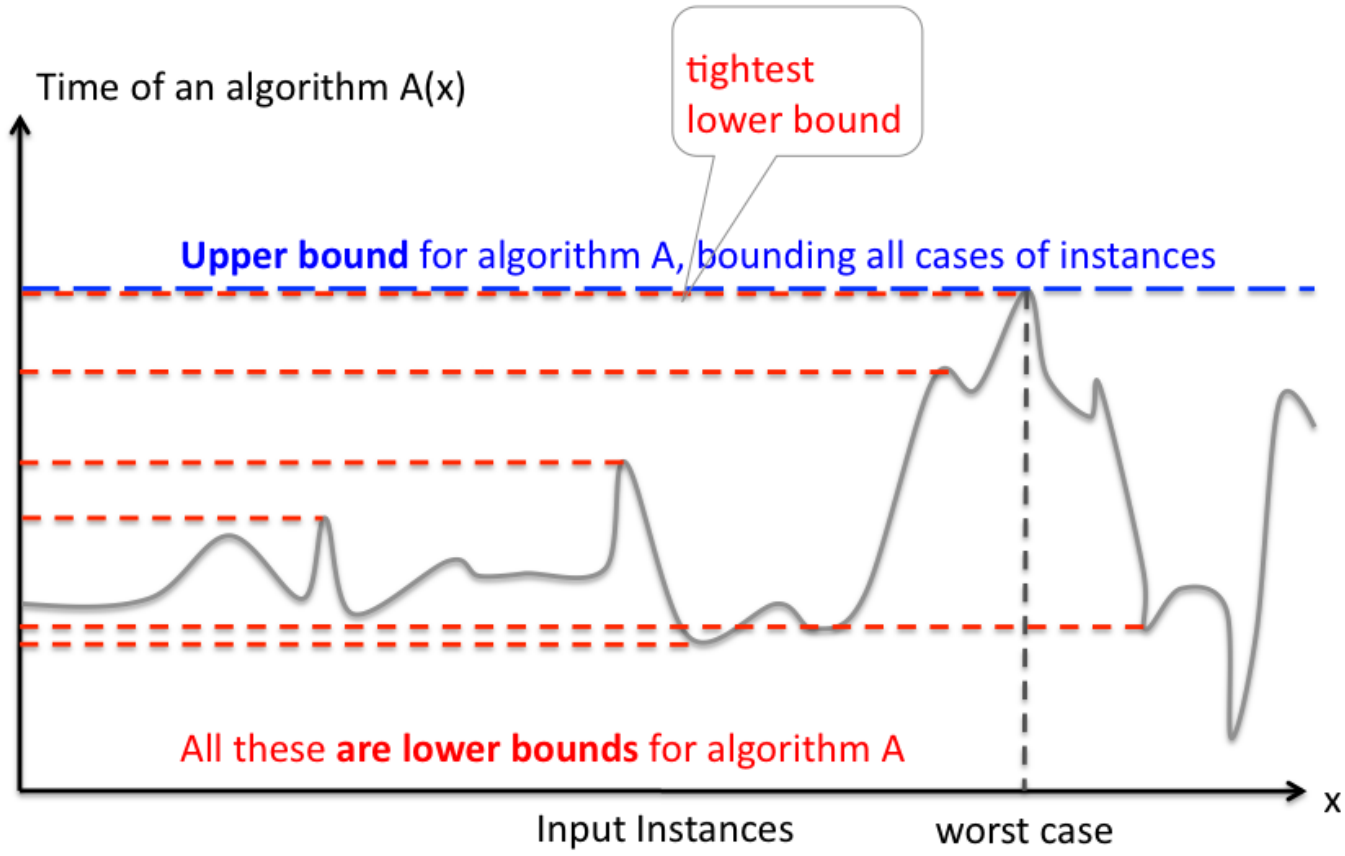

lower bounds ≤ worst case time ≤ upper bounds

Chapter 2. Getting Started

Resource measurement based on

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- speed between different machines: a constant multiplicative factor.

Chapter 2. Getting Started

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Assume t_j to be the number of times **while** is executed for every j.

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

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$$T(n) \leq a\frac{n}{2}(n+1) + bn + c - a \leq xn^2 + yn + z$$

for some constants x, y, z.

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  To sort the 4 items, a constant number of comparisons is needed, each taking a time linear in N (i.e., total time is linear in n).

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   order of growth: $T(n) = an^2 + bn + c$ grows the same rate as an^2 (if $a > 0$).

Chapter 3. Growth of Functions

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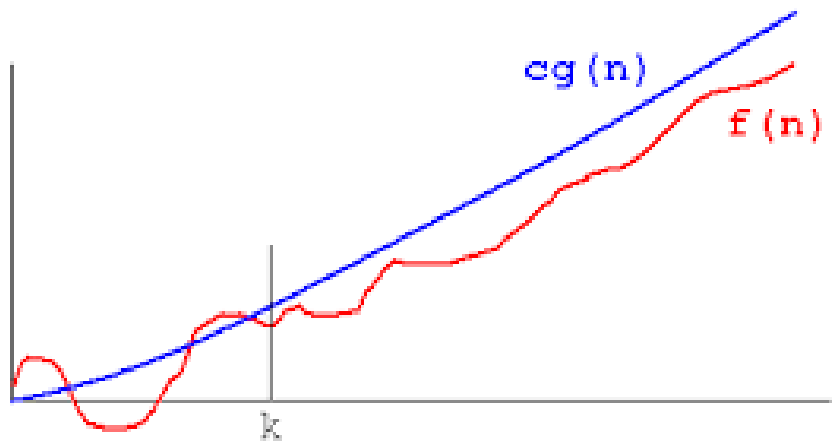

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(9) $n^{2.001}$

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- if B is not constant, for example, $B = \lceil \log_2 n \rceil$, the time is $O(n^2/(log_2^n)^2)$.

Chapter 3. Growth of Functions

Could you also correct the time complexities for INSERTION SORT and REC-FIBONACCI measured by the input size?

Chapter 4. Solving Recurrences

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Algorithm MERGE SORT(A, p, r)

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2. **then** $q = \lfloor \frac{p+r}{2} \rfloor$
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base case: $T(1) \leq c$.

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Chapter 4. Solving Recurrences

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To prove a property $\mathcal{P}(n)$ for every natural number $n \geq 1$, it suffices to prove

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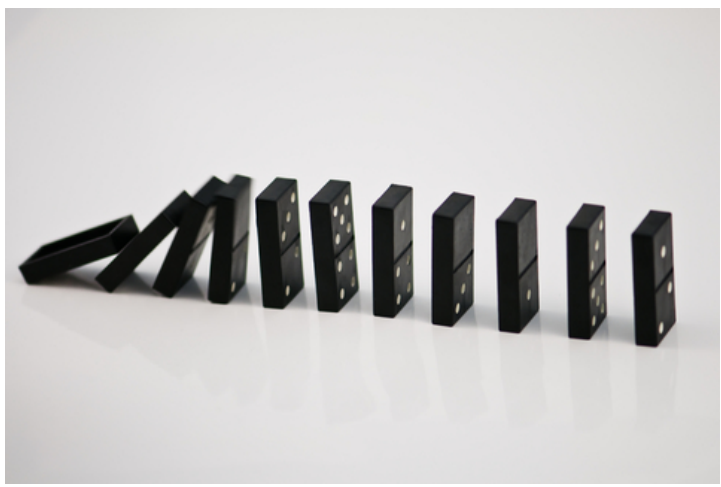
"The principle for dominos to fall".



Chapter 4. Solving Recurrences

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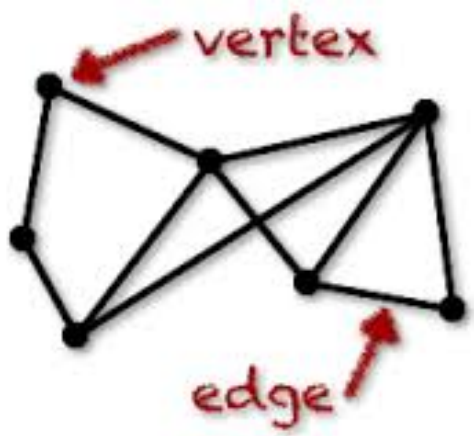

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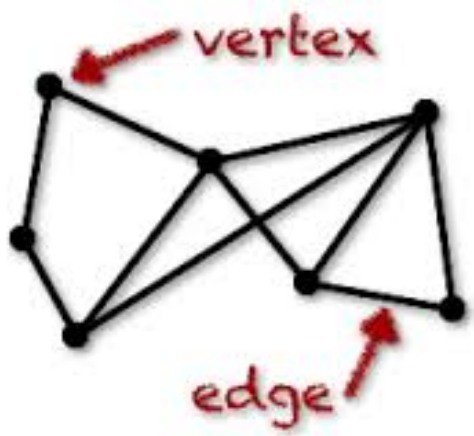

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Let vertex counts of $G_1, \ldots, G_t$ be $m_1, \ldots, m_t$, respectively.
Note that $\sum_{i=1}^{t} m_i = k$ (why?).

By the assumption, these components contain at least $m_1 - 1, \ldots m_t - 1$ edges, respectively.

The total count of edges in G is

$$\geq \sum_{i=1}^{t}(m_i - 1) + r = \sum_{i=1}^{t} m_i - \sum_{i=1}^{t} 1 + r = \sum_{i=1}^{t} m_i + r - t$$

Chapter 4. Solving Recurrences

Lemma: every *connected graph* of n vertices has $\geq n-1$ edges.

Proof:

Base case: $n = 1$, a connected graph of single vertex contains zero edges.

Assumption: Assume the lemma holds for every i, $i \leq k$, where $k \geq 1$. That is, a connected graph of i vertices contains at least $i-1$ edges.

Induction: Let G be any connected graph G of $k+1$ vertices and v be an arbitrary vertex in it. Assume v shares edges with r other vertices, $r \geq 1$ (why?).

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Chapter 4. Solving Recurrences

Example: binary search algorithm

Algorithm BINARY SEARCH(A, key, i, j)

1. **if** $j < i$ **return** (NULL)
2. **else**
3. $k = \lfloor \frac{i+j}{2} \rfloor$
4. **if** $A[k] = key$ **return** (k)
5. **else**
6. **if** $A[k] > key$ BINARY SEARCH $(A, key, i, k-1)$
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Let $n = j - i + 1$. It has the time with recurrence,

$$T(n) \leq T(\lfloor \frac{n}{2} \rfloor) + c \text{ and } T(1) = d$$

where $c > 0, d > 0$ are constants.

Chapter 4. Solving Recurrences

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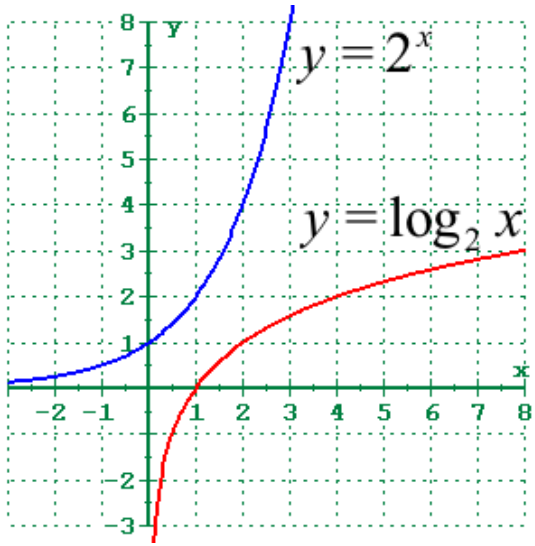
Chapter 4. Solving Recurrences

A little review on logarithm functions:



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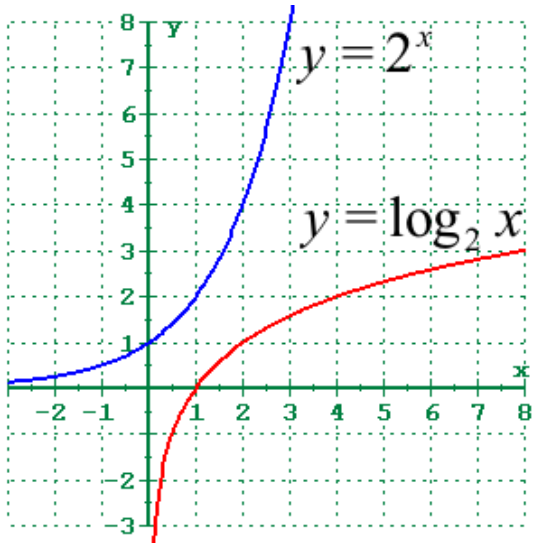

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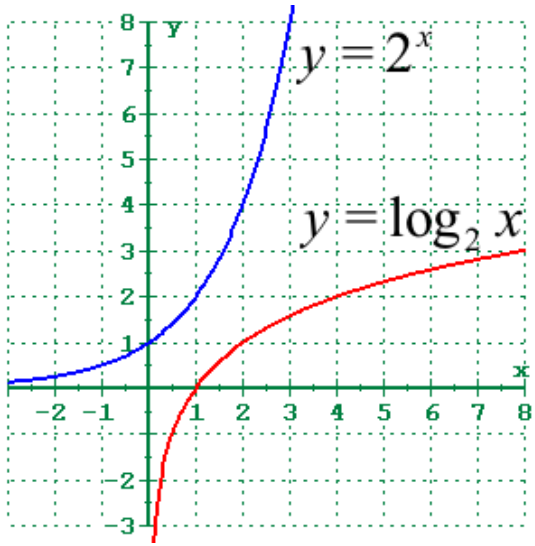

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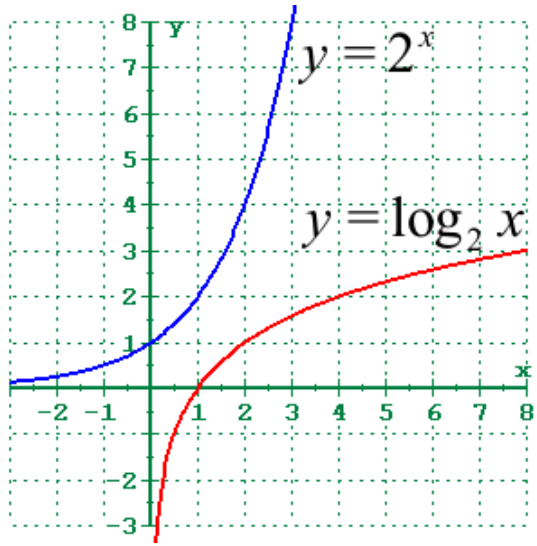

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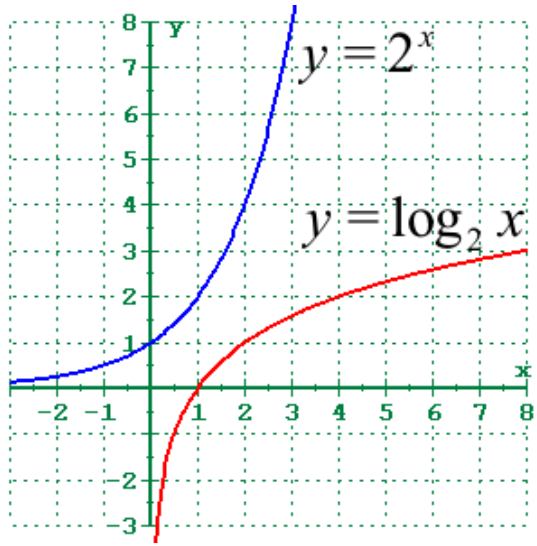

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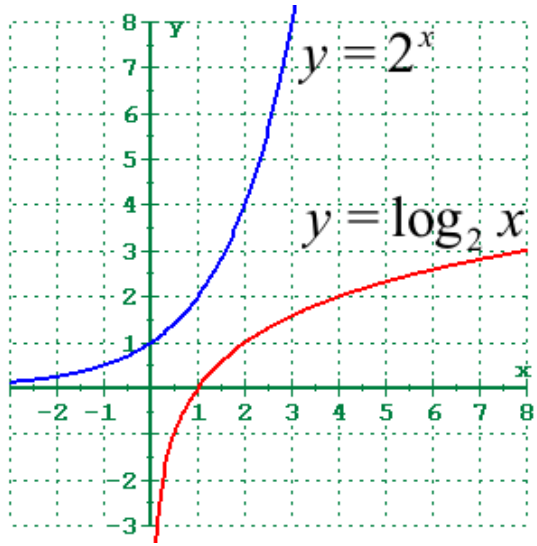

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Solving recurrence with the substitution method (guess then verify)

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We instead **guess** $T(n) \leq cn\log_2 n + 2$, so it holds for $n = 1$.

Now assume the guessed upper bound holds for $\lfloor\frac{2n}{3}\rfloor$, i.e.,

$$T(\lfloor\tfrac{2n}{3}\rfloor) \leq c\lfloor\tfrac{2n}{3}\rfloor\log_2\lfloor\tfrac{2n}{3}\rfloor + 2$$

substitute it in the recurrence, we get

$$T(n) = \frac{3}{2}T(\lfloor\frac{2n}{3}\rfloor) + n \leq \frac{3}{2}(c\lfloor\frac{2n}{3}\rfloor\log_2\lfloor\frac{2n}{3}\rfloor + 2) + n \leq cn\log_2\frac{2n}{3} + 3 + n$$

$$= cn(\log_2 n + \log_2\tfrac{2}{3}) + 3 + n = cn\log_2 n + 3 + n - cn\log_2\tfrac{3}{2}$$

$$T(n) = \frac{3}{2}T(\lfloor\frac{2n}{3}\rfloor)+n \leq \frac{3}{2}(c\lfloor\frac{2n}{3}\rfloor \log_2\lfloor\frac{2n}{3}\rfloor+2)+n \leq cn\log_2\frac{2n}{3}+3+n$$

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To allow that the last term $\leq cn\log_2 n + 2$, the assumed upper bound,

$$T(n) = \frac{3}{2}T(\lfloor\frac{2n}{3}\rfloor)+n \leq \frac{3}{2}(c\lfloor\frac{2n}{3}\rfloor \log_2\lfloor\frac{2n}{3}\rfloor+2)+n \leq cn\log_2\frac{2n}{3}+3+n$$

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$$=\leq cn\log_2 n + 2 + 1 + n - cn\log_2\tfrac{3}{2}$$

To allow that the last term $\leq cn\log_2 n + 2$, the assumed upper bound, we have choose c such that $1 + n - cn\log_2\frac{3}{2} \leq 0$.

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for example, we can choose $c = 2/\log_2\frac{3}{2}$, such that

$$1 + n - cn\log_2\frac{3}{2} = 1 + n - 2n = 1 - n \leq 0$$

for all $n \geq 1$.

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So we have shown $T(n) \leq cn\log_2 n + 2$, for all $n \geq 1$.

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So we have shown $T(n) \leq cn\log_2 n + 2$, for all $n \geq 1$. Are you sure?

$$T(n) = \frac{3}{2}T(\lfloor\frac{2n}{3}\rfloor)+n \leq \frac{3}{2}(c\lfloor\frac{2n}{3}\rfloor \log_2\lfloor\frac{2n}{3}\rfloor+2)+n \leq cn\log_2\frac{2n}{3}+3+n$$

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So we have shown $T(n) \leq cn\log_2 n + 2$, for all $n \geq 1$. Are you sure?

Does this also imply

$$T(n) = O(n\log_2 n) \text{ or } T(n) = O(n\log_2 n + 2)\,?$$

Chapter 4. Solving Recurrences

2. Changing variables

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$T(2^m) = 2T(2^{m/2}) + m$

rename the function: $S(m) = T(2^m)$

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solve it, we have $S(m) = O(m \log m)$

so $T(n) = T(2^m) = O(m \log m) = O(\log n \log \log n)$.

Chapter 4. Solving Recurrences

3. Recursive tree method

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(1) $T(n)$ is a tree with non-recursive terms as the root and recursive terms as its children.

(2) for each child, replace it with then non-recursive terms and produce children that are then recursive terms

(3) repeat (2), expand the tree until all children are the base case.

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l_0: $T(n)$

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l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	

Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

$$
\begin{array}{lccccccc}
l_0: & & & T(n) & & & & n^2 \\
l_1: & & T(n/4) & & T(n/4) & & T(n/4) & 3(\frac{n}{4})^2 \\
l_2: & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & &
\end{array}
$$

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l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	$3(\frac{n}{4})^2$
l_2:	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$3^2(\frac{n}{4^2})^2$
l_3:	$\ldots\ldots$			

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l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	$3(\frac{n}{4})^2$
l_2:	$T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$	$T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$	$T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$	$3^2(\frac{n}{4^2})^2$
l_3:	.....			$3^3(\frac{n}{4^3})^2$
l_4:	.....			

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l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	$3(\frac{n}{4})^2$
l_2:	$T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$	$T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$	$T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$	$3^2(\frac{n}{4^2})^2$
l_3:	.....			$3^3(\frac{n}{4^3})^2$
l_4:	.....			
l_{m-1}:	.....			

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$$
\begin{array}{lccccccc}
l_0: & & & T(n) & & & & n^2 \\
l_1: & & T(n/4) & & T(n/4) & & T(n/4) & 3(\frac{n}{4})^2 \\
l_2: & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & & 3^2(\frac{n}{4^2})^2 \\
l_3: & \ldots\ldots & & & & & & 3^3(\frac{n}{4^3})^2 \\
l_4: & \ldots\ldots & & & & & & \\
l_{m-1}: & \ldots\ldots & & & & & & 3^{m-1}(\frac{n}{4^{m-1}})^2
\end{array}
$$

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Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

l_0: $T(n)$ n^2

l_1: $T(n/4)$ $T(n/4)$ $T(n/4)$ $3(\frac{n}{4})^2$

l_2: $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $T(\frac{n}{4^2})$ $3^2(\frac{n}{4^2})^2$

l_3: ..... $3^3(\frac{n}{4^3})^2$

l_4: .....

l_{m-1}: ..... $3^{m-1}(\frac{n}{4^{m-1}})^2$

l_m: $T(1), T(1), T(1), T(1), T(1), \ldots,$ $T(1)$

Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	$3(\frac{n}{4})^2$
l_2:	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$3^2(\frac{n}{4^2})^2$
l_3:	$\ldots\ldots$			$3^3(\frac{n}{4^3})^2$
l_4:	$\ldots\ldots$			
l_{m-1}:	$\ldots\ldots$			$3^{m-1}(\frac{n}{4^{m-1}})^2$
l_m:	$T(1), T(1), T(1), T(1), T(1), \ldots,$			$T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

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Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	$3(\frac{n}{4})^2$
l_2:	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$3^2(\frac{n}{4^2})^2$
l_3:	$\ldots\ldots$			$3^3(\frac{n}{4^3})^2$
l_4:	$\ldots\ldots$			
l_{m-1}:	$\ldots\ldots$			$3^{m-1}(\frac{n}{4^{m-1}})^2$
l_m:	$T(1), T(1), T(1), T(1), T(1), \ldots,$			$T(1)$

where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

$$T(n) = n^2[1 + 3(\tfrac{1}{4})^2 + 3^2(\tfrac{1}{4^2})^2 + 3^3(\tfrac{1}{4^3})^2 + \cdots + 3^{m-1}(\tfrac{1}{4^{m-1}})^2] + 3^m T(1)$$

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$$
\begin{array}{lcccccc}
l_0: & & & T(n) & & & n^2 \\
l_1: & & T(n/4) & & T(n/4) & & T(n/4) \quad 3(\frac{n}{4})^2 \\
l_2: & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & & T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2}) & 3^2(\frac{n}{4^2})^2 \\
l_3: & \ldots\ldots & & & & & 3^3(\frac{n}{4^3})^2 \\
l_4: & \ldots\ldots & & & & & \\
l_{m-1}: & \ldots\ldots & & & & & 3^{m-1}(\frac{n}{4^{m-1}})^2 \\
l_m: & T(1), T(1), T(1), T(1), T(1), \ldots, & & & & & T(1)
\end{array}
$$

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$$T(n) = n^2[1 + 3(\tfrac{1}{4})^2 + 3^2(\tfrac{1}{4^2})^2 + 3^3(\tfrac{1}{4^3})^2 + \cdots + 3^{m-1}(\tfrac{1}{4^{m-1}})^2] + 3^m \times 1$$

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l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	$3(\frac{n}{4})^2$
l_2:	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$3^2(\frac{n}{4^2})^2$
l_3:	$\ldots\ldots$			$3^3(\frac{n}{4^3})^2$
l_4:	$\ldots\ldots$			
l_{m-1}:	$\ldots\ldots$			$3^{m-1}(\frac{n}{4^{m-1}})^2$
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$$= n^2[1 + 3(\tfrac{1}{4})^2 + 3^2(\tfrac{1}{4^2})^2 + 3^3(\tfrac{1}{4^3})^2 + \cdots + 3^{m-1}(\tfrac{n}{4^{m-1}})^2] + 3^m(\tfrac{n}{4^m})^2$$

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l_0:		$T(n)$		n^2
l_1:	$T(n/4)$	$T(n/4)$	$T(n/4)$	$3(\frac{n}{4})^2$
l_2:	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$T(\frac{n}{4^2})\ T(\frac{n}{4^2})\ T(\frac{n}{4^2})$	$3^2(\frac{n}{4^2})^2$
l_3:	$\ldots\ldots$			$3^3(\frac{n}{4^3})^2$
l_4:	$\ldots\ldots$			
l_{m-1}:	$\ldots\ldots$			$3^{m-1}(\frac{n}{4^{m-1}})^2$
l_m:	$T(1), T(1), T(1), T(1), T(1), \ldots,$			$T(1)$

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$$T(n) = n^2[1 + 3(\tfrac{1}{4})^2 + 3^2(\tfrac{1}{4^2})^2 + 3^3(\tfrac{1}{4^3})^2 + \cdots + 3^{m-1}(\tfrac{1}{4^{m-1}})^2] + 3^m \times 1$$
$$= n^2[1 + 3(\tfrac{1}{4})^2 + 3^2(\tfrac{1}{4^2})^2 + 3^3(\tfrac{1}{4^3})^2 + \cdots + 3^{m-1}(\tfrac{n}{4^{m-1}})^2] + 3^m(\tfrac{n}{4^m})^2$$
$$= n^2[1 + \tfrac{3}{16} + (\tfrac{3}{16})^2 + (\tfrac{3}{16})^3 + \cdots + (\tfrac{3}{16})^m]$$

Chapter 4. Solving Recurrences

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l_0: $T(n)$ n^2

l_1: $T(n/4)$ $T(n/4)$ $T(n/4)$ $3(\frac{n}{4})^2$

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where $\frac{n}{4^m} = 1$, i.e., $m = \log_4 n$.

Then $T(n)$ is the sum

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Chapter 4. Solving Recurrences

Example $T(n) = 3T(n/4) + n^2$, with base case $T(1) = 1$

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$$= \tfrac{16}{13}n^2$$

for all $n > 0$.

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Recursive tree notation in the textbook:

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Chapter 5. Probabilistic analysis and randomized algorithms

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A good example: QUICK SORT algorithm [Hoare'1959]

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- Las Vegas algorithms is as powerful as Monte Carlo algorithms, if not more.