CSCI 4470/6470 Algorithms, Fall 2015

Lecture Note V

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Part VII. Selected Topics

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Chapter 34 NP-Completeness

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Part VII. Selected Topics

Chapter 34 NP-Completeness

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   - reduction, polynomial-time reduction

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   - decision versions of optimization problems
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Chapter 34 NP-Completeness

1. Intractable problems
   - decision versions of optimization problems
2. Nondeterministic computational models
   - nondeterministic computation = certificate + verification
3. NP-completeness framework
   - reduction, polynomial-time reduction
4. NP-completeness proof
   - NP-complete problems, reduction techniques.

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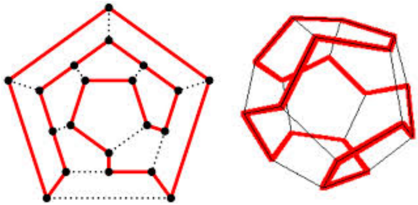

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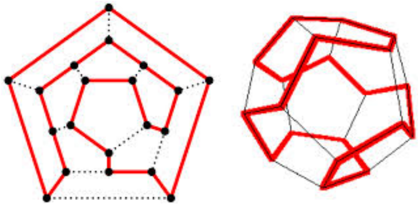

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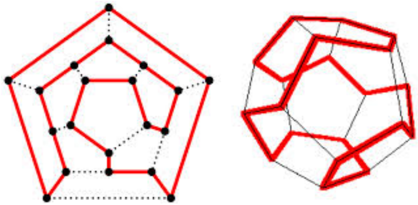

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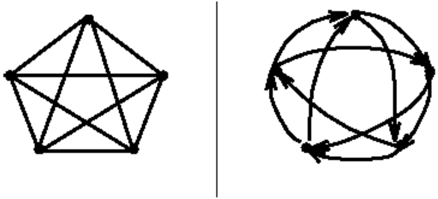

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$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \geq n \times (n-1) \cdots \times \frac{n}{2} \geq (\frac{n}{2})^{\frac{n}{2}}$$

- all known algorithms (solving TSP) are of exponential-time.

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Proof: (P-time algorithms for $\mathrm{TSP} \Longleftarrow$ P-time algorithms for HCW)

- assume P-time algorithm A for HCW such that $A(G, K) =$"YES/"NO"
- construct a P-time algorithm $B(G)$ for TSP to behave as follows:

1. on input G, **for** every possible values of K, **call** $A(G, K)$;
   remember the smallest k_{min} such that $A(G, k_{min}) =$"YES".
2. mark all edges in G as "unvisited";
   while there are "unvisited" edges in G
   pick an "unvisited" edge (u, v), mark it "visited";
   let $G' = G - \{(u, v)\}$;
   if $A(G', k_{min}) =$ "YES"
   then $G = G'$;
   else mark (u, v) "critical";
   return (all "critical" edges)

- show algorithm B runs in P-time. How to make Step 1 P-time?

Theorem 1 says problems HCW and TSP are "polynomially equivalent".

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4. We need a rigorous framework to study tractability via the notion "Polynomial equivalency".

Chapter 34. NP-Completeness

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- All algorithms we have seen so far are deterministic.
- Every deterministic algorithm can be unfolded into a linear sequence of steps (when the input is given).

$M = -\infty$
$n = 3$
$i = 1$
check $1 \leq 3$
check $-\infty < 10$
$M = 10$
$i = 2$
check $2 \leq 3$
check $10 < 30$
$M = 30$
$i = 3$
check $3 \leq 3$
check $30 < 20$
$i = 4$
check $4 \leq 3$
return (30)

Unfolded when input $L = (10, 30, 20)$

MaxOfList(L)
$M = -\infty$
$n = length(L)$
for $i = 1$ **to** n
  if $M < L[i]$
    $M = L[i]$
return (M)

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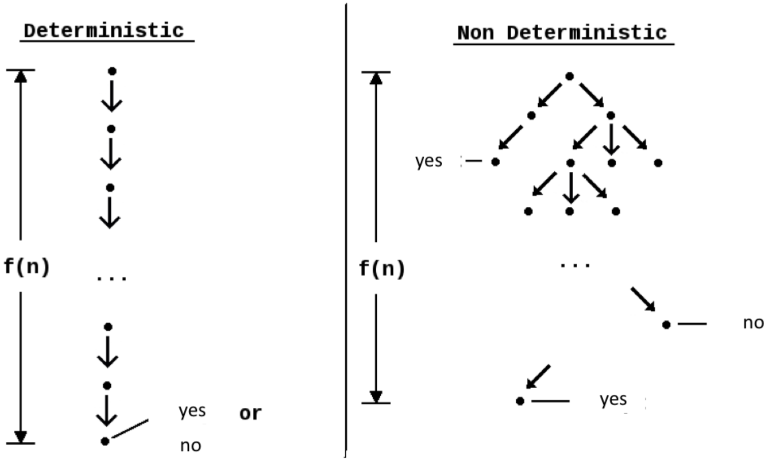
Let us call this **tree model** of nondeterministic algorithms.

Chapter 34. NP-Completeness



Use nondeterministic algorithms to solve problem HAMILTONIAN CYCLE

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- The algorithm runs in polynomial time as each path takes $O(n)$ steps.

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Can you prove the claim?

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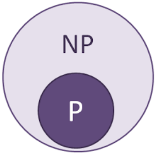

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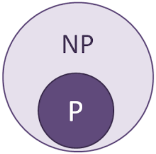

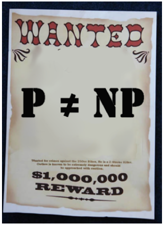

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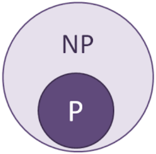

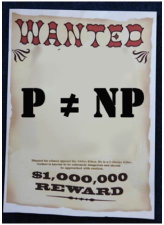

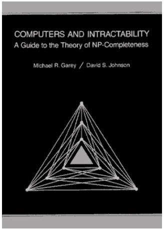

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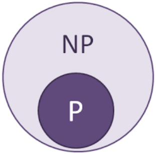
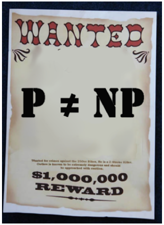
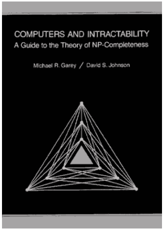
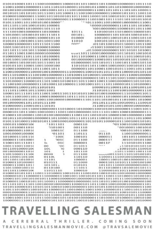
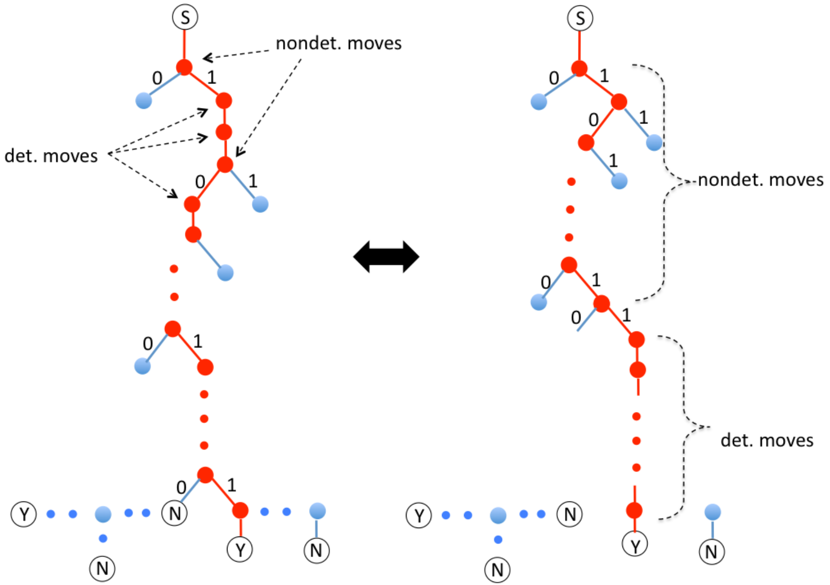








Chapter 34. NP-Completeness



Chapter 34. NP-Completeness

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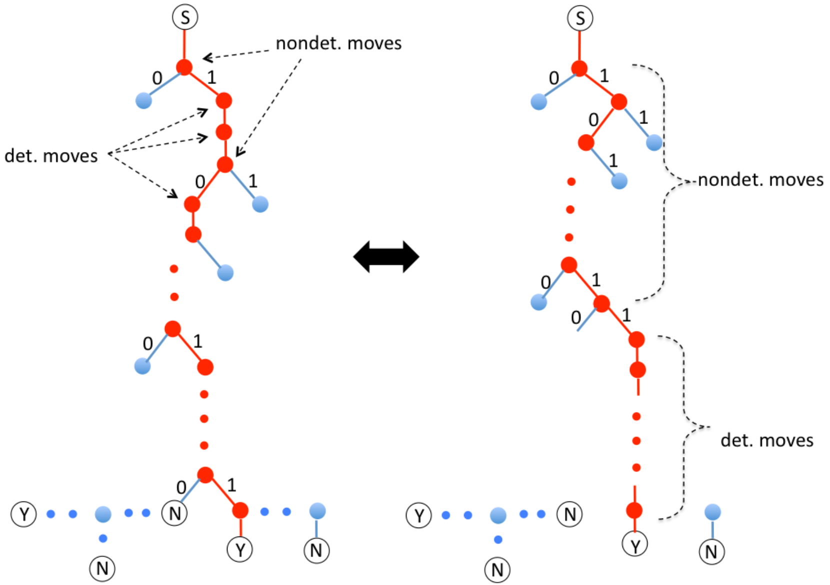

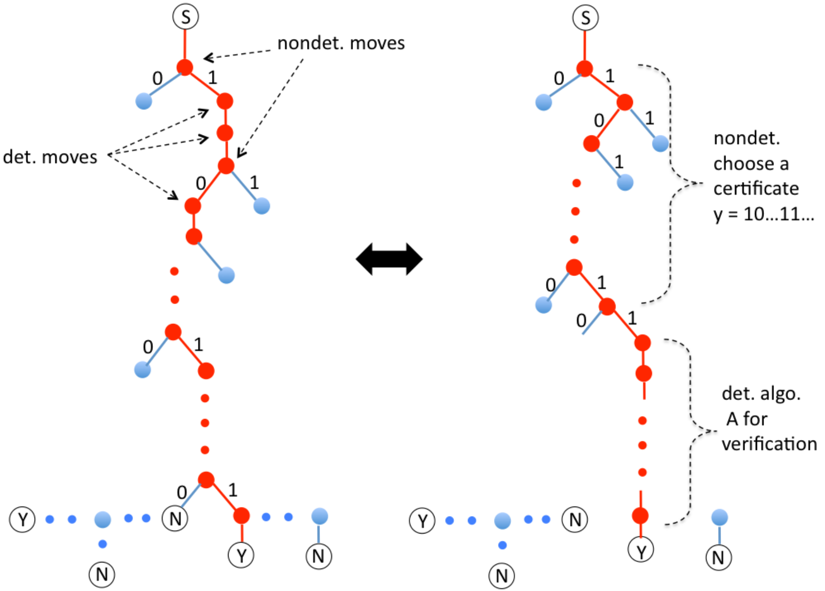
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Chapter 34. NP-Completeness

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Chapter 34. NP-Completeness

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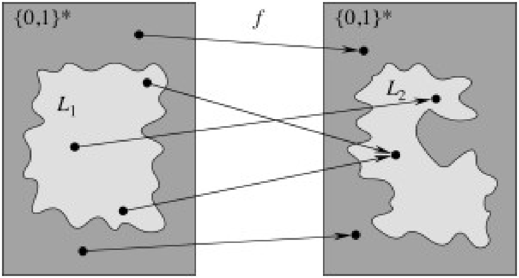

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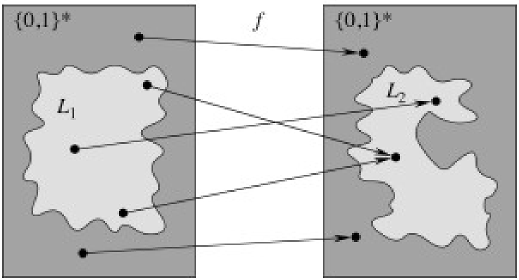

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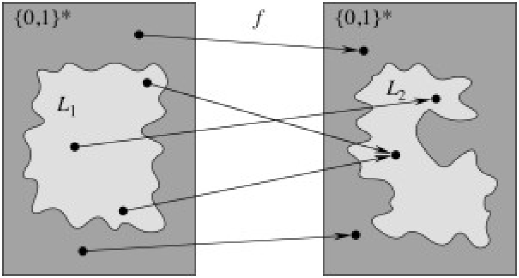

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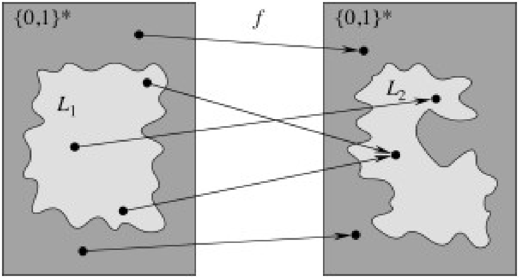

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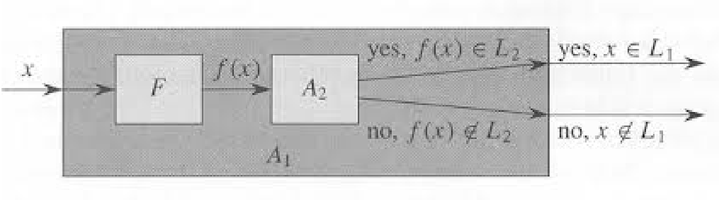

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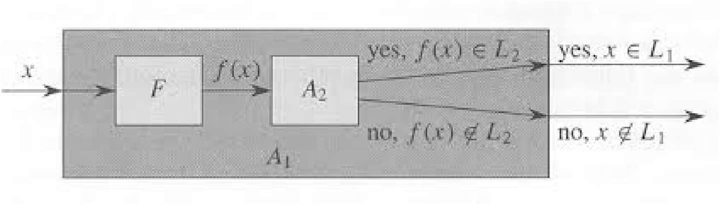

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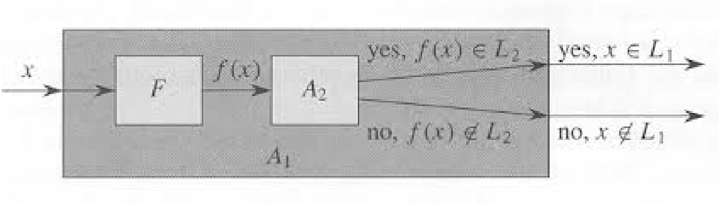

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So the combined algorithm (gray-color box) solves for L_1.

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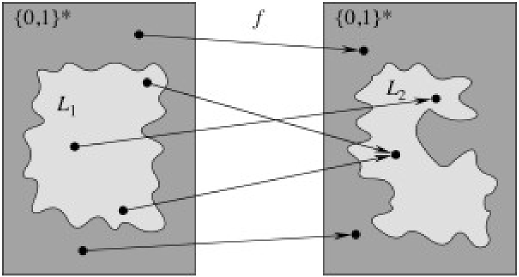

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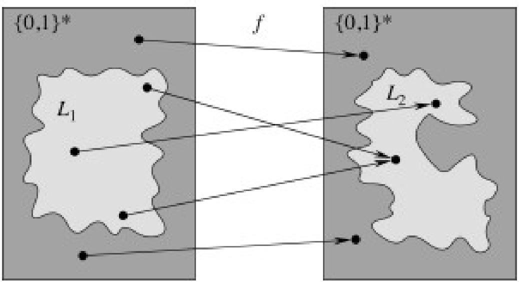

For example, $L_{IS} \leq_p L_V$.

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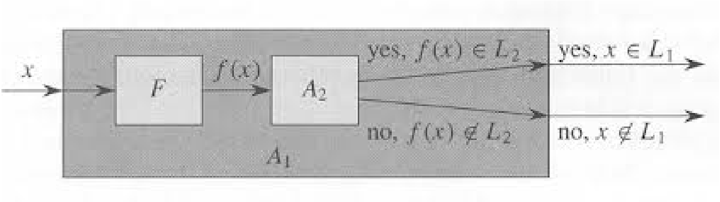

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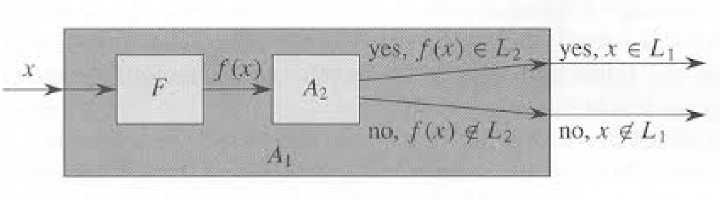

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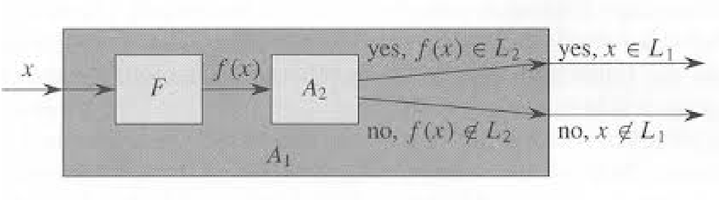

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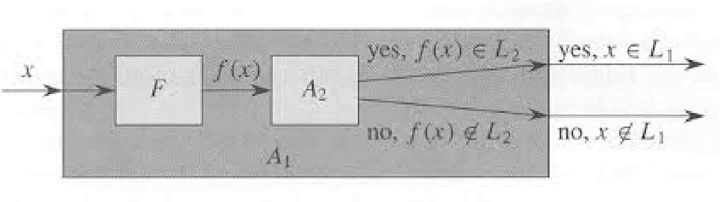

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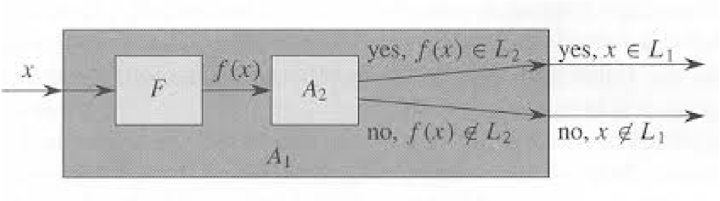

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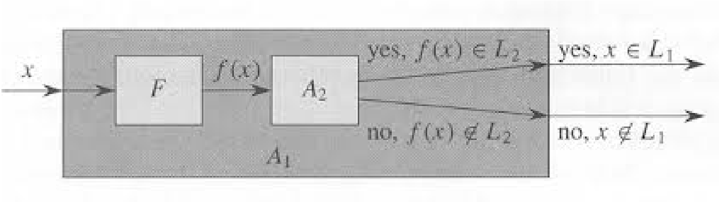

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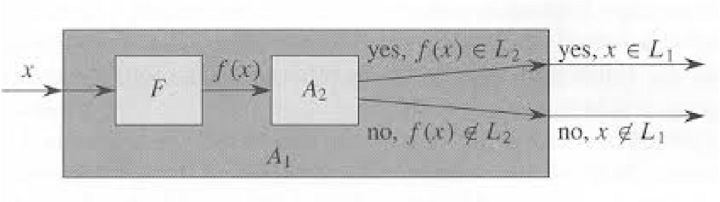

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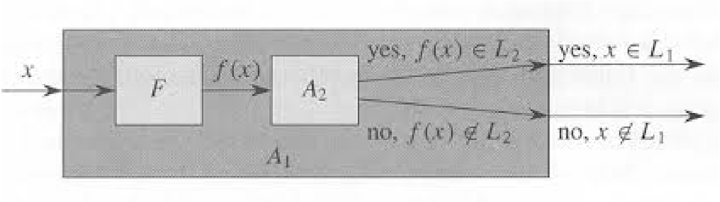

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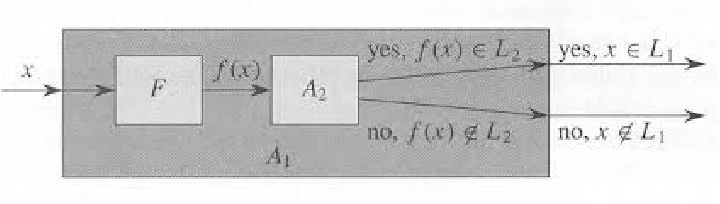

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- Informally, those at the end of a $\leq_p$ chain are called NP-hard.

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How to prove a language is NP-hard?

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- Instead, formulate a generic language that represents all languages in NP and prove that every language in $\mathcal{NP}$ can be reduced to the generic language in polynomial time.
- To obtain such a generic language, we need to consider the definition of languages in NP.

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Can we identify L_{tbd} and f ?

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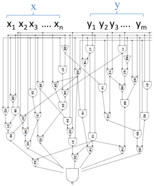

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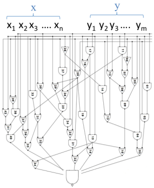
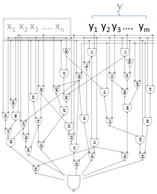


B_L



C_L^x

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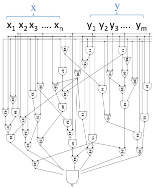
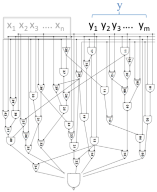
- A_L is a deterministic algorithm can be implemented with a boolean circuit B_L with two sets of input gates $x = x_1 x_2 \dots x_n$ and $y = y_1 y_2 \dots y_m$ such that

$$A_L(x, y) = 1 \text{ if and only if } B_L(x, y) = 1 \tag{2}$$



B_L



C_L^x

- Because x is given, circuit B_L can be made into circuit C_L^x such that

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Chapter 34. NP-Completeness

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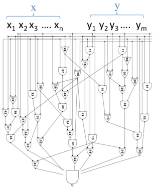

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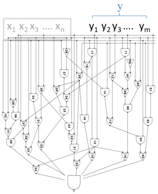

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- From (1), (2), and (3), we have

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Chapter 34. NP-Completeness

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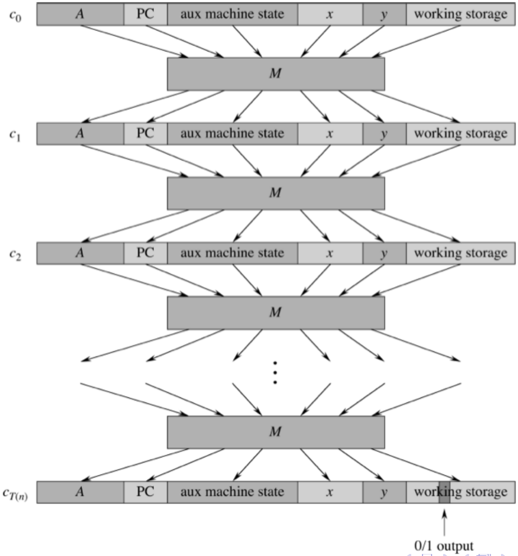

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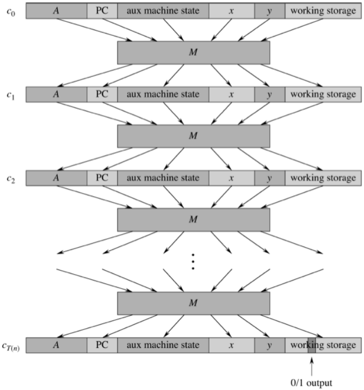

Chapter 34. NP-Completeness

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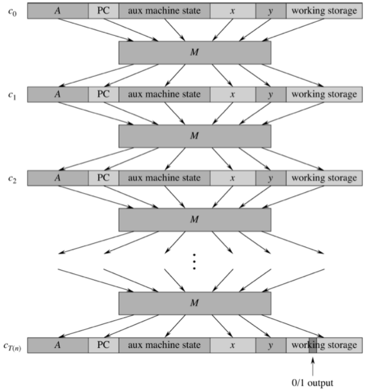

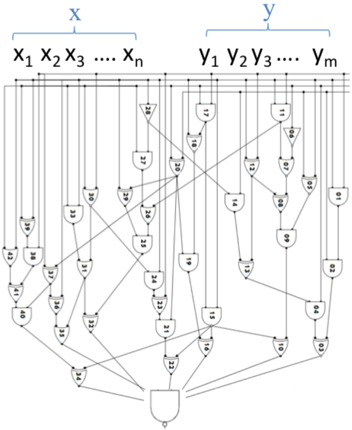

Chapter 34. NP-Completeness

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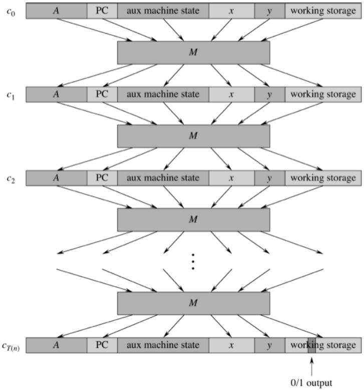

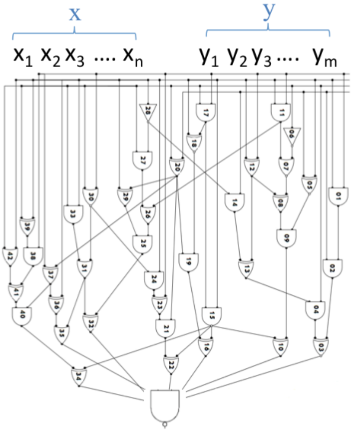

And the circuit can be built from the algorithm in polynomial time.

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Chapter 34. NP-Completeness

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Cook's Theorem: SAT is NP-complete.

Cook's reduction: characterizing a polynomial-time computation on nondeterministic Turing machine with a boolean formula, such that a nondeterministic path leading to the accept state **corresponds** to an assignment to the variables making the the formula TRUE.

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Will be discussed in CSCI 8470 Advanced Algorithms.

Chapter 34. NP-Completeness

Landscape of NP problems and beyond

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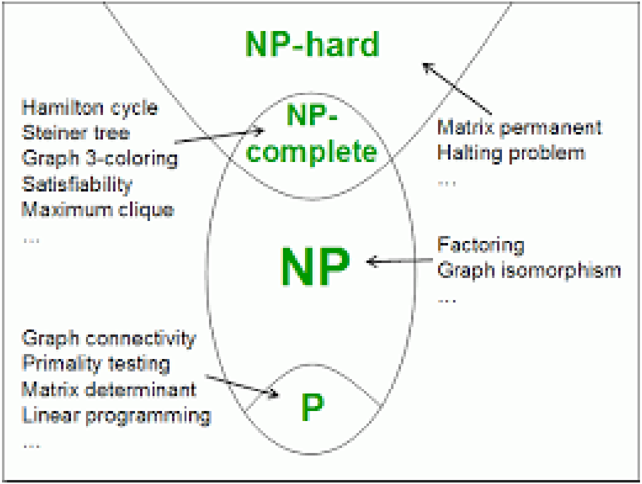
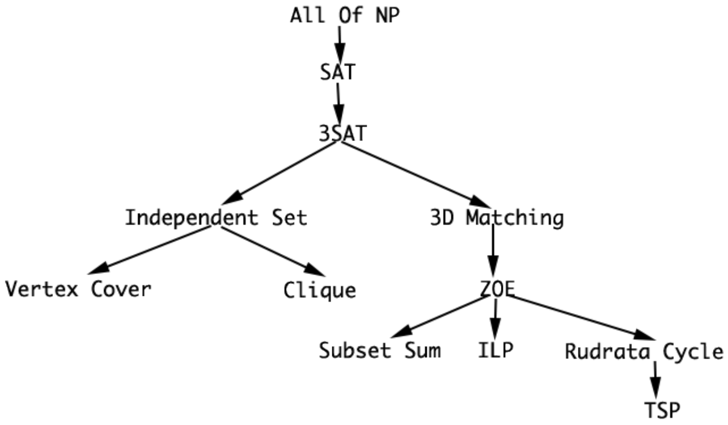


Chapter 34. NP-Completeness

Many problems/languages have been proved NP-complete (Karp70s)



Examples of reduction techniques

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Example 1: $\mathrm{SAT} \leq_p \mathrm{3SAT}$

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Chapter 34. NP-Completeness

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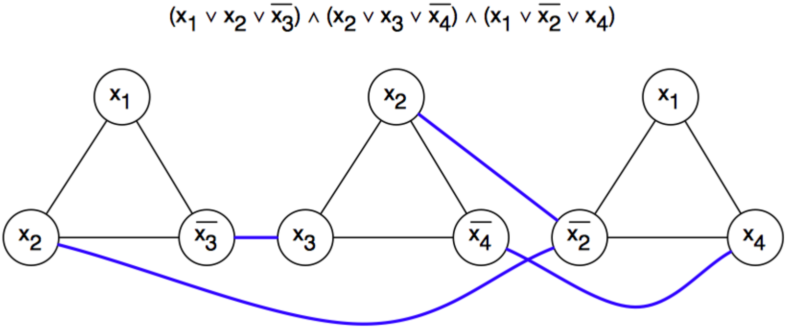
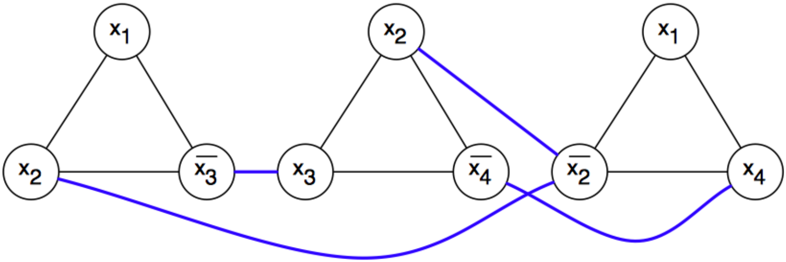


Example 2: $3\mathrm{SAT} \leq_p \mathrm{IS}$

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_2 \vee x_3 \vee \overline{x_4}) \wedge (x_1 \vee \overline{x_2} \vee x_4)$$



An assignment TRUE to one literal in each clause

Chapter 34. NP-Completeness

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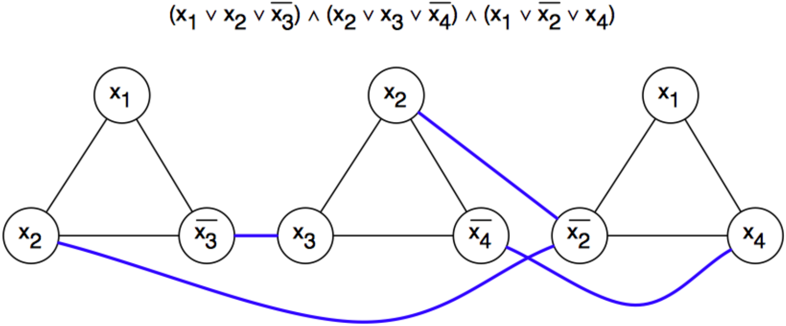

An assignment TRUE to one literal in each clause corresponds to an independent set in the transformed graph.

Summary

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Scope of the Final Exam

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- Minimum spanning tree

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- NP-completeness theory

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Scope of the Final Exam (cont')

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  NP-completeness proofs (simple, assembly of previous known reductions)

More abou algorithms and complexity theory

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What will be in CSCI 8470 Advanced Algorithms

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More abou algorithms and complexity theory

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2. Coping with the intractability
   - Exact algorithms via parameterization (parameterized computation)
   - Randomized algorithms (Monte Carlo algorithms)
   - Approximation algorithms (an introduction)