CSCI 4470/6470 Algorithms, Spring 2017

Lecture Note II

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Part II Sorting and Order Statistics

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- Chapter 6. Heapsort, the use of priority queue
- Chapter 7. Quicksort, probabilistic analysis, randomized algorithms
- Chapter 8. Sorting in linear time, lower bounds
- Chapter 9. Medians and order statistics

Chapter 6. Heapsort

Chapter 6. Heapsort and the use of priority queue

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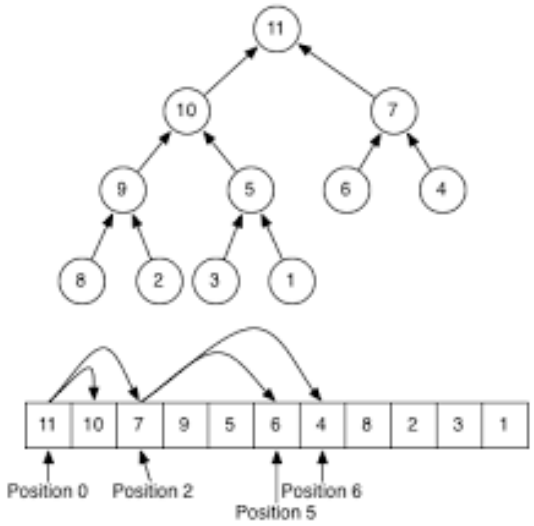

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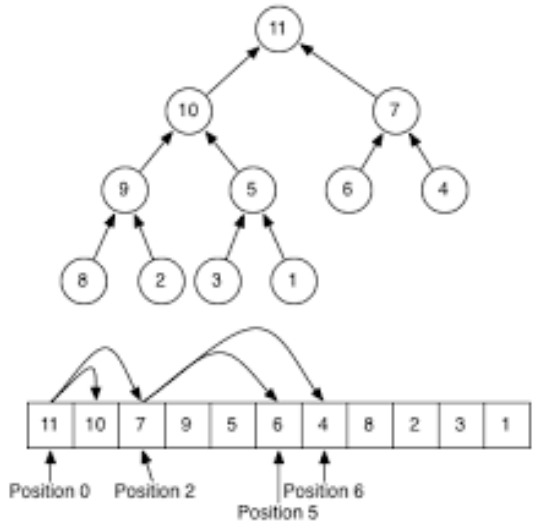

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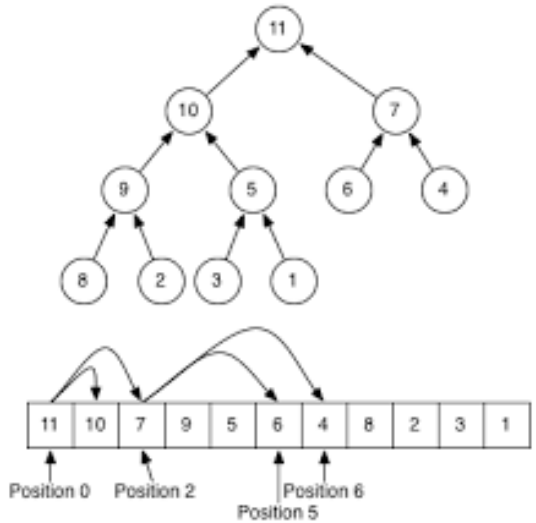

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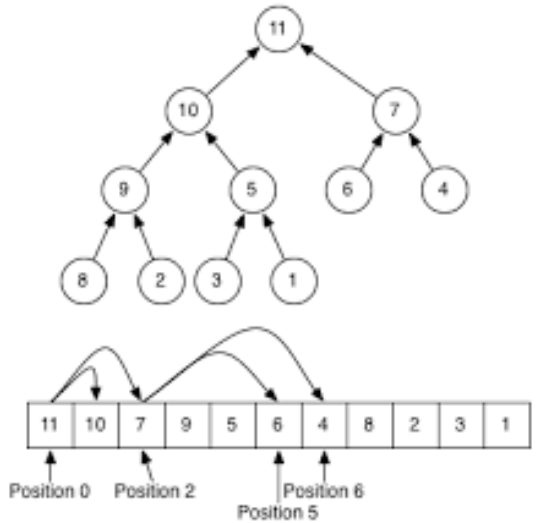

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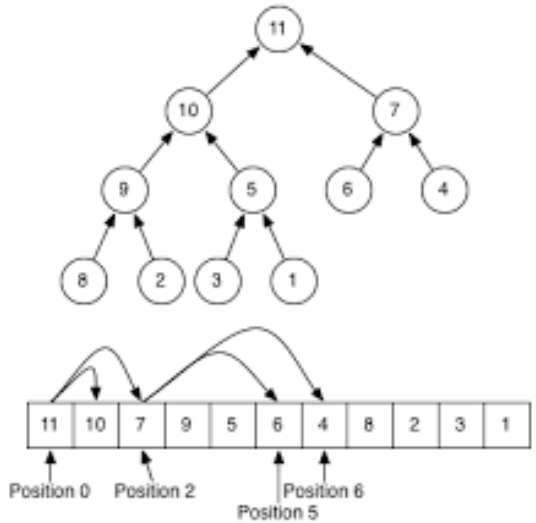

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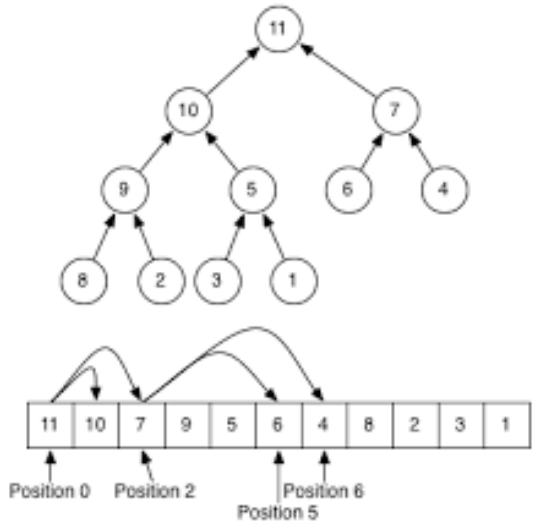

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Chapter 6. Heapsort

Subroutine $\textsc{Max-Heapify}(A, i)$

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3. **if** $(l \leq heapsize[A])$ **and** $(A[l] > A[i])$

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1. $l = LEFT[i]$
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3. **if** $(l \leq heapsize[A])$ **and** $(A[l] > A[i])$
4. **then** $largest = l$

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$T_{MH}(n, i) \leq c \log_2 n$, for all $i = 1, 2, \ldots, n$.

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$$T_{HS}(n) = T_{BMH}(n) + (n-1)T_{MH}(n, 1) \leq c\frac{n}{2} \log_2 n + (n-1)c \log_2 n \leq O(n \log n)$$

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Operations on heaps:

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Function HEAP-MAXIMUM(A) — obtain the maximum

1. **return** ($A[1]$)

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Function HEAP-MAXIMUM(A) — obtain the maximum

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Function HEAP-EXTRACT-MAX(A) — obtain and remove the maximum

1. **if** $heapsize[A] < 1$
2. **then return** ("heap underflow")
3. $max = A[1]$
4. $A[1] = A[heapsize[A]]$
5. $heapsize[A] = heapsize[A] - 1$
6. MAX-HEAPIFY($A, 1$)
7. **return** (max)

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Function HEAP-INCREASE-KEY(A, i, key) — replace a key with a larger value

1. **if** $key < A[i]$
2. **then return** ("new key is smaller than current key")
3. $A[i] = key$
4. **while** $i > 1$ **and** $A[PARENT[i]] < A[i]$
5. exchange $A[i] \longleftrightarrow A[PARENT[i]]$
6. $i = PARENT[i]$

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5. exchange $A[i] \longleftrightarrow A[PARENT[i]]$
6. $i = PARENT[i]$

Function MAX-HEAP-INSERT(A, key) — insert a new key to heap

1. $heapsize[A] = heapsize[A] + 1$
2. $A[heapsize[A]] = -\infty$
3. HEAP-INCREASE-KEY($A, heapsize[A], key$)

Chapter 7. Quicksort and randomized algorithms

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Idea of the Quicksort: divide-and-conquer

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Idea of the Quicksort: divide-and-conquer

- divide: re-organize list $A[p, r]$ into two sublists $A[p, q-1]$ and $A[q+1, r]$ based on pivot $A[q]$, such that

Chapter 7. Quicksort and randomized algorithms

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  (b) $A[i] \geq A[q]$ for all $i = q+1, \cdots, r$

Chapter 7. Quicksort and randomized algorithms

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- divide: re-organize list $A[p, r]$ into two sublists $A[p, q-1]$ and $A[q+1, r]$ based on pivot $A[q]$, such that

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  (b) $A[i] \geq A[q]$ for all $i = q+1, \cdots, r$

- conquer: sort $A[p, q-1]$ and $A[q+1, r]$, recursively.

Chapter 7. Quicksort and Randomized algorithms

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Algorithm $\textsc{QuickSort}\ (A, p, r)$

Chapter 7. Quicksort and Randomized algorithms

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1. **if** $p < r$

Chapter 7. Quicksort and Randomized algorithms

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Chapter 7. Quicksort and Randomized algorithms

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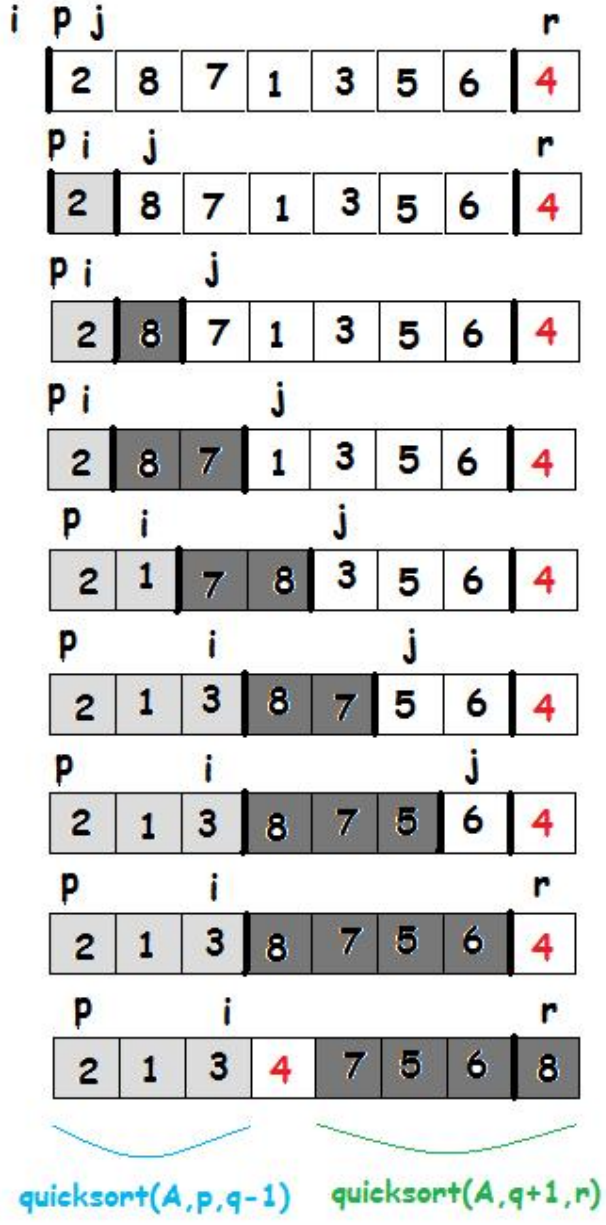
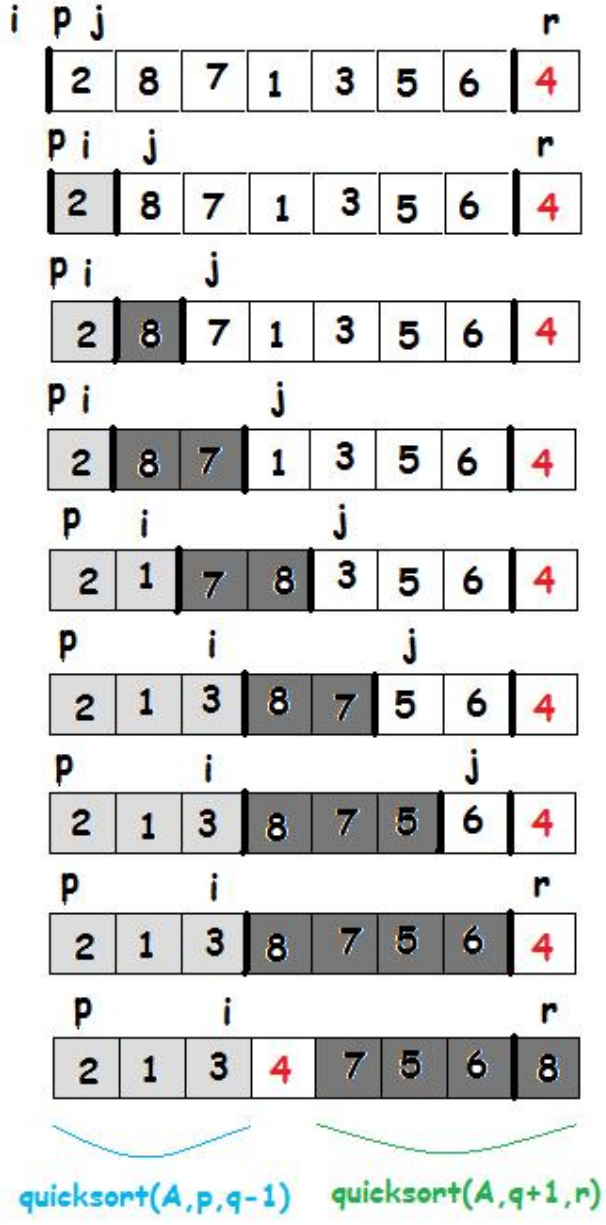
How can we identify such a pivot?

Chapter 7. Quicksort



Chapter 7. Quicksort



PARTITION(A, p, r)

```
1  x ← A[r]
2  i ← p − 1
3  for j ← p to r − 1
4       do if A[j] ≤ x
5            then i ← i + 1
6                 exchange A[i] ↔ A[j]
7  exchange A[i + 1] ↔ A[r]
8  return i + 1
```

Chapter 7. Quicksort

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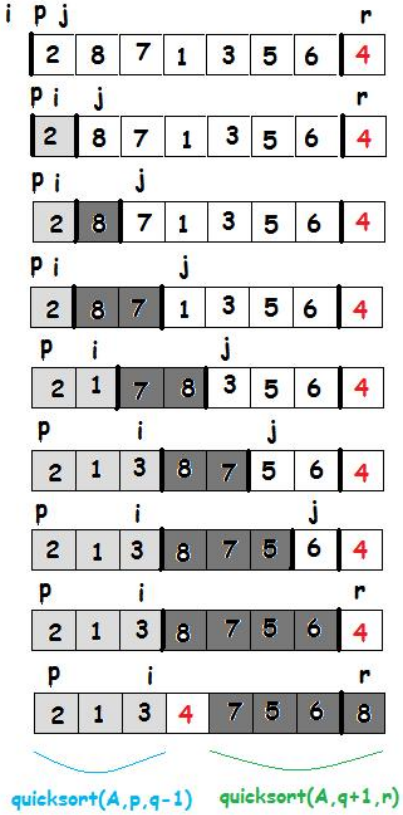
So the idea of Quicksort may work well on a majority of data.

Chapter 7. Quicksort

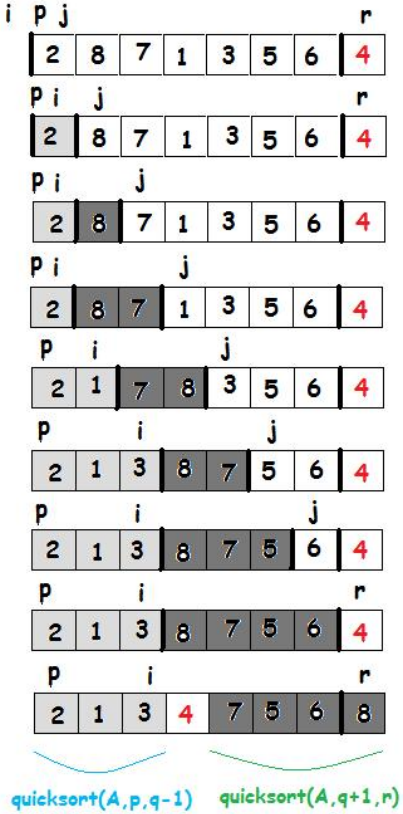


Assume that the equal likely chance for every number to be in the last position, what is the chance to partition the list into

$$x\% \text{ vs } (100-x)\%$$

fragments, for $10 \leq x \leq 90$?

Chapter 7. Quicksort



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The chance is $= 80\%$

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Chapter 7. Quicksort

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3. **return** (PARTITON(A, p, r))

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3. RANDOMIZED QUICKSORT $(A, p, q - 1)$

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Chapter 7. Quicksort

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Chapter 7. Quicksort

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Chapter 7. Quicksort

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Chapter 7. Quicksort

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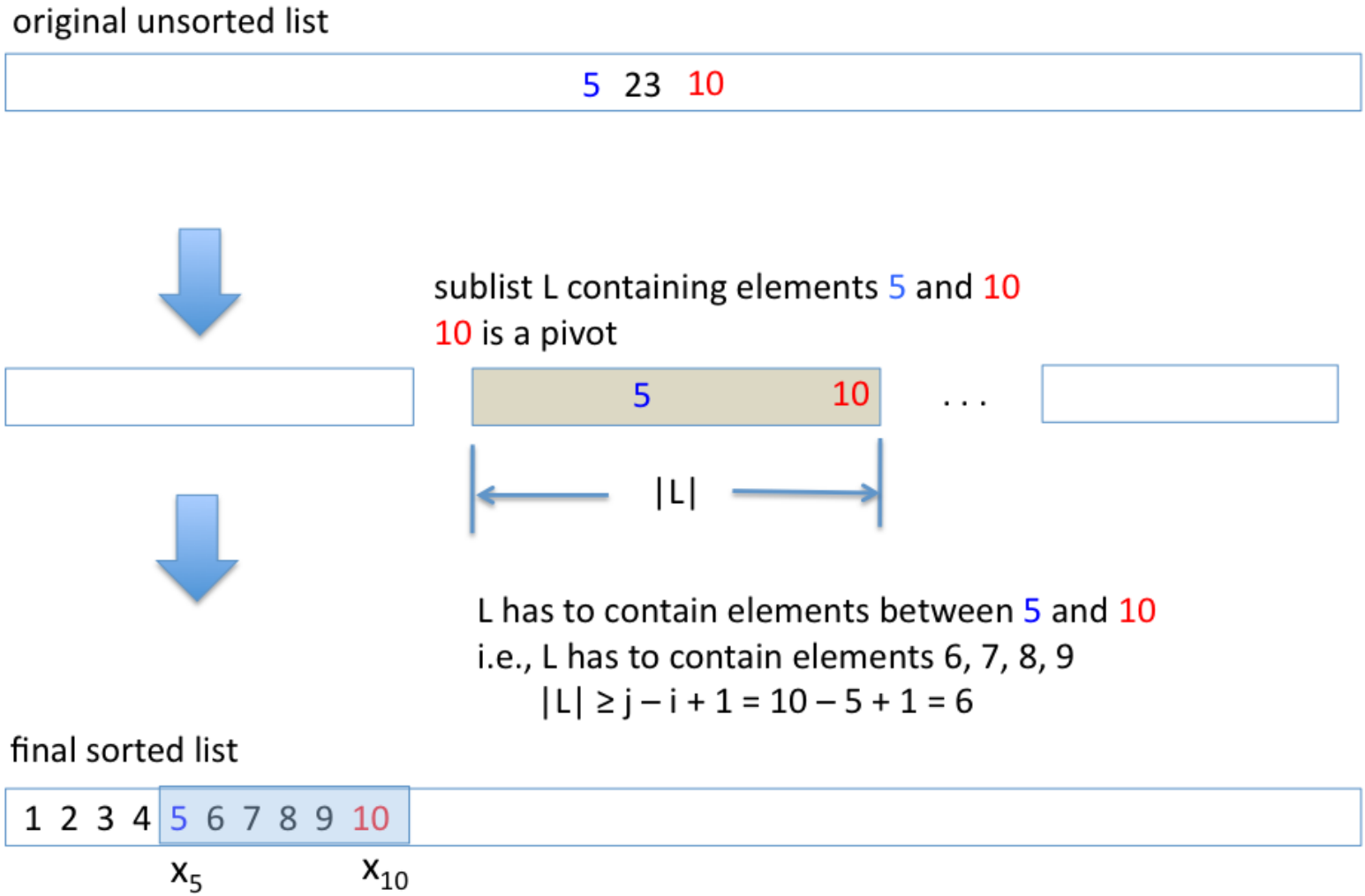
So $P(X_{i,j} = 1) \leq 2\frac{1}{|L|} \leq 2\frac{1}{j-i+1}$

Chapter 7. Quicksort



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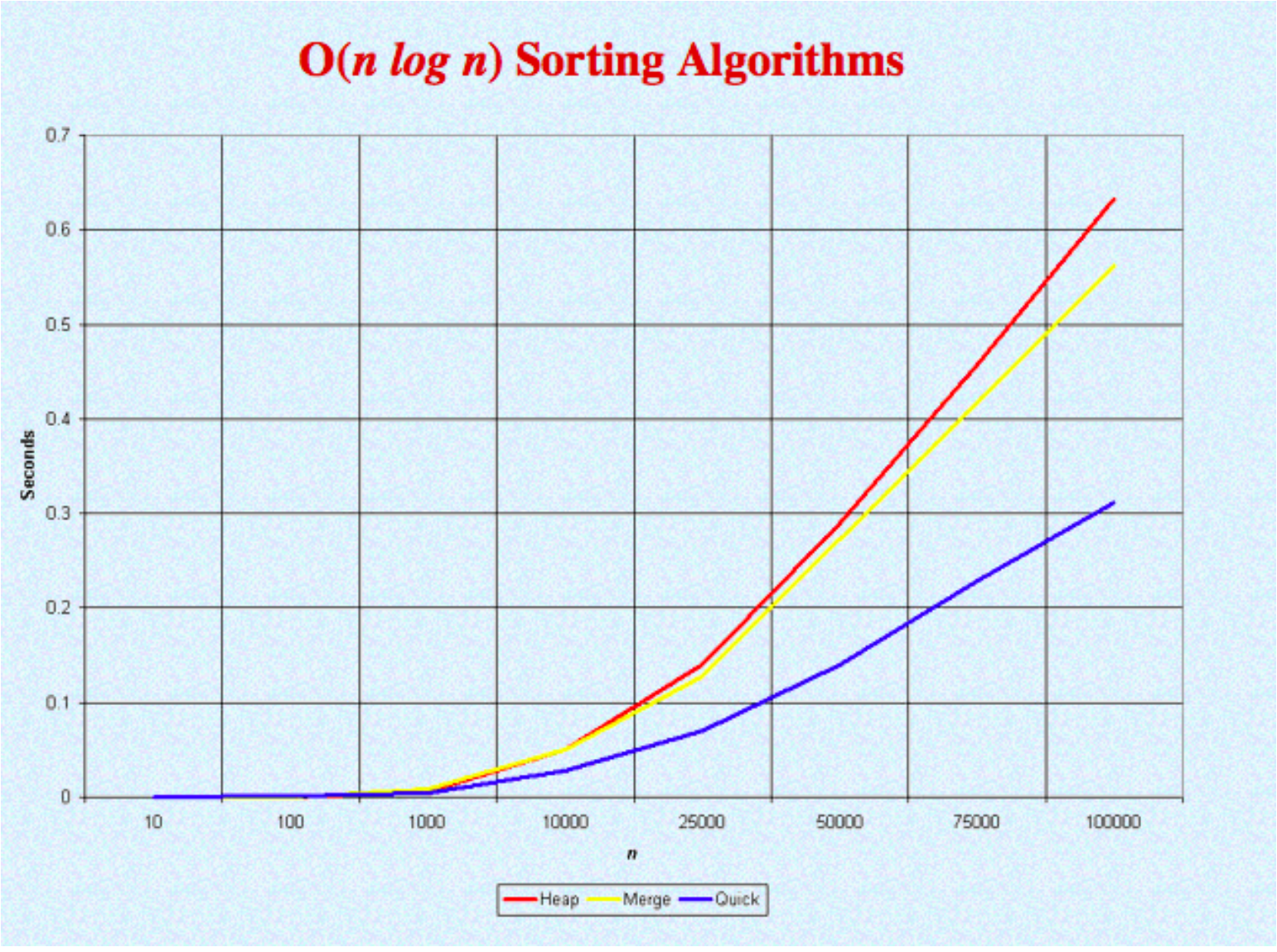
So $E(X) = O(n\log_2 n)$.

Chapter 7. Quicksort



Chapter 8. Lower bounds and sorting in linear time

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- We have used Big-O for upper bounds.

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In other word, $\lim_{n\to\infty} \frac{f(n)}{g(n)} =$

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For example, we have shown $T(n) = \Omega(n^2)$ for Insertion Sort.

Proof techniques for Big-Ω are similar to those for Big-O.

Chapter 8. Lower Bounds and Sorting in Linear Time

Important notes on lower bound and upper bound

Chapter 8. Lower Bounds and Sorting in Linear Time

Important notes on lower bound and upper bound

Bounds	Algorithms	Sorting Problem

Chapter 8. Lower Bounds and Sorting in Linear Time

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Bounds	Algorithms	Sorting Problem
upper bound	Insertion Sort $O(n^2)$	

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$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ if and only if $f(n) = \Theta(g(n))$.

Chapter 8. Lower Bounds and Sorting in Linear Time

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So we say

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Chapter 8. Lower Bounds and Sorting in Linear Time

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So we say

- INSERTION SORT runs in time $\Theta(n^2)$,
- MERGESORT runs in time $\Theta(n \log_2 n)$,
- the sorting problem has $\Theta(n \log_2 n)$ time complexity.

Chapter 8. Lower Bounds and Sorting in Linear Time

Deriving a lower bound for sorting

with decision tree as algorithm/computation model

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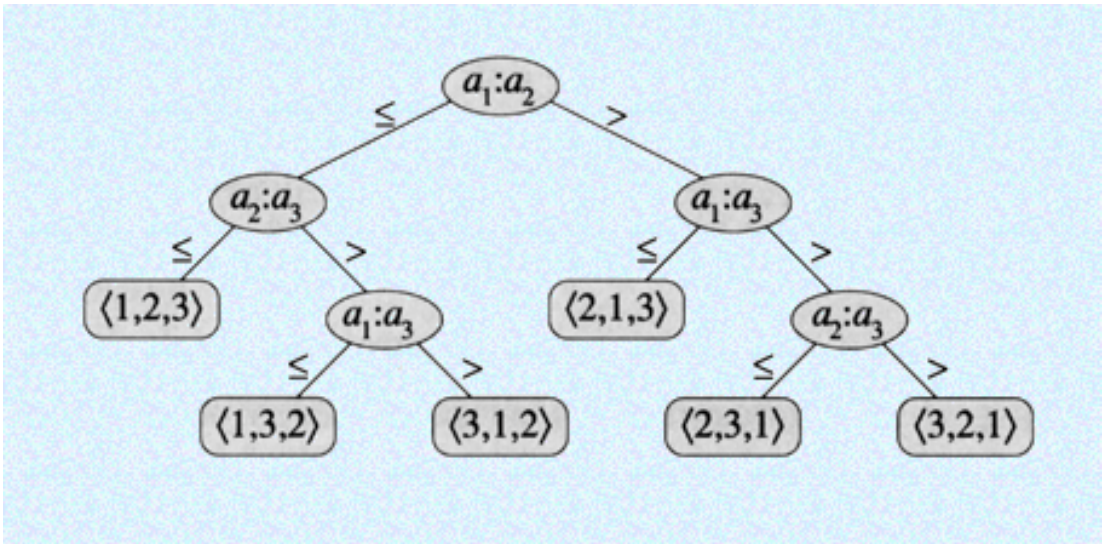

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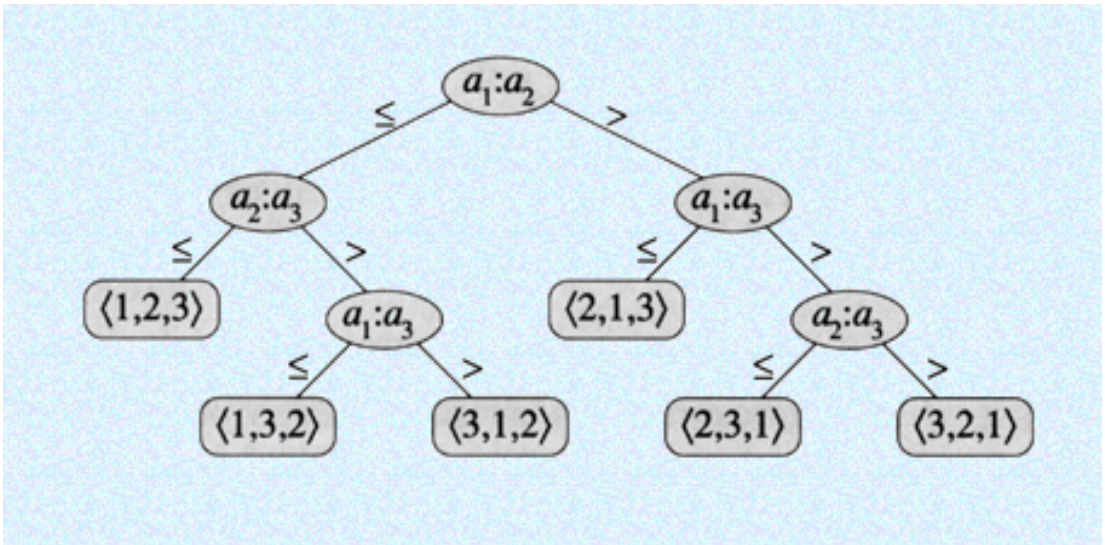

Claim 1: total number of leaves is $\geq n!$.

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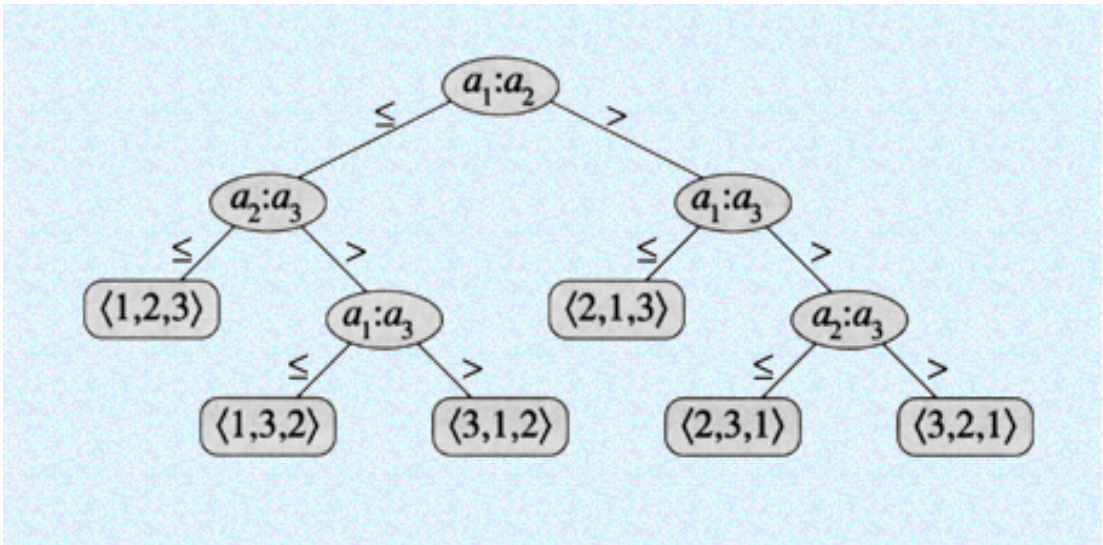
- each internal node denotes $(x_i \leq x_j)$, with two outcomes
- each path corresponds to one possible outcome of the algorithm
- each path is for one permutation of generic list $(1, 2, \ldots, n)$



Claim 1: total number of leaves is $\geq n!$.

Claim 2: the height of the tree at least $\geq \log n!$.
(The minimum of heights of all such trees!)

Chapter 8. Lower Bounds and Sorting in Linear Time

Theorem: Sorting needs $\Omega(n \log n)$ comparisons on comparison-based computation models.

Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Prove.
The longest path from the root to a leave is $\Omega(\log n!)$. I.e.,
the number of comparisons needed in the worst case is $\Omega(\log n!)$.

$$n! = n(n-1)(n-2)\cdots(n-\frac{n}{2})(n-\frac{n}{2}-1)\cdots 2\times 1$$

$$\geq (\frac{n}{2})^{\frac{n}{2}} \times 2^{\frac{n}{2}-1} \geq \frac{1}{2}n^{\frac{n}{2}}$$

or by Stirling's formula:

$$n! = \sqrt{2\pi n}(n/e)^n(1+O(1/n))$$

$$\Omega(\log(n!)) = \Omega(n \log n)$$

Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time

Chapter 8. Lower Bounds and Sorting in Linear Time

Sorting algorithms with worst case linear time

- count sort
- radix sort
- bucket sort

Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Chapter 8. Lower Bounds and Sorting in Linear Time

Count sort

Algorithm COUNTING-SORT (A, B, k) $\{A$ contains n integers; k is the max$\}$

Chapter 8. Lower Bounds and Sorting in Linear Time

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Algorithm COUNTING-SORT (A, B, k) $\{A$ contains n integers; k is the max$\}$

1. **for** $i = 0$ **to** k

Chapter 8. Lower Bounds and Sorting in Linear Time

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Algorithm COUNTING-SORT (A, B, k) $\{A$ contains n integers; k is the max$\}$

1. **for** $i = 0$ **to** k
2. $\quad C[i] = 0$

Chapter 8. Lower Bounds and Sorting in Linear Time

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Algorithm COUNTING-SORT (A, B, k) $\{A$ contains n integers; k is the max$\}$

1. **for** $i = 0$ **to** k
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3. **for** $j = 1$ **to** $length[A]$

Chapter 8. Lower Bounds and Sorting in Linear Time

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1. **for** $i = 0$ **to** k
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4. $\quad C[A[j]] = C[A[j]] + 1$

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6. **for** $i = 1$ **to** k
7. $\quad C[i] = C[i] + C[i-1]$

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Example: A: 2 5 3 0 2 3 0 3, $\quad k = 5$,

Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Chapter 8. Lower Bounds and Sorting in Linear Time

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Example: A: 2 5 3 0 2 3 0 3, $k = 5$, C: 2 0 2 3 0 1

analysis: $T(n) = O(k + n)$

Chapter 8. Lower Bounds and Sorting in Linear Time

Radix Sort:

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Radix Sort:

329	720	720	329
457	355	329	355
657	436	436	436
839	457	839	457
436	657	355	657
720	329	457	720
355	839	657	839

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355	839	657	839

Algorithm RADIX-SORT(A, d)

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Lemma. Given n b-bit binary numbers and any positive $r \leq b$. RADIX-SORT uses $\Theta(\lceil b/r \rceil (n + 2^r))$ time.

Chapter 8. Lower Bounds and Sorting in Linear Time

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Proof. Each b-digit binary number can be regarded as $\lceil b/r \rceil$ r-digit binary numbers. These r-digit binary numbers are of integer values in the range of $\{0, 1, \ldots, 2^r - 1\}$.

Chapter 8. Lower Bounds and Sorting in Linear Time

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Run RADIX-SORT on the original binary numbers assumed to be $\lceil b/r \rceil$ columns.

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For every column, sorting by COUNTING-SORT with $2^r - 1$ being the maximum.

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Since all steps in the two algorithms are mandatory, the total time is also $\Omega(\lceil b/r \rceil (n + 2^r))$, thus $\Theta(\lceil b/r \rceil (n + 2^r))$.

Chapter 8. Lower Bounds and Sorting in Linear Time

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Since all steps in the two algorithms are mandatory, the total time is also $\Omega(\lceil b/r \rceil (n + 2^r))$, thus $\Theta(\lceil b/r \rceil (n + 2^r))$.

Once b and n are given, we can choose r to minimize the quantity $\lceil b/r \rceil (n + 2^r)$.

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Bucket Sort (assuming uniform distribution of inputs)

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Bucket Sort (assuming uniform distribution of inputs)

Algorithm BUCKET-SORT(A)

Chapter 8. Lower Bounds and Sorting in Linear Time

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Algorithm BUCKET-SORT(A)
1. $n = length[A]$

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3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$

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A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

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1 $\rightarrow$ .12 $\rightarrow$ .17

Chapter 8. Lower Bounds and Sorting in Linear Time

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1 → .12 → .17
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7 → .72 → .78
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2. **for** $i = 1$ **to** n
3. insert $A[i]$ into list $B[\lfloor nA[i] \rfloor]$
4. **for** $i = 0$ **to** $n - 1$
5. sort list $B[i]$ with INSERTION SORT
6. concatenate the lists $B[0], B[1], ..., B[n-1]$

A: .78 .17 .39 .26 .72 .94 .21 .12 .23 .68

B: 0 /
1 → .12 → .17
2 → .21 → .23 → .26
3 → .39
4 /
5 /
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7 → .72 → .78
8 /
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Chapter 8. Lower Bounds and Sorting in Linear Time

Bucket Sort (assuming uniform distribution of inputs)

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Chapter 9. Medians and order statistics

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- find the maximum: linear time

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Chapter 9. Medians and order statistics

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  Can we do better?

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INPUT: a list A of elements, an integer i;

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    else recursively do for A_u with $i - k$;

Chapter 9. Medians and Order Statistics

Algorithm RANDOMIZED-SELECT (A, p, r, i)

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assuming $r \geq 2$,

$$T(n) \leq cn\frac{(r-1)}{r} + cn(\frac{(r-1)}{r})^2 + cn(\frac{(r-1)}{r})^4 + \ldots cn(\frac{(r-1)}{r})^m = O(n)$$

where $(\frac{(r-1)}{r})^m n = 1$, $m = \log_{\frac{r}{r-1}} n$

Chapter 9. Medians and Order Statistics

Performance analysis

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The worst case: running time $\Theta(n^2)$.

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Average case: $E[T(n)]$ (cont')

- so the expected time $E[T(n)]$ needs to include the average time of recursion on the case when sublist $A[p..q]$ possibly has lengths $k = 0, 1, 2, \ldots, n-1$

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$$E[T(n)] \leq \sum_{k=1}^{n} \frac{1}{n} \cdot E[T(\max\{k-1, n-k\})] + an, \text{ for some constant } a > 0$$

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because $\max\{k-1, n-k\} = k-1$ if $k > n/2$ and
$\max\{k-1, n-k\} = n-k$ if $k \leq n/2$

$$E[T(n)] \leq 2/n \Sigma_{k=n/2}^{n-1} E[T(k)] + an$$

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$$E[T(k)] \leq 2/n\Sigma_{k=n/2}^{n-1} E[T(k)] + an \leq 2/n\Sigma_{k=n/2}^{n-1} ck + an$$

$$= \frac{2c}{n}[\sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k] + an$$

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- Base case: assume $T(n) \leq cn$, for $n < 2c/(c-4a)$

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- both S_1 and S_2 are guaranteed only a fraction of S;
- the ith smallest element is either x, or in S_1 or in S_2 (but not both);
- in either of the latter two cases, the algorithm is applied recursively.

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Selection in worst case linear time

INPUT: set S of n elements and i;
OUTPUT: the ith smallest element in S;

Main idea:

- find a pivot x to partition the list S into two sublists S_1 and S_2, such that $\forall y \in S_1 \; y < x$ and $\forall z \in S_2 \; z > x$
- both S_1 and S_2 are guaranteed only a fraction of S;
- the ith smallest element is either x, or in S_1 or in S_2 (but not both);
- in either of the latter two cases, the algorithm is applied recursively.
- $T(n) \leq T(\beta n) + cn$ where $0 < \beta < 1$

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- the size of the sublist to find the pivot is also a fraction αn of the original list S, $|S| = n$;
- the total time actually is

$$T(n) \leq T(\alpha n) + T(\beta n) + cn$$

where $\alpha + \beta < 1$

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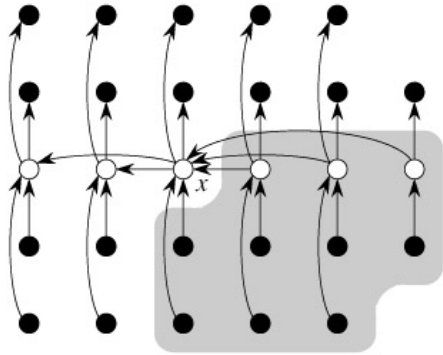

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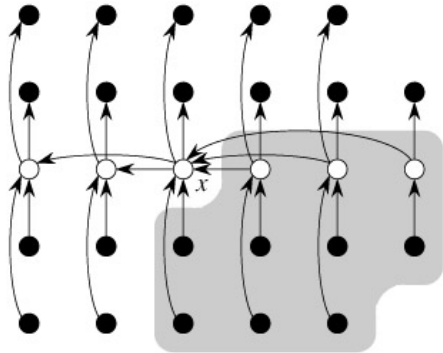

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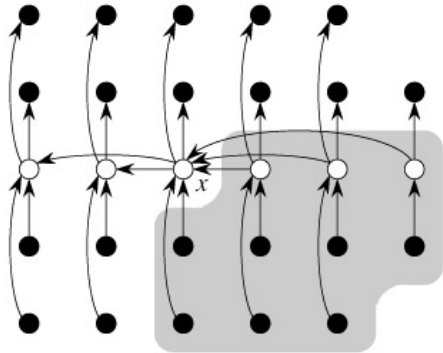

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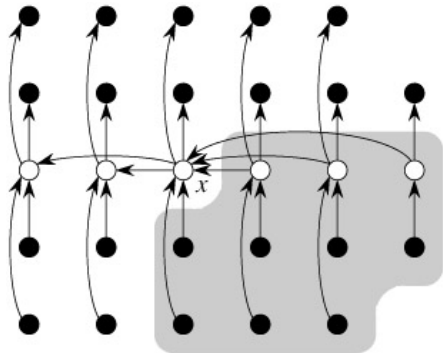

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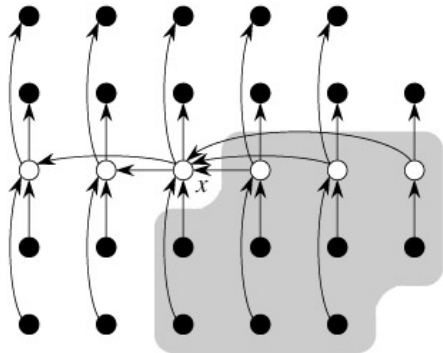

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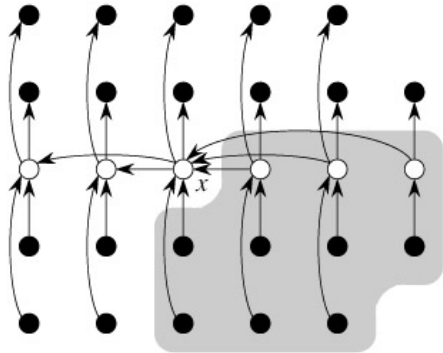

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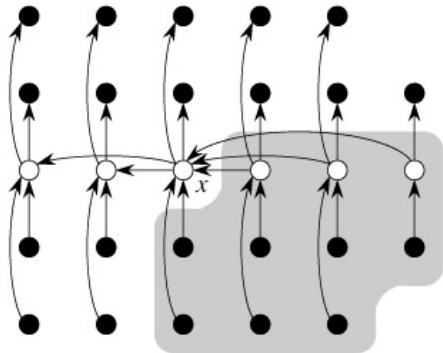

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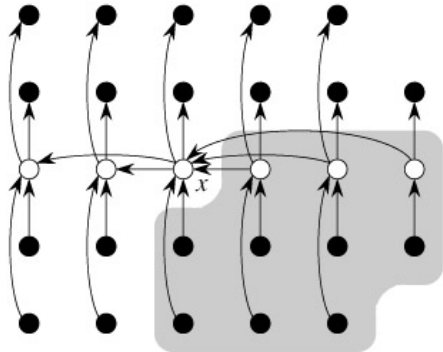

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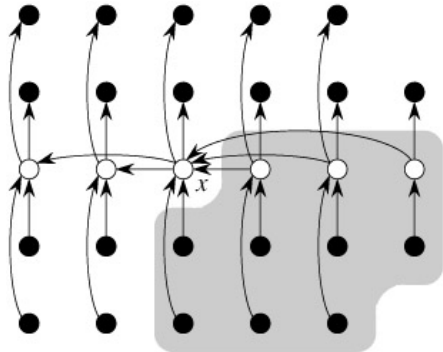

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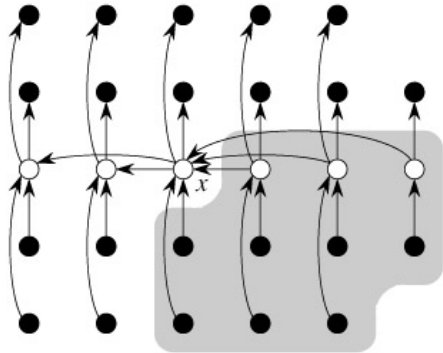

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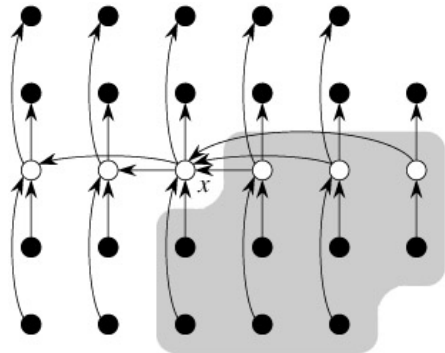

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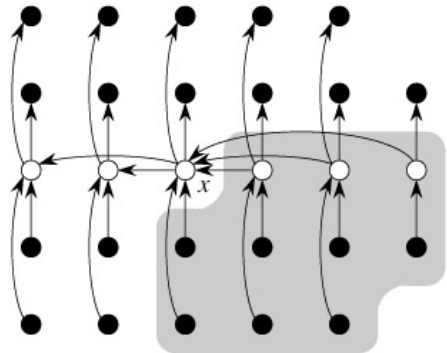

Note: the number of elements $\leq x$ is at least:

$$|S_1| = 3(\frac{\lceil n/5 \rceil}{2}) \geq 3n/10$$

similarly, the number of elements $\geq x$ is at least:

$$|S_2| \geq 3(\frac{\lceil n/5 \rceil}{2} - 2) \geq 3n/10 - 6$$

So a time upper bound for SELECT is

$$T(n) \leq T(\lceil n/5 \rceil) + T(\lceil 7n/10 + 6 \rceil) + O(n)$$

when $n \geq 140$

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For example, given INSERTION SORT:

- we first analyzed the algorithm
  and obtained

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

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