

Writing Sample of Algebraic Topology

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It is known that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups; nevertheless, their universal covers do not. Show that the universal covers have non-isomorphic homology.

Solution:

Since \mathbb{R} is the universal cover of S^1 , \mathbb{R}^2 is the universal cover of $S^1 \times S^1$. Since \mathbb{R}^2 is contractible, then

$$H_n(\mathbb{R}^2) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Suppose Y is the universal cover of $S^1 \vee S^1 \vee S^2$. Note that S^2 is path-connected and locally path-connected, also $\pi_1(S^2) = 0$. Choose the wedge point $x_0 \in S^2$ as the basepoint. Since $f_*(\pi_1(S^2, x_0)) = \{0\} \subset p_*(\pi_1(Y, \tilde{x}_0))$, by the lifting criterion there exists a lift \tilde{f} such that the following diagram commutes, i.e. $f = p \circ \tilde{f}$, where f is the inclusion map and p is the covering map. And let $\tilde{x}_0 \in p^{-1}(f(x_0))$. Then $\tilde{f}(x_0) = \tilde{x}_0$.

$$\begin{array}{ccc} & & (Y, \tilde{x}_0) \\ & \nearrow \tilde{f} & \downarrow p \\ (S^2, x_0) & \xrightarrow{f} & S^1 \vee S^1 \vee S^2 \end{array}$$

Since H_n is a functor, $f_* = (p \circ \tilde{f})_* = p_* \circ \tilde{f}_*$, which gives the following commutative diagram:

$$\begin{array}{ccc} & & H_n(Y) \\ & \nearrow \tilde{f}_* & \downarrow p_* \\ H_n(S^2) & \xrightarrow{f_*} & H_n(S^1 \vee S^1 \vee S^2) \end{array}$$

Note that when $n = 2$, $H_2(S^2) \cong \mathbb{Z}$. Suppose $H_2(Y) = 0$. Then for every $x \in H_2(S^2)$, $f_*(x) = p_*(\tilde{f}_*(x)) = p_*(0) = 0$, so $\text{im}(f_*) = 0$. If we can prove that f_* is injective, then $\text{im}(f_*) \cong \mathbb{Z}$, contradiction. We get $H_2(Y) \neq 0$. But $H_2(\mathbb{R}^2) = 0$, so the homology of \mathbb{R}^2 and Y are non-isomorphic when $n = 2$. \square

Claim: f_* is injective.

Proof: Define $r : S^1 \vee S^1 \vee S^2 \rightarrow S^2$ sending two S^1 to the wedge point, clearly it is a retraction. Again, H_n is a functor. Then $r \circ f = \text{Id}_{S^2}$ gives $r_* \circ f_* = \text{Id}_{H_n(S^2)}$, which implies f_* is injective. \square