Universal covers distinguish spaces with the isomorphic homology

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Question. *It is known that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups; nevertheless, their universal covers do not. Show that the universal covers have non-isomorphic homology.* (Inspired by an exercise in Hatcher, Algebraic Topology.)

Proof. Since $\mathbb{R}$ is the universal cover of S^1, $\mathbb{R}^2$ is the universal cover of $S^1 \times S^1$. Since $\mathbb{R}^2$ is contractible, then

$$H_n(\mathbb{R}^2) \cong \begin{cases} \mathbb{Z}, & n = 0, \\ 0, & n \neq 0 \,. \end{cases}$$

Suppose Y is the universal cover of $S^1 \vee S^1 \vee S^2$. Note that S^2 is path-connected and locally path-connected, also $\pi_1(S^2) = 0$. Choose the wedge point $x_0 \in S^2$ as the basepoint. Since $f_*\big(\pi_1(S^2, x_0)\big) = \{0\} \subset p_*\big(\pi_1(Y, \tilde{x}_0)\big)$, by the lifting criterion there exists a lift $\tilde{f}$ such that the following diagram commutes, i.e. $f = p \circ \tilde{f}$, where f is the inclusion map and p is the covering map. And let $\tilde{x}_0 \in p^{-1}(f(x_0))$. Then $\tilde{f}(x_0) = \tilde{x}_0$.

$$\begin{tikzcd} & (Y, \tilde{x}_0) \arrow[d, "p"] \\ (S^2, x_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \vee S^1 \vee S^2 \end{tikzcd}$$

Since H_n is a functor, $f_* = (p \circ \tilde{f})_* = p_* \circ \tilde{f}_*$, which gives the following commutative diagram:

$$\begin{tikzcd} & H_n(Y) \arrow[d, "p_*"] \\ H_n(S^2) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & H_n(S^1 \vee S^1 \vee S^2) \end{tikzcd}$$

Note that when $n = 2$, $H_2(S^2) \cong \mathbb{Z}$. Suppose $H_2(Y) = 0$. Then for every $x \in H_2(S^2)$, $f_*(x) = p_*(\tilde{f}_*(x)) = p_*(0) = 0$, so $\mathrm{im}(f_*) = 0$. If we can prove that f_* is injective, then $\mathrm{im}(f_*) \cong \mathbb{Z}$, contradiction. We get $H_2(Y) \neq 0$. But $H_2(\mathbb{R}^2) = 0$, so the homology of $\mathbb{R}^2$ and Y are non-isomorphic when $n = 2$. □

Claim: f_* is injective.

Proof: Define $r : S^1 \vee S^1 \vee S^2 \to S^2$ sending two S^1 to the wedge point, clearly it is a retraction. Again, H_n is a functor. Then $r \circ f = \mathrm{Id}_{S^2}$ gives $r_* \circ f_* = \mathrm{Id}_{H_n(S^2)}$, which implies f_* is injective. □