Another example: constant prop

- Set D = $\mathcal{P}(\{x \to c \mid x \in Var, c \in Const\})$

in

X := N

out

$F_{X := N}(in) = in \cup \{x \to N\}$

$- \{x \to *\}$

in

X := Y op Z

out

$F_{X := Y \text{ op } Z}(in) = in - \{x \to *\}$

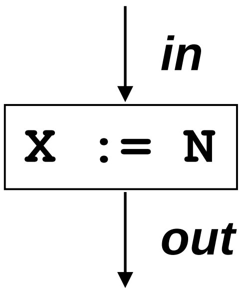
$\cup \{ \quad \mid y \to N$

Another example: constant prop

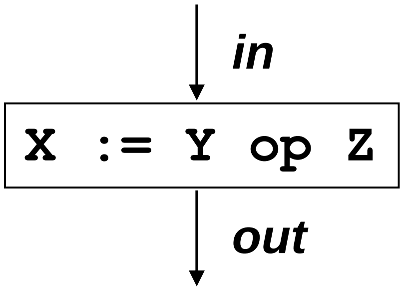
- Set D = $2^{\{ x \to N \mid x \in \text{Vars} \wedge N \in Z \}}$

a := 0

b := d/a;



$$F_{X := N}(in) = in - \{ X \to * \} \cup \{ X \to N \}$$



$$F_{X := Y \text{ op } Z}(in) = in - \{ X \to * \} \cup \{ X \to N \mid (Y \to N_1) \in in \wedge (Z \to N_2) \in in \wedge N = N_1 \text{ op } N_2 \}$$

Another example: constant prop

$mapt(y)$

in

`X := *Y`

out

$F_{X := *Y}(in) =$ $in - \{x \to *\}$

$\cup \{x \to N \mid \forall v \in mapt(y).\ v \to N \in in\}$

$z \to N \quad y \to N$

in

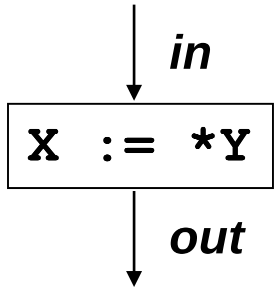
`*X := Y`

out

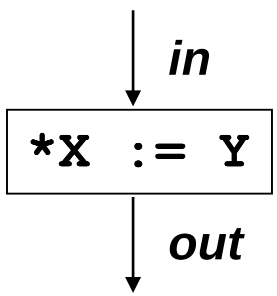
$z \to N$

$F_{*X := Y}(in) =$ $in - \{v \to N \mid v \in mapt(x)\}$

Another example: constant prop



$$
\begin{aligned}
F_{X\ :=\ *Y}(\mathit{in}) = \mathit{in} - \{\, X \to * \,\} \\
\cup \{\, X \to N \mid \forall\, Z \in \text{may-point-to}(Y) \,.\ \\
(Z \to N) \in \mathit{in} \,\}
\end{aligned}
$$



$$
\begin{aligned}
F_{*X\ :=\ Y}(\mathit{in}) = \mathit{in} - \{\, Z \to * \mid Z \in \text{may-point}(X) \,\} \\
\cup \{\, Z \to N \mid Z \in \text{must-point-to}(X) \wedge \\
Y \to N \in \mathit{in} \,\} \\
\cup \{\, Z \to N \mid (Y \to N) \in \mathit{in} \wedge \\
(Z \to N) \in \mathit{in} \,\}
\end{aligned}
$$

Another example: constant prop

in

`*X := *Y + *Z`

out

$F_{*X := *Y + *Z}(in) =$ in -

in

`X := G(...)`

out

$F_{X := G(...)}(in) = \emptyset$

Another example: constant prop

in

```
*X := *Y + *Z
```

out

$$F_{*X\ :=\ *Y\ +\ *Z}(in) = F_{a\ :=\ *Y; b\ :=\ *Z; c\ :=\ a\ +\ b;\ *X\ :=\ c}(in)$$

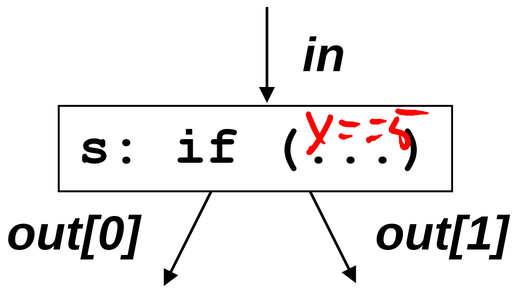
in

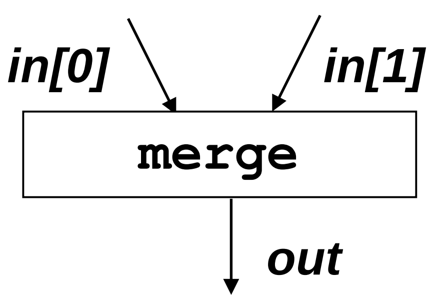
```
X := G(...)
```

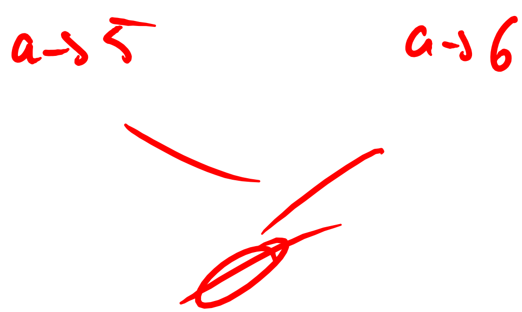
out

$$F_{X\ :=\ G(...)}(in) = \emptyset$$

Another example: constant prop







FS = { x -> c }

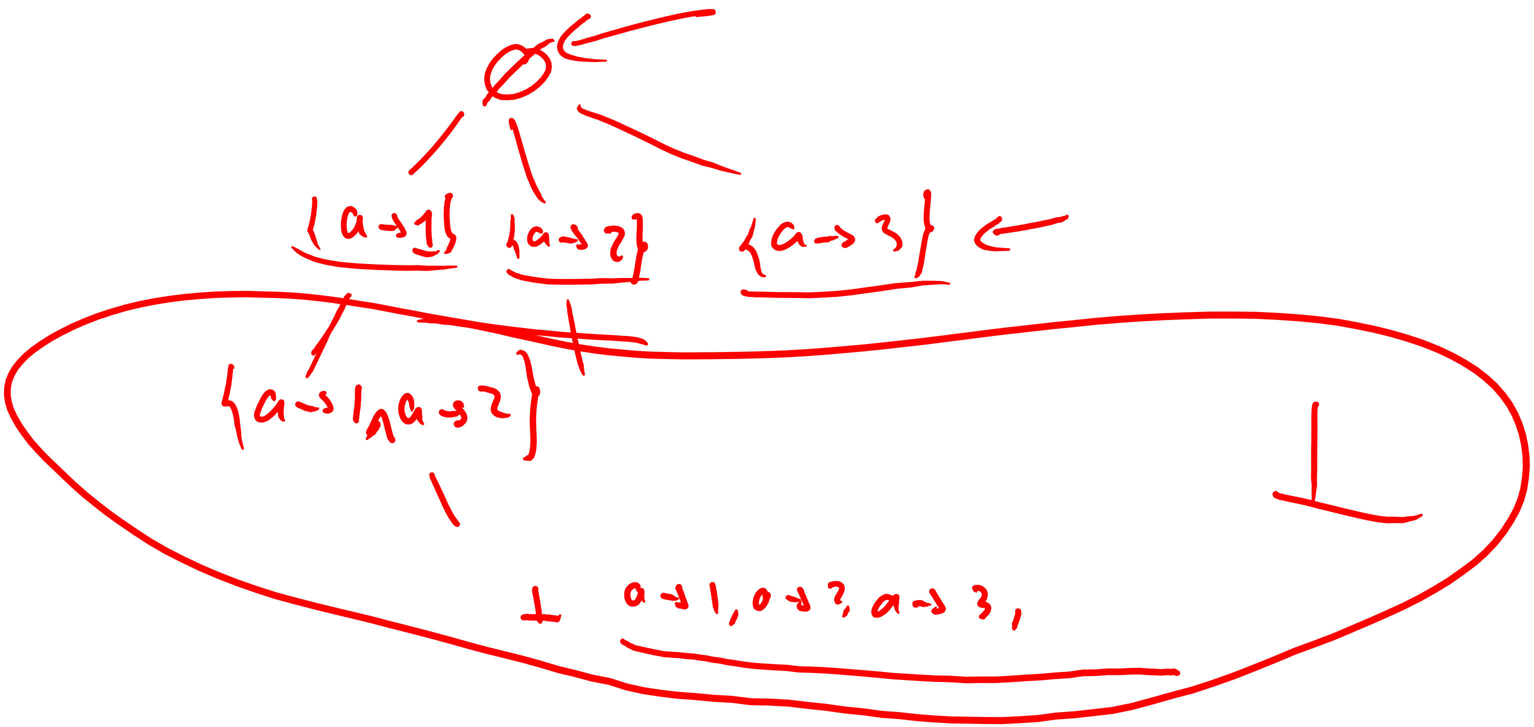
Lattice

- $(D, \sqsubseteq, \bot, \top, \sqcup, \sqcap) =$

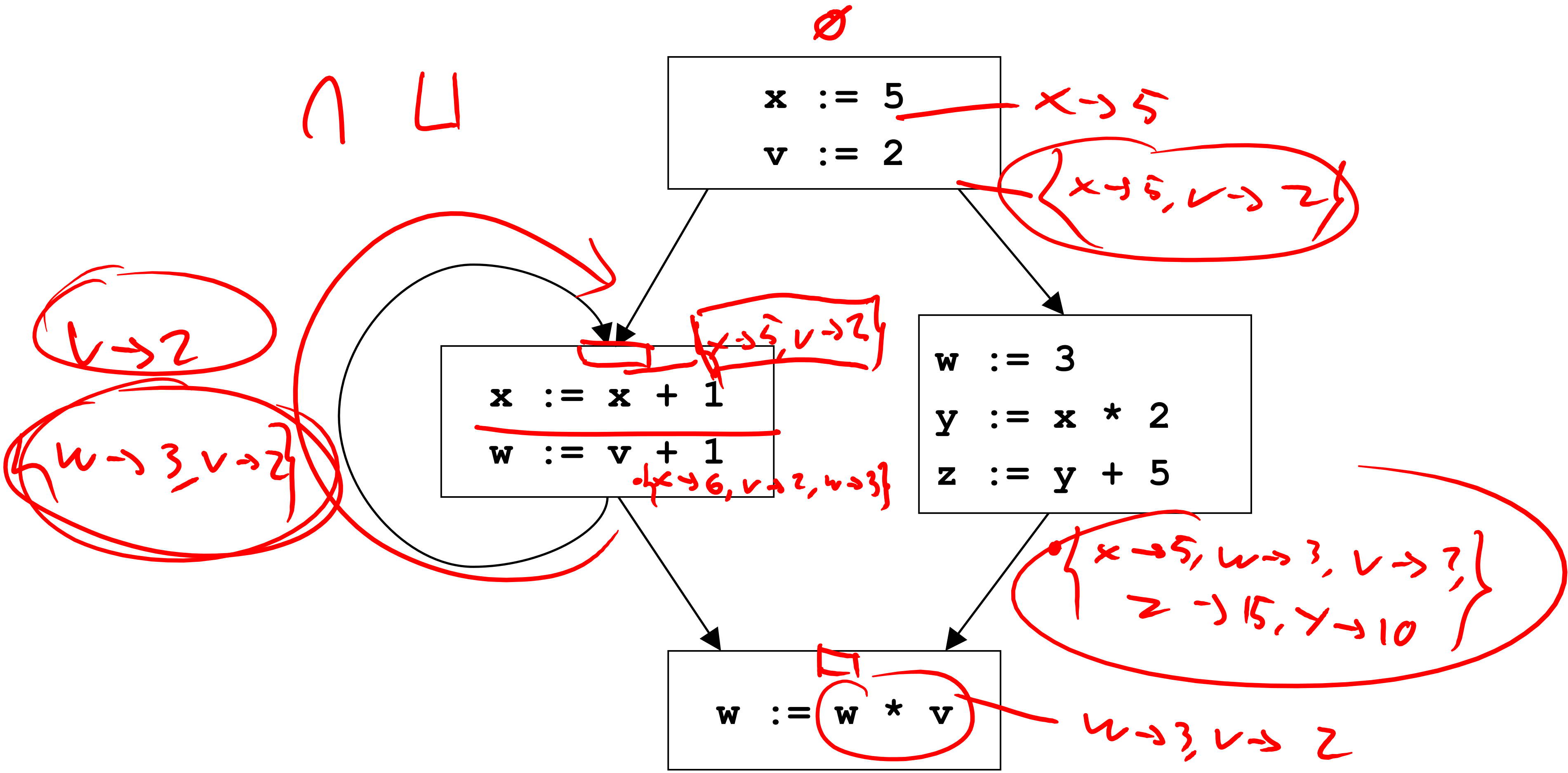
  $\supseteq$ FS $\emptyset$ $\cap$ $\cup$

Lattice

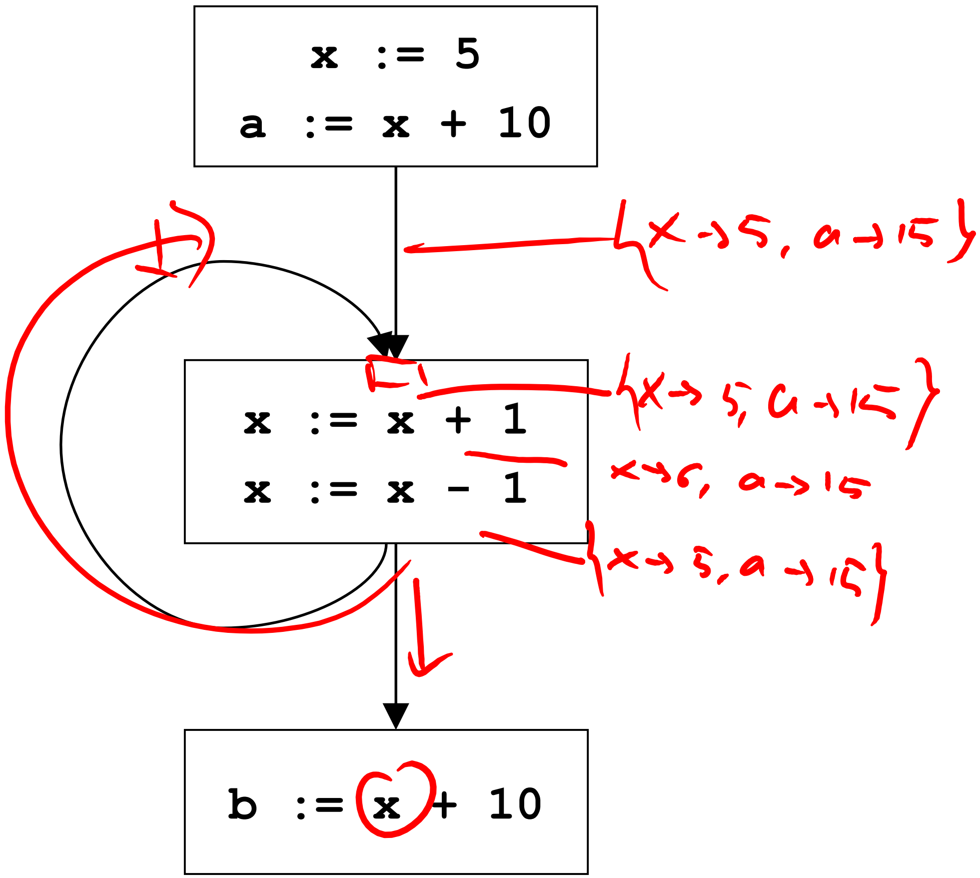
- $(D, \sqsubseteq, \perp, \top, \sqcup, \sqcap) =$
  $(2^A, \supseteq, A, \emptyset, \cap, \cup)$
  where $A = \{ x \to N \mid x \in Vars \wedge N \in Z \}$



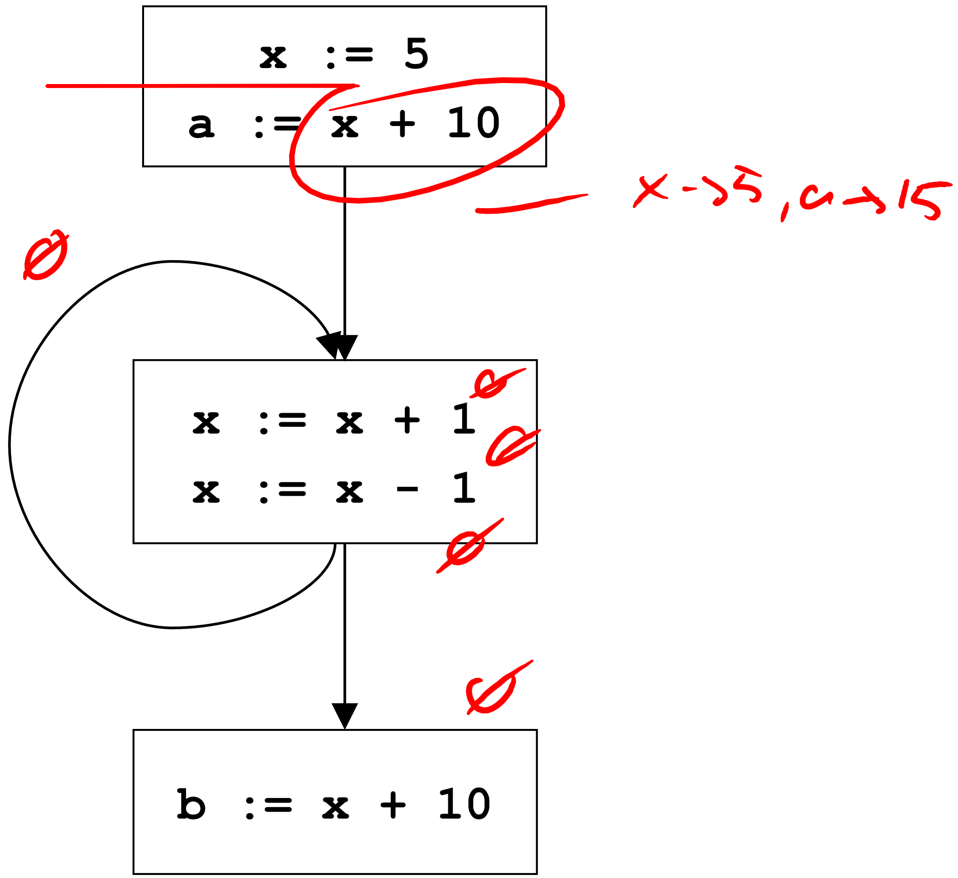
Example


∅
x := 5
v := 2
x → 5
{x → 5, v → 2}
x := x + 1
w := v + 1
{x → 5, v → 2}
{x → 6, v → 2, w → 3}
{w → 3, v → 2}
v → 2
w := 3
y := x * 2
z := y + 5
{x → 5, w → 3, v → 2, z → 15, y → 10}
w := w * v
w → 3, v → 2
⊓
⊔

Another Example



Another Example starting at top



Back to lattice

- $(D, \sqsubseteq, \bot, \top, \sqcup, \sqcap) =$
  $(2^A, \supseteq, A, \emptyset, \cap, \cup)$
  where $A = \{ x \to N \mid x \in Vars \wedge N \in Z \}$

- What's the problem with this lattice?

Back to lattice

- $(D, \sqsubseteq, \bot, \top, \sqcup, \sqcap) =$
  $(2^A, \supseteq, A, \emptyset, \cap, \cup)$
  where $A = \{ x \to N \mid x \in Vars \wedge N \in Z \}$

- What's the problem with this lattice?

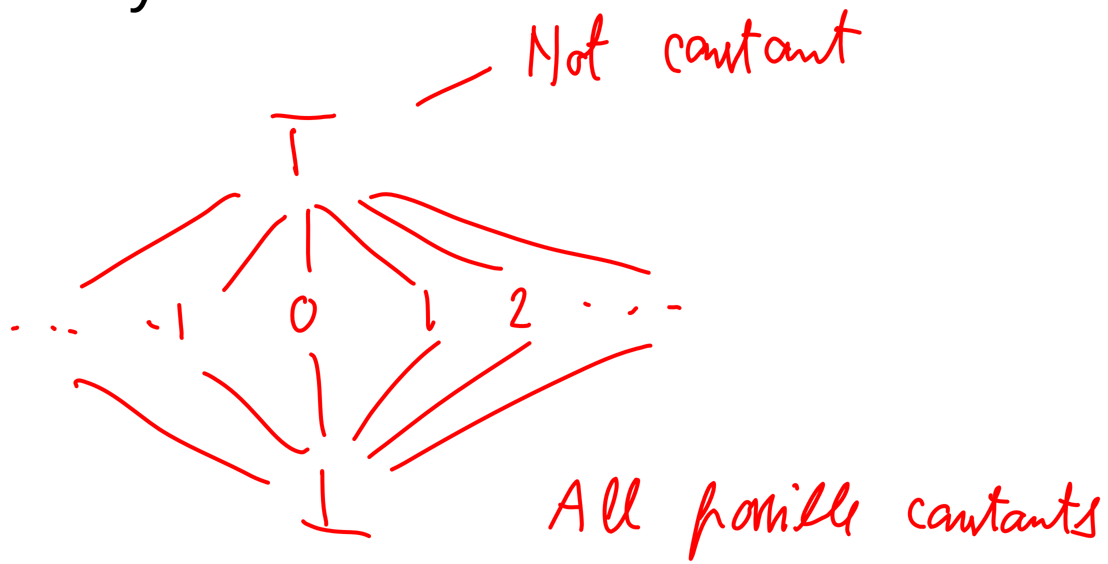
- Lattice is infinitely high, which means we can't guarantee termination

Better lattice

- Suppose we only had one variable

Better lattice

- Suppose we only had one variable



- $D = \{\bot, \top\} \cup Z$
- $\forall\, i \in Z \,.\, \bot \sqsubseteq i \wedge i \sqsubseteq \top$
- height = 3

For all variables

- Two possibilities
- Option 1: Tuple of lattices
- Given lattices $(D_1, \sqsubseteq_1, \bot_1, \top_1, \sqcup_1, \sqcap_1)$ ... $(D_n, \sqsubseteq_n, \bot_n, \top_n, \sqcup_n, \sqcap_n)$ create:

tuple lattice D^n =

For all variables

- Two possibilities
- Option 1: Tuple of lattices
- Given lattices $(D_1, \sqsubseteq_1, \bot_1, \top_1, \sqcup_1, \sqcap_1)$ ... $(D_n, \sqsubseteq_n, \bot_n, \top_n, \sqcup_n, \sqcap_n)$ create:

tuple lattice $D^n = ((D_1 \times ... \times D_n), \sqsubseteq, \bot, \top, \sqcup, \sqcap)$ where

$\bot = (\bot_1, ..., \bot_n)$

$\top = (\top_1, ..., \top_n)$

$(a_1, ..., a_n) \sqcup (b_1, ..., b_n) = (a_1 \sqcup_1 b_1, ..., a_n \sqcup_n b_n)$

$(a_1, ..., a_n) \sqcap (b_1, ..., b_n) = (a_1 \sqcap_1 b_1, ..., a_n \sqcap_n b_n)$

height = height(D_1) + ... + height(D_n)

For all variables

- Option 2: Map from variables to single lattice
- Given lattice $(D, \sqsubseteq_1, \bot_1, \top_1, \sqcup_1, \sqcap_1)$ and a set V, create:

$$\text{map lattice } V \to D = (V \to D, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$$

For all variables

- Option 2: Map from variables to single lattice
- Given lattice $(D, \sqsubseteq_1, \bot_1, \top_1, \sqcup_1, \sqcap_1)$ and a set V, create:

map lattice $V \to D = (V \to D, \sqsubseteq, \bot, \top, \sqcup, \sqcap)$

$\bot = \lambda v. \bot_1$

$\top = \lambda v. \top_1$

$a \sqcup b = \lambda v. (a(v) \sqcup_1 b(v))$

$a \sqcap b = \lambda v. (a(v) \sqcap_1 b(v))$

$a \sqsubseteq b \iff \forall v. a(v) \sqsubseteq_1 b(v)$

Back to example

$$in(Y)\ op\ in(Z)$$

↓ ***in***

`X := Y op Z`

↓ ***out***

$$F_{X\ :=\ Y\ op\ Z}(in) = in[X \mapsto in(Y)\ \hat{op}\ in(Z)]$$

$$\square\ \hat{op}\ \square$$

$\hat{op}$	$\perp$	c_1	$\top$
$\perp$	$\perp$	$\top$	$\top$
c_2	$\perp$	$c_1\ op\ c_2$	$\top$
$\top$	$\top$	$\top$	$\top$

Back to example

in

```
X := Y op Z
```

out

$$F_{X\ :=\ Y\ op\ Z}(in) = in\,[\,X \to in(Y)\ \widehat{op}\ in(Z)\,]$$

where $a\ \widehat{op}\ b$ =

$\widehat{op}$	$\bot$	d_1	$\top$
$\bot$	$\bot$	$\bot$	$\top$
d_2	$\bot$	$d_1\ op\ d_2$	$\top$
$\top$	$\top$	$\top$	$\top$

General approach to domain design

- Simple lattices:
  - boolean logic lattice
  - powerset lattice
  - incomparable set: set of incomparable values, plus top and bottom (eg const prop lattice)
  - two point lattice: just top and bottom
- Use combinators to create more complicated lattices
  - tuple lattice constructor
  - map lattice constructor

May vs Must

- Has to do with definition of computed info

- Set of $x \rightarrow y$ must-point-to pairs
  - if we compute $x \rightarrow y$, then, then during program execution, x must point to y
- Set of $x\rightarrow y$ may-point-to pairs
  - if during program execution, it is possible for x to point to y, then we must compute $x \rightarrow y$

$A \Rightarrow B$

$\neg B \Rightarrow \neg A$

May vs must

	May	Must
most optimistic (bottom)	$\emptyset$	FS
most conservative (top)	FS	$\emptyset$
safe	add	smaller
merge	$\cup$	$\cap$

May vs must

	May	Must
most optimistic (bottom)	empty set	full set
most conservative (top)	full set	empty set
safe	overly big	overly small
merge	$\cup$	$\cap$

Common Sub-expression Elim

- Want to compute when an expression is available in a var
- Domain:

Common Sub-expression Elim

- Want to compute when an expression is available in a var
- Domain:

$$S = \{ X \to E \mid X \in Var, E \in Expr \}$$

$$D = 2^S$$

$$\sqcup = \cap$$

$$\top = \emptyset$$

$$\sqcap = \cup$$

Flow functions

$a \to b + c$

in

`X := Y op Z`

out

$F_{X\ :=\ Y\ op\ Z}(in) =$ $in - \{X \to *\} - \{* \to \ldots X \ldots\} \cup \{X \to Y \text{ op } Z\}$

$X \neq Y \wedge X \neq Z$

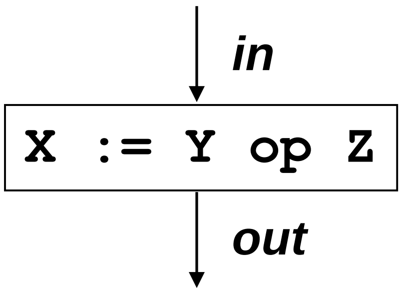
in

`X := Y`

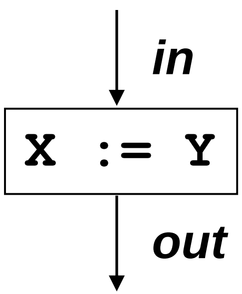
out

$F_{X\ :=\ Y}(in) =$

Flow functions

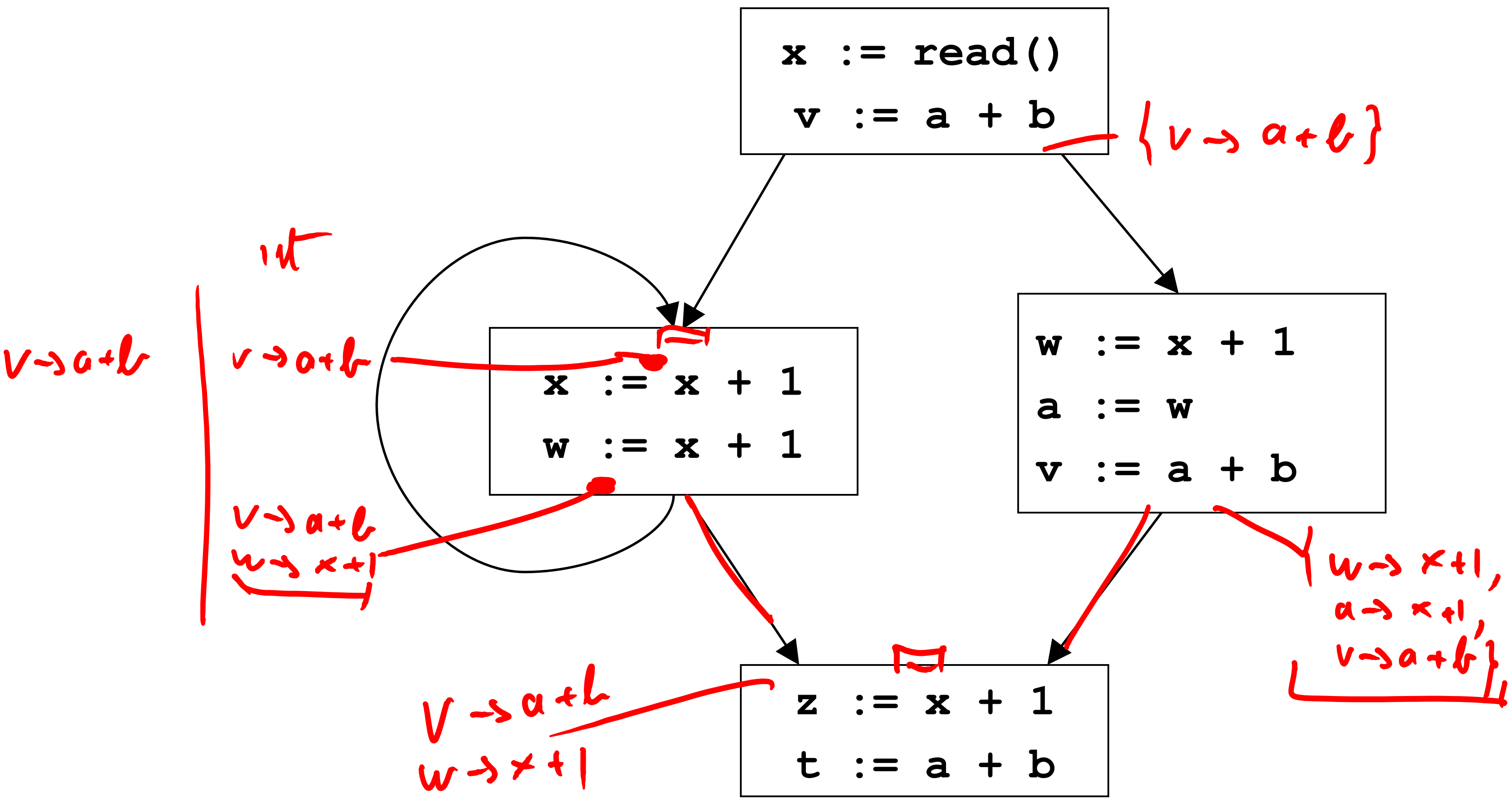


$$\begin{aligned} F_{X\,:=\,Y \text{ op } Z}(\text{in}) = \text{in} &- \{ X \to * \} \\ &- \{ * \to \ldots X \ldots \} \cup \\ &\{ X \to Y \text{ op } Z \mid X \neq Y \wedge X \neq Z \} \end{aligned}$$



$$\begin{aligned} F_{X\,:=\,Y}(\text{in}) = \text{in} &- \{ X \to * \} \\ &- \{ * \to \ldots X \ldots \} \cup \\ &\{ X \to E \mid Y \to E \in \text{in} \} \end{aligned}$$

Example



x := read()
v := a + b
{v → a+b}
1st
v → a+b
v → a+b
x := x + 1
w := x + 1
v → a+b
w → x+1
w := x + 1
a := w
v := a + b
{w → x+1,
a → x+1,
v → a+b}
z := x + 1
t := a + b
v → a+b
w → x+1

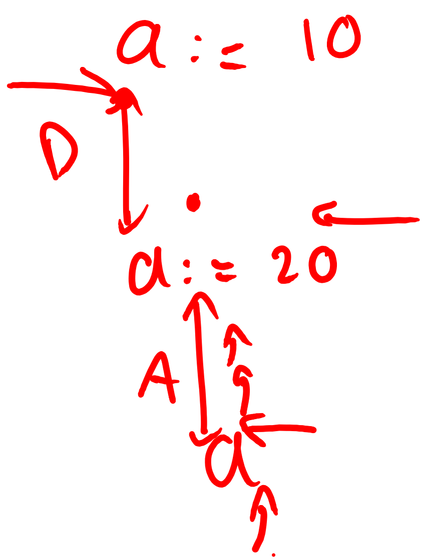
Direction of analysis

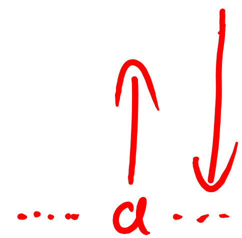
- Although constraints are not directional, flow functions are
- All flow functions we have seen so far are in the forward direction
- In some cases, the constraints are of the form

  in = F(out)
- These are called backward problems.
- Example: live variables
  - compute the set of variables that may be live

Live Variables

- A variable is live at a program point if it will be used before being redefined
- A variable is dead at a program point if it is redefined before being used





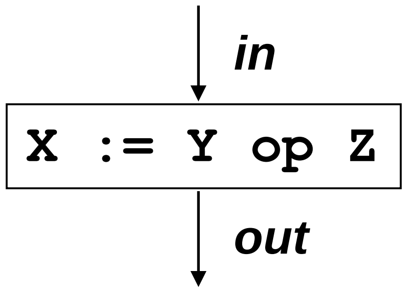
Example: live variables

- Set D = $\mathcal{P}(Vars)$
- Lattice: (D, $\sqsubseteq$, $\bot$, $\top$, $\sqcup$, $\sqcap$) =

$\emptyset$ Va $\cup$ $\cap$

Example: live variables

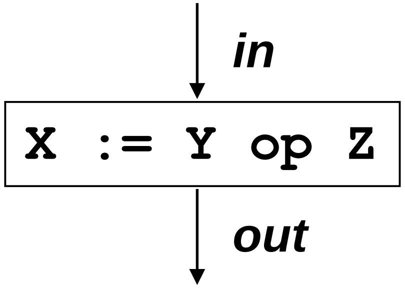
- Set $D = 2^{\text{Vars}}$
- Lattice: $(D, \sqsubseteq, \bot, \top, \sqcup, \sqcap) = (2^{\text{Vars}}, \subseteq, \emptyset, \text{Vars}, \cup, \cap)$



$F_{X \text{ := } Y \text{ op } Z}(\text{out}) = \text{out} - \{X\} \cup \{Y, Z\}$

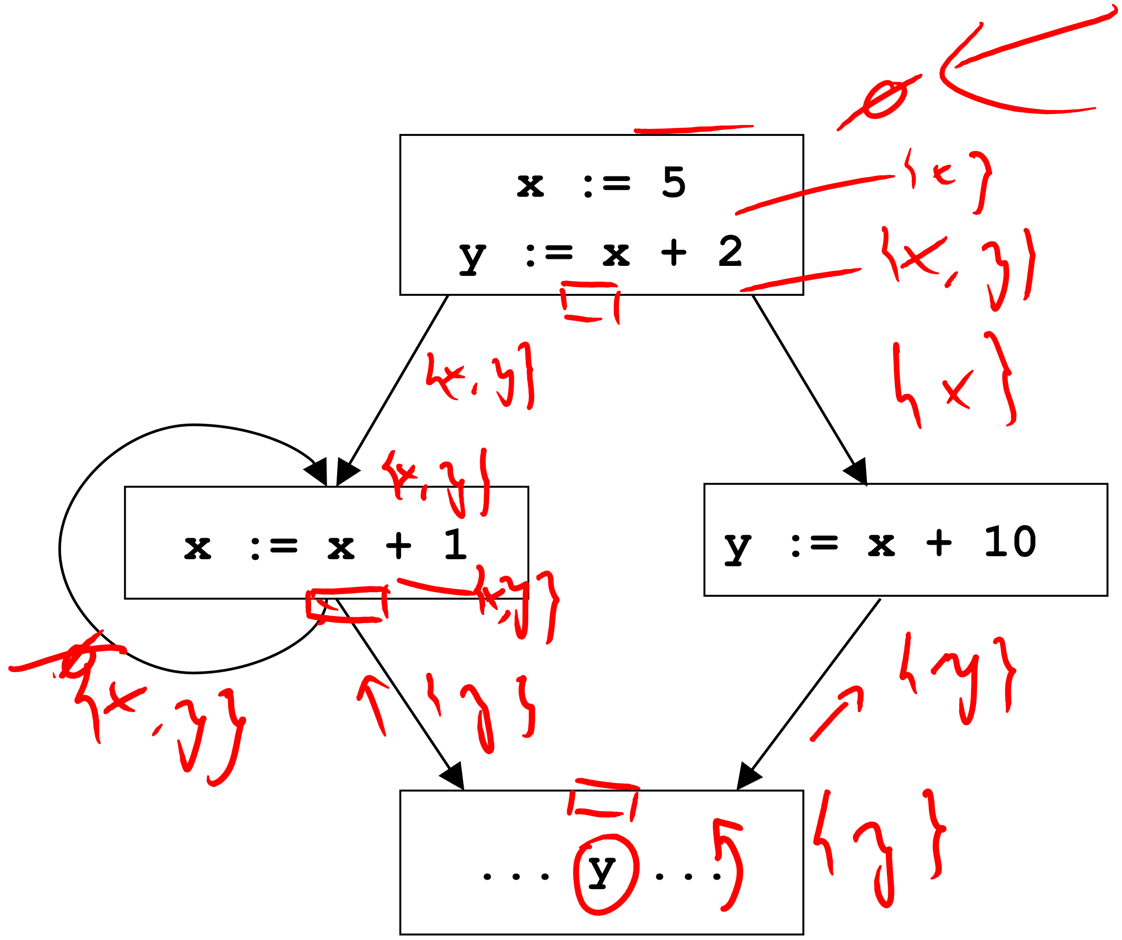
Example: live variables

- Set D = 2^{Vars}
- Lattice: $(D, \sqsubseteq, \perp, \top, \sqcup, \sqcap) = (2^{\text{Vars}}, \subseteq, \emptyset, \text{Vars}, \cup, \cap)$



$F_{X := Y \text{ op } Z}(\text{out}) = \text{out} - \{ X \} \cup \{ Y, Z\}$

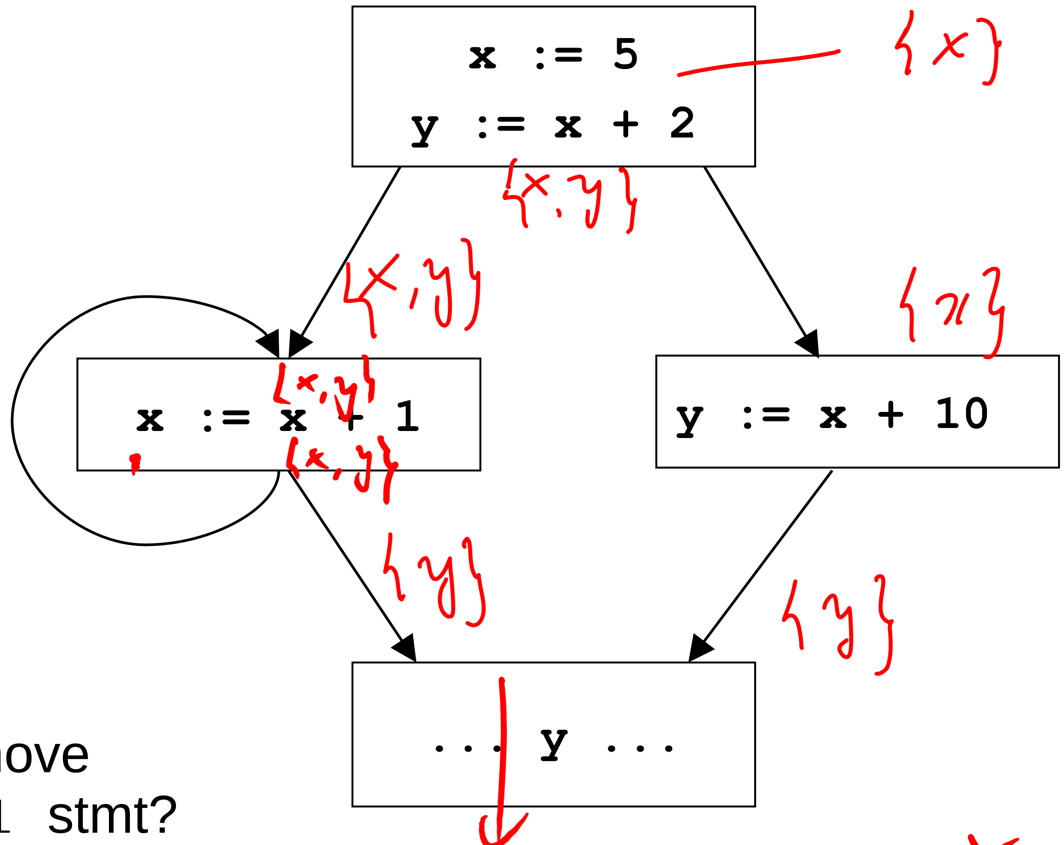
Example: live variables



Example: live variables

1st iter | 2nd iter

{y} | {x, y}



How can we remove the `x := x + 1` stmt?

x := E

Revisiting assignment

b := e + f
a := b + c

in

~~`X := Y op Z`~~

out

$F_{X\ :=\ Y\ op\ Z}(out) = out - \{\ X\ \} \cup \{\ Y, Z\}$

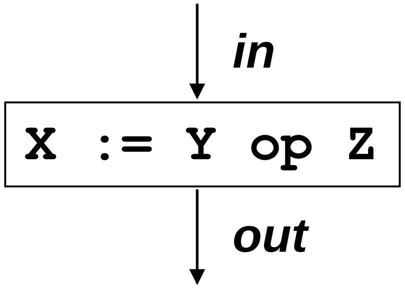
$X \notin out\ ?\ out :$

$(out - \{X\} \cup \{Y, Z\})$

x
Y := X + 1
X := X + 1

{y}

Revisiting assignment



$$F_{X := Y \text{ op } Z}(\text{out}) = \text{out} - \{ X \} \cup \{ Y, Z \}$$

$\text{out} - \{x\} \cup$

$x \notin \text{out} \; ? \; \emptyset : \{Y, Z\}$

Theory of backward analyses

- Can formalize backward analyses in two ways
- Option 1: reverse flow graph, and then run forward problem
- Option 2: re-develop the theory, but in the backward direction

Precision

- Going back to constant prop, in what cases would we lose precision?

```
if (p) {
    a := 7
} else {
    a := 4
}

if (p) {
    b := a - 2
} else {
    b := a + 1
}
b := 5
```

Precision

- Going back to constant prop, in what cases would we lose precision?

```
x := 5
if (<expr>) {
    x := 6
}
... x ...

where <expr> is
equiv to false
```

```
if (p) {
    x := 5;
} else
    x := 4;
}
...
if (p) {
    y := x + 1
} else {
    y := x + 2
}
... y ...
```

7

```
if (...) {
    x := -1;
} else
    x := 1;
}
y := x * x;
... y ...
```

Precision

- The first problem: Unreachable code
  - solution: run unreachable code removal before
  - the unreachable code removal analysis will do its best, but may not remove all unreachable code
- The other two problems are path-sensitivity issues
  - Branch correlations: some paths are infeasible
  - Path merging: can lead to loss of precision

MOP: meet over all paths

- Information computed at a given point is the meet of the information computed by each path to the program point

```
if (...) {
   x := -1;
} else
   x := 1;
}
y := x * x;
... y ...
```

x := -1

y := x * x

y → 1

x := 1

y := x * x

y → 1

y → 1 ⊔

MOP

- For a path p, which is a sequence of statements $[s_1, ..., s_n]$, define: $F_p(\text{in}) = F_{s_n}(\ldots F_{s_1}(\text{in}) \ldots)$
- In other words: $F_p =$ $F_{s_1} \circ \cdots \circ F_{s_n}$
- Given an edge e, let paths-to(e) be the (possibly infinite) set of paths that lead to e
- Given an edge e, MOP(e) = $\bigsqcup_{p \in \text{paths-to}(e)} F_p(\perp)$
- For us, should be called JOP (ie: join, not meet)

MOP vs. dataflow

- MOP is the "best" possible answer, given a fixed set of flow functions
  - This means that MOP $\sqsubseteq$ dataflow at edge in the CFG
- In general, MOP is not computable (because there can be infinitely many paths)
  - vs dataflow which is generally computable (if flow fns are monotonic and height of lattice is finite)
- And we saw in our example, in general, MOP $\neq$ dataflow

MOP vs. dataflow

$a * (b + c) = a * b + a * c$

- However, it would be great if by imposing some restrictions on the flow functions, we could guarantee that dataflow is the same as MOP. What would this restriction be?

Dataflow

```
x := -1;        x := 1;
```

a   b

$\sqcup$

Merge

$a \sqcup b$

```
y := x * x;
... y ...
```

F

$F(a \sqcup b)$

$y \to T$

MOP

```
x := -1;        x := 1;
y := x * x;     y := x * x;
... y ...       ... y ...
```

a   b

F(a)   F(b)

Merge

$F(a) \sqcup F(b)$

$y \to 1$

MOP vs. dataflow

- However, it would be great if by imposing some restrictions on the flow functions, we could guarantee that dataflow is the same as MOP. What would this restriction be?

- Distributive problems. A problem is distributive if:

$$\forall\ a, b\ .\ F(a \sqcup b) = F(a) \sqcup F(b)$$

- If flow function is distributive, then MOP = dataflow

Summary of precision

- Dataflow is the basic algorithm
- To basic dataflow, we can add path-separation
  - Get MOP, which is same as dataflow for distributive problems
  - Variety of research efforts to get closer to MOP for non-distributive problems
- To basic dataflow, we can add path-pruning
  - Get branch correlation
- To basic dataflow, can add both:
  - meet over all feasible paths