UM–SJTU Joint Institute

Physics Laboratory

(Vp141)

Laboratory Report

Exercise 5

Damped and Driven Oscillations. Mechanical Resonance

Name: Kang Jiaming	ID: 518021911220	Group: 18
Name: Luo Chenhao	ID: 518370910038	Group: 18

Contents

1 Introduction

The objective of this exercise is to further the knowledge about damped and driven mechanical oscillations, and also the mechanical resonance phenomenon. How to operate on the Pohl resonator will also be explored.

For a simple harmonic oscillator, its equation of motion is $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$. If the oscillator experiences a linear drag, its motion becomes a damped harmonic oscillation, and the equation of motion is $\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{k}{m}x = 0$. If further, a driving force is applied to the damped harmonic oscillator, the resulting motion is called driven (or forced) oscillations. Assuming that the driving force is of the form $F = F_0 \cos \omega t$, the equation of motion is then:

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \frac{k}{m}x = F_0 \cos \omega t. \tag{1}$$

It can be observed that the driven oscillation will be a periodic motion with the same angular frequency ω after it stabilizes.

In this experiment, however, the oscillation system takes the angular form instead of linear form. A balance wheel, the oscillator in our system, rotates about its central axis, with the restoring torque $\tau = -k\theta$, a damping torque $\tau_f = -b\frac{d\theta}{dt}$ and a periodic driving torque $\tau_{\mathrm{dr}} = \tau_0 \cos \omega t$. Its equation of motion is then

$$I\frac{d^2\theta}{dt^2} = -k\theta - b\frac{d\theta}{dt} + \tau_0 \cos \omega t. \tag{2}$$

Simplifying the notions using$\omega_0^2 = \frac{k}{I}, \quad 2\beta = \frac{b}{I}, \quad \mu = \frac{\tau_0}{I}$, Eq.2 can be written as

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \omega_0^2\theta = \mu \cos \omega t. \tag{3}$$

The solution to Eq.3 is

$$\theta(t) = \theta_{\mathrm{tr}}(t) + \theta_{\mathrm{st}} \cos(\omega t + \varphi). \tag{4}$$

θ_{tr} represents the influence of initial conditions on the motion, which vanishes exponentially as $t \to \infty$. $\theta_{\mathrm{st}} \cos(\omega t + \varphi)$ describes the motion of the steady states. Its amplitude θ_{st} and the phase shift φ are found as

$$\theta_{\mathrm{st}} = \frac{\mu}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2,}} \tag{5}$$

and

$$\tan \varphi = \frac{2\beta\omega}{\omega^2 - \omega_0^2} \; (-\pi \leq \varphi < 0), \tag{6}$$

The figures of θ_{st} vs. ω (left) and φ vs. ω (right)are shown in Figure 1.

From Eq.5, it can be derived that when ω becomes closed to ω_0, the amplitude will sharply increase. Such a phenomenon is called resonance. It can also be obtained that when $\omega = \omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}$, which is called the resonance angular frequency, the amplitude reaches its maximum

$$\theta_{\mathrm{res}} = \theta_{\mathrm{st}}\left(\omega_{\mathrm{res}}\right) = \frac{\mu}{2\beta\sqrt{\omega_0^2 - \beta^2}} \tag{7}$$

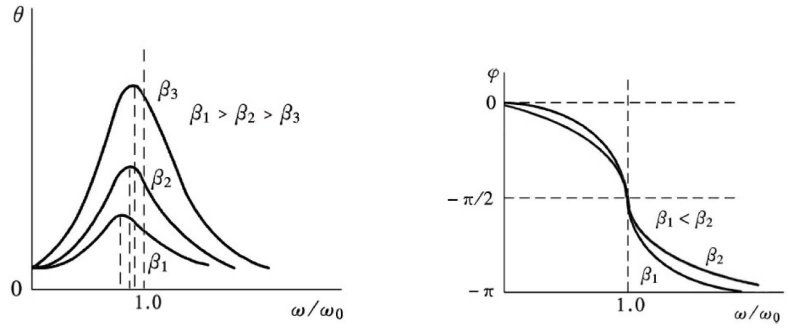


Figure 1: The dependence of the amplitude (left) and phase shift (right) of steady-state driven oscillations.

As for the basic ideas for measurement, period T and angular frequency ω are related by the formula

$$\omega = \frac{2\pi}{T}. \tag{8}$$

Therefore, the angular frequency can be found by measuring the period of oscillations. The solution to Eq.3 combining the initial state and the final steady state is $\theta(t) = \theta e^{-\beta t}\cos(\omega_{\mathrm{f}} t + \alpha)$. Hence, if we measure the amplitude every other period and obtain $\theta_0 = \theta e^{-\beta T}, \theta_1 = \theta e^{-\beta(2T)}, \ldots, \theta_n = \theta e^{-\beta(nT)}$, we have

$$\ln\frac{\theta_i}{\theta_j} = \ln\frac{\theta e^{-\beta(iT)}}{\theta e^{-\beta(jT)}} = (j-i)\beta T \tag{9}$$

The damping coefficient β can be then calculated as

$$\beta = \frac{1}{(j-i)T}\ln\frac{\theta_i}{\theta_j} \tag{10}$$

In our experiment, we will measure ten items of data, i.e. $\theta_0, \theta_1, \ldots, \theta_9$. But how should we process the measurement data? Shall we simply takes $j = i+1$, achieve ten β, and calculate the average as the result?

In that case, the average value is expressed by

$$\beta = \frac{\frac{1}{T}\ln\frac{\theta_0}{\theta_1} + \frac{1}{T}\ln\frac{\theta_1}{\theta_2} + \cdots + \frac{1}{T}\ln\frac{\theta_8}{\theta_9}}{10} = \frac{\ln\frac{\theta_0}{\theta_9}}{10T} \tag{11}$$

It can be seen that actually only θ_0 and θ_9 are taken into account. How to deal with the data so that all the data can be used?

The successive difference method is an effective method to reduce the error of the average value calculated from a series of measurement data. Applying the idea to our experiment, we take $j = i+5$ and the damping coefficient is thus calculated as

$$\beta = \frac{1}{5T}\ln\frac{\theta_i}{\theta_{i+5}} \tag{12}$$

In this case, the average value is expressed by

$$\beta = \frac{\frac{1}{T}\ln\frac{\theta_0}{\theta_5} + \frac{1}{T}\ln\frac{\theta_1}{\theta_6} + \cdots + \frac{1}{T}\ln\frac{\theta_4}{\theta_9}}{5} = \frac{\ln\frac{\theta_0\theta_1\cdots\theta_4}{\theta_5\theta_6\cdots\theta_9}}{5T} \tag{13}$$

In Eq.13, all the data measured is used. The error can be reduced in this way.

2 Experimental Setup

The whole experimental is conducted through a BG-2 Pohl resonator which consists of a vibrometer (Figure 2) and a control box (Figure 3 and 4). Period can be measured by the timer in the control box. Phase lag can be read off from the angle scale on the balance wheel which a flash generated by the strobe shines onto. Amplitude can also be displayed by the control box. The knobs on the control box can change the magnitude of the damping force and the frequency of the driving force.

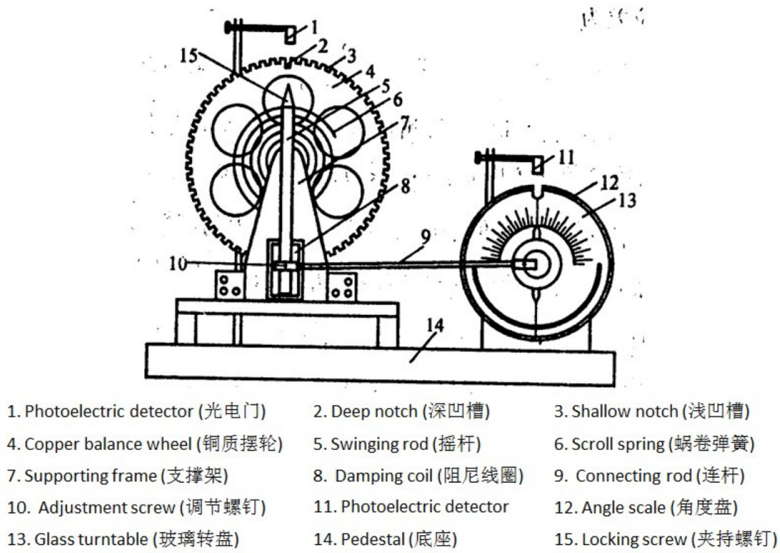
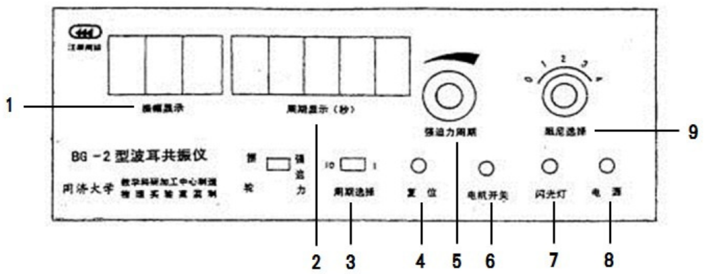


Figure 2: The vibrometer.

The type-B uncertainties of the measured physical quantities are summarized in Table 1.

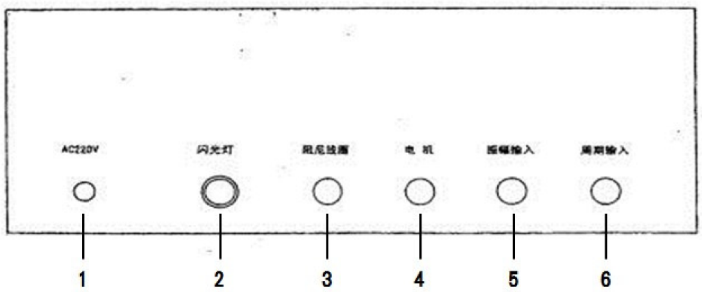
Physical quantities	Type-B uncertainties
Ten periods	$0.001\,\mathrm{s}$
Amplitude	$1\,^{\circ}$
Phase lag	$1\,^{\circ}$

Table 1: Type-B uncertainties.



1. Amplitude display (振幅显示) 2. Period display (周期显示) 3. Period selection (周期选择)
4. Reset (复位) 5. Period of driving force (策动力周期) 6. Motor switch (电机开关)
7. Flash (闪光灯) 8. Power (电源) 9. Damping selection (阻尼选择)

Figure 3: The front panel of the control box.



1. 220V AC power (AC220V) 2. Flash (闪光灯) 3. Damping coil (阻尼线圈)
4. Motor (电机) 5. Amplitude input(振幅输入) 6. Period input (周期输入)

Figure 4: The rear panel of the control box.

3 Measurement Procedure

CAUTION

1. Check the position of the photoelectric gate above the balanced wheel. Make sure there is enough space between the gate and the wheel.
2. Pohl resonator is a very delicate device, you should operate the apparatus according to the manual and instructions of the teaching assistant.
3. The motor must be turned off during steps 3.1 and 3.2.
4. To ensure accuracy of the measurement, you should not change the Damping Selection until the entire measurement is completed.

3.1 Natural Angular Frequency

The natural angular frequency is obtained by simply measuring the period of the oscillation without damping and driving force and applying Eq.8.

1. Turn the Damping Selection knob to ”0”.
2. Carefully rotate the balance wheel to the initial angular position $\theta_0 \approx 150^\circ$ and release it. Record the time of 10 periods.
3. Repeat for four times and calculate the natural angular frequency ω_0.

3.2 Damping Coefficient

The damping coefficient is found by measuring the amplitude every other period and applying Eq. Note that the first measured amplitude should be neglected since the oscillation at first may be affected by the operators. 12.

1. Turn the Damping Selection knob to ”2”, and the selection should not be changed during this part.
2. Carefully rotate the balance wheel to the initial amplitude of approximately 150° and release it. Record the amplitude of each period (start from the second amplitude after you release the wheel) and the time of 10 periods.
   Tip. In case you miss some readings, repeat the measurement recording it with your phone camera.
3. Apply Eq.12 and calculate the average of the results to find β

3.3 θ_{st} vs. ω and φ vs. ω Characteristics of Forced Oscillations

In this part, driving force will be applied to the oscillation system. Its frequency should be adjusted in a well-designed way as instructed in step 2 in order to get sufficient data to plot the θ_{st} vs. ω and φ vs. ω graphs.

Please be patient enough to wait for the oscillation stabilizes. It is recommended to wait for at least ten seconds after the amplitude stays unchanged.

When measuring the phase lag, the flash will not occur exactly at the same position. Read a medium scale in that case.

1. Keep the Damping Selection at "2", and set the speed of the motor (record the position of the motor knob in case you need to repeat the measurement). Record the amplitude θ_{st}, the period T, and the phase shift φ when the oscillation reaches a steady state.
2. Repeat the steps above by changing the speed of the motor. It will result in a change of the phase shift φ (referred to as $\delta\varphi$). To make your plots more accurate, you should collect more data when φ and θ_{st} change rapidly (*e.g.* near to the resonance point). At least 15 data should be collected for plotting.
3. Choose Damping Selection "1" or "3". Repeat the above steps.
4. Plot the $\theta_{st}(\omega)$ characteristics, with ω/ω_0 on the horizontal axis and θ_{st} on the vertical axis. Two sets of data should be plotted on the same graph.
   Plot the $\varphi(\omega)$ characteristics, with ω/ω_0 on the horizontal axis and φ on the vertical axis. Two sets of data should be plotted on the same graph.

4 Results

4.1 Natural Angular Frequency

The measured time of 10 periods is presented in Table 2

Trial	T_{10} [s] $\pm$ 0.001 [s]
1	15.881
2	15.881
3	15.879
4	15.879

Table 2: Measurement results of the natural frequency.

The average value of T_{10} is

$$\overline{T_{10}} = \frac{1}{4}\sum_{i=1}^{4} T_{10_i} = 15.880 \pm 0.00208\,[\mathrm{s}].$$

The average value of T is thus

$$\overline{T} = \frac{1}{10}\overline{T_{10}} = 1.5880 \pm 0.0002\,[\mathrm{s}].$$

Hence, applying Eq.8, the natural frequency is

$$\omega_0 = \frac{2\pi}{\overline{T}} = \frac{2\pi}{1.5880} = 3.9567 \pm 0.0005\,[\mathrm{s}^{-1}],$$

with relative uncertainty 0.01 %

4.2 Damping Coefficient

The time of ten periods measured is $T_{10} = 15.892 \pm 0.001\,\mathrm{s}$.
The measured data of amplitude is presented in Table 3

Item	Amplitude[°] ± 1[°]	Item	Amplitude[°] ± 1[°]	$q_i = \ln(\theta_i/\theta_{i+5})$
θ_0	128	θ_5	78	0.495 ± 0.015
θ_1	116	θ_6	70	0.505 ± 0.017
θ_2	104	θ_7	62	0.517 ± 0.019
θ_3	95	θ_8	56	0.529 ± 0.021
θ_4	87	θ_9	51	0.534 ± 0.023

Table 3: Measurement results of the damping coefficient.

From Table 3 we can calculate

$$\overline{q} = \frac{1}{5}\sum_{i=1}^{5} q_i = 0.516 \pm 0.030.$$

The period T is

$$T = \frac{1}{10} \times 15.892 = 1.5892 \pm 0.0001\,[\mathrm{s}].$$

Thus, we can calculate the damping coefficient β by applying Eq.12:

$$\beta = \frac{1}{5T}\overline{q} = 0.064938 \pm 0.004\,[\mathrm{s}^{-1}],$$

with relative uncertainty 6.16 %

4.3 θ vs. ω and φ vs. ω Characteristics of Forced Oscillations

In the first trial, the Damping Selection is "2" and the data is presented in Table 4; in the second trial, the Damping Selection is "1" and the data is presented in Table 5

Calculate $T_{10_{\text{natural}}}/T_{10}$, which is equal to ω/ω_0, and the corresponding uncertainties. The results are shown in Table 6 and 7.

With the data in Table 4 $\sim$ 7, plot θ_{st} vs. ω/ω_0 and φ vs. ω/ω_0 using Origin. The results are presented in Figure 5 and Figure 6 respectively.

Trial	T_{10} [s] $\pm$ 0.001 [s]	φ [°] $\pm$ 1 [°]	θ [°] $\pm$ 1 [°]
1	16.613	20	48
2	16.408	27	63
3	16.274	34	80
4	16.152	44	99
5	16.095	51	110
6	16.023	62	123
7	15.962	73	133
8	15.925	81	138
9	15.906	85	138
10	15.893	88	138
11	15.886	89	138
12	18.879	91	138
13	15.864	95	138
14	15.855	97	138
15	15.848	99	137
16	15.843	100	136
17	15.783	113	127
18	15.719	126	112
19	15.655	135	98
20	15.592	142	85
21	15.524	147	73
22	15.407	154	58
23	15.255	159	46
24	15.045	163	35

Table 4: Measurement results when Damping Selection is“2”.

Trial	T_{10} [s] $\pm$ 0.001 [s]	φ [°] $\pm$ 1 [°]	θ [°] $\pm$ 1 [°]
1	16.575	20	51
2	16.411	25	64
3	16.227	36	89
4	16.139	44	104
5	16.072	53	118
6	16.015	62	131
7	15.964	72	140
8	15.918	82	146
9	15.901	86	148
10	15.893	88	148
11	15.887	89	148
12	15.878	91	148
13	15.865	94	148
14	15.843	100	146
15	15.786	114	136
16	15.709	130	114
17	15.603	143	89
18	15.448	154	64
19	15.269	161	47
20	15.077	164	36

Table 5: Measurement results when Damping Selection is “1”.

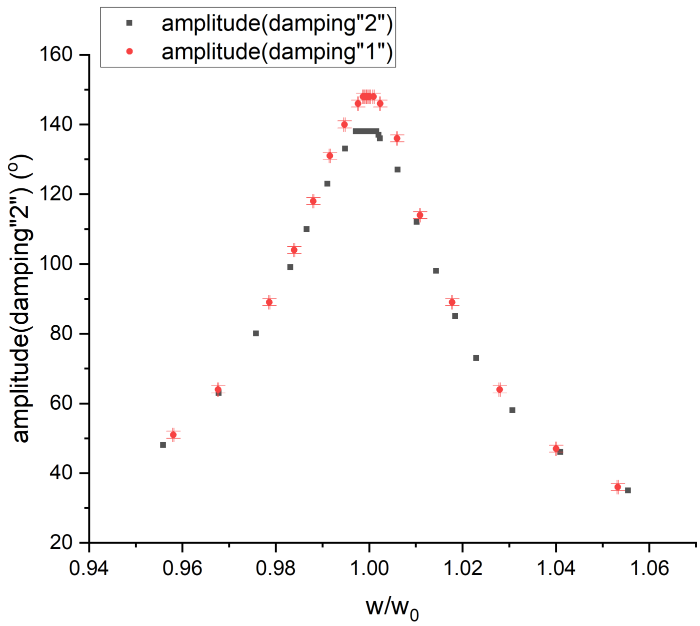


Figure 5: θ_{st} vs. ω/ω_0.

Trial	T_{10} [s] ± 0.001 [s]	φ [°] ± 1 [°]	θ [°] ± 1 [°]	ω/ω_0	u_{ω/ω_0}
1	16.613	20	48	0.9559	0.0001
2	16.408	27	63	0.9678	0.0001
3	16.274	34	80	0.9758	0.0001
4	16.152	44	99	0.9832	0.0001
5	16.095	51	110	0.9866	0.0001
6	16.023	62	123	0.9911	0.0001
7	15.962	73	133	0.9949	0.0001
8	15.925	81	138	0.9972	0.0001
9	15.906	85	138	0.9984	0.0001
10	15.893	88	138	0.9992	0.0001
11	15.886	89	138	0.9996	0.0001
12	18.879	91	138	0.8411	0.0001
13	15.864	95	138	1.0010	0.0001
14	15.855	97	138	1.0016	0.0001
15	15.848	99	137	1.0020	0.0001
16	15.843	100	136	1.0023	0.0001
17	15.783	113	127	1.0061	0.0001
18	15.719	126	112	1.0102	0.0001
19	15.655	135	98	1.0144	0.0001
20	15.592	142	85	1.0185	0.0001
21	15.524	147	73	1.0229	0.0001
22	15.407	154	58	1.0307	0.0002
23	15.255	159	46	1.0410	0.0002
24	15.045	163	35	1.0555	0.0002

Table 6: ω/ω_0 and corresponding uncertainties when Damping Selection is "2".

Trial	T_{10} [s] $\pm$ 0.001 [s]	φ [°] $\pm$ 1 [°]	θ [°] $\pm$ 1 [°]	ω/ω_0	u_{ω/ω_0}
1	16.575	20	51	0.9581	0.0001
2	16.411	25	64	0.9676	0.0001
3	16.227	36	89	0.9786	0.0001
4	16.139	44	104	0.9840	0.0001
5	16.072	53	118	0.9881	0.0001
6	16.015	62	131	0.9916	0.0001
7	15.964	72	140	0.9947	0.0001
8	15.918	82	146	0.9976	0.0001
9	15.901	86	148	0.9987	0.0001
10	15.893	88	148	0.9992	0.0001
11	15.887	89	148	0.9996	0.0001
12	15.878	91	148	1.0001	0.0001
13	15.865	94	148	1.0009	0.0001
14	15.843	100	146	1.0023	0.0001
15	15.786	114	136	1.0060	0.0001
16	15.709	130	114	1.0109	0.0001
17	15.603	143	89	1.0178	0.0001
18	15.448	154	64	1.0280	0.0002
19	15.269	161	47	1.0400	0.0002
20	15.077	164	36	1.0533	0.0002

Table 7: ω/ω_0 and corresponding uncertainties when Damping Selection is "1".

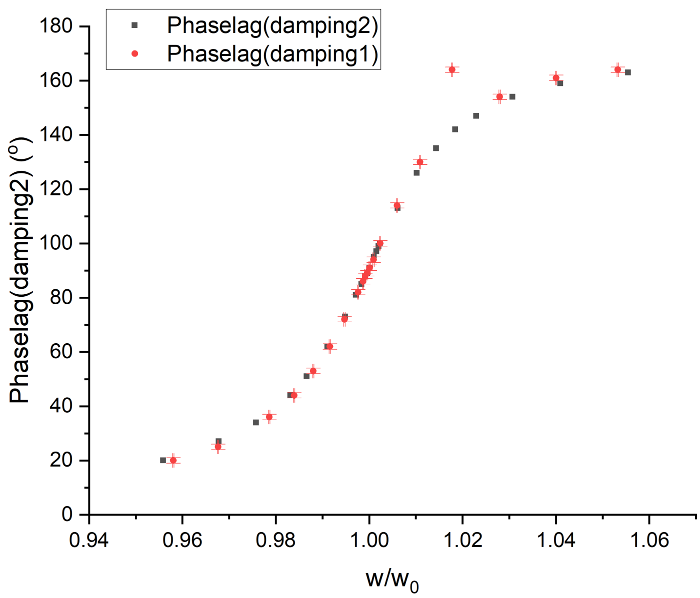


Figure 6: φ vs. ω/ω_0.

5 Conclusions and Discussion

5.1 Measurement of natural frequency

The relative uncertainty of the measurement of this part is 0.01 %, which suggests that our experiment is of high precision. However, the existing air drag may make the natural frequency of our system a bit 'unnatural' and thus cause error to our measurement.

5.2 Measurement of damping coefficient

The relative uncertainty of the measured β is 6.16 %, which is rather high. Some factors may result in the uncertainty and cause error:

1. The air drag may influence our oscillation system and thus cause error.

2. The low resolution of the device we used is 1°, which is relatively large. It may result in the great uncertainty.

5.3 Measurement of θ vs. ω and φ vs. ω characteristics

The plotted graphs are shown in Figure 5 and 6. And the corresponding uncertainties are showed in Table 6 and 7. The shape of the two plottings basically conforms to the theoretical graph as shown in Figure 1. However, note that the fourth circle point from the left in Figure 6 deviates from the plotting obviously. This is because we perform measurement before the oscillation becomes steady. Also, the amplitude about $\omega/\omega_0 = 1$ is measured to be the same. This is because the low resolution of our device, which is $1\,^\circ$.

5.4 Suggestions

Here are some tips for the improvement:

1. Use the device with higher resolution. Also, the flash is rather harmful to the eyes.

2. Try to reduce the air drag, which may cause error to our experiment.

6 Reference

[1] Qin Tian, et al. editor."VP141 Exercise 5: Damped and Driven Oscillations. Mechanical Resonance".
[2] Young and Freedman. *University Physics with Modern Physics.* Chapter 14.

A Uncertainty Analysis

Please find in the attached pages.

B Data Sheet

Please find in the attached pages.