JOINT INSTITUTE
交大密西根学院

Physics Laboratory
Vp241

Exercise 5

RC, RL, and RLC Circuits

This lab manual is based on materials provided by the Department of Physics, Shanghai Jiaotong University.

Edited by: Qin Tian, Feng Yaming, Gu Yichen, Mateusz Krzyzosiak rev. 2.6

1 Pre-lab Reading

Chapter 31 (Young and Freedman)

2 Objectives

The objective of this exercise is to understand the physics of alternating-current circuits, in particular the processes of charging/discharging of capacitors, the phenomenon of electromagnetic induction in inductive elements, and other dynamic processes in RC, RL, and RLC series circuits. Moreover, methods for measuring the amplitude-frequency and the phase-frequency characteristics of RC, RL, and RLC series circuits will be studied. The resonance frequency of a RLC circuit as well as the quality factor of the circuit will be found from the amplitude-frequency curve.

3 Theoretical Background

Resistors, capacitors, and inductors are basic elements of electric circuits. Depending on a particular arrangement of these elements, RC, RL, RLC alternating-current (AC) circuits may display various features, including transient, steady state, and resonant behavior.

3.1 Transient Processes in RC, RL, RLC Series Circuits

3.1.1 RC Series Circuits

In a RC circuit, the process of charging or discharging of the capacitor is an example of a transient process. Figure 1 shows a RC series circuit in which a square-wave signal is used as the source signal. In the first half of the cycle, the square-wave voltage is $U(t) = \mathcal{E}$

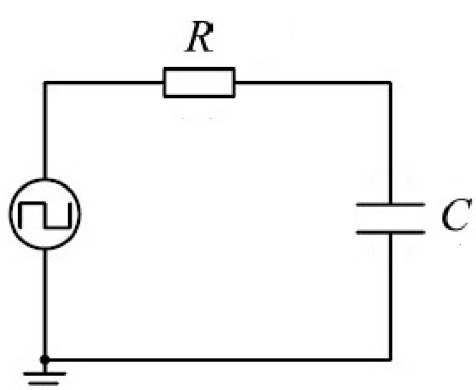

Figure 1. RC series circuit.

and it charges the capacitor. In the second half-cycle, the square-wave voltage is zero,

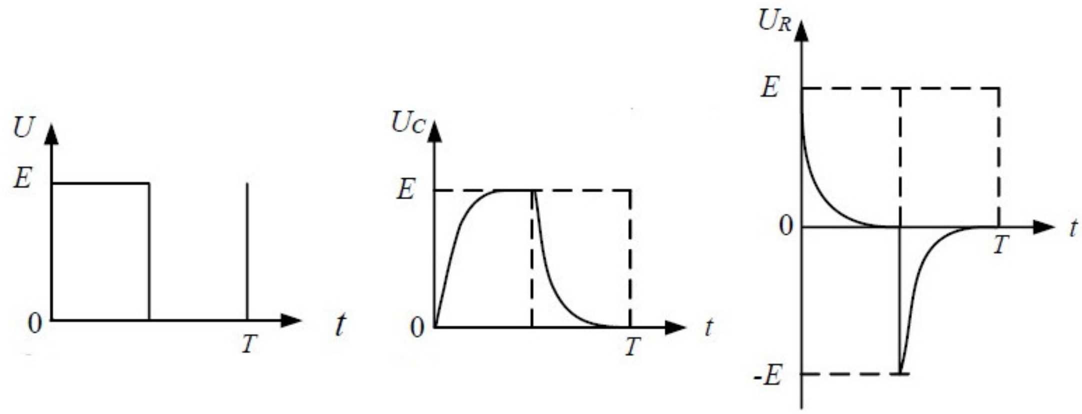

Figure 2. Charging/discharging curves for a RC series circuit.

and the capacitor discharges through the resistor. The loop equation (Kirchhoff's loop rule) for the charging process is

$$RC\,\frac{\mathrm{d}U_C}{\mathrm{d}t} + U_C = \mathcal{E}. \tag{1}$$

With the initial condition $U_C(t=0) = 0$, the solution of Eq. (1) can be found as

$$U_C = \mathcal{E}(1 - e^{-\frac{t}{RC}}) \qquad \text{and} \qquad U_R = iR = \mathcal{E}e^{-\frac{t}{RC}}.$$

Hence the voltage across the capacitor U_C increases exponentially with time t, whereas the voltage on the resistor U_R decreases exponentially with time. The curves $U(t)$, $U_C(t)$, and $U_R(t)$ are shown in Figure 2.

For the discharging process, the loop rule gives

$$RC\,\frac{\mathrm{d}U_C}{\mathrm{d}t} + U_C = 0. \tag{2}$$

The solution of Eq. (2), with the initial condition $U_C(t=0) = \mathcal{E}$, is

$$U_C = \mathcal{E}e^{-\frac{t}{RC}},$$

and, consequently,

$$U_R = iR = -\mathcal{E}e^{-\frac{t}{RC}},$$

where the magnitudes of both U_C and U_R decrease exponentially with time. Since $RC = \tau$ has the units of time, it is called the *time constant* of the circuit, and characterizes the dynamics of the transient process. There is another characteristics related to the time constant, easier to measure in experiments, which is called the *half-life period* $T_{1/2}$. The half-life period is the time needed for U_C to decrease to a half of the initial value (or increase to a half of the terminal value), and may be also used to characterize the dynamics of the transient process. Both quantities, in the process with exponential dynamics discussed above, are related by the equation

$$T_{1/2} = \tau \ln 2 \approx 0.693\tau.$$

3.1.2 RL Series Circuit

A similar analysis can be carried out for a RL series circuit. In this case,

$$\tau = \frac{L}{R} \qquad \text{and} \qquad T_{1/2} = \frac{L}{R}\ln 2.$$

3.1.3 RLC Series Circuit

First, let us discuss the situation when a power source is suddenly plugged into a RLC circuit. Then the voltage across the capacitor satisfies the differential equation

$$LC\frac{\mathrm{d}^2 U_C}{\mathrm{d}t^2} + RC\frac{\mathrm{d}U_C}{\mathrm{d}t} + U_C = \mathcal{E}, \tag{3}$$

following again from the loop rule. Dividing both sides of the equation by LC and introducing the symbols

$$\beta = \frac{R}{2L}, \qquad \text{and} \qquad \omega_0 = \frac{1}{\sqrt{LC}}, \tag{4}$$

Eq. (3) can be rewritten as

$$\frac{\mathrm{d}^2 U_C}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}U_C}{\mathrm{d}t} + \omega_0^2 U_C = \omega_0^2\mathcal{E}. \tag{5}$$

Note that Eq. (5) is an inhomogeneous differential equation and it is mathematically equivalent to the equation of motion of a damped harmonic oscillator with a constant driving force. Therefore, the complementary homogeneous equation is fully analogous to the equation of motion of a damped harmonic oscillator, with β being the damping coefficient, and ω_0 — the natural angular frequency. Moreover, after a specific solution to the inhomogeneous equation is found, a unique solution to the initial value problem consisting of Eq. (5) and the initial conditions

$$U_C(t=0) = 0 \qquad \text{and} \qquad \left.\frac{\mathrm{d}U_C}{\mathrm{d}t}\right|_{t=0} = 0. \tag{6}$$

can be found.

Exactly as for mechanical oscillations, depending on the relation between β and ω_0, there are three regimes, as implied by the solution of the complementary homogeneous equation:

- If $\beta^2 - \omega_0^2 < 0$ (weak damping), the system is in the underdamped regime and the solution to the initial value problem is of the form

$$U_C = \mathcal{E} - \mathcal{E}e^{-\beta t}\left(\cos\omega t + \frac{\beta}{\omega}\sin\omega t\right),$$

where $\omega = \sqrt{\omega_0^2 - \beta^2}$.

- If $\beta^2 - \omega_0^2 > 0$ (strong damping), the system is in the overdamped regime with the solution of the form

$$U_C = \mathcal{E} - \frac{\mathcal{E}}{2\gamma} e^{-\beta t} \left[(\beta + \gamma) e^{\gamma t} - (\beta - \gamma) e^{-\gamma t} \right],$$

where $\gamma = \sqrt{\beta^2 - \omega_0^2}$.

- Finally, if $\beta^2 - \omega_0^2 = 0$, the system is said to be critically damped, and

$$U_C = \mathcal{E} - \mathcal{E}(1 + \beta t) e^{-\beta t}. \tag{7}$$

When the circuit reaches a steady state, the power source is suddenly removed ($\mathcal{E} = 0$). The differential equation for the discharging process is similar to that of the charging process, and there are also three regimes of the process.

The above discussion is valid for an ideal circuit and a step-signal source with zero internal resistance. In the experiment, the ideal system is replaced by a square-wave source with a small internal resistance. The period of the square-signal must be much greater than the time constant of the circuit. Note that, according to the above equations, the voltage across the capacitor U_C will finally reach $\mathcal{E}$ regardless of the regime (Figure 7efrlcdisc, charging phase).

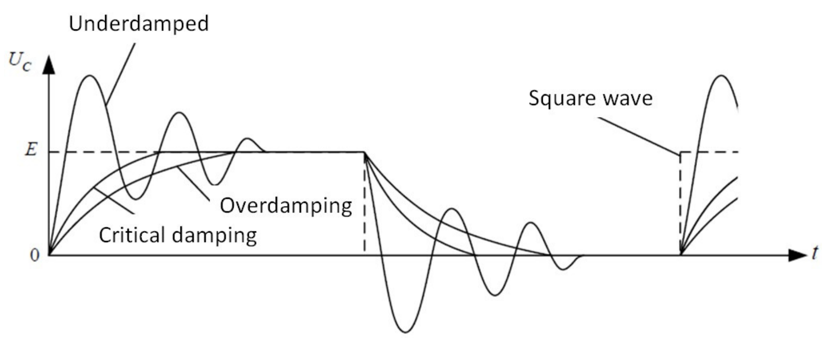

Figure 3. Three different regimes of transient processes in a RLC series circuit.

3.2 RC, RL Steady–State Circuits

When a sinusoidal alternating input voltage is provided to a RC (or RL) series circuit, the amplitude and the phase of the voltage across the capacitor and the resistor will change with the frequency of the input voltage. Then the amplitude vs. frequency relation and the phase vs. frequency relation can be obtained by measuring the voltage across the elements in the circuit for different input signal frequencies

$$\varphi = \tan^{-1}\left(\frac{U_L}{U_R}\right) = \tan^{-1}\left(\frac{\omega L}{R}\right), \qquad \varphi = \tan^{-1}\left(-\frac{U_C}{U_R}\right) = \tan^{-1}\left(-\frac{1}{\omega RC}\right).$$

3.3 *RLC* Resonant Circuit

3.3.1 *RLC* Series Circuit

A generic *RLC* series circuit is shown in Figure 4. The impedance and the phase difference in the *RLC* circuit can be calculated, *e.g.*, by using the phasors technique. Representing the current I by a vector along the horizontal axis, the phase differences

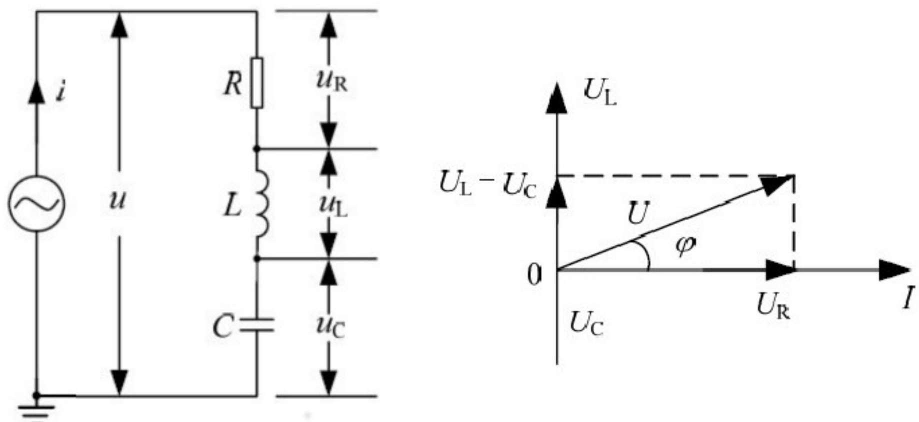

Figure 4. *RLC* series circuit.

between the current and the voltages across the resistor, coil, and capacitor are

$$\varphi_R = 0, \qquad \varphi_L = \frac{\pi}{2}, \qquad \varphi_C = -\frac{\pi}{2},$$

respectively. The corresponding voltage amplitudes across the elements are

$$U_R = IZ = IR, \qquad U_L = IZ_L = I\omega L, \qquad U_C = IZ_C = \frac{I}{\omega C}.$$

Hence, the voltage amplitude

$$U = \sqrt{U_R^2 + (U_L - U_C)^2} \qquad \text{or} \qquad U = I\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}, \tag{8}$$

and the total impedance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}, \tag{9}$$

with the phase difference between the current and the voltage in the circuit

$$\varphi = \tan^{-1}\left(\frac{U_L - U_C}{U_R}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right).$$

3.3.2 Resonance

error analysis

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If the frequency of the input signal provided by the source satisfies the condition

$$\omega_0 L = \frac{1}{\omega_0 C}, \qquad \text{or, equivalently,} \qquad \omega_0 = \frac{1}{\sqrt{LC}},$$

the total impedance will reach a minimum, $Z_0 = R$. Note that the resistance R in a real circuit includes the internal resistance and all kinds of alternating-current power losses, so its actual value will be greater than the theoretical one.

When the current reaches its maximum, $I_\mathrm{m} = U/R$, the circuit is said to be at resonance. The frequency

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}},$$

at which the resonance phenomenon occurs, is called the *resonance frequency*.

The total impedance Z, the current I, and the phase difference $\varphi = \varphi_u - \varphi_i$ all depend on the frequency, with generic shapes of the three curves shown in Figure 5. According to Eqs. (8) and (9), when the frequency is low ($f < f_0$, *i.e.* $1/\omega C > \omega L$), then $\varphi < 0$. In this situation the total voltage lags behind the current and the circuit is said to be *capacitive*.

When the circuit is resonant ($f = f_0$, *i.e.* $1/\omega C = \omega L$), then $\varphi = 0$ and the voltages across the capacitor and the inductor should be equal. The circuit is said to be *resistive*.

Finally, when the frequency is high ($f > f_0$, *i.e.* $\omega L > 1/\omega C$), then $\varphi > 0$. In this situation the total voltage leads the current, and the circuit is said to be *inductive*.

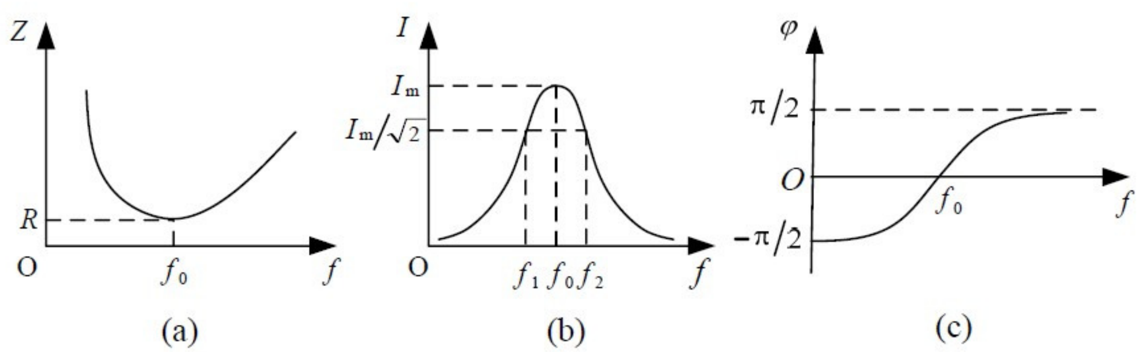

Figure 5. The impedance, the current and the phase difference as functions of the frequency for a RLC series circuit (generic sketches).

3.3.3 Quality Factor in Resonant Circuits

Since $I_\mathrm{m} = U/R$, the voltages across the resistor, the inductor, and the capacitor are

$$U_R = I_\mathrm{m} R = U,$$

$$U_L = I_m Z_L = \frac{U}{R}\omega L,$$
$$U_C = I_m Z_C = \frac{U}{R}\frac{1}{\omega_0 C} = U_L,$$

respectively. For a circuit driven at the resonance frequency, the ratio of U_L (or U_C) to U is called the quality factor Q of a resonant circuit

$$Q = \frac{U_L}{U} = \frac{\omega_0 L}{R} \qquad \text{or} \qquad Q = \frac{U_C}{U} = \frac{1}{\omega_0 RC}.$$

When the total voltage is fixed, the greater Q is, the greater U_L and U_C are. The value of Q can be used to quantify the efficiency of resonant circuits.

The quality factor can also be found as

$$Q = \frac{f_0}{f_2 - f_1},$$

where f_1 and f_2 are two frequencies such that $I(f_1) = I(f_2) = I_m/\sqrt{2}$ (see Figure 5b).

4 Measurement Setup and Procedure

4.1 Apparatus

The measurement setup consists of the following main elements: a signal generator, an oscilloscope, a digital multimeter, a wiring board, a fixed resistor 100 Ω (2 W), a variable resistor 2 kΩ (2 W), two capacitors 0.47 μF and 0.1 μF, as well as two inductors (10 mH and 33 mH).

4.2 Measurement Procedure

4.2.1 RC, RL Series Circuit

1. Choose a capacitor and an inductor to assemble a circuit with the fixed-resistance 100 Ω resistor. Adjust the output frequency of the square-wave signal provided by the signal generator. Observe the change of the waveform when the time constant is smaller or greater than the period of the square–wave. Choose the frequency that allows the capacitor to fully charge/discharge. Use the `PRINT` function of the oscilloscope to store the waveforms.

2. Adjust display parameters of the oscilloscope and measure $T_{1/2}$ for the studied circuits. Then, calculate the time constant and compare it with the theoretical value. In order to find the time constant accurately, only one period should be displayed on the oscilloscope screen.

4.2.2 *RLC* Series Circuit

1. Choose a capacitor and an inductor to assemble a *RLC* series circuit with the variable resistor. Observe the waveform of the capacitor voltage in the underdamped, critically damped, and overdamped regimes. Use the `PRINT` function of the oscilloscope to store the waveforms.

2. Adjust the variable resistor to the critically damped regime. According to the definition of the half-life period $T_{1/2}$, we have $\beta T_{1/2} = 1.68$. By finding the value of $T_{1/2}$, the time constant can be found as $\tau = 1/\beta = T_{1/2}/1.68$. Compare your result with the theoretical value.

4.2.3 *RLC* Resonant Circuit

Apply a sinusoidal input voltage U_i to the *RLC* series circuit, change the frequency, then observe the change of the voltage U_R for a fixed resistor R, as well as the phase difference between U_R and U_i. Measure how U_R changes with U_i and calculate the phase difference according to Figure 4. Plot the graphs I/I_m vs. f/f_0 and φ vs. f/f_0. Estimate the resonance frequency and calculate the quality factor Q.

5 Cautions

- Read manuals carefully before operating the instruments.
- The circuit should be grounded to the same point as the instruments used in the measurements.

6 Preview Questions

- What elements does a *RLC* series circuit consist of?
- For a *RLC* circuit in a transient state, if we change the resistance of the resistor, will we get different time dependence of the voltage across the elements of the circuit?
- What are the three regimes a *RLC* series circuit may be in depending on the relationship between β and ω_0?
- What characteristics of a *RLC* series circuit can we use to judge whether the circuit is resonant or not?
- When a *RLC* series circuit is close to resonance, what is the phase difference between the voltages across the inductor and the capacitor?
- For *RC*, *RL* circuits, if the resistance of *C* or *L* cannot be ignored, can we still use the experiment to measure the time constant of the circuit?