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Early Termination of Convex QP Solvers in Mixed-Integer Programming for Real-Time Decision Making

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***Abstract*—The branch-and-bound optimization algorithm for mixed-integer model predictive control (MI-MPC) solves several convex quadratic program relaxations, but often the solutions are discarded based on already known integer feasible solutions. This letter presents a projection and early termination strategy for infeasible interior point methods to reduce the computational effort of finding a globally optimal solution for MI-MPC. The method is shown to be also effective for infeasibility detection of the convex relaxations. We present numerical simulation results with a reduction of the total number of solver iterations by 42% for an MI-MPC example of decision making for automated driving with obstacle avoidance constraints.**

***Index Terms*—Cyber-physical systems, mixed-integer programming, optimization algorithms.**

I. Introduction

MODEL predictive control (MPC) is a model-based control approach which allows to handle performance optimization and constraints by design [1]. This framework can be further extended to hybrid dynamical systems [2] that include both continuous and discrete decision variables, providing a powerful control design for a large class of problems, e.g., switched dynamical systems [3], discrete or quantized actuation [4], logic rules and temporal logic specifications [5]. The resulting optimization problems are non-convex because they contain variables that only take integer values. For a linear-quadratic objective, (piecewise) linear dynamics and linear constraints, the optimization problem can be formulated as a mixed-integer quadratic program (MIQP).

In this letter, we aim at solving MIQPs of the form:

$$\min_{X,\,U} \quad \sum_{k=0}^{N} \frac{1}{2}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top H_k \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \begin{bmatrix} x_k \\ u_k \end{bmatrix}^\top \begin{bmatrix} q_k \\ r_k \end{bmatrix} \tag{1a}$$

$$\text{s.t.} \quad x_0 = \hat{x}_0, \tag{1b}$$

$$x_{k+1} = \begin{bmatrix} A_k & B_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + a_k, \quad k \in \mathbb{Z}_0^{N-1}, \tag{1c}$$

$$\underline{y}_k \le \begin{bmatrix} C_k & D_k \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} \le \bar{y}_k, \quad k \in \mathbb{Z}_0^{N}, \tag{1d}$$

$$u_{k,j} \in \mathbb{Z}, \quad j \in \mathcal{I}_k, \quad k \in \mathbb{Z}_0^{N}, \tag{1e}$$

where the notation $\mathbb{Z}_a^b$ denotes the range of integers $a, a+1, \dots, b$, and the optimization variables are the state $X = [x_0^\top, \dots, x_N^\top]^\top$ and control trajectory $U = [u_0^\top, \dots, u_N^\top]^\top$. For simplicity of notation, the set $\mathcal{I}_k$ denotes the indices of the discrete control variables for each step $k \in \mathbb{Z}_0^N$ of the prediction horizon. The Hessian matrices in (1a) are assumed to be positive semi-definite $H_k \succeq 0$, but not necessarily positive definite. Unlike standard MPC [1], our MIQP (1) can include discrete and/or continuous control variables on the terminal step of the MPC horizon.

MPC for hybrid dynamical systems aims at solving the MIQP (1) at every sampling time instant. In order to solve MIQPs to global optimality, most algorithms are based on a variant of the branch-and-bound (B&B) method [6]. The B&B strategy has been combined with various methods for solving the relaxed convex quadratic programs (QPs) in mixed-integer MPC (MI-MPC), such as dual active-set [7], primal active-set [8], interior point method (IPM) [9], dual projected gradient [10], and the alternating direction method of multipliers (ADMM) [11]. An advantage of a dual QP solver is that a dual feasible starting point for the child problems can be computed from the dual solution of the parent. This allows for early termination of the relaxed problem whenever the dual objective becomes larger than the current upper bound in the B&B [7], [10], [12].

IPMs are appealing for real-time optimization because they generally exhibit better worst-case performance than active-set type methods [13]. An early termination strategy for a dual feasible IPM is described in [14, Sec. 5.1]. Instead, here we focus on an infeasible IPM [15] that does not need a starting point that is dual feasible, and is applied to solve the convex relaxations within the B&B method. In general, all iterates of an infeasible primal-dual IPM are infeasible, although its limit points are both feasible and optimal [15]. Infeasible IPMs are appealing in B&B [16] because they can be effectively warm started, see [17], [18].

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This letter proposes a first early termination strategy for an infeasible IPM to solve each convex QP relaxation in the B&B method, without the assumption of strict positive definiteness of the Hessian in (1) and without the need to compute a dual feasible starting point. We developed a computationally efficient projection to compute a dual feasible solution guess for early termination before the optimum of the QP is found. We additionally show that our early termination strategy is effective for infeasibility detection for the convex relaxations in the B&B method.

This letter is organized as follows. Section II summarizes background materials on optimization and B&B methods. Section III presents our projection and early termination strategy for primal-dual IPMs, and the effectiveness for infeasibility detection is motivated in Section IV. Section V presents numerical results for an MI-MPC example of automated driving using the active-set based IPM (ASIPM) method with warm starting in [18]. The conclusions are summarized in Section VI.

II. Problem Formulation

We reformulate the MIQP in (1) in the compact notation

$$\min_{z} \quad \frac{1}{2} z^\top H z + h^\top z \tag{2a}$$
$$\text{s.t.} \quad Cz \le c, \quad Fz = f, \tag{2b}$$
$$z_j \in \mathbb{Z}, \quad j \in \mathcal{I}, \tag{2c}$$

where z includes all primal optimization variables and the set $\mathcal{I}$ indexes the integer variables. For many practical MPC formulations, the objective can directly be written as

$$\frac{1}{2} z^\top H z + h^\top z = \frac{1}{2} \boldsymbol{x}^\top Q \boldsymbol{x} + h_{\boldsymbol{x}}^\top \boldsymbol{x} + h_{\boldsymbol{y}}^\top \boldsymbol{y}, \tag{3}$$

where $H \succeq 0$ and $Q \succ 0$, by partitioning z into $\boldsymbol{x}$ and $\boldsymbol{y}$, entering in the linear-quadratic and linear-only terms, respectively. The objective of any MIQP (2) can always be reformulated as in (3) by a change of variables, e.g., based on the eigenvalue decomposition for the Hessian matrix $H \succeq 0$.

A. Branch-and-Bound Optimization

The standard B&B algorithm [6] sequentially creates partitions for the search space of the original mixed-integer program and compute bounds for those partitions. Each integer-feasible solution provides a global upper bound to the optimum. Local lower bounds can be computed from convex relaxations of the MIQP for a particular partition or node of the B&B tree. The B&B effectiveness stems from pruning certain partitions or nodes whenever the local lower bound exceeds the current global upper bound.

The B&B partitions are constructed through branching, and selecting an appropriate integer variable for branching is a key B&B step, e.g., studied extensively in [19]. Another ingredient is the pre-solve routine to construct tighter convex relaxations leading to less branching, see the overview in [20]. For lowering the upper bound, heuristics are often used to compute integer-feasibile solutions, e.g., by a feasibility pump [21] or warm starting [22]. The B&B method used in this letter for numerical tests is similar to [8] and exploits domain propagation in the pre-solve routine, reliability branching based on pseudo-cost, and warm starting.

B. Convex QP Relaxations

We focus on the efficient solution of convex QP relaxations in a B&B optimization method. For a particular node of the B&B tree, a convex QP is obtained by relaxing the integer equality constraint in (2c) as

$$\underline{z}_j \le z_j \le \bar{z}_j, \quad j \in \tilde{\mathcal{I}}, \tag{4}$$

where the index set $\tilde{\mathcal{I}}$ denotes each integer variable that has not been fixed due to branching, or due to the pre-solve routine, in the current node of the B&B tree. The values $\underline{z}_j$ and $\bar{z}_j$ denote the lower and upper bound values for each integer variable $j \in \tilde{\mathcal{I}}$, respectively. As discussed in [20], fixed variables, i.e., $\underline{z}_j = \bar{z}_j$, and redundant constraints should be removed from each convex QP for efficiency.

We do not need to solve a convex subproblem if

1) the convex QP relaxation is infeasible,
2) the optimal solution is detected to have an objective value that exceeds the current global upper bound.

In both cases, the node, and hence the corresponding subtree, can be pruned from the B&B tree to find the optimal solution. A considerable computational effort can be avoided if the above scenarios are detected early, i.e., more quickly than the solving time of the convex QP relaxations. In this letter, we propose a tailored early termination strategy for infeasible primal-dual IPMs to handle both cases and to reduce the computational effort of the B&B method without affecting the quality of the optimal solution.

C. Dual QP Problem Formulation

We consider the primal convex QP of the form

$$\min_{\boldsymbol{x},\boldsymbol{y}} \quad \phi(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{2} \boldsymbol{x}^\top Q \boldsymbol{x} + h_{\boldsymbol{x}}^\top \boldsymbol{x} + h_{\boldsymbol{y}}^\top \boldsymbol{y} \tag{5a}$$
$$\text{s.t.} \quad G_{\boldsymbol{x}} \boldsymbol{x} + G_{\boldsymbol{y}} \boldsymbol{y} \le g, \tag{5b}$$
$$F_{\boldsymbol{x}} \boldsymbol{x} + F_{\boldsymbol{y}} \boldsymbol{y} = f, \tag{5c}$$

where $Q \succ 0$ in the primal objective $\phi(\boldsymbol{x},\boldsymbol{y})$, and the inequality constraints (5b) include both the original inequalities from (2b) and the convex relaxations (4) of the integer constraints. We additionally use the compact notation $z := [\boldsymbol{x}^\top \boldsymbol{y}^\top]^\top$, $G := [G_{\boldsymbol{x}} | G_{\boldsymbol{y}}]$, $F := [F_{\boldsymbol{x}} | F_{\boldsymbol{y}}]$, $h := [h_{\boldsymbol{x}}^\top h_{\boldsymbol{y}}^\top]^\top$ and $H := \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$. The dual QP of (5) reads as

$$\max_{\mu,\lambda} \quad \psi(\mu,\lambda) := -\frac{1}{2} \|\hat{h}(\mu,\lambda)\|^2_{Q^{-1}} - \begin{bmatrix} g \\ f \end{bmatrix}^\top \begin{bmatrix} \mu \\ \lambda \end{bmatrix} \tag{6a}$$
$$\text{s.t.} \quad G_{\boldsymbol{y}}^\top \mu + F_{\boldsymbol{y}}^\top \lambda = -h_{\boldsymbol{y}}, \tag{6b}$$
$$\mu \ge 0, \tag{6c}$$

where λ and μ denote the Lagrange multipliers for the equality and inequality constraints, respectively, and

$$\hat{h}(\mu,\lambda) := h_{\boldsymbol{x}} + G_{\boldsymbol{x}}^\top \mu + F_{\boldsymbol{x}}^\top \lambda, \tag{7}$$

for simplifying the dual objective function $\psi(\mu,\lambda)$ in (6a).

D. Primal-Dual Interior Point Method

A primal-dual IPM uses a Newton-type method to solve a sequence of relaxed Karush-Kuhn-Tucker (KKT) conditions for the convex QP in (5). An iteration of the IPM usually solves the reduced linear system

$$\begin{bmatrix} H & F^\top & G^\top \\ F & 0 & 0 \\ G & 0 & -W^k \end{bmatrix}\begin{bmatrix} \Delta z^k \\ \Delta\lambda^k \\ \Delta\mu^k \end{bmatrix} = -\begin{bmatrix} r_z^k \\ r_\lambda^k \\ \bar{r}_\mu^k \end{bmatrix}, \tag{8}$$

where $W^k = \operatorname{diag}(w^k) \succ 0$ and $w_i^k = s_i^k/\mu_i^k > 0$. The right-hand side in (8) denotes the residual value for the optimality conditions and reads as

$$\begin{aligned} r_z^k &= Hz^k + F^\top\lambda^k + G^\top\mu^k + h, \\ r_\lambda^k &= Fz^k - f, \quad r_\mu^k = Gz^k - g + s^k, \\ r_s^k &= M^k S^k 1 - \tau^k 1, \quad \bar{r}_\mu^k = r_\mu^k - {M^k}^{-1} r_s^k, \end{aligned} \tag{9}$$

based on the barrier parameter $\tau^k \to 0$ for $k \to \infty$, where $M^k = \operatorname{diag}(\mu^k)$, $S^k = \operatorname{diag}(s^k)$ and the slack variables are updated as $\Delta s^k = -(W^k\Delta\mu^k + {M^k}^{-1} r_s^k)$. We consider an infeasible primal-dual IPM for which the starting point $\{(z^0, \mu^0, \lambda^0, s^0)\}$ may not be primal and/or dual feasible, but the slack variables and Lagrange multipliers are positive at each iteration, i.e., $s^k \geq 0$ and $\mu^k \geq 0$. See [15] for details on properties and implementation of primal-dual IPMs. For the numerical tests in this letter, we use the active-set based inexact Newton implementation (ASIPM) [18], which allows for reduced computations, warm starting and improved numerical conditioning.

III. Early Termination of Primal-Dual Interior Point Methods

Due to duality properties, see, e.g., [23], for a dual feasible point (μ, λ) that satisfies (6b)-(6c) and a primal feasible point $(\boldsymbol{x}, \boldsymbol{y})$ that satisfies (5b)-(5c)

$$\psi(\mu, \lambda) \leq \psi^\star \leq \phi^\star \leq \phi(\boldsymbol{x}, \boldsymbol{y}), \tag{10}$$

where $\phi^\star$ and $\psi^\star$ are the primal and dual optima, respectively. Based on (10), we propose an approach to find a dual feasible point that allows for early termination when $\psi(\mu, \lambda) > \texttt{UB}$ for the current upper bound (UB) to the optimum of the MIQP.

A. Projection Strategy for Dual Feasibility

A dual feasible solution, i.e., (μ^k, λ^k) satisfying (6b)-(6c) is required in order to perform early termination based on the duality result in (10). Since an infeasible IPM generally does not compute a solution that satisfies the equality constraint in (6b) until convergence, we propose a projection step to compute new values $(\mu^+, \lambda^+) = (\mu^k + \Delta\mu, \lambda^k + \Delta\lambda)$ satisfying (6b) and (6c). A standard minimum-norm projection would compute $(\Delta\mu, \Delta\lambda)$ as

$$\begin{aligned} \min_{\Delta\lambda, \Delta\mu} \quad & \frac{1}{2}\|\Delta\lambda\|^2 + \frac{1}{2}\|\Delta\mu\|^2 \\ \text{s.t.} \quad & F_{\boldsymbol{y}}^\top \Delta\lambda + G_{\boldsymbol{y}}^\top \Delta\mu = -r_{\boldsymbol{y}}^k, \end{aligned} \tag{11}$$

where $r_{\boldsymbol{y}}^k := F_{\boldsymbol{y}}^\top\lambda^k + G_{\boldsymbol{y}}^\top\mu^k + h_{\boldsymbol{y}}$. Instead, here we propose to perform the projection to a dual feasible point by solving the optimization problem

$$\begin{aligned} \min_{\Delta\boldsymbol{x}, \Delta\lambda, \Delta\mu} \quad & \frac{1}{2}\|\Delta\boldsymbol{x}\|_Q^2 + \frac{1}{2}\|\Delta\lambda\|_{\epsilon_{\text{dual}}}^2 + \frac{1}{2}\|\Delta\mu\|_{W^k}^2 \\ \text{s.t.} \quad & \begin{bmatrix} Q \\ 0 \end{bmatrix}\Delta\boldsymbol{x} + F^\top\Delta\lambda + G^\top\Delta\mu = -\begin{bmatrix} 0 \\ r_{\boldsymbol{y}}^k \end{bmatrix}, \end{aligned} \tag{12}$$

where $W^k = \operatorname{diag}(w^k)$ and $w_i^k = \frac{s_i^k}{\mu_i^k} > 0$.

There are three advantages from the projection (12) over the projection (11). Neither (12) nor (11) directly enforce the positivity constraints $\mu^k + \Delta\mu > 0$, which would require solving an inequality constrained QP. However, a first advantage of (12) is that, since $w_i^k = \frac{s_i^k}{\mu_i^k} > 0$, the term $\|\Delta\mu\|_{W^k}^2 = \sum_i (\frac{s_i^k}{\mu_i^k}\Delta\mu_i^2)$ in the objective of (12) penalizes the step $\Delta\mu_i$ to remain small when $\frac{s_i^k}{\mu_i^k}$ is relatively large, i.e., when $\mu_i^k > 0$ is close to zero. This makes it more likely to satisfy $\mu_i^k + \Delta\mu_i > 0$ without forcing the update step to be always small, since the step is forced to be smaller when approaching the positivity constraint. Second, the solution $(\Delta\mu, \Delta\lambda)$ to the optimization problem (12) is equivalent to solving the symmetric system

$$\begin{bmatrix} H & F^\top & G^\top \\ F & -\epsilon_{\text{dual}} I & 0 \\ G & 0 & -W^k \end{bmatrix}\begin{bmatrix} \Delta z \\ \Delta\lambda \\ \Delta\mu \end{bmatrix} = -\begin{bmatrix} \begin{bmatrix} 0 \\ r_{\boldsymbol{y}}^k \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}, \tag{13}$$

which corresponds to an IPM iteration similar to (8) with only differences being the right-hand side and the augmented Lagrangian type regularization $\epsilon_{\text{dual}} > 0$ as in [18]. A derivation of the equivalence between (12) and (13) can be found in Proposition 1, here below. The latter means that the same matrix factorization procedure from [18] can be reused for our projection step (13) as follows

$$\left(H + \frac{1}{\epsilon_{\text{dual}}}F^\top F + G^\top {W^k}^{-1} G\right)\Delta z = -\begin{bmatrix} 0 \\ r_y^k \end{bmatrix}, \tag{14}$$

such that $\Delta\lambda = \frac{1}{\epsilon_{\text{dual}}}F\Delta z$ and $\Delta\mu = {W^k}^{-1}G\Delta z$. Third, the projection (13) aims at retaining the IPM progress towards optimum, since the projection step does not increase the residual value for the remaining optimality conditions in (9), due to the zero elements in the right-hand side of (13).

Proposition 1: The solution to the linear system in (13) forms a solution to the optimization problem in (12).

Proof: Let us write down the KKT optimality conditions for the equality constrained QP in (12) as follows

$$\begin{bmatrix} Q & & & [Q \;\; 0] \\ & \epsilon_{\text{dual}} I & & F \\ & & W^k & G \\ \begin{bmatrix} Q \\ 0 \end{bmatrix} & F^\top & G^\top & 0 \end{bmatrix}\begin{bmatrix} \Delta\boldsymbol{x} \\ \Delta\lambda \\ \Delta\mu \\ \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 0 \\ \begin{bmatrix} 0 \\ r_{\boldsymbol{y}}^k \end{bmatrix} \end{bmatrix}, \tag{15}$$

where $\nu := \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}$ denotes the Lagrange multipliers for the equality constraints in (12). Due to $Q \succ 0$, Eq. (15) shows that $\nu_1 = -\Delta\boldsymbol{x}$ can be eliminated and, after rearranging the equations and variables, results in

$$\begin{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} & F^\top & G^\top \\ -F & \epsilon_{\text{dual}} I & \\ -G & & W^k \end{bmatrix}\begin{bmatrix} \begin{bmatrix} \Delta\boldsymbol{x} \\ -\nu_2 \end{bmatrix} \\ \Delta\lambda \\ \Delta\mu \end{bmatrix} = -\begin{bmatrix} \begin{bmatrix} 0 \\ r_{\boldsymbol{y}}^k \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}. \tag{16}$$

A solution to (16) provides a solution to (13) by the change of variables $\nu_2 = -\Delta\boldsymbol{y}$, which concludes the proof. ■

Algorithm 1 Early Termination for IPM in B&B Method

1: **Input:** Warm start $\left\{\left(z^0, \mu^0, \lambda^0, s^0\right)\right\}$, *tol*, and $\texttt{UB}$.
2: **while** $\max\{\tau^k, \|r^k\|\} > tol$ **do**
3:   **if** $\psi(\mu^k, \lambda^k) > \texttt{UB}$ & $\texttt{dual\_feasible}$ **then**
4:     **break** while loop. ▷ Early termination
5:   **else if** $\psi(\mu^k, \lambda^k) > \texttt{UB}$ **then**
6:     Compute projection step $(\Delta\mu, \Delta\lambda)$ in (13).
7:     $\mu \leftarrow \mu^k + \Delta\mu$, $\lambda \leftarrow \lambda^k + \Delta\lambda$, and
8:     $r_y \leftarrow F_y^\top \lambda + G_y^\top \mu + h_y$.
9:     **if** $\mu > 0$ & $\|r_y\| < tol$ **then**
10:       $\mu^k \leftarrow \mu$, $\lambda^k \leftarrow \lambda$, $r_y^k \leftarrow r_y$, and
11:       $\texttt{dual\_feasible} \leftarrow 1$.
12:       **if** $\psi(\mu^k, \lambda^k) > \texttt{UB}$ **then**
13:         **break** while loop. ▷ Early termination
14:       **end if**
15:     **end if**
16:   **end if**
17:   Perform an IPM iteration (8), e.g., see [18].
18: **end while**

B. Early Termination Strategy

Based on the projection step in (13), we detail our proposed early termination strategy in Algorithm 1. As discussed in the next subsection, one dual objective evaluation is computationally cheaper than a projection on a dual feasible point. Therefore, Algorithm 1 performs the projection in (13) if and only if the dual objective $\psi(\mu^k, \lambda^k)$ is larger than the current UB (line 5). Since in this letter for numerical tests we use the ASIPM method [18], we note that multiple evaluations, reusing the same matrix factorization, of the projection step in (13) may be needed to ensure dual feasibility $\|F_y^\top \lambda + G_y^\top \mu + h_y\| < tol$ when using the inexact Newton implementation of [18].

C. Computational Complexity

The proposed early termination strategy requires two computational steps, i.e., the projection and the dual objective evaluation, which are typically not needed in a standard IPM. Considering the optimal control structured program (1), the evaluation of the dual objective value (6a) requires

$$N(n^2 + 2\,n\,m + 2\,n\,p), \tag{17}$$

operations to compute $L^{-1}(G_x^\top \mu + F_x^\top \lambda)$ based on a block-diagonal Cholesky factorization $Q = LL^\top$, in which n, m and p are the number of variables $\boldsymbol{x}$, the number of equality and inequality constraints per control interval, respectively.

Based on (13), we can perform one projection step at the computational cost of one IPM iteration. This allows the reuse of the corresponding matrix factorization in the subsequent IPM iteration if the projection is not successful. Based on the particular ASIPM implementation, as proposed recently in [18], one iteration requires a block-tridiagonal Cholesky factorization for the matrix in (14), for which the dominant terms in the computational cost are

$$N\left(\frac{7}{3}n_{\mathrm{x}}^3 + 4n_{\mathrm{x}}^2 n_{\mathrm{u}} + 2n_{\mathrm{x}} n_{\mathrm{u}}^2 + \frac{1}{3}n_{\mathrm{u}}^3\right), \tag{18}$$

where n_{x} and n_{u} denote the number of state and control variables per interval in (1), respectively. Then, the linear system in (13) can be solved by

$$N\Big(6\,n_{\mathrm{x}}^2 + 8\,n_{\mathrm{x}} n_{\mathrm{u}} + 2\,n_{\mathrm{u}}^2 + 2(n_{\mathrm{x}} + n_{\mathrm{u}})(m + p)\Big), \tag{19}$$

operations for the resulting block-structured forward and backward substitution. Since $n \le (n_{\mathrm{x}}+n_{\mathrm{u}})$, and often $n \ll (n_{\mathrm{x}}+n_{\mathrm{u}})$ due to many auxiliary variables in hybrid systems [2] for which the Hessian contribution is zero, the cost for a dual objective evaluation (17) is considerably smaller than the projection cost (19), as anticipated.

IV. Primal Infeasibility Detection

We first present a certificate of primal infeasibility and then justify why our proposed early termination strategy can be also effective for primal infeasibility detection. We confine ourselves to the compact form of the QP relaxation in Eq. (5) and a primal-dual IPM described in Section II-D.

A. Certificate of Primal Infeasibility

Let the primal problem (5) with strict inequalities,

$$Gz < g, \quad Fz = f, \tag{20}$$

be infeasible. Using Farkas' lemma [15], we can derive the following certificate of strict infeasibility for (20),

$$G^\top \tilde{\mu} + F^\top \tilde{\lambda} = 0, \quad g^\top \tilde{\mu} + f^\top \tilde{\lambda} < 0, \quad \tilde{\mu} > 0. \tag{21}$$

Eq. (20) is strictly infeasible if and only if there exists a pair $(\tilde{\mu}, \tilde{\lambda})$ satisfying (21). If $(\hat{\mu}, \hat{\lambda})$ is a dual feasible solution satisfying (6b) and (6c), then $(\mu, \lambda) = (\hat{\mu} + \alpha\tilde{\mu}, \hat{\lambda} + \alpha\tilde{\lambda})$ also satisfies (6b) and (6c) for $\alpha > 0$, and the dual objective in (6a) $\to \infty$ as $\alpha \to \infty$, i.e., the dual problem is unbounded.

B. Early Termination for Infeasible QPs

As discussed in detail in [24], the standard iterates of an IPM can be used to generate certificates of infeasibility (21). Intuitively, since the dual objective is unbounded in case of primal infeasibility, we can terminate the solver whenever the dual objective of a dual feasible solution is "large enough". We motivate why our proposed early termination strategy is effective for infeasibility detection and, given a tight UB from the B&B optimization method, it may lead to termination before a certificate of infeasibility is found.

Proposition 2 will prove that the dual objective $\psi(\mu^k, \lambda^k)$ for the primal-dual IPM iterates is unbounded in the limit. Before, we need to prove two technical lemmas.

Lemma 1: The following statements about the IPM iterates (8) hold for $k \ge 0$:

a) for some $\beta_k \in [0, 1]$, iterates z^k and s^k satisfy

$$M\begin{bmatrix} z^k \\ s^k \end{bmatrix} = \beta_k M \begin{bmatrix} z^0 \\ s^0 \end{bmatrix} + (1 - \beta_k)\begin{bmatrix} g \\ f \end{bmatrix} =: r_{z,s}^k, \tag{22}$$

where $M := \begin{bmatrix} G & I \\ F & 0 \end{bmatrix}$, and $r_{z,s}^k$ is bounded;

b) $h^k := Hz^k + G^\top \mu^k + F^\top \lambda^k$ is bounded.

Proof: a) Eq. (22) holds for $k = 0$ with $\beta_0 = 1$. Next, assuming that (22) holds for $k \geq 0$, it follows from Eq. (8) and (9), and $\Delta s^k = -(W^k \Delta\mu^k + M^{k^{-1}} r_s^k)$ that

$$M\begin{bmatrix} z^{k+1} \\ s^{k+1} \end{bmatrix} = (1-\alpha^k)M\begin{bmatrix} z^k \\ s^k \end{bmatrix} + \alpha^k \begin{bmatrix} g \\ f \end{bmatrix}, \quad (23)$$

where $(z^{k+1}, s^{k+1}) = (z^k + \alpha^k \Delta z^k, s^k + \alpha^k \Delta s^k)$, and $\alpha^k \in (0, 1]$ is the stepsize taken in the IPM. Hence, it follows from (23) that Eq. (22) holds for $k+1$ with $\beta_{k+1} = \beta_k(1-\alpha^k)$. It follows from its definition in (22) that $r_{z,s}^k$ is bounded as a convex combination of two bounded vectors.

b) This statement follows from similar arguments as for the derivations in the proof of (a). ■

It is easy to see from (22) that $\beta_k > 0$ in case of strict primal infeasibility. The following lemma confirms this observation and the boundedness of s^k and z^k.

Lemma 2: If the primal problem (20) is strictly infeasible, then the following statements hold for $k \geq 0$:

a) there exists $\bar{\mu} > 0$ such that

$$\beta_k \geq (1 + \bar{\mu}^\top s^0)^{-1} > 0, \quad \bar{\mu}^\top s^k \leq \bar{\mu}^\top s^0; \quad (24)$$

b) assuming that $[G^\top F^\top]^\top$ has full rank, then the iterates s^k and z^k are bounded.

Proof: a) Certificate (21) is equivalent to the existence of $\bar{\mu}$ and $\bar{\lambda}$ satisfying

$$G^\top \bar{\mu} + F^\top \bar{\lambda} = 0, \; g^\top \bar{\mu} + f^\top \bar{\lambda} = -1, \; \bar{\mu} > 0. \quad (25)$$

Following a similar argument as in the proof [24, Prop. 5.1], i.e., multiplying (22) by $[\bar{\mu}^\top \, \bar{\lambda}^\top]$ from left, and using (25) and the fact that $s^k > 0$, we have

$$\beta_k\left(1 + \bar{\mu}^\top s^0\right) - 1 = \bar{\mu}^\top s^k > 0, \quad (26)$$

and hence the first inequality in (24) holds. Moreover, the second inequality in (24) follows from (26) and $0 < \beta_k \leq 1$.

b) Using the second inequality in (24) and $\bar{\mu} > 0$, $s^k > 0$ and $s^0 > 0$, we know that s^k is bounded. From (22), using Lemma 1(a) and s^k being bounded, z^k is also bounded because $[G^\top F^\top]^\top$ has full rank. ■

We can now prove the main result of this section, namely, if the primal problem is infeasible, the dual objective of the IPM tends to infinity, which allows for early termination.

Proposition 2: If the sequence of IPM iterates $\{(z^k, \mu^k, \lambda^k, s^k)\}$ satisfy $\mu^{k^\top} s^k \leq \mu^{0^\top} s^0$ and $\|\mu^k\| \to \infty$, then the dual objective $\psi(\mu^k, \lambda^k) \to \infty$.

Proof: Lemmas 1(b) and 2(b) imply that $\hat{h}(\mu^k, \lambda^k)$ in (7) is bounded. As a result, in order to show that the dual objective $\psi(\mu^k, \lambda^k)$ (see (6a)) is unbounded for the IPM iterates, it suffices to show

$$\ell^k := g^\top \mu^k + f^\top \lambda^k \to -\infty. \quad (27)$$

Multiplying (22) by $[\mu^{k^\top} \lambda^{k^\top}]$ from the left, using the assumption that $\mu^{k^\top} s^k \leq \mu^{0^\top} s^0$, the definition of ℓ^k in (27), and the identity $G^\top \mu^k + F^\top \lambda^k = h^k - Hz^k$ (see Lemma 1(b)), after rearranging the terms, we obtain

$$(1-\beta_k)\ell^k \leq t^k + \left(\mu^0 - \beta_k \mu^k\right)^\top s^0, \quad (28)$$

where $t^k := (z^k - \beta_k z^0)^\top (h^k - Hz^k)$. Lemmas 1(b) and 2(b) imply that t^k is bounded. For $\|\mu^k\| \to \infty$, since $\mu^k > 0$ and $s^0 > 0$, we have $\mu^{k^\top} s^0 \to \infty$. It follows from $\beta_k > 0$ in Lemma 2(a) that the right-hand side of (28) tends to $-\infty$. Finally, since $\beta_k < 1$ in (22), inequality (28) implies (27) and concludes the proof. ■

V. Numerical Example

Next, we provide numerical simulation results to illustrate the performance of our early termination strategy applied to the ASIPM QP solver with warm starting [18], as part of the B&B implementation from [8] in MATLAB.

A. Problem Formulation: Vehicle Decision Making

The numerical example involves an MI-MPC formulation of a high-level motion planning task for an autonomous vehicle, including discrete decisions resulting from lane changes, static and dynamic obstacles. Similar to the problem formulation in [5], the vehicle motion is modeled in curvilinear coordinates resulting in affine system dynamics with 3 state variables and 2 control inputs. Additional decision variables are defined to model lane changes and obstacle avoidance, resulting in an MIQP formulation of the form in (1) with $N = 15$ intervals, $n_x = 6$ state variables and $n_u = 18$ control variables, including 14 binary decision variables per control interval. Due to the pre-solve routine based on domain propagation [8], many of the 224 binary decision variables can be eliminated in practice.

B. Numerical Simulation Results

The performance of the early termination strategy in Algorithm 1 for this MI-MPC example is presented in Fig. 1, including the total number of QP calls and solver iterations at each time step, as well as the reduction of computational effort in percentage. The x-axis of each subplot is the simulation time in [0, 170] s, using a sampling period of 2 s. It can be observed that the reduction in QP iterations is most considerable when a larger number of QPs need to be solved in the B&B method to compute the global solution of the MIQP. This is where it matters the most for real-time applications that must consider the worst-case execution time. Overall, in this particular numerical example, early termination reduces 42% of the total QP iterations.

Fig. 2 presents a detailed comparison between early terminated and fully solved QPs to solve the MIQP at each time step. Subplot 1 compares the number of early terminated and fully solved QPs, and shows that early termination of the IPM happens often. More specifically, about 36% of all QPs are early terminated in this particular example. Subplot 2 compares the average number of iterations in early terminated QPs with and without warm starting, and shows that early termination benefits from warm starting. In fact, the number of QP iterations is 0 at times 112 − 134 s, i.e., the warm start leads to immediate termination of the IPM. Subplot 3 presents the average number of iterations in the solved QPs with and without warm starting. Comparing subplots 2 and 3, one can observe that early terminated QPs take considerably less iterations than fully solved QPs.

Table I finally illustrates the performance of our early termination strategy for infeasibility detection of three infeasible QPs from the MI-MPC example. We compare the number of IPM iterations to obtain a certificate of infeasibility (21) versus the early

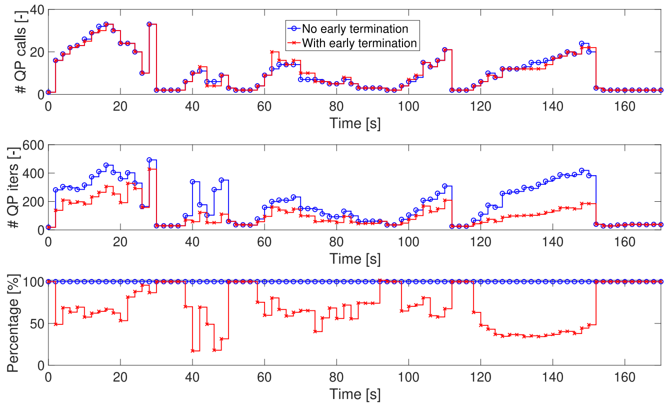

Fig. 1. QP calls and total iterations with and without early termination: early termination reduces total number of QP iterations by 42%.

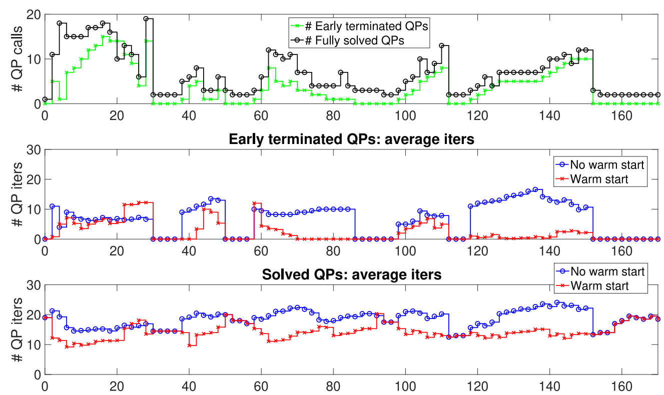

Fig. 2. Early terminated versus fully solved QPs: average number of QP iterations in both cases, with and without warm starting of ASIPM.

TABLE I
INFEASIBILITY DETECTION: # OF QP ITERATIONS FOR CERTIFICATE VERSUS EARLY TERMINATION WITH AND WITHOUT WARM STARTING

	QP # 1	QP # 2	QP # 3
Certificate of primal infeasibility	40	45	38
Early termination: cold started	10	12	10
Early termination: warm started	0	0	11

termination strategy from Algorithm 1, with and without warm starting of ASIPM [18]. Table I shows that early termination requires considerably less IPM iterations than the computation of a certificate of infeasibility. Note that early termination does not distinguish between a case of primal infeasibility or a case where the objective value exceeds the current UB. In addition, Table I shows that warm starting can reduce the number of IPM iterations further and it can lead to immediate termination, i.e., without the need to perform any IPM iterations.

VI. CONCLUSION

We proposed an efficient early termination strategy based on a projection step tailored to IPMs, in order to reduce the computational cost within the B&B method in solving MI-MPC problems. In addition, we proved that our early termination strategy can be effective for infeasibility detection and often it terminates the IPM before a certificate of infeasibility is found. Finally, we illustrated the performance of this early termination technique based on numerical simulation results for an MI-MPC example of vehicle decision making.

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