

Gradient Methods

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1 Convex smooth functions

1.1 Smoothness

Definition 1. A differentiable function is called L -smooth on \mathbb{R}^n if its gradient is L -Lipschitz continuous on \mathbb{R}^n , i.e.,

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

Lemma 1. Let f be a convex and differentiable function. Then, the following statements are equivalent:

(i) f is L -smooth;

(ii) for all $x, y \in \mathbb{R}^n$,

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle \leq \frac{L}{2}\|x - y\|^2;$$

(iii) for all $x, y \in \mathbb{R}^n$,

$$f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2L}\|\nabla f(x) - \nabla f(y)\|^2 \leq f(y);$$

(iv) for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{L}\|\nabla f(x) - \nabla f(y)\|^2 \leq \langle \nabla f(x) - \nabla f(y), x - y \rangle;$$

(v) for all $x, y \in \mathbb{R}^n$,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq L\|x - y\|^2.$$

Proof. (i) \implies (ii)

Consider $g(t) = f(x + t(y - x))$ for $t \in [0, 1]$. Observe that $g(0) = f(x)$, $g(1) = f(y)$, and

$$g'(t) = \langle \nabla f(x + t(y - x)), y - x \rangle.$$

Then, we have

$$g(1) = g(0) + \int_0^1 g'(\tau)d\tau,$$

equivalently,

$$f(y) = f(x) + \int_0^1 \langle \nabla f(x + \tau(y - x)), y - x \rangle d\tau. \quad (1)$$

It follows from the Cauchy-Schwarz inequality and the assumption that f is L -smooth that

$$\begin{aligned} f(y) - f(x) - \langle \nabla f(x), y - x \rangle &= \int_0^1 \langle \nabla f(x + \tau(y - x)) - \nabla f(x), y - x \rangle d\tau \\ &\leq \int_0^1 L\tau \|y - x\|^2 d\tau = \frac{L}{2} \|y - x\|^2. \end{aligned}$$

(ii) \implies (iii)

Fix $x_0 \in \mathbb{R}^n$ and consider a convex function $\phi(y) = f(y) - \langle \nabla f(x_0), y \rangle$. Note that the optimal solution is $y_* = x_0$. Using (ii), we have

$$\begin{aligned} \phi(y_*) &= \min_{x \in \mathbb{R}^n} \phi(x) \leq \min_{x \in \mathbb{R}^n} \left\{ \phi(y) + \langle \nabla \phi(y), x - y \rangle + \frac{L}{2} \|x - y\|^2 \right\} \\ &= \min_{r \geq 0} \left\{ \phi(y) - r \|\nabla \phi(y)\| + \frac{L}{2} r^2 \right\} = \phi(y) - \frac{1}{2L} \|\nabla \phi(y)\|^2. \end{aligned}$$

Hence, (iii) holds in view of $\nabla \phi(y) = \nabla f(y) - \nabla f(x_0)$.

(iii) \implies (iv)

We obtain (iv) by adding two copies of (iii) with x and y interchanged.

(iv) \implies (i)

This is simply by (iv) and Cauchy-Schwarz inequality.

(ii) \implies (v)

We obtain (v) by adding two copies of (ii) with x and y interchanged.

(v) \implies (ii)

Using the identity (1) and (v), we have

$$\begin{aligned} f(y) - f(x) - \langle \nabla f(x), y - x \rangle &= \int_0^1 \langle \nabla f(x + \tau(y - x)) - \nabla f(x), y - x \rangle d\tau \\ &\leq \int_0^1 L\tau \|y - x\|^2 d\tau = \frac{L}{2} \|y - x\|^2. \end{aligned}$$

□

Remark 1. Note that the convexity of f is explicitly used in the proof of (ii) \implies (iii), and implicitly used in the proof of (iii) \implies (iv). The convexity is necessary in Lemma 1, since even if smoothness disappears, i.e., $L = \infty$, the resulting inequalities of both (iii) and (iv) still imply that f is convex.

1.2 Gradient method

Algorithm 1 Gradient method

Input: Initial point $x_0 \in \mathbb{R}^n$

for $k \leftarrow 0, \dots, K-1$ **do**

 Step 1. Choose $h_k > 0$.

 Step 2. Compute $x_{k+1} = x_k - h_k \nabla f(x_k)$.

end for

Output: x_K

Theorem 1. Assume f is convex and L -smooth and choose $h_k = h \in (0, 2/L)$ for every $k \geq 0$. Then, the Gradient Method generates a sequence of points $\{x_k\}$ satisfying

$$f(x_k) - f_* \leq \frac{2[f(x_0) - f_*]\|x_0 - x_*\|^2}{2\|x_0 - x_*\|^2 + kh(2 - Lh)[f(x_0) - f_*]}, \quad \forall k \geq 0.$$

Proof. Let $r_k := \|x_k - x_*\|$. Then, we get

$$\begin{aligned} r_{k+1}^2 &= \|x_k - x_* - h \nabla f(x_k)\|^2 \\ &= r_k^2 - 2h \langle \nabla f(x_k), x_k - x_* \rangle + h^2 \|\nabla f(x_k)\|^2 \\ &= r_k^2 - 2h \langle \nabla f(x_k) - \nabla f(x_*), x_k - x_* \rangle + h^2 \|\nabla f(x_k)\|^2. \end{aligned}$$

Using Lemma 1(iv), we have

$$r_{k+1}^2 \leq r_k^2 - h \left(\frac{2}{L} - h \right) \|\nabla f(x_k)\|^2.$$

Therefore, $r_k \leq r_0$. Using Lemma 1(i), we have

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + \frac{L}{2} \|x_{k+1} - x_k\|^2 \\ &= f(x_k) - \alpha \|\nabla f(x_k)\|^2 \end{aligned}$$

where $\alpha = h(1 - Lh/2)$. This inequality gives the descent property of the function value. Define $\Delta_k = f(x_k) - f_*$. Then,

$$\Delta_k \leq \langle \nabla f(x_k), x_k - x_* \rangle \leq r_k \|\nabla f(x_k)\| \leq r_0 \|\nabla f(x_k)\|.$$

Thus,

$$\Delta_{k+1} \leq \Delta_k - \frac{\alpha}{r_0^2} \Delta_k^2.$$

Dividing the above inequality by $\Delta_{k+1}\Delta_k$, we have

$$\frac{1}{\Delta_{k+1}} \geq \frac{1}{\Delta_k} + \frac{\alpha}{r_0^2} \frac{\Delta_k}{\Delta_{k+1}} \geq \frac{1}{\Delta_k} + \frac{\alpha}{r_0^2}.$$

Summing up these inequalities, we obtain

$$\frac{1}{\Delta_k} \geq \frac{1}{\Delta_0} + \frac{\alpha k}{r_0^2}.$$

The conclusion follows by inverting the above inequalities. \square

Choosing $h = 1/L$ maximizes $h(2 - Lh)$ and hence the denominator, so it is the optimal stepsize. We have the following convergence rate of the Gradient Method:

$$f(x_k) - f_* \leq \frac{2L[f(x_0) - f_*]\|x_0 - x_*\|^2}{2L\|x_0 - x_*\|^2 + k[f(x_0) - f_*]}, \quad \forall k \geq 0.$$

Again, using the smoothness of f , we have

$$f(x_0) \leq f_* + \langle \nabla f(x_*) , x_0 - x_* \rangle + \frac{L}{2} \|x_0 - x_*\|^2 = f_* + \frac{L}{2} \|x_0 - x_*\|^2.$$

Thus, we have the following result.

Corollary 1. *Assume f is convex and L -smooth and choose $h_k = h = 1/L$ for every $k \geq 0$. Then,*

$$f(x_k) - f_* \leq \frac{2L\|x_0 - x_*\|^2}{k+4}, \quad \forall k \geq 0.$$

Theorem 2. *If f is differentiable and convex on \mathbb{R}^n and $\nabla f(x_*) = 0$, then x_* is the global minimum of f on \mathbb{R}^n .*

Proof. It follows from the convexity of f that for every $x \in \mathbb{R}^n$,

$$f(x) \geq f(x_*) + \langle \nabla f(x_*) , x - x_* \rangle = f(x_*).$$

\square

Hence, it is also meaningful in finding a point with a small norm of the gradient: $\|\nabla f(x)\| \leq \varepsilon$.

2 Strongly convex and smooth functions

Definition 2. *A proper extended real-valued function f is μ -strongly convex if*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) - \lambda(1 - \lambda)\frac{\mu}{2}\|x - y\|^2 \quad \text{for all } x, y \in \mathbb{R}^n, \lambda \in [0, 1].$$

Lemma 2. *A function f is μ -strongly convex if and only if $f - \frac{\mu}{2}\|\cdot\|_2^2$ is convex.*

Theorem 3. *A continuously differentiable function f is μ -strongly convex on \mathbb{R}^n if and only if for any $x, y \in \mathbb{R}^n$ we have*

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\mu}{2}\|x - y\|^2.$$

Proof. This theorem follows from Lemma 2 and Theorem 2 of Lecture 2. \square

Theorem 4. A twice continuously differentiable function f is μ -strongly convex on \mathbb{R}^n if and only if for any $x \in \mathbb{R}^n$ we have

$$\nabla^2 f(x) \succeq \mu I.$$

Proof. This theorem follows from Lemma 2 and Theorem 3 of Lecture 2. \square

Lemma 3. If a continuously differentiable function f is μ -strongly convex on \mathbb{R}^n , then we have

(i) for all $x, y \in \mathbb{R}^n$,

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{1}{2\mu} \|\nabla f(x) - \nabla f(y)\|^2;$$

(ii) for all $x, y \in \mathbb{R}^n$,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \leq \frac{1}{\mu} \|\nabla f(x) - \nabla f(y)\|^2;$$

(iii) for all $x, y \in \mathbb{R}^n$,

$$\mu \|x - y\| \leq \|\nabla f(x) - \nabla f(y)\|.$$

Proof. The proof is left as a HW problem. \square

Lemma 4. Assume f is μ -strongly convex and L -smooth. For every $x, y \in \mathbb{R}^n$, we have

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{\mu L}{\mu + L} \|x - y\|^2 + \frac{1}{\mu + L} \|\nabla f(x) - \nabla f(y)\|^2.$$

Proof. The proof is left as a HW problem. \square

We are now ready to estimate the performance of the Gradient Method on the class of strongly convex functions.

Theorem 5. Assume f is μ -strongly convex and L -smooth and choose $0 \leq h \leq 2/(\mu + L)$. Then, the Gradient Method generates a sequence $\{x_k\}$ such that

$$\|x_k - x_*\|^2 \leq \left(1 - \frac{2h\mu L}{\mu + L}\right)^k \|x_0 - x_*\|^2.$$

If $h = 2/(\mu + L)$, then

$$\|x_k - x_*\| \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^k \|x_0 - x_*\|,$$

and

$$f(x_k) - f_* \leq \frac{L}{2} \left(\frac{\kappa - 1}{\kappa + 1}\right)^{2k} \|x_0 - x_*\|^2, \quad \forall k \geq 0$$

where $\kappa = L/\mu$.

Proof. Let $r_k := \|x_k - x_*\|$. Then, we get

$$\begin{aligned} r_{k+1}^2 &= \|x_k - x_* - h \nabla f(x_k)\|^2 \\ &= r_k^2 - 2h \langle \nabla f(x_k), x_k - x_* \rangle + h^2 \|\nabla f(x_k)\|^2 \\ &= r_k^2 - 2h \langle \nabla f(x_k) - \nabla f(x_*), x_k - x_* \rangle + h^2 \|\nabla f(x_k)\|^2. \end{aligned}$$

Using Lemma 4, we have

$$r_{k+1}^2 \leq \left(1 - \frac{2h\mu L}{\mu + L}\right) r_k^2 + h \left(h - \frac{2}{\mu + L}\right) \|\nabla f(x_k)\|^2.$$

It follows from the assumption that $0 \leq h \leq 2/(\mu + L)$ that

$$r_{k+1}^2 \leq \left(1 - \frac{2h\mu L}{\mu + L}\right) r_k^2.$$

So the first conclusion holds by applying the above inequality recursively. The second conclusion holds by plugging in $h = 2/(\mu + L)$. The last conclusion follows from Lemma 1(ii) and the second conclusion. \square

Note that the fastest rage of convergence is achieved for $h = 2/(\mu + L)$. In this case, we have

$$\|x_k - x_*\| \leq \left(\frac{L - \mu}{L + \mu}\right)^k \|x_0 - x_*\|.$$

3 Optimization with constraints

Let us consider now a smooth optimization problem with the set constraint:

$$\min_{x \in Q} f(x) \tag{2}$$

where Q is a closed convex set.

In the unconstrained case, the optimality condition is

$$\nabla f(x) = 0.$$

But this condition does not work with the set constraint. Consider the following univariate minimization problem:

$$\min_{x \geq 0} x.$$

Here $Q = \{x \in \mathbb{R} : x \geq 0\}$ and $f(x) = x$. Note that $x_* = 0$ but $f'(x_*) = 1 > 0$.

Theorem 6. Let f be convex and differentiable and Q be closed and convex. A point x_* is a solution to (2) if and only if

$$\langle \nabla f(x_*), x - x_* \rangle \geq 0 \quad (3)$$

for all $x \in Q$.

Proof. Indeed, if (3) is true, then

$$f(x) \geq f(x_*) + \langle \nabla f(x_*), x - x_* \rangle \geq f(x_*)$$

for all $x \in Q$. On the other hand, let x_* be a solution to (2). Assume that there exists some $x \in Q$ such that

$$\langle \nabla f(x_*), x - x_* \rangle < 0.$$

Consider the function

$$\phi(\alpha) = f(x_* + \alpha(x - x_*)), \quad \alpha \in [0, 1].$$

Note that

$$\phi(0) = f(x_*), \quad \phi'(0) = \langle \nabla f(x_*), x - x_* \rangle < 0.$$

Therefore, for α small enough we have

$$f(x_* + \alpha(x - x_*)) = \phi(\alpha) < \phi(0) = f(x_*).$$

This is a contradiction. \square

The next statement is often addressed as the growth property of strongly convex functions.

Theorem 7. If f is μ -strongly convex, then for any $x \in Q$, we have

$$f(x) \geq f(x_*) + \frac{\mu}{2} \|x - x_*\|^2.$$

Proof. Indeed, by strong convexity and Theorem 6, we have

$$\begin{aligned} f(x) &\geq f(x_*) + \langle \nabla f(x_*), x - x_* \rangle + \frac{\mu}{2} \|x - x_*\|^2 \\ &\geq f(x_*) + \frac{\mu}{2} \|x - x_*\|^2. \end{aligned}$$

\square

Theorem 8. Let f be μ -strongly convex with $\mu > 0$ and the set Q is closed and convex. Then there exists a unique solution x_* to (2).

Proof. The proof is left as a HW problem. \square

3.1 Minimization over simple sets

Let us consider the following minimization problem over a set

$$\min_{x \in Q} f(x)$$

where f is μ -strongly convex and L -smooth, and Q is a closed convex set. We assume that Q is simple enough so that projection onto Q is easy to compute.

Definition 3. Let Q be a closed set and $x_0 \in \mathbb{R}^n$. Define

$$\text{proj}_Q(x_0) = \arg \min_{x \in Q} \|x - x_0\|.$$

We call $\text{proj}_Q(x_0)$ the Euclidean projection of the point x_0 onto the set Q .

Lemma 5. For any two points x_1 and $x_2 \in \mathbb{R}^n$, we have

$$\|\text{proj}_Q(x_1) - \text{proj}_Q(x_2)\| \leq \|x_1 - x_2\|.$$

Proof. The proof is left as a HW problem. \square

Theorem 9. Let x_* be an optimal solution to (2). Then, for any $h > 0$, we have

$$\text{proj}_Q(x_* - h\nabla f(x_*)) = x_*.$$

Proof. The proof is left as a HW problem. \square

Examples

- nonnegative orthant $Q = \mathbb{R}_+^n$,

$$\text{proj}_Q(x) = [x]_+;$$

- box $Q = \text{Box}[\ell, u]$,

$$\text{proj}_Q(x) = (\min \{\max \{x_i, \ell_i\}, u_i\})_{i=1}^n;$$

- affine set $Q = \{x \in \mathbb{R}^n : Ax = b\}$,

$$\text{proj}_Q(x) = x - A^T (AA^T)^{-1} (Ax - b);$$

- l_2 ball $Q = B_{\|\cdot\|_2}[c, r]$,

$$\text{proj}_Q(x) = c + \frac{r}{\max \{\|x - c\|_2, r\}} (x - c);$$

- half-space $Q = \{x : a^T x \leq \alpha\}$,

$$\text{proj}_Q(x) = x - \frac{[a^T x - \alpha]_+}{\|a\|^2} a.$$

Algorithm 2 Gradient method for simple set

Input: Initial point $x_0 \in Q$
for $k \leftarrow 0, \dots, K-1$ **do**
 Step 1. Choose h_k .
 Step 2. Compute $x_{k+1} = \text{proj}_Q(x_k - h_k \nabla f(x_k))$.
end for
Output: x_K

Theorem 10. Assume f is μ -strongly convex and L -smooth and choose $h_k = h \in (0, 2/(\mu + L)]$. Then, the Gradient Method generates a sequence $\{x_k\}$ such that

$$\|x_k - x_*\| \leq (1 - \mu h)^k \|x_0 - x_*\|.$$

If $h = 2/(\mu + L)$, then

$$\|x_k - x_*\| \leq \left(\frac{\kappa - 1}{\kappa + 1} \right)^k \|x_0 - x_*\|$$

and

$$f(x_k) - f_* \leq \frac{L}{2} \left(\frac{\kappa - 1}{\kappa + 1} \right)^{2k} \|x_0 - x_*\|^2, \quad \forall k \geq 0$$

where $\kappa = L/\mu$.

Proof. Let $r_k := \|x_k - x_*\|$. Then, using Theorem 9 and step 2 of Algorithm 2, we get

$$\begin{aligned} r_{k+1}^2 &= \|\text{proj}_Q(x_k - h \nabla f(x_k)) - \text{proj}_Q(x_* - h \nabla f(x_*))\|^2 \\ &\leq \|x_k - x_* - h[\nabla f(x_k) - \nabla f(x_*)]\|^2 \\ &= r_k^2 - 2h \langle \nabla f(x_k) - \nabla f(x_*), x_k - x_* \rangle + h^2 \|\nabla f(x_k) - \nabla f(x_*)\|^2 \end{aligned}$$

where the inequality is due to Lemma 5. Using Lemma 4, we have

$$r_{k+1}^2 \leq \left(1 - \frac{2h\mu L}{\mu + L} \right) r_k^2 + h \left(h - \frac{2}{\mu + L} \right) \|\nabla f(x_k) - \nabla f(x_*)\|^2.$$

Using Lemma 3(iii), we have

$$\mu \|x - y\| \leq \|\nabla f(x) - \nabla f(y)\|$$

and the assumption that $0 \leq h \leq 2/(\mu + L)$, we further have

$$r_{k+1}^2 \leq \left(1 - \frac{2h\mu L}{\mu + L} + \mu^2 h \left(h - \frac{2}{\mu + L}\right)\right) r_k^2 = (1 - \mu h)^2 r_k^2.$$

So the first conclusion holds by applying the above inequality recursively. The second conclusion holds by plugging in $h = 2/(\mu + L)$. The last conclusion follows from the second conclusion, Lemma 1(ii), and Theorem 6. \square

Note that the convergence rate here is the same as in the unconstrained one.