A Single Cut Proximal Bundle Method for Stochastic Convex Composite Optimization

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Introduction

Main problem

$$\phi_* := \min \left\{ \phi(x) := f(x) + h(x) : x \in \mathbb{R}^n \right\}, \quad f(x) = \mathbb{E}_\xi [F(x, \xi)]$$

E.g., two-stage convex stochastic program

$$\min\{f_1(x) + \mathbb{E}[Q(x, \xi)] : x \in X\}$$

where $Q(x, \xi) = \min\{f_2(x, y, \xi) : g_2(x, y, \xi) \le 0, y \in Y\}$.

An instance of the main problem with

$$h(x) = \delta_X(x), \quad F(x, \xi) = f_1(x) + Q(x, \xi).$$

Goal: SA-type algorithm based on the proximal bundle (PB) method

Assumptions

Stochastic convex composite optimization

$$\phi_* := \min\left\{\phi(x) := f(x) + h(x) : x \in \mathbb{R}^n\right\}, \quad f(x) = \mathbb{E}_\xi[F(x,\xi)]$$

Black-box model

(A1) f is closed convex and $\operatorname{dom} f \supset \operatorname{dom} h$;

(A2) for almost every $\xi \in \Xi$, there exist a functional oracle $F(\cdot,\xi) : \operatorname{dom} h \to \mathbb{R}$ and a stochastic subgradient oracle $s(\cdot,\xi) : \operatorname{dom} h \to \mathbb{R}^n$ satisfying

$$f(x) = \mathbb{E}[F(x,\xi)], \quad f'(x) := \mathbb{E}[s(x,\xi)] \in \partial f(x);$$

(A3) for every $x \in \operatorname{dom} h$, we have $\mathbb{E}[\|s(x,\xi)\|^2] \le M^2$;

(A4) the set of optimal solutions X^* is nonempty.

Review of Deterministic PB

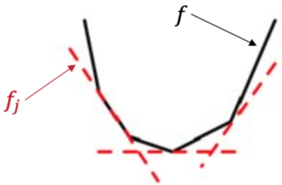
Proximal point method: constructs a sequence of proximal problems.
E.g., Chambolle-Pock for saddle point, ADMM for distributed optimization.

Approximately solve the proximal problem by an iterative process

$$x^+ \leftarrow \min_{u \in \mathbb{R}^n} \left\{ f(u) + \frac{1}{2\lambda} \|u - x^c\|^2 \right\}.$$

Recursively build up a cutting-plane model

$$f_j(u) = \max\{f(x) + \langle f'(x), u - x \rangle : x \in C_j\}, \quad C_{j+1} = C_j \cup \{x_j\}$$



Review of Deterministic PB

Algorithm 1 PB (one cycle)

1. Construct a proximal problem

$$\min_{u \in \mathbb{R}^n} \left\{ f(u) + h(u) + \frac{1}{2\lambda} \|u - x^c\|^2 \right\};$$

2. **If** find an $(\varepsilon/2)$-solution to the current proximal problem, **then** change the prox-center; $\leftarrow$ serious

Otherwise, keep the prox-center, update the cutting-plane model and solve the prox subproblem based on the current model, i.e., $\leftarrow$ null

$$x_j = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ f_j(u) + \frac{1}{2\lambda} \|u - x^c\|^2 \right\}.$$

Review of Deterministic PB

- Proximal bundle method $\mathcal{O}(\varepsilon^{-3})$ [1] $\to \mathcal{O}(\varepsilon^{-2})$ [2]
- Lower complexity bound $\Omega(\varepsilon^{-2})$

Proximal bundle method is optimal for black-box model.

[1] Kiwiel, 2000. Efficiency of proximal bundle methods.

[2] Liang and Monteiro, 2020. A proximal bundle variant with optimal iteration-complexity for a large range of prox stepsizes.

Other bundle models

(E1) **one-cut update[3]:** $\Gamma^+ = \Gamma^+_\tau := \tau\Gamma + (1-\tau)[\ell_f(\cdot;x) + h]$ with $\bar{\Gamma} = \Gamma$.

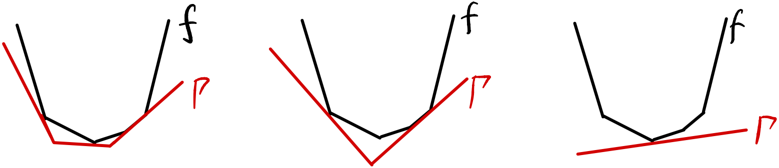
(E2) **two-cuts update:** assume $\Gamma = \max\{A_f, \ell_f(\cdot;x^-)\} + h$ where A_f is an affine function satisfying $A_f \le f$, set

$$\Gamma^+ = \max\{A_f^+, \ell_f(\cdot;x)\} + h$$

where $A_f^+ = \theta A_f + (1-\theta)\ell_f(\cdot;x^-)$. Also set $\bar{\Gamma} = A_f^+ + h$.

Bundle of past information $\{(x, f(x), f'(x)) : x \in C\}$

$f \geq \Gamma$



Multiple cuts Two cuts One cut

[3]Liang and Monteiro, 2021. A unified analysis of a class of proximal bundle methods for solving hybrid convex composite optimization problems.

Cutting-plane Model in the Stochastic Setting

A straightforward fact:

$$\mathbb{E}[\max\{X, Y\}] \geq \max\{\mathbb{E}[X], \mathbb{E}[Y]\}.$$

For a fixed u,

$$\mathbb{E}[\Gamma_j(u)] \geq \max\{\mathbb{E}[F(x,\xi) + \langle s(x,\xi), u - x\rangle] : x \in C_j\}.$$

On the other hand,

$$\begin{aligned}&\max\{\mathbb{E}[F(x,\xi) + \langle s(x,\xi), u - x\rangle] : x \in C_j\}\\ =&\max\{f(x) + \langle f'(x), u - x\rangle : x \in C_j\} \leq f(u)\end{aligned}$$

So

$$\mathbb{E}[\Gamma_j(u)] \ ? \ f(u)$$

A Single Cut Model

Aggregate all cuts into a single one

$$\Gamma_j(u) = \tau\Gamma_{j-1}(u) + (1-\tau)[F(x_{j-1}, \xi_{j-1}) + \langle s(x_{j-1}, \xi_{j-1}), u - x_{j-1}\rangle].$$

Since

$$\mathbb{E}[F(x,\xi) + \langle s(x,\xi), u - x\rangle] = f(x) + \langle f'(x), u - x\rangle \leq f(u),$$

we have by induction

$$\mathbb{E}[\Gamma_j(u)] \leq f(u).$$

1. Let $\lambda, \theta > 0$, integer $K \geq 1$, and $x_0 \in \operatorname{dom} h$ be given, and set $x_0^c = x_0$, $j = k = 1$, $j_0 = 0$, and
$$\tau = \frac{\theta K}{\theta K + 1};$$
2. Take an independent sample ξ_{j-1} of r.v. ξ, set
$$x_j^c = \begin{cases} x_{j_{k-1}}, & \text{if } j = j_{k-1} + 1, \\ x_{j-1}^c, & \text{otherwise,} \end{cases}$$
and compute
$$x_j = \underset{u \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ h(u) + \langle S_j, u \rangle + \frac{1}{2\lambda} \|u - x_j^c\|^2 \right\},$$
where
$$S_j := \begin{cases} s(x_{j_{k-1}}, \xi_{j_{k-1}}), & \text{if } j = j_{k-1} + 1, \\ (1-\tau) s(x_{j-1}, \xi_{j-1}) + \tau S_{j-1}, & \text{otherwise,} \end{cases}$$

Stochastic Composite Proximal Bundle Framework

2. Compute

$$y_j = \begin{cases} x_j, & \text{if } j = j_{k-1}+1, \\ (1-\tau)x_j + \tau y_{j-1}, & \text{otherwise;} \end{cases}$$

3. Choose an integer $j_k \ge j_{k-1}+1$, and set $\hat{y}_k = y_{j_k}$ when the k-th cycle ends;
4. if $k = K$ then **stop** and output

$$\hat{y}_K^a = \frac{1}{\lceil K/2 \rceil} \sum_{k=\lfloor K/2 \rfloor + 1}^{K} \hat{y}_k;$$

otherwise, set $k \leftarrow k+1$ and $j \leftarrow j+1$, and go to step 1.

Remarks on SCPB

- An aggregated single cut
- No termination criterion for a cycle

Define a cycle

$$\mathcal{C}_k := \{i_k, \ldots, j_k\}, \quad \text{where} \quad i_k := j_{k-1} + 1$$

Two ways of setting j_k:

(B1) the smallest integer $j_k \geq i_k$ and $\lambda k \tau^{j_k - i_k} \leq C$;

(B2) the smallest integer $j_k \geq i_k + 1$ and $\lambda k \tau^{j_k - i_k} t_{i_k} \leq C$.

(B1) is deterministic and (B2) is stochastic

Main Results – SCPB1

Assume that conditions (A1)-(A4) hold and $\mathrm{dom}\, h$ has a finite diameter $D_h \geq 0$.

SCPB with (B1) satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (\theta K + 1) \ln \left(\frac{\lambda k}{C} + 1 \right) \right\rceil + 1.$$

- Convergence of SCPB1

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{1}{K} \left(\frac{D^2}{\lambda} + \frac{6C \min\{\lambda M^2, MD\}}{\lambda} + \frac{2\lambda M^2}{\theta} \right).$$

A Practical Variant of SCPB1

Let pair (λ, K) and constant $m \geq 1$ be given, and define

$$\theta = \frac{m}{K}, \quad C = \frac{D}{6M},$$

SCPB based on (B1) satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (m+1) \ln\left(\frac{\lambda k}{C} + 1\right) \right\rceil + 1.$$

- Convergence of SCPB1

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + \frac{2\lambda M^2}{m}.$$

- Its expected overall iteration complexity is $\tilde{\mathcal{O}}(mK)$.

Robust Stochastic Approximation (RSA) [4]

$$x_j = \operatorname*{argmin}_{u \in X} \left\{ \langle s(x_{j-1}, \xi_{j-1}), u \rangle + \frac{1}{2\lambda} \|u - x_{j-1}\|^2 \right\} \qquad \forall j = 1, \ldots, N.$$

- Convergence of RSA

$$\mathbb{E}[\phi(x_K^a)] - \phi_* \le \frac{2D^2}{\lambda K} + 2\lambda M^2, \quad x_N^a = \frac{1}{\lceil N/2 \rceil} \sum_{j=\lfloor N/2 \rfloor + 1}^{N} x_j.$$

Taking $\lambda = \frac{\sqrt{m} D}{M \sqrt{K}}$, given $\varepsilon > 0$, to obtain $x \in \operatorname{dom} h$ such that $\mathbb{E}[\phi(x)] - \phi_* \le \varepsilon$,

- RSA has iteration complexity $\mathcal{O}\left(\frac{mM^2D^2}{\varepsilon^2}\right)$;
- SCPB1 has iteration complexity $\tilde{\mathcal{O}}\left(\frac{M^2D^2}{\varepsilon^2}\right)$.

[4] Nemirovski, Juditsky, Lan and Shapiro, 2009. Robust stochastic approximation approach to stochastic programming.

Relationship between SCPB1 and RSA

Recall (B1) the smallest integer $j_k \geq i_k$ and $\lambda k \tau^{j_k - i_k} \leq C$.
Choosing

$$C = \frac{\alpha D \sqrt{K}}{M}, \quad \lambda = \frac{\alpha D}{M\sqrt{K}},$$

then (B1) is satisfied with $j_k = i_k$, since

$$\frac{C}{\lambda k} \geq \frac{C}{\lambda K} = 1 = \tau^{j_k - i_k}.$$

In summary,

- RSA performs one iteration per cycle
- RSA $\rightarrow$ SCPB1 is analogous to Subgradient method $\rightarrow$ PB
- RSA is restricted to small stepsizes, while SCPB1 can use large ones
- SCPB1 implicitly reduces the variance and the sample complexity by m

Main Results – SCPB2

Recall (B2) the smallest integer $j_k \geq i_k + 1$ and $\lambda k \tau^{j_k - i_k} t_{i_k} \leq C$.

Assume that conditions (A1)-(A4) hold and $\operatorname{dom} h$ has a finite diameter $D_h \geq 0$.

SCPB with (B2) satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (\theta K + 1) \ln \left(\frac{2M^2 \lambda^2 k}{C} + 1 \right) \right\rceil + 1.$$

- Convergence of SCPB2

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{1}{K} \left(\frac{3C + D^2}{\lambda} + \frac{2\lambda M^2}{\theta} + \frac{2\lambda M^2}{\theta^2 K} \right).$$

A Practical Variant of SCPB2

Let pair (λ, K) and constant $m \geq 1$ be given, and define

$$\theta = \frac{m}{K}, \quad C = \frac{D^2}{3},$$

SCPB based on (B2) satisfies the following statements:

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (m+1) \ln \left(\frac{6M^2\lambda^2 k}{D^2} + 1 \right) \right\rceil + 1.$$

- Convergence of SCPB2

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + \frac{4\lambda M^2}{m}.$$

- Its expected overall iteration complexity is $\tilde{\mathcal{O}}(mK)$.

Test 1 – Two-stage Stochastic Program

$$\begin{cases} \min \; c^T x_1 + \mathbb{E}[Q(x_1,\xi)] \\ x_1 \in \mathbb{R}^n : x_1 \geq 0, \sum_{i=1}^n x_1(i) = 1 \end{cases}$$

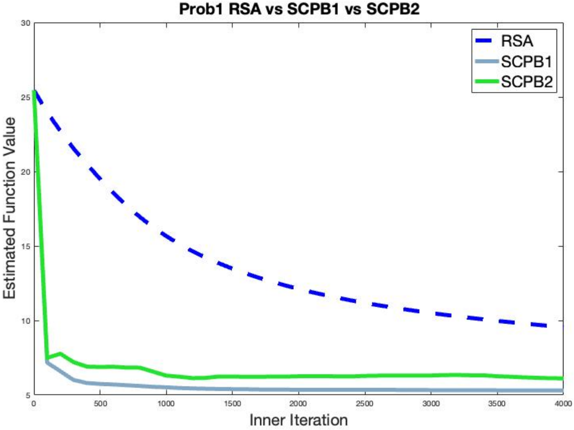
where the second stage recourse function is given by

$$Q(x_1,\xi) = \begin{cases} \min\limits_{x_2 \in \mathbb{R}^n} \; \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \left(\xi\xi^T + \lambda_0 I_{2n}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \xi^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ x_2 \geq 0, \sum\limits_{i=1}^n x_2(i) = 1. \end{cases}$$

Table: $n = 50, \; N = 4000$

Statistics	RSA	SCPB1	SCPB2
λ	7.4×10^{-7}	10^{-3}	10^{-3}
Min Inner	1	9	2
Max Inner	1	52	43
Avg Inner	1	43	5

Test 1 – Two-stage Stochastic Program



Test 2 – Two-stage Stochastic Program

$$\begin{cases} \min \ c^T x_1 + \mathbb{E}[\mathfrak{Q}(x_1, \xi)] \\ x_1 \in \mathbb{R}^n : \|x_1 - x_0\|_2 \leq 1 \end{cases}$$

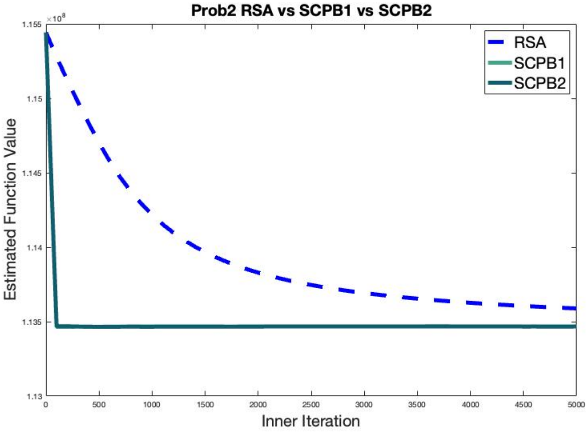
where the second stage recourse function is given by

$$Q(x_1, \xi) = \begin{cases} \min\limits_{x_2 \in \mathbb{R}^n} \ \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \left(\xi\xi^T + \lambda_0 I_{2n}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \xi^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \|x_2 - y_0\|_2^2 + \|x_1 - x_0\|_2^2 - R^2 \leq 0. \end{cases}$$

Table: $n = 50$, $N = 5000$

Statistics	RSA	SCPB1	SCPB2
λ	8.9×10^{-10}	10^{-3}	10^{-3}
Min Inner	1	71	54
Max Inner	1	109	89
Avg Inner	1	100	77

Test 2 – Two-stage Stochastic Program



Test 3 – One-stage Stochastic Program

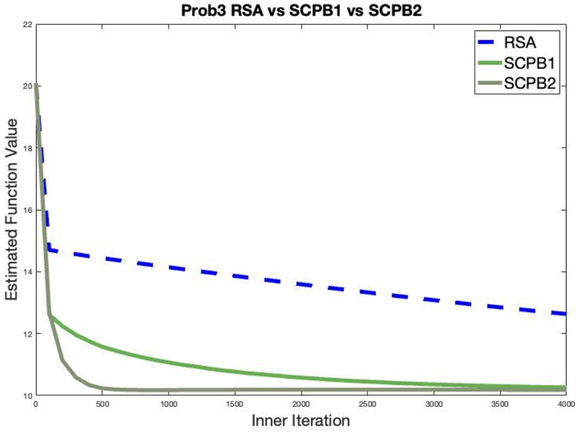
$$\min_{x\in X} \mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(\frac{i}{n}+\xi_i)x_i\right)\right]$$

where X is the unit simplex.

Table: $n = 100,\ N = 4000$

Statistics	RSA	SCPB1	SCPB2
λ	2.8×10^{-5}	10^{-3}	10^{-3}
Min Inner	1	1	2
Max Inner	1	26	6
Avg Inner	1	17	2

Test 3 – One-stage Stochastic Program



Take-away

- A parameter-free single cut proximal bundle method for stochastic programming
- Aggregating all past information by convex combination
- $\mathcal{O}(1/K)$ convergence rate
- Includes RSA as an instance, outperforms RSA in theory and practice
- Variance reduction

References

- J. Liang, V. Guigues and R. D. C. Monteiro. A single cut proximal bundle method for stochastic convex composite optimization. Available on arXiv:2207.09024, 2022.
- J. Liang and R. D. C. Monteiro. A unified analysis of a class of proximal bundle methods for smooth-nonsmooth convex composite optimization. Available on arXiv:2110.01084, 2021.
- J. Liang and R. D. C. Monteiro. A proximal bundle variant with optimal iteration-complexity for a large range of prox stepsizes. SIAM Journal on Optimization, 31(4):2955-2986, 2021.

Thank you!

Supplementary Materials

A generic bundle update scheme

Definition

Let $\mathcal{C}_\mu(\phi)$ denote a class of convex functions Γ satisfying $\Gamma \le \phi$ and Γ is μ-convex.

For a given quadruple $(\Gamma, x_0, \lambda, \tau) \in \mathcal{C}_\mu(\phi) \times \mathbb{R}^n \times \mathbb{R}_{++} \times (0,1)$, the generic bundle update scheme returns $\Gamma^+ \in \mathcal{C}_\mu(\phi)$ satisfying

$$\tau\bar{\Gamma} + (1-\tau)[\ell_f(\cdot; x) + h] \le \Gamma^+ \tag{1}$$

where $\ell_f(\cdot; x) = f(x) + \langle f'(x), \cdot - x\rangle$,

$$x = \underset{u\in\mathbb{R}^n}{\operatorname{argmin}} \left\{ \Gamma(u) + \frac{1}{2\lambda}\|u - x_0\|^2 \right\}$$

and $\bar{\Gamma} \in \mathcal{C}_\mu(\phi)$ is such that

$$\bar{\Gamma}(x) = \Gamma(x), \quad x = \underset{u\in\mathbb{R}^n}{\operatorname{argmin}} \left\{ \bar{\Gamma}(u) + \frac{1}{2\lambda}\|u - x_0\|^2 \right\}. \tag{2}$$

SCPB vs. other SA methods

Comparison with Dual Averaging (DA) [5]:

- DA uses a fixed prox-center throughout the process
- DA uses variable prox stepsizes

Comparison with Robust Stochastic Approximation (RSA) [6]:

- RSA does not use previous cuts
- RSA performs one iteration per cycle
- RSA $\rightarrow$ SCPB is analogous to Subgradient method $\rightarrow$ PB

[5] Nesterov, 2009. Primal-dual subgradient methods for convex problems.

[6] Nemirovski, Juditsky, Lan and Shapiro, 2009. Robust stochastic approximation approach to stochastic programming.