A Universal Proximal Framework for Optimization and Sampling

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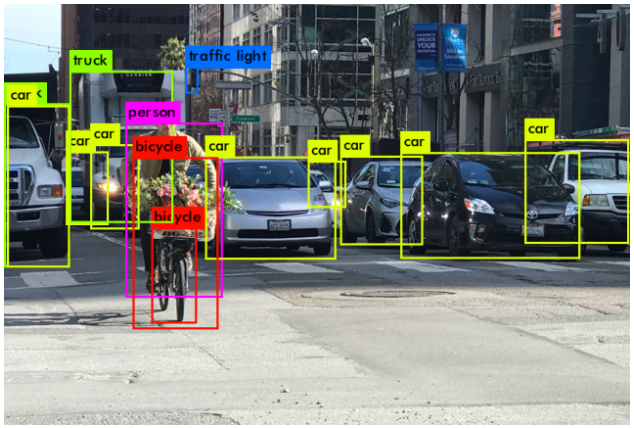
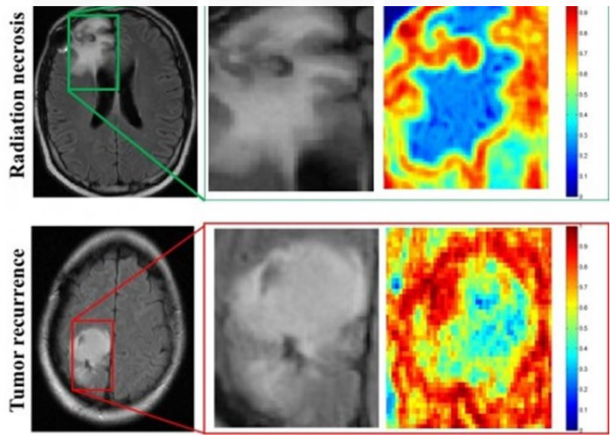
Introduction



(a) Object Detection



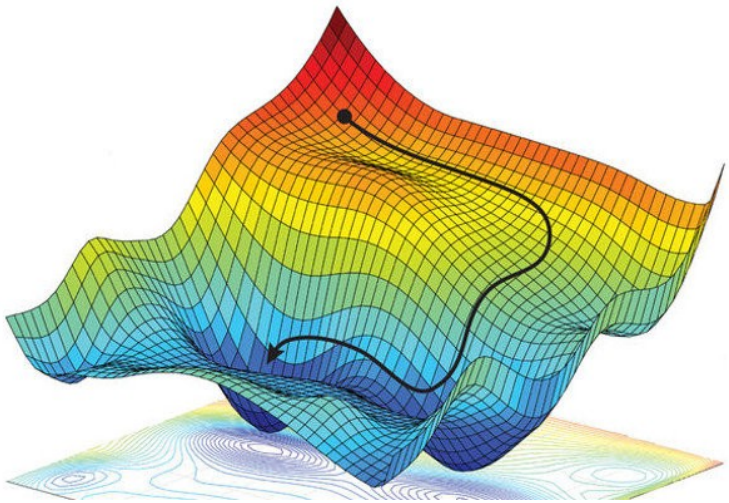
(b) Medical Image Analysis

(c) ChatGPT

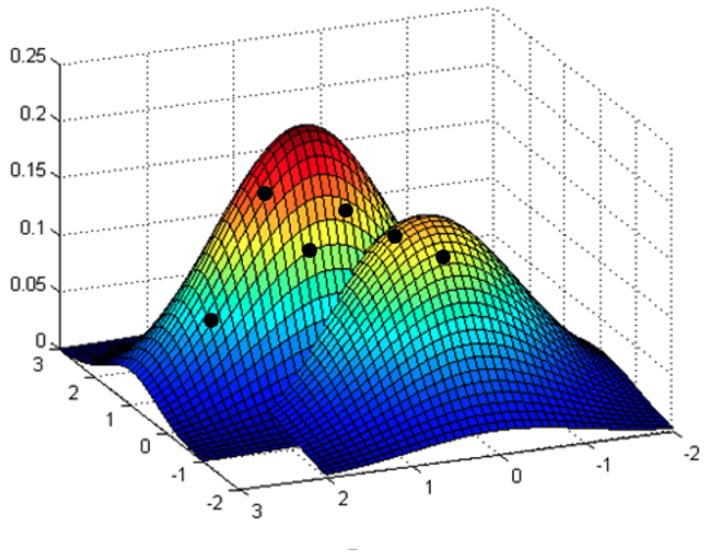
(d) DALL-E 2

Optimization and Sampling

I design and analyze fast algorithms for solving fundamental Optimization and Sampling problems arising from Data Science.



(e) Optimization, $\min f(x)$



(f) Sampling, samp $\exp(-f(x))$

Algorithms for Optimization and Sampling

- Stochastic gradient descent, $\min_x \mathbb{E}_\xi[F(x,\xi)]$

$$x_{k+1} = x_k - \lambda_k s(x_k,\xi_k), \quad s(x_k,\xi_k) \in \partial F(x_k,\xi_k)$$

- Accelerated gradient descent, $\min_x f(x)$

$$\tilde{x}_k = \frac{A_k y_k + a_k x_k}{A_{k+1}}, \quad y_{k+1} = \tilde{x}_k - \lambda_k \nabla f(\tilde{x}_k), \quad x_{k+1} = \frac{A_{k+1}}{a_k} y_{k+1} - \frac{A_k}{a_k} y_k$$

- Unadjusted Langevin algorithm, sample from $\nu(x) \propto \exp(-f(x))$

$$x_{k+1} = x_k - \lambda_k \nabla f(x_k) + \sqrt{2\lambda_k} z, \quad z \sim \mathcal{N}(0, I)$$

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A Universal Proximal Framework

Optimization

Algorithm Proximal Point Framework

1. $y_k \leftarrow \operatorname{argmin}_x \frac{1}{2\lambda}\|x - x_k\|^2 = x_k$
2. $x_{k+1} \leftarrow \operatorname{argmin}_x \left\{ f(x) + \frac{1}{2\lambda}\|x - y_k\|^2 \right\}$

E.g., GD, SGD, AGD, Newton, Chambolle-Pock, ADMM, proximal bundle ...

Sampling

Algorithm Alternating Sampling Framework

1. Sample $y_k \sim \pi^{Y|X}(y \mid x_k) \propto \exp[-\frac{1}{2\lambda}\|x_k - y\|^2]$
2. Sample $x_{k+1} \sim \pi^{X|Y}(x \mid y_k) \propto \exp[-f(x) - \frac{1}{2\lambda}\|x - y_k\|^2]$

E.g., ULA, proximal Langevin algorithm, symmetric Langevin algorithm ...

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My Research: The Big Picture

- Nonsmooth Optimization

  The first **optimal** complexity result for a proximal bundle type method.

- Stochastic Optimization

  The **best** stochastic approximation method in both theory and practice.

- High-dimensional Sampling

  A proximal algorithm for sampling from nonconvex and semi-smooth densities with the **best** complexity.

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Outline

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Assumptions

Convex nonsmooth composite problem

$$\phi_* := \min \{\phi(x) := f(x) + h(x) : x \in \mathbb{R}^n\}$$

(A1) bounded subgradient

$$\|f'(x)\| \leq M;$$

(A2) h is μ-strongly convex ($\mu \geq 0$).

Motivation - Proximal Bundle Method

Goal: find $\hat{x}$ such that $\phi(\hat{x}) - \phi_* \leq \varepsilon$

- Subgradient, Mirror descent, Bundle-level, and Prox Level method are optimal.
- Proximal bundle method $\mathcal{O}(\varepsilon^{-3})$ $\leftarrow$ previously best, improvable?
- Lower complexity bound $\Omega(\varepsilon^{-2})$

Proximal bundle method is not optimal in general

We close the gap by showing the tight upper bound $\mathcal{O}(\varepsilon^{-2})$
through a new proximal bundle method and a refined analysis

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Review of the Proximal Bundle Method

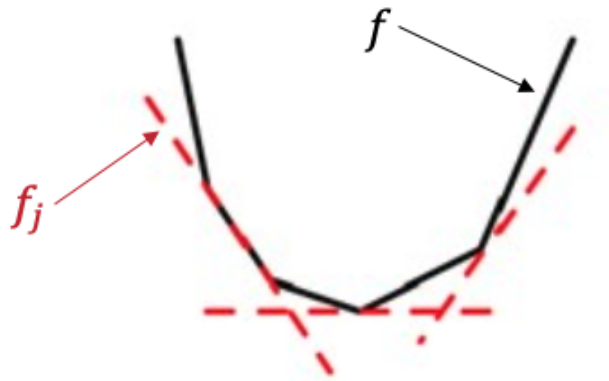
Proximal point framework: constructs a sequence of proximal problems.

Approximately solve the proximal problem by an iterative process

$$x^+ \leftarrow \min_{z \in \mathbb{R}^n} \left\{ f(z) + h(z) + \frac{1}{2\lambda} \|z - x^c\|^2 \right\}.$$

Recursively build up a cutting-plane model

$$f_j(z) = \max\{f(z_i) + \langle f'(z_i), z - z_i \rangle : 0 \leq i \leq j-1\}$$



Relaxed Proximal Bundle Method (L. and Monteiro, 2021)

Consider a proximal problem

$$\min_{u\in\mathbb{R}^n}\left\{f(u)+h(u)+\frac{1}{2\lambda}\|u-x^c\|^2\right\}$$

Algorithm RPB (one stage)

If find an $(\varepsilon/2)$-solution to the current proximal problem, **then** change the prox-center; $\leftarrow$ serious

Otherwise, keep the prox-center, update the cutting-plane model and solve the prox subproblem based on the current model, i.e., $\leftarrow$ null

$$x_j=\operatorname*{argmin}_{u\in\mathbb{R}^n}\left\{f_j(u)+h(u)+\frac{1}{2\lambda}\|u-x^c\|^2\right\}.$$

Main Results (L. and Monteiro, 2021)

We establish **improved** upper bounds and **matching** lower bounds.

Table: Upper and lower complexity bounds

	Convex	Strongly convex
Upper bound	$\mathcal{O}\left(\frac{M^2 d_0^2}{\varepsilon^2}\right)$	$\mathcal{O}\left(\frac{M^2}{\mu\varepsilon}\log\frac{\mu d_0^2}{\varepsilon}\right)$
Lower bound	$\Omega\left(\frac{M^2 d_0^2}{\varepsilon^2}\right)$	$\Omega\left(\frac{M^2}{\mu\varepsilon}\right)$

Optimal for convex and nearly optimal for strongly convex

Outline

Motivation

Main problem

$$\phi_* := \min_{x \in \mathbb{R}^n} \{\phi(x) := f(x) + h(x)\}, \quad f(x) = \mathbb{E}_\xi[F(x, \xi)]$$

Applications: Two-stage SP, Statistical learning, Statistical inference

$$\min_{P_\theta \in \mathcal{P}_2(\mathbb{R}^n)} KL(P_{\theta_0} || P_\theta) = \min_{P_\theta \in \mathcal{P}_2(\mathbb{R}^n)} \int \log \frac{P_{\theta_0}}{P_\theta} P_{\theta_0}(x) dz$$

$$= \int \log P_{\theta_0} P_{\theta_0}(z) dz - \max_{\theta \in \Theta} \mathbb{E}_{z \sim P_{\theta_0}}[\log P_\theta(z)].$$

Maximum likelihood estimation (MLE) is a sample average approximation (SAA)

$$\max_{\theta \in \Theta} \left\{ \ell(\theta | Z) := \frac{1}{N} \sum_{i=1}^{N} \log P_\theta(Z_i) \right\} \quad \leftarrow \text{offline}$$

Goal: stochastic approximation (SA) based on proximal bundle $\leftarrow$ online

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Assumptions

Stochastic convex composite optimization

$$\phi_* := \min \{\phi(x) := f(x) + h(x) : x \in \mathbb{R}^n\}, \quad f(x) = \mathbb{E}_\xi[F(x, \xi)]$$

(A1) unbiased estimators

$$\mathbb{E}[F(x, \xi)] = f(x), \quad \mathbb{E}[s(x, \xi)] = f'(x) \in \partial f(x);$$

(A2) bounded variance

$$\mathbb{E}[\|s(x, \xi)\|^2] \leq M^2.$$

A Motivating Question

- Stochastic gradient descent, $\min_x \mathbb{E}_\xi[F(x,\xi)]$

$$x_{k+1} = x_k - \lambda_k s(x_k,\xi_k), \quad s(x_k,\xi_k) \in \partial F(x_k,\xi_k)$$

Approximation by a single cut: $\mathbb{E}[f(y) + \langle s(y;\xi), x-y\rangle] \leq f(x)$

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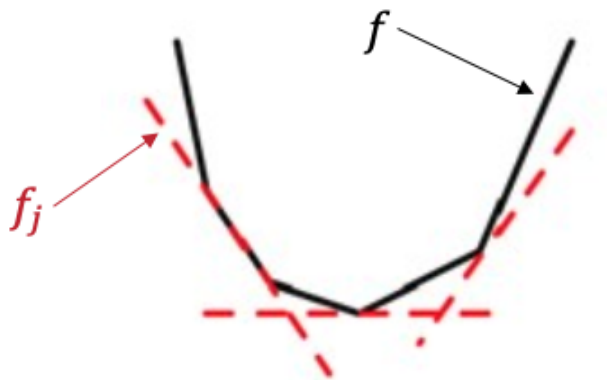

- In the stochastic setting, is it still true?

$$\mathbb{E}[f_j(x)] \leq f(x)?$$

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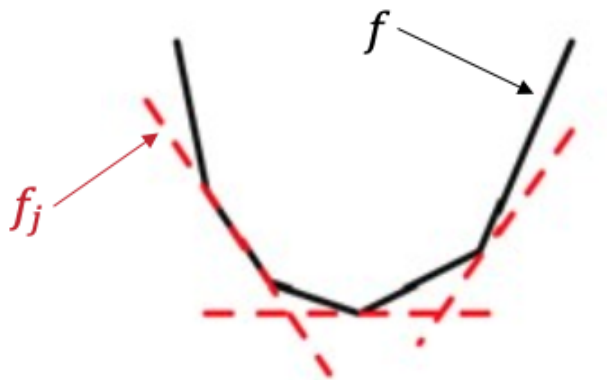

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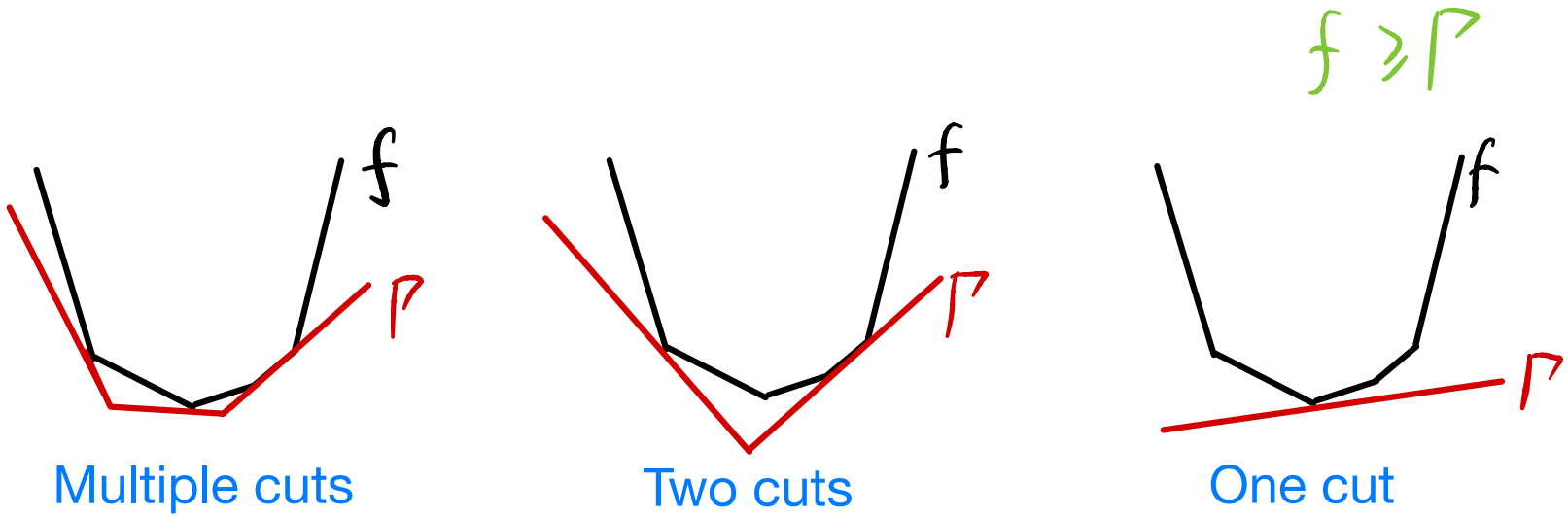
Other bundle models

(E1) **single cut update**[1]**:** $\Gamma^+ = \Gamma_\tau^+ := \tau\Gamma + (1-\tau)\ell_f(\cdot; x)$.

(E2) **two cuts update:** $\Gamma^+ = \max\{A_f^+, \ell_f(\cdot; x)\}$ where $A_f^+ = \theta A_f + (1-\theta)\ell_f(\cdot; x^-)$.

Bundle of past information $\{(x_i, f(x_i), f'(x_i))\}$



Multiple cuts Two cuts One cut

[1] Liang and Monteiro, 2021. A unified analysis of a class of proximal bundle methods for solving hybrid convex composite optimization problems.

Convergence of SCPB

Let pair (λ, K) and constant $m \geq 1$ be given

- Number of iterations within $\mathcal{C}_k$, or number of null steps

$$|\mathcal{C}_k| \leq \left\lceil (m+1) \ln \left(\frac{\lambda k}{C} + 1 \right) \right\rceil + 1.$$

- Convergence of SCPB

$$\mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + \frac{2\lambda M^2}{m}.$$

- Its expected overall iteration complexity is $\tilde{\mathcal{O}}(mK)$.

Comparison with Robust Stochastic Approximation [2]

RSA is basically SGD with constant stepsize λ

$$\text{RSA: } \mathbb{E}[\phi(x_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + 2\lambda M^2$$

$$\text{SCPB: } \mathbb{E}[\phi(\hat{y}_K^a)] - \phi_* \leq \frac{2D^2}{\lambda K} + \frac{2\lambda M^2}{m}$$

Taking the optimal stepsize for SCPB $\lambda = \frac{\sqrt{m}D}{M\sqrt{K}}$

- RSA has iteration complexity $\mathcal{O}\left(\frac{mM^2D^2}{\varepsilon^2}\right)$;
- SCPB has iteration complexity $\tilde{\mathcal{O}}\left(\frac{M^2D^2}{\varepsilon^2}\right)$.

[2] Nemirovski, Juditsky, Lan and Shapiro, 2009. Robust stochastic approximation approach to stochastic programming.

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Two-stage Stochastic Program

$$\begin{cases} \min\ c^T x_1 + \mathbb{E}[Q(x_1, \xi)] \\ x_1 \in \mathbb{R}^n : x_1 \geq 0, \sum_{i=1}^n x_1(i) = 1 \end{cases}$$

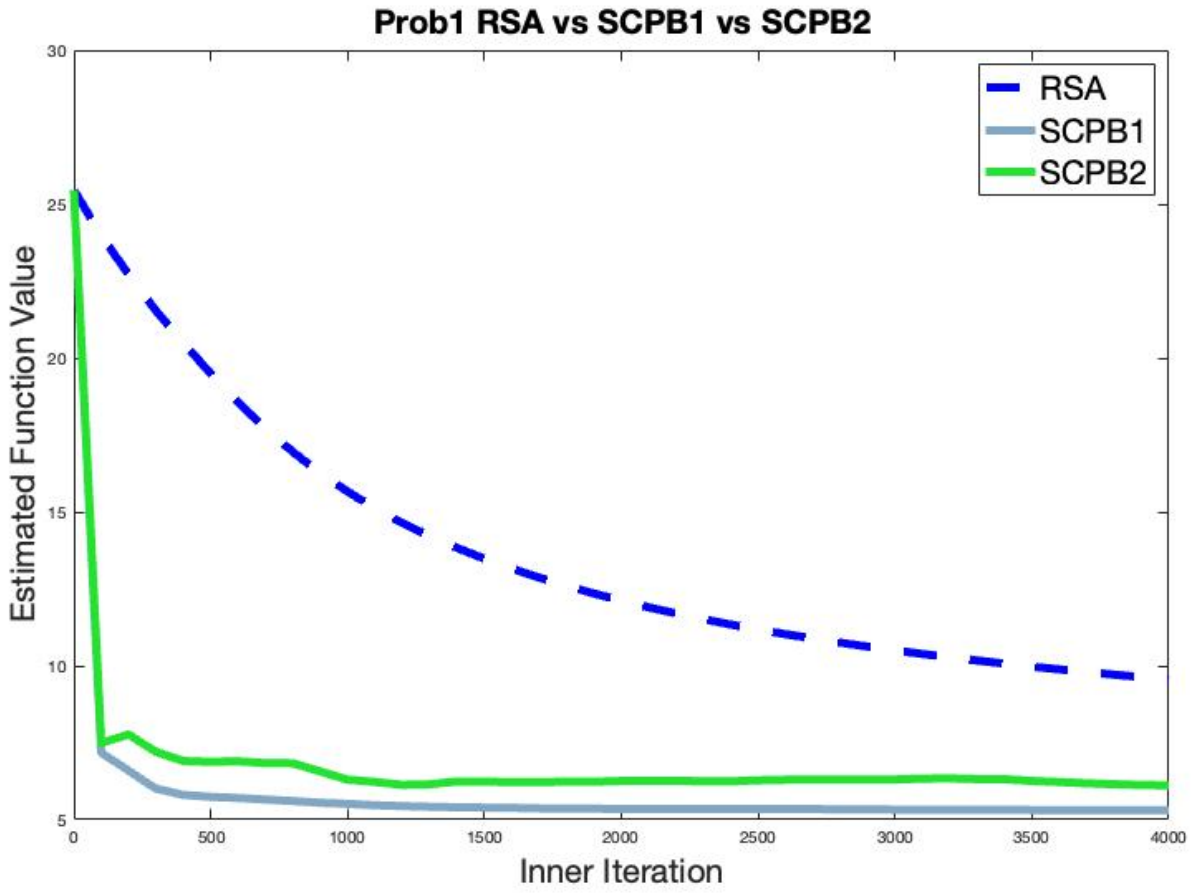
where the second stage recourse function is given by

$$Q(x_1, \xi) = \begin{cases} \min\limits_{x_2 \in \mathbb{R}^n} \dfrac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \left(\xi\xi^T + \lambda_0 I_{2n}\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \xi^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ x_2 \geq 0, \sum\limits_{i=1}^{n} x_2(i) = 1. \end{cases}$$

Table: $n = 50,\ N = 4000$

Statistics	RSA	SCPB1	SCPB2
λ	7.4×10^{-7}	10^{-3}	10^{-3}
Min Inner	1	9	2
Max Inner	1	52	43
Avg Inner	1	43	5

Two-stage Stochastic Program



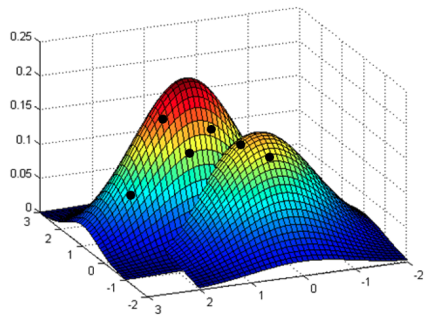
Take-away Message

- Optimal complexity for large stepsizes
- Non-trivial variance reduction by PPF

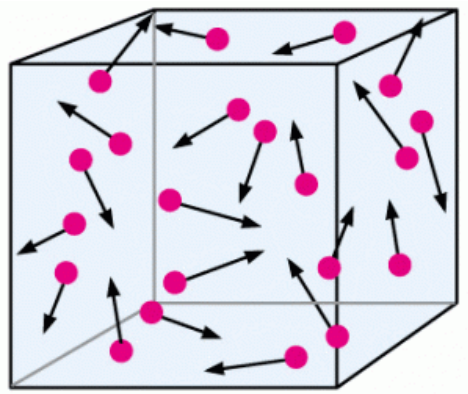
Outline

Sampling - Generation from Data

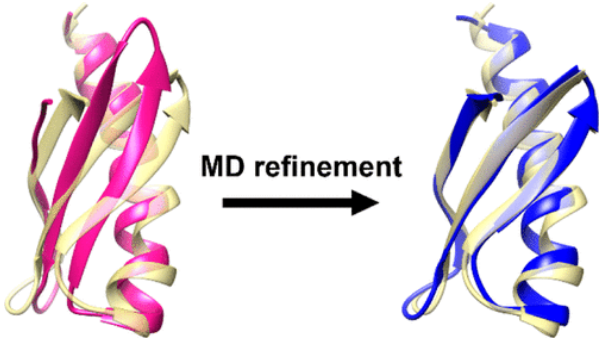
Sample from a probability distribution $\propto \exp(-f(x))$ where f has certain properties, such as convexity and smoothness



Extensively used in Bayesian inference and scientific computing

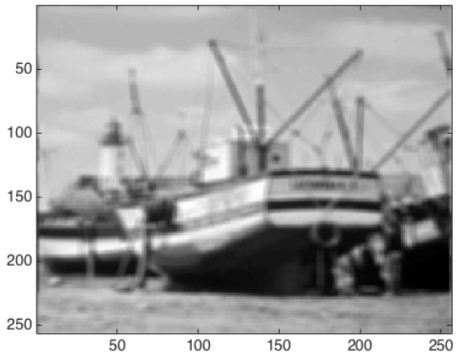


(g) Statistical Mechanics



(h) Molecular Dynamics

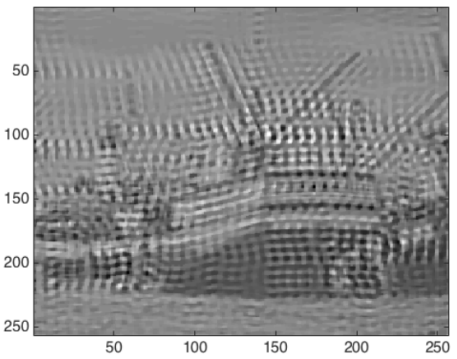
Image Deconvolution – Bayesian Model Selection



(a)

(b)

(c)



(d)

$$p(\mathcal{M}_1|y) = 0.964, \quad p(\mathcal{M}_2|y) = 0.036, \quad p(\mathcal{M}_3|y) < 0.001$$

Assumptions

Problem: sample from $\nu(x) \propto \exp(-f(x))$

(A1) f is semi-smooth, i.e., there exist $\alpha_i \in [0,1]$ and $L_{\alpha_i} > 0$, $i = 1, \ldots, n$, s.t.

$$\|f'(u) - f'(v)\| \leq \sum_{i=1}^{n} L_{\alpha_i} \|u - v\|^{\alpha_i}, \quad \forall u, v \in \mathbb{R}^d$$

Examples: $n = 1$
1) $\alpha_1 = 1$, smooth, 2) $\alpha_1 = 0$, nonsmooth, 3) $0 < \alpha_1 < 1$, weakly smooth

(A2) ν satisfies log-Sobolev inequality (LSI) or Poincaré inequality (PI).

LSI: $H_\nu(\rho) \leq \frac{C_{LSI}}{2} J_\rho(\nu)$, PI: $\mathbb{E}_\nu[(\psi - \mathbb{E}_\nu[\psi])^2] \leq C_{PI} \mathbb{E}_\nu[\|\nabla \psi\|^2]$

Observations: ν is not necessarily log-concave, f is not necessarily convex.

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Comparison

Source	Complexity	Assumption	Metric
Chewi et al.	$\tilde{\mathcal{O}}\left(\frac{C_{\mathrm{PI}}^{1+1/\alpha} L_\alpha^{2/\alpha} d^{2+1/\alpha}}{\varepsilon^{1/\alpha}}\right)$	weakly smooth $\alpha > 0$, PI	Rényi
This work	$\tilde{\mathcal{O}}\left(C_{\mathrm{PI}} L_\alpha^{2/(1+\alpha)} d^2\right)$	semi-smooth, PI	Rényi

Table: Complexity bounds for sampling from non-convex semi-smooth potentials.

Source	Complexity	Assumption	Metric
Nguyen et al.	$\tilde{\mathcal{O}}\left(C_{\mathrm{LSI}}^{1+\max\{\frac{1}{\alpha_i}\}} \left[\frac{n \max\{L_{\alpha_i}^2\} d}{\varepsilon}\right]^{\max\{\frac{1}{\alpha_i}\}}\right)$	weakly smooth $\alpha_i > 0$, LSI	KL
This work	$\tilde{\mathcal{O}}\left(C_{\mathrm{LSI}} \sum_{i=1}^n L_{\alpha_i}^{2/(\alpha_i+1)} d\right)$	semi-smooth, LSI	KL
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Table: Complexity bounds for sampling from non-convex composite potentials.

Alternating Sampling Framework

Joint distribution $\pi(x, y) \propto \exp[-f(x) - \frac{1}{2\eta}\|x - y\|^2]$

Algorithm ASF (Shen, Tian and Lee 2021)

---

1. Sample $y_k \sim \pi^{Y|X}(y \mid x_k) \propto \exp[-\frac{1}{2\eta}\|x_k - y\|^2]$
2. Sample $x_{k+1} \sim \pi^{X|Y}(x \mid y_k) \propto \exp[-f(x) - \frac{1}{2\eta}\|x - y_k\|^2]$

Restricted Gaussian Oracle (RGO)
Given y, sample from

$$\pi^{X|Y}(\cdot|y) \propto \exp\left(-f(\cdot) - \frac{1}{2\eta}\|\cdot - y\|^2\right).$$

Without an implementable and provable RGO, ASF is only conceptual.

Nontrivial

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RGO Implementation

RGO: given y, sample from $\exp(-f_y^\eta(x))$

Algorithm RGO Rejection Sampling

1. Compute an approximate stationary point w of f_y^η
2. Generate sample $X \sim \exp(-h_1(x))$
3. Generate sample $U \sim \mathcal{U}[0,1]$
4. If
$$U \leq \frac{\exp(-f_y^\eta(X))}{\exp(-h_1(X))},$$
then accept/return X; otherwise, reject X and go to step 2.

Proposal: $\exp(-h_1(x))$ where $h_1(x) \leq f_y^\eta(x)$, construct the proposal as a Gaussian

Rejection Sampling Efficiency (L. and Chen, 2022)

Proposition

Assume

$$\eta \leq \frac{1}{Md} = \frac{[(\alpha+1)\delta]^{\frac{1-\alpha}{\alpha+1}}}{L_\alpha^{\frac{2}{\alpha+1}} d},$$

then the expected number of rejection steps in RGO Rejection Sampling is at most $\exp\left(\frac{3(1-\alpha)\delta}{2} + 3\right)$.

Proposition

Assume $\eta \leq \frac{1}{Md}$, *then the iteration-complexity to find the approx. stat. pt.* w *s.t.* $\left\| f'(w) + \frac{1}{\eta}(w-y) \right\| \leq \sqrt{Md}$ *by Nesterov acceleration is* $\tilde{\mathcal{O}}(1)$.

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ASF Complexity

Another ingredient for total complexity: **Convergence rate analysis of ASF**

Theorem (Chen, Chewi, Salim and Wibisono 2022)

If $\nu \propto \exp(-f)$ satisfies PI with $C_{\mathrm{PI}} > 0$, then x_k of ASF $\sim \rho_k$, which satisfies

$$\chi^2_\nu(\rho_k) \le \frac{\chi^2_\nu(\rho_0)}{\left(1 + \frac{\eta}{C_{\mathrm{PI}}}\right)^{2k}}.$$

Main Result (L. and Chen, 2022)

Theorem

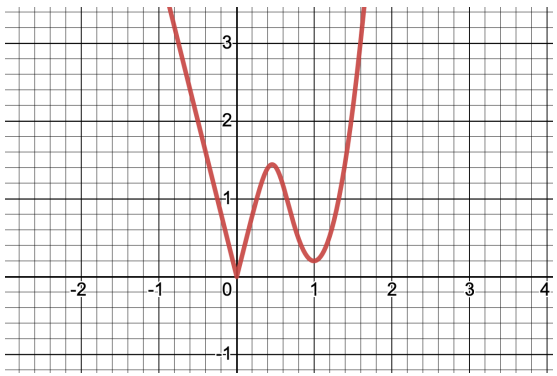
Suppose f is L_α-semi-smooth and ν satisfies PI. With $\eta \asymp 1/(L_\alpha^{\frac{2}{\alpha+1}} d)$, then ASF with RGO by rejection has complexity bound

$$\tilde{\mathcal{O}}\left(C_{\mathrm{PI}} L_\alpha^{\frac{2}{\alpha+1}} d\right)$$

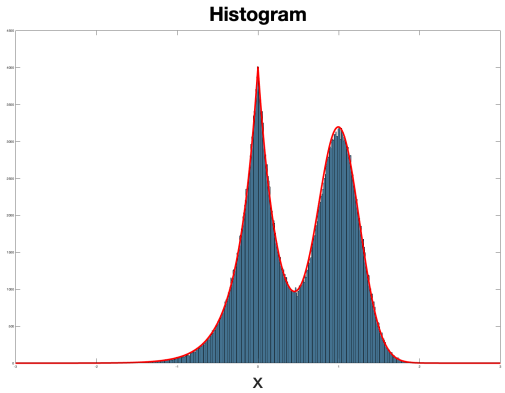
to achieve ε error to ν in terms of χ^2 divergence. Each iteration queries $\tilde{\mathcal{O}}(1)$ subgradients of f and generates $\mathcal{O}(1)$ samples in expectation from Gaussian distribution.

Gaussian-Laplace Mixture

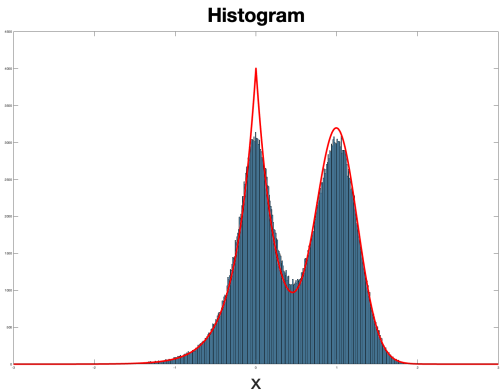
$$\nu(x) = 0.5(2\pi)^{-d/2}\sqrt{\det Q}\exp\left(-\frac{1}{2}(x-\mathbf{1})^\top Q(x-\mathbf{1})\right) + 0.5(2^d)\exp(-\|4x\|_1)$$



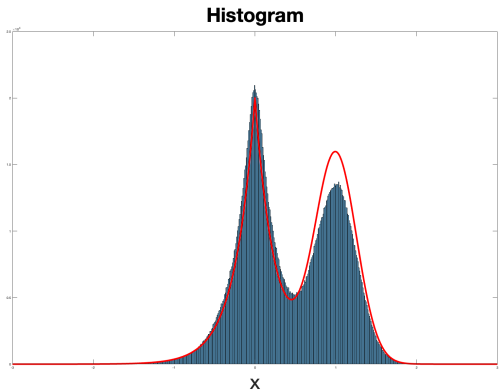
(i) $f(x) = -\ln \nu(x)$



(j) Histogram ASF



(k) Histogram ULA



(l) Histogram ULA with small η

Future Directions

Stochastic Optimization

- High Probability

$$\mathbb{P}\left(f\left(x_{\varepsilon,p}\right) - f_* \leq \varepsilon\right) \geq 1 - p$$

By Markov's inequality, a sufficient condition is $\mathbb{E}[f\left(x_{\varepsilon,p}\right) - f_*] \leq p\varepsilon$

Sample complexity becomes $\mathcal{O}(1/(p\varepsilon)^2)$ or
sub-Gaussian assumptions on noise for $\mathcal{O}(\log(1/p))$ complexity

[Davis et al. 2019] PPF diminishing stepsize, not parameter free

- Statistical Efficiency
  - Asymptotic normality

$$\frac{1}{\sqrt{k}} \sum_{i=1}^{k} \left(x_i - x_*\right) \xrightarrow{d} \mathcal{N}\left(0, \nabla^2 f\left(x_*\right)^{-1} \operatorname{Cov}\left(s\left(x_*; \xi\right)\right) \nabla^2 f\left(x_*\right)^{-1}\right)$$

  - Convergence rate in KL divergence $KL(p_k||\nu)$?

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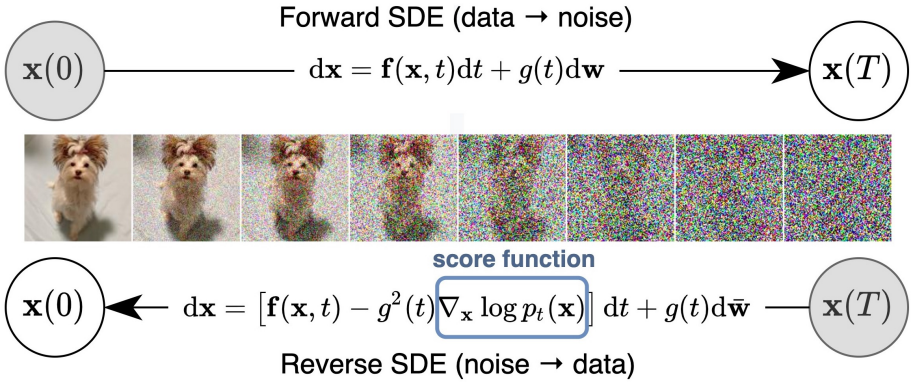
Diffusion Generative Model

(m) DALL-E 2



(n) Forward backwars SDE

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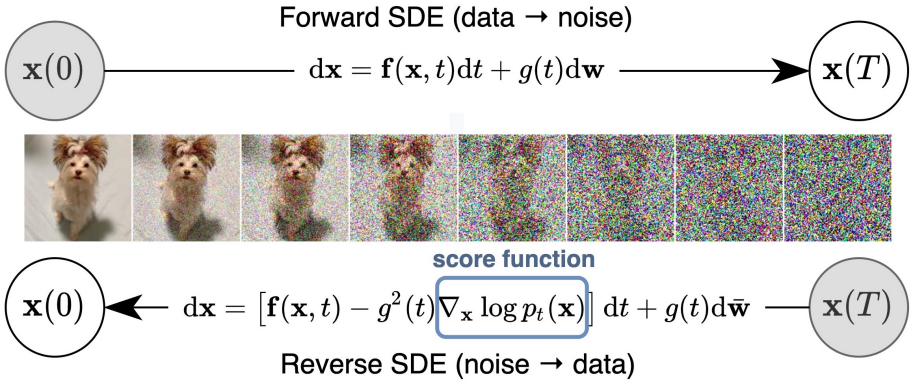

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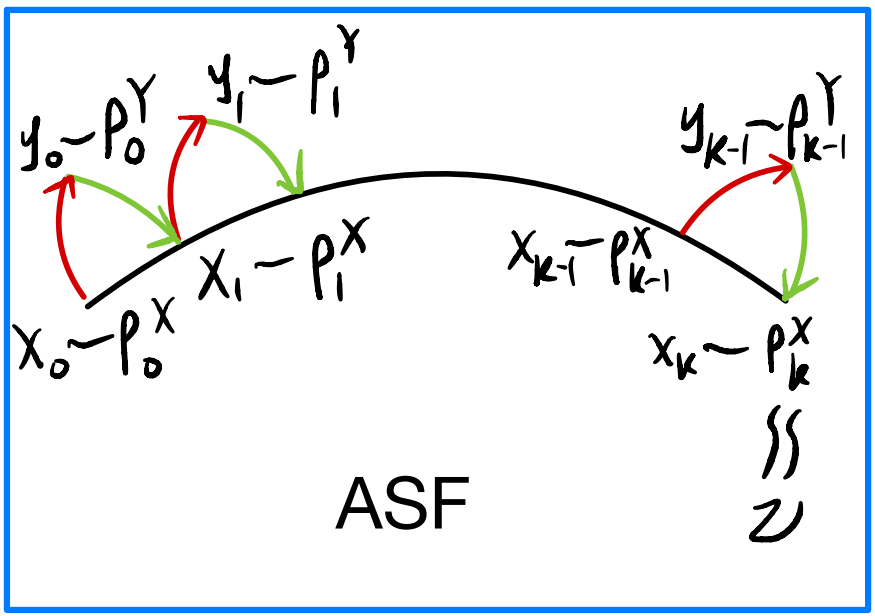

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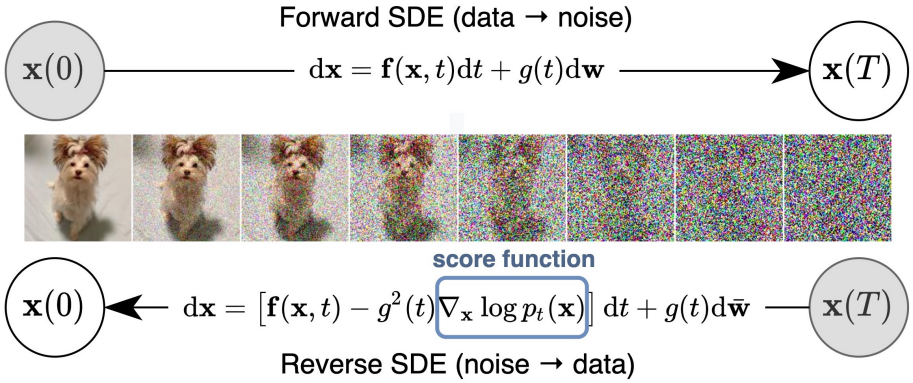

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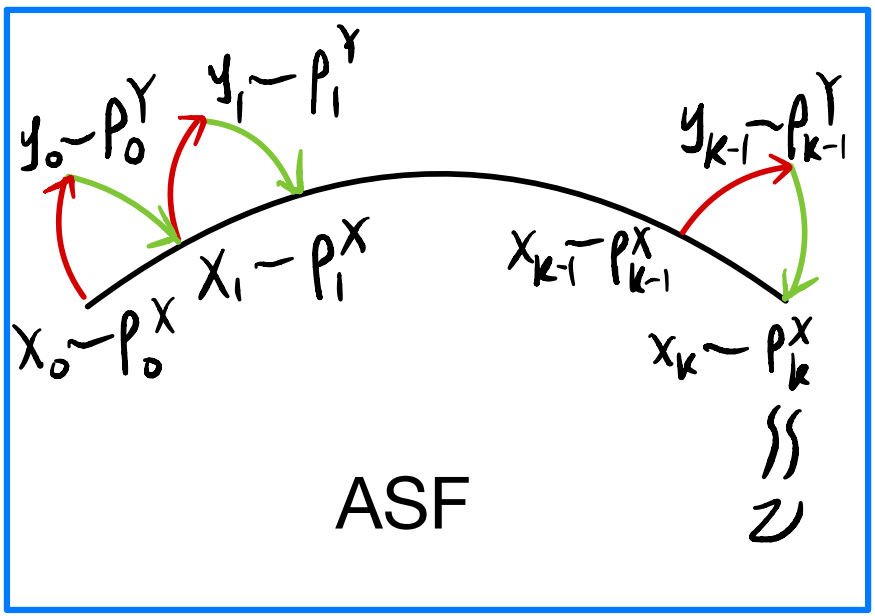

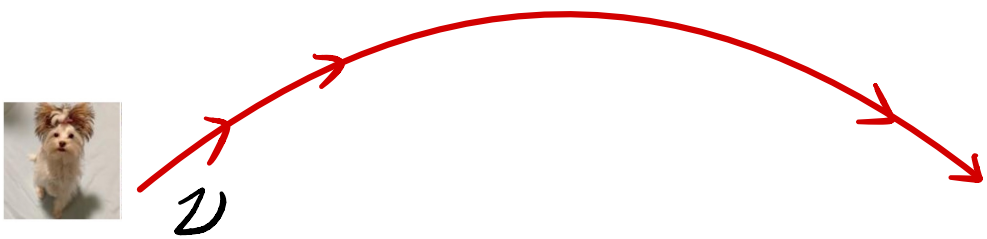

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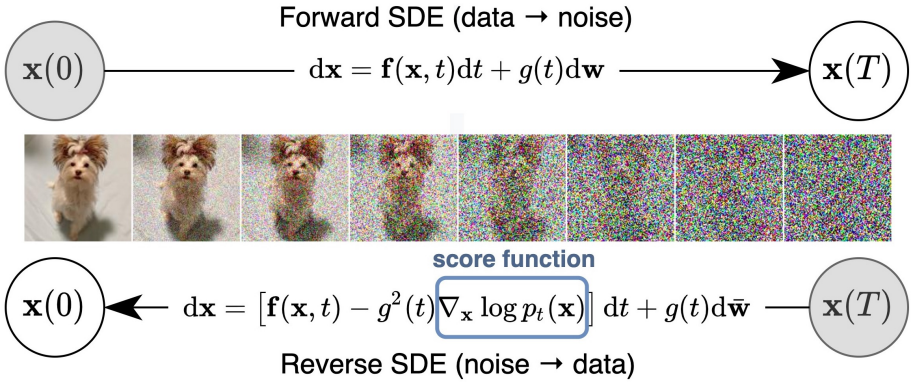

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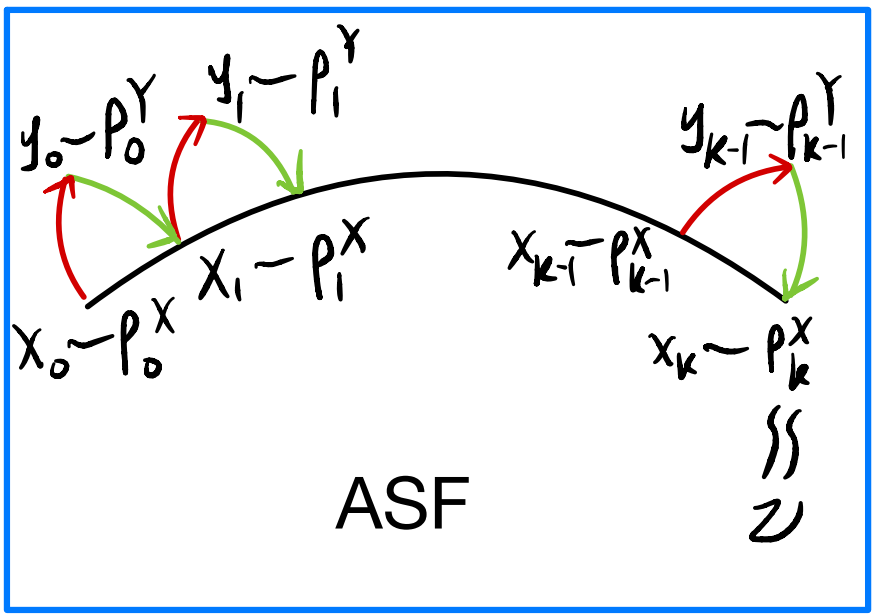

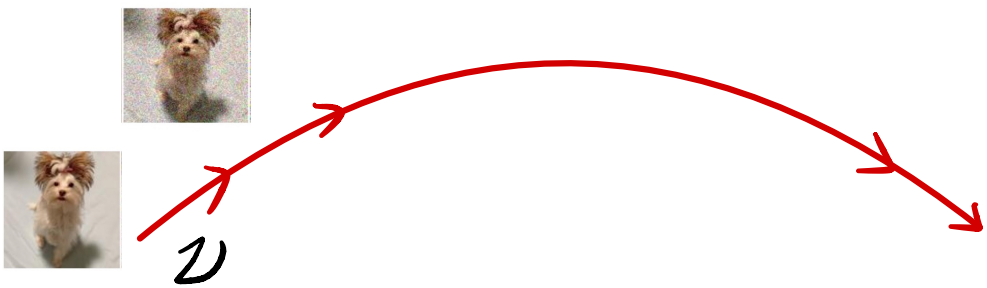

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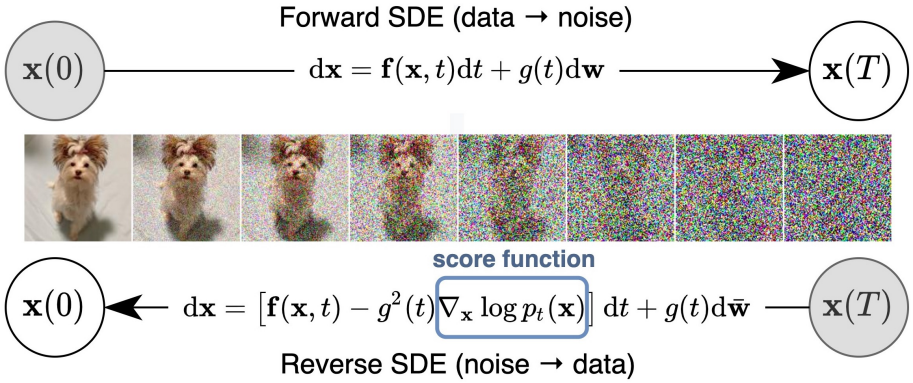

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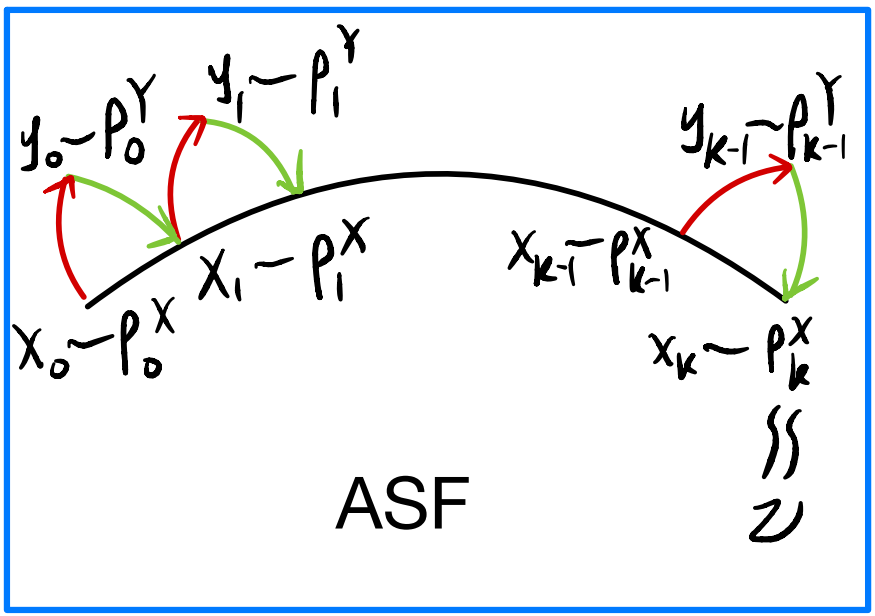

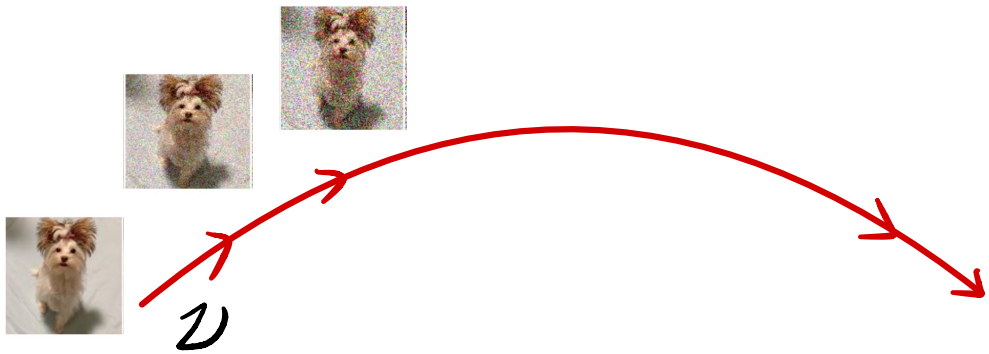

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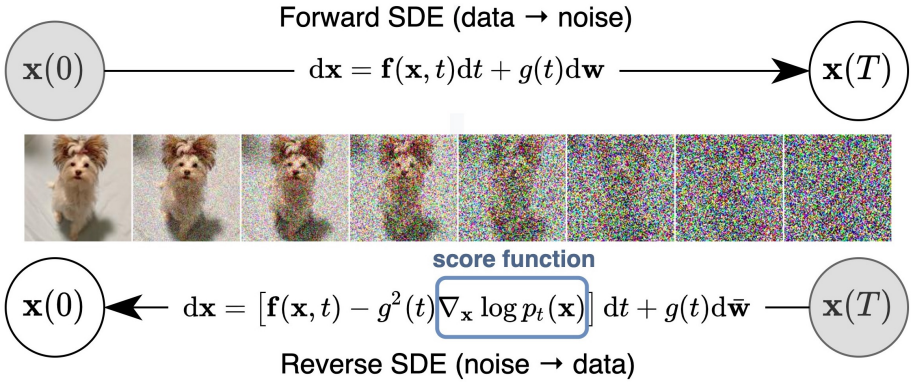

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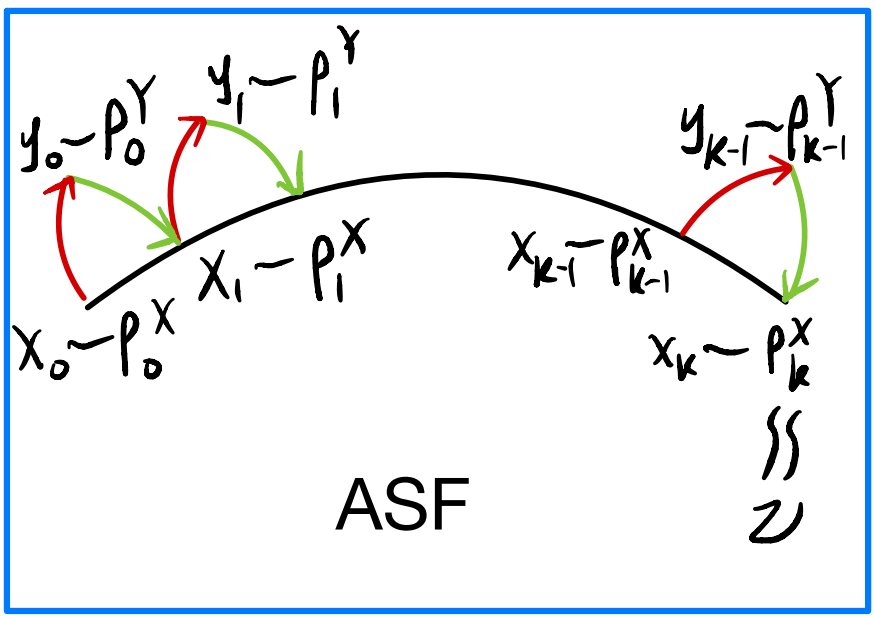

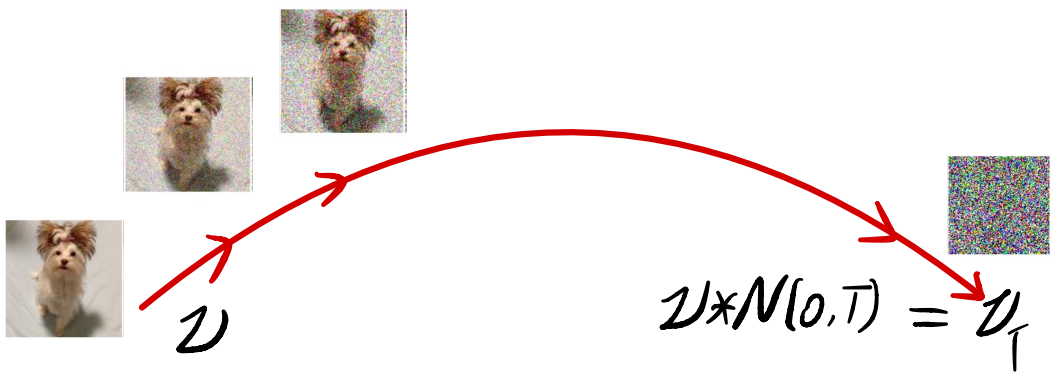

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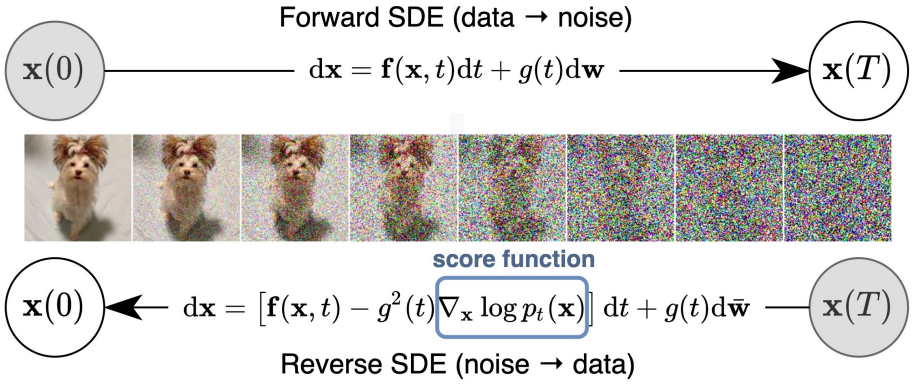
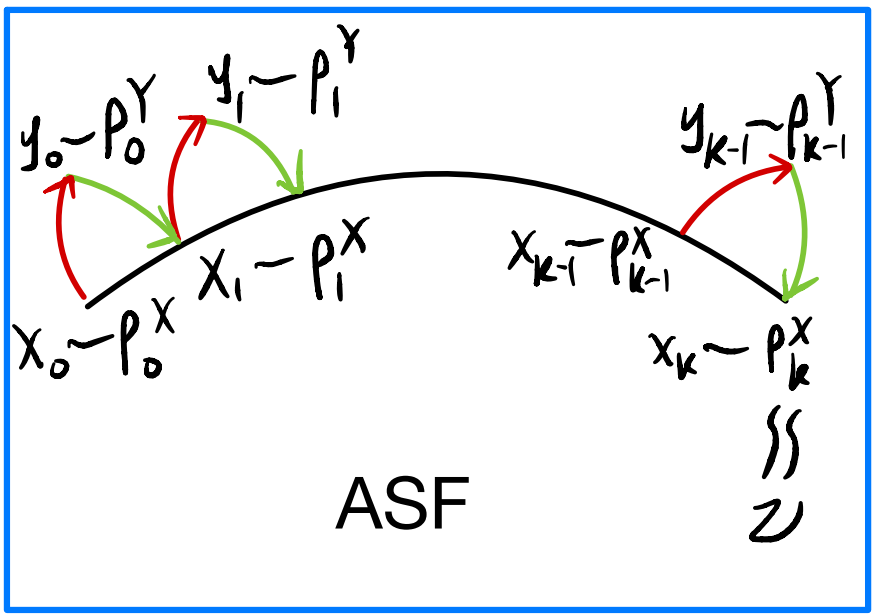
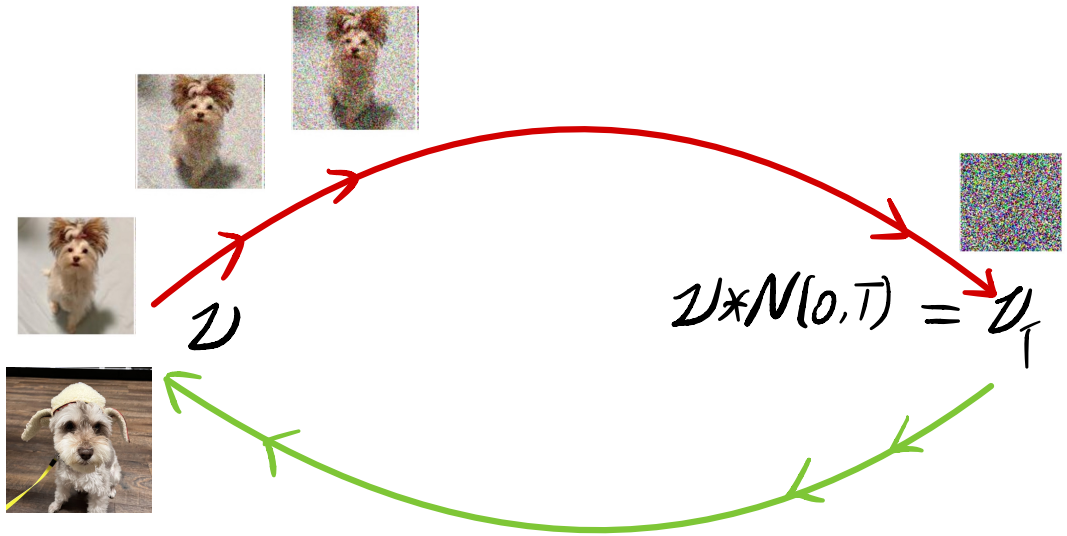


(n) Forward backwars SDE





Conclusion

- A universal proximal framework
  - Nonsmooth optimization
  - Stochastic optimization
  - High-dimensional sampling
- Future directions
  - Stochastic optimization beyond classical theory
  - Diffusion generative model
- Optimization and sampling + X

  statistical signal processing, medical imaging, biostatistics, ...

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Thank you!

Verification of GD as (inexact) PPF

Proximal mapping in optimization

$$\text{prox}_{\lambda f}(y) = \underset{x}{\text{argmin}} \left\{ f(x) + \frac{1}{2\lambda}\|x - y\|^2 \right\}$$

Approximate $f(x) \approx f(y) + \langle \nabla f(y), x - y \rangle$

$$\text{GD:} \quad x_{k+1} = x_k - \lambda \nabla f(x_k)$$

Algorithm Gradient Descent

---

1. $y_k \leftarrow \underset{x}{\text{argmin}} \frac{1}{2\lambda}\|x - x_k\|^2 = x_k$
2. $x_{k+1} \leftarrow \underset{x}{\text{argmin}} \left\{ f(y_k) + \langle \nabla f(y_k), x - y_k \rangle + \frac{1}{2\lambda}\|x - y_k\|^2 \right\}$

$$x_{k+1} = y_k - \lambda \nabla f(y_k) = x_k - \lambda \nabla f(x_k)$$

Verification of ULA as (inexact) ASF

Restricted Gaussian oracle in sampling

$$\mathsf{RGO}_{\lambda f}(y) \sim \exp\left(-f(x) - \frac{1}{2\lambda}\|x-y\|^2\right)$$

Approximate $f(x) \approx f(y) + \langle \nabla f(y), x - y\rangle$

$$\text{ULA:} \quad y_{k+1} = y_k - \eta \nabla f(y_k) + \sqrt{2\eta} z, \quad z \sim \mathcal{N}(0, I)$$

Algorithm Unadjusted Langevin Algorithm

---

1. Sample $y_k \sim \pi^{Y|X}(y \mid x_k) \propto \exp[-\frac{1}{2\eta}\|x_k - y\|^2]$
2. Sample $x_{k+1} \sim \exp[-\frac{1}{2\eta}\|x - y_k + \eta\nabla f(y_k)\|^2]$

$$x_{k+1} = y_k - \eta \nabla f(y_k) + \sqrt{\eta} z_k, \quad z_k \sim N(0, I)$$
$$y_{k+1} = x_{k+1} + \sqrt{\eta} z'_k, \quad z'_k \sim \mathcal{N}(0, I)$$

Hence,

$$y_{k+1} = y_k - \eta \nabla f(y_k) + \sqrt{\eta}(z_k + z'_k) \overset{d}{=} y_k - \eta \nabla f(y_k) + \sqrt{2\eta} z, \quad z \sim \mathcal{N}(0, I)$$

Exploration-Exploitation Trade-Off

- RPB consists of a sequence of proximal problems. $\leftarrow$ Exploration
- Each proximal problem is solved by an iterative procedure. $\leftarrow$ Exploitation
- Exploration: $\mathcal{O}\left(\frac{d_0^2}{\lambda\varepsilon}\right)$, exploitation: $\mathcal{O}\left(\frac{\lambda M^2}{\varepsilon}\right)$.
- Smaller $\lambda \implies$ more exploration and less exploitation.
- If $\lambda = \frac{\varepsilon}{M^2}$, then RPB reduces to the subgradient method.

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Extension - Universal Method (L. and Monteiro, 2021)

The nonsmooth setting:

$$\|f'(u) - f'(v)\| \leq 2M$$

A more general setting:

$$\|f'(u) - f'(v)\| \leq 2M_\nu + L_\nu \|u - v\|^\nu, \quad \nu \in [0, 1]$$

Upper bound for convex

$$\mathcal{O}\left(\frac{M_\nu^2 d_0^2}{\varepsilon^2} + \left(\frac{L_\nu}{\varepsilon}\right)^{\frac{2}{\nu+1}} d_0^2\right)$$

Upper bound for strongly convex

$$\mathcal{O}\left(\left(\frac{M_\nu^2}{\mu\varepsilon} + \frac{L_\nu^{\frac{2}{\nu+1}}}{\mu\varepsilon^{\frac{1-\nu}{1+\nu}}}\right)\log\frac{\mu d_0^2}{\varepsilon}\right)$$

Cutting-plane Model in the Stochastic Setting

A straightforward fact:

$$\mathbb{E}[\max\{X, Y\}] \geq \max\{\mathbb{E}[X], \mathbb{E}[Y]\}.$$

For a fixed u,

$$\mathbb{E}[f_j(u)] \geq \max_{0 \leq i \leq j-1} \{\mathbb{E}[F(x_i, \xi_i) + \langle s(x_i, \xi_i), u - x_i \rangle]\}.$$

On the other hand,

$$\begin{aligned} f(u) &\geq \max_{0 \leq i \leq j-1} \{f(x_i) + \langle f'(x_i), u - x_i \rangle\} \\ &\geq \max_{0 \leq i \leq j-1} \{\mathbb{E}[F(x_i, \xi_i) + \langle s(x_i, \xi_i), u - x_i \rangle]\} \end{aligned}$$

So

$$\mathbb{E}[f_j(u)] \ ? \ f(u)$$

Single Cut Model in the Stochastic Setting

Aggregate all cuts into a single one

$$\Gamma^+(u) = \tau\Gamma(u) + (1-\tau)[F(x,\xi) + \langle s(x,\xi), u - x\rangle].$$

Since

$$\mathbb{E}[F(x,\xi) + \langle s(x,\xi), u - x\rangle] = f(x) + \langle f'(x), u - x\rangle \leq f(u),$$

we have by induction

$$\mathbb{E}[\Gamma^+(u)] \leq f(u).$$

Proximal Interpretation

Recall in the deterministic setting

$$x^+ \leftarrow \min_{u \in \mathbb{R}^n} \left\{ f(u) + h(u) + \frac{1}{2\lambda} \|u - x^c\|^2 \right\}.$$

Now in the stochastic setting

$$\Gamma_j(u) = \sum_{i=0}^{j-1} [F(x_i, \xi_i) + \langle s(x_i, \xi_i), u - x_i \rangle], \quad Q_j(u) = \sum_{i=0}^{j-1} F(u; \xi_i),$$

and we have

$$\Gamma_j(u) \leq Q_j(u), \quad \mathbb{E}\left[Q_j(u)\right] = f(u).$$

Approximately solve the proximal problem by an iterative process

$$x_j = \operatorname{argmin} \left\{ \Gamma_j(u) + h(u) + \frac{1}{2\lambda} \|u - x_j\|^2 \right\}.$$

Test 2 – One-stage Stochastic Program

$$\min_{x \in X} \mathbb{E}\left[\phi\left(\sum_{i=1}^{n}(\frac{i}{n} + \xi_i)x_i\right)\right]$$

where X is the unit simplex.

Table: $n = 100,\ N = 4000$

Statistics	RSA	SCPB1	SCPB2
λ	2.8×10^{-5}	10^{-3}	10^{-3}
Min Inner	1	1	2
Max Inner	1	26	6
Avg Inner	1	17	2

Test 2 – One-stage Stochastic Program

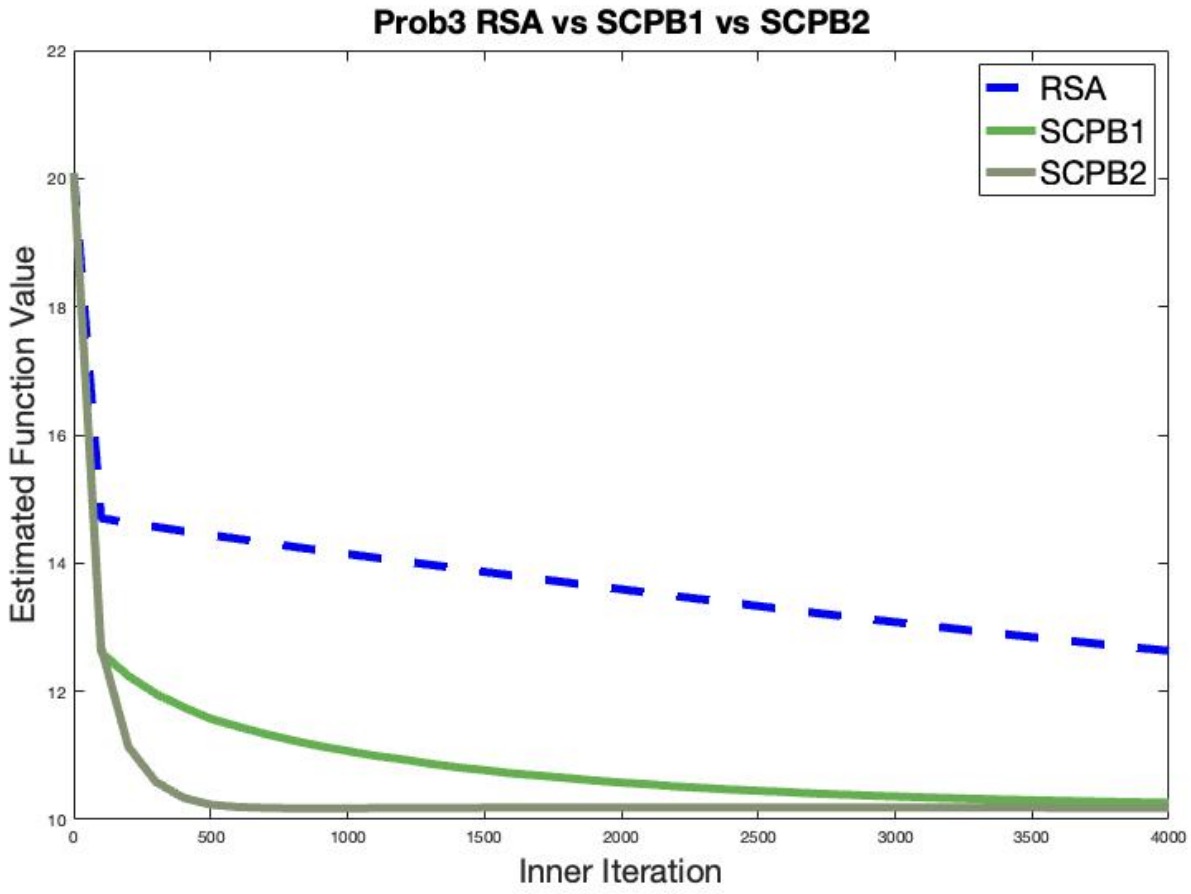

Image Deconvolution – Bayesian Model Selection

Goal: Recover an image x from noisy observation $y = Hx + w$

- Total variation prior $p(x)$, statistical model $\mathcal{M}$ with likelihood function $p(y|x)$, Bayes' rule

$$\pi(x) \triangleq p(x|y) = \frac{p(y|x)p(x)}{\int p(y|x)p(x)\mathrm{d}x}.$$

- Posterior distribution $p(x|y)$

$$\pi(x) \propto \exp\left[-\frac{1}{2\sigma^2}\|y - Hx\|^2 - \beta\|x\|_{\mathsf{TV}}\right] = \exp(-f(x)).$$

Generate samples from $\pi(x)$ to compute $P(M_i|y)$ by Monte Carlo.