A Proximal Bundle Type Method for Smooth and Nonsmooth Convex Optimization and Stochastic Programming

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This talk is based on the following papers:

- J. Liang and R. D. C. Monteiro. A unified analysis of a class of proximal bundle methods for solving hybrid convex composite optimization problems. Available on arXiv:2110.01084, 2021.
- J. Liang and R. D. C. Monteiro. A proximal bundle variant with optimal iteration-complexity for a large range of prox stepsizes. To appear in SIAM Journal on Optimization, available on arXiv:2003.11457, 2020.

The first **optimal** complexity result for a PB type method.

Introduction

Main problem:

$$\phi_* := \min \{\phi(x) := f(x) + h(x) : x \in \mathbb{R}^n\} \tag{1}$$

Main goal:

To present a framework consisting of most proximal bundle methods for hybrid convex composite optimization.

Hybrid (smooth-nonsmooth) convex composite problem

Consider (1), where

(A1) $f, h \in \overline{\mathrm{Conv}}\,(\mathbb{R}^n)$ are such that $\mathrm{dom}\, h \subset \mathrm{dom}\, f$ and a subgradient oracle $f' : \mathrm{dom}\, h \to \mathbb{R}^n$ satisfying $f'(x) \in \partial f(x)$ for every $x \in \mathrm{dom}\, h$ is available;

(A2) the set of optimal solutions X^* of problem (1) is nonempty;

(A3) $\|f'(u) - f'(v)\| \leq 2M_f + L_f\|u - v\|$ for every $u, v \in \mathrm{dom}\, h$;

(A4) h is μ-convex.

In a previous paper [1], we proposed a relaxed proximal bundle (RPB) method that is optimal for convex (and strongly convex) nonsmooth optimization.

In this work, we generalize and improve RPB in the following aspects:

1. hybrid cases;
2. a general framework including 3 bundle update schemes;
3. a unified and much simpler analysis;
4. stronger complexity results;
5. an adaptive PB method.

[1] J. Liang and R. D. C. Monteiro. A proximal bundle variant with optimal iteration-complexity for a large range of prox stepsizes. To appear in SIAM Journal on Optimization, available on arXiv:2003.11457, 2020.

Proximal bundle method

Proximal point method: constructs a sequence of proximal subproblems. E.g., Chambolle-Pock for saddle point, ADMM for distributed OPT.

Solving the proximal problem

$$x^+ \leftarrow \min_{u \in \mathbb{R}^n} \left\{ \phi(u) + \frac{1}{2\lambda}\|u - x\|^2 \right\} \tag{2}$$

can be as difficult as solving $\min\{\phi(u) : u \in \mathbb{R}^n\}$.

Proximal bundle method approximately solves (2) and recursively builds up a model by using a standard cutting-plane approach.

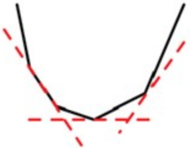
Proximal **bundle** method

The **bundle method** solves a sequence of prox subproblems of the form

$$x_j = \underset{u\in\mathbb{R}^n}{\operatorname{argmin}} \left\{ \Gamma_j^\lambda(u) := f_j(u) + h(u) + \frac{1}{2\lambda}\|u - x_{j-1}^c\|^2 \right\}, \qquad (3)$$

where x_{j-1}^c is the **prox-center**, f_j is the **cutting-plane** model defined as

$$f_j(u) = \max\{f(x) + \langle f'(x), u - x\rangle : x \in C_j\} \quad \forall u \in \mathbb{R}^n.$$



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where x_{j-1}^c is the **prox-center**, f_j is the **cutting-plane** model defined as

$$f_j(u) = \max\{f(x) + \langle f'(x), u - x \rangle : x \in C_j\} \quad \forall u \in \mathbb{R}^n,$$

and decides to perform a **serious** or **null** iteration based on the **descent condition** $\phi\left(x_j\right) \leq (1-\gamma)\phi\left(x_{j-1}^c\right) + \gamma\left(f_j + h\right)\left(x_j\right)$ for some $\gamma \in (0,1)$.

A generic bundle update scheme

Definition

Let $\mathcal{C}_\mu(\phi)$ denote a class of convex functions Γ satisfying $\Gamma \le \phi$ and Γ is μ-convex.

For a given quadruple $(\Gamma, x_0, \lambda, \tau) \in \mathcal{C}_\mu(\phi) \times \mathbb{R}^n \times \mathbb{R}_{++} \times (0,1)$, the generic bundle update scheme returns $\Gamma^+ \in \mathcal{C}_\mu(\phi)$ satisfying

$$\tau \bar{\Gamma} + (1-\tau)[\ell_f(\cdot; x) + h] \le \Gamma^+ \tag{4}$$

where

$$x = \operatorname*{argmin}_{u \in \mathbb{R}^n} \left\{ \Gamma(u) + \frac{1}{2\lambda}\|u - x_0\|^2 \right\}$$

and $\bar{\Gamma} \in \mathcal{C}_\mu(\phi)$ is such that

$$\bar{\Gamma}(x) = \Gamma(x), \quad x = \operatorname*{argmin}_{u \in \mathbb{R}^n} \left\{ \bar{\Gamma}(u) + \frac{1}{2\lambda}\|u - x_0\|^2 \right\}. \tag{5}$$

Examples

(E1) **one-cut scheme:** $\Gamma^+ = \Gamma^+_\tau := \tau\Gamma + (1-\tau)[\ell_f(\cdot;x) + h]$ with $\bar\Gamma = \Gamma$.

(E2) **two-cuts scheme:** assume $\Gamma = \max\{A_f, \ell_f(\cdot;x^-)\} + h$ where A_f is an affine function satisfying $A_f \le f$, set

$$\Gamma^+ = \max\{A_f^+, \ell_f(\cdot;x)\} + h$$

where $A_f^+ = \theta A_f + (1-\theta)\ell_f(\cdot;x^-)$ and

$$\theta \begin{cases} = 1, & \text{if } A_f(x) > \ell_f(x;x^-), \\ = 0, & \text{if } A_f(x) < \ell_f(x;x^-), \\ \in [0,1], & \text{if } A_f(x) = \ell_f(x;x^-). \end{cases}$$

Also set $\bar\Gamma = A_f^+ + h$.

Examples

(E3) **multiple-cuts scheme:** assume $\Gamma = \Gamma(\cdot; C)$ where $C \subset \mathbb{R}^n$ is the current bundle set and $\Gamma(\cdot; C) := \max\{\ell_f(\cdot; c) : c \in C\} + h$, choose the next bundle set C^+ satisfying

$$C(x) \cup \{x\} \subset C^+ \subset C \cup \{x\}, \quad C(x) := \{c \in C : \ell_f(x; c) + h(x) = \Gamma(x)\},$$

and then set $\Gamma^+ = \Gamma(\cdot; C^+)$ and $\bar{\Gamma} = \Gamma(\cdot; C(x))$.

Generic proximal bundle (GPB) framework

0. Let $x_0 \in \operatorname{dom} h$, $\lambda > 0$, $\bar\varepsilon > 0$ and $\tau \in [\bar\tau, 1)$ be given where

$$\bar\tau = \left[1 + \frac{(1+\lambda\bar\mu)\bar\varepsilon}{8\lambda T_{\bar\varepsilon}}\right]^{-1}, \quad T_{\bar\varepsilon} := \left(\bar M_f^2 + \bar\varepsilon \bar L_f\right)^{1/2}, \tag{6}$$

and set $y_0 = x_0$, $t_0 = 0$ and $j = 0$;

1. if $t_j \le \bar\varepsilon/2$, then perform a **serious update**, i.e., set $x_{j+1}^c = x_j$ and find $\Gamma_{j+1} \in \mathcal{C}_\mu(\phi)$ such that $\Gamma_{j+1} \ge \ell_f(\cdot; x_j) + h$; else, perform a **null update**, i.e., set $x_{j+1}^c = x_j^c$ and find $\Gamma_{j+1} \in \mathcal{C}_\phi(\Gamma_j, x_j^c, \lambda, \tau)$;

2. compute

$$x_{j+1} = \operatorname*{argmin}_{u \in \mathbb{R}^n} \left\{ \Gamma^{\lambda}_{j+1}(u) := \Gamma_{j+1}(u) + \frac{1}{2\lambda}\|u - x^c_{j+1}\|^2 \right\}, \qquad (7)$$

choose $y_{j+1} \in \{x_{j+1}, y_j\}$ such that

$$\phi^{\lambda}_{j+1}(y_{j+1}) = \min\left\{\phi^{\lambda}_{j+1}(x_{j+1}), \phi^{\lambda}_{j+1}(y_j)\right\} \qquad (8)$$

where ϕ^{λ}_j is defined as

$$\phi^{\lambda}_j := \phi + \frac{1}{2\lambda}\|\cdot - x^c_j\|^2, \qquad (9)$$

and set

$$m_{j+1} = \Gamma^{\lambda}_{j+1}(x_{j+1}), \quad t_{j+1} = \phi^{\lambda}_{j+1}(y_{j+1}) - m_{j+1}; \qquad (10)$$

3. set $j \leftarrow j+1$ and go to step 1.

GPB vs. standard bundle method

- introduce an auxiliary iterate y_j, convergence in $\{y_j\}$
- null/serious iterate decision making based on t_j
- motivation for y_j and t_j:
  define $m_j^* := \min\{\phi_j^\lambda(u) : u \in \mathbb{R}^n\}$, then we have

$$m_j \le m_j^* \le \phi_j^\lambda(y_j),$$

  and hence

$$\phi_j^\lambda(y_j) - m_j^* \le t_j \le \frac{\bar{\varepsilon}}{2}$$

  where $t_j = \phi_j^\lambda(y_j) - m_j$.

RPB can be viewed as GPB with bundle update scheme (E3).

While RPB only deals with the nonsmooth case ($L_f = 0$), GPB extends the analysis to the hybrid case ($L_f \geq 0$).

Complexity for GPB variants

Theorem

Let $x_0 \in \operatorname{dom} h$, $\bar\varepsilon > 0$ and $C > 0$ be given. Then, any variant of GPB with input $(x_0, \lambda, \bar\varepsilon, \tau)$ satisfying

$$\tau = \left[1 + \frac{(1+\lambda\mu)\bar\varepsilon}{8\lambda(M_f^2 + \bar\varepsilon L_f)}\right]^{-1}, \quad \frac{\bar\varepsilon}{C(M_f^2 + \bar\varepsilon L_f)} \le \lambda \le \frac{C d_0^2}{\bar\varepsilon}, \tag{11}$$

has $\bar\varepsilon$-iteration complexity given (up to a logarithmic term) by

$$\mathcal{O}_1\left(\min\left\{\frac{(M_f^2 + \bar\varepsilon L_f)d_0^2}{\bar\varepsilon^2}, \left(\frac{M_f^2 + \bar\varepsilon L_f}{\mu\bar\varepsilon} + 1\right)\log\left(\frac{\mu d_0^2}{\bar\varepsilon} + 1\right)\right\}\right). \tag{12}$$

Complexity for τ-free GPB variants in the nonsmooth case

Theorem

Let $x_0 \in \operatorname{dom} h$, $\bar{\varepsilon} > 0$ and $C > 0$ be given. Then, any variant of the τ-free GPB subclass with input $(x_0, \lambda, \bar{\varepsilon})$ satisfying

$$\frac{\bar{\varepsilon}}{CM_f^2} \leq \lambda \leq \frac{Cd_0^2}{\bar{\varepsilon}}, \tag{13}$$

has $\bar{\varepsilon}$-iteration complexity given (up to a logarithmic term) by

$$\mathcal{O}_1\left(\min\left\{\frac{M_f^2 d_0^2}{\bar{\varepsilon}^2}, \left(\frac{M_f^2}{\mu\bar{\varepsilon}} + 1\right)\log\left(\frac{\mu d_0^2}{\bar{\varepsilon}} + 1\right)\right\}\right). \tag{14}$$

Complexity of RPB in the strongly convex case

Theorem

Let $x_0 \in \mathrm{dom}\, h$, $\bar{\varepsilon} > 0$ *and* $C > 0$ *be given, and assume*

$$\frac{CM_f d_0}{\bar{\varepsilon}} \geq 1, \qquad 0 \leq \mu \leq \frac{CM_f}{d_0}. \tag{15}$$

Then, RPB with input $(x_0, \lambda, \bar{\varepsilon})$ *satisfying*

$$\frac{d_0}{M_f} \leq \lambda \leq \frac{Cd_0^2}{\bar{\varepsilon}} \tag{16}$$

has $\bar{\varepsilon}$*-iteration complexity given by*

$$\mathcal{O}_1\left(\min\left\{\frac{M_f^2 d_0^2}{\bar{\varepsilon}^2}, \left(\frac{M_f^2}{\mu\bar{\varepsilon}} + 1\right)\log\left(\frac{\mu d_0^2}{\bar{\varepsilon}} + 1\right)\right\}\right). \tag{17}$$

Complexity of RPB in the convex case

Theorem

Let $x_0 \in \mathrm{dom}\, h$, $\bar{\varepsilon} > 0$ *and* $C > 0$ *be given, and assume*

$$\frac{CM_f d_0}{\bar{\varepsilon}} \geq 1, \qquad M_h \leq CM_f, \qquad \mu = 0. \tag{18}$$

Then, RPB with input $(x_0, \lambda, \bar{\varepsilon})$ *satisfying*

$$\frac{\bar{\varepsilon}}{CM_f^2} \leq \lambda \leq \frac{Cd_0^2}{\bar{\varepsilon}} \tag{19}$$

has $\bar{\varepsilon}$*-iteration complexity given by* $\mathcal{O}_1(M_f^2 d_0^2/\bar{\varepsilon}^2)$.

One-cut Adaptive Proximal Bundle (1C-APB)

0. Let $x_0 \in \mathrm{dom}\, h$, $\lambda > 0$, $\tau_0 = 0$ and $\bar{\varepsilon} > 0$ be given, and set $y_0 = x_0$, $t_0 = 0$ and $j = 0$;

1. set $\tau = \tau_j$;

2. if $t_j \le \bar{\varepsilon}/2$, then perform a **serious update**, i.e., set $x^c_{j+1} = x_j$ and $\Gamma_{j+1} = \ell_f(\cdot; x_j) + h$; else, perform a **null update**, i.e., set $x^c_{j+1} = x^c_j$ and $\Gamma_{j+1} = \tau\Gamma_j + (1-\tau)[\ell_f(\cdot; x_j) + h]$;

3. compute x_{j+1}, y_{j+1}, m_{j+1} and t_{j+1} as in step 2 of GPB;

4. if $t_j > \bar{\varepsilon}/2$ and $t_{j+1} > \tau t_j + (1-\tau)\bar{\varepsilon}/4$, then set $\tau = (1+\tau)/2$ and go to step 2; else, set $\tau_{j+1} = \tau$ and $j \leftarrow j+1$, and go to step 1.

Complexity of 1C-APB

The general $\bar{\varepsilon}$-iteration complexity for 1C-APB is

$$\tilde{\mathcal{O}}\left(\left[\frac{\lambda_\mu(M_f^2+\bar{\varepsilon}L_f)}{\bar{\varepsilon}}+1\right]\left[\min\left\{\frac{d_0^2}{\lambda\bar{\varepsilon}},\frac{1}{\mu\lambda_\mu}\right\}+1\right]\right).$$

Under same assumptions as in previous theorems of GPB, 1C-APB has the same iteration complexity as GPB.
The total number of times τ is updated in step 4 is at most

$$\left\lceil\log\left(1+\frac{8\lambda_\mu(M_f^2+\bar{\varepsilon}L_f)}{\bar{\varepsilon}}\right)\right\rceil.$$

Extension: Stochastic Proximal Bundle Method

Assume f is deterministic and f has stochastic first-order oracle $s(x;\xi)$ such that

(A5) $\mathbb{E}\left[s\left(x;\xi\right)\right] = f'(x) \in \partial f(x)$ and $\mathbb{E}\left[\left\|s\left(x;\xi\right) - f'(x)\right\|^2\right] \le \sigma^2$.

It is uncertain whether $\mathbb{E}[\Gamma] \le \phi$ if Γ is as in (E2) or (E3), hence the standard PB method is unsuitable to handle stochastic problems.

On the other hand, it is easy to verify that $\mathbb{E}[\Gamma] \le \phi$ if Γ is as in (E1), i.e., one-cut model.

Another extension: sampling from a density $\propto \exp(-f(x))$ where f is a convex and Lipschitz continuous function.
Complexity $\tilde{\mathcal{O}}(d\varepsilon^{-1})$, the best in the high accuracy regime, i.e., $\varepsilon < d^{-1/3}$.

Concluding remarks

- A generic GPB framework for hybrid convex composite optimization
- Including most proximal bundle variants such as RPB, and a novel one based on the one-cut model
- A unified and simple analysis and stronger complexity results
- An adaptive variant that requires no prior knowledge of problem parameters
- A stochastic proximal bundle method based on the one-cut model

THE END

Thanks!