The Fine Structure of L

In this note we will briefly outline the fine structure theory of L. As a preliminary we have the well-known Condensation Lemma:

Lemma 1. *There exists a finite consistent $T \subseteq \mathcal{L}_\in$ such that, for any transitive $M \models T$, $M = L_\alpha$ for $\alpha \in Ord \cup \{Ord\}$. In particular, if α is a limit ordinal and $\pi : M \to L_\alpha$ is a Σ_1-elementary embedding between transitive models, then $M = L_\beta$ for some $\beta \leqslant \alpha$.*

Proof. ... □

As an example, consider the following case: Assume $V = L$ and $\pi : L_\beta \to L_{\omega_1}$ be fully elementary, with β countable. Then there exists $\gamma \geqslant \beta$ such that $L_{\gamma+1} \models |\beta| = \omega$. Let γ be the least such ordinal.

We start by switching the L-hierarchy to another, more convenient J-hierarchy. I would believe that this name is made to honor Ronald Jensen, the inventor of this theory:

Definition 1. Let $k < \omega$ and V be a class. A function $f : {}^kV \to V$ is called rudimentary, if it is generated by the following schemata:

- $f(\langle x_1, ..., x_k\rangle) = x_i$;
- $f(\langle x_1, ..., x_k\rangle) = x_i - x_j$;
- $f(\langle x_1, ..., x_k\rangle) = \{x_i, x_j\}$;
- $f(\langle x_1, ..., x_k\rangle) = h(g_1(\langle x_1, ..., x_k\rangle), ..., g_n(\langle x_1, ..., x_k\rangle))$;
- $f(\langle x_1, ..., x_k\rangle) = \bigcup_{y \in x_i} g(y, \langle x_1, ..., x_k\rangle)$.

Let R be a k-ary relation defined on V. We say R is rudimentary iff there exists a rudimentary function $f : {}^kV \to V$ such that $R(\langle x_1, ..., x_k\rangle) \iff f(\langle x_1, ..., x_k\rangle) \neq \varnothing$.

If X is a set, we say $V = rud(X \cup \{X\})$, the rudimentary closure of X, iff V is the minimal set which contains $X \cup \{X\}$ and closed under all rudimentary functions. In other words:

$$rud(X) = \{f(\bar{x}) : f \text{ rudimentary}, \bar{x} \in {}^{<\omega}(X \cup \{X\})\}.$$

Remark. There is a notion of "rudimentary relative to A", where A is a class. That is to say we add the following scheme to the definition:

- $f(\langle x_1, ..., x_k\rangle) = x_1 \cap A$.

We shall not use this definition in this note, but the reader can check that many results in this note can be generalized to this "relative to A" notion.

Observation.
- If X is transitive, then $rud(X)$ is transitive which is a proper superset of X.
- ...

Our first lemma would show that rudimentary functions adds all definable subsets of X:

Lemma 2. $rud(X) \cap \mathcal{P}(X) = \mathbf{\Sigma}_\omega^X \cap \mathcal{P}(X)$.

Proof. First notice that $\mathbf{\Sigma}_\omega^X \cap \mathcal{P}(X) = \Sigma_0^{X \cup \{X\}} \cap \mathcal{P}(X)$. I.e., we just add X into the set of parameters.
$\subseteq$: By an induction on rudimentary functions, we can show that each rudimentary f is **simple**, i.e., for each Δ_0 formula $\phi(\bar{x})$ over a transitive set, there exists a Δ_0 formula $\psi(\bar{x})$ such that $\psi(y, \bar{x}) \iff \phi(y, f(\bar{x}))$ in this transitive set. Now: notice that each set in the lefthand side is of the form $\{y : y \in f(\bar{x})\}$. Let $\phi(y, f(\bar{x}))$ be the formula $y \in f(\bar{x})$ so we are done.
$\supseteq$: Followed by induction we can show that each Δ_0 class relation is rudimentary. $\square$

We now turn to the definition of the J-hierarchy:

Definition 2. The J-hierarchy is defined recursively on all limit ordinals as follows:

- $J_0 = \varnothing$;
- $J_{\alpha+1} = rud(J_\alpha)$;
- If β is a limit ordinals, $J_\beta = \bigcup_{\alpha<\beta} J_\alpha$.
- $J = \bigcup_{\alpha \in \mathsf{Lim}} J_\alpha$.

Observation. Clearly J is a transitive class, and for each $\alpha \in Ord$,

- $Ord \cap J_\alpha = \omega\alpha$;
- $L_\alpha \subseteq J_\alpha \subseteq L_{\omega\alpha}$;
- $L_\alpha = J_\alpha$ iff $\alpha = \omega\alpha$.

As a corollary: $J = L$.[1]

It is then clear that $<_L$ is a well-ordering of J, but we shall further define a well-ordering that respect the J-hierarchy, after we defined a more refined hierarchy as follows:

Definition 3. The base functions are the following 9 rudimentary functions:

- $F_0(x, y) = \{x, y\}$;
- $F_1(x, y) = x - y$;
- $F_2(x, y) = x \times y$;
- $F_3(x, y) = \{(a, b, c) : b \in x \wedge (a, c) \in y\}$;

[1] See pp.255-256 of *Constructibility* by Keith Devlin. It is also mentioned in this book that $J_\alpha = L_{\omega\alpha}$ is not always true.

- $F_4(x, y) = \{(a, b, c) : c \in x \wedge (a, b) \in y\}$;
- $F_5(x, y) = \bigcup x$;
- $F_6(x, y) = dom(x)$;
- $F_7(x, y) = \{(a, b) \in x \times x : a \in b\}$;
- $F_8(x, y) = \{\{b : (a, b) \in x\} : a \in y\}$.

Lemma 3. *Let X be a transitive set, then $\boldsymbol{S}(X)$ is the image of $X \cup \{X\}$ under F_0 to F_8. We have:*

- *$rank(\boldsymbol{S}(X)) = rank(X) + 4$;*
- *$rud(X) = \bigcup_{n\in\omega} \boldsymbol{S}^n(X)$;*
- *There exists a Σ_1-definable*

One advantage that we use the J-hierarchy is that now each level adds more rank to the earlier one. Another advantage is that the Condensation Lemma expands to every level of the hierarchy:

Lemma 4. *There exists a finite consistent $T \subseteq \mathcal{L}_\in$ such that, for any transitive $M \models T$, $M = J_\alpha$ for $\alpha \in Ord$. In particular, if α is a limit ordinal and $\pi : M \to J_\alpha$ is a Σ_1-elementary embedding between transitive models, then $M = J_\beta$ for some $\beta \leqslant \alpha$.*

Now we start to navigate into the fine structure theory, starting from several definitions:

Definition 4. Let β be a limit ordinal. The first projectum $\rho_1^{J_\beta} \leqslant \beta$ is the smallest ordinal such that there exists $A \notin J_\beta$ and $A \in \boldsymbol{\Sigma}_1^{J_\beta} \cap \mathcal{P}(\omega \cdot \rho_1^{J_\beta})$.

Sometimes we shall omit the superscript and just write ρ_1. In the case where $\rho_1 = \beta$, by the well-ordering we just defined, J_β satisfies the Σ_1-collection scheme. In the case where $\rho_1 < \beta$, we have that J_β is **acceptable** defined as follows:

Proposition. There exists a $\boldsymbol{\Sigma_1}^{J_\beta}$ surjection $f : \omega \cdot \rho_1 \to J_\beta$. In particular, its in $J_{\beta+1}$.

This will be a corollary followed easily by the soundness of ρ_1 as we shall show in a while. Also, by the well-ordering on J_β, there exists a uniformly Σ_1-definable surjection $f : [\beta]^{<\omega} \to J_\beta$, and thus we can let A(as in the definition of projectum) to be defined relative to some $p \in [\beta]^{<\omega}$. Since the natural lexicographic ordering on $[\beta]^{<\omega}$ is a well-order, we have:

Definition 5. $p_1 \in [\beta]^{<\omega}$ is the standard parameter of J_β iff p is the $<_{lex}$-least member such that there exists $A \in \Sigma_1^{J_\beta}(p) \cap \mathcal{P}(\rho_1)$ and $A \notin J_\beta$.

Now we state some important results:

Lemma 5. *$\rho_1 = \tau = \sigma$, where:*

- *$\sigma \leqslant \beta$ is the least ordinal such that there exists $r \in J_\beta$, $J_\beta = Hull_1^{J_\beta}(\sigma \cup r)$;*
- *$\tau \leqslant \beta$ is the least ordinal such that there exists $r \in J_\beta$, $Th_1^{J_\beta}(\tau \cup r) \notin J_\beta$.*

Proof.

- $\sigma \leqslant \rho_1$: We need to show that:

$$Hull_1^{J_\beta}(\rho_1 \cup \{p_1\}) = J_\beta.$$

This is called the 1-soundness property. Let π be the inverse of the transitive collapse of $Hull_1^{J_\beta}(\rho_1 \cup \{p_1\})$. Then by condensation, there exists $\bar{\beta}$ such that $Hull_1^{J_{\bar{\beta}}}(\rho_1 \cup \{\pi^{-1}(p_1)\}) = J_{\bar{\beta}}$ holds. We argue that $\bar{\beta} = \beta$ and $\pi^{-1}(p_1) = p_1$: For the first part, notice that if $\bar{\beta} < \beta$, the missing $\Sigma_1(\rho_1 \cup \{p_1\})$ set $A \subseteq \rho_1$ is in $J_{\bar{\beta}+1}$ by the first theorem, which is impossible. The second part is followed by the minimality of p_1.

- $\rho_1 \leqslant \tau$: Notice that if we hide the single parameter r in $Th_1^{J_\beta}(\tau \cup \{r\})$ and just look at the parameters in τ, we would get some A which can be seen as a subset of τ. Notice that A is $\Sigma_1^{J_\beta}(\{r\})$ since $Th_1^{J_\beta}(\tau \cup \{r\})$ is, and from A we can recover $Th_1^{J_\beta}(\tau \cup \{r\})$, so $A \notin J_\beta$, which witnesses that $\rho_1 \leqslant \tau$.

- $\tau \leqslant \sigma$: Otherwise, we can find some $r \in J_\beta$ such that $Th_1^{J_\beta}(\sigma \cup \{r\}) \in J_\beta$ and $Hull_1^{J_\beta}(\sigma \cup \{r\}) = J_\beta$. We shall see later that from $Th_1^{J_\beta}(\sigma \cup \{r\})$ we can build a structure which is isomorphic to J_α which leads to a contradiction.

□

Using similar techniques respectively, we have:

Lemma 6. *$p_1 = q = r$, where:*

- *$q \in [\beta]^{\leqslant\omega}$ is the least such that $J_\beta = Hull_1^{J_\beta}(\rho_1 \cup q)$;*
- *$r \in [\beta]^{\leqslant\omega}$ is the least such that $Th_1^{J_\beta}(\rho_1 \cup r) \notin J_\beta$.*

Using the projectum and the standard parameter, we can code J_β into J_{ρ_1} by adding a predicate:

Definition 6. Let φ be a formula and let $\ulcorner\varphi\urcorner$ be its Gödel number. Let

$$A_1^{J_\beta} = \{(\ulcorner\varphi\urcorner, x) \in \omega \times [\rho_1^{J_\beta}]^{<\omega} : J_\beta \models \phi(x, p_1^{J_\beta}\}.$$

$A_1^{J_\beta}$ is called the Σ_1 master code of J_β. $\mathfrak{C}_1^{J_\beta} = (J_{\rho_1^{J_\beta}}, \in, A_1^{J_\beta})$ is called the Σ_1 coding structure of J_β.

We shall now finish the two proofs above by stating the decoding process: Suppose we now have a Σ_1-coding structure, how should we recover the original structure? We shall define an equivalent relation $\sim$ over $\omega \times \rho_1$ and a membership relation $\dot{\in}$ such that

$$((\omega \times \rho)/\sim, \dot{\in}) \cong (J_\beta, \in),$$

and we would like to define these two relations using just $A_1^{J_\beta}$. Notice that for every element $x \in J_\beta$, it corresponds to a Σ_1 fact $\phi(\bar{\alpha}, p_1)$, such that x is the $<_{J_\beta}$-minimal witness. This is followed by the soundness property of J_β.

Now the problem is: This correspondence is neither injective nor surjective, so we need to deal with it. Now, flip the correspondence for the moment and look at any pair $(n, \bar{\alpha}) \in \omega \times \rho^{<\omega}$. We can ask ourselves the following two questions:

- Let $\ulcorner\phi\urcorner = n$ for some Σ_1 formula ϕ. Does $J_\beta \models \phi(\bar{\alpha}, p_1)$?
- Let $(\ulcorner\psi\urcorner, \bar{\beta})$ be another pair, and suppose the above question has a positive answer. Let x be the $<_{J_\beta}$-least witness of the fact $\phi(\bar{\alpha}, p_1)$. Is x also the $<_{J_\beta}$-least witness of the fact $\psi(\bar{\beta}, p_1)$?

It turns out that the answers are already contained in A: More precisely, for the first question,

$$J_\beta \models \phi(\bar{\alpha}, p_1) \iff (\ulcorner\phi\urcorner, \bar{\alpha}) \in A_1^{J_\beta},$$

and the answer of the second one depends on whether $(\ulcorner\chi_{\phi,\psi}\urcorner, \bar{\alpha}^\frown\bar{\beta}) \in A_1^{J_\beta}$, where

$$\chi_{\phi,\psi}(\bar{\alpha}, \bar{\beta}, p_1) : \phi(\bar{\alpha}, p_1) \wedge \phi(\bar{\beta}, p_1) \wedge \mu_{\phi(\bar{\alpha}, p_1)} = \mu_{\psi(\bar{\beta}, p_1)},$$

where $\mu_{\phi(\bar{\alpha}, p_1)}$ is the $<_{J_\beta}$-least element witnessing the Σ_1-fact $\phi(\bar{\alpha}, p_1)$. It is clear that $\chi_{\phi,\psi}$ is also a Σ_1-formula[2] , and the function $(\ulcorner\phi\urcorner, \ulcorner\psi\urcorner) \mapsto \chi_{\phi,\psi}$ is computable.

By answering these two questions, we can define $\sim$ over $\omega \times \rho^{<\omega}$ as follows: Let $(n, \bar{\alpha}) \simeq (m, \bar{\beta})$ if one of the two cases happens:

- The answer of the first question is false, or the minimal witness is $\varnothing$;
- Both of the two questions are correct.

Using a similar technique, we can define $\dot{\in}$ on all equivalent classes, and by the soundness property of J_β, the defined structure is well-founded, and its transitive collapse is just J_β.

Remark. The decoding process gives a proof of the following lemma naturally:

Lemma 7. *Suppose $\mathfrak{C} = (J_\rho, \in, A)$ and $\mathfrak{D} = (J_{\rho'}, \in, A')$ are two coding structures, and $\pi : \mathfrak{C} \to \mathfrak{D}$ is a Σ_0-embedding. Let the decoding of $\mathfrak{C}$ be J_β, and the decoding of $\mathfrak{D}$ be $J_{\beta'}$. Then there exists a unique Σ_1-embedding $\tilde{\pi} : J_\beta \to J_{\beta'}$ extending π, and $\tilde{\pi}$ sends the standard parameter of J_β to the standard parameter of $J_{\beta'}$.*

This is called the **downward extension lemma**. Similarly we have the **upward extension lemma**:

[2] In order to give $\mu_{\phi(\bar{\alpha}, p_1)}$ a Σ_1-formulation, one needs to use the S-hierarchy as follows: Say $x = \mu_{\phi(\bar{\alpha}, p_1)}$, iff

Lemma 8. *Suppose $\mathfrak{C} = (J_\rho, \in, A)$ is a coding structure and $\mathfrak{D} = (J_{\rho'}, \in, A')$ is any structure with a unary predicate. Let $\pi : \mathfrak{C} \to \mathfrak{D}$ be a Σ_1-embedding such that, if R is a well-founded rudimentary relation defined in $\mathfrak{C}$, then R' is also well-founded, where R' is the rudimentary relation defined in $\mathfrak{D}$ with the same definition.*

Then $\mathfrak{D}$ is also a coding structure.

In order to show the upward extension lemma, one needs more effort. Not only should we define the equivalence relation and the membership relation, we also need to code the Σ_1-satisfaction relation in $\mathfrak{C}$, and send it to $\mathfrak{D}$ using π. It turns out that this would be enough.