Gambler's ruin probability - a general formula

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Abstract

We derive an explicit formula for the probability of ruin of a gambler
playing against an infinitely-rich adversary, when the games have payoff
given by a general integer-valued probability distribution.

1   Introduction

We consider the classical Gambler's Ruin Problem. A gambler, who starts with
an initial wealth M ($M \in \mathbb{N}$), plays a series of games, where in game t the
gambler's payoff is a random variable X_t (independent for different values of t)
with

$$P(X_t = k) = p_k, \ \ k \in \mathbb{Z}, \ \ -\nu \leq k < \infty.$$

$\nu > 0$ is thus the maximal possible loss, and we assume $p_{-\nu} \neq 0$. The series
of games proceeds until the gambler's wealth is less than ν, in which case the
gambler must stop playing, and we say that the gambler is ruined. We are inter-
ested in the probability of ruin P_{ruin}, which depends on the payoff distribution
$\{p_k\}_{k=-\nu}^{\infty}$ and on the initial wealth M. It is well-known that if the expected
value of X_t is non-positive then $P_{ruin} = 1$, so that we can assume that we are
dealing with a favorable game

$$E(X_t) = \sum_{k=-\nu}^{\infty} k p_k > 0. \tag{1}$$

Another trivial case is when the gambler's initial wealth is less than ν, so that
ruin occurs immediately, so we henceforth assume $M \geq \nu$.

We will derive a formula for P_{ruin}, which, surprisingly, seems not to be available
in the literature, except for some very special cases (see discussion below).

We define the generating function

$$p(z) = \sum_{k=-\nu}^{\infty} p_k z^k.$$

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Note that, since $\sum_{k=-\nu}^{\infty} p_k = 1$, the series $p(z)$ defines a meromorphic function in the unit disk $|z| < 1$ of the complex plane, with a unique pole, of order ν, at $z = 0$.

For integers $n > 0, r \geq 0$, the complete symmetric polynomial of order r in the variables $z_1, \cdots, z_n$ is defined as the sum of all products of the variables $z_1, \cdots, z_n$ of degree r, that is

$$\Phi_{n,r}(z_1, \cdots, z_n) = \sum_{i_j \geq 0, \ i_1 + \cdots + i_n = r} \prod_{j=1}^{n} z_j^{i_j}.$$

Theorem 1 *The equation $p(z) = 1$ has ν solutions (counting multiplicities) in the unit disk $|z| < 1$ of the complex plane, which we denote by η_j $(1 \leq j \leq \nu)$. The ruin probability is given by*

$$P_{ruin}(M) = \sum_{n=1}^{\nu} \Phi_{n,M-n+1}(\eta_1, \cdots, \eta_n) \prod_{j=1}^{n-1} (1 - \eta_j). \tag{2}$$

When the roots $\eta_1, \cdots, \eta_\nu$ are distinct, one can use the alternative expression

$$P_{ruin}(M) = \sum_{j=1}^{\nu} \eta_j^M \prod_{i \neq j} \frac{1 - \eta_i}{\eta_j - \eta_i}. \tag{3}$$

We note that empty products, as occurs in (2) for $n = 1$, are to be interpreted as 1. The two expressions (2),(3) are algebraically equivalent, but each has its advantages. (2) allows for the case of multiple roots, and shows that the expression on the right-hand side of (3) is in fact a polynomial in $\eta_1, \cdots, \eta_\nu$. (3) clearly shows that $P_{ruin}(M)$ is is exponentially decaying as a function of M (a well-known fact), with the rate of decay determined by the root η_j with maximal absolute value.

The special case of the gambler's ruin problem in which

$$p_{-1} > 0, \ p_1 > 0, \quad p_k = 0, \ |k| \geq 2, \tag{4}$$

is treated in every textbook of probability theory, and it is shown that

$$P_{ruin}(M) = \left(\frac{p_{-1}}{p_1}\right)^M \tag{5}$$

(note that the assumption (1) implies $p_{-1} < p_1$). (5) is obtained as an especially simple case of (2), because when (4) holds we have $\nu = 1$, $p(z) = p_{-1}z^{-1} + p_0 + p_1 z$, so that $\eta_1 = \frac{p_{-1}}{p_1}$, and (2) gives

$$P_{ruin}(M) = \Phi_{1,M}(\eta_1) = \eta_1^M = \left(\frac{p_{-1}}{p_1}\right)^M.$$

In general, computing P_{ruin} using (2) requires numerically solving $p(z) = 1$ in the unit disk $|z| < 1$.

The gambler's ruin problem, going back to Pascal and Huygens (see [1] and [2], Sec. 7.5 for history), is usually considered together with a version in which the

games also stop if the gambler's fortune exceeds a threshold W (the gambler wins, or the casino is ruined). Our problem corresponds to $W = \infty$, and is often called gambling against an infinitely rich adversary. The result (5) is usually obtained as a limit of an expression for finite W when $W \to \infty$.

Feller ([4], Sec. XIV.8) and Ethier ([2], Sec. 7.2) discuss the problem with an upper threshold W in the case of a payoff distribution $\{p_k\}_{k=-\nu}^{\mu}$, supported both from below and from above. They show that the ruin probabilities can be computed by solving a system of linear equations of size $W - \nu + 1$ for $P_{ruin}(M)$ ($\nu \leq M \leq W$). It is also possible to express $P_{ruin}(M)$ in the form

$$P_{ruin}(M) = \sum_{j=1}^{\nu+\mu} a_j \lambda_j^M, \quad \nu \leq M \leq W, \tag{6}$$

where λ_j ($1 \leq j \leq \nu + \mu$) are *all* the roots of $p(z) = 1$, with the coefficients a_j determined by a linear system of equations of size $\nu + \mu$. Here again there is no general explicit expression for the solution, so that efforts have been devoted to proving upper and lower bounds for P_{ruin}, either in terms of the largest root of the equation $p(z) = 1$ [2, 4], or more recently in terms of the moments of the payoff distribution [3, 5].

One import of Theorem 1 is that the problem with $W = \infty$ is simpler than the case $W < \infty$, in the sense that an explicit formula for the ruin probability can be derived. Comparing (6) (for the case $W < \infty$) and (3) (for the case $W = \infty$), we see that in both formulas we have linear combinations of the M-th powers of roots of the equation $p(z) = 1$, but in (6) the coefficients a_j must be determined by solving a (potentially large) system of linear equations, while in (3) everything is given explicitly. Note also that in (3) *only* those roots which are inside the unit disk are involved. Our derivation does not use (6) or any other result for the case $W < \infty$ - we deal directly with the case $W = \infty$. Our approach relies on the construction of appropriate generating functions and their analysis.

A previous result which comes closest to the formula (3), in a very special case, was obtained by Skala [8], who considered the case

$$p_{-\nu} > 0, \quad p_\mu > 0, \quad p_k = 0, \quad k \neq -\nu, \mu,$$

and showed that the formula (3) holds. Skala used the system of linear equations mentioned above for the case $W < \infty$, and a process of going to the limit $W \to \infty$.

Let us remark that in some contexts 'ruin' would be defined as the event in which the gambler's fortune drops to 0 or less, rather than below ν. Translating the results to this case is trivial: on the right-hand sides of (2),(3), M will be replaced by $M + \nu - 1$.

We conclude this introduction with a numerical example. Let us consider a game in which participation costs \$$\nu$ and with a prize which is Poisson distributed with mean $\nu + \epsilon$ ($\epsilon > 0$). Then $p(z) = z^{-\nu} e^{(\nu+\epsilon)(1-z)}$. We take $\nu = 3$ and $\epsilon = 0.01$. The solutions of $p(z) = 1$ in the open unit disk (found numerically using MAPLE) are then given by: $z_1 = 0.993362$, $z_2 = -.202699 + .220049i$, $z_3 = \bar{z}_2$.

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Using (3) we obtain

$$P_{ruin}(3) = 0.9900, \quad P_{ruin}(10) = 0.9456, \quad P_{ruin}(50) = 0.7245,$$

$$P_{ruin}(100) = 0.5193, \quad P_{ruin}(200) = 0.2668, \quad P_{ruin}(500) = 0.0361.$$

2 Proof of the ruin probability formula

Defining S_t to be the gambler's wealth after the t-th game, we have $S_0 = M$, and, for all integer $t \geq 0$

$$S_{t+1} = \begin{cases} S_t + X_t, & S_t \geq \nu \\ S_t, & 0 \leq S_t < \nu \end{cases} . \tag{7}$$

We define the sequence S_t so that if ruin occurs at some time t_0, then $S_t = S_{t_0} \in \{0, 1, \cdots, \nu - 1\}$ for all $t \geq t_0$, or, in other words, $\{0, 1, \cdots, \nu - 1\}$ are absorbing states of the Markov process S_t. Thus the probability of ruin at or before time t is given by $\sum_{k=0}^{\nu-1} P(S_t = k)$, and the probability of ultimate ruin is

$$P_{ruin} = \lim_{t \to \infty} \sum_{k=0}^{\nu-1} P(S_t = k). \tag{8}$$

For later use, we remark that another way to express (7) is

$$P(S_{t+1} = k) = \begin{cases} \sum_{l=-\nu}^{k-\nu} p_l \cdot P(S_t = k - l), & k \geq \nu \\ P(S_t = k) + \sum_{l=-\nu}^{k-\nu} p_l \cdot P(S_t = k - l), & 0 \leq k < \nu \end{cases} . \tag{9}$$

With each of the random variables S_t we associate its generating function

$$f_t(z) = E(z^{S_t}) = \sum_{k=0}^{\infty} P(S_t = k) z^k. \tag{10}$$

Note that $f_0(z) = z^M$. Since the coefficients $P(S_t = k)$ are non-negative and their sum is 1, the series (10) converges uniformly in the closed unit disk $|z| \leq 1$ of the complex plane, and thus defines a continuous function in the closed unit disk, which is holomorphic in the open unit disk, with

$$|z| \leq 1 \quad \Rightarrow \quad |f_t(z)| \leq 1. \tag{11}$$

Denoting by $\mathcal{P}$ the vector space of all power series, $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we define the truncation operator $T : \mathcal{P} \to \mathcal{P}$,

$$T \left[\sum_{k=0}^{\infty} c_k z^k \right] = \sum_{k=0}^{\nu-1} c_k z^k.$$

Lemma 1 *The sequence of polynomials (of degree $\nu - 1$) $\{T[f_t]\}_{t=0}^{\infty}$ converges to a polynomial $Q(z)$*

$$Q(z) = \lim_{t \to \infty} T[f_t](z), \quad \forall z \in \mathbb{C}, \tag{12}$$

and

$$P_{ruin} = Q(1). \tag{13}$$

Proof: We have

$$T[f_t](z) = \sum_{k=0}^{\nu-1} P(S_t = k)z^k, \tag{14}$$

and since, for each $0 \le k \le \nu - 1$, $P(S_k = t)$ is increasing as a function of t and bounded by 1, we have convergence of each of the coefficients of $T[f_t](z)$ as $t \to \infty$, so that the limiting degree-$(\nu - 1)$ polynomial exists, and (12) holds. In view of (8), (12) and (14), we have (13). ∎

We will find P_{ruin} by finding an explicit expression for $Q(z)$ and taking $z = 1$. We note that the coefficient of z^k ($0 \le k \le \nu - 1$) in $Q(z)$ is the probability that ruin occurs with final fortune k.

We now derive a recursive formula relating f_{t+1} to f_t.

Lemma 2 *For all $t \ge 0$,*

$$f_{t+1}(z) = T[f_t](z) + p(z)\Big[f_t(z) - T[f_t](z)\Big]. \tag{15}$$

Proof: We show that, for each integer $k \ge 0$, the coefficient of z^k in the power series $f_{t+1}(z)$, is equal to the coefficient of z^k in the power series on the right-hand of (15). Using the notation $[z^k]g(z)$ to refer to the coefficient of z^k in a power series $g(z)$, we have

$$[z^k]f_{t+1}(z) = P(S_{t+1} = k), \tag{16}$$

$$[z^k](T[f_t](z)) = \begin{cases} 0, & k \ge \nu \\ P(S_t = k), & 0 \le k \le \nu - 1 \end{cases}, \tag{17}$$

$$[z^k]\left(p(z)\Big[f_t(z) - T[f_t](z)\Big]\right) = \sum_{l=-\nu}^{k-\nu} p_l \cdot P(S_t = k - l), \tag{18}$$

and using (9) we see that the right-hand side of (16) is equal to the sum of the right-hand sides of (17) and (18). ∎

Introducing the linear operator $L : \mathcal{P} \to \mathcal{P}$ defined by

$$L[f](z) = T[f](z) + p(z)\Big[f(z) - T[f](z)\Big], \tag{19}$$

Lemma 2 tells us that, for all $t \ge 0$,

$$f_{t+1} = L[f_t]. \tag{20}$$

For each $w \in [0, 1)$, we define the function

$$F_w(z) = \sum_{t=0}^{\infty} w^t f_t(z).$$

By (11), the above series converges uniformly in $|z| \le 1$ for each $w \in [0, 1)$, and since each f_t is a continuous function in the closed unit disk, holomorphic in its interior, F_w also has these properties. We note also that the functions $w \to F_w(z)$ (z fixed) are continuous, uniformly for z in compact subsets of the unit disk, which in particular implies that the mapping $w \to T[F_w](z)$ from $[0, 1)$ to the space of degree-$(\nu - 1)$ polynomials is continuous, a fact that will be used below.

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Lemma 3 *For any $z \in \mathbb{C}$ we have*

$$Q(z) = \lim_{w \to 1-} (1-w)T[F_w](z), \qquad (21)$$

where $Q(z)$ is the polynomial defined by (12).

Proof: Fix z. Set
$$c_t = T[f_t](z) - T[f_{t-1}](z), \quad t \geq 1.$$

By Lemma 1, we have

$$\sum_{t=1}^{\infty} c_t = -T[f_0](z) + \lim_{t \to \infty} T[f_t](z) = Q(z) - T[f_0](z).$$

Therefore Abel's theorem on power series ([7] Theorem 8.2) implies that

$$\lim_{w \to 1-} \sum_{t=1}^{\infty} c_t w^t = Q(z) - T[f_0](z). \qquad (22)$$

For any $w \in [0,1)$ we have

$$
\begin{aligned}
(1-w)\sum_{t=0}^{\infty} w^t T[f_t](z) &= T[f_0](z) + \sum_{t=1}^{\infty} w^t \Big[T[f_t](z) - T[f_{t-1}](z)\Big] \\
&= T[f_0](z) + \sum_{t=1}^{\infty} c_t w^t. \qquad (23)
\end{aligned}
$$

Taking the limit $w \to 1-$ in (23), using (22), we get

$$\lim_{w \to 1-} (1-w)\sum_{t=0}^{\infty} w^t T[f_t](z) = Q(z).$$

Since

$$T[F_w](z) = \sum_{t=0}^{\infty} w^t T[f_t](z),$$

we have (21). ∎

In view of Lemma 3, we now want to find explicit expressions for the polynomials $T[F_w](z)$ ($w \in [0,1)$). We need the following result on the solutions of the equation

$$p(z) = w^{-1}. \qquad (24)$$

Lemma 4 *For any $w \in (0,1]$, (24) has ν roots (counted with multiplicities) $z = z_j(w)$ ($1 \leq j \leq \nu$) in the open unit disk $|z| < 1$.*

Proof: We first derive an a-priori bound for all solutions. Restricting the function $p(z)$ to the interval $(0,1]$, we have, using the fact that p_k are non-negative, that $p(z)$ a convex function with $\lim_{z \to 0+} p(z) = +\infty$, $p(1) = 1$, $p'(1) = E(X_t) > 0$, so that elementary calculus implies that there is a unique $z^* \in (0,1)$ with

$$p(z^*) = 1, \quad p(z) < 1 \text{ for } z \in (z^*, 1). \qquad (25)$$

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We claim that any solution $|z| < 1$ of $p(z) = w^{-1}$, $w \in (0, 1]$, satisfies $|z| \leq z^*$. Indeed if z satisfies $p(z) = w^{-1}$ then

$$p(|z|) = \sum_{k=-\nu}^{\infty} p_k |z|^k \geq \left| \sum_{k=-\nu}^{\infty} p_k z^k \right| = |p(z)| = w^{-1} \geq 1,$$

which, by (25), implies $|z| \leq z^*$.

Defining

$$h(z) = z^\nu p(z) = \sum_{k=0}^{\infty} p_{k-\nu} z^k,$$

which is continuous on the closed unit disk and holomorphic in the open unit disk, we can write the equation $p(z) = w^{-1}$ in the form

$$z^\nu = wh(z). \tag{26}$$

When $w = 0$, $z = 0$ is a zero of multiplicity ν of (26). Since, as shown above, all solutions of (26) in the unit disk satisfy the $|z| < z^* < 1$, we conclude by the argument principle that (26) has ν solutions for all $w \in [0, 1]$. ∎

Lemma 5 *For any $w \in (0, 1)$, $|z| \leq 1$,*

$$T[F_w](z) = \frac{1}{1-w} \sum_{n=1}^{\nu} \Phi_{n,M-n+1}(z_1(w), \cdots, z_n(w)) \prod_{j=1}^{n-1} (z - z_j(w)), \tag{27}$$

where $z_j(w)$ ($1 \leq j \leq \nu$) are the solutions of $p(z) = w^{-1}$ in the unit disk $|z| < 1$, ordered arbitrarily.

If we further assume that $z_j(w)$ ($1 \leq j \leq \nu$) are different from each other then we can also write

$$T[F_w](z) = \frac{1}{1-w} \sum_{j=1}^{\nu} (z_j(w))^M \prod_{i \neq j} \frac{z - z_i(w)}{z_j(w) - z_i(w)}. \tag{28}$$

Proof: Using the linearity of the operator L defined by (19), and the relation (20), we have

$$L[F_w](z) = \sum_{t=0}^{\infty} w^t L[f_t](z) = \sum_{t=0}^{\infty} w^t f_{t+1}(z) = w^{-1} \sum_{t=1}^{\infty} w^t f_t(z) = w^{-1}[F_w(z) - f_0(z)],$$

which we can write as

$$F_w(z) = wL[F_w](z) + f_0(z),$$

or, using the definition of L,

$$F_w(z) = wT[F_w](z) + wp(z)\Big[F_w(z) - T[F_w](z)\Big] + f_0(z).$$

Solving for F_w, we have

$$F_w(z) = \frac{w(1 - p(z))T[F_w](z) + f_0(z)}{1 - p(z)w}. \tag{29}$$

Since $F_w(z)$ is holomorphic for $|z| < 1$, the numerator of the right-hand side of (29) must vanish whenever the denominator does (otherwise we would have a pole), that is

$$|z| < 1, \ 1 - p(z)w = 0 \ \Rightarrow \ w(1 - p(z))T[F_w](z) + f_0(z) = 0.$$

In other words, denoting by $z_j(w)$ $(1 \le j \le \nu)$ the solutions of (24) (recall Lemma 4) we have

$$w(1 - p(z_j(w)))T[F_w](z_j(w)) + f_0(z_j(w)) = 0, \quad 1 \le j \le \nu,$$

and using $wp(z_j(w)) = 1$, we can rewrite this as

$$T[F_w](z_j(w)) = \frac{1}{1-w} f_0(z_j(w)), \quad 1 \le j \le \nu. \tag{30}$$

We will now temporarily assume that w is chosen so that $z_j(w)$ $(1 \le j \le \nu)$ are *distinct*. This holds for all but a discrete set of values of $w \in [0,1)$, since if $p(z) = w^{-1}$ has a multiple root z then $p'(z) = 0$, so that w^{-1} is a critical value of p, and the set of critical values of a nonconstant holomorphic function is discrete.

$T[F_w](z)$ is a polynomial of degree $\nu - 1$, and since (30) prescribes the values of this polynomial at the ν points $z_j(w)$ $(1 \le j \le \nu)$, we can use Newton's interpolation formula [6] to write this polynomial as

$$T[F_w](z) = \frac{1}{1-w} \sum_{n=1}^{\nu} f_0[z_1(w), \cdots, z_n(w)] \prod_{j=1}^{n-1} (z - z_j(w)), \tag{31}$$

where $f[z_1(w), \cdots, z_n(w)]$ is the n-th divided difference. We now use the following result ([6], Theorem 1.2.1): when $f_0(z) = z^M$ and $0 \le n \le M$,

$$f_0[z_1, \cdots, z_n] = \Phi_{n,M-n+1}(z_1, \cdots, z_n). \tag{32}$$

Substituting (32) into (31), we obtain (27).

We have proved (27) under the assumption that $z_j(w)$ are distinct, but we may now approximate any value of w for which some $z_j(w)$ coincide by values of w for which $z_j(w)$ are distinct, and use the continuity of both sides of (27) to conclude that it is valid for all w.

If instead of using Newton's interpolation formula we use Lagrange's formula, we obtain (again under the assumption that $z_j(w)$ are distinct)

$$T[F_w](z) = \frac{1}{1-w} \sum_{j=1}^{\nu} f_0(z_j(w)) \prod_{i \ne j} \frac{z - z_i(w)}{z_j(w) - z_i(w)}.$$

Recalling that $f_0(z) = z^M$, we get (27). This formula, of course, does not make sense when $z_j(w)$ are not distinct. ∎

We are now ready to obtain the

Proof of Theorem 1: Fixing any $|z| \leq 1$, (21) and (27) imply,

$$
\begin{aligned}
Q(z) &= \lim_{w \to 1-} (1-w)T[F_w](z) \tag{33} \\
&= \lim_{w \to 1-} \sum_{n=1}^{\nu} \Phi_{n,M-n+1}(z_1(w), \cdots, z_n(w)) \prod_{j=1}^{n-1}(z - z_j(w)).
\end{aligned}
$$

By Lemma 4, we have

$$
\begin{aligned}
&\lim_{w \to 1-} \sum_{n=1}^{\nu} \Phi_{n,M-n+1}(z_1(w), \cdots, z_n(w)) \prod_{j=1}^{n-1}(z - z_j(w)) \tag{34} \\
&= \sum_{n=1}^{\nu} \Phi_{n,M-n+1}(\eta_1, \cdots, \eta_n) \prod_{j=1}^{n-1}(z - \eta_j),
\end{aligned}
$$

where η_j $(1 \leq j \leq \nu)$ are the roots of $p(z) = 1$ in the open unit disk. Combining (33) and (34) we have

$$
Q(z) = \sum_{n=1}^{\nu} \Phi_{n,M-n+1}(\eta_1, \cdots, \eta_n) \prod_{j=1}^{n-1}(z - \eta_j),
$$

and setting $z = 1$ and using (13) gives (2).

If we assume that $\eta_1, \cdots, \eta_\nu$ are distinct, which implies that $z_1(w), \cdots, z_\nu(w)$ are distinct for w sufficiently close to 1, then the same argument, using (28) instead of (27), leads to

$$
Q(z) = \sum_{j=1}^{\nu} \eta_j^M \prod_{i \neq j} \frac{z - \eta_i}{\eta_j - \eta_i},
$$

and substituting $z = 1$ gives (3). ∎

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