MATA35 Course Notes

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Claude Shannon once told me (Manuel Blum) that as a kid,
he remembered being stuck on a jigsaw puzzle.
His brother, who was passing by, said to him:
"You know: I could tell you something."
That's all his brother said.
Yet that was enough hint to help Claude solve the puzzle.
The great thing about this hint…
is that you can always give it to yourself!
I advise you, when you're stuck on a hard problem,
to imagine a little birdie or an older version of yourself whispering
"… I could tell you something…"

——*Manuel Blum*

Contents

1. Preliminaries **6**
- 1.1. Sets 6
- 1.2. Functions 6
  - 1.2.1. Polynomials 6
  - 1.2.2. Trigonometric Functions 7
- 1.3. Limits 7

2. Integration **9**
- 2.1. U-Substitution 11
- 2.2. Integration By Parts 12
  - 2.2.1. Tabular Method 13
- 2.3. Initial Value Problem (IVP) 14

3. Matrix Operations **15**
- 3.1. Eigenvalues and Eigenvectors 15

4. Multi-Variable Calculus **15**

5. Regression Analysis **15**

6. Differential Equations (DE) **15**
- 6.1. First Order DE 15
- 6.2. Second Order DE 15
- 6.3. Stability of DE Solutions 15

7. Non-linear System of Equations **15**

8. Appendix **16**
- 8.1. Commonly Used Integration Formula 16
- 8.2. Trigonometric Identities 17

1. Preliminaries

Definition is essential. Without giving a precise, clear definition of objects, it is impossible to have an academic conversation or develop rigorous concepts. However, the imperfection of reality imposes restriction on our ability of giving objects definitions: *we cannot give everything a definition.* Our journey begins as such.

1.1. Sets

Definition. A *set* is a "collection" of objects.[1]

Definition. Objects that are comprise the set are *element.* If X is a set, we denote x is an element of X as $x \in X$.

We may describe a set by either listing all the elements of the set, for example

$$F = \{\text{apple}, \text{banana}, \text{orange}\},$$

or characterizing the properties of the elements in the set, for example

$$A = \{x \in \mathbb{R} : x < 1\}.$$

Remark. We can also use *interval* notation to describe sets of real numbers. For any $a, b \in \mathbb{R}, a < b$,

$$\begin{aligned} (a, b) &= \{x \in \mathbb{R} : a < x < b\} \\ [a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \\ (a, b] &= \{x \in \mathbb{R} : a < x \leq b\} \\ [a, b) &= \{x \in \mathbb{R} : a \leq x < b\}. \end{aligned}$$

Particularly, (a, b) is called an *open interval* and $[a, b]$ is called a *closed interval.*

Definition. Given two sets X, Y, X is a *subset* of Y, written as $X \subseteq Y$, if for every element $x \in X$, $x \in Y$.

Definition. The *complement* of a set $X \subseteq Y$, denoted by X^c, is given by

$$X^c = \{x \in Y : x \notin X\}$$

1.2. Functions

Naively speaking, function is a black box that gives an output for every input. We often write $f : X \to Y$ to denote f is a function maps elements in X to some element in Y, where X is called the domain and Y is a called the codomain (aka. range) of f. There are some functions you may get familiar with:

1.2.1. Polynomials

Definition. A *polynomial* is function of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

for some natural number n. Each of $a_0, a_1, \ldots, a_n$ are called the *coefficients* of the polynomial p. The highest degree of monomial with non-zero coefficient is called the *degree* of the polynomial.

[1] This clearly, is not a definition —— it defines set using another notion, "collection". We may ask ourself: What is a collection? What is an objects? We can continue this Socratic-styled interrogation and use up words eventually. Therefore, we will take the definition of sets for granted.

Definition. Given a polynomial $p : \mathbb{R} \to \mathbb{R}$, a real number r is called the *root* of p if $p(r) = 0$.

Example. $f(x) = x^2 + 5x - 6$ is a polynomial if degree 2. It has roots $x = -6, 1$.

Moreover, I want to emphasis on the root finding problem of polynomials in degree 2, that is, of the form $p(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ with $a \neq 0$. There are three ways of finding the root of $p(x)$ that we should know:

1. **Quadratic Formula.** Let the discriminant $\Delta = b^2 - 4ac$. If $\Delta > 0$, then the root r of $p(x)$ satisfies:
$$r = \frac{-b \pm \sqrt{\Delta}}{2a}.$$
When $\Delta > 0$, $p(x)$ has two distinct roots, on the contrary of $\Delta = 0$, $p(x)$ has one repeated root.
2. **Complete the Square.** Write the polynomial into the form $p(x) = a(x + \alpha)^2 + \beta$. The root is given by
$$r = \pm \frac{\beta}{a} - \alpha.$$
3. **Factorization.** Write the the polynomial in the form of $p(x) = a(x - \alpha)(x - \beta)$. Then the roots are given by $r = \alpha, \beta$.

1.2.2. Trigonometric Functions

Trigonometric functions are defined using the unite circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

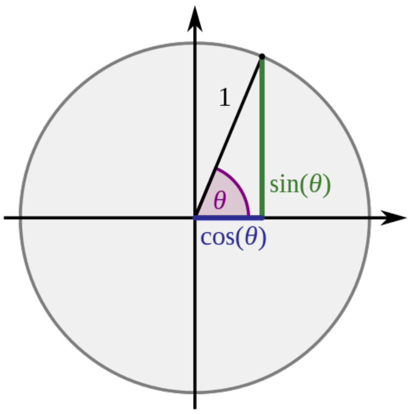

Figure 1: Definition of $\sin(\theta)$ and $\cos(\theta)$ on unite circle S^1.

Let $p = (x, y)$ be a point on the unit circle S^1. The line goes through p and the origin forms an angle θ. We define $\cos(\theta) \equiv x$ and $\sin(\theta) \equiv y$. There are some basic trigonometric identities you should familiar with, check **Appendix 3.2** for details.

1.3. Limits

Definition. Let $f : A \to \mathbb{R}, x \mapsto f(x)$ be a function with $a \in \mathbb{R}$ be a limit point of A. Given $\ell \in \mathbb{R}$, we write
$$\lim_{x \to a} f(x) = \ell$$
if and only if
$$\forall \varepsilon > 0, \exists \delta > 0, s.t. 0 < |x - a| < \delta \Rightarrow |f(x) - \ell\ | < \varepsilon.$$

Remark.

- The limit point a does not necessarily be in the domain A of f. Considering the example

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

  The function $f(x) = \frac{x^2-1}{x-1}$ has domain $A = \{x : x \in \mathbb{R} - \{1\}\}$, we can immediately see the fact that $1 \notin A$ but the it is a valid limit point of f.
- FYI, we may seen this definition as "if x is infinitesimally approaching to a, then $f(x)$ is infinitesimally approaching to ℓ." However, this definition leads to the *Second Mathematical Crisis*.

Proposition. Basic limit properties.

Let $f, g : A \to \mathbb{R}$ be two functions, let $a \in \mathbb{R}$ be a limit point. If

$$\lim_{x \to a} f(x) = \ell, \lim_{x \to a} g(x) = m$$

then we have

- $\lim_{x \to a} f(x) \pm g(x) = \ell \pm m$,
- $\lim_{x \to a} f(x)g(x) = \ell m$,
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\ell}{m}$.

2. Integration

** The continuous version of summation.*

Given a linear function, can easily compute its underline area, which is just trapezoid or triangle.

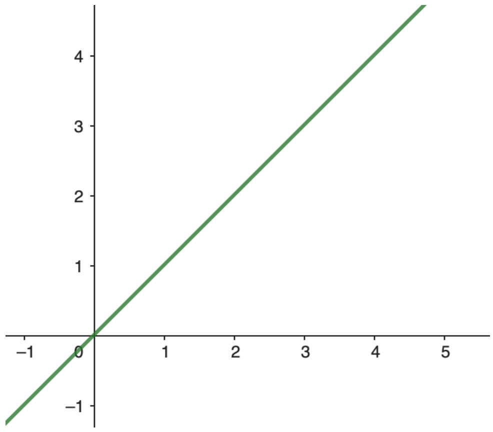

Figure 2: A linear function $f(x) = x$

Problem of area finding arises when the function of interest is no longer linear: what if we want to find the underline area of an sinusoid, or an exponential function?

Motivation. Consider a function $f : [a, b] \to \mathbb{R}$ defined on a closed interval $[a, b]$, where $a < b$. We define the *signed area* A of f by the following:

- area above the x-axis are positive, and
- area below the x-axis are negative.

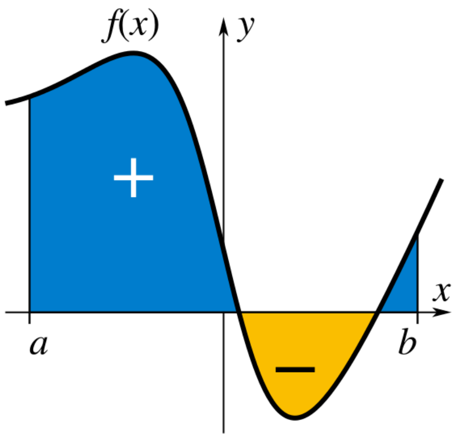

Figure 3: Signed area of $f(x)$.

We write

$$A = \int_a^b f(x)\,\mathrm{d}x.$$

The symbol $\int$ is meaning for "integration", and $\mathrm{d}x$, also known as *differential*, is used to reminding us x is the variable we are integrating with. Irrigorously speaking, we may seen $\mathrm{d}x$ as an infinitesimal increment of x.

Remark. Integration and differentiation are inverse to each other. Consider a function $f(x)$, it is always feasible† to have

$$f(x) = \int \left[\frac{\mathrm{d}}{\mathrm{d}x} f(x) \right] \mathrm{d}x.$$

(†) Unless f is not differentiable for all points on its domain.

Definition. Define $i \in \{1, ..., n\}$, $\Delta x = \frac{b-a}{n}$, and for every $x_i^* \in [x_i, x_i + \Delta x]$, the infinite integral of f can be approximated by *Riemann sum*:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} f(x_i^*) \Delta x.$$

As $n \to \infty$, we can accurately calculate the definite integral of f, that is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \Delta x.$$

Remark. This formula characterized the intuition of *Riemann integral*: we can approximate the value of the indefinite integral of a given function f by chopping the underline area of f into n small rectangles, summing up the area of each rectangle gives the approximation. The more rectangles we chopped, the more accurate the approximation we have.

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$, we call

$$F(x) \equiv \int f(x)\,\mathrm{d}x$$

the *antiderivative* of f.

Example. Given $f(x) = \sin(x)$,

$$F(x) = -\cos(x) + c$$

is the antiderivative of f, where $c \in \mathbb{R}$ is an arbitrary constant. We can verify this fact by computing $F'(x)$.

Proposition (Properties of Integration).

- For arbitrary continuous function $f, g : \mathbb{R} \to \mathbb{R}$,

$$\int f(x) + g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

- For arbitrary continuous function $f : \mathbb{R} \to \mathbb{R}$ and scalar $a \in \mathbb{R}$,

$$\int a f(x)\,\mathrm{d}x = a \int f(x)\,\mathrm{d}x.$$

- For arbitrary integrable function $f : \mathbb{R} \to \mathbb{R}$, interval $[a, b] \subseteq \mathbb{R}$, $c \in [a, b]$,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Example. Consider $f(x) = e^{-x}$ and $g(x) = 6x^2$. We can see that

$$\begin{aligned}\int f(x) + 2g(x)\,\mathrm{d}x &= \int e^{-x} + 12x^2\,\mathrm{d}x \\ &= -e^{-x} + 4x^3 + c,\end{aligned}$$

and

$$\begin{aligned}\int f(x)\,\mathrm{d}x + 2\int g(x)\,\mathrm{d}x &= \int e^{-x}\,\mathrm{d}x + 2\int 6x^2\,\mathrm{d}x \\ &= -e^{-x} + c_1 + 2 \times 2x^3 + c_2 \\ &= -e^{-x} + 4x^3 + c\end{aligned}$$

Theorem (Fundamental Theorem of Calculus). If $f : [a, b] \to \mathbb{R}$ is continuous over $[a, b]$ and F is an antiderivative of f, then

$$\int_b^a f(x)\,\mathrm{d}x = F(b) - F(a).$$

2.1. U-Substitution

Theorem (U-Substitution). Let $f : [a, b] \to \mathbb{R}$ and $G : [\alpha, \beta] \to \mathbb{R}$, by carefully choosing a function $u(x)$, we can write the definite integral of f in terms of G and u, that is:

$$\int_a^b f(x)\,\mathrm{d}x = \int_\beta^\alpha \frac{G(u)}{u'(x)}\,\mathrm{d}u.$$

Example. Consider

$$\int_0^{\frac{\pi}{3}} \tan(x)\,\mathrm{d}x.$$

We will compute this definite integral step by step:

1. **Choose an appropriate $u(x)$** : Let $u(x) = \cos(x)$.
2. **Compute write $f(x)$ in terms of $\frac{G(u)}{u'(x)}$** : $\frac{G(u)}{u'(x)} = \frac{1}{u}$.
3. **Find the changed upper and lower bound of definite integral:** $\alpha = u(0) = 1, \beta = u\left(\frac{\pi}{3}\right) = \frac{1}{2}$.
4. **Compute the definite integral** by putting all together:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \tan(x)\,\mathrm{d}x &= \int_0^{\frac{\pi}{3}} \tan(x)\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{3}} \frac{\sin(x)}{\cos(x)}\,\mathrm{d}x \\ &= \int_{\frac{1}{2}}^1 \frac{1}{u}\,\mathrm{d}u \\ &= [\ln|u|]_{\frac{1}{2}}^0 \\ &= \ln(1) - \ln\left(\frac{1}{2}\right) \\ &= \ln(2)\end{aligned}$$

2.2. Integration By Parts

Theorem (Integration By Parts). Sometime is simply written "by parts" for shorthand.

$$\int_a^b u(x)v'(x)\,\mathrm{d}x = [u(x)v(x)]_a^b - \int_a^b u'(x)v(x)\,\mathrm{d}x.$$

Remark. To apply integration by parts, we need to select appropriate $u(x)$ and $v'(x)$ before we actually begin applying the formula. Sometime it may seem hard to have an idea of assign which to which in the first place, but there's a general rule:

inverse trigs ——— logs ——— polys ——— trigs/exps

Functions on the right are likely to be selected as $v'(x)$, in which we are going to integrate with, in the later calculation; whereas functions on the left are likely to be selected as $u(x)$, in which we are going to differentiate.

The intuition behind is because inverse functions and logarithmic functions, like $\arctan(\cdot)$ and $\ln(\cdot)$, are hard to integrates, you are encouraged to try to find the antiderivative of those by yourself (either by applying u-sub or by parts is doable) and it will be an excellent exercise. But the key is that, once we differentiate them, they become a fraction which we can seek techniques already learnt to find a solution.

Example. Compute the integral by using integration by parts

$$\int x \arctan(x)\,\mathrm{d}x.$$

By applying the technique introduced above, define:

$$u(x) = \arctan(x), v'(x) = x,$$

and it follows that

$$u'(x) = \frac{1}{1+x^2}, v(x) = \frac{1}{2}x^2.$$

By substituting the result into the formula of integration by parts gives:

$$\begin{aligned}
\int x \arctan(x)\,\mathrm{d}x &= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,\mathrm{d}x \\
&= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,\mathrm{d}x \\
&= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,\mathrm{d}x \\
&= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + c
\end{aligned}$$

Example. Compute the indefinite integral

$$I = \int e^x \sin(x)\,\mathrm{d}x.$$

Since trigonometric and exponential functions are interchangeably selected as $u(x)$, so we are going to do one case here, the remaining is left as exercise.

Define

$$u(x) = e^x, v'(x) = \sin(x),$$

therefore we have

$$u'(x) = e^x, v(x) = -\cos(x)$$

which follows that

$$I = -e^x \cos(x) + \int e^x \cos(x) \, \mathrm{d}x.$$

Furthermore, apply by parts twice, with $u(x) = e^x, v'(x) = \cos(x)$, we have

$$\begin{aligned} I &= -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x) \, \mathrm{d}x + c \\ I &= -e^x \cos(x) + e^x \sin(x) - I + c \\ 2I &= e^x \sin(x) - e^x \cos(x) + c \\ I &= \frac{1}{2} e^x \sin(x) - \frac{1}{2} e^x \cos(x) + c \end{aligned}$$

2.2.1. Tabular Method

Sometime we find it is hard to solve an integral with applying integration by one or two times. In this case, we can draw a table to assist us, to make the computation procedure more clear and well-organized.

Example. Compute the following integral:

$$\int x^4 \sin(x) \, \mathrm{d}x$$

We have seen this pattern quite a lot as we have some intuition that thinking this can be solved by applying integration by parts. As the general trend of choosing $u(x)$ and $v'(x)$, let's define

$$u(x) = x^4, v'(x) = \sin(x).$$

Then, we can draw a table to trace the computation steps while we are doing by parts:

Sign	$u(x)$	$v'(x)$
$+$	x^4	$\sin(x)$
$-$	$4x^3$	$-\cos(x)$
$+$	$12x^2$	$-\sin(x)$
$-$	$24x$	$\cos(x)$
$+$	24	$\sin(x)$
$-$	0	$-\cos(x)$

Figure 4: Tabular Method

Firstly make a table contains three columns, where

- the first column indicates the sign in front to the integration we have on each stage,
- the second column indicates the $u(x)$ in each stage of the integration we have,
- the third column indicates the $v'(x)$ in each stage of the integration we have.

Then, by multiplying each term as the red arrows indicate, we end up with

$$\begin{aligned}\int x^4 \sin(x)\,\mathrm{d}x = &+ x^4 \times (-\cos(x)) \\ &- 4x^3 \times (-\sin(x)) \\ &+ 12x^2 \times \cos(x) \\ &- 24x \times \sin(x) + c \\ = &-x^4\cos(x) + 4x^3\sin(x) + 12x^2\cos(x) - 24x\sin(x) - 24\cos(x) + c.\end{aligned}$$

2.3. Initial Value Problem (IVP)

** Solve the undetermined constant c with given constraint(s).*

Example. What is the antiderivative of $f(x) = \frac{1}{2x} + \sec^2(\pi x)$, with $F(1) = 0$?

Step 1: Find the antiderivative of f.

$$\begin{aligned}F(x) &= \int f(x)\,\mathrm{d}x \\ &= \int \frac{1}{2x} + \sec^2(\pi x)\,\mathrm{d}x \\ &= \frac{1}{2}\ln(x) + \frac{1}{\pi}\tan(\pi x) + c\end{aligned}$$

Step 2: Solve c with the given constraint $F(1) = 0$.

$$\begin{aligned}F(1) &= 0 \\ \frac{1}{2}\ln(1) + \frac{1}{\pi}\tan(\pi) + c &= 0 \\ c &= 0\end{aligned}$$

Combining results above, we have

$$F(x) = \frac{1}{2}\ln(x) + \frac{1}{\pi}\tan(\pi x)$$

3. Matrix Operations

3.1. Eigenvalues and Eigenvectors

4. Multi-Variable Calculus

5. Regression Analysis

6. Differential Equations (DE)

6.1. First Order DE

6.2. Second Order DE

6.3. Stability of DE Solutions

7. Non-linear System of Equations

8. Appendix

8.1. Commonly Used Integration Formula

Basic Integration Rules

$f(x)$	$\int f(x)\,\mathrm{d}x$
$x^n, n \neq 1$	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\ln(x) + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$
$\sin(x)$	$-\cos(x) + c$
$\cos(x)$	$\sin(x) + c$
$\sec^2(x)$	$\tan(x) + c$
$\sec(x)\tan(x)$	$\sec(x) + c$
$\csc^2(x)$	$-\cot(x) + c$
$\csc(x)\cot(x)$	$-\csc(x) + c$
$\frac{1}{x^2 + a^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$

8.2. Trigonometric Identities

$$\sin(-x) = -\sin(x)$$

$$\cos(-x) = \cos(x)$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$1 + \tan^2(x) = \sec^2(x)$$