Elementary Row Operations and Row Echelon Form

Row echelon form (REF) of a matrix can be obtained by using Gaussian elimination, it plays significant role in the problem of finding solution with respect to a linear system. Let's consider the following system of equations with n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

It can be simplified into matrix form

$$Ax = b$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_1n \\ a_{21} & a_{22} & \ldots & a_2n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Where A is the coefficient matrix, b is the constant matrix, and x is the matrix of variables.

Elementary Row Operations (ERO)

To solve the system of linear equations, we are perform elementary row operations:

1. Swapping two rows:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \ldots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \xrightarrow{r'_i = r_j, r'_j = r_i} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \ldots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

1. Multiplying a row by a nonzero number:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \xrightarrow{r'_i = c \times r_i} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ c \times a_{i1} & c \times a_{i2} & \ldots & c \times a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

1. Adding a multiple of one row to another row.

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \ldots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \xrightarrow{r'_i = r_i + c r_j} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} + c a_{j1} & a_{i2} + c a_{j2} & \ldots & a_{in} + c a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \ldots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

The greatest advantage of elementary row operations is: they **do not change the solution space of our system of equations**, that is, the set of solutions of REF generated by using elementary row operations is identical to the origin system.

Find REF by Gaussian Elimination

Definition. Given a matrix $A \in M_{n \times m}(\mathbb{R})$, A is in *row echelon form (REF)* if

- All rows with entry all zero are at the bottom.
- The pivot (the first non-zero entry) is 1 and is on the right of the pivot of every row above.

Definition. A matrix is in *reduced row echelon form (RREF)* if it is in REF, and

- Each column containing a pivot has zeros in all entries above the pivot.

Example. The following matrix is in REF, but not in RREF:

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 1 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the Solution Set in RREF

Define the augmented matrix

$$\begin{aligned} B &= [A \mid b] \\ &= \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right], \end{aligned}$$

our goal is reduce B to its RREF

$$\left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & s_1 \\ 0 & 1 & \dots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & s_n \end{array}\right]$$

if possible. The solution of our original system $Ax = b$ is just

$$\begin{cases} x_1 = s_1 \\ x_2 = s_2 \\ \quad\vdots \\ x_n = s_n \end{cases}$$

Example. Lets be more specific, consider the following linear system

$$\begin{cases} 2x_1 + x_2 - x_3 = 8 \\ -3x_1 - x_2 + 2x_3 = -11 \\ -2x_1 + x_2 + 2x_3 = -3 \end{cases}$$

To solve this system, firstly we are going to write it into matrix form:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ -11 \\ 3 \end{bmatrix}.$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & 3 \end{array}\right]$$

By doing Gaussian elimination, we have

$$\begin{aligned}
\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & 3 \end{array}\right] &\overset{r_2'=r_2+\frac{3}{2}r_j}{\longrightarrow} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ -2 & 1 & 2 & 3 \end{array}\right] \\
&\overset{r_2'=r_2+r_1}{\longrightarrow} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 2 & 1 & 5 \end{array}\right] \\
&\overset{r_3'=r_3-4r_2}{\longrightarrow} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \\
&\overset{r_2'=r_2+\frac{1}{2}r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right] \\
&\overset{r_1'=r_1-r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right] \\
&\overset{r_1'=r_1-2r_2}{\longrightarrow} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right] \\
&\overset{r_1'=\frac{1}{2}r_1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right] \\
&\overset{r_2'=2r_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right] \\
&\overset{r_3'=-r_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right].
\end{aligned}$$

And this gives us the solution

$$\begin{cases} x_1 = 2 \\ x_2 = 3 \\ x_3 = -1 \end{cases}.$$

Set of Solution of a System of Linear Equations

However, it is not alway possible to find an unique solution for all linear system. There are three scenarios of a linear system:

- The system has a unique solution.
- The system has infinitely many solutions.

- The system has no solution (aka. inconsistent).

The best way of distinguish the three cases above is to take the advantage of REF. In general, for the augmented matrix $[A \mid b]$ of a linear system with n variable and n equations, the system has:
- a unique solution if every row has a pivot in its REF,
- infinitely many solution if there exists a row of all zeros in its REF,
- no solution if there exists a row of $[0 \ \ 0 \ \ \dots \ \ 0 \mid c]$, for some nonzero constant in its REF.

Example. Suppose

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & \alpha \end{bmatrix}, x = [x_1 \ \ x_2 \ \ x_3] \text{ and } b = [4 \ \ -2 \ \ \beta].$$

Find conditions on α and β so that $Ax = b$ has
- no solution,
- a unique solution,
- infinitely many solution.

Before answering this question directly, we should write the argument matrix $[A \mid b]$ in REF. As you can verify, the following is the REF of $[A \mid b]$

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 4 \\ 0 & 1 & -3 & 10 \\ 0 & 0 & \alpha + 7 & \beta - 18 \end{array}\right]$$

Therefore, from the REF[1] of the matrix, we can conclude $Ax = b$
- has no solution if $\alpha + 7 = 0$ and $\beta - 18 \neq 0$,
- has a unique solution if $\alpha + 7 = 1$ and β can be arbitrary real number,
- has infinitely many solution if $\alpha + 7 = 0$ and $\beta - 18 = 0$.

[1]In rigorous speaking, we cannot claim this matrix is in REF because there is no guarantee for $\alpha + 7 = 1$, or $\alpha + 7 = 0$ and $\beta - 18 = 1$, which are required for REF.