Introduction

In practice assignment 2 (PA 2), we are asked to check if result of the integral

$$I = \int_3^\infty \frac{1}{x(x-1)(x-2)} \, \mathrm{d}x$$

is finite. With the support of basic logarithmic and limit properties, we are able to discuss the convergency using partial fraction decomposition as well as comparison test.

Preliminaries

Theorem 1 (Limit property). Let $f : X \to \mathbb{R}, g : Y \to \mathbb{R}$ be functions with $f(X) \subseteq Y$, and let $g \circ f : A \to \mathbb{R}$ be the composition. Suppose

$$\lim_{x \to a} f(x) = \ell \text{ and} \lim_{y \to \ell} = g(\ell),$$

Then

$$\lim_{x \to a} g(f(x)) = g(\ell).$$

Theorem 2 (logarithm property). For every $a, b \in \mathbb{R}$, the followings always hold

$$\begin{aligned} \ln(a) + \ln(b) &= \ln(ab) \\ \ln(a) - \ln(b) &= \ln\left(\frac{a}{b}\right) \\ a \ln(b) &= \ln(b^a) \end{aligned}$$

Approach 1: Partial Fraction Decomposition

Let's suppose

$$\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

for some constant $A, B, C \in \mathbb{R}$.

By multiplying both sides by $x(x-1)(x-2)$, we have

$$\begin{aligned} 1 &= A(x-1)(x-2) + Bx(x-2) + Cx(x-1) \\ 0x^2 + 0x + 1 &= (A + B + C)x^2 + (-3A - 2B + -C)x + 2A \end{aligned}$$

By matching the coefficients, we can obtain a system of equations with three variables

$$\begin{cases} 0 = A + B + C \\ 0 = -3A - 2B - C \\ 1 = 2A \end{cases}.$$

As you can verify, this system of equation has solution

$$\begin{cases} A = \frac{1}{2} \\ B = -1 \\ C = \frac{1}{2} \end{cases}$$

This leads to

$$
\begin{aligned}
\int_3^\infty \frac{1}{x(x-1)(x-2)}\,\mathrm{d}x &= \lim_{a\to\infty}\int_3^a \frac{1}{x(x-1)(x-2)}\,\mathrm{d}x \\
&= \lim_{a\to\infty}\int_3^a \frac{1}{2x}-\frac{1}{x-1}+\frac{1}{2(x-2)}\,\mathrm{d}x \\
&= \lim_{a\to\infty}\left[\frac{1}{2}\ln|x| - \ln|x-1| + \frac{1}{2}\ln|x-2|\right]_3^a \\
&= \lim_{a\to\infty}\left[\ln\left|\sqrt{x}\right| + \ln\left|\frac{1}{x-1}\right| + \frac{1}{2}\ln\left|\sqrt{x-2}\right|\right]_3^a & \text{[By Thm 2]} \\
&= \lim_{a\to\infty}\left[\ln\left(\frac{\sqrt{x^2-2x}}{x-1}\right)\right]_3^a \\
&= \lim_{a\to\infty}\ln\left(\frac{\sqrt{a^2-2a}}{a-1}\right) - \ln\left(\frac{\sqrt{3}}{2}\right) & \text{[By Thm 2]} \\
&= \ln\left(\lim_{a\to\infty}\frac{\sqrt{a^2-2a}}{a-1}\right) - \ln\left(\frac{\sqrt{3}}{2}\right) & \text{[By Thm 1]} \\
&= \ln\left(\lim_{a\to\infty}\frac{\sqrt{1-\frac{2}{a}}}{1-\frac{1}{a}}\right) - \ln\left(\frac{\sqrt{3}}{2}\right) & \text{[Divides by } a\text{]} \\
&= \ln(1) - \ln\left(\frac{\sqrt{3}}{2}\right) \\
&= \ln\left(\frac{\sqrt{3}}{2}\right)
\end{aligned}
$$

Approach 2: Comparison Test

Notice the following:

$$
\begin{aligned}
x \geq x-4 &\Rightarrow \frac{1}{x} \leq \frac{1}{x-4} \\
x-1 \geq x-4 &\Rightarrow \frac{1}{x-1} \leq \frac{1}{x-4} \\
x-2 \geq x-4 &\Rightarrow \frac{1}{x-2} \leq \frac{1}{x-4}
\end{aligned}
$$

By multiplying all the left hand side term together and all the right hand side together, it gives us

$$\frac{1}{x(x-1)(x-2)} \leq \frac{1}{(x-4)^3}.$$

Therefore, consider the integral

$$\int_3^\infty \frac{1}{(x-4)^3}\,\mathrm{d}x,$$

as you can verify, the result of integral evaluation is equal to 1. So by comparison test, integral I converges.

Remark. Comparison test only tells us the convergency of the integral of interest. The theorem, however, does not make any claim regarding the actual value that the integral converges to.