Introduction

In practice assignment 2 (PA 2), we are asked to check if result of the integral

$$I = \int_3^\infty \frac{1}{x(x-1)(x-2)}\,\mathrm{d}x$$

is finite. With the support of basic logarithmic and limit properties, we are able to discuss the convergency using partial fraction decomposition as well as comparison test.

Preliminaries

Theorem 1 (Limit property). Let $f : X \to \mathbb{R}, g : Y \to \mathbb{R}$ be functions with $f(X) \subseteq Y$, and let $g \circ f : A \to \mathbb{R}$ be the composition. Suppose

$$\lim_{x\to a} f(x) = \ell \text{ and} \lim_{y\to \ell} = g(\ell),$$

Then

$$\lim_{x\to a} g(f(x)) = g(\ell).$$

Theorem 2 (logarithm property). For every $a, b \in \mathbb{R}$, the followings always hold

$$\ln(a) + \ln(b) = \ln(ab)$$
$$\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$$
$$a\ln(b) = \ln(b^a)$$

Approach 1: Partial Fraction Decomposition

Let's suppose

$$\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

for some constant $A, B, C \in \mathbb{R}$.

By multiplying both sides by $x(x-1)(x-2)$, we have

$$1 = A(x-1)(x-2) + Bx(x-2) + Cx(x-1)$$
$$0x^2 + 0x + 1 = (A+B+C)x^2 + (-3A - 2B + -C)x + 2A$$

By matching the coefficients, we can obtain a system of equations with three variables

$$\begin{cases} 0 = A + B + C \\ 0 = -3A - 2B - C. \\ 1 = 2A \end{cases}$$

As you can verify, this system of equation has solution

$$\begin{cases} A = \frac{1}{2} \\ B = -1 \\ C = \frac{1}{2} \end{cases}$$

This leads to

$$
\begin{aligned}
\int_3^\infty \frac{1}{x(x-1)(x-2)}\,\mathrm{d}x &= \lim_{a\to\infty}\int_3^a \frac{1}{x(x-1)(x-2)}\,\mathrm{d}x \\
&= \lim_{a\to\infty}\int_3^a \frac{1}{2x}-\frac{1}{x-1}+\frac{1}{2(x-2)}\,\mathrm{d}x \\
&= \lim_{a\to\infty}\left[\frac{1}{2}\ln|x|-\ln|x-1|+\frac{1}{2}\ln|x-2|\right]_3^a \\
&= \lim_{a\to\infty}\left[\ln\left|\sqrt{x}\right|+\ln\left|\frac{1}{x-1}\right|+\frac{1}{2}\ln\left|\sqrt{x-2}\right|\right]_3^a && \text{[By Thm 2]} \\
&= \lim_{a\to\infty}\left[\ln\left(\frac{\sqrt{x^2-2x}}{x-1}\right)\right]_3^a \\
&= \lim_{a\to\infty}\ln\left(\frac{\sqrt{a^2-2a}}{a-1}\right)-\ln\left(\frac{\sqrt{3}}{2}\right) && \text{[By Thm 2]} \\
&= \ln\left(\lim_{a\to\infty}\frac{\sqrt{a^2-2a}}{a-1}\right)-\ln\left(\frac{\sqrt{3}}{2}\right) && \text{[By Thm 1]} \\
&= \ln\left(\lim_{a\to\infty}\frac{\sqrt{1-\frac{2}{a}}}{1-\frac{1}{a}}\right)-\ln\left(\frac{\sqrt{3}}{2}\right) && \text{[Divides by } a\text{]} \\
&= \ln(1)-\ln\left(\frac{\sqrt{3}}{2}\right) \\
&= \ln\left(\frac{\sqrt{3}}{2}\right)
\end{aligned}
$$

Approach 2: Comparison Test

Notice the following:

$$
\begin{aligned}
x \geq x-2 &\Rightarrow \frac{1}{x} \leq \frac{1}{x-2} \\
x-1 \geq x-2 &\Rightarrow \frac{1}{x-1} \leq \frac{1}{x-2} \\
x-2 \geq x-2 &\Rightarrow \frac{1}{x-2} \leq \frac{1}{x-2}
\end{aligned}
$$

By multiplying all the left hand side term together and all the right hand side together, it gives us

$$
\frac{1}{x(x-1)(x-2)} \leq \frac{1}{(x-2)^3}.
$$

Therefore, consider the integral

$$
\int_3^\infty \frac{1}{(x-2)^3}\,\mathrm{d}x,
$$

we have

$$\begin{aligned}\int_3^\infty \frac{1}{(x-2)^3}\,\mathrm{d}x &= \lim_{a\to\infty}\int_3^a \frac{1}{(x-2)^3}\,\mathrm{d}x \\ &= \lim_{a\to\infty}\left[-\frac{1}{2(x-2)^2}\right]_3^a \\ &= \lim_{a\to\infty}\left(-\frac{1}{2(a-2)^2}+\frac{1}{2(3-2)^2}\right) \\ &= \lim_{a\to\infty}\left(-\frac{1}{2(a-2)^2}+\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

Therefore, because $\frac{1}{x(x-1)}(x-2) \leq \frac{1}{(x-2)^3}$ and $\int_3^\infty \frac{1}{(x-2)^3}\,\mathrm{d}x < \infty$, so integral I convergent.

Remark. Comparison test only tells us the convergency of the integral of interest. The theorem, however, does not make any claim regarding the actual value that the integral converges to.

Error Corrections

In last week's tutorial, I claimed

$$J = \int_3^\infty \frac{1}{(x-4)^3}\,\mathrm{d}x$$

converges, which is false.

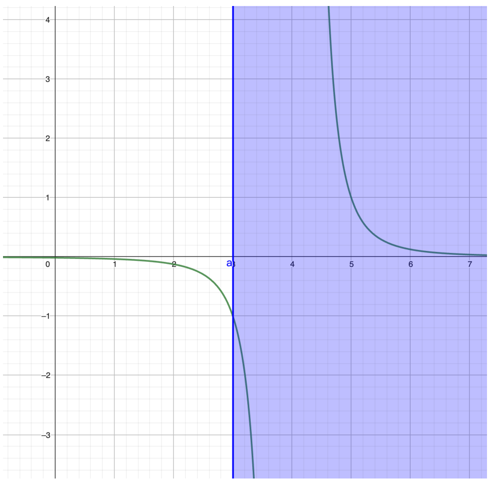

Figure 1: Discontinuity at $x = 4$.

As you can see, there exists a discontinuity at $x = 4$. Since $4 \in [3, \infty)$, so we cannot apply the fundamental integral rule to compute the integral value directly as we requires integrand to be continuous within the interval of interest. So to discuss the convergency of J, we need to apply some ticks to manipulate the integrand.

Definition. Function $f : \mathbb{R} \to \mathbb{R}$ is an *odd function* if $f(-x) = -f(x)$ for every $x \in \mathrm{dom}(f)$.

Theorem 3. For every odd function $f : \mathbb{R} \to \mathbb{R}$ and $a > 0$ such that $a, -a \in \mathrm{dom}(f)$

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

Therefore, back to our discussion on

$$J = \int_{3}^{\infty} \frac{1}{(x-4)^3}\,\mathrm{d}x.$$

Claim: $\frac{1}{x^3}$ is an odd function.

proof. Let $f(x) = \frac{1}{x^3}$, therefore

$$\begin{aligned} f(-x) &= \frac{1}{(-x)^3} \\ &= -\frac{1}{x^3} \\ &= -f(x) \end{aligned}$$

□

Then, we are going to apply u-sub:

1. Let $u = x - 4$, it implies $\mathrm{d}u = \mathrm{d}x$.
2. Notice that when $x = 3$, $u = -1$. And when $x \to \infty$, $u \to \infty$.

It leaves us

$$\begin{aligned} \int_{3}^{\infty} \frac{1}{(x-4)^3}\,\mathrm{d}x &= \int_{-1}^{\infty} \frac{1}{u^3}\,\mathrm{d}u \\ &= \int_{-1}^{1} \frac{1}{u^3}\,\mathrm{d}u + \int_{1}^{\infty} \frac{1}{u^3}\,\mathrm{d}x \\ &= 0 + \int_{1}^{\infty} \frac{1}{u^3}\,\mathrm{d}x && \text{[By Thm 3]} \\ &= \lim_{a\to\infty} \int_{1}^{a} \frac{1}{u^3}\,\mathrm{d}x \\ &= \lim_{a\to\infty} \left[-\frac{1}{2u^2}\right]_{1}^{a} \\ &= -\frac{1}{2} \lim_{a\to\infty} \left(\frac{1}{a^2} - 1\right) \\ &= \frac{1}{2} < \infty \end{aligned}$$