Row Echelon Form and Gaussian Elimination

Row echelon form (REF) of a matrix can be obtained by using Gaussian elimination, it plays significant role in the problem of finding solution with respect to a linear system. Let's consider the following system of equations with n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n \end{cases}$$

It can be simplified into matrix form

$$Ax = b$$

with

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_1n \\ a_{21} & a_{22} & \dots & a_2n \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Elementary Row Operations

To solve the system of linear equations, we are perform elementary row operations:

1. Swapping two rows:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \xrightarrow{r'_i = r_j, r'_j = r_i} \begin{pmatrix} a_{11} & a_{12} & \dots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

2. Multiplying a row by a nonzero number:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \xrightarrow{r'_i = c \times r_i} \begin{pmatrix} a_{11} & a_{12} & \dots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ c \times a_{i1} & c \times a_{i2} & \dots & c \times a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

3. Adding a multiple of one row to another row.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \xrightarrow{r'_i = r_i + c r_j} \begin{pmatrix} a_{11} & a_{12} & \dots & a_1n \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} + c a_{j1} & a_{i2} + c a_{j2} & \dots & a_{in} + c a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

The greatest advantage of elementary row operations is: they **do not change the solution space of our system of equations**, that is, the set of solutions of REF generated by using elementary row operations is identical to the origin system.

Find REF by Gaussian Elimination

Define the augmented matrix

$$A = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_1 n & b_1 \\ a_{21} & a_{22} & \dots & a_2 n & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{array}\right),$$

our goal is reduce A to its REF

$$\left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & s_1 \\ 0 & 1 & \dots & 0 & s_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & s_n \end{array}\right)$$

if possible[1]. The solution of our original system $Ax = b$ is just

$$\begin{cases} x_1 = s_1 \\ x_2 = s_2 \\ \quad \vdots \\ x_n = s_n \end{cases}$$

Example

Lets be more specific, consider the following linear system

$$\begin{cases} 2x_1 + x_2 - x_3 \quad = 8 \\ -3x_1 - x_2 + 2x_3 = -11 \\ -2x_1 + x_2 + 2x_3 = -3 \end{cases}$$

To solve this system, firstly we are going to write it into matrix form:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ -11 \\ 3 \end{pmatrix}.$$

The corresponding augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & 3 \end{array}\right)$$

By doing Gaussian elimination, we have

[1] For more details, see **Appendix**

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & 3 \end{array}\right) \overset{r_2'=r_2+\frac{3}{2}r_j}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ -2 & 1 & 2 & 3 \end{array}\right)$$

$$\overset{r_2'=r_2+r_1}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 2 & 1 & 5 \end{array}\right)$$

$$\overset{r_3'=r_3-4r_2}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{r_2'=r_2+\frac{1}{2}r_3}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{r_1'=r_1-r_3}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 7 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{r_1'=r_1-2r_2}{\longrightarrow} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{r_1'=\frac{1}{2}r_1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{r_2'=2r_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$\overset{r_3'=-r_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

And this gives us the solution

$$\begin{cases} x_1 = 2 \\ x_2 = 3 \\ x_3 = -1 \end{cases}$$

Appendix

We cannot always find a unique set of solutions for arbitrary given linear system of equations. You can formally prove that a **consistent** linear system with n variables has unique solution if and only if there exists n **linearly independent** constraints.

In terms of consistency, consider the following system of equations:

$$\begin{cases} 2x_1 + x_2 - x_3 & = 8 \\ -3x_1 - x_2 + 2x_3 & = -11 \\ -3x_1 - x_2 + 2x_3 & = -10 \end{cases}$$

Try to find its solution using Gaussian elimination, what tells you that it does not have solution?

In terms of linearly independent constraints, consider the following system of equations:

$$\begin{cases} 2x_1 + x_2 - x_3 & = 8 \\ -3x_1 - x_2 + 2x_3 & = -11 \\ 4x_1 + 2x_2 - 2x_3 & = 16 \end{cases}$$

Try to find its solution using Gaussian elimination, this is the case that we does not provides enough constraints on the system of equation. It turns out the system is **consistent** but has **infinity** number of solutions.