Differential Equations

Introduction

Definition. *Differential equation (DE)* is an equation involving functions and their derivatives.

The study of differntial equations plays a significant role in the modern study of physics, engineering, economics, and etc. To dive deep in the mathematically rigorous dicussion of differnetial equation, it is inevitable to introduce a diverse branches of mathematics, for instance, analysis and numerical methods. In light of the limitation to the person who made the note, we are only going to talk about the most fundamental part of differneitlal equation.

First Order Differential Equations

Definition. The *order* of a differential equation is defined by the highest degree of derivative of the differential equation has.

Definition. let $y : \mathbb{R} \to \mathbb{R}$ be a function with variable t, and $f : \mathbb{R}^2 \to \mathbb{R}$. A *first order differential equatoin* has the the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y, t) \tag{1}$$

Example. *Acceleration* describes the rate of change of velocity of an object. Suppose a ball is dropped from air and undergoes a free fall, that is, only gravity is acting on the ball, its acceleration $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t}$ is characterized by a first order DE

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g. \tag{2}$$

Where g is the *gravitational constant*

There is a special class of DE we are interested to study: separable DE.

Definition. A *separable differential equation* can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} = M(y)N(t). \tag{3}$$

To solve a separable, we follows the following strategy in general:
1. Re-write the given DE into the form given in (3).
2. Separable varable, so each side of equation only contains one type of variable

$$\frac{1}{M(y)}\frac{\mathrm{d}y}{\mathrm{d}t} = N(t). \tag{4}$$

3. Then integrate both sides of the equation at the same time.

$$\int \frac{1}{M(y)} \frac{\mathrm{d}y}{\mathrm{d}t} \,\mathrm{d}t = \int N(t) \,\mathrm{d}t. \tag{5}$$

Example (Newton's Law of Cooling). Let $T(t)$ be the temperature of an object at time t, and T_s be the temperature of surrounding environment. The rate of change of temperature of $T(t)$ is described as

$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T(t) - T_s), \tag{6}$$

where both k, T_s are a constant in real number.

To solve (6), we are going to follow the strategy stated before:

$$\begin{aligned}
\frac{1}{T(t) - T_s} \frac{\mathrm{d}T}{\mathrm{d}t} &= k \\
\int \frac{1}{T(t) - T_s} \frac{\mathrm{d}T}{\cancel{\mathrm{d}t}} \cancel{\mathrm{d}t} &= \int k \,\mathrm{d}t \\
\int \frac{1}{T(t) - T_s} \,\mathrm{d}T &= \int k \,\mathrm{d}t \\
\ln|T(t) - T_s| &= kt + c \\
T(t) - T_s &= e^{kt+c} \\
T(t) - T_s = e^{kt} \times e^{c} \\
T(t) = Ae^{kt} + T_s
\end{aligned} \tag{7}$$

Remark. Notice that it is irrigorous to treat the derivative $\frac{\mathrm{d}T}{\mathrm{d}t}$ as a fraction and cancel it with the term $\mathrm{d}t$. The definition of derivative involving limit, and is to find the flactuation of the original function in the infidecimal change in t, that is

$$\frac{\mathrm{d}T}{\mathrm{d}t} \equiv \lim_{h \to 0} \frac{T(t+h) - T(t)}{h}, \tag{8}$$

which is a unity becuase the limit is not gaurantee to be congervent. Whereas when we write the whole term

$$\frac{1}{T(t) - T_s} \frac{\mathrm{d}T}{\mathrm{d}t} \,\mathrm{d}t, \tag{9}$$

it is a 1-form of differential form, which is a smooth section of the co-tangent bundle on a manifold. Under no circumstance should those two notion to be inter-changibly used.

However, what if the DE is not separable? For instace, how can we solve for a differenial equation has the form

$$A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + B(x)y = C(x), \tag{10}$$

where function A, B, C functions over real. In this senario, we need introduce some prior knowledge.

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function over $\mathbb{R}$ with respect to variable $x_1, x_2, ..., x_n$. The *total derivative* $\mathrm{d}f$ can be written as

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \,\mathrm{d}x_i \tag{11}$$

Example. The total derivative of $f(x, y) = x^2 + y^2$ is given by

$$\begin{aligned}\mathrm{d}f &= \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y \\ &= \frac{\partial}{\partial x}(x^2 + y^2)\,\mathrm{d}x + \frac{\partial}{\partial y}(x^2 + y^2)\,\mathrm{d}y \\ &= 2x\,\mathrm{d}x + 2y\,\mathrm{d}y\end{aligned} \tag{12}$$

Definition. A differential form α is *exact* if there exists some differential form β such that

$$\mathrm{d}\beta = \alpha \tag{13}$$

Proposition. Let $P, Q : \mathbb{R}^2 \to \mathbb{R}$ be multi-variable functions with respect to variable x and y. Then the differential form

$$P\,\mathrm{d}x + Q\,\mathrm{d}y \tag{14}$$

is exact if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial y}{\partial x} \tag{15}$$

Remark. The notion of exactness is telling us, if a differential form α is exact, then it can be obtained by computing the total derivative of another differential form β.

Furthermore, the proposition above provides us an easy, swift way of verifying if a given differential form is exact. The intuition behinds the proposition is the following: if equation (12) were exact, then P and Q are the derivative of same function but with respect to different variable. That is

$$P = \frac{\partial f}{\partial x}, \quad Q = \frac{\partial f}{\partial y}. \tag{16}$$

Therefore,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}, \tag{17}$$

which are essentially equal.

In light the the introductory of differential form and exactness, we can take their advantages in the discussion of solving DEs. Given a first order differential equation with the form:

$$A\frac{\mathrm{d}y}{\mathrm{d}x} + By = C, \tag{18}$$

where $A, B, C : \mathbb{R} \to \mathbb{R}$ are functions over real with respect to variable x, we can re-write it by using differential forms

$$\underbrace{A\,\mathrm{d}y + By\,\mathrm{d}x}_{\alpha} = C\,\mathrm{d}x. \tag{19}$$

Hence, if α were exact, then by equation (11) there exists some other form β such that

$$\mathrm{d}\beta = \alpha. \tag{20}$$

It implies that (17) is equivalent to

$$\mathrm{d}\beta = C\,\mathrm{d}x. \tag{21}$$

And by integrating both sides of (19), we can obtain the solution of our DE (16) in implicit form:

$$
\begin{aligned}
\int \mathrm{d}\beta &= \int C \,\mathrm{d}x \\
\beta &= \int C \,\mathrm{d}x
\end{aligned}
\tag{22}
$$

Example. Solve the following differential equation

$$
(4+t^2)\frac{\mathrm{d}y}{\mathrm{d}t} + 2ty = 4t. \tag{23}
$$

Let's firstly write it using differential form and check if it is exact:

$$
\underbrace{(4+t^2)\,\mathrm{d}y}_{P} + \underbrace{2ty\,\mathrm{d}t}_{Q} = 4t\,\mathrm{d}t. \tag{24}
$$

$$
\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{\partial}{\partial t}(4+t^2) \\
&= 2t \\
\frac{\partial Q}{\partial y} &= \frac{\partial}{\partial y}(2ty) \\
&= 2t.
\end{aligned}
\tag{25}
$$

So we can see the left part of (22) is an exact differential form, which is implying that

$$
P = \frac{\partial f}{\partial y}, \quad Q = \frac{\partial f}{\partial y}. \tag{26}
$$

Hence

$$
\begin{aligned}
f &= \int P \,\mathrm{d}y \\
&= \int 4 + t^2 \,\mathrm{d}y \\
&= 4y + t^2 y + h(t)
\end{aligned}
\tag{27}
$$

And we can solve the unknown function $h(t)$ by computing the derivative of f with respect to t:

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial t}(4y + t^2 y + h(t)) \\
&= 2ty + h'(t)
\end{aligned}
\tag{28}
$$

Therefore, by using the fact that

$$\begin{aligned} \frac{\partial f}{\partial t} &= Q \\ 2ty + h'(t) &= 2ty \\ h'(t) &= 0 \end{aligned} \tag{29}$$

And it follows that $h(t) = c$, where c is an aribitrary constant in $\mathbb{R}$. Hence, we have the final answer:

$$f = 2y + t^2 y + c \tag{30}$$

However, what if the given DE is not exact? Then we need to develop some trick to modify the DE and change to exact.

Definition. Given a DE of the from

$$A \,\mathrm{d}y + By \,\mathrm{d}x = C \tag{31}$$

A *integrating factor* is an auxiliary function $I(x)$ such that when we multiply it to the both sides of the eqaution, making

$$IA \,\mathrm{d}y + IB \,\mathrm{d}x \tag{32}$$

to be an exact differential form.

Proposition. If a DE has the form

$$A \,\mathrm{d}y + By \,\mathrm{d}x = C, \tag{33}$$

where $A, B, C : \mathbb{R} \to \mathbb{R}$ are function with respect to variable x, and A is non-zero over its domain. Then the DE has an integrating factor

$$I(x) = e^{\int P(x) \,\mathrm{d}x}, \tag{34}$$

where $P(x) = \frac{B(x)}{A(x)}$.

Example. Solve the following DE

$$x \,\mathrm{d}y + 2y \,\mathrm{d}x = 4x^3 \,\mathrm{d}x \tag{35}$$

As you can verify, this DE is not exact, so we are going to find the integrating factor

$$\begin{aligned} I(x) &= e^{\int P(x) \,\mathrm{d}x} \\ &= e^{\int \frac{2}{x} \,\mathrm{d}x} \\ &= e^{2 \ln(x)} \\ &= x^2 \end{aligned} \tag{36}$$

By multiplying both sides by the integrating factor, we have

$$
\begin{aligned}
x\,\mathrm{d}y + 2y\,\mathrm{d}x &= 4x^3\,\mathrm{d}x \\
\mathrm{d}y + \frac{2y}{x}\,\mathrm{d}x &= 4x^4\,\mathrm{d}x \\
x^2\left(\mathrm{d}y + \frac{2y}{x}\,\mathrm{d}x\right) &= x^2 \cdot 4x^4\,\mathrm{d}x \\
x^2\,\mathrm{d}y + 2yx\,\mathrm{d}x &= 4x^6\,\mathrm{d}x \\
d(x^2y) &= 4x^6\,\mathrm{d}x \\
\int d(x^2y) &= \int 4x^6\,\mathrm{d}x \\
x^2y &= \frac{4}{7}x^7 + c
\end{aligned}
\tag{37}
$$