# Generative Adversarial Learning of Markov Chains

Jiaming Song, Shengjia Zhao and Stefano Ermon

Computer Science Department, Stanford University

## Introduction

**Setting**: Learn the **transition operator** for a Markov chain.

**Goal**: Match **its stationary distribution** to a **target distribution**.

We propose a novel **adversarial training procedure** that

- avoids sampling from the stationary distribution directly, and
- capable of reaching the target distribution asymptotically

The learned Markov chain:

- can start from **random noise**
- can be trained in a **likelihood free** way
- is able to generate **multiple distinct samples** during a single run
- **does not** require detailed balance
- has nice mixing properties

## Problem Setup

Let $\mathcal{S}$ be the state space for the sequence of random variables $\mathcal{X} = \{\mathbf{X}_t\}_{t=0}^{t=\infty}$, $\mathbf{X}_t \in \mathcal{S}$. Let $\pi^0$ be an initial probability distribution for $\mathbf{X}_0$, and $T_\theta(\cdot|\mathbf{x})$ be a transition kernel parameterized by $\theta$.

Every $T_\theta$ defines a time-homogeneous Markov chain over $\mathcal{X}$. We denote $\pi_\theta^t(\mathbf{x})$ the resulting probability distribution at time step $t$.

If we assume that $T_\theta(\mathbf{x}_t|\mathbf{x}_{t-1}) > 0$ for all $\mathbf{x}_t, \mathbf{x}_{t-1} \in \mathcal{S}$, then the Markov chain defined by $T_\theta$ has a unique stationary distribution

$$\pi_\theta = \lim_{t\to\infty} \pi_\theta^t$$

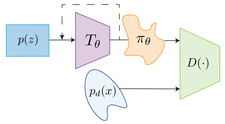
For all $\mathbf{x} \in \mathcal{S}$, $\pi_\theta$ satisfies

$$\pi_\theta(\mathbf{x}) = \int_{\mathcal{S}} T_\theta(\mathbf{x}|\mathbf{x}')\pi_\theta(\mathbf{x}')\mathrm{d}\mathbf{x}'$$

## Adversarial Training

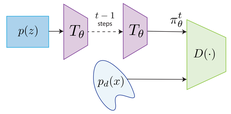
For a Markov chain, it is difficult to compute the marginal likelihood but easy to sample from it, so we consider **adversarial training** (Goodfellow et al., 2014), which is **a likelihood-free method** that trains deep generative models using a two player game between a **discriminator** $D$ and a **generator** $G$.

This cannot be done directly for $\pi_\theta$ since it would require running the chain until convergence, resulting in optimization difficulties.

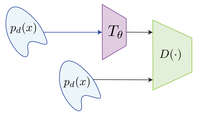



We tackle this problem by splitting the objective into two parts.

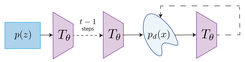
- From a noise distribution, the model should reach the data distribution in $T$ steps.
  (when outside the data distribution, move closer)




- From a data distribution, the model should remain in the data distribution after 1 step.
  (when within the data distribution, remain there)




The resulting generative process is illustrated as follows:




## Results on MNIST

With a simple MLP as the transition operator, our model is able to generate clear samples from noise in a few steps, and has fast mixing between samples. The next sample **is still related to** the previous sample in terms of digit identity and/or style.

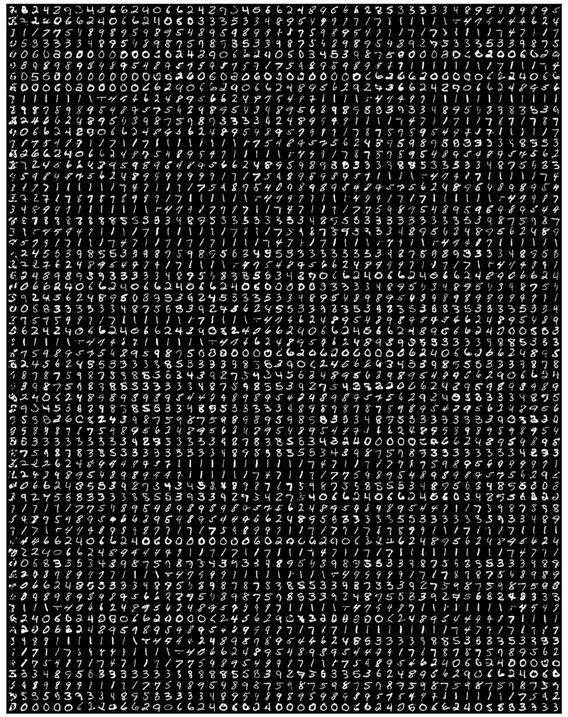


Figure 1: Samples generated from the Markov Chain trained on the MNIST dataset. The transition operator is an MLP. Each row denotes a chain sampled from step 1 to step 50.

## Results on CelebA

Samples from model trained on the CelebA dataset.

Figure 2: Samples generated from the Markov Chain trained on the CelebA dataset.

Samples from an operator where **shortcut connections** are added to the same MLP.

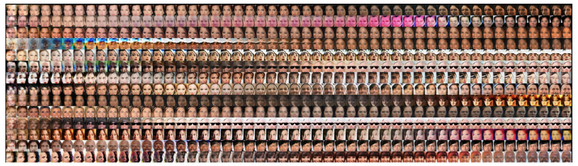


Figure 3: The transition operator is an MLP with shortcut connections, which allows slower mixing.

## Conclusion

We presented an efficient adversarial training method for learning the transition operator in a Markov chain, with few conditions enforced on the model. The chain possesses fast mixing properties, which can be controlled by modifying the architecture.

In extension, we are interested in applying this method to larger datasets, as well as performing detailed analysis over the effect of hyperparameters. It is also interesting to consider chains with certain desirable properties, such as detailed balance.