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Notes

Some results on the multivariate truncated normal distribution

William C. Horrace[a, b, *]

[a] Center for Policy Research, Syracuse University, 426 Eggers Hall, Syracuse, NY 13244-1020, USA
[b] National Bureau of Economic Research, 1050 Massachusetts Avenue, Cambridge, MA 02138, USA

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Abstract

This note formalizes some analytical results on the n-dimensional multivariate truncated normal distribution where truncation is one-sided and at an arbitrary point. Results on linear transformations, marginal and conditional distributions, and independence are provided. Also, results on log-concavity, A-unimodality and the MTP_2 property are derived.

JEL classification: C16; C24

AMS 1991 subject classification: 60E05

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1. Introduction and definitions

This note formalizes some analytical results on the n-dimensional multivariate truncated normal distribution where truncation is one-sided and at an arbitrary point. Using the characteristic function derived in [4], results on linear transformations, marginal and conditional

* Fax: 1315 443 1081.
E-mail address: whorrace@maxwell.syr.edu.

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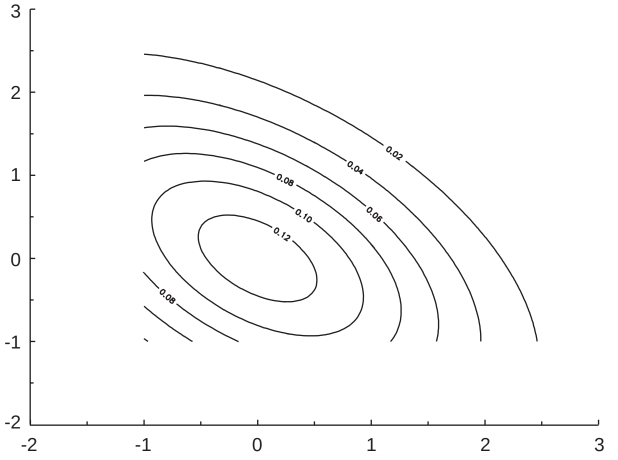

Fig. 1. Standard bivariate truncated normal contour, $\sigma_{12} = -0.5$, $\mathbf{c} = -\mathbf{1}$.

distributions, and independence are provided. Also, results on log-concavity, A-unimodality and the MTP_2 property are derived. Basic definitions follow.

Definition 1. Let $\mathbf{W}^{*\prime} = [W_1^*, \ldots, W_n^*]$ be an n-dimensional random variable with $n \geqslant 2$. $\mathbf{W}^*$ has a non-singular n-variate normal distribution with mean vector $\boldsymbol{\mu}' = [\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_n]$ and $(n \times n)$ positive definite correlation matrix $\boldsymbol{\Sigma} = \{\sigma_{ij}\}$, if it has density:

$$f_{\mathbf{W}^*}(w, \boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-n/2}(\det \boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\right\}; \mathbf{w} \in \mathbb{R}^n.$$

We use the standard notation $\mathbf{W}^* \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Now, let $\mathbf{W}' = [W_1, \ldots, W_n]$ be the truncation of $\mathbf{W}^*$ below $\mathbf{c}' = [c_1, \ldots, c_n] \in \mathbb{R}^n$.

Definition 2. W has an n-variate truncated normal distribution given by

$$\begin{aligned} f_{\mathbf{W}}(\mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) &= \frac{(2\pi)^{-n/2}(\det \boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}}{(2\pi)^{-n/2}(\det \boldsymbol{\Sigma})^{-\frac{1}{2}} \int_{\mathbf{c}}^{\infty n} \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}\, d\mathbf{w}}; \mathbf{w} \in \mathbb{R}^n_{\geqslant \mathbf{c}}; \\ &= \frac{\exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}}{\int_{\mathbf{c}}^{\infty n} \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}\, d\mathbf{w}}; \mathbf{w} \in \mathbb{R}^n_{\geqslant \mathbf{c}}, \end{aligned}$$

where $\int_{\mathbf{c}}^{\infty n}$ is a n-dimensional Riemann integral from $\mathbf{c}$ to ∞, and $\mathbb{R}^n_{\geqslant \mathbf{c}} = \{\mathbf{w} \in \mathbb{R}^n : \mathbf{w} \geqslant \mathbf{c}\}$ (the non-strict inequality ensures right-continuity of the cumulations of the probabilities of $\mathbf{W}$). Figs. 1–4 are contour plots for standard bivariate truncated normals ($\boldsymbol{\mu} = \mathbf{0}$, $\sigma_{11} = \sigma_{22} = 1$) for a few combinations of σ_{12} and $\mathbf{c}$.

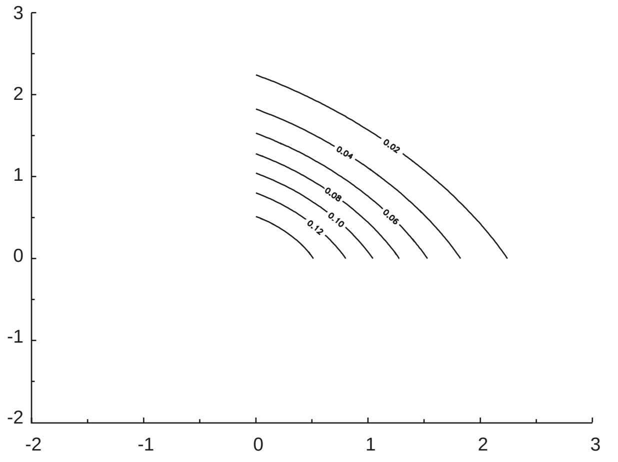

Fig. 2. Standard bivariate truncated normal contour, $\sigma_{12} = -0.5$, $\mathbf{c} = \mathbf{0}$.

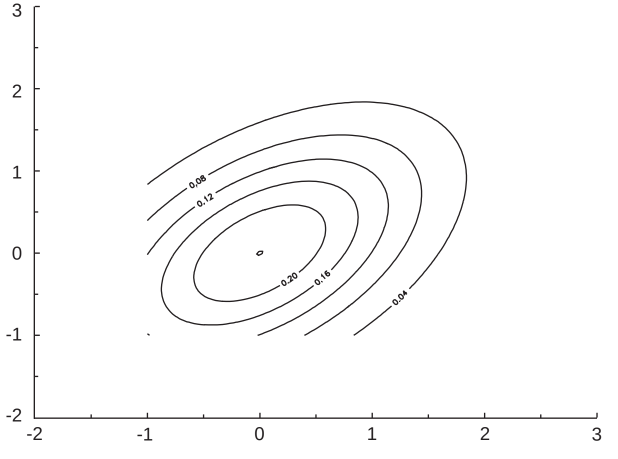

Fig. 3. Standard bivariate truncated normal contour, $\sigma_{12} = 0.5$, $\mathbf{c} = -\mathbf{1}$.

This note is concerned with truncation below $\mathbf{c}$ for each element of $\mathbf{W}^*$, however one could envision truncation of a subset of $\mathbf{W}^*$; this just requires that for certain W_j^*, the c_j go to $-\infty$ in the limit. There are also other forms of truncation that have been suggested. For example, [8] considers "elliptical truncation", where $\mathbf{W}^*$ is restricted by the condition $a < \mathbf{W}^{*\prime}\mathbf{\Sigma}^{-1}\mathbf{W}^* < b$, while [9] considers truncation of the form $\sum_{j=1}^n a_{t_j} W_j^* > a_{t_j}$. Finally, [2] considers truncation of the pair (W_1^*, W_2^*) from below, so that a specified portion of the original distribution is retained and $\sum_{j=1}^n w_j E(W_j^*)$ is maximized. In what follows it will be useful to define: $\mathbf{M} = \mathbf{c} - \boldsymbol{\mu}$, $\mathbf{t}_{(n\times 1)} \in \mathbb{R}^n$ and $\mathbf{P}_{(n\times 1)} = \mathbf{c} - \boldsymbol{\mu} - \imath\mathbf{\Sigma}\mathbf{t}$ with typical elements $M_j, t_j, P_j;\ j \in N$, $N = [1, \ldots, n]$, respectively, and $\imath = \sqrt{-1}$.

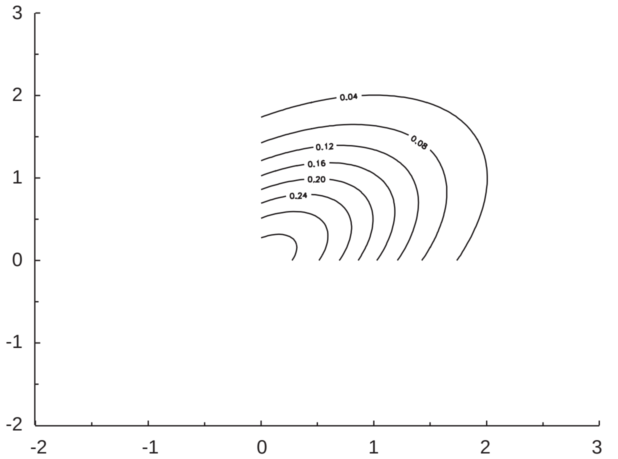

Fig. 4. Standard bivariate truncated normal contour, $\sigma_{12} = 0.5$, $\mathbf{c} = \mathbf{0}$.

Definition 3. The characteristic function of W is given by

$$CF_{\mathbf{W}}(\mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = \frac{\int_{\mathbf{M}}^{\infty} {}^{n} \exp\{-\frac{1}{2}(\mathbf{w} - \imath\boldsymbol{\Sigma}\mathbf{t})'\boldsymbol{\Sigma}^{-1}(\mathbf{w} - \imath\boldsymbol{\Sigma}\mathbf{t})\}\, d\mathbf{w}}{\int_{\mathbf{M}}^{\infty} {}^{n} \exp\{-\frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}^{-1}\mathbf{w}\}\, d\mathbf{w}} \exp\left\{\imath\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\right\}$$

$$= \frac{\int_{\mathbf{P}}^{\infty} {}^{n} \exp\{-\frac{1}{2}\mathbf{u}'\boldsymbol{\Sigma}^{-1}\mathbf{u}\}\, d\mathbf{u}}{\int_{\mathbf{M}}^{\infty} {}^{n} \exp\{-\frac{1}{2}\mathbf{u}'\boldsymbol{\Sigma}^{-1}\mathbf{u}\}\, d\mathbf{u}} \exp\left\{\imath\mathbf{t}'\boldsymbol{\mu} - \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}\mathbf{t}\right\}.$$

The result is derived in [4]. A similar formula for the moment generating function of W is derived in [7]. The univariate case ($n = 1$) was first suggested in a problem posed by Horrace and Hernandez [5]. The characteristic function is used to derive some results in the next section.

2. Distributional properties

Interest centers on determining which of the desirable properties of the multivariate normal (if any) are preserved after truncation. Let $\mathbf{D}$ and $\mathbf{b}$ be real matrices of dimension $(n \times n)$ and $(n \times 1)$, respectively, with $\det \mathbf{D} \neq \mathbf{0}$. Define the linear transformation: $\mathbf{Y} = \mathbf{DW} + \mathbf{b}$, then:

Theorem 4. *For* $\mathbf{W}$ *with general correlation structure,* $\mathbf{Y}$ *has a truncated normal distribution based on truncation of* $\mathbf{Y}^* \sim N(\mathbf{D}\boldsymbol{\mu} + \mathbf{b}, \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}')$ *below* $\mathbf{c}_{\mathbf{Y}} = \mathbf{c} + \mathbf{b}$, *if and only if* $\mathbf{D} = \mathbf{I}_n$.

The proof is contained in the mathematical appendix. Hence, the family of truncated normal distributions is *not* closed to general linear transformations (but it is closed to relocation by **b**). Notice that this result is different from the problem of transforming $\mathbf{W}^*$ to $\mathbf{Y}^* = \mathbf{D}\mathbf{W}^* + \mathbf{b}$, and then truncating $\mathbf{Y}^*$, which produces a truncated normal distribution. This is also different from the problem of the distribution of $\mathbf{W}^*$ subject to linear inequality constraints $\mathbf{c} < \mathbf{D}\mathbf{W}^* + \mathbf{b}$, which also produces a truncated normal distribution. For example see [3]. For the special case where $\boldsymbol{\Sigma}$ is diagonal, the condition in Theorem 4 for $\mathbf{Y}$ to be truncated normal is that $\mathbf{D}$ be a diagonal matrix (or more generally, a matrix formed from the permuted columns or rows of a diagonal matrix).

Theorem 4 has implications for the marginal distributions. Partition

$$\mathbf{W}' = [\underset{1\times k}{\mathbf{W}_1'} \quad \underset{1\times(n-k)}{\mathbf{W}_2'}], \quad \boldsymbol{\mu}' = [\underset{1\times k}{\boldsymbol{\mu}_1'} \quad \underset{1\times(n-k)}{\boldsymbol{\mu}_2'}], \quad \mathbf{t}' = [\underset{1\times k}{\mathbf{t}_1'} \quad \underset{1\times(n-k)}{\mathbf{t}_2'}],$$
$$\mathbf{M}' = [\underset{1\times k}{\mathbf{M}_1'} \quad \underset{1\times(n-k)}{\mathbf{M}_2'}], \quad \mathbf{P}' = [\underset{1\times k}{\mathbf{P}_1'} \quad \underset{1\times(n-k)}{\mathbf{P}_2'}], \quad \mathbf{c}' = [\underset{1\times k}{\mathbf{c}_1'} \quad \underset{1\times(n-k)}{\mathbf{c}_2'}]$$

and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \underset{k\times k}{\boldsymbol{\Sigma}_{11}} & \underset{k\times(n-k)}{\boldsymbol{\Sigma}_{12}} \\ \underset{(n-k)\times k}{\boldsymbol{\Sigma}_{21}} & \underset{(n-k)\times(n-k)}{\boldsymbol{\Sigma}_{22}} \end{bmatrix}; k < n.$$

Then the $CF_{\mathbf{W}_1}(\mathbf{t}_1, \boldsymbol{\mu}, \boldsymbol{\Sigma}, c) = CF_{\mathbf{W}}(\mathbf{t}_1, \mathbf{t}_2 = \mathbf{0}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c})$ or

$$CF_{\mathbf{W}_1}(\mathbf{t}_1, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = \frac{\int_{\mathbf{P}_1}^{\infty} {}_k \int_{\mathbf{M}_2}^{\infty} {}_{n-k} \exp\{-\frac{1}{2}\mathbf{u}'\boldsymbol{\Sigma}^{-1}\mathbf{u}\}\, d\mathbf{u}_1\, d\mathbf{u}_2}{\int_{\mathbf{M}_1}^{\infty} {}_n \int_{\mathbf{M}_2}^{\infty} {}_{n-k} \exp\{-\frac{1}{2}\mathbf{u}'\boldsymbol{\Sigma}^{-1}\mathbf{u}\}\, d\mathbf{u}_1\, d\mathbf{u}_2} \exp\left\{\imath \mathbf{t}_1'\boldsymbol{\mu}_1 - \frac{1}{2}\mathbf{t}_1'\boldsymbol{\Sigma}_{11}\mathbf{t}_1\right\},$$

which is not the characteristic function of a truncated normal distribution in general, because the probabilities in the numerator and denominator will still be a function of all of $\boldsymbol{\Sigma}$. In fact, the marginal distributions are given by

$$f_{\mathbf{W}_1}(\mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = \frac{\int_{\mathbf{c}_2}^{\infty} {}_{n-k} \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}\, d\mathbf{w}_2}{\int_{\mathbf{c}}^{\infty} {}_n \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}\, d\mathbf{w}}; \mathbf{w}_1 \in \mathbb{R}^n_{\geqslant \mathbf{c}_1}$$

and

$$f_{\mathbf{W}_2}(\mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = \frac{\int_{\mathbf{c}1}^{\infty} {}_k \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}\, d\mathbf{w}_1}{\int_{\mathbf{c}}^{\infty} {}_n \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}\, d\mathbf{w}}; \mathbf{w}_2 \in \mathbb{R}^n_{\geqslant \mathbf{c}_2}$$

implying conditional distributions:

$$f_{\mathbf{W}_1|\mathbf{W}_2}(\mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = \frac{\exp\{-\frac{1}{2}(\mathbf{w}-\mu)'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\mu)\}}{\int_{\mathbf{c}_1}^{\infty} {}_k \exp\{-\frac{1}{2}(\mathbf{w}-\mu)'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\mu)\}\, d\mathbf{w}}; \mathbf{w}_1 \in \mathbb{R}^n_{\geqslant \mathbf{c}_1}$$

and

$$f_{\mathbf{W}_2|\mathbf{W}_1}(\mathbf{w}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = \frac{\exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}}{\int_{\mathbf{c}_2}^{\infty}{}_{n-k} \exp\{-\frac{1}{2}(\mathbf{w}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{w}-\boldsymbol{\mu})\}\, d\mathbf{w}}; \mathbf{w}_2 \in \mathbb{R}^n_{\geqslant \mathbf{c}_2},$$

which are truncated normal distributions.

Conclusion 5. *The marginal distributions from a truncated normal distribution are not truncated normal distributions, in general. However, the conditional distributions are truncated normal distributions.*

It is a well-known fact that $\mathbf{W}_1^*$ and $\mathbf{W}_2^*$ are independent if and only if $\boldsymbol{\Sigma}_{12} = 0$, but is this the case for their truncations?

Theorem 6. *Define $\mathbf{W}_1$ and $\mathbf{W}_2$ as above, then $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent if and only if $\boldsymbol{\Sigma}_{12} = 0$.*

The proof is in Appendix and follows from the fact that when $\boldsymbol{\Sigma}_{12} = 0$, then

$$CF_W(\mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = CF_{\mathbf{W}_1}(\mathbf{t}_1, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c})CF_{\mathbf{W}_2}(\mathbf{t}_2, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}).$$

Therefore, it follows that:

Corollary 7. *If $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent, then the marginal distribution of $\mathbf{W}_1$ is that of a $\mathbf{W}_1^* \sim N(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ random variable truncated at $\mathbf{c}_1$, and the marginal distribution of $\mathbf{W}_2$ is that of a $\mathbf{W}_2^* \sim N(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$ random variable truncated at $\mathbf{c}_2$.*

The intuition is that, in the independent case, the marginal scalings $\Pr\{\mathbf{W}_1^* > \mathbf{c}_1\}$ and $\Pr\{\mathbf{W}_1^* > \mathbf{c}_1\}$ are preserved by the joint scaling $\Pr\{\mathbf{W}^* > \mathbf{c}\}$.

2.1. Log-concavity, A-unimodalty and the MTP$_2$ property

It is well known that multivariate normal distributions possess certain properties that make them useful for economic theory and probability theory. Our purpose here is to see if a few of these properties hold up after truncation.

Definition 8. A multivariate density function $f : \mathbb{R}^n \to [0, \infty)$ is log-concave if:

$$f(\alpha\mathbf{w} + (1-\alpha)\mathbf{y}) \geqslant [f(\mathbf{w})]^{\alpha}[f(\mathbf{y})]^{1-\alpha}$$

holds for all $w, \mathbf{y} \in \mathbb{R}^n$ and all $\alpha \in [0, 1]$.

This is known to hold for multivariate normal distributions. The following theorem proves that it holds for truncated normals as well.

Theorem 9. *If $\mathbf{W}^*$ is multivariate normal, then the distribution of the truncation of W^* below $\mathbf{c}$ is log-concave.*

The proof is in Appendix. Log-concavity leads to several important probabilistic results. For example, [6] shows that for log-concave density function and any sets $A, B \in \mathbb{R}^n$:

$$\Pr(\alpha A + (1-\alpha)B) \geqslant [\Pr(A)]^{\alpha}[\Pr(B)]^{1-\alpha}.$$

Theorem 9 formalizes the result for the truncated case.

Definition 10. A density function $f : \mathbb{R}^n \to [0, \infty)$ is A-unimodal if the set:

$$A_{\lambda} = \{\mathbf{w} : f(\mathbf{w}) \geqslant \lambda\}$$

is convex for all $\lambda > 0$.

A-unimodality is just an n-dimensional generalization of scalar unimodality. Since all log-concave functions are A-unimodal (see [10, p. 72]), it follows that:

Conclusion 11. *If $\mathbf{W}^*$ is multivariate normal, then the distribution of the truncation of $\mathbf{W}^*$ below $\mathbf{c}$ is A-unimodal.*

Unfortunately, $\mathbf{W}$ does not possess a symmetric distribution, so many of the properties that hinge on A-unimodality *and* symmetry are lost. For example, [1] presents a theorem involving the monotonicity property of integrals over A-unimodal symmetric (about the origin) functions. However, it would be useful to determine any special cases that may still hold. The theorem is:

Theorem 12. *Let E be a convex set in $\mathbb{R}^n$, symmetric about the origin. Let $f(w)$ be a function such that (i) $f(w) = f(-w)$ (ii) $A_{\lambda} = \{\mathbf{w} : f(\mathbf{w}) \geqslant \lambda\}$ is convex for all $\lambda > 0$, and (iii) $\int_{\mathbf{E}^n} f(w)\,dw < \infty$ (in the Lebesgue sense). Then,*

$$\int_{\mathbf{E}^n} f(w + \alpha y)\,dw \geqslant \int_{\mathbf{E}^n} f(w + y)\,dw,$$

$\alpha \in [0, 1]$.

The proof is in [1]. Clearly, this holds for $\mathbf{W}^* \sim N(\mathbf{0}, \mathbf{\Sigma})$. We now present a version of this theorem that holds for truncations of $\mathbf{W}^*$. That is, we relax the condition above that $f(\mathbf{w}) = f(-\mathbf{w})$.

Theorem 13. *Let $\mathbf{W} \in \mathbb{R}^n_{\geqslant c}$ be the truncation of $\mathbf{W}^* \sim N(\mathbf{0}, \mathbf{\Sigma})$ below $\mathbf{c}$. Further, let $f_{\mathbf{W}}$ be the density function of $\mathbf{W}$, and let $y \in \mathbb{R}^n_-$, then Theorem 12 holds for $f_{\mathbf{W}}$.*

The proof is in Appendix and hinges on the translation, y, being negative ($y \in \mathbb{R}^n_-$). If the translation is unrestricted, then Theorem 12 only holds for *some* cases but not all. In particular:

Corollary 14. *Let* $\mathbf{W} \in \mathbb{R}^n_{\geqslant c}$ *be the truncation of* $\mathbf{W}^* \sim N(\mathbf{0}, \mathbf{\Sigma})$ *below* $\mathbf{c}$. *Further, let* $f_{\mathbf{W}}$ *be the density function of* $\mathbf{W}$, *and let* $y \in \mathbb{R}^n$, *then Theorem 12 holds for* $f_{\mathbf{W}}$ *if* $\{(E + \alpha y) \cap \mathbb{R}^n_{<c}\} = \emptyset$, *where* $\mathbb{R}^n_{<c}$ *is the compliment of* $\mathbb{R}^n_{\geqslant c}$.

The result follows simply from arguments in the proof of Theorem 13. The implication is that as long as the (non-strictly) smaller translation, αy, does not produce a truncation in the support of $\mathbf{W}$, then Anderson's monotonicity property holds.

Definition 15. A density function $f : \mathbb{R}^n \to [0, \infty)$ is multivariate-totally-positive-of-order-2 (MTP_2) if:

$$f(\mathbf{y}) f(\mathbf{y}^*) \leqslant f(\mathbf{w}) f(\mathbf{w}^*)$$

holds for all $\mathbf{y}, \mathbf{y}^*$ in the domain of f, where

$$\begin{aligned} w_j &= \max\{y_j, y_j^*\} \quad \text{and} \\ w_j^* &= \min\{y_j, y_j^*\}, \ \ j = 1, \ldots, n. \end{aligned}$$

If a multivariate normal distribution satisfies the MTP_2 property then the off-diagonal elements of the covariance matrix are all non-negative. (See [10, p. 77].)

Theorem 16. *If* $\mathbf{W}^*$ *is multivariate normal and satisfies the* MTP_2 *property, then the distribution of the truncation of* W^* *below* $\mathbf{c}$ *satisfies the* MTP_2 *property also.*

The proof is Appendix and formalizes the normal result for the truncated case.

3. Conclusions

This note presents a few results for the multivariate truncated normal distribution. The results may be particularly useful in economic applications where truncated random variables are used to describe data generation processes (e.g., see [4]). Results on linear transformations imply that sums of independent truncated normals are not truncated normal, so the simple average from a random sample of truncated normal variates is not truncated normal. Additionally, the MTP_2 result implies many other useful properties for the truncated normal: conditionally increasing in sequence, positively associated, positively dependent in increasing sets, positively upper orthant-dependent, and non-negatively correlated. Finally, the results may be extended to truncations of scale mixtures of multivariate normal distributions.

Acknowledgment

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Appendix A.

Proof of Theorem 4. By a well-known result on linear transformation of random variable:

$$CF_{\mathbf{Y}}(\mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = e^{\imath \mathbf{t}' b} E(e^{\imath \mathbf{t}'\mathbf{DW}}),$$

$$CF_{\mathbf{Y}}(\mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = e^{\imath \mathbf{t}' b} \frac{\int_{\mathbf{c}}^{\infty}{}_n \exp\{\imath \mathbf{t}'\mathbf{D}w\} \exp\{-\frac{1}{2}(w-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(w-\boldsymbol{\mu})\}\, dw}{\int_{\mathbf{c}}^{\infty}{}_n \exp\{-\frac{1}{2}(w-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(w-\boldsymbol{\mu})\}\, dw}.$$

Let $\mathbf{W} = \mathbf{W} - \boldsymbol{\mu}$, then $\mathbf{W} \in [\mathbf{M}, \infty]$:

$$CF_{\mathbf{Y}}(\mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = \frac{\int_{\mathbf{M}}^{\infty}{}_n \exp\{\imath \mathbf{t}'\mathbf{D}w - \frac{1}{2}w'\boldsymbol{\Sigma}^{-1}w\}\, dw}{\int_{\mathbf{M}}^{\infty}{}_n \exp\{-\frac{1}{2}w'\boldsymbol{\Sigma}^{-1}w\}\, dw} \exp\{\imath \mathbf{t}'(\mathbf{D}\boldsymbol{\mu} + \mathbf{b})\}.$$

Now $\imath t'\mathbf{D}w - \frac{1}{2}w'\boldsymbol{\Sigma}^{-1}w = -\frac{1}{2}t'\mathbf{D}\boldsymbol{\Sigma}\mathbf{D}'t - \frac{1}{2}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'t)'\boldsymbol{\Sigma}^{-1}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'t)$ and:

$$\begin{aligned} CF_{\mathbf{Y}}(\mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) &= \frac{\int_{\mathbf{M}}^{\infty}{}_n \exp\{-\frac{1}{2}\mathbf{t}'\mathbf{D}\boldsymbol{\Sigma}\mathbf{D}'\mathbf{t} - \frac{1}{2}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'\mathbf{t})'\boldsymbol{\Sigma}^{-1}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'\mathbf{t})\}\, dw}{\int_{\mathbf{M}}^{\infty}{}_n \exp\{-\frac{1}{2}w'\boldsymbol{\Sigma}^{-1}w\}\, dw} \\ &\quad \times \exp\left\{\imath \mathbf{t}'(\mathbf{D}\boldsymbol{\mu} + \mathbf{b})\right\} \\ &= \frac{\int_{\mathbf{M}}^{\infty}{}_n \exp\{-\frac{1}{2}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'\mathbf{t})'\boldsymbol{\Sigma}^{-1}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'\mathbf{t})\}\, dw}{\int_{\mathbf{M}}^{\infty}{}_n \exp\{-\frac{1}{2}w'\boldsymbol{\Sigma}^{-1}w\}\, dw} \\ &\quad \times \exp\left\{\imath \mathbf{t}'(\mathbf{D}\boldsymbol{\mu} + b) - \frac{1}{2}\mathbf{t}'\mathbf{D}\boldsymbol{\Sigma}\mathbf{D}'\mathbf{t}\right\}. \end{aligned}$$

Now $\mathbf{Y}^* \sim N(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{Y}})$ with $\boldsymbol{\mu}_{\mathbf{Y}} = \mathbf{D}\boldsymbol{\mu} + \mathbf{b}$, $\boldsymbol{\Sigma}_{\mathbf{Y}} = \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}'$ and $\det \mathbf{D} > 0$, so $-\frac{1}{2}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'t)'\boldsymbol{\Sigma}^{-1}(w - \imath\boldsymbol{\Sigma}\mathbf{D}'t) = -\frac{1}{2}(\mathbf{D}w - \imath\boldsymbol{\Sigma}_{\mathbf{Y}}t)'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}(\mathbf{D}w - \imath\boldsymbol{\Sigma}_{\mathbf{Y}}t)$ and $-\frac{1}{2}w'\boldsymbol{\Sigma}^{-1}w = -\frac{1}{2}(\mathbf{D}w)'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}\mathbf{D}w$, so the characteristic function becomes

$$\begin{aligned} CF_{\mathbf{Y}}(\mathbf{t}, \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{Y}}, \mathbf{c}) &= \frac{\int_{\mathbf{M}}^{\infty}{}_n \exp\{-\frac{1}{2}(\mathbf{D}w - \imath\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{t})'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}(\mathbf{D}w - \imath\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{t})\}\, dw}{\int_{\mathbf{M}}^{\infty}{}_n \exp\{-\frac{1}{2}(\mathbf{D}w)'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}\mathbf{D}w\}\, dw} \\ &\quad \times \exp\left\{\imath \mathbf{t}'\boldsymbol{\mu}_{\mathbf{Y}} - \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}\mathbf{t}\right\}. \end{aligned}$$

Let $\mathbf{M} = \mathbf{M}_{\mathbf{Y}} = \mathbf{c}_{\mathbf{Y}} - \boldsymbol{\mu}_{\mathbf{Y}}$, where $\mathbf{c}_{\mathbf{Y}} = \mathbf{c} + \mathbf{b} + (\mathbf{I}_n - \mathbf{D})\boldsymbol{\mu}$, then

$$\begin{aligned} CF_{\mathbf{Y}}(\mathbf{t}, \boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{Y}}, \mathbf{c}_{\mathbf{Y}}) &= \frac{\int_{\mathbf{M}_{\mathbf{Y}}}^{\infty}{}_n \exp\{-\frac{1}{2}(\mathbf{D}w - \imath\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{t})'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}(\mathbf{D}w - \imath\boldsymbol{\Sigma}_{\mathbf{Y}}\mathbf{t})\}\, dw}{\int_{\mathbf{M}_{\mathbf{Y}}}^{\infty}{}_n \exp\{-\frac{1}{2}(\mathbf{D}w)'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}\mathbf{D}w\}\, dw} \\ &\quad \times \exp\left\{\imath \mathbf{t}'\boldsymbol{\mu}_{\mathbf{Y}} - \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_{\mathbf{Y}}^{-1}\mathbf{t}\right\} \end{aligned}$$

which is the not the characteristic function of a truncated normal variate in general. If $\mathbf{D} = \mathbf{I}_n$, then by the *uniqueness theorem of characteristic functions* this is the characteristic function of $Y^* \sim N(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{Y}})$ truncated below $\mathbf{c}_{\mathbf{Y}} = \mathbf{c} + \mathbf{b}$. Also, if this is the characteristic function of $Y^* \sim N(\boldsymbol{\mu}_{\mathbf{Y}}, \boldsymbol{\Sigma}_{\mathbf{Y}})$ truncated below $\mathbf{c_Y} = \mathbf{c} + \mathbf{b}$ then by the *uniqueness theorem of characteristic functions* $\mathbf{D}$ must equal $\mathbf{I}_n$. $\square$

Proof of Theorem 6. Let $\boldsymbol{\Sigma}_{12} = 0$, then:

$$
\begin{aligned}
CF_{\mathbf{W}_1}(\mathbf{t}_1, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) &= \frac{\int_{\mathbf{P}_1}^{\infty} {}^{k} \int_{\mathbf{M}_2}^{\infty} {}^{n-k} \exp\{-\frac{1}{2}[u_1'\boldsymbol{\Sigma}_{11}^{-1}u_1 + u_2'\boldsymbol{\Sigma}_{22}^{-1}u_2]\}\, du_1\, du_2}{\int_{\mathbf{M}_1}^{\infty} {}^{k} \int_{\mathbf{M}_2}^{\infty} {}^{n-k} \exp\{-\frac{1}{2}[u_1'\boldsymbol{\Sigma}_{11}^{-1}u_1 + u_2'\boldsymbol{\Sigma}_{22}^{-1}u_2]\}\, du_1\, du_2} \\
&\quad \times \exp\left\{\imath \mathbf{t}_1'\boldsymbol{\mu}_1 - \frac{1}{2}\mathbf{t}_1'\boldsymbol{\Sigma}_{11}\mathbf{t}_1\right\} \\
&= \frac{\int_{\mathbf{P}_1}^{\infty} {}^{k} \exp\{-\frac{1}{2}u_1'\boldsymbol{\Sigma}_{11}^{-1}u_1\}\, du_1}{\int_{\mathbf{M}_1}^{\infty} {}^{k} \exp\{-\frac{1}{2}u_1'\boldsymbol{\Sigma}_{11}^{-1}u_1\}\, du_1} \exp\left\{\imath \mathbf{t}_1'\boldsymbol{\mu}_1 - \frac{1}{2}\mathbf{t}_1'\boldsymbol{\Sigma}_{11}\mathbf{t}_1\right\}
\end{aligned}
$$

which happens to be a truncated normal characteristic function, as is

$$
\begin{aligned}
CF_{\mathbf{W}_2}(\mathbf{t}_2, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) &= \frac{\int_{\mathbf{M}_1}^{\infty} {}^{k} \int_{p_2}^{\infty} {}^{n-k} \exp\{-\frac{1}{2}[u_1'\boldsymbol{\Sigma}_{11}^{-1}u_1 + u_2'\boldsymbol{\Sigma}_{22}^{-1}u_2]\}\, du_1\, du_2}{\int_{\mathbf{M}_1}^{\infty} {}^{k} \int_{\mathbf{M}_2}^{\infty} {}^{n-k} \exp\{-\frac{1}{2}[u_1'\boldsymbol{\Sigma}_{11}^{-1}u_1 + u_2'\boldsymbol{\Sigma}_{22}^{-1}u_2]\}\, du_1\, du_2} \\
&\quad \times \exp\left\{\imath \mathbf{t}_2'\boldsymbol{\mu}_2 - \frac{1}{2}\mathbf{t}_2'\boldsymbol{\Sigma}_{11}\mathbf{t}_2\right\}, \\
&= \frac{\int_{p_2}^{\infty} {}^{n-k} \exp\{-\frac{1}{2}u_2'\boldsymbol{\Sigma}_{22}^{-1}u_2\}\, du_2}{\int_{\mathbf{M}_2}^{\infty} {}^{n-k} \exp\{-\frac{1}{2}u_2'\boldsymbol{\Sigma}_{22}^{-1}u_2\}\, du_2} \exp\left\{\imath \mathbf{t}_2'\boldsymbol{\mu}_2 - \frac{1}{2}\mathbf{t}_2'\boldsymbol{\Sigma}_{11}\mathbf{t}_2\right\}.
\end{aligned}
$$

Hence, the joint characteristic function equals the product of the marginal characteristic functions:

$$
CF_{\mathbf{W}}(\mathbf{t}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}) = CF_{\mathbf{W}_1}(\mathbf{t}_1, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c})CF_{\mathbf{W}_2}(\mathbf{t}_2, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \mathbf{c}).
$$

Since W_1 and W_2 are independent if and only if their joint characteristic is the product of the marginal characteristic functions and W_1^* and W_2^* are independent if and only if $\boldsymbol{\Sigma}_{12} = 0$, and the proof is complete. $\square$

Proof of Theorem 9. Notice that $f_{\mathbf{W}}(\mathbf{w}) > 0 \Longrightarrow f_{\mathbf{W}^*}(\mathbf{w}) > 0$. Consider two cases.

Case 1: $w_j, y_j \geqslant c_j$ for all $j \Longrightarrow f_{\mathbf{W}}(\mathbf{w}), f_{\mathbf{W}}(\mathbf{y}) > 0 \Longrightarrow f_{\mathbf{W}^*}(\mathbf{w}), f_{\mathbf{W}^*}(\mathbf{y}) > 0$. Now $\alpha \in [0, 1]$ implies

$$
\begin{aligned}
\alpha w_j &\geqslant \alpha c_j, \\
(1-\alpha)y_j &\geqslant (1-\alpha)c_j
\end{aligned}
$$

or $\alpha w_j + (1-\alpha)y_j \geqslant c_j$ for all $j \Longrightarrow f_{\mathbf{W}}(\alpha\mathbf{w} + (1-\alpha)\mathbf{y}) > 0 \Longrightarrow f_{\mathbf{W}^*}(\alpha\mathbf{w} + (1-\alpha)\mathbf{y}) > 0$. Then we are always in the range above $\mathbf{c}$ where the condition holds for the multivariate normal $f_{\mathbf{W}^*}$:

$$\begin{aligned} f_{\mathbf{W}^*}(\alpha\mathbf{w}+(1-\alpha)\mathbf{y}) &\geqslant [f_{\mathbf{W}^*}(\mathbf{w})]^{\alpha}[f_{\mathbf{W}^*}(\mathbf{y})]^{1-\alpha} \\ \frac{f_{\mathbf{W}^*}(\alpha\mathbf{w}+(1-\alpha)\mathbf{y})}{\int_{\underset{c}{n}}^{\infty}\exp\{-\frac{1}{2}(w-\mu)'\mathbf{\Sigma}^{-1}(w-\mu)\}\,dw} &\geqslant \frac{[f_{\mathbf{W}^*}(\mathbf{w})]^{\alpha}[f_{\mathbf{W}^*}(\mathbf{y})]^{1-\alpha}}{\int_{\underset{c}{n}}^{\infty}\exp\{-\frac{1}{2}(w-\mu)'\mathbf{\Sigma}^{-1}(w-\mu)\}\,dw} \\ f_{\mathbf{W}}(\alpha\mathbf{w}+(1-\alpha)\mathbf{y}) &\geqslant [f_{\mathbf{W}}(\mathbf{w})]^{\alpha}[f_{\mathbf{W}}(\mathbf{y})]^{1-\alpha} \end{aligned}$$

Case 2: $w_j < c_j$ for some j or $y_j < c_j$ for some $j \Longrightarrow [f_{\mathbf{W}}(\mathbf{w})]^{\alpha}[f_{\mathbf{W}}(\mathbf{y})]^{1-\alpha} = 0$. The condition holds, since $f_{\mathbf{W}}(\alpha w_j + (1-\alpha)y_j) \geqslant 0$ for all j. $\square$

Proof of Theorem 13. The probability statement in Theorem 12 is equivalent to:

$$\int_{\underset{\mathbf{E}+\alpha y}{n}} f_{\mathbf{W}^*}(w)\,dw \geqslant \int_{\underset{\mathbf{E}+y}{n}} f_{\mathbf{W}^*}(w)\,dw,$$

which certainly holds, when the condition $E(\mathbf{W}) = \mathbf{0}$ holds. We want to show that this holds for $f_{\mathbf{W}}$ under the same condition when y is negative. Define the following partitions:

$$\begin{aligned} A &= \{(E+y)\cap\mathbb{R}^n_{\geqslant c}\}, \\ B &= \{(E+y)\cap\mathbb{R}^n_{<c}\}, \\ C &= \{(E+\alpha y)\cap\mathbb{R}^n_{\geqslant c}\}, \\ D &= \{(E+\alpha y)\cap\mathbb{R}^n_{<c}\}. \end{aligned}$$

Notice that for the truncation below $\mathbf{c}$,

$$\int_{\underset{B}{n}} f_{\mathbf{W}}(w)\,dw = \int_{\underset{D}{n}} f_{\mathbf{W}}(w)\,dw = 0,$$

so that, for $Q = \Pr\{\mathbf{W}^* \geqslant \mathbf{c}\}$, the following statements are true:

$$\int_{\underset{\mathbf{E}+y}{n}} f_{\mathbf{W}^*}(w)\,dw \geqslant Q\int_{\underset{\mathbf{E}+y}{n}} f_{\mathbf{W}}(w)\,dw, \tag{1}$$

$$\int_{\underset{\mathbf{E}+\alpha y}{n}} f_{\mathbf{W}^*}(w)\,dw \geqslant Q\int_{\underset{\mathbf{E}+\alpha y}{n}} f_{\mathbf{W}}(w)\,dw. \tag{2}$$

The inequalities are strict when $B \neq \emptyset$ and $D = \emptyset$, respectively, and become equalities when $B = \emptyset$ and $D = \emptyset$, respectively. We consider four cases which exhaust the possibilities for the content of B and D. We show that in Cases 1 and 3 the integral condition holds for any translation (positive or negative) of the set E, but Cases 2 and 4 require the translation to be negative.

Case 1: $B = D = \emptyset$. In this case, Eqs. (1) and (2) hold with equality.

$$\begin{aligned} Q\int_{\underset{\mathbf{E}+\alpha y}{n}} f_{\mathbf{W}}(w)\,dw &= \int_{\underset{\mathbf{E}+\alpha y}{n}} f_{\mathbf{W}^*}(w)\,dw \\ &\geqslant \int_{\underset{\mathbf{E}+y}{n}} f_{\mathbf{W}^*}(w)\,dw \geqslant Q\int_{\underset{\mathbf{E}+y}{n}} f_{\mathbf{W}}(w)\,dw \end{aligned}$$

or

$$Q\int_{\mathbf{E}+\alpha y}^{n} f_{\mathbf{W}}(w)\,dw \geqslant Q\int_{\mathbf{E}+y}^{n} f_{\mathbf{W}}(w)\,dw.$$

Therefore, the theorem holds for any y.

Case 2: $B \neq \emptyset$, $D \neq \emptyset$. In this case, both $(E+y)$ and $(E+\alpha y)$ are truncated from below at $\mathbf{c}$. If y is negative then $A \subseteq C$, and

$$\int_{\mathbf{E}+\alpha y}^{n} f_{\mathbf{W}}(w)\,dw = \int_{\mathbf{C}}^{n} f_{\mathbf{W}}(w)\,dw \geqslant \int_{\mathbf{A}}^{n} f_{\mathbf{W}}(w)\,dw = \int_{\mathbf{E}+y}^{n} f_{\mathbf{W}}(w)\,dw.$$

Therefore, the theorem holds for y negative.

Case 3: $B \neq \emptyset$, $D = \emptyset$. In this case, the inequality in Eq. (1) is strict while Eq. (2) holds with equality. Hence,

$$\begin{aligned} Q\int_{\mathbf{E}+\alpha y}^{n} f_{\mathbf{W}}(w)\,dw &= \int_{\mathbf{E}+\alpha y}^{n} f_{\mathbf{W}^*}(w)\,dw \\ &\geqslant \int_{\mathbf{E}+y}^{n} f_{\mathbf{W}^*}(w)\,dw \\ &> Q\int_{\mathbf{E}+y}^{n} f_{\mathbf{W}}(w)\,dw \end{aligned}$$

and the theorem holds for any y.

Case 4: $B = \emptyset$, $D \neq \emptyset$. If y is negative, then this case is precluded. If B is empty, then the negative translation of E by y resulted in no truncation of the set $E + y$. Therefore, the negative translation of E by the (non-strictly) smaller αy will not produce truncation of $E + \alpha y$, which contradicts the condition that $D \neq \emptyset$.

Clearly, Theorem 12 only holds more generally (for any y) when $D = \emptyset$. □

Proof of Theorem 16. Consider two cases.

Case 1: $y_j, y_j^* \geqslant c_j$ for all $j \implies f_{\mathbf{W}}(\mathbf{y}), \mathbf{f}_{\mathbf{W}}(\mathbf{y}^*) > \mathbf{0}$. This also implies: $w_j = \max\{y_j, y_j^*\} \geqslant c_j$ and $w_j^* = \min\{y_j, y_j^*\} \geqslant c_j$ for all $j \implies f_{\mathbf{W}}(\mathbf{w}),\ f_{\mathbf{W}}(\mathbf{w}^*) > 0$. Therefore we are in the non-truncated range of $f_{\mathbf{W}}$. Since the condition holds in this range for $f_{\mathbf{W}^*}$, it must hold for $f_{\mathbf{W}}$.

Case 2: $y_j^* < c_j$ or $y_j < c_j$ for some $j \implies f_{\mathbf{W}}(\mathbf{y}^*) = 0$ or $f_{\mathbf{W}}(\mathbf{y}) = 0$. This also implies that $w_j^* = \min\{y_j, y_j^*\} < c_j$ for some $j \implies f_{\mathbf{W}}(\mathbf{w}^*) = 0$, and the condition holds with equality at zero. □

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