Ch 3. Various Kinds of Options

I. Path Independent Options

II. Path Dependent Options

III. Exotic Options

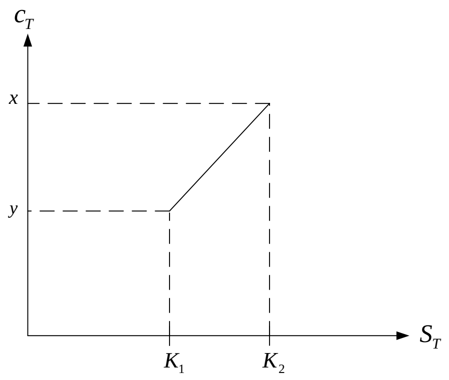
- This chapter is devoted to introduce different kinds of option contracts. For most cases, only call options are considered, and it is straightforward to infer the counterpart of put options.
- There are two goals of this chapter. First, students can learn a variety of features of options and understand it is possible to modify these features or combine them arbitrarily to create new option contracts to satisfy customers' needs. Second, for each option contract, students should think about how to price this contract with the martingale pricing method and Monte Carlo simulation.

I. Path Independent Options

- A path independent option is an option whose payoff depends only on the prices of the underlying assets at one particular future time point, which is the maturity date in most cases.
- Proportional payoff calls: $c_T = \alpha[\max(S_T - K, 0)]$, where α is usually smaller than 1
- Capped calls: $c_T = \begin{cases} 0 & \text{if } S_T < K_1 \\ S_T - K_1 & \text{if } K_1 \leq S_T < K_2 \\ K_2 - K_1 & \text{if } K_2 \leq S_T \end{cases}$

- Payoff segment calls (局部支付權證): $c_T = \begin{cases} 0 & \text{if } S_T < K_1 \\ y + \alpha(S_T - K_1) & \text{if } K_1 \leq S_T < K_2 \\ 0 & \text{if } K_2 \leq S_T \end{cases}$

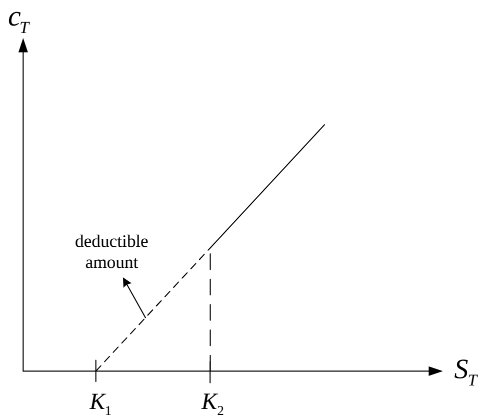
Figure 3-1



- Cash-or-nothing calls (CNC): Payoff segment calls with $\alpha = 0$

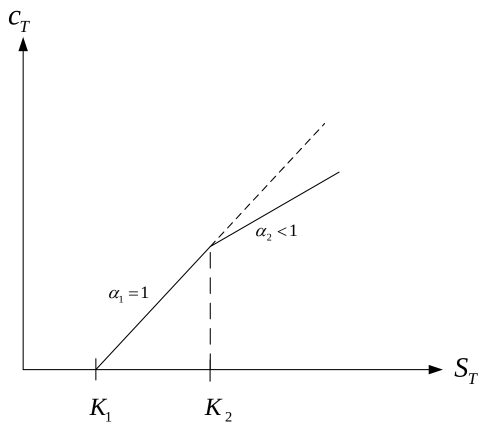
- Deductible calls (DC): $c_T = \begin{cases} 0 & \text{if } S_T < K_1 \\ 0 & \text{if } K_1 \leq S_T < K_2 \\ S_T - K_1 & \text{if } K_2 \leq S_T \end{cases}$

Figure 3-2



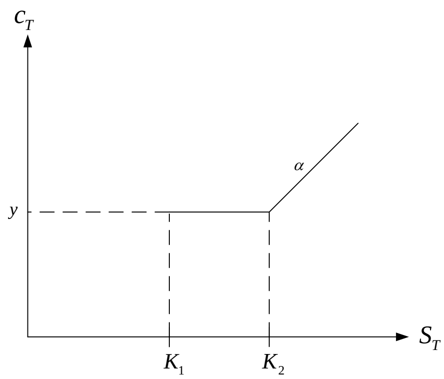
- Variation 1 of DC: $c_T = \begin{cases} 0 & \text{if} \quad S_T < K_1 \\ S_T - K_1 & \text{if} \quad K_1 \le S_T < K_2 \\ (K_2 - K_1) + \alpha_2 (S_T - K_2) \text{ for } \alpha_2 < 1 & \text{if} \quad K_2 \le S_T \end{cases}$

Figure 3-3



- Variation 2 of DC: $c_T = \begin{cases} 0 & \text{if} \quad S_T < K_1 \\ y & \text{if} \quad K_1 \le S_T < K_2 \\ y + \alpha(S_T - K_2) & \text{if} \quad K_2 \le S_T \end{cases}$

Figure 3-4



- Digit calls: $c_T = \begin{cases} y & \text{if} \quad S_T \geq K \\ 0 & \text{o/w} \end{cases}$

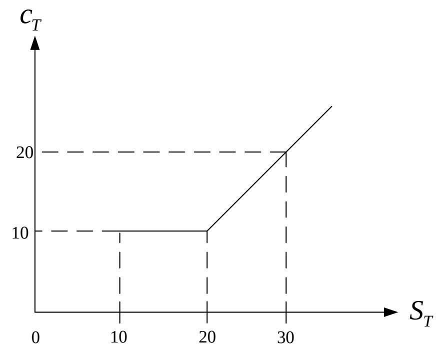
- Binary options: $\begin{cases} \text{Cash-or-nothing call (CNC):} & \text{the same as the digit call} \\ \text{Asset-or-nothing call (ANC):} & c_T = \begin{cases} S_T & \text{if} \quad S_T \geq K \\ 0 & \text{o/w} \end{cases} \end{cases}$

- C-brick options: $c_T = \begin{cases} 1 & \text{if} \quad K_1 \leq S_T < K_2 \\ 0 & \text{o/w} \end{cases}$

- A-brick option: $c_T = \begin{cases} S_T & \text{if} \quad K_1 \leq S_T < K_2 \\ 0 & \text{o/w} \end{cases}$

- The combinations of the payoffs of C-brick and A-brick can replicate the payoffs of many other options. The first example is to replicate the plain vanilla call with 1 share of A-brick (K, ∞) and $(-K)$ shares of C-brick (K, ∞). The second example is to replicate the variation 2 of DC $(K_1 = 10, K_2 = 20, y = 10, \alpha = 1)$ through combining 10 shares of C-brick $(10, 20)$, 1 share of A-brick $(20, \infty)$, and (-10) shares of C-brick $(20, \infty)$. Due to the law of one price, the value of the synthetic option portfolio should be identical to the value of the option to be replicated.

Figure 3-5



- The underlying assets of options could be other derivatives, e.g., futures, options, or swaps.

- Futures options: $c_T = \max(F_T - K, 0)$ and $p_T = \max(K - F_T, 0)$ where F_T is the futures price (not futures value) on the maturity date T. At the time point T, the call holder will receive one unit of the long position of the futures plus the cash $F_T - K$ if $F_T \geq K$, and the put holder will receive one unit of the short position of the futures plus the cash $K - F_T$ if $K \geq F_T$.

  Advantages of futures options: 1) Reduce the settlement cost due to the cash settlement rather than physical delivery. 2) When the underlying asset is not tradable, e.g., various kinds of weather options, almost all of these kinds options are futures options.

- Chooser options:

  Standard form: $\text{payoff}_t = \max(c_t(K, T), p_t(K, T))$. At time t, option holders choose bwtween the European call or put with the strike price K and the maturity date T.

  Variation 1: Option holders can choose between a series of European calls and puts at $0 < t_1, t_2, \cdots, t_n \leq T$, payoff $= \sum_{i=1}^{n} \max(c_{t_i}(K, T), p_{t_i}(K, T))$ if we ignore the time value of money.

  Varaition 2: It is possible that the European call or put are with different K and T in chooser options: $\text{payoff}_t = \max(c_t(K_1, T_1), p_t(K_2, T_2))$.

- Compound options:

  $$\text{Call on call}: cc_t = \max(0, e^{-r(T-t)}E[\max(S_T - K, 0)] - k)$$
  $$\text{Put on call}: pc_t = \max(0, k - e^{-r(T-t)}E[\max(S_T - K, 0)])$$

  (Following the same logic, you can deduce the payoffs of the call on put or put on put.)

- Rainbow option: option involving two or more risky assets are referred as rainbow option, for example, basket options or calls on the maximum of multiple assets.

- Options on the maximum or the minimum of multiple risky assets:

  $c_{\max} = \max(\max(S_{1T}, S_{2T}, S_{3T}, \cdots, S_{nT}) - K, 0)$

  $c_{\min} = \max(\min(S_{1T}, S_{2T}, S_{3T}, \cdots, S_{nT}) - K, 0)$

- Basket options: $c_T = \max(P_T - K, 0)$, where $P_T = \sum_{i=1}^{n} w_i S_{iT}$ and $\sum_{i=1}^{n} w_i = 1$

- Exchange options: $\text{payoff}_T = \max(S_{1T} - S_{2T}, 0)$ (A common application of the exchange option is to design the compensation plan for fund managers such that it is dependent on the excess return over the market return, i.e., $\max(r_p - r_m, 0)$.)

- Bivariate CNC (BCNC): $$c_T = \begin{cases} y & \text{if } \; S_{1T} \underset{<}{>} K_1 \text{ and } S_{2T} \underset{<}{>} K_2 \qquad \text{(BCNP)} \\ & \\ 0 & \quad \text{o/w} \end{cases}$$

- Bivariate ANC (BANC): $$c_T = \begin{cases} S_{1T} & \text{if } \; S_{1T} \underset{<}{>} K_1 \text{ and } S_{2T} \underset{<}{>} K_2 \qquad \text{(BANP)} \\ & \\ 0 & \quad \text{o/w} \end{cases}$$

- Exchange rate options: $$\begin{aligned} c_T &= \max(X_T - K, 0) \\ &= \max(X_T B'(T,T) - KB(T,T), 0) \\ c_t &= \max(X_t B'(t,T) - KB(t,T), 0) \end{aligned}$$

  (This kind of option actually takes both the interest rate and exchange rate into account.)

  X_t: exchange rate at t (the value of one dollar foreign currency in units of the domestic currency)

  K: the strike price for the foreign exchange rate

  $B(t,T)$: the value of the zero coupon bond in domestic country (denominated in the domestic currency) ($B(T,T) = 1$)

  $B'(t,T)$: the value of the zero coupon bond in foreign country (denominated in the foreign currency) ($B'(T,T) = 1$)

  (The consideration of $B(t,T)$ and $B'(t,T)$ rather than $e^{-r(T-t)}$ and $e^{-r'(T-t)}$ is to reflect the stochastic feature of the interest rates.)

- Cross-currency options (also called cross or composite options)

  (i) $c_T = X_T \max(S'_T - K', 0)$, where S'_T is the foreign stock price, K' is the strike price denominated in the foreign currency, and X_T is the exchange rate at T and defined as the value of one dollar foreign currency in units of the domestic currency.

  (ii) $c_T = \max(S'_T X_T - K, 0)$, where K is the strike price denominated in the domestic currency.

  (iii) $c_T = \overline{X} \max(S'_T - K', 0)$, where $\overline{X}$ is a fixed exchange rate.

  (iv) $c_T = S'_T \max(X_T - K, 0)$, where K is the strike price for the exchange rate.

- Quanto (Quantity Adjusting Option) options (set $\overline{X}$ to be an arbitrary constant in the case (iii)):

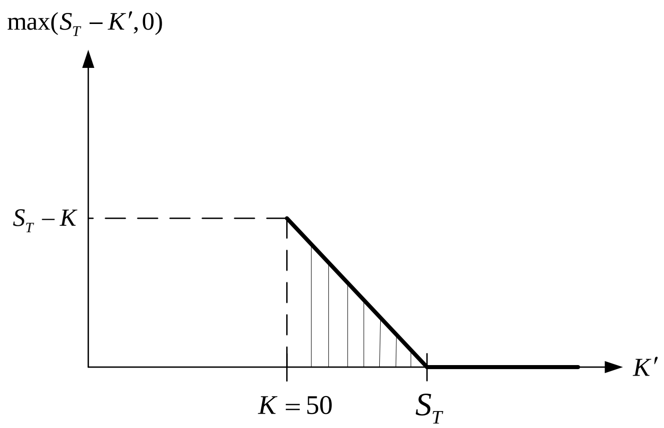
  Generally speaking, for a quanto derivative, the underlying asset is denominated in a currency other than the currency in which the derivative is settled. The CME Nikkei 225 futures and options are typical examples of quanto derivatives. They are derivative contracts for which the underlying asset–the Nikkei 225 Stock Average Index which is denominated in Japanese yen–settled in U.S. dollars, as opposed to Japanese yen. To calculate the payoffs of the CME Nikkei 225 futures and options, $\overline{X}$ is set to be 5 such that $c_T = 5 \cdot \max(S'_T - K', 0)$, where S'_T is the Nikkei 225 Stock Average Index level, K' is the strike price denominated in Japanese yen, but c_T is denominated in the domestic currency (US\$). The price of the quanto option, taking a Nikkei 225 call option for example, can be expressed as $c_0 = e^{-rT} E[\max(S'_T - K', 0)]$, where r is the domestic risk free rate in the U.S., but the growth rate of S'_T is r_f, which is the foreign risk free rate in Japan.

- Polynomial options: $\text{payoff}_T = \max(\alpha_n S_T^n + \alpha_{n-1} S_T^{n-1} + ... + \alpha_1 S_T + \alpha_0 - K, 0)$. It is useful to hedge a portfolio which is exposed to a risk factor in a nonlinear way.

- Continuous strike option (CSC) (連續履約價選擇權)

  Suppose $K = S_0 = 50$ and the actual strike price K' is defined as all numbers above K:

Figure 3-6



$$C_T = \begin{cases} \int_K^\infty \max(S_T - K', 0) dK' = \frac{1}{2}(S_T - K)^2 & \text{if} \quad S_T \geq K \\ 0 & \text{o/w} \end{cases}$$

- Continuous strike range call (CSRC) ($K_1 \leq K' < K_2$)

  CSRC = CSC(K_1) − CSC(K_2)

- Power options (次方選擇權): $c_T = \begin{cases} S_T^n - K & \text{if } S_T \geq K \\ 0 & \text{o/w} \end{cases}$

- Credit risk puts or puts on the credit spread

  The value of a i-th class rated zero-coupon bond at t is $B_i(t,T)$, and the corresponding zero rate is $r_i(t,T)$, where $r_i(t,T) = -\frac{\ln B_i(t,T)}{T-t}$

  Define the credit spread as $\Delta_i(t,T) = r_i(t,T) - r_0(t,T) = -\frac{1}{T-t}\ln\frac{B_i(t,T)}{B_0(t,T)}$, the payoff for puts on the credit spread with the strike price K at maturity t is as follows:

$$\text{Payoff}_t = \begin{cases} K - \Delta_i(t,T) & \text{if } \Delta_i(t,T) \leq K \\ 0 & \text{o/w} \end{cases}$$

  (For an investor who needs to buy a i-th class rated zero-coupon bond at t, if he is afraid of that the credit spread $\Delta_i(t,T)$ is not large enough to cover the credit risk of holding the i-th class rated zero-coupon bond, he can buy this credit risk puts today to hedge the risk if $\Delta_i(t,T)$ is smaller than K at t.)

II. Path Dependent Options

- A path dependent option is an option whose payoff depends on the realzied paths of the underlying assets' prices during the option life.

- Arithmetic average options: $c_T = \max(\overline{S} - K, 0)$, where $\overline{S} = \frac{1}{m+1}\sum_{i=0}^{m} S(t_i)$, $0 \leq t_i \leq T$

- Asian options: $c_T = \max(S_T - \overline{K}, 0)$, where $\overline{K} = \frac{1}{m+1}\sum_{i=0}^{m} S(t_i)$, $0 \leq t_i \leq T$

- Forward-starting Asian options: $c_T = \max(S_T - \overline{K}, 0)$, where $\overline{K} = \frac{1}{m+1}\sum_{i=0}^{m} S(t_i)$, $t \leq t_i \leq T$

- Reset puts: K is reset upward at t, if $\begin{cases} S_t > K & \Rightarrow & K \leftarrow S_t \\ S_t \le K & \Rightarrow & K \leftarrow K \end{cases}$ (OTM put $\rightarrow$ ATM put)

$$p_T = \begin{cases} S_t - S_T & \text{if} \quad S_t > K(\text{reset}) \text{ and } S_T < S_t \\ K - S_T & \text{if} \quad S_t \le K(\text{no reset}) \text{ and } S_T < K \\ 0 & \quad \text{o/w} \end{cases}$$

$*$ For calls, K is reset downward.

- Bear market warrant (BMW) (重設型熊市認售權證) (offered by NYSE and CBOE since 1996)

$$\text{BMW}_T = \begin{cases} \frac{S_t - S_T}{S_t} \times \$50 & \text{if} \quad S_t > K(\text{reset}) \text{ and } S_T < S_t \\ \frac{K - S_T}{K} \times \$50 & \text{if} \quad S_t \le K(\text{no reset}) \text{ and } S_T < K \\ 0 \times \$50 & \quad \text{o/w} \end{cases}$$

(The lower the S_T, the higher the payoff for option holders.)

- Multiple reset options:

(i) Call options (the strike price is reset downward)

$$K(t_i) = \begin{cases} S_{t_i} & \text{if} \quad S_{t_i} < K(t_{i-1}) \quad \Rightarrow \quad \text{reset} \\ K(t_{i-1}) & \text{if} \quad S_{t_i} \ge K(t_{i-1}) \quad \Rightarrow \quad \text{no reset} \end{cases}$$

$$\begin{aligned} \Rightarrow K(t_i) &= \min(K(t_{i-1}), S_{t_i}) \\ &= \min(\min(K(t_{i-2}), S_{t_{i-1}}), S_{t_i}) \\ &\vdots \\ &= \min(K, S_{t_1}, S_{t_2}, \cdots, S_{t_i}) \end{aligned}$$

(ii) Put options (the strike price is reset upward)

$$K(t_i) = \begin{cases} S_{t_i} & \text{if} \quad S_{t_i} > K(t_{i-1}) \quad \Rightarrow \quad \text{reset} \\ K(t_{i-1}) & \text{if} \quad S_{t_i} \le K(t_{i-1}) \quad \Rightarrow \quad \text{no reset} \end{cases}$$

$$\begin{aligned} \Rightarrow K(t_i) &= \max(K(t_{i-1}), S_{t_i}) \\ &= \max(\max(K(t_{i-2}), S_{t_{i-1}}), S_{t_i}) \\ &\vdots \\ &= \max(K, S_{t_1}, S_{t_2}, \cdots, S_{t_i}) \end{aligned}$$

- Lookback option

  (i) $c_T = \max(S_T - m_0^T, 0)$, where $m_0^T = \min\limits_{0 \le u \le T} S_u$

  (ii) $p_T = \max(M_0^T - S_T, 0)$, where $M_0^T = \max\limits_{0 \le u \le T} S_u$

  (iii) $c_T = \max(M_0^T - K, 0)$

  (iv) $p_T = \max(K - m_0^T, 0)$

- Barrier options: $\left\{\begin{array}{rl} \text{up-and-out} & \text{call} \\ \text{down-and-out} & \text{call} \end{array}\right.$ $\left\{\begin{array}{rl} \text{up-and-in} & \text{call} \\ \text{down-and-in} & \text{call} \end{array}\right.$

  $\left\{\begin{array}{rl} \text{up-and-out} & \text{put} \\ \text{down-and-out} & \text{put} \end{array}\right.$ $\left\{\begin{array}{rl} \text{up-and-in} & \text{put} \\ \text{down-and-in} & \text{put} \end{array}\right.$

III. Exotic Options

- Soft barrier options (軟著界限選擇權): the barrier is not a specified price level, and instead it corresponds to a range.

  For example, up-and-out call with a barrier to be 34

  $31 \to 34 \to 32 \Rightarrow$ The call is knocked out and worthless

  For a soft barrier = [34, 35.5],

  $31 \to 34.5 \Rightarrow$ the call holder loses $\frac{1}{3} = \frac{34.5-34}{1.5}$ of his right, and is still with $\frac{2}{3}$ of his right

  $34.5 \to 32$

  $32 \to 35 \Rightarrow$ the call holder loses $\frac{2}{3} = \frac{35-34}{1.5}$ of his current right, so he is still with $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$ of his original right

  (In a word, the holder of this kind of option will not lose all his right if the stock price only touches rather than passes through the range of the barrier.)

- A Bermudan option is an option where the buyer has the right to exercise at a set (always discretely spaced) number of times. This is an intermediate between a European option, which allows exercise at a single time, namely the maturity date, and an American option, which allows exercise at any time. Finally, the name is a pun: Bermuda is between America and Europe.

- A Canary option is an option whose exercise style lies somewhere between European options and Bermudan options. Typically, the holder can exercise the option at quarterly dates, but not before a time period (typically one year) has elapsed. (The name is also a pun due to the geographic position of the Canary Islands.)

- A capped-style option is not an interest rate cap but a conventional option with a predefined profit cap written into the contract. A capped-style option is automatically exercised when the underlying security closes at a pre-specified price. This feature is common in the contracts of structure notes.

- A shout option allows its holders effectively two exercise dates: during the life of the option they can (at any time) “shout” to the seller that they are locking-in the current price, and if this gives them a better deal than the payoff at maturity, they will use the underlying price on the shout date rather than the price at maturity to calculate their final payoff.

- A Russian option is an American-style lookback option which runs for perpetuity. The payoff is given as the discounted maximum price at which the underlying asset has ever traded during the life of the option (Shepp and Shiryaev (1993)).

- A game option or Israeli option is an option in which the writer has the opportunity to cancel the option he has offered but must pay the exercise value at that point plus a penalty fee.

- The Parisian option is like the barrier option, but its payoff depends on the length of time period for which the underlying asset has stayed continuously above or below a pre-specified price.

- Mountain ranges are exotic options originally marketed by Societe Generale (法國興業銀行) in 1998. The options combine the characteristics of basket options and range options by linking the value of the option on several underlying assets and by setting a time frame for the option. The mountain range options can be further subdivided into different types, depending on the specific terms of the options. Examples include:

  ⊙ Altiplano - in which a vanilla call option is combined with a compensatory coupon payment if the security price never reaches its strike price during a given period.

  ⊙ Annapurna - in which the option holder is rewarded if all securities in the basket never fall below a certain price during the relevant time period.

  ⊙ Atlas - in which the best- and worst-performing securities are removed from the basket prior to execution of the option. For a call option on a basket of n stocks at maturity T, n_1 of the best performer and n_2 of the worst performer are removed before calculating the payoff. (strike price $= K$ and $n > n_1 + n_2$)

$$\text{payoff}_T = \max\left(\frac{1}{n-(n_1+n_2)} \sum_{j=n_1+1}^{n-n_2} \frac{S_{jT}}{S_{j0}} - K, 0\right)$$

  ⊙ Everest - a long-term option in which the option holder gets a payoff based on the worst-performing securities in the basket.

  ⊙ Himalayan - a multi-year option on the best performers in a basket. Until each examined time point t_i, for $i = 1$ to n, the top performer, denoted as the $B(i)$-th asset, is removed (and its return recorded) from the basket sequentially, and the option payoff is the average of all recorded returns.

$$\text{payoff}_T = \begin{cases} \frac{1}{n}\sum_{i=1}^{n} \max\left(\frac{S_{B(i),t_i}}{S_{B(i),t_0}} - 1, 0\right) & \text{if a periodic floor is considered} \\ \max\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{S_{B(i),t_i}}{S_{B(i),t_0}} - 1\right), 0\right) & \text{if a global floor is considered} \end{cases}$$