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【Basic Requirement】

(1) Derive the closed-form formula using the martingale pricing method by hands.

First we find the payoff function from the diagram, define:

$$f(S_t) \coloneqq \begin{cases} 0 \ \text{ if } \ S_t < K_1 \ \text{ or } \ S_t > K4 \\ S_t - K_1 \ \text{ if } \ K_1 \le S_t < K_2 \\ K_2 - K_1 \ \text{ if } \ K_2 \le S_t < K_3 \\ (K_2 - K_1) - (S_t - K_3)\dfrac{K_2 - K_1}{K_4 - K_3} \ \text{ if } \ K_3 \le S_t \le K_4 \end{cases}$$

If we further define:

$$\mathrm{I_A}(S_t) = \begin{cases} 1 \ \text{ if } \ K_1 \le S_t < K_2 \\ 0 \ \text{ otherwise} \end{cases}, \mathrm{I_B}(S_t) = \begin{cases} 1 \ \text{ if } \ K_2 \le S_t < K_3 \\ 0 \ \text{ otherwise} \end{cases}, \mathrm{I_C}(S_t) = \begin{cases} 1 \ \text{ if } \ K_3 \le S_t \le K_4 \\ 0 \ \text{ otherwise} \end{cases}$$

Then we have:

$$f(S_t) = (S_t - K_1)I_A(S_t) + (K_2 - K_1)I_B(S_t) + \left[(K_2 - K_1) - (S_t - K_3)\frac{K_2 - K_1}{K_4 - K_3}\right] I_C(S_t)$$

To find the present value of such an option, we must evaluate the expected value of $f(S_t)$, i.e. $\mathrm{E}[f(S_t)]$. By the linearity of expected value, we have:

$$\begin{aligned} \mathrm{E}[f(S_t)] &= \mathrm{E}[(S_t - K_1)I_A(S_t)] + \mathrm{E}[(K_2 - K_1)I_B(S_t)] \\ &\quad + \mathrm{E}\left[\left[(K_2 - K_1) - (S_t - K_3)\frac{K_2 - K_1}{K_4 - K_3}\right] I_C(S_t)\right] \\ &= \mathrm{E}[S_t I_A(S_t)] - K_1\mathrm{E}[I_A(S_t)] + (K_2 - K_1)\mathrm{E}[I_B(S_t)] + (K_2 - K_1)\mathrm{E}[I_C(S_t)] \\ &\quad + K_3\frac{K_2 - K_1}{K_4 - K_3}\mathrm{E}[I_C(S_t)] - \frac{K_2 - K_1}{K_4 - K_3}\mathrm{E}[S_t I_C(S_t)] \end{aligned}$$

Now, we evaluate each term on the right-hand-side of the equal sign. Since we want to apply ***Girsanov Theorem*** on some of the upcoming computation. Let the probability measure we use above be "measure Q".

Let's start with $\mathrm{E}[S_t I_A(S_t)]$:

$$\mathrm{E}[S_t I_A(S_t)] \coloneqq \mathrm{E^Q}[S_t I_A(S_t)] = \mathrm{E^Q}\left[S_0 \cdot e^{\left(r - q - \frac{\sigma^2}{2}\right)T + \sigma\cdot\Delta Z^Q(T)} \cdot I_A(S_T)\right]$$

$$= S_0 e^{(r-q)T} \cdot \mathrm{E^Q}\left[e^{-\frac{\sigma^2}{2}T + \sigma\cdot\Delta Z^Q(T)} \cdot I_A(S_T)\right] \ \ldots \ (*)$$

If we setting $H(t)$ in ***Girsanov Theorem*** equals to $H(t) =-\sigma$, we have:

$$\Lambda = \frac{\mathrm{d}R}{\mathrm{d}Q} = e^{-\frac{1}{2}\int_0^T \sigma^2 d\tau - \int_0^T -\sigma dZ^Q(\tau)} = e^{-\frac{\sigma^2}{2}T + \sigma\Delta Z^Q(T)}$$

Plug the result above in $(*)$, we have:

$$\begin{aligned} \mathrm{E^Q}[S_t I_A(S_t)] &= S_0 e^{(r-q)T} \cdot \mathrm{E^Q}[\Lambda \cdot I_A(S_T)] = S_0 e^{(r-q)T} \cdot E^R[I_A(S_T)] \\ &= S_0 e^{(r-q)T} \cdot \mathrm{Pr^R}(\{K_1 \le S_t < K_2\}) \end{aligned}$$

After apply nature log function on S_t, K, replace S_t with $\ln S_0 + \left(r - q + \frac{\sigma^2}{2}\right)T + \sigma\Delta Z^R(T)$,

and last normalization, we have:

$$\mathrm{E}^{\mathrm{Q}}[S_t I_A(S_t)] = S_0 e^{(r-q)T} \cdot \Pr{}^{\mathrm{R}}\left(\left\{-\frac{\Delta Z^R(T)}{\sqrt{T}} \le \frac{\ln\frac{S_0}{K_1} + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right\} \setminus \left\{\frac{\Delta Z^R(T)}{\sqrt{T}} \le \frac{\ln\frac{S_0}{K_2} + \left(r - q + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right\}\right)$$

Define $d_{1,R} \coloneqq \frac{\ln\frac{S_0}{K_1}+\left(r-q+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, d_{2,R} \coloneqq \frac{\ln\frac{S_0}{K_2}+\left(r-q+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, we obtain the result of:

$$\mathrm{E}^{\mathrm{Q}}[S_t I_A(S_t)] = S_0 e^{(r-q)T} \cdot \left[N\left(d_{2,R}\right) - N\left(d_{1,R}\right)\right]$$

As for $\mathrm{E}[I_A(S_t)]$, there's NO need of applying ***Girsanov Theorem***, we simply use the step above, i.e. nature log, replace S_t and then normalize, we can obtain:

$$\mathrm{E}[I_A(S_t)] = \left[N\left(d_{2,Q}\right) - N\left(d_{1,Q}\right)\right]$$

Where $d_{1,Q} \coloneqq \frac{\ln\frac{S_0}{K_1}+\left(r-q-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, d_{2,Q} \coloneqq \frac{\ln\frac{S_0}{K_2}+\left(r-q-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

With similar method, and define $d_{i,Q} \coloneqq \frac{\ln\frac{S_0}{K_i}+\left(r-q-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, d_{i,R} \coloneqq \frac{\ln\frac{S_0}{K_i}+\left(r-q+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ we can also obtain:

$$\mathrm{E}[I_B(S_t)] = \left[N\left(d_{3,Q}\right) - N\left(d_{2,Q}\right)\right]$$
$$\mathrm{E}[I_C(S_t)] = \left[N\left(d_{4,Q}\right) - N\left(d_{3,Q}\right)\right]$$
$$\mathrm{E}[S_t I_C(S_t)] = S_0 e^{(r-q)T} \cdot \left[N\left(d_{4,R}\right) - N\left(d_{3,R}\right)\right]$$

With these result, we can finally conclude that:

$$\begin{aligned}
\mathrm{E}[f(S_t)] &= S_0 e^{(r-q)T} \cdot \left[N\left(d_{2,R}\right) - N\left(d_{1,R}\right)\right] - K_1\left[N\left(d_{2,Q}\right) - N\left(d_{1,Q}\right)\right] \\
&\quad +(K_2 - K_1)\left[N\left(d_{3,Q}\right) - N\left(d_{2,Q}\right)\right] + (K_2 - K_1)\left[N\left(d_{4,Q}\right) - N\left(d_{3,Q}\right)\right] \\
&\quad +K_3\frac{K_2 - K_1}{K_4 - K_3}\left[N\left(d_{4,Q}\right) - N\left(d_{3,Q}\right)\right] - \frac{K_2 - K_1}{K_4 - K_3}S_0 e^{(r-q)T} \cdot \left[N\left(d_{4,R}\right) - N\left(d_{3,R}\right)\right] \\
&= S_0 e^{(r-q)T} \cdot \left[N\left(d_{2,R}\right) - N\left(d_{1,R}\right)\right] \\
&\qquad +K_1 N\left(d_{1,Q}\right) - K_2 N\left(d_{2,Q}\right) + K_2 N\left(d_{4,Q}\right) - K_1 N\left(d_{4,Q}\right) \\
&\qquad +K_3\frac{K_2 - K_1}{K_4 - K_3}\left[N\left(d_{4,Q}\right) - N\left(d_{3,Q}\right)\right] - \frac{K_2 - K_1}{K_4 - K_3}S_0 e^{(r-q)T} \cdot \left[N\left(d_{4,R}\right) - N\left(d_{3,R}\right)\right]
\end{aligned}$$

The last step is to discount the expected value with r, finally we can Derive the closed-form formula of the value of this derivative

$e^{-rt} \cdot \mathrm{E}[f(S_t)] =$

$$\begin{aligned}
&e^{-rt}\Big\{S_0 e^{(r-q)T} \cdot \left[N\left(d_{2,R}\right) - N\left(d_{1,R}\right)\right] + K_1 N\left(d_{1,Q}\right) - K_2 N\left(d_{2,Q}\right) + K_2 N\left(d_{4,Q}\right) - K_1 N\left(d_{4,Q}\right) \\
&\qquad + K_3\frac{K_2 - K_1}{K_4 - K_3}\left[N\left(d_{4,Q}\right) - N\left(d_{3,Q}\right)\right] - \frac{K_2 - K_1}{K_4 - K_3}S_0 e^{(r-q)T} \cdot \left[N\left(d_{4,R}\right) - N\left(d_{3,R}\right)\right]\Big\}
\end{aligned}$$

(2) blahblahblah

(3)