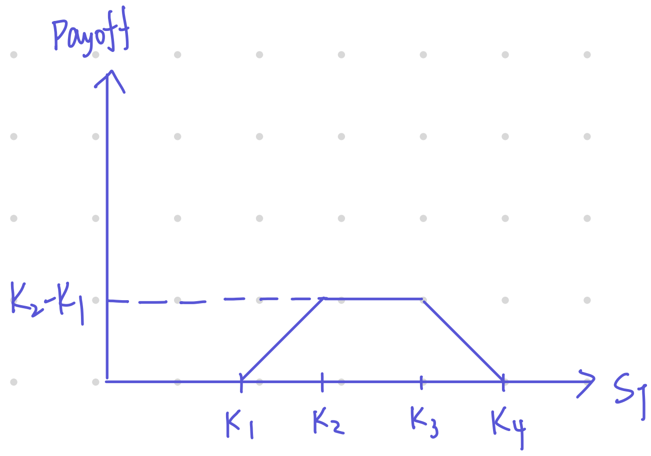
* HW1 B05703029 陳怡蓁



$$\text{Payoff} = \begin{cases} 0 & \text{if } S_T < K_1 \\ S_T-K_1 & \text{if } K_1 \le S_T < K_2 \\ K_2-K_1 & \text{if } K_2 < S_T < K_3 \\ \frac{K_2-K_1}{K_3-K_4}(S_T-K_4) & \text{if } K_3 \le S_T < K_4 \\ 0 & \text{if } S_T \ge K_4 \end{cases}$$

Let $1_A = \begin{cases} 1 & \text{if } K_1 \le S_T < K_2 \\ 0 & \text{otherwise} \end{cases}$, $1_B = \begin{cases} 1 & \text{if } K_2 \le S_T < K_3 \\ 0 & \end{cases}$

$1_C = \begin{cases} 1 & \text{if } K_3 \le S_T < K_4 \\ 0 & \text{otherwise} \end{cases}$

Then $c = e^{-rT} E^Q\left[(S_T-K_1)\cdot 1_A + (K_2-K_1) 1_B + \left(\frac{K_2-K_1}{K_3-K_4}\right)(S_T-K_4)\cdot 1_C \right]$

$= e^{-rT} E^Q[(S_T-K_1)_A] + e^{-rT} E^Q[(K_2-K_1) 1_B] + e^{-rT} E^Q\left[\frac{K_2-K_1}{K_3-K_4}(S_T-K_4)\cdot 1_C \right]$

For S_T, we have $\frac{dS}{S} = (r-q)dt + \sigma dz^Q \Rightarrow \ln S_T = \ln S_0 + (r-q-\frac{\sigma^2}{2})T + \sigma \Delta z^Q(T)$

$\Rightarrow S_T = S_0 \exp\left((r-q-\frac{\sigma^2}{2})T + \sigma \Delta z^Q(T)\right)$

- Part I: $E^Q[(S_T-K_1)1_A] = E^Q[S_T 1_A] - K_1 E^Q[1_A]$

1. $E^Q[1_A] = P^Q(K_1 \le S_T < K_2) = P^Q(\ln K_1 \le \ln S_T < \ln K_2) = P^Q(\ln K_1 \le \ln S_0 + (r-q-\frac{\sigma^2}{2})T + \sigma \Delta z^Q(T) < \ln K_2)$

$= P^Q\left(\frac{\ln(S_0/K_2)+(r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < -\frac{\Delta z^Q(T)}{\sqrt{T}} \le \frac{\ln(S_0/K_1)+(r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \right)$

Let $d_3 = \frac{\ln(S_0/K_1)+(r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_4 = \frac{\ln(S_0/K_2)+(r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

And $\Delta z^Q(T) \sim ND(0,T) \Rightarrow \frac{-\Delta z^Q(T)}{\sqrt{T}} \sim ND(0,1)$

$\therefore E^Q[1_A] = N(d_3) - N(d_4)$

2. $E^Q[S_T \cdot 1_A] = E^Q[S_0 \exp[(r-q-\frac{\sigma^2}{2})T + \sigma dz^Q(T)]\cdot 1_A] = S_0 e^{(r-q)T} E^Q[e^{-\frac{\sigma^2}{2}T + \sigma dz^Q(T)} \cdot 1_A]$

Let z^R be standard Wiener process under measure R and $H(t) = -\sigma$

Then $\Lambda = \frac{dR}{dQ} = e^{-\frac{1}{2}\int_0^T \sigma^2 dt - \int_0^T \sigma z^Q(\tau)} = e^{-\frac{\sigma^2}{2}T + \sigma\Delta z^Q(T)} \Rightarrow dz^R = dz^Q - \sigma dt$ by Girsanov Thm.

$\Rightarrow \Delta z^R(T) = \Delta z^Q(T) - \sigma T$

$\Rightarrow \ln S_T = \ln S_0 + (r-q-\frac{\sigma^2}{2})T + \sigma\Delta z^Q(T) = \ln S_0 + (r-q-\frac{\sigma^2}{2})T + \sigma(\Delta z^R(T) + \sigma T)$

$= \ln S_0 + (r-q+\frac{\sigma^2}{2})T + \sigma\Delta z^R(T)$

$$\therefore E^Q[S_T \cdot 1_A] = S_0 e^{(r-q)T} E^Q[e^{-\frac{\sigma^2}{2}T + \sigma dZ^Q(T)} \cdot 1_A]$$

$$= S_0 e^{(r-q)T} E^R[1_A] = S_0 e^{(r-q)T} P^R(\ln K_1 \le \ln S_T < \ln K_2) = S_0 e^{(r-q)T} P^R(\ln K_1 \le \ln S_0 + (r-q+\frac{\sigma^2}{2})T + \sigma \Delta Z^R(T) < \ln K_2)$$

$$= S_0 e^{(r-q)T} P^R\left(\frac{\ln(K_1/S_0) - (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \le \frac{\Delta Z^R(T)}{\sqrt{T}} < \frac{\ln(K_2/S_0) - (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$= S_0 e^{(r-q)T} P^R\left(\frac{\ln(S_0/K_2) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}} < -\frac{\Delta Z^R(T)}{\sqrt{T}} \le \frac{\ln(S_0/K_1) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

Let $d_1 = \dfrac{\ln(S_0/K_1) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_2 = \dfrac{\ln(S_0/K_2) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$. and $-\dfrac{\Delta Z^R(T)}{\sqrt{T}} \sim ND(0,1)$

$$\Rightarrow E^Q[S_T \cdot 1_A] = S_0 e^{(r-q)T}[N(d_1) - N(d_2)]$$

$$\therefore E^Q[(S_T - K) 1_A] = S_0 e^{(r-q)T}[N(d_1) - N(d_2)] - (N(d_3) - N(d_4))$$

- Part II: $E^Q[(K_2 - K_1) \cdot 1_B]$

$$E^Q[(K_2-K_1)\cdot 1_B] = (K_2 - K_1) E^Q[1_B].$$

Similar to Part I-1, $d_4 = \dfrac{\ln(S_0/K_2) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, Let $d_5 = \dfrac{\ln(S_0/K_3) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$$\Rightarrow E^Q[(K_2-K_1)\cdot 1_B] = (K_2 - K_1)[N(d_4) - N(d_5)]$$

- Part III: $E^Q\left[\frac{K_2-K_1}{K_3-K_4}(S_T - K_4)\cdot 1_C\right] = \left(\frac{K_2-K_1}{K_3-K_4}\right)\left[E^Q[S_T \cdot 1_C] - K_4 E^Q[1_C]\right]$

Similar to Part I.

$E^Q[1_C] = N(d_5) - N(d_6)$, where $d_6 = \dfrac{\ln(S_0/K_4) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$E^Q[S_T \cdot 1_C] = S_0 e^{(r-q)T}[N(d_7) - N(d_8)]$, where $d_7 = \dfrac{\ln(S_0/K_3) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$d_8 = \dfrac{\ln(S_0/K_4) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$$\Rightarrow E^Q\left[\frac{K_2-K_1}{K_3-K_4}(S_T - K_4)\cdot 1_C\right] = \left(\frac{K_2-K_1}{K_3-K_4}\right)\left[S_0 e^{(r-q)T}[N(d_7) - N(d_8)] - K_4(N(d_5) - N(d_6))\right]$$

From Part I to Part III,

$$C = E^Q\left[(S_T - K_1)\cdot 1_A + (K_2-K_1)\cdot 1_B + \frac{K_2-K_1}{K_3-K_4}(S_T-K_4)\cdot 1_C\right] = S_0 e^{(r-q)T}\left\{[N(d_1) - N(d_2)] + \left(\frac{K_2-K_1}{K_3-K_4}\right)[N(d_7) - N(d_8)]\right\}$$

$$-[N(d_3) - N(d_4)] + (K_2-K_1)[N(d_4) - N(d_5)] - \frac{K_2-K_1}{K_3-K_4}\cdot K_4[N(d_5) - N(d_6)]$$, where

$d_1 = \dfrac{\ln(S_0/K_1) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_2 = \dfrac{\ln(S_0/K_2) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_3 = \dfrac{\ln(S_0/K_1) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$d_4 = \dfrac{\ln(S_0/K_2) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_5 = \dfrac{\ln(S_0/K_3) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ $d_6 = \dfrac{\ln(S_0/K_4) + (r-q-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$d_7 = \dfrac{\ln(S_0/K_3) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_8 = \dfrac{\ln(S_0/K_4) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ #