Ch 2. Black-Scholes Model

- This chapter introduces two methods to derive the Black-Scholes formula. The traditional method solves a partial differential equation and thus calculates the integral over the log-normally (or normally) distributed underlying variable. However, it is difficult to extend this method to price other options, e.g., path dependent options or rainbow options.

- Another method, the martingale pricing method (MPM), will be introduced in this chapter as well. Since this method does not involve any integration, the calculation process is simple. Furthermore, it is straightforward to extend this method to price other options. Although the calculation process of the MPM is simple, it is difficult to understand this method because the MPM employs the technique of changing measure for stochastic processes.

I. Partial Differential Equation for Derivatives

- The partial differential equation (PDE) for derivatives:
  $\frac{dS}{S} = \mu dt + \sigma dZ$
  $\Rightarrow dS = \mu S dt + \sigma S dZ$
  If $f(S,t)$ is the price for any derivative, according to the Itô's Lemma,
  $$df = (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt + \frac{\partial f}{\partial S}\sigma S dZ.$$

Construct a portfolio π:

-1 derivative

$+\frac{\partial f}{\partial S}$ shares

$\Rightarrow \pi = -f + \frac{\partial f}{\partial S} \cdot S \Rightarrow d\pi = -df + \frac{\partial f}{\partial S} dS = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt$

(Since there is no dZ in $d\pi$, holding π is without risk and should earn the risk free rate for an infinitesimal time period dt due to the no-arbitrage argument.)

$$\begin{aligned}
&\Rightarrow d\pi = r\pi dt \\
&\Rightarrow (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt = r(-f + \frac{\partial f}{\partial S}S)dt \\
&\Rightarrow \frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\end{aligned}$$

Recall that I mentioned in Chapter 1 that df is not what we should care about, and instead we are interested in the behavior of f given the time point t and the stock price S_t, i.e., to derive the solution of $f(S_t, t)$.

Taking the call option for example, it should satisfy the boundary condition that $f(S_T, T) = \max(S_T - K, 0)$ when $t = T$. The Black-Scholes formula is to find the analytic solution $f(S_t, t)$ to satisfy the above partial differential equation at any time point t as well as the boundary condition at T.

In addition to the Black-Scholes formula, it is possible to solve this PDE via other numerical methods, such as the finite difference method introduced in Chapter 5.

- Note that since the underlying asset is tradable, we can construct a portfolio to eliminate the terms including dZ in the derivative and the underlying asset. Therefore, we can introduce r into the the partial differential equation. If the underlying asset is not tradable, we need to use two types of derivatives to form a risk free portfolio by eliminating dZ terms. During this process, the "market price of the risk" of the underlying asset can be introduced as well.

- (Advanced content) The PDE for derivatives under the jump-diffusion process.

$$\frac{dS}{S} = (\mu - \lambda K_Y)dt + \sigma dZ + (Y_S - 1)dq$$

If $f(S, t)$ is the price for any derivative, according to the Itô's Lemma,

$$\begin{aligned}
df = &\{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(\mu - \lambda K_Y)S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \lambda E[f(SY_S, t) - f(S, t)]\}dt \\
&+ \frac{\partial f}{\partial S}\sigma S dZ + (Y_f - 1)f dq.
\end{aligned}$$

(Recall that the total jump effect on f (from $(Y_S - 1)dq$) equals the sum of $\lambda dt E[f(SY_S, t) - f(S, t)]$ and $(Y_f - 1)f dq$, where the mean of $(Y_f - 1)f dq$ is zero.)

Construct a portfolio $\pi = -f + \frac{\partial f}{\partial S}S$

$$\Rightarrow d\pi = -df + \frac{\partial f}{\partial S}dS = \{-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - \lambda E[f(SY_S,t) - f(S,t)]\}dt$$
$$-(Y_f - 1)f dq + \frac{\partial f}{\partial S}(Y_S - 1)S dq$$

Note that both $-(Y_f - 1)f dq$ and $\frac{\partial f}{\partial S}(Y_S - 1)S dq$ depend on the identical Poisson process, but these two terms cannot offset each other perfectly.

$\Rightarrow$ π is NOT an instantaneous riskless portfolio

$\Rightarrow$ We cannot obtain the instantaneous expected rates of return of any derivatives (e.g. f and S) to be r

$\odot$ Suppose the intantaneous expected rate of return of f is $g(S,t)$.

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}(\mu - \lambda K_Y)S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \lambda E[f(SY_S,t) - f(S,t)]\} = g(S,t)f$$

(The PDE for derivatives under the jump-diffusion process if the no-arbitrage argument cannot be used.)

$\odot$ Suppose the jump is a type of firm-specific risk, and the firm-specific risk is not priced according to the CAPM, so the instantaneous expected rate of return of π (with a drift term and the firm-specific risk) should be r.

$\Rightarrow$ Expected change in π during the following dt period

$$= \{-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 - \lambda E[f(SY_S,t) - f(S,t)] + \frac{\partial f}{\partial S}\lambda K_Y S\}dt$$

(Note that the mean of $(Y_f - 1)dq$ is zero, and the mean of $(Y_S - 1)dq$ is $\lambda K_Y dt$.)

$$= r(-f + \frac{\partial f}{\partial S}S)dt$$

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \lambda E[f(SY_S,t) - f(S,t)] + \frac{\partial f}{\partial S}(r - \lambda K_Y)S = rf$$

(The PDE for derivatives under the jump-diffusion process is identical to Eq. (14) in Merton (1976).)

II. Market Price of Risk and Degree of Risk Aversion

- The market price of risk

$\frac{d\theta}{\theta} = mdt + sdZ$ (θ is not necessary to be tradable. It can be a state variable.)

Find two derivatives, $f_1 = f_1(\theta,t)$ and $f_2 = f_2(\theta,t)$, apply the Itô's Lemma, and rewrite df_1 and df_2 in the form similar to the geometric Brownian motions:

$df_1 = \mu_1 f_1 dt + \sigma_1 f_1 dZ,$
$df_2 = \mu_2 f_2 dt + \sigma_2 f_2 dZ,$

where μ_i and σ_i are interpreted as the expected value and the volatility of the return of f_i. That is, $\mu_i = (\frac{\partial f_i}{\partial t} + m\theta\frac{\partial f_i}{\partial \theta} + \frac{1}{2}s^2\theta^2\frac{\partial^2 f_i}{\partial \theta^2})/f_i$ and $\sigma_i = \frac{\partial f_i}{\partial \theta}s\theta/f_i$.

Construct a portfolio $\pi = (\sigma_2 f_2)f_1 - (\sigma_1 f_1)f_2$

$\Rightarrow d\pi = (\sigma_2 f_2)df_1 - (\sigma_1 f_1)df_2$

Substitute df_1 and df_2 into the above equation $\Rightarrow d\pi = (\mu_1\sigma_2 f_1 f_2 - \mu_2\sigma_1 f_1 f_2)dt = r\pi dt$
$= (\sigma_2 f_2 f_1 r - \sigma_1 f_1 f_2 r)dt$

$\Rightarrow \mu_1\sigma_2 - \mu_2\sigma_1 = \sigma_2 r - \sigma_1 r$

$\Rightarrow \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda$ (the market price of risk of θ)

Rewrite the above equation to obtain $\mu_i - r = \sigma_i\lambda \Rightarrow df_i = (r + \lambda\sigma_i)f_i dt + \sigma_i f_i dZ$.

(For bearing σ_i percent of risk, which is caused by the dZ of θ, the holder of f_i can earn more excess return by $\lambda\sigma_i\%$.)

The PDE for f_i is $(\frac{\partial f_i}{\partial t} + m\theta\frac{\partial f_i}{\partial \theta} + \frac{1}{2}s^2\theta^2\frac{\partial^2 f_i}{\partial \theta^2})/f_i = r + \lambda\sigma_i$

(If θ is an investment asset, i.e., it is tradable, we can further obtain $\frac{m-r}{s} = \lambda$.)

- $\lambda = 0 \Rightarrow \mu_i = r \Rightarrow$ risk neutral world

  $\lambda > 0 \Rightarrow \mu_i > r \Rightarrow$ risk averse world

  $\lambda < 0 \Rightarrow \mu_i < r \Rightarrow$ risk loving world

  $*$ different values of $\lambda \Rightarrow$ different expected return $\Rightarrow$ different worlds
  $\Rightarrow$ different probability measures

  (Later I will show that under different probability measures, the mean of a random variable or the drift of a stochastic process should change.)

- Multiple state variables

  $\frac{d\theta_i}{\theta_i} = m_i dt + s_i dZ^i$, and $dZ^i \cdot dZ^j = \rho_{ij}dt$

  $\Rightarrow \frac{df}{f} = \mu dt + \sum_{i=1}^{n}\sigma_i dZ^i$ (which is the result by the multi-variable Itô's Lemma)

  $\Rightarrow \mu - r = \sum_{i=1}^{n}\lambda_i\sigma_i$

  (Note that the expected growth rate μ is a function of ρ_{ij}, which means ρ_{ij} influences the excess return and thus the market price of risk λ_i.)

  (Please refer to Chapter 27 or Technical Note 30 in Hull (2011) for details.)

III. RNVR and Black-Scholes Formula

- Suppose we consider to replace μ with r for the process S on page 2-1. (The intuition for this replacement is that there are no terms including μ in the final PDE.)

  First, it is equivalent to consider the underlying stock prices in the risk neutral world. Second, for the stochastic process df, its drift term becomes $(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt$ $= (\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}rS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)dt = rfdt$, where the last equality is due to the partial differential equation for the derivative f.

  Therefore, the expected growth rate of the derivative f is also r, so you can treat f to be in the risk neutral world as well. As a result, to solve option prices based on the partial differential equation on page 2-2 is equivalent to considering both S and f to be in the risk neutral world, i.e., the expected growth rates of both the underlying asset and its derivatives are equal to r, and thus the payoff of any derivative f should be discounted with the risk free rate r.

  This result is called the Risk Neutral Valuation Relationship (RNVR).

- Feynman-Kac formula: to price any derivative, one needs to calculate only the expectation of the present value of the payoff (with the risk free rate as the discount rate).

  Given $dX(t)$=$\mu(X(t),t)dt + \sigma(X(t),t)dZ(t)$ and $X(0) = x$, then $f(X,0) =$ $E[e^{-\int_0^T r(X(\tau),\tau)d\tau} g(X(T))]$ is the unique solution of the following PDE.

  $$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial X} \cdot \mu(X,t) + \frac{1}{2}\sigma^2(X,t)\frac{\partial^2 f}{\partial X^2} = r(X,t)f(X,t),$$

  where $g(X(T))$ is the boundary condition (or said the payoff function) at T of $f(X,t)$, i.e., $f(X,T) = g(X(T))$.

  $*$ If r is constant, $f(X,0) = e^{-rT}E[g(X(T))]$.
  $*$ The formula was formally proposed after the introduction of the Black-Scholes formula.

- Apply the Feynman-Kac formula and RNVR to deriving the Black-Scholes formula:

  Based on the RNVR and the Feynman-Kac formula, the unique solution of the target PDE can be obtained by calculating the expectation of the present value of the derivative payoff at the maturity in the risk neutral world. Considering a constant risk free rate and taking a call option for example, the option price today is

  $$\begin{aligned} c(S_0,0) &= e^{-rT} \cdot E[\text{payoff at } T|\text{in the risk neutral world}] \\ &= e^{-rT} \cdot E[\max(S_T - K, 0)|\text{in the risk neutral world}] \\ &= e^{-rT} \int_0^\infty \max(S_T - K, 0) f(S_T|\text{in the risk neutral world}) dS_T, \end{aligned}$$

  where $f(S_T|\text{in the risk neutral world})$ is the probability density function of S_T in the risk neutral world.

After performing some staightforward calculation and the technique of changing variables (see the appendix in Chapter 14 in Hull (2011)), we can derive the famous Black-Scholes formula.

$$c(S_0, 0) = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

where $d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, and $d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$.

- Another method without the technique of changing variables:

  To calculate the integral directly over the lognormally distributed probability density function.

$$c(S_0, 0) = e^{-rT} \int_K^\infty (S_T - K) \cdot f(S_T | \text{in the risk neutral world}) dS_T$$

> lognormal probability density function under the risk neutral measure:
> $f(S_T|\text{in the risk neutral world}) \equiv f(S_T) = \frac{1}{S_T} \cdot \frac{1}{\sigma\sqrt{T}\sqrt{2\pi}} \exp\left[\frac{-(\ln S_T - E^Q[\ln S_T])^2}{2\sigma^2 T}\right]$

$$= e^{-rT} \int_K^\infty S_T \cdot f(S_T) dS_T - K \cdot e^{-rT} \cdot \int_K^\infty f(S_T) dS_T$$

  (This method can derive an identical pricing formula as the Black-Scholes formula.)

- To implement the computer program for the Black-Scholes formula, two methods to calculate the cumulative distribution function $N(x)$ are introduced:

1. Call the NORMSDIST function in Excel.

In Excel, you can insert NORMSDIST into a cell on a worksheet to calculate $N(x)$. However, in the VBA environment of Excel, you need the following statement to call this Excel-providing function, "Application.WorksheetFunction.NormSDist(x)".

2. A polynomial approximation:

$$N(x) = \begin{cases} 1 - N'(x)(a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 + a_5 k^5) & \text{when } x \geq 0 \\ 1 - N(-x) & \text{when } x < 0 \end{cases}$$

where

$$k = \frac{1}{1+\gamma x}, \gamma = 0.2316419, a_1 = 0.319381530, a_2 = -0.356563782,$$
$$a_3 = 1.781477937, a_4 = -1.821255978, a_5 = 1.330274429, N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

(Useful features of polynomial functions: integrable and differentiable, and the corresponding calculus calculations are simple.)

IV. Martingale Pricing Method

- This method can derive Black-Scholes-like formulae for many different types of derivatives without evaluating the complicated and tedious integration to derive $N(\cdot)$ terms. However, how to change the probability measure for stochastic processes is not easy to understand.

- $\frac{dS}{S} = \mu dt + \sigma dZ^P$, where dZ^P is the Wiener process under the probability measure P.

  If the dividend yield q is taken into consideration: $\frac{dS}{S} = (\mu - q)dt + \sigma dZ^P$.

  (The probability measure P is also called the physical measure, which is the probability measure in our real world, i.e., in a risk averse world.)

  ⊙ Transform to the Wiener process under the risk neutral measure Q

  $dZ^P = dZ^Q - \lambda dt = dZ^Q - (\frac{\mu - r}{\sigma})dt$, where dZ^Q is the Wiener process under Q.

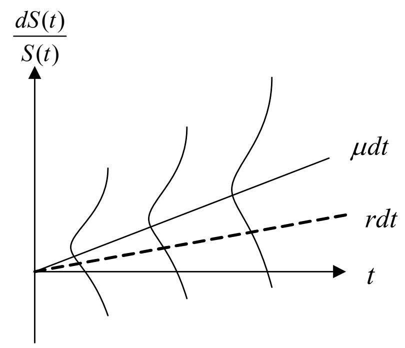
  Multiplying both sides of the equation with $\sigma \Rightarrow \sigma dZ^P = \sigma dZ^Q - (\mu - r)dt$.

  After rearranging $\Rightarrow \mu dt + \sigma dZ^P = rdt + \sigma dZ^Q$. (If the dividend yield q is considered, $(\mu - q)dt + \sigma dZ^P = (r - q)dt + \sigma dZ^Q$.)

  (Since it is known that for all security prices in the risk neutral world are with returns to be the risk free rate, we can infer that the measure Q is the risk neutral measure.)

  (It also can be observed that changing probability measure affects only the drift term.)

Figure 2-1



- In conclusion, corresponding to the measure P, there is a measure Q, under which the drift term of the stock price process is the risk free rate. We call this measure Q to be the risk neutral measure.

- Under the risk neutral measure Q, the stock price process becomes as follows.
  $\frac{dS}{S} = (\mu - q)dt + \sigma dZ^P$ (replacing dZ^P with $dZ^Q - (\frac{\mu - r}{\sigma})dt$)
  $\Rightarrow \frac{dS}{S} = (r - q)dt + \sigma dZ^Q$
  $\Rightarrow d\ln S = (r - q - \frac{\sigma^2}{2})dt + \sigma dZ^Q \Rightarrow \ln S_T = \ln S_0 + (r - q - \frac{\sigma^2}{2})T + \sigma \Delta Z^Q(T)$,
  where $\Delta Z^Q(T) \sim ND^Q(0, T)$

- Definition of measure

  Ω: universal set (the set of all possible events)

  $\mathcal{F}$: a set of "events" (For stochastic processes, $\mathcal{F}$ changes over time and is called the filtration, which will be introduced in Appendix A.)

  A measure is a nonnegative and countable additive real-number function, which assigns each subset a real number, intuitively interpreted as the size of the subset.

  That is, a function μ: $\mathcal{F} \to R$ with the following two properties is a measure:

  (i) (Non-negativity) $\mu(A) \geq \mu(\phi) = 0$ for all $A \in \mathcal{F}$

  (ii) (Countable additivity) If $A_i \in \mathcal{F}$ are countable disjoint sets (i.e., $A_i \bigcap A_j = \phi$ if $i \neq j$), $\mu(\bigcup_i A_i) = \sum\limits_i \mu(A_i)$

- Examples of measures:

  1. A typical example of the measure is the function to count the number of items in each set A_i.

  2. Lebesque measure m on the real line R is defined as $m((a, b)) = b - a$ (or $m([a, b]) = b - a$), where (a, b) (or $[a, b]$) is an open (or closed) interval on the real number axis.

- If $\mu(\Omega) = 1$, we call μ a probability measure. The probability measure of a random variable is its cumulative distribution function. For example, $\mu(x \in [-\infty, 0]) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$ if x follows the standard normal distribution. However, note that in this chapter we focus on the probability measure for a stochastic process, which represents a series of random variables.

- The effect of changing probability measures is to change the mean of a random variable or the drift of a stochastic process (see Appendix B).

- The RNVR holds in both our risk averse and the risk neutral worlds $\Rightarrow$ Measures Q and P are equivalent measures

  1. Risk neutral valuation relationship (RNVR): we construct a risk free portfolio and this portfolio should earn the risk free rate based on the no-arbitrage argument. Therefore, the risk free rate is introduced in option pricing, and we can price options as if they were in the risk neutral world. It is worth noting that even under RNVR, we actually derive the option prices in the risk averse world.

  2. Definition of equivalent measures: Two measures are equivalent as long as they return zero probability for zero probability events. Of course, for sure events, two equivalent measures both return 100% probability.

  3. It is known that the arbitrage profit is a sure event, and a no-arbitrage portfolio in our risk averse world (corresponding to the measure P) is also a no-arbitrage portfolio in the risk neutral world (corresponding to the measure Q), so we can infer that the risk neutral measures Q and the physical measure P are equivalent.

  $*$ We can change probability measures only between equivalent measures.

- The existence of the risk neutral measure Q is equivalent to excluding any arbitrage opportunity.

  ($\Leftarrow$) The no arbitrage argument implies that there is a measure Q under which the stock return changes from μ to r.

  ($\Rightarrow$) If the Q measure exists such that the drift term changes from μ to r under the measure Q, it is implied that the no-arbitrage argument holds.

- After employing the no arbitrage argument to obtain the RNVR, we can price options as if they were in the virtual risk-neutral world, in which all security returns are the risk free rate. Since the drift term μ is changed to be r under the measure Q, it implies that considering the risk neutral world is equivalent to considering the measure Q. As a result, the option price today is the present value of its expected payoff under the measure Q, i.e., $c(S_0, 0) = e^{-rT} E^Q[c(S_T, T)]$. ($E^Q[\cdot]$ denotes the expectation in the risk neutral world.)

- Martingale (平賭): A process $Y = Y(t)$ is a martingale under any probability measure P if $E^P[Y(s)|\mathcal{F}_t] = Y(t)$, where $E^P[\cdot|\mathcal{F}_t]$ is the expectation under P conditional on $\mathcal{F}_t$.

  For example, in the risk neutral world, if the stock price follows the geometric Wiener process, $E^Q[S_t|\mathcal{F}_0] = S_0 e^{rt}$ and we can infer that $e^{-rt} S_t$ is a martingale process under the measure Q. In fact, in the risk neutral world, since the expected return of all securities is the risk free rate r, for the price of any security f_t (including all derivatives), $e^{-rt} f_t$ is a martingale process under the measure Q.

- Girsanov theorem (to change measure for stochastic processes)

  Given Z^Q and Z^R to be standard Wiener processes under the measure Q and R. If $E[e^{\frac{1}{2}\int_0^t H^2(\tau)d\tau}] < \infty$, and define the Radon-Nikodym derivative as

  $$\Lambda = \frac{dR}{dQ} = e^{-\int_0^T H(\tau)dZ^Q(\tau) - \frac{1}{2}\int_0^T H^2(\tau)d\tau},$$

  then $dZ^R = dZ^Q + H(t)dt$ or $Z^R(t) = Z^Q(t) + \int_0^t H(\tau)d\tau$. In addition, Q and R are equivalent measures.

  $*$ An important application of the Girsanov Theorem: $E^Q[\Lambda \cdot X] = E^R[X]$.

  Pf: $E^Q[\Lambda \cdot X] = \int \Lambda X dQ(X) = \int X \frac{dR}{dQ} dQ(X) = \int X dR(X) = E^R[X]$

  $*$ Furthermore, if $X = 1_A = \begin{cases} 1 & \text{if the event } A \text{ occurs} \\ 0 & \text{o/w} \end{cases}$

  $\Rightarrow E^Q[\Lambda \cdot 1_A] = E^R[1_A]$.

- $c(S_0, 0) = e^{-rT}E^Q[\max(S_T - K, 0)] = e^{-rT}E^Q[(S_T - K) \cdot 1_A]$,

  where $A = \{S_T \mid S_T \geq K\}$, and $1_A = \begin{cases} 1 & \text{if } \ S_T \geq K \\ 0 & \text{o/w} \end{cases}$

  $$\Rightarrow c(S_0, 0) = e^{-rT}\underbrace{E^Q[S_T \cdot 1_A]}_{(1)} - Ke^{-rT}\underbrace{E^Q[1_A]}_{(2)}$$

  $(2) = E^Q[1_A] = Pr^Q(S_T \geq K) = Pr^Q(\ln S_T \geq \ln K)$

  $= Pr^Q(\ln S_0 + (r - q - \frac{\sigma^2}{2})T + \sigma \cdot \Delta Z^Q(T) \geq \ln K)$

  $= Pr^Q(\underbrace{-\frac{\Delta Z^Q(T)}{\sqrt{T}}}_{ND(0,1)} \leq \underbrace{\frac{\ln(\frac{S_0}{K}) + (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}_{d_2})$

  $= N(d_2)$

$$
\begin{aligned}
(1) = E^Q[S_T \cdot 1_A] &= E^Q[S_0 \cdot e^{(r-q-\frac{\sigma^2}{2})T+\sigma\cdot\Delta Z^Q(T)} \cdot 1_A] \\
&= S_0 e^{(r-q)T} \cdot E^Q[\underbrace{e^{-\frac{\sigma^2}{2}T+\sigma\Delta Z^Q(T)} \cdot 1_A}] \ (= S_0 e^{(r-q)T} \textstyle\int \Lambda 1_A dQ(S_T))
\end{aligned}
$$

> Apply the Girsanov theorem
>
> Setting $H(t) = -\sigma$
>
> $\Rightarrow \Lambda = \frac{dR}{dQ} = e^{-\frac{1}{2}\int_0^T \sigma^2 d\tau - \int_0^T -\sigma dZ^Q(\tau)} = e^{-\frac{\sigma^2}{2}T+\sigma\Delta Z^Q(T)}$
>
> $\Rightarrow dZ^R = dZ^Q - \sigma dt$ (or $dZ^Q = dZ^R + \sigma dt$)
>
> (Note that when changing measure P to measure Q, $H(t) = \lambda = \frac{\mu - r}{\sigma}$.)
>
> Replace dZ^Q in the stock price process in the risk neutral world, we can obtain
>
> $\frac{dS}{S} = (r-q)dt + \sigma(dZ^R + \sigma dt) = (r-q+\sigma^2)dt + \sigma dZ^R$
>
> $\Rightarrow \frac{dS}{S} = (r-q+\sigma^2)dt + \sigma dZ^R$
>
> $\Rightarrow d\ln S = (r-q+\frac{\sigma^2}{2})dt + \sigma dZ^R \Rightarrow \ln S_T = \ln S_0 + (r-q+\frac{\sigma^2}{2})T + \sigma\Delta Z^R(T)$,
>
> where $\Delta Z^R(T) \sim ND^R(0,T)$.

$$
\begin{aligned}
&= S_0 e^{(r-q)T} \cdot E^R[1_A] \ (= S_0 e^{(r-q)T} \textstyle\int 1_A dR(S_T)) \\
&= S_0 e^{(r-q)T} \cdot Pr^R(S_T \geq K) \\
&= S_0 e^{(r-q)T} \cdot Pr^R(\underbrace{\ln S_T}_{\ln S_0 + (r-q+\frac{\sigma^2}{2})T + \sigma\triangle Z^R(T)} \geq \ln K) \\
&= S_0 e^{(r-q)T} \cdot Pr^R(\underbrace{-\frac{\Delta Z^R(T)}{\sqrt{T}}}_{ND(0,1)} \leq \underbrace{\frac{\ln(\frac{S_0}{K}) + (r-q+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}_{d_1}) \\
&= S_0 e^{(r-q)T} \cdot N(d_1)
\end{aligned}
$$

$$
\begin{aligned}
c(S_0, 0) &= e^{-rT} \cdot (1) - Ke^{-rT} \cdot (2) \\
&= e^{-rT} \cdot S_0 e^{(r-q)T} N(d_1) - Ke^{-rT} \cdot N(d_2) \\
&= S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)
\end{aligned}
$$

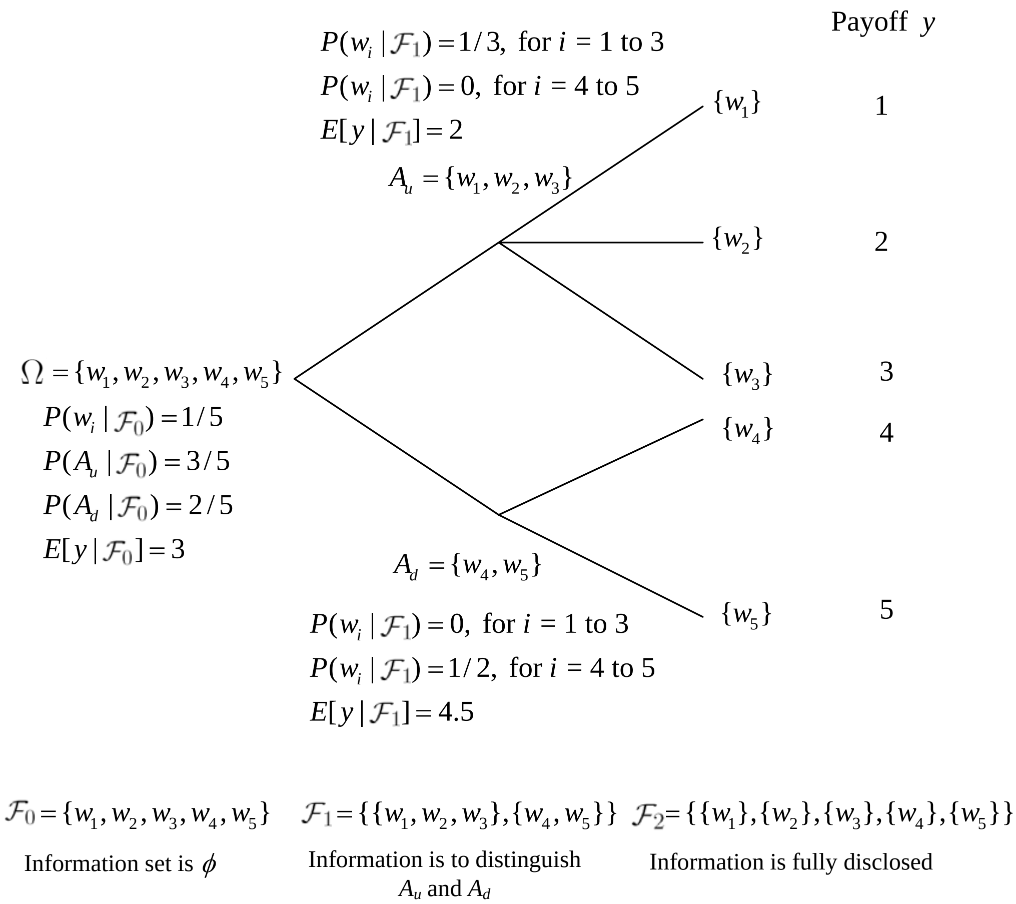
"Financial Calculus: An Introduction to Derivative Pricing," Bazter and Rennie, 1996.

"金融工程學: 金融商品創新與選擇權理論," 陳松男, 2002.

Appendix A. Illustration of Filtration and Probability Measure

- Here a two-period, discrete-value process is employed to illustrate the filtration (or the infomation structure) $\mathcal{F}_t$ and the probability measure.

Figure 2-2



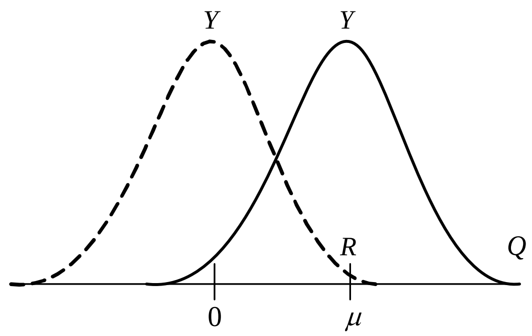
Appendix B. Changing Measure for Random Variables

- Given $X \sim ND(0,1)$ under a measure Q, examine the effect of changing measure for this random variable X.

  Suppose $Y = X + \mu$, and it is obvious that $Y \sim ND(\mu, 1)$ under Q. Find an equivalent probability measure R such that $Y (= X + \mu) \sim ND(0,1)$ under R.

  Define $\Lambda = \frac{dR}{dQ} = e^{-\mu X - \frac{\mu^2}{2}}$ (Λ plays the role of Radon-Nikodym derivative in the Girsanov theorem.)

  Figure 2-3



- Suppose $f_Q(X)$ is the probability density function of X under Q.

  $E^Q[X] = \int_{-\infty}^{\infty} X f_Q(X) dX$

  || where $f_Q(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2}$ because X is normally distirbuted.
  || In addition, define $Q(X) \equiv \int f_Q(X)dX$
  || $\Rightarrow dQ(X) = f_Q(X)dX$, and $E^Q[X] = \int X f_Q(X)dX = \int X dQ(X)$

  $= \int_{-\infty}^{\infty} X dQ(X)$

  $= 0$ (because the mean of X is zero)

- $E^Q[Y] = \int_{-\infty}^{\infty} Y f_Q(X) dX$

  $= \int_{-\infty}^{\infty} (X + \mu) f_Q(X) dX$

  $= \int_{-\infty}^{\infty} X f_Q(X) dX + \int_{-\infty}^{\infty} \mu f_Q(X) dX$

  $= 0 + \mu$

  $= \mu$

- Consider $f_R(X) = f_Q(X) \cdot \frac{dR}{dQ} = f_Q(X) \cdot \Lambda = f_Q(X) \cdot e^{-\mu X - \frac{\mu^2}{2}}$. Since $\int_{-\infty}^{\infty} f_R(X) dX = 1$, by the definition of the probability measure that $\mu(\Omega) = 1$ on page 2-8, we can conclude that $f_R(X)$ is still a probability density function.

- We thus can regard $f_R(X)$ as the probability density function of X for an equivalent measure R.

- $E^R[Y] = \int_{-\infty}^{\infty} Y f_R(X) dX = \int_{-\infty}^{\infty} (X + \mu) f_R(X) dX$

$$= \int_{-\infty}^{\infty} (X + \mu) f_Q(X) e^{-\mu X - \frac{\mu^2}{2}} dX$$

$$\left\| \begin{aligned} &= \textstyle\int_{-\infty}^{\infty} (X + \mu) e^{-\mu X - \frac{\mu^2}{2}} f_Q(X) dX \\ &= \textstyle\int_{-\infty}^{\infty} (X + \mu) \Lambda dQ(X) \ (= E^Q[(X + \mu) \cdot \Lambda]) \\ &= \textstyle\int_{-\infty}^{\infty} (X + \mu) \frac{dR}{dQ} dQ(X) \\ &= \textstyle\int_{-\infty}^{\infty} (X + \mu) dR(X) \ (= E^R[X + \mu]) \end{aligned} \right.$$

$$= \int_{-\infty}^{\infty} (X + \mu) \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(X^2 + 2\mu X + \mu^2)} dX$$

$$= \int_{-\infty}^{\infty} (X + \mu) \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(X + \mu)^2} dX$$

(define $W = X + \mu$, and thus $dW = dX$)

$$= \int_{-\infty}^{\infty} W \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}W^2} dW = 0$$

- Changing measure for a random variable is a special case of changing measure for a stochastic process:
  - ⊙ Suppose the $H(t)$ in the Girsanov Theorem to be μ.
    $\Rightarrow \Lambda = \frac{dR}{dQ} = e^{-\frac{1}{2}\int_0^T \mu^2 d\tau - \int_0^T \mu dZ^Q(\tau)} = e^{-\frac{1}{2}\mu^2 T - \mu \Delta Z^Q(T)} = e^{-\mu \Delta Z^Q(T) - \frac{1}{2}\mu^2 T}$
    (which is similar to $\Lambda = e^{-\mu X - \frac{\mu^2}{2}}$ in the above example based on the assumption of $T = 1$ and thus $\Delta Z^Q(T) \sim ND(0, 1)$ can act a similar role as X.)
    $\Rightarrow dZ^R = dZ^Q + \mu dt$ (or $dZ^Q = dZ^R - \mu dt$)
  - ⊙ Consider $\Delta Y = \mu \Delta t + \Delta Z^Q(\Delta t)$ ($\Delta Y \sim ND(\mu, 1)$ under the measure Q given the assumption $\Delta t = 1$). If we replace $\Delta Z^Q(\Delta t)$ with $\Delta Z^R(\Delta t) - \mu \Delta t$, we can derive $\Delta Y = \Delta Z^R(\Delta t)$ ($\Delta Y \sim ND(0, 1)$ under the measure R given the assumption $\Delta t = 1$).

Appendix C. Option Values Under the Jump-Diffusion Model

- The content in this appendix belongs to the advanced content.

- Under the risk neutral measure Q, given

$$d\ln S = (r - q - \tfrac{\sigma^2}{2} - \lambda K_Y)dt + \sigma dZ^Q + \ln Y^Q dq^Q,$$

where dq^Q is a Poisson counting process with the jump intensity λ, $\ln Y^Q \sim ND(\mu_J, \sigma_J^2)$, $K_Y = E[Y^Q - 1] = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$, and dZ^Q, dq^Q, and Y^Q are mutually independent, how to evaluate

$$C(S_0, 0) = e^{-rT}E^Q[(S_T - K)\cdot 1_A] = e^{-rT}E^Q[S_T \cdot 1_A] - Ke^{-rT}E^Q[1_A],$$

where $1_A = \begin{cases} 1 & \text{if} \quad S_T \geq K \\ 0 & \quad \text{o/w} \end{cases}$?

$$\begin{aligned}
\odot\ E^Q[1_A] &= Pr^Q(S_T \geq K) = Pr^Q(\ln S_T \geq \ln K) \\
&= Pr^Q(\ln S_0 + (r - q - \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma\Delta Z^Q(T) + \sum_{i=1}^{N_T^Q} \ln Y^Q \geq \ln K) \\
&\quad \Big\| \ N_T^Q \text{ is a Poisson variable with the jump intensity } \lambda T \text{ under } Q \\
&= \tfrac{e^{-\lambda T}(\lambda T)^0}{0!}\cdot Pr^Q(\ln S_0 + (r - q - \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma\Delta Z^Q(T) \geq \ln K) \ + \\
&\quad \tfrac{e^{-\lambda T}(\lambda T)^1}{1!}\cdot Pr^Q(\ln S_0 + (r - q - \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma\Delta Z^Q(T) + \ln Y^Q \geq \ln K) \ + \\
&\quad \vdots \\
&\quad \vdots \\
&\quad \tfrac{e^{-\lambda T}(\lambda T)^n}{n!}\cdot \underbrace{Pr^Q(\ln S_0 + (r - q - \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma\Delta Z^Q(T) + \sum_{i=1}^{n} \ln Y^Q \geq \ln K) \ +}_{\text{(II)}} \\
&\quad \vdots \\
&\quad \vdots \\
&\quad \tfrac{e^{-\lambda T}(\lambda T)^\infty}{\infty!}\cdot Pr^Q(\ln S_0 + (r - q - \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma\Delta Z^Q(T) + \sum_{i=1}^{\infty} \ln Y^Q \geq \ln K)
\end{aligned}$$

$*$ For (II):

$$Pr^Q(-\sigma\Delta Z^Q(T) - \sum_{i=1}^{n} \ln Y^Q \leq \ln(\tfrac{S_0}{K}) + (r - q - \tfrac{\sigma^2}{2} - \lambda K_Y)T)$$

$$\begin{array}{ll} \because & -\sigma\Delta Z^Q(T) \sim ND(0,\sigma^2 T) \\ & -\sum_{i=1}^{n}\ln Y^Q \sim ND(-n\mu_J, n\sigma_J^2) \\ \therefore & -\sigma\Delta Z^Q(T) - \sum_{i=1}^{n}\ln Y^Q \sim ND(-n\mu_J, \sigma^2 T + n\sigma_J^2) \end{array}$$

$= Pr^Q\left(\frac{-\sigma\Delta Z^Q(T)-\sum_{i=1}^{n}\ln Y^Q+n\mu_J}{\sqrt{\sigma^2T+n\sigma_J^2}} \le \frac{\ln(\frac{S_0}{K})+(r-q-\frac{\sigma^2}{2}-\lambda K_Y)T+n\mu_J}{\sqrt{\sigma^2T+n\sigma_J^2}}\right)$

$= N\left(\frac{\ln(\frac{S_0}{K})+(r+n\mu_J/T-\lambda K_Y-q-\frac{\sigma^2}{2})T}{\sqrt{\sigma^2+n\sigma_J^2/T}\sqrt{T}}\right)$

$= N\left(\frac{\ln(\frac{S_0}{K})+(r_n-q-\frac{v_n^2}{2})T}{\sqrt{v_n^2}\sqrt{T}}\right) = N(d_{2n})$,

where $r_n \equiv r + n(\mu_J + \frac{1}{2}\sigma_J^2)/T - \lambda K_Y$, and $v_n^2 \equiv \sigma^2 + n\sigma_J^2/T$

$\odot\ E^Q[S_T \cdot 1_A] = E^Q[S_0 e^{(r-q-\frac{\sigma^2}{2}-\lambda K_Y)T+\sigma\Delta Z^Q(T)+\sum_{i=1}^{N_T^Q}\ln Y^Q} \cdot 1_A]$

$= S_0 e^{(r-q)T} E^Q[e^{-(\frac{\sigma^2}{2}+\lambda K_Y)T+\sigma\Delta Z^Q(T)+\sum_{i=1}^{N_T^Q}\ln Y^Q} \cdot 1_A]$

The Girsanov theorem for the jump-diffusion process:

Consider the Radon-Nikodym derivative,

$$\Lambda = \frac{dR}{dQ} = e^{-\int_0^T(\frac{\sigma^2}{2}+\lambda K_Y)d\tau-\int_0^T -\sigma dZ^Q(\tau)+\sum_{i=1}^{N_T^Q}\ln Y^Q},$$

we can obtain

$dZ^R = dZ^Q - \sigma dt,$

and under the measure R,

N_T^Q is still a Poisson variable but with a different jump intensity $\lambda' T = \lambda(K_Y+1)T$, and $\ln Y^Q \sim ND(\mu_J+\sigma_J^2, \sigma_J^2)$.

By defining dq^R to be a Possion process with the jump and intensity λ', the corresponding jump size to be $\ln Y^R \sim ND\left(\mu_J+\sigma_J^2, \sigma_J^2\right)$ under the measure R, the stochastic differentiation equation for S should be

$$d\ln S = \left(r - q - \frac{\sigma^2}{2} - \lambda K_Y\right)dt + \sigma\left(dZ^R + \sigma dt\right) + \ln Y^R dq^R.$$

$$
\begin{aligned}
&= S_0 e^{(r-q)T} E^R[1_A] \\
&= S_0 e^{(r-q)T} Pr^R(S_T \geq K) \\
&= S_0 e^{(r-q)T} Pr^R(\ln S_T \geq \ln K) \\
&= S_0 e^{(r-q)T} Pr^R(\ln S_0 + (r - q + \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma \Delta Z^R(T) + \sum_{i=1}^{N_T^R} \ln Y^R \geq \ln K)
\end{aligned}
$$

$\Big\|$ N_T^R is a Poisson variable with the jump intensity $\lambda' T$ under R.

$$
\begin{aligned}
= \; & S_0 e^{(r-q)} T \cdot \tfrac{e^{-\lambda' T}(\lambda' T)^0}{0!} \cdot Pr^R(\ln S_0 + (r - q + \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma \Delta Z^R(T) \geq \ln K) \; + \\
& S_0 e^{(r-q)} T \cdot \tfrac{e^{-\lambda' T}(\lambda' T)^1}{1!} \cdot Pr^R(\ln S_0 + (r - q + \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma \Delta Z^R(T) + \ln Y^R \geq \ln K) \; + \\
& \vdots \\
& \vdots \\
& S_0 e^{(r-q)} T \cdot \tfrac{e^{-\lambda' T}(\lambda' T)^n}{n!} \cdot \underbrace{Pr^R(\ln S_0 + (r - q + \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma \Delta Z^R(T) + \sum_{i=1}^{n} \ln Y^R \geq \ln K)}_{\text{(I)}} \; + \\
& \vdots \\
& \vdots \\
& S_0 e^{(r-q)} T \cdot \tfrac{e^{-\lambda' T}(\lambda' T)^\infty}{\infty!} \cdot Pr^R(\ln S_0 + (r - q + \tfrac{\sigma^2}{2} - \lambda K_Y)T + \sigma \Delta Z^R(T) + \sum_{i=1}^{\infty} \ln Y^R \geq \ln K)
\end{aligned}
$$

$*$ For (I):

$$Pr^R(-\sigma \Delta Z^R(T) - \sum_{i=1}^{n} \ln Y^R \leq \ln(\tfrac{S_0}{K}) + (r - q + \tfrac{\sigma^2}{2} - \lambda K_Y)T)$$

$$
\left\|
\begin{aligned}
\because \quad & -\sigma \Delta Z^R(T) \sim ND(0, \sigma^2 T) \\
& -\sum_{i=1}^{n} \ln Y^R \sim ND(-n(\mu_J + \sigma_J^2), n\sigma_J^2) \\
\therefore \quad & -\sigma \Delta Z^R(T) - \sum_{i=1}^{n} \ln Y^R \sim ND(-n(\mu_J + \sigma_J^2), \sigma^2 T + n\sigma_J^2)
\end{aligned}
\right.
$$

$$= Pr^R\left(\frac{-\sigma \Delta Z^R(T) - \sum_{i=1}^{n} \ln Y^R + n(\mu_J + \sigma_J^2)}{\sqrt{\sigma^2 T + n\sigma_J^2}} \leq \frac{\ln(\frac{S_0}{K}) + (r - q + \frac{\sigma^2}{2} - \lambda K_Y)T + n(\mu_J + \sigma_J^2)}{\sqrt{\sigma^2 T + n\sigma_J^2}}\right)$$

$$= N\left(\frac{\ln(\frac{S_0}{K}) + (r + n\mu_J/T - \lambda K_Y - q + \frac{\sigma^2}{2} + n\sigma_J^2/T)T}{\sqrt{\sigma^2 + n\sigma_J^2/T}\sqrt{T}}\right)$$

$$= N\left(\frac{\ln(\frac{S_0}{K}) + (r_n - q + \frac{\upsilon_n^2}{2})T}{\sqrt{\upsilon_n^2}\sqrt{T}}\right) = N(d_{1n})$$

(Note that $d_{1n} = d_{2n} + \upsilon_n \sqrt{T}$.)

- Combining everything leads to

$$C(S_0) = S_0 e^{-qT} \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} N(d_{1n}) - K e^{-rT} \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} N(d_{2n}),$$

where $\lambda' = \lambda(K_Y + 1) = \lambda e^{\mu_J + \frac{1}{2}\sigma_J^2}$,

$$d_{1n} = \frac{\ln(\frac{S_0}{K}) + (r_n - q + \frac{\upsilon_n^2}{2})T}{\upsilon_n \sqrt{T}},$$

$$d_{2n} = \frac{\ln(\frac{S_0}{K}) + (r_n - q - \frac{\upsilon_n^2}{2})T}{\upsilon_n \sqrt{T}} = d_{1n} - \upsilon_n \sqrt{T},$$

$$r_n = r + n(\mu_J + \tfrac{1}{2}\sigma_J^2)/T - \lambda K_Y,$$

$$\upsilon_n^2 = \sigma^2 + n\sigma_J^2/T.$$

(The above formula is exactly the same as Eq. (19) in Merton (1976) due to the fact that $e^{-rT} \frac{e^{-\lambda T}(\lambda T)^n}{n!} = e^{-r_n T} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!}$.)

Appendix D. Option Values Under the SVJ Model

- The content in this appendix belongs to the advanced content.

- Stochastic volatility and jump (SVJ) process (Bakshi, Cao, and Chen (1997))

$$\begin{aligned} \tfrac{dS}{S} &= (r - \lambda K_Y)dt + \sqrt{V}dZ_S + (Y_S - 1)dq, \\ dV &= \kappa(\theta - V)dt + \sigma_V \sqrt{V} dZ_V, \end{aligned}$$

  where $\ln Y_S \sim ND(\mu_J, \sigma_J^2)$, dq, dZ_S, dZ_V and Y_S are mutually independent except that $\text{corr}(dZ_S, dZ_V) = \rho$.

- For a European call written on S with a strike price K and a time to maturity τ, $c(S, V, t)$, it must satisfy the following PDE:

$$\begin{aligned} (r - \lambda K_Y)S\tfrac{\partial c}{\partial S} + \kappa(\theta - V)\tfrac{\partial c}{\partial V} - \tfrac{\partial c}{\partial \tau} + \tfrac{1}{2}VS^2\tfrac{\partial^2 c}{\partial S^2} + \tfrac{1}{2}\sigma_V^2 V\tfrac{\partial^2 c}{\partial V^2} \\ +\rho\sigma_V VS\tfrac{\partial^2 c}{\partial S \partial V} + \lambda E[c(SY_S, V, t) - c(S, V, t)] = rc, \end{aligned}$$

  subject to the boundary condition $c(S_{t+\tau}, V_{t+\tau}, t+\tau) = \max(S_{t+\tau} - K, 0)$.

- The value of the call option today can be expressed as

$$c(S_t, V_t, t) = S_t\Pi_1(S_t, V_t, t) - Ke^{-rT}\Pi_2(S_t, V_t, t),$$

  where Π_1 and Π_2 are risk-neutral probabilities and can be recovered from inverting the respective characteristic functions.

  (Note that Π_1 and Π_2 play similar roles as the cumulative distribution probabilities $N(d_1)$ and $N(d_2)$ in the Black-Scholes formula.)

- A characteristic function of any real-valued random variable completely defines its probability distribution. If a random variable admits a probability density function, then the characteristic function is the inverse Fourier transform of the probability densiy function.

- Two equivalent approaches to determine behavior and properties of the probability distribution of a random variable X:

  Cumulative distribution function: $F_X(x) \equiv E[1_{\{X \le x\}}]$

  Characteristic function: $f_X(\phi) \equiv E[e^{i\phi X}] = \int_X e^{i\phi x} dF_X(x)$

- If X follows $\begin{cases} ND(\mu,\sigma^2) \qquad\quad , f_X(\phi) = e^{i\phi\mu - \frac{1}{2}\phi^2\sigma^2} \\ Poisson(\lambda) \qquad , f_X(\phi) = e^{\lambda(e^{i\phi}-1)} \\ Exponential(\lambda), f_X(\phi) = (1 - i\phi\lambda^{-1})^{-1} \end{cases}$.

  (However, under a SVJ model, the distribution of S is unknown.)

- Due to the one-to-one correspondence between $F_X(x)$ and $f_X(\phi)$, it is always possible to find one of these functions if we know the other one. The relation is expressed as follows.

$$F'_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x} f_X(\phi) d\phi$$

  $*$ For example, if $f_X(\phi) = e^{i\phi\mu - \frac{1}{2}\phi^2\sigma^2}$, then

$$\begin{aligned} F'_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\phi x} e^{i\phi\mu - \frac{1}{2}\phi^2\sigma^2} d\phi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\phi(x-\mu) - \frac{1}{2}\phi^2\sigma^2} d\phi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}(\phi + \frac{i(x-\mu)}{\sigma^2})^2 + k} d\phi \\ &\quad \Big\| \; k = \frac{\sigma^2}{2}(\frac{i(x-\mu)}{\sigma^2})^2 = -\frac{1}{2}(\frac{x-\mu}{\sigma})^2 \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &\quad \Big\| \; -\frac{y^2}{2} = -\frac{\sigma^2}{2}(\phi + \frac{i(x-\mu)}{\sigma^2})^2 \\ &\quad \Big\| \; \Rightarrow \; dy = \sigma d\phi \\ &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \text{ (the probability density function for } X \sim ND(\mu,\sigma^2)) \end{aligned}$$

- Suppose we know the characteristic function $f_j(\phi)$ corresponding to Π_j, for $j = 1, 2$. Then Π_j can be derived as

$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re[\frac{e^{-i\phi\ln(K)} f_j(\phi)}{i\phi}] d\phi,$$

  where $Re[\,\cdot\,]$ denotes the real part of a complex number.

  $*$ For most cases, the above integral does not have an analytical solution. For instance, even for the normal distribution, although we can obtain the probability density function as shown above, we connot obtain the analytical formula for its cumulative distributionfunction. So, it is usual to employ the technique of numerical integration to solve Π_j.

  $*$ Therefore, the only remaining task is to solve $f_j(\phi)$.

- Define $X = \ln(S)$ and rewrite the PDE for $c(S,V,t)$ to be

$$(r - \tfrac{1}{2}V - \lambda K_Y)\tfrac{\partial c}{\partial X} + \kappa(\theta - V)\tfrac{\partial c}{\partial V} - \tfrac{\partial c}{\partial \tau} + \tfrac{1}{2}V\tfrac{\partial^2 c}{\partial X^2} + \tfrac{1}{2}\sigma_V^2 V\tfrac{\partial^2 c}{\partial V^2} + \rho\sigma_V V\tfrac{\partial^2 c}{\partial X\partial V}$$
$$+\lambda E[c(X + \ln Y_S, V, t) - c(X, V, t)] = rc$$

⊙ By replacing $c = e^X\Pi_1 - Ke^{-rT}\Pi_2$, one can obtain

  * $\frac{\partial c}{\partial X} = e^X\frac{\partial \Pi_1}{\partial X} + e^X\Pi_1 - Ke^{-rT}\frac{\partial \Pi_2}{\partial X}$,
  * $\frac{\partial^2 c}{\partial X^2} = e^X\Pi_1 + 2\cdot e^X\cdot\frac{\partial \Pi_1}{\partial X} + e^X\frac{\partial^2 \Pi_1}{\partial X^2} - Ke^{-rT}\frac{\partial^2 \Pi_2}{\partial X^2}$,
  * $\frac{\partial c}{\partial V} = e^X\frac{\partial \Pi_1}{\partial V} - Ke^{-rT}\frac{\partial \Pi_2}{\partial V}$,
  * $\frac{\partial^2 c}{\partial V^2} = e^X\frac{\partial^2 \Pi_1}{\partial V^2} - Ke^{-rT}\frac{\partial^2 \Pi_2}{\partial V^2}$,
  * $\frac{\partial^2 c}{\partial X\partial V} = e^X\frac{\partial^2 \Pi_1}{\partial X\partial V} + e^X\frac{\partial \Pi_1}{\partial V} - Ke^{-rT}\frac{\partial^2 \Pi_2}{\partial X\partial V}$,
  * $\frac{\partial c}{\partial \tau} = e^X\frac{\partial \Pi_1}{\partial \tau} - Ke^{-rT}\frac{\partial \Pi_2}{\partial \tau} + rKe^{-rT}\Pi_2$,
  * $\lambda E[c(X + \ln Y_S, V, t) - c(X, V, t)]$
  $= \lambda E[e^{X+\ln Y_S}\Pi_1(X + \ln Y_S, V, t) - Ke^{-rT}\Pi_2(X + \ln Y_S, V, t)$
  $\quad -e^X\Pi_1(X, V, t) + Ke^{-rT}\Pi_2(X, V, t)]$
  $= e^X\{\lambda E[Y_S\Pi_1(X + \ln Y_S, V, t) - \Pi_1(X, V, t)]\}$
  $\quad -Ke^{-rT}\{\lambda E[\Pi_2(X + \ln Y_S, V, t) - \Pi_2(X, V, t)]\}$,
  * $rc = r(e^X\Pi_1 - Ke^{-rT}\Pi_2)$.

⊙ Insert the above equations into the PDE and separate Π_1 and Π_2 to derive the PDEs for Π_1 and Π_2, respectively.

$(r + \frac{1}{2}V - \lambda K_Y)\frac{\partial \Pi_1}{\partial X} + [\kappa(\theta - V) + \rho\sigma_V V]\frac{\partial \Pi_1}{\partial V} - \frac{\partial \Pi_1}{\partial \tau} + \frac{1}{2}V\frac{\partial^2 \Pi_1}{\partial X^2} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2 \Pi_1}{\partial V^2} + \rho\sigma_V V\frac{\partial^2 \Pi_1}{\partial X\partial V}$
$-\lambda K_Y\Pi_1 + \lambda E[Y_S\Pi_1(X + \ln Y_S, V, t) - \Pi_1(X, V, t)] = 0$,

and

$(r - \frac{1}{2}V - \lambda K_Y)\frac{\partial \Pi_2}{\partial X} + \kappa(\theta - V)\frac{\partial \Pi_2}{\partial V} - \frac{\partial \Pi_2}{\partial \tau} + \frac{1}{2}V\frac{\partial^2 \Pi_2}{\partial X^2} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2 \Pi_2}{\partial V^2} + \rho\sigma_V V\frac{\partial^2 \Pi_2}{\partial X\partial V} + \lambda E[\Pi_2$
$(X + \ln Y_S, V, t) - \Pi_2(X, V, t)] = 0$,

with boundary conditions $\Pi_j(X_{t+\tau}, V_{t+\tau}, t+\tau) = 1_{\{X_{t+\tau}\geq \ln K\}}$, for $j = 1, 2$.

⊙ PDEs for f_1 and f_2 (see Bakshi, Cao, and Chen (1997)):

$(r + \frac{1}{2}V - \lambda K_Y)\frac{\partial f_1}{\partial X} + [\kappa(\theta - V) + \rho\sigma_V V]\frac{\partial f_1}{\partial V} - \frac{\partial f_1}{\partial \tau} + \frac{1}{2}V\frac{\partial^2 f_1}{\partial X^2} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2 f_1}{\partial V^2} + \rho\sigma_V V\frac{\partial^2 f_1}{\partial X\partial V}$
$-\lambda K_Y f_1 + \lambda E[Y_S f_1(\phi, X + \ln Y_S, V, t) - f_1(\phi, X, V, t)] = 0$,

and

$(r - \frac{1}{2}V - \lambda K_Y)\frac{\partial f_2}{\partial X} + \kappa(\theta - V)\frac{\partial f_2}{\partial V} - \frac{\partial f_2}{\partial \tau} + \frac{1}{2}V\frac{\partial^2 f_2}{\partial X^2} + \frac{1}{2}\sigma_V^2 V\frac{\partial^2 f_2}{\partial V^2} + \rho\sigma_V V\frac{\partial^2 f_2}{\partial X\partial V} + \lambda E[f_2$
$(\phi, X + \ln Y_S, V, t) - f_2(\phi, X, V, t)] = 0$,

with boundary conditions $f_j(\phi, X_{t+\tau}, V_{t+\tau}, t+\tau) = e^{i\phi X_{t+\tau}}$, for $j = 1, 2$.

⊙ Conjecture the solutions of f_1 and f_2 as follows.

$f_1(\phi, X_t, V_t, t) = \exp(\alpha(\tau) + \alpha_V(\tau)V_t + i\phi X_t)$,

$f_2(\phi, X_t, V_t, t) = \exp(\beta(\tau) + \beta_V(\tau)V_t + i\phi X_t)$,

with $\alpha(0) = \alpha_V(0) = \beta(0) = \beta_V(0) = 0$ such that the boundary conditions of f_1 and f_2 can be satisfied.

⊙ Solve $\alpha(\tau)$ and $\alpha_V(\tau)$ in $f_1(\phi, X_t, V_t, t)$:

$$\frac{\partial f_1}{\partial X} = i\phi f_1,\ \frac{\partial f_1}{\partial V} = \alpha_V(\tau) f_1,\ \frac{\partial f_1}{\partial \tau} = [\alpha'(\tau) + \alpha'_V(\tau)V]f_1,$$
$$\frac{\partial^2 f_1}{\partial X^2} = -\phi^2 f_1,\ \frac{\partial^2 f_1}{\partial V^2} = [\alpha_V(\tau)]^2 f_1,\ \frac{\partial^2 f_1}{\partial X \partial V} = i\phi\alpha_V(\tau) f_1.$$

Replacing the above partial derivatives into the PDE of f_1 yields

$$(r + \tfrac{1}{2}V - \lambda K_Y)i\phi f_1 + [\kappa(\theta - V) + \rho\sigma_V V]\alpha_V(\tau) f_1 - [\alpha'(\tau) + \alpha'_V(\tau)V]f_1 + \tfrac{1}{2}V(-\phi^2 f_1)$$
$$+\tfrac{1}{2}\sigma_V^2 V[\alpha_V(\tau)]^2 f_1 + \rho\sigma_V V[i\phi\alpha_V(\tau) f_1] - \lambda K_Y f_1 + \lambda f_1[e^{(i\phi+1)\mu_J + \frac{1}{2}(i\phi+1)^2\sigma_J^2} - 1] = 0$$

$$\begin{aligned}
&\lambda E[Y_S f_1(\phi, X + \ln Y_S, V, t) - f_1(\phi, X, V, t)] \\
&= \lambda E[Y_S \exp(\alpha(\tau) + \alpha_V(\tau)V_t + i\phi(X_t + \ln Y_S)) - f_1] \\
&= \lambda E[Y_S \exp(i\phi(\ln Y_S)) f_1 - f_1] \\
&= \lambda f_1 E[Y_S^{i\phi+1} - 1] \\
&\because \ln Y_S \sim ND(\mu_J, \sigma_J^2) \\
&\therefore \ln Y_S^{i\phi+1} \sim ND((i\phi+1)\mu_J, (i\phi+1)^2\sigma_J^2) \\
&\quad \text{and thus } E[Y_S^{i\phi+1}] = e^{(i\phi+1)\mu_J + \frac{1}{2}(i\phi+1)^2\sigma_J^2}
\end{aligned}$$

$$\Rightarrow (r + \tfrac{1}{2}V - \lambda K_Y)i\phi + [\kappa(\theta - V) + \rho\sigma_V V]\alpha_V(\tau) - \alpha'(\tau) - \alpha'_V(\tau)V - \tfrac{1}{2}V\phi^2$$
$$+\tfrac{1}{2}\sigma_V^2 V[\alpha_V(\tau)]^2 + i\phi\rho\sigma_V V\alpha_V(\tau) - \lambda K_Y + \lambda[e^{(i\phi+1)\mu_J + \frac{1}{2}(i\phi+1)^2\sigma_J^2} - 1] = 0$$

Next, two ordinary differential equations (ODEs) for $\alpha_V(\tau)$ (based on V-terms) and $\alpha(\tau)$ (based on other terms) can be derived.

$$\alpha'_V(\tau) = \tfrac{1}{2}\sigma_V^2[\alpha_V(\tau)]^2 + [\tfrac{\kappa(\theta-V)}{V} + \rho\sigma_V(1 + i\phi)]\alpha_V(\tau) + \tfrac{1}{2}\phi(i - \phi),$$
$$\alpha'(\tau) = (r - \lambda K_Y)i\phi - \lambda K_Y + \lambda[e^{(i\phi+1)\mu_J + \frac{1}{2}(i\phi+1)^2\sigma_J^2} - 1].$$

For the above two ODEs, there exist analytical solutions (see Bakshi, Cao, and Chen (1997)). One can also refer to Appendix A in Nielsen and Schwartz (2004) for the detailed steps of solving $\alpha'_V(Z)$. If analytical solutions are not available, the Runge-Kutta method (with the fourth-order being enough) can be employed to solve ODEs numerically.

$\odot$ As for $\beta(\tau)$ and $\beta_V(\tau)$ in $f_2(\phi, X_t, V_t, t)$, they can be solved by performing similar steps.