Ch 6. Option Pricing When Volatility is Non-Constant

I. Volatility Smile

II. Option Pricing When Volatility is a Function of S and t

III. Option Pricing Under Stochastic Volatility Process

● It is convincingly believed that the constant volatility assumption of the Black-Scholes model is rejected by many empirical facts. Therefore, this chapter introduces two option pricing models to deal with the non-constant volatility.

● The first option pricing model is proposed by Nelson and Ramaswamy (1990) to consider the volatility of the underlying asset being a function of the time and the price of the underlying asset. The second option pricing model, developed by Ritchken and Trevor (1999), is to price options when the underlying asset price follows the GARCH or stochastic volatility processes.
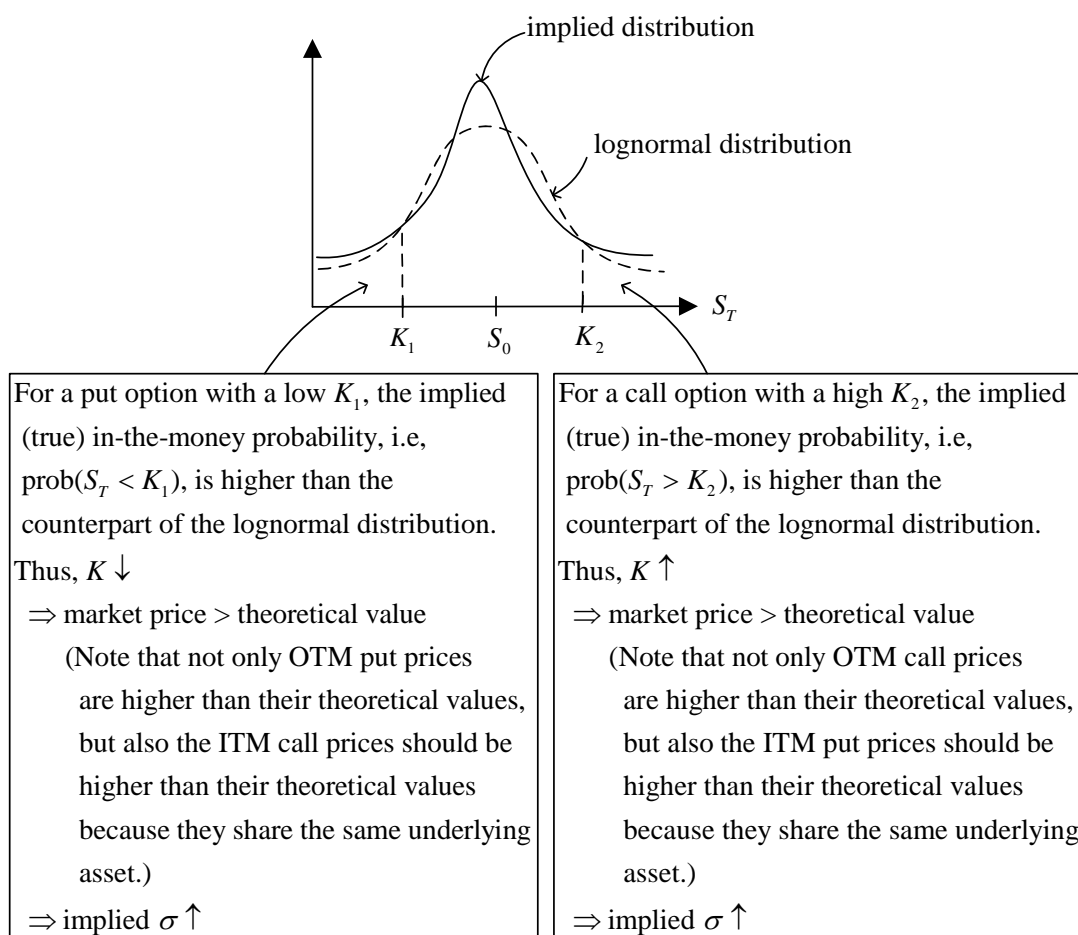
I. Volatility Smile

● The phenomenon of volatility smile shows the variation of the Black-Scholes implied volatility with respect to the strike price. In other words, the constant volatility assumption in the Black-Scholes is not so correct. Next, I will show the volatility smile for currency and equity options, respectively.

● Volatility smile for currency options

Figure 6-1 Illustration of the Volatility Smile for Currency Options
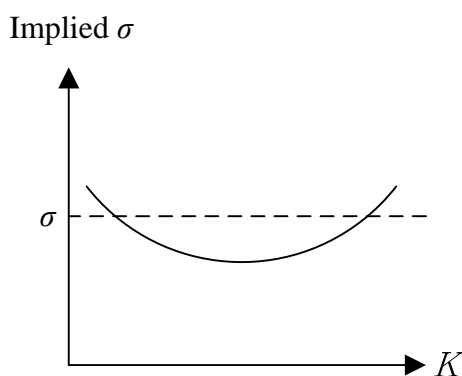
- It is intuitive that both implied volatilities of call or put options should be the same as long as they share a common underlying asset. Thus, either the implied volatilities of calls or the implied volatilities of puts could be employed to explain the volatility smile phenomenon.

Figure 6-2 Explanation of the Volatility Smile for Currency Options

For a put option with a low K_1, the implied (true) in-the-money probability, i.e, $\text{prob}(S_T < K_1)$, is higher than the counterpart of the lognormal distribution.
 	
Thus, $K \downarrow$
$\Rightarrow$ market price > theoretical value
   (Note that not only OTM put prices
    are higher than their theoretical values,
    but also the ITM call prices should be
    higher than their theoretical values
    because they share the same underlying
    asset.)
$\Rightarrow$ implied $\sigma \uparrow$
	For a call option with a high K_2, the implied (true) in-the-money probability, i.e, $\text{prob}(S_T > K_2)$, is higher than the counterpart of the lognormal distribution.
	
	Thus, $K \uparrow$
	$\Rightarrow$ market price > theoretical value
	   (Note that not only OTM call prices
	    are higher than their theoretical values,
	    but also the ITM put prices should be
	    higher than their theoretical values
	    because they share the same underlying
	    asset.)
	$\Rightarrow$ implied $\sigma \uparrow$

- Possible reasons for the volatility smile of currency options:

(i) The volatility of the underlying asset is never to be constant.

(ii) The existence of price jump causes extreme returns and thus increase the probabilities when S_T is fairly high or low. Furthermore, for longer maturity, the jumps are averaged out and thus the jumps have weaker impacts on the probabilities for S_T being fairly high or low. So, the volatility smile dies out with the increase of maturity, which is consistent with empirical studies.

(iii) The stochastic or GARCH process also can generate the probability distribution of the underlying assets with fatter tails.

- Volatility smile for equity options (including index options and individual stock options):

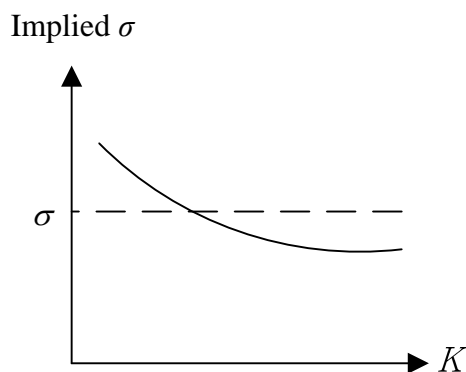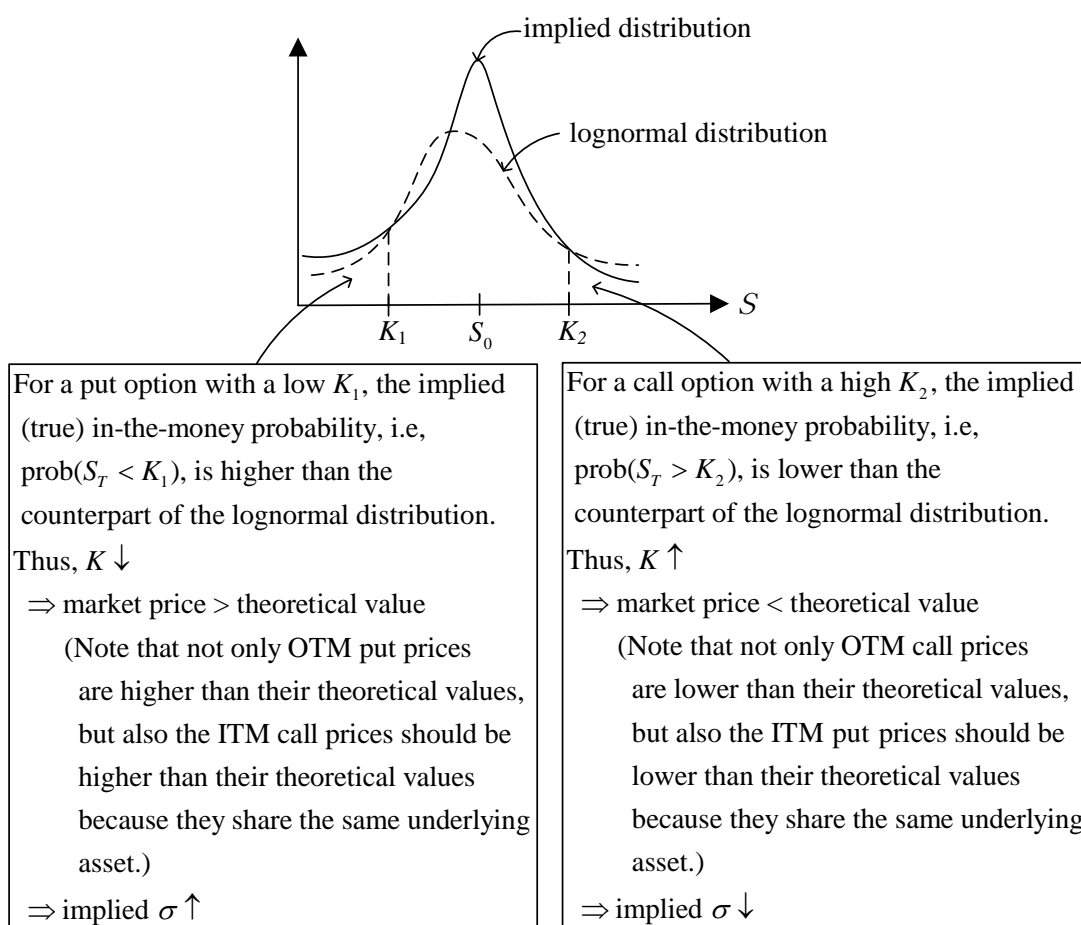**Figure 6-3 Illustration of the Volatility Smile for Equity Options**

Figure 6-4 Explanation of the Volatility Smile for Equity Options

For a put option with a low K_1, the implied (true) in-the-money probability, i.e, $\text{prob}(S_T < K_1)$, is higher than the counterpart of the lognormal distribution.

Thus, $K \downarrow$

$\Rightarrow$ market price > theoretical value

(Note that not only OTM put prices are higher than their theoretical values, but also the ITM call prices should be higher than their theoretical values because they share the same underlying asset.)

$\Rightarrow$ implied $\sigma \uparrow$

For a call option with a high K_2, the implied (true) in-the-money probability, i.e, $\text{prob}(S_T > K_2)$, is lower than the counterpart of the lognormal distribution.

Thus, $K \uparrow$

$\Rightarrow$ market price < theoretical value

(Note that not only OTM call prices are lower than their theoretical values, but also the ITM put prices should be lower than their theoretical values because they share the same underlying asset.)

$\Rightarrow$ implied $\sigma \downarrow$

- Possible reasons for the volatility smile of equtiy options:

  (i) The above volatility smile pattern for equity options is ture after the crash of stock markets in October of 1987. Before October of 1987, the volatility is almost constant for different strike prices. Thus, Mark Rubinstein, a famous academic in the finance field and the inventor of the CRR binomial tree model, terms the reason underlying this volatility pattern as "crashophobia," meaning investors are afraid that the stock markets crash again.

  (ii) Leverage effect: it is well known that when a firm employs more leverage, the financial risk of the firm increases and thus the volatility of the stock price increases to reflect this risk.

  Stock price $\downarrow$, leverage ratio $\uparrow$, volatility $\uparrow$, which make the stock price further decline.

  Stock price $\uparrow$, leverage ratio $\downarrow$, volatility $\downarrow$, which make the stock price further rise.

  (iii) The volatility should follow an asymmetric GARCH or stochastic process to reflect the countercyclical variation of the volatility, i.e., when the stock price rises (declines), it volatility decreases (increases).

- The relationship between the implied volatility and the time to maturity can be expressed as a volatility term structure. If both the stirke price and the time to maturity are considered, a 3-D implied volatility function can be derived, which is called the volatility surface. By observing the volatility surface, it can be found that the volatility smile is pronounced for shorter times to maturity, but becomes minor for longer times to maturity.

- Option pricing models for non-constant volatilities:

  (i) Volatility is a function of K:

  Since the strike price K can be measured relative to the stock price S, the volatility to be a function of K implies the volatility to be a function of S.

  $*$ Nelson and Ramaswamy (1990) deals with the volatility as a function of S and t.

  (ii) Volatility is a function of T:

  Since the current time point t and the time to maturity T always change by the same magnitude but in opposite directions, the volatility to be a function of T implies the volatility to be a function of t.

  $*$ Nelson and Ramaswamy (1990) deals with the volatility as a function of S and t.

  (iii) Volatility movements conform the GARCH or stochastic variance processes:

 $*$ Ritchken and Trevor (1999) deals with the GARCH or the stochastic variance process.

II. Option Pricing When Volatility is a Function of S and t

• The setting of the stock price process and the binomial tree model in Nelson and Ramaswamy (1990), "Simple Binomial Processes as Diffusion Approximations in Financial Models," *Review of Financial Studies* 3, pp. 393–430.

⊙ The underlying asset price is assumed to be an Itô proces: $dS = \mu(S,t)dt + \sigma(S,t)dZ_t$. (Note that both the drift and the volatility terms are functions of S and t and should be stochastic. In addition, S can be any kind of underlying asset rather than only the stock price.)

⊙ The binomial tree considered in Nelson and Ramaswamy (1990) is the model in Cox and Rubinstein (1985) as follows.

Figure 6-5 Non-recombination of the binomial tree model when the volatility is not a constant
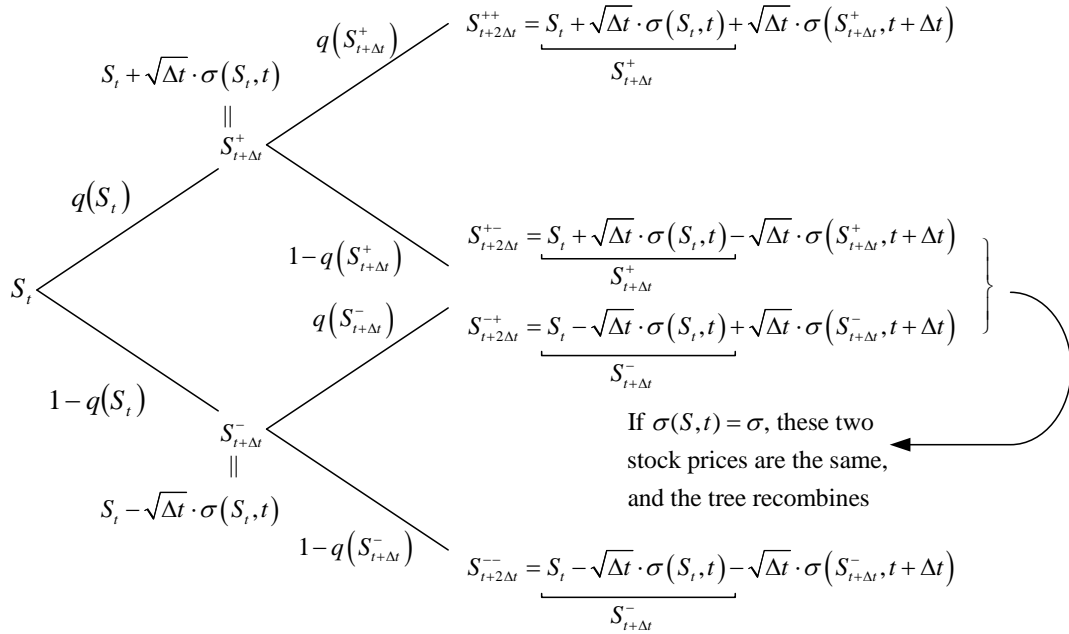
where $q(S_t) = \frac{E[S_{t+\Delta t}] - S_{t+\Delta t}^-}{S_{t+\Delta t}^+ - S_{t+\Delta t}^-} = \frac{S_t + \mu(S_t,t)\Delta t - S_{t+\Delta t}^-}{S_{t+\Delta t}^+ - S_{t+\Delta t}^-}$.

(It is obvious that the non-constant $\sigma(S,t)$ is the reason to make the binomial tree non-recombined. As for $\mu(S,t)$, it only affects the upward and downward branching probabilities for each node.)

6-5

- Main idea of Nelson and Ramaswamy (1990):

⊙ Suppose X is a function of S and t, and X is twice differentiable with respect to S and once differentiable with respect to t. According to the Itô's Lemma:

$$dX(S,t) = (\mu(S,t)\tfrac{\partial X(S,t)}{\partial S} + \tfrac{1}{2}\sigma^2(S,t)\tfrac{\partial^2 X(S,t)}{\partial S^2} + \tfrac{\partial X(S,t)}{\partial t})dt + (\sigma(S,t)\tfrac{\partial X(S,t)}{\partial S})dZ_t.$$

Deliberately set the volatility term to be 1:

$$\Rightarrow \tfrac{\partial X(S,t)}{\partial S}\sigma(S_t,t) = 1$$
$$\Rightarrow X(S,t) = \int_S \tfrac{1}{\sigma(K,t)}dK \text{ (a mapping between } X \text{ and } S)$$

⊙ Three-step process to build a recombined tree for S:

(i) Build the binomial tree for X first. Since the volatility term of X is a constant and equal to 1, the binomial tree for X recombines (see Figure 6-6).

(ii) For each node, transform the value of X to the corresponding value of S such that $S(X,t) = \{S : X(S,t) = X\}$.

(iii) The resulting binomial tree for S will recombine, and the upward and downward probability for each node can be derived via $q(S_t) = \dfrac{E[S_{t+\Delta t}] - S_{t+\Delta t}^-}{S_{t+\Delta t}^+ - S_{t+\Delta t}^-} = \dfrac{S_t + \mu(S_t,t)\Delta t - S_{t+\Delta t}^-}{S_{t+\Delta t}^+ - S_{t+\Delta t}^-}.$

Figure 6-6

※ After the transformation of deriving the S-tree, the risk neutral upward probabilities for each node can be calculated via

$$q(S_t) = \frac{E[S_{t+\Delta t}] - S_{t+\Delta t}^-}{S_{t+\Delta t}^+ - S_{t+\Delta t}^-}$$
$$= \frac{S_t + \mu(S_t,t)\Delta t - S_{t+\Delta t}^-}{S_{t+\Delta t}^+ - S_{t+\Delta t}^-}$$
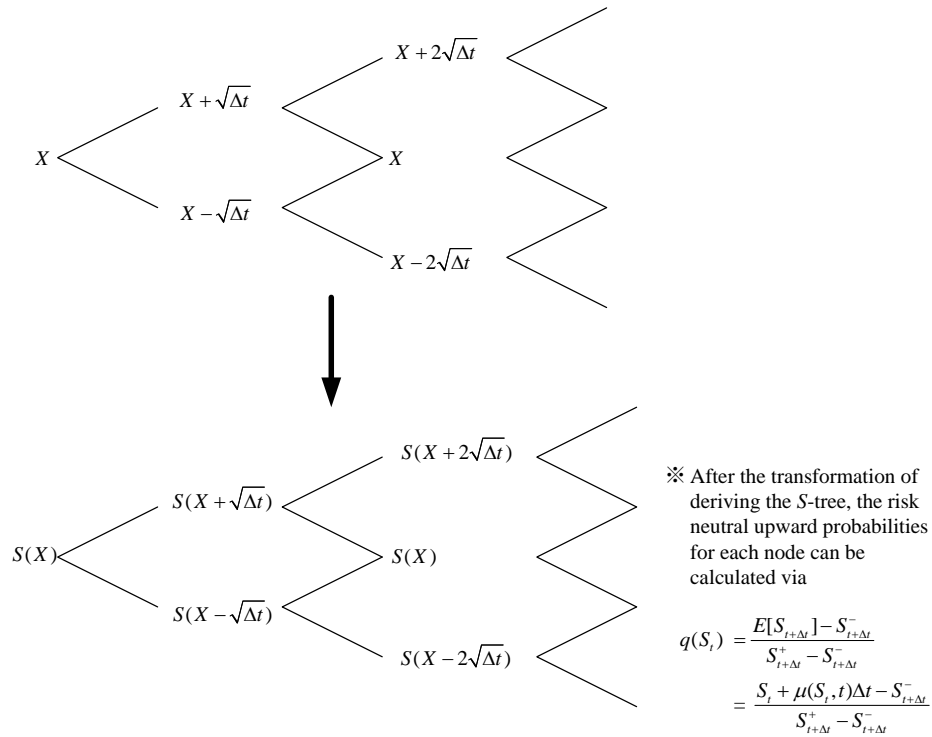
- Other implementation issues:

  (i) $\sigma(S, t)$ cannot be zero, otherwise the value of X will approach infinity.

  (Since S is a stochastic process, $\sigma(S, t)$ should not be zero. However, when the value of S is very small, sometimes $\sigma(S, t)$ is very close to 0, e.g., the CIR interest rate process.)

  (ii) Δt should be small enough to ensure that $0 < q < 1$ and thus guarantee the binomial tree of S to model the process $S(t)$ properly.

  (iii) There is no constraint for the negative value of X. However, for the underlying asset price S, it should be nonnegative.

  (If you need to avoid negative S or sometimes the transformation fucntion $S(X, t)$ only accepts positive values of X, e.g., $S(X, t) = \ln(X)$, the simplest modification you can try is to let the volatility term of X to be a number far smaller than 1 and thus restrict the variation of X. More specifically, intead of setting $\frac{\partial X(S,t)}{\partial S}\sigma(S_t, t) = 1$, you can consider, for example, $\frac{\partial X(S,t)}{\partial S}\sigma(S_t, t) = 0.01$.)

- Several examples for Nelson and Ramaswamy (1990):

  ⊙ Example 1: Constant elasticity of variance (CEV) stock price process:

  $dS = \mu S dt + \sigma S^\gamma dZ$, $0 < \gamma \le 1$

  $\left\|\begin{array}{l} \text{The feature of CEV process:} \\ \left\{ \begin{array}{l} S > 1 \Rightarrow S^\gamma < S \text{ (higher } S\text{, lower volatility than the lognormal distribution)} \\ S < 1 \Rightarrow S^\gamma > S \text{ (extermely lower } S\text{, higher volatility than the lognormal distribution)} \end{array} \right. \end{array}\right.$

  $\Rightarrow X(S,t) = \sigma^{-1} \int_S K^{-\gamma} dK = \frac{S^{1-\gamma}}{\sigma(1-\gamma)}$

  $\left\| \text{ If } \gamma = 1, \ dS = \mu S dt + \sigma S dZ, \ X(S,t) = \frac{1}{\sigma} \ln S.\right.$

  $\Rightarrow S(X,t) = \left\{ \begin{array}{ll} [\sigma(1-\gamma)X]^{\frac{1}{1-\gamma}} & \text{if } X > 0 \\ \\ 0 & \text{o/w} \end{array} \right.$

  $\Rightarrow \left\{ \begin{array}{l} S_t \equiv S(X,t) \\ S_{t+\Delta t}^+ \equiv S(X + J^+\sqrt{\Delta t}, t+\Delta t) \\ S_{t+\Delta t}^- \equiv S(X - J^-\sqrt{\Delta t}, t+\Delta t) \\ q* = \frac{S_t + \mu\Delta t - S_{t+\Delta t}^-}{S_{t+\Delta t}^+ - S_{t+\Delta t}^-} \\ q = \left\{ \begin{array}{ll} q* & \text{if } 0 \le q* \le 1 \\ 0 & \text{if } q* < 0 \\ 1 & \text{if } q* > 1 \end{array} \right. \end{array} \right.$

  ⊙ J^+ and J^- are introduced to allow multiple jumps such that in most cases, the $E[S_{t+\Delta t}]$ is inbetween two following branches and thus the upward probability q^* is in $[0,1]$. The rules to decide J^+ and J^- are as follows.

  $J^+ = \left\{ \begin{array}{ll} \text{the smallest, odd, positive, integer } j \text{ s.t.} \\ \quad S(X + j\sqrt{\Delta t}, t+\Delta t) - S(X,t) \ge \mu(S,t)\Delta t & \text{if } X < X_L \\ \\ 1 & \text{if } X \ge X_L \end{array} \right.$

  $\left\|\begin{array}{l} \text{To prevent the explosive growth of the number of nodes on the binomial tree, the} \\ \text{upward multiple jumps are allowed to reach the nodes already on the binomial tree} \\ \text{only when } X < X_L. \text{ In addition, the increment of the underlying asset price of the} \\ \text{upward branch should be higher than the expected growth of the underlying asset} \\ \text{price. (For } X \ge X_L, \text{ if the upward multiple jumps are allowed, new nodes should} \\ \text{be generated, which could result in the unexpected grow of the binomial tree.)} \end{array}\right.$

$$J^- = \begin{cases} \text{the smallest, odd, positive, integer } j \text{ s.t.} \\ \quad S(X,t) - S(X - j\sqrt{\Delta t}, t + \Delta t) \le \mu(S,t)\Delta t \\ \text{or} \\ \quad S(X - j\sqrt{\Delta t}, t + \Delta t) = 0 \ (S \text{ should be nonnegative}) \end{cases}$$

‖ The decrement of the underlying asset price of the downward branch should be smaller
‖ than the expected growth (may be negative) of the underlying asset price. However, the
‖ smallest underlying asset price which can be reached is 0 becuase S is nonnegative.

⊙ Example 2: $dS = \mu(S,t)dt + \sigma(S,t)dZ$, where $\mu(S,t) = \mu S$ and $\sigma(S,t) = \sigma S$

$$\Rightarrow \sigma \cdot S \cdot \frac{\partial X}{\partial S} = 1 \Rightarrow \partial X = \frac{1}{\sigma} \cdot \frac{\partial S}{S}$$

$$\Rightarrow X(S) = \frac{1}{\sigma} \cdot \ln S \Rightarrow S = e^{\sigma X}$$

Therefore, for $dX(S) = (\frac{\mu}{\sigma} - \frac{1}{2}\sigma)dt + 1 \cdot dZ$, X-tree can be built first. Next, S-tree can
be derived through the transformation $S = e^{\sigma X}$. In fact, the resulting S-tree is exactly
indentical to the CRR binomial tree.

⊙ Example 3: $dS = \mu(S,t)dt + \sigma(t)SdZ$, where $\sigma(t)$ is a stepwise function of t defined as follows.

Suppose the whole period is partitioned into 3 subperiods, and

$$\sigma(t) = \begin{cases} \sigma_1 & 0 \le t \le t_1 \\[2ex] \sigma_2 & t_1 < t \le t_2 \\[2ex] \sigma_3 & t_2 < t \le T \end{cases}$$

(The step function of σ_t is useful to discribe the different stages of a growing firm, i.e., seed, growing, and matured stages of a firm.)

$$\text{Nelson and Ramaswamy (1990)} \Rightarrow X(S,t) = \begin{cases} \frac{1}{\sigma_1}\ln S \Rightarrow S = e^{\sigma_1 X} & 0 \le t \le t_1 \\[2ex] \frac{1}{\sigma_2}\ln S \Rightarrow S = e^{\sigma_2 X} & t_1 < t \le t_2 \\[2ex] \frac{1}{\sigma_3}\ln S \Rightarrow S = e^{\sigma_3 X} & t_2 < t \le T \end{cases}$$

(It is worth noting that for the 3 subperiods, the volatility terms of the X prcoess are all equal to 1. Therefore, the X-tree is identical to the one in Figure 6-7 regardless of different subperiods. For different subperiods, only the transformation functions listed above are not the same.)

Figure 6-7
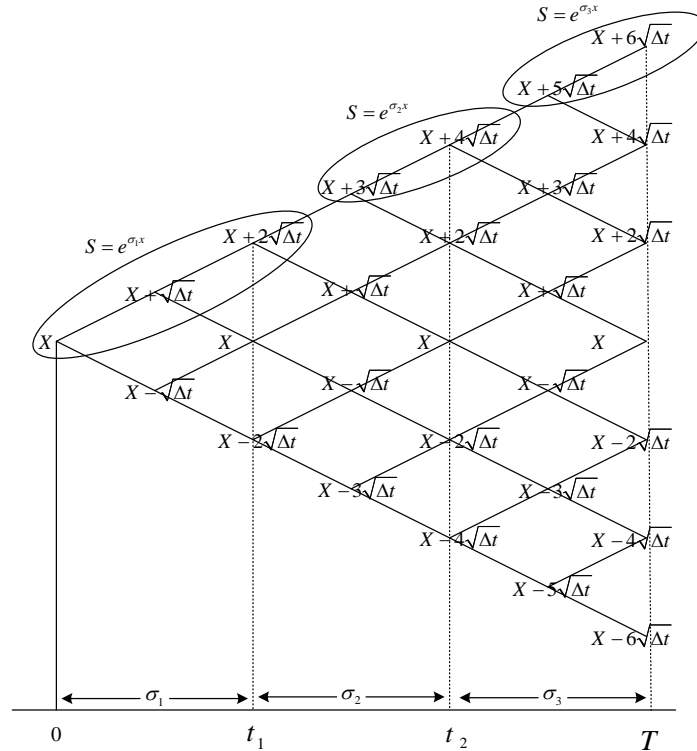
III. Option Pricing under Stochastic Volatility Process

- Ritchken and Trevor (1999),"Pricing Option under Generalized GARCH and Stochastic Volatility," *Journal of Finance* 54, pp. 377–402.

⊙ Main idea:

(i) A trinomial tree framework for the log stock price is considered.

(ii) Unlike the standard trinomial tree framework introduced in Chapter 4, a grid structure of log stock price is constructed and thus the upward and downward tick changes are fixed initially.

(iii) The most importnat originality of this paper: The change of each branching log stock price implies an innovation of the standard normally distributed variable in the Wiener process. In addition, the innovation is employed to update the conditional variance process.

(iv) Instead of recording all possible values of variances on each node, only several selected representative variances are recorded on each node. During the backward induction, the linear interpolation method is employed to find the corresponding option values for the missing variances.

⊙ General settings:

(i) Nonlinear asymmetric GARCH (NGARCH)

$$\ln(\tfrac{S_{t+1}}{S_t}) = r + \lambda\sqrt{h_t} - \tfrac{1}{2}h_t + \sqrt{h_t}\upsilon_{t+1}$$
$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t (\upsilon_{t+1} - C)^2$$

(The above two equations represent the NGARCH process under the physical measure, in which h_t represents the conditional variance process, λ is the market price of risk of S, β_0, β_1, β_2, and C are constants, and υ_{t+1} follows the standard normal distribution)

Under the risk-neutral measure, the corresponding NGARCH process is as follows.

$$\ln(\tfrac{S_{t+1}}{S_t}) = (r - \tfrac{1}{2}h_t) + \sqrt{h_t}\varepsilon_{t+1}$$
$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t (\varepsilon_{t+1} - C^*)^2$$

where $C^* = C + \lambda$, and ε_{t+1} follows the standard normal distribution.

$\left\|\begin{array}{l} \text{Change variable to } y_t \text{ by defining } y_t = \ln(S_t) \\ \Rightarrow E_t[y_{t+1}] = y_t + r - \tfrac{1}{2}h_t \text{ and } \mathrm{var}(y_{t+1}) = h_t \end{array}\right.$

(ii) For each Δt, which is assumed to be 1 day in Ritchken and Trevor (1999),

(i) $2n + 1$ branches are employed to span the normal distribution for y_{t+1}, e.g., 3 branches are used if $n = 1$. Futhermore, the vertical spacing parameter between node is defined as $\gamma_n \equiv \frac{\gamma}{\sqrt{n}} \equiv \frac{\sqrt{h_0}}{\sqrt{n}}$. The illustration of the case of $n = 1$ is shown in Figure 6-8.

(ii) The value of h_{t+1} is updated at the end of each Δt.

Figure 6-8
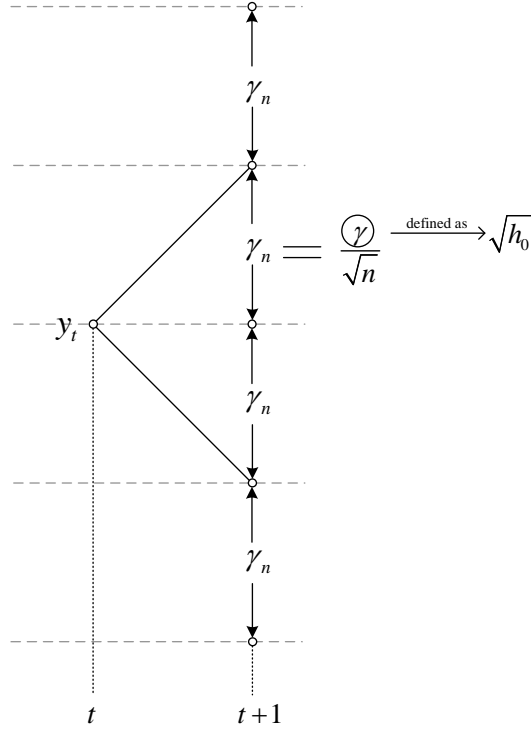
(iii) η "multiple-sized" jumps are allowed such that the probabilities of the upward, middle, and downward branches are guaranteed to be in $[0, 1]$. The rule to decide η for any variance h_t is as follows.

$$\eta - 1 < \frac{\sqrt{h_t}}{\gamma} \leq \eta, \text{ where } \gamma \equiv \sqrt{h_0}$$

(iv) For y_{t+1}, $2n + 1$ branches span the corresponding normal distribution, i.e.,

$$y_{t+1} = y_t + \theta \cdot \eta \cdot \gamma_n \ , \ \theta = 0, \pm 1, \pm 2, ..., \pm n$$

Therefore, each possible value of y_{t+1} implies a realized value of ε_{t+1} through

$$\varepsilon_{t+1} = \frac{\theta \cdot \eta \cdot \gamma_n - (r - \frac{h_t}{2})}{\sqrt{h_t}},$$

and thus leads to an update of h_{t+1} as follows.

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t [\varepsilon_{t+1} - C^*]^2.$$

(v) How to decide the probability for each branch?

(1) Partition Δt (1 day in Ritchken and Trevor (1999)) into n intervals, and the trinomial tree model is employed to model the movements of y for each interval.

(2) Therefore, if $n = 2$, there are $2n + 1 = 5$ branches for the period of Δt. Note that although n intervals are considered in the period of Δt, we do not consider the intermediate value of y during Δt, and instead we use directly the $2n + 1$ branches for each Δt.

(3) For each of the outgoing $2n + 1$ branches of y_t, its probability equals the sum of conditional probabilities of n-interval paths starting from y_t and reaching that node. The details to decide $P(\theta) \equiv \text{Prob}(y_{t+1} = y_t + \theta \eta \gamma_n)$, for $\theta = 0, \pm 1, \pm 2, ..., \pm n$, are as follows.

$$p(\theta) = \sum_{j_u, j_m, j_d} \binom{n}{j_u j_m j_d} p_u^{j_u} p_m^{j_m} p_d^{j_d}$$

$$\text{s.t. } n = j_u + j_m + j_d$$

$$\theta = j_u - j_d$$

where

$$p_u = \frac{h_t}{2\eta^2 \gamma^2} + \frac{(r - \frac{h_t}{2}) \cdot \sqrt{\frac{1}{n}}}{2 \cdot \eta \cdot \gamma}$$

$$p_m = 1 - \frac{h_t}{\eta^2 \gamma^2}$$

$$p_d = \frac{h_t}{2\eta^2 \gamma^2} - \frac{(r - \frac{h_t}{2}) \sqrt{\frac{1}{n}}}{2 \cdot \eta \cdot \gamma}$$

(Note that there are typos in the formuale for p_u, p_m, and p_d in Ritchken and Trevor (1999). To correct their formulae for p_u, p_m, and p_d, you need to replace γ_n with γ and then you can derive the above equations.)

6-13

* Illustration of the n-subperiod paths to reach $y_{t+1} = y_t + \theta \cdot \eta \cdot \gamma_n$ for different θ.

If $n = 1$, it is not necessary to make partitions for Δt and the Ritchken and Trevor's model reduces to a trinomial tree model since for each Δt, $2n + 1 = 3$ branches are considered.

$$\text{For } \theta = 1, \ (j_u, j_m, j_d) = (1, 0, 0)$$
$$\text{For } \theta = 0, \ (j_u, j_m, j_d) = (0, 1, 0)$$
$$\text{For } \theta = -1, \ (j_u, j_m, j_d) = (0, 0, 1)$$

If $n = 2$, the Ritchken and Trevor's model becomes a pentanomial tree model since for each Δt, $2n + 1 = 5$ branches are considered.

$$\text{For } \theta = 2, \ (j_u, j_m, j_d) = (2, 0, 0)$$
$$\text{For } \theta = 1, \ (j_u, j_m, j_d) = (1, 1, 0)$$
$$\text{For } \theta = 0, \ (j_u, j_m, j_d) = (0, 2, 0) \text{ and } (j_u, j_m, j_d) = (1, 0, 1)$$
$$\text{For } \theta = -1, \ (j_u, j_m, j_d) = (0, 1, 1)$$
$$\text{For } \theta = -2, \ (j_u, j_m, j_d) = (0, 0, 2)$$

(vi) Forward induction process to build the stock price tree and derive the possible variances reaching each node.

⊙ Note that there could be so many conditional variances for each node because different paths reaching that node generates different variances.

⊙ Since the total number of paths is at least 3^N, where $T/N = \Delta t$, in the case of $n = 1$, we can infer that the number of paths grows exponentially.

⊙ It is infeasible to record all conditional variances reaching each node due to the availability of memory space in a PC and the concern of the efficiency problem.

⊙ The solution proposed by Ritchken and Trevor (1999):
For each node, record only the maximum and minimum conditional variances among all conditional variances generated by the paths reaching that node. In addition, M interpolated representative conditional variances are equally-spaced placed from the maximum to minimum conditional variances. The table of representative conditional variances are constructed as follows.

$$h(i, j, k) = \frac{M - k}{M - 1} h_{\max}(i, j) + \frac{k - 1}{M - 1} h_{\min}(i, j), \ \text{for } k = 1, ..., M,$$

where $h_{\max}(i, j)$ and $h_{\min}(i, j)$ denote the maximum and minimum conditional variances reaching node(i, j).

* Based on these M interpolated representative conditional variances, the conditional variances h_t are updated in the next period of Δt.

* When M approaches infinity, the error caused by the above approximation can be ignored.

Figure 6-9 Lattice Model of Ritchken and Trevor (1999) Over Three Days.
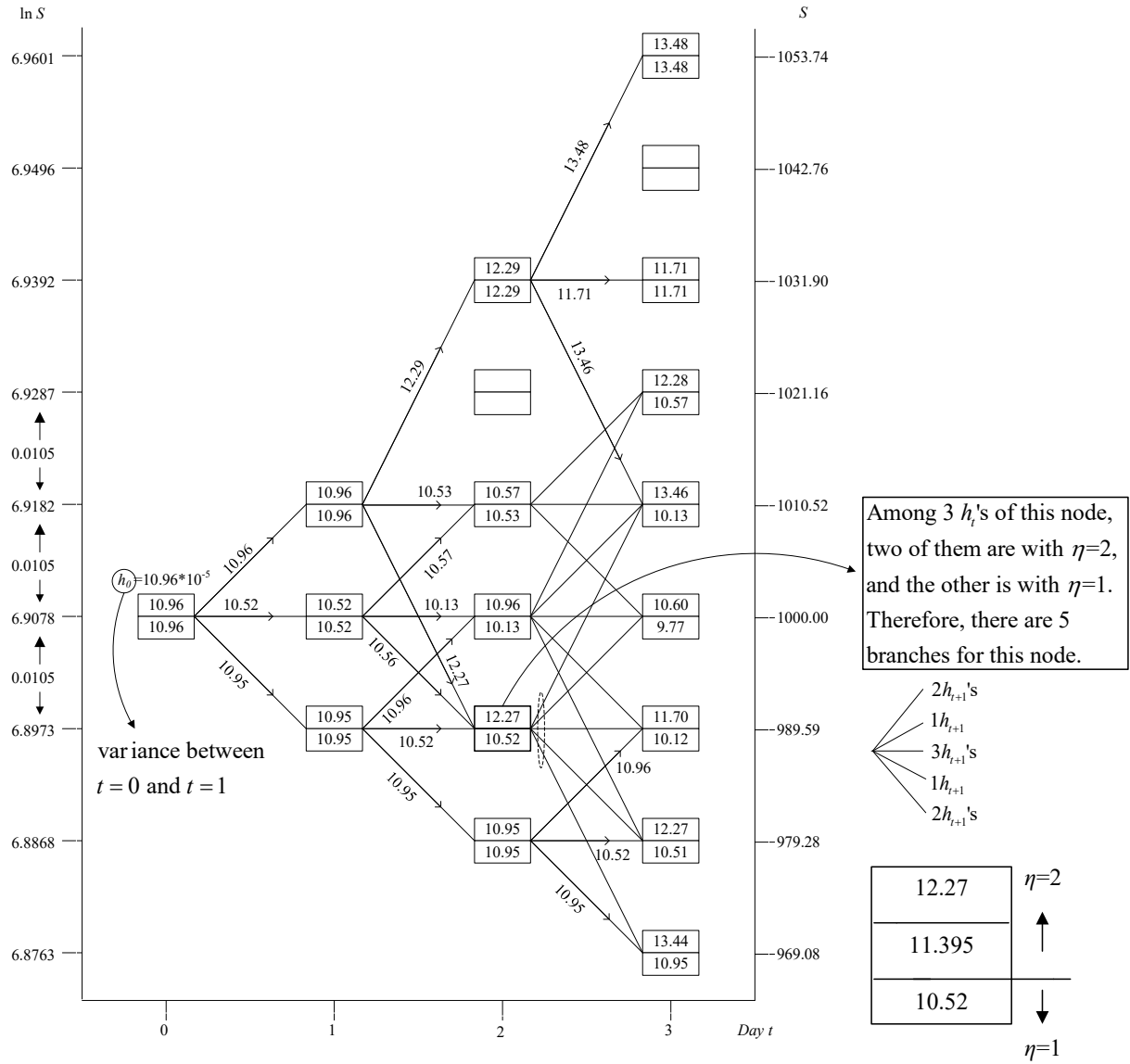
This figures shows the first three days of the first phase of the lattice for an NGARCH model with parameters $r = 0$, $\lambda = 0$, $\beta_0 = 6.575 \times 10^{-6}$, $\beta_1 = 0.90$, $\beta_2 = 0.04$, and $C = 0$. The grid of values for the logarithmic price of the underlying, $y = \ln S$ is determined by taking intervals of size $\gamma = \sqrt{h_0} = 0.0105$ around the log of the initial price $S_0 = 1000$. In this example, $n=1$, giving three possible paths from each node for a given conditional variance. Each node is represented by a box containing two numbers. The top (bottom) number is the maximum (minimum) conditional variance (multiplied by 10^5) of all paths reaching that node. In this example $M = 3$, so for each node, one additional representative conditional variance is inserted bwteen the maximum and minimum conditional variances. All these three variances determine whether the successor nodes are one or more units of γ apart on the gard. The formulae to update the conditional variance are as follows.

$$h_{t+1} = \beta_0 + \beta_1 \cdot h_t + \beta_2 \cdot h_t(\varepsilon_{t+1} - C^*)^2, \ h_0 = 0.0001096,$$

where $\varepsilon_{t+1} = \frac{j \cdot \eta \cdot \gamma_n - (r - \frac{h_t}{2})}{\sqrt{h_t}}$, $C^* = C + \lambda$

6-15

⊙ Backward induction process for option pricing:

Step 1. Decide the payoff of each conditional variance of each terminal node. Since the conditional variance is indepdent of the option value, the payoffs of different conditional variances on each node are the same. See Figure 6-10.

Step 2. For every conditional variance $h(i,j,k)$ on node(i,j) for $i = N-1, N-2, \ldots, 0$,

(i) Find the evolutions of the conditional variance on the next time step to be

$$h^{next}(\theta) = \beta_0 + \beta_1 h(i,j,k) + \beta_2 h(i,j,k) \left(\frac{\theta \eta \gamma_n - (r - h(i,j,k)/2)}{\sqrt{h(i,j,k)}} - c^* \right)^2,$$

for $\theta = 0, \pm 1, \pm 2, \ldots, \pm n$.

(ii) Suppose that $h^{next}(\theta)$ is inside the range $[h(i+1, j+\theta\eta, k_\theta), h(i+1, j+\theta\eta, k_\theta-1)]$. By the linear interpolation method, the option value C_θ for the conditional variance $h^{next}(\theta)$ can be approximated as $C_\theta = w_\theta C(i+1, j+\theta\eta, k_\theta) + (1-w_\theta)C(i+1, j+\theta\eta, k_\theta-1)$, where $w_\theta = (h(i+1, j+\theta\eta, k_\theta-1) - h^{next}(\theta))/(h(i+1, j+\theta\eta, k_\theta-1) - h(i+1, j+\theta\eta, k_\theta))$.

(iii) The continuation value for each $h(i,j,k)$ is

$$C(i,j,k) = e^{-r} \sum_{\theta=-n}^{n} p(\theta)C_\theta.$$

If the feature of early exercise is taken into account, taking vanilla call options as examples, the option value corresponding to $h(i,j,k)$ becomes

$$\max(C(i,j,k), e^{y(i,j)} - K).$$

Step 3. Repeat Step 2 for all $h(i,j,k)$'s backward over the lattice model, the value of $C(0,0,1)$ will be the GARCH option price derived by Ritchken and Trevor (1999).

∗ See Figure 6-10 for a numerical example of the above backward induction process.

Figure 6-10 Lattice for Pricing Three-Period At-The-Money Call Option.

Period 4 (right-most) column, with underlying value S:

h	c		S
13.48	53.74		
	53.74		1053.74
13.48	53.74		
			1042.76
11.71	31.90		
	31.90		1031.90
11.71	31.90		
12.28	21.16		
	21.16		1021.16
10.57	21.16		
13.46	10.52		
	10.52		1010.52
10.13	10.52		
10.60	0.00		
	0.00		1000.00
9.77	0.00		
11.70	0.00		
	0.00		989.59
10.12	0.00		
12.27	0.00		
	0.00		979.28
10.51	0.00		
13.44	0.00		
	0.00		969.08
10.95	0.00		

Period 3 column nodes:

h	c
12.29	31.90
	31.90
12.29	31.90

10.57	10.52
	10.52
10.53	10.52

10.96	2.62
	5.04
10.13	4.84

12.27	1.46
	1.35
10.52	0.00

10.95	0.00
	0.00
10.95	0.00

Period 2 column nodes:

h	c
10.96	12.02
	12.02
10.96	12.02

10.52	5.25
	5.25
10.52	5.25

10.95	1.30
	1.30
10.95	1.30

Period 1 (root) node:

h	c
10.96	6.63
	6.63
10.96	6.63

This figures shows the valuation of a three-period at-the-money European call option. Each node is represented by a box containing five mumbers. The top (bottom) number in the first column is the maximum (minimum) variance (multiplied by 10^5) of all paths reaching that node, as shown in Figure 6-9. As for the second column, the top number is the option value corresponding to the maximum variance, the bottom number is the value corresponding to the minimum variance, and the middle number corresponds to the midpoint variance.