Basic

Decision problem: Answer yes/no
P: Decision problems for which there is a poly-time algorithm.
NP: Decision problems for which there exists a poly-time certifier.

- Algorithm $C(s,t)$ is a certifier for problem X if for every string s, $s \in X$ iff there exists a string t such that $C(s,t) = \text{yes}$.

co-NP: Complements of decision problems in NP.
PSPACE: Decision problems solvable in polynomial **space**.
EXP: Decision problems for which there is an exponential-time algorithm.
Claim: $\text{P} \subseteq \text{NP}, \text{co-NP} \subseteq \text{EXP} \qquad \text{NP} \subseteq \text{PSPACE} \subseteq \text{EXP}$

Reduction: Problem X **polynomial-time reduces to** problem Y if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps.
- Polynomial number of calls to oracle(A black box that solves instances of Y in a single step) that solves problem Y.

Notation: $X \leq_P Y$.
Common approach: polynomial transformation

- Given any input x to X, **construct** an input y in poly-time such that x is a yes instance of X iff y is a yes instance of Y.

NP-Completeness

NP-complete. A problem Y in NP with the property that for every problem X in NP, $X \leq_P Y$.
Recipe to establish NP-completeness of problem Y.

1. Show that Y is in NP.
2. Choose an NP-complete problem X.
3. Prove that $X \leq_P Y$

NPC problem

- **SAT:** Given CNF formula Φ, does it have a satisfying truth assignment.
- **3-SAT:** SAT where each clause contains exactly 3 literals.
- **INDEPENDENT SET:** Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?
- **VERTEX COVER:** Given a graph $G = (V, E)$ and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?
- **SET COVER:** Given a set U of elements, a collection $S_1, S_2, \cdots, S_m$ of subsets of U, and an integer k, does there exist a collection of $\leq k$ of these sets whose union is equal to U?
- **DIR-HAM-CYCLE:** Given a digraph $G = (V, E)$, does there exists a simple directed cycle Γ that contains every node in V?
- **HAM-CYCLE:** Given an undirected graph $G = (V, E)$, does there exist a simple cycle C that visits every node.

- **TSP:** Given a set of n cities and a pairwise distance function $d(u, v)$, is there a tour of length $\leq D$.
- **GRAPH 3-COLOR:** Given an **undirected** graph G does there exist a way to color the nodes red, green, and blue so that no adjacent nodes have the same color.
- **3D-MATCHING:** Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples.
- **SUBSET-SUM:** Given natural numbers $w_1, \cdots, w_n$ and an integer W, is there a subset that adds up to exactly W.
- **SCHEDULING:** Given a set of n jobs with processing time t_i , release time r_i, and deadline d_i, is it possible to schedule all jobs on a single machine such that job i is processed with a contiguous slot of t_i time units in the interval $[r_i, d_i]$

PSPACE-Complete

Problem Y is PSPACE-complete if (i) Y is in PSPACE and (ii) for every problem X in PSPACE, $X \leq_P Y$. PSPACE-Complete problems

- **QSAT(Quantified 3-SAT)** Let $\Phi(x_1, \cdots, x_n)$ be a Boolean CNF formula. Is the following propositional formula true:

$$\exists x_1 \ \forall x_2 \ \exists x_3 \ \forall x_4 \cdots \forall x_{n-1} \ \exists x_n \ \Phi(x_1, \cdots, x_n) \qquad n \text{ is odd}$$

- Competitive Facility Location
  - **Input**. Graph with positive node weights, and target number B.
  - **Game**. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors has been selected.
  - **Competitive facility location**. Can second player guarantee at least B units of profit?

Randomized Algorithms

- $(1 - 1/x)x \leq 1/e \qquad \ln(n + 1) < \sum\limits_{i=1}^{n} 1/n < 1 + \ln n$
- $(1 - 1/n)^n$ converges monotonically from 1/4 up to $1/e$(n increases from 2)
- $(1 - 1/n)^{n-1}$ converges monotonically from 1/2 down to $1/e$
- **Union bound:** Given events $E_1, \cdots, E_n$, $\Pr[\bigcup\limits_{i=1}^{n} E_i] \leq \sum\limits_{i=1}^{n} \Pr[E_i]$
- $E[X] = \sum\limits_{j=0}^{\infty} \Pr[X = j]$
- Given two random variables X and Y (not necessarily independent) defined over the same probability space, $E[X + Y] = E[X] + E[Y]$.

Monte Carlo algorithm: Guaranteed running time, likely to find correct answer.

- Contraction algorithm for global min cut

Las Vegas algorithm: Guaranteed to find correct answer, likely to run in certain time.

- Johnson's MAX-3SAT algorithm
- Randomized quicksort