Quadratically Constrained Quadratic Programming

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***Abstract*—This project mainly focus on the non-convex quadratically constrained quadratic programming (QCQP) problems. We find and summarize some relaxations for solving non-convex QCQPs including the widely used semidefinite relaxation and Lagrangian relaxation. Then we find a sufficient condition under which there exists hidden convexity in the QCQP problem. Finally, we compare the performance of the semidefinite relaxation and the non-convex algorithm - Concensus-ADMM applied to a simple multicast beamforming problem.**

I. INTRODUCTION

A. Background

Quadratically Constrained Quadratic Programming (QCQP) is an optimization problem in which both constraints and the object function are quadratic functions. It consists of many computationally hard optimization problems and has the general form

$$\begin{aligned} &minimize && \frac{1}{2}x^T P_0 x + q_0^T x \\ &subject\ to && \frac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0 \quad i = 1, ...m \\ & && Ax = b \end{aligned} \tag{1}$$

where P_0,... P_m are n-by-n matrices and $x \in R^n$.

In general, solving nonconvex QCQPs is known to be a NP-hard problem. However, there exists some reasonable approaches to approximate them and compute close estimation of the original problems. When P_0,... P_m are all positive definite, the problem becomes convex and are easy to solve or give approximation of high accuracy. In practice, nonconvex QCQPs have numerous applications in a broad spectrum of fields including wireless communications (e.g., MAXCUT problem [1], waveform radar design [2], [3], multicast beamforming [4]–[6]), signal processing (e.g., phase retrieval [7]), power systems (e.g., optimal power flow(OPF) [8]), machine learning [9], [10] and financial engineering (e.g., portfolio risk management [11]).

B. Objective

There are several objectives in this project to accomplish. The first one is to find and summarize all the relaxations of non-convex QCQP such that convex programming method can be applied. The second one is to discover conditions under which the relaxed convex program will give the same solution (or at least a good approximation) of the original form. The third one is to find other possible conditions except that P_0,... P_m are positive definite for original QCQP to be convex (hidden convexity). The fourth one is to construct an application of non-convex QCQP and solve it by applying the relaxations proposed in the second part. At last, present a literature review of non-convex algorithms for solving QCQP and compare the result of using non-convex algorithm with that in the fourth part.

II. RELAXATIONS FOR SOLVING NON-CONVEX QCQPS

For simplicity, the form of QCQP we consider is shown below

$$\begin{aligned} &\underset{x \in R^n}{minimize} && \frac{1}{2}x^T P_0 x + q_0^T x \\ &subject\ to && \frac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0 \quad i = 1, ...m \end{aligned} \tag{2}$$

This is because the equality constraint $h_i(x) = 0$ can be expressed as

$$h_i \leq 0, \quad h_i \geq 0$$

from which we can see that (1) is equivalent to (2) if we eliminate the equality constraint in (1) and introduce two additional inequality constraints.

A. Semidefinite relaxation

Semidefinite is a powerful, computationally efficient approximation technique for a bunch of very hard optimization problems.

First, let us focus on the real-valued homogeneous QCQP, which has the form as follows:

$$\begin{aligned} &\underset{x \in R^n}{minimize} && x^T P_0 x \\ &subject\ to && x^T P_i x \leq r_i, \ i = 1, ..m \end{aligned} \tag{3}$$

Following the same procedure in [12],by introducing a new variable $X = xx^T$ which is a symmetric positive semidefinite matrix with rank one, we can rewrite (3) as the following equivalent form:

$$\begin{aligned} &\underset{X \in S^n}{minimize} && Tr(P_0 X) \\ &subject\ to && Tr(P_i X) \leq r_i, \ i = 1, ...m \\ & && X \succeq 0, \quad rank(X) = 1 \end{aligned} \tag{4}$$

Here, we use $X \succeq 0$ to suggest that X is PSD.

The rank constraint of X is the only difficult part and all other constraints of X are convex in X. Then the semidefinite relaxation (SDR) of (3) is formulated by dropping the rank constraint in (4):

$$\begin{aligned} &\underset{X \in S^n}{minimize} && Tr(P_0 X) \\ &subject\ to && Tr(P_i X) \leq r_i, \ i = 1, ...m \\ & && X \succeq 0 \end{aligned} \tag{5}$$

As a result, (5) can be solved to any arbitrary accuracy, in a numerically reliable an efficient way (under polynomial time complexity determined by the problem size n and the number of constraints m). Denote f^* be the optimal value for (3), f^{sdr} be the optimal value for (5), then the value γ that satisfies

$$f^* \leq f^{sdr} \leq \gamma f^*$$

is the so-called approximation ratio. There many types of homogenous QCQP that have known approximation ratio γ. E.g. in Boolean QP problem,

$$\begin{aligned} \underset{x \in R^n}{maximize} \quad & x^T P x \\ subject\ to \quad & x_i^2 = 1, \quad i = 1, ...n \end{aligned}$$

$\gamma = 0.87856$ when $P \succeq 0$, $P_{ij} \leq 0\ \forall i \neq j$, $\gamma = \frac{2}{\pi}$ when $C \succeq 0$, and $\gamma = 1$ (optimal) when $C_{ij} \geq 0, i \neq j$ [13]–[15]. More examples can be found in [12]. Besides, when the non-convex QCQP with Toeplitz-Hermition quadratics, SDR will obtain the exact solution [16]. (See in Section 3)

For a specific subclass of (2), when the Slater regularity condition is satisfied and then number of constraints is two, then if any of the following conditions is achieved, then the SDR of (2) gives the exact solution, i.e. there is no gap between the optimal value between the original problem and SDR of the original problem [17]:

- One of the two constraints in the SDR is not binding
- The two constraint functions and the objective function are all homogeneous quadratic functions
- One ellipsoidal and a linear complementarity constraint

For the general inhomogeneous QCQP , we can rewrite (2) as homogeneous QCQP :

$$\begin{aligned} \underset{x \in R^n, t \in R}{minimize} \quad & \begin{bmatrix} x^T & t \end{bmatrix} \begin{bmatrix} P_0 & q_0 \\ q_0^T & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ subject\ to \quad & t^2 = 1, \\ & \begin{bmatrix} x^T & t \end{bmatrix} \begin{bmatrix} P_i & q_i \\ q_i^T & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + r_i \leq 0 \\ & i = 1, ...m \end{aligned} \tag{6}$$

We can see that we can change inhomogeneous QCQPs to homogenized form with dimension $n + 1$ and $m + 1$ constraints. Thus, we can apply SDR to the homogenized form of inhomogeneous QCQPs.

B. Lagrangian relaxation

The Lagrangian relaxation is a simple method to find the lower bound of (2). For simplicity, define:

$$\begin{aligned} \tilde{P}(\lambda) &= P_0 + \sum_{i=1}^{m} \lambda_i P_i \\ \tilde{q}(\lambda) &= q_0 + \sum_{i=1}^{m} \lambda_i q_i \\ \tilde{r}(\lambda) &= \sum_{i=1}^{m} \lambda_i r_i \end{aligned} \tag{7}$$

where $\lambda \in R_+^m$. The Lagrangian dual problem can be expressed as a semidefinite program (SDP) [18] :

$$\begin{aligned} maximize \quad & \alpha \\ subject\ to \quad & \lambda_i \geq 0, \quad i = 1, ...m \\ & \begin{bmatrix} \tilde{P}(\lambda) & (1/2)\tilde{q}(\lambda) \\ (1/2)\tilde{q}(\lambda)^T & \tilde{r}(\lambda) - \alpha \end{bmatrix} \succeq 0 \end{aligned} \tag{8}$$

In most cases, the strong duality doesn't hold. However, there exists some QCQPs that there is no gap between (8) and (2) under certain conditions. For example:

- When the number of constraint is one (i.e., $m = 1$). Also when the QCQP has one interval constraint
$$l \leq f_1(x) \leq u$$
- In the case that the number of constraints is two. Suppose both problems (2) and its Lagrangian dual (8) are strictly feasible and that
$$\exists \alpha, \beta \in R\ such\ that\ \alpha P_1 + \beta P_2 \succ 0$$
Let $(\lambda_1^*, \lambda_2^*)$ be an optimal solution for the dual problem (8). If
$$d = dim(\mathcal{N}(A_0 - \lambda_1 P_1 - \lambda_2 P_2) \neq 1$$
Then the solution of Lagrangian relaxation (8) is the exact solution to the original form (2) [19].
- Let $\mathcal{F}$ denote the feasible set of (2). Denote
$$I(x) = \{i \in \{1, ...m\} \mid \frac{1}{2} x^T P_i x + q_i^T x + r_i = 0\}$$
Let $x^* \in \mathcal{F}$ and $\lambda^{x*} \in R_+^m$, then there is no duality gap between (2) and (8) if the following conditions hold [20]:
  - $\tilde{P} \succeq 0$, $\tilde{P}$ is defined in (7);
  - $\tilde{P}(\lambda^*)x^* + \tilde{q}(\lambda^*) = 0 \quad \tilde{q}$ is defined in (7);
  - For each i=1,...m, $\lambda_i^* > 0$ implies $i \in I(x^*)$

C. Other relaxations

The semidefinite relaxation and the Lagrangian relaxation are useful and popular tools to solve non-convex QCQPs. In general, there exists some other relaxations for non-convex QCQPs

- Reformulation Linearization Technique (RLT) [21]
- Second Order Cone Constrained Convex Relaxations [22]
- Spectral Relaxation [23]–[25]

III. Hidden convexity in QCQP with Toeplitz-Hermitian Quadratics

It is well known that if $P_0, P_1, ...Pm$ are positive definite, the QCQP is convex. When the QCQP is convex, the problem can be solved very computationally efficiently. In fact, when all the matrices $P_0, ...P_m$ possess Toeplitz-Hermitian structure and $P_0 \succeq 0$, then first the feasibility of the QCQP can always be determined in polynomial-time using SDR and second if the problem is feasible, then SDR can give a globally optimal solution in polynomial time [13].

A. QCQPs with Toeplitz quadratic structure

When the $\{P_i\}_{i=0}^m$ are all Toeplitz, each P_i can be expressed as

$$P_i = \sum_{k=-p}^{p} a_{m,k}\Theta_k$$

where $p = n-1$, and $\Theta_k \in R^{n\times n}$ is an elementary Toeplitz matrix with ones on the k^{th} diagonal and zeros elsewhere ($k=0$: main diagonal, $k>0$:super-diagonal), $a_{i,k}$ represents the entries along the k^{th} diagonal. Also note that the domain of the problem in this section is changed to $\mathcal{C}$.

For those QCQPs with Toeplitz Quadratic forms, we can check the feasibility in polynomial time and solve them to obtain the global optimality in polynomial time.

B. QCQPs with Circulant Quadratic forms

Now we consider a more special case of QCQP with Toeplitz quadratic forms. When all the matrices $\{P_i\}_{i=0}^m$ are circulant, i.e. $\{P_i\}_{i=0}^m$ are diagonalized by the discrete Fourier Transform (DFT) matrix and can be written as $P_i = F\Lambda_i F^H$, where $F \in \mathcal{C}^{n\times n}$ is the unitary DFT matrix:

$$F = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \dots & & & & \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)(n-1)} \end{bmatrix}$$

where $\omega_n = \exp -j\frac{2\pi}{N}$ is the N^{th} root of unity, Λ_i is a diagonal matrix of the eigenvalues of P_i obtained by taking the DFT of the first row of P_i. By applying the above equations, we can change the variable to rewrite (3) as a linear program problem.

$$\begin{aligned} \underset{z\in R^n}{minimize} \quad & \lambda_0^T z \\ subject\ to \quad & \lambda_i^T z \le r_i, \quad i = 1,...m \\ & z \ge 0 \end{aligned} \tag{9}$$

which can be solved to the global optimality very efficiently.

IV. Application

In this section, we will consider the single-group multicast beamforming, which has the following form [26]

$$\begin{aligned} \underset{\omega\in\mathcal{C}^n}{minimize} \quad & \parallel \omega \parallel^2 \\ subject\ to \quad & \mid h_i^H \omega \mid^2 \ge 1, \quad i = 1,2...m \end{aligned} \tag{10}$$

Here, $h_i \in \mathcal{C}^n$ are given problem data, which corresponds to the channels coefficients determined by the noise power and the required receive SNR, and m represents the number of users. The problem can be described as minimizing the transmit power $\parallel \omega \parallel^2$ while conforming to the receive SNR requirements. We will apply SDR and ADMM (a non-convex algorithm) to solve the single group multicast beamforming problem.

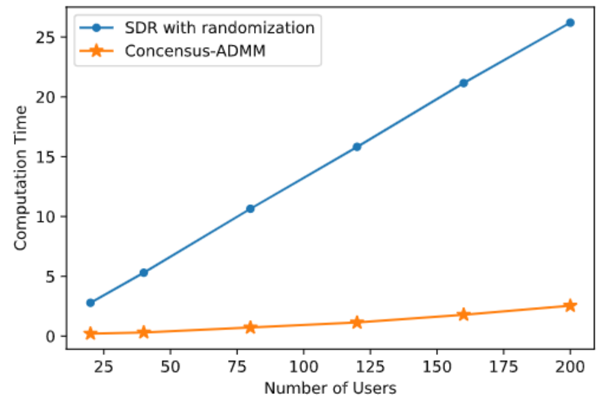

Fig. 1. Average computation time

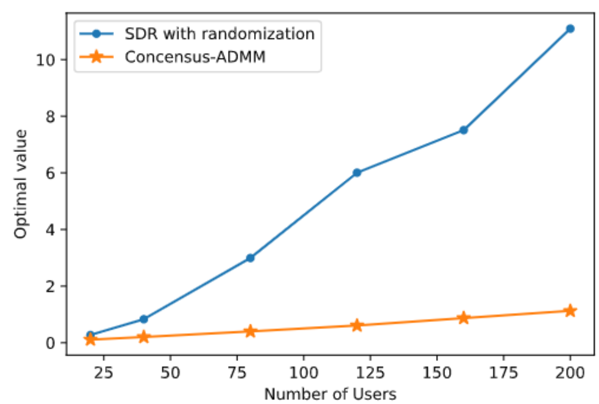

Fig. 2. Average optimal value

A. Semidefinite relaxation

The software we use to solve this problem is $CVXPY$ [27]. Note that the domain of this problem has changed to $\mathcal{C}^n$. Since $CVXPY$ doesn't support problems in complex value, following [18], we convert the problem to an equivalent form with real numbers only. Let:

$$a_i = (Re(h_i), Img(h_i)), \quad b_i = (-Img(h_i), Re(h_i)),$$

where $a_i, b_i \in R^{2n}$. ($Re(h_i)$ and $Img(h_i)$ denote the real and imaginary parts of h_i). Then the equivalent problem is written as

$$\begin{aligned} \underset{x\in R^{2n}}{minimize} \quad & \parallel x \parallel_2^2 \\ subject\ to \quad & (a_i^T x)^2 + (b_i^T x)^2 \ge 1, \quad i = 1,2,...m \end{aligned} \tag{11}$$

B. Non-convex algorithms

Besides relaxations, there exists several non-convex algorithms that can solve QCQPs efficiently.

- Alternating directions method of multipliers (ADMM). [28]

- Feasible Point Pursuit and Successive Approximation (FPP-SCA) [29]
- Conjugate gradient-based algorithm [30]

In this project, we implement an ADMM based algorithm - Concensus-ADMM [31] and in the experiment section, we show some numerical results of this non-convex algorithm.

V. Experiment results

A. Problem Settings

We consider a problem instance with fixed $n = 50$, $m \in \{20, 40, 80, 120, 160, 200\}$. The real and imaginary parts of $\{h_i\}_1^m$ are drawn IID from $\mathcal{N}(0, 1)$. $CVXPY$ is used to solve the SDR in the experiment and the Gaussian randomization [12] with 50 trials is then applied to obtain the optimal point of SDR. For concensus-ADMM, we choose the parameter ρ the same with the author which is $2\sqrt{m}$. For each $m \in \{20, 40, 80, 120, 160, 200\}$, we conduct 50 experiments with standard Gaussian random initialization for h_i and take the average results of them as the final experiment results.

B. Numerical Results

The average computation time and the average optimal transmit power are shown in the Fig. 1 and Fig. 2. Note that the CPU processor is 2.3 GHz Quad-Core Intel Core i5.

We can see that in the case of the single-group multicast beamforming problem, the Concensus-ADMM algorithm achieves a better performance compared to SDR with lower running time and better optimality.

VI. Conclusion

In this project, we mainly consider the non-convex QCQPs and summarize some relaxations for obtaining reasonable approximations. Besides, we find some conditions under which the relaxations will give the same solution of the original form. Furthermore, there exists hidden convexity of those QCQPs with Toeplitz-Hermition quadratics. Finally, we compare the performance of the SDR and Concensus-ADMM for a simple single-group multicast beamforming problem and find that the non-convex algorithm gives a significantly better results.

References

[1] Goemans, Michel X. and David P. Williamson. “Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming.” J. ACM 42 (1995): 1115-1145.

[2] A. De Maio, S. De Nicola, Y. Huang, S. Zhang, and A. Farina, “Code design to optimize radar detection performance under accuracy and similarity constraints,” IEEE Trans. Signal Process., vol. 56, no. 11, pp. 5618–5629, Nov. 2008

[3] A. De Maio, S. De Nicola, Y. Huang, Z.-Q. Luo, and S. Zhang, “Design of phase codes for radar performance optimization with a similarity constraint,” IEEE Trans. Signal Process., vol. 57, no. 2, pp. 610–621, Feb. 2009.

[4] N. Sidiropoulos, T. Davidson, and Z.-Q. Luo, “Transmit beamforming for physical-layer multicasting,” IEEE Trans. Signal Process., vol. 54, no. 6, pp. 2239–2251, June 2006.

[5] E. Karipidis, N. D. Sidiropoulos, and Z.-Q. Luo, “Quality of Service and Max-min-fair Transmit Beamforming to Multiple Co-channel Multicast Group,” IEEE Trans. Signal Pro- cess., vol. 56, no. 3, pp. 1268–1279, Mar. 2008.

[6] A. B. Gershman, N. D. Sidiropoulos, S. Shahbazpanahi, M. Bengtsson, and B. Ottersten, “Convex optimization based beamforming,” IEEE Signal Process. Mag., vol. 27, no. 3, pp. 62–75, May 2010.

[7] J. R. Fienup, “Reconstruction of an object from the modulus of its fourier transform,” Optics Letters, vol. 3, no. 1, pp. 27–29, 1978.

[8] H. Godard, S. Elloumi, A. Lambert, J. Maeght and M. Ruiz, ”Novel Approach Towards Global Optimality of Optimal Power Flow Using Quadratic Convex Optimization,” 2019 6th International Conference on Control, Decision and Information Technologies (CoDIT), Paris, France, 2019, pp. 1227-1232, doi: 10.1109/CoDIT.2019.8820584.

[9] Bishop C M 2006 Pattern recognition and machine learning. Springer, Berlin.

[10] Duda R O, Hart P E and Stork D G 2001 Pattern classification. Wiley.

[11] Qian Li, Yanqin Bai (2016) Optimal trade-off portfolio selection between total risk and maximum relative marginal risk, Optimization Methods and Software, 31:4, 681-700, DOI: 10.1080/10556788.2015.1041946

[12] Z. Luo, W. Ma, A. M. So, Y. Ye and S. Zhang, ”Semidefinite Relaxation of Quadratic Optimization Problems,” in IEEE Signal Processing Magazine, vol. 27, no. 3, pp. 20-34, May 2010, doi: 10.1109/MSP.2010.936019.

[13] M. X. Goemans and D. P. Williamson, “Improved approximation algorithms for maximum cut and satisfiability problem using semi-definite programming,” J. ACM, vol. 42, pp. 1115–1145, 1995.

[14] Yu. E. Nesterov, “Semidefinite relaxation and nonconvex quadratic optimization,” Optim. Meth. Softw., vol. 9, pp. 140–160, 1998.

[15] S. Zhang, “Quadratic maximization and semidefinite relaxation,” Math. Program., vol. 87, pp. 453–465, 2000.

[16] A. Konar and N. D. Sidiropoulos, ”Hidden Convexity in QCQP with Toeplitz-Hermitian Quadratics,” in IEEE Signal Processing Letters, vol. 22, no. 10, pp. 1623-1627, Oct. 2015, doi: 10.1109/LSP.2015.2419571.

[17] Y. Ye and S. Zhang, “New Results on Quadratic Minimization”, SIAM J. Optim., vol. 14, no. 1, pp. 245-267, 2003.

[18] Park, Jaehyun, Boyd, Stephen. (2017). General Heuristics for Nonconvex Quadratically Constrained Quadratic Programming.

[19] Strong Duality in Nonconvex Quadratic Optimization with Two Quadratic Constraints Amir Beck and Yonina C. Eldar SIAM Journal on Optimization 2006 17:3, 844-860

[20] Zheng, X.J., Sun, X.L., Li, D. et al. On zero duality gap in nonconvex quadratic programming problems. J Glob Optim 52, 229–242 (2012). https://doi.org/10.1007/s10898-011-9660-y

[21] Anstreicher, K.M. Semidefinite programming versus the reformulation-linearization technique for nonconvex quadratically constrained quadratic programming. J Glob Optim 43, 471–484 (2009). https://doi.org/10.1007/s10898-008-9372-0

[22] Jiang, R., Li, D. Second order cone constrained convex relaxations for nonconvex quadratically constrained quadratic programming. J Glob Optim 75, 461–494 (2019). https://doi.org/10.1007/s10898-019-00793-y

[23] C. Delorme and S. Poljak. Laplacian eigenvalues and the maximum cut prob- lem. Mathematical Programming, 62(1-3):557–574, 1993.

[24] C. Delorme and S. Poljak. The performance of an eigenvalue bound on the max- cut problem in some classes of graphs. Discrete Mathematics, 111(1-3):145–156, 1993.

[25] S. Poljak and F. Rendl. Solving the max-cut problem using eigenvalues. Dis- crete Applied Mathematics, 62(1):249–278, 1995.

[26] N. D. Sidiropoulos, T. N. Davidson, and Z.-Q. Luo. Transmit beamforming for physical-layer multicasting. IEEE Transactions on Signal Processing, 54(6):2239–2251, 2006.

[27] S. Diamond and S. Boyd. CVXPY: A Python-embedded modeling language for convex optimization. Journal of Machine Learning Research, 17(83):1–5, 2016.

[28] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statis- tical learning via the alternating direction method of multipliers. Foundations and TrendR in Machine Learning, 3(1):1–122, 2011.

[29] O. Mehanna, K. Huang, B. Gopalakrishnan, A. Konar, and N. D. Sidiropoulos. Feasible point pursuit and successive approximation of non-convex QCQPs. IEEE Signal Processing Letters, 22(7):804–808, 2015.

[30] Taati, A., Salahi, M. A conjugate gradient-based algorithm for large-scale quadratic programming problem with one quadratic constraint. Comput Optim Appl 74, 195–223 (2019). https://doi.org/10.1007/s10589-019-00105-w

[31] K. Huang and N. D. Sidiropoulos, ”Consensus-ADMM for General Quadratically Constrained Quadratic Programming,” in IEEE Transactions on Signal Processing, vol. 64, no. 20, pp. 5297-5310, 15 Oct.15, 2016, doi: 10.1109/TSP.2016.2593681.