For a general linear elastic material, strain ϵ is related to stress σ by the compliance matrix $[S]$: $\epsilon = [S]\sigma$. In orthotropic materials aligned with principal axes, normal stresses decouple from shear stresses.

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

E, ν: Young's modulus and Poisson's ratio in the transverse (x-z) plane.

E', ν': Young's modulus and Poisson's ratio along the longitudinal (y) axis.

μ: Shear modulus in the transverse (x-z) plane.

μ': Shear modulus in longitudinal planes (x-y and y-z).

We determine the matrix coefficients S_{ij} based on the definitions of the moduli and the symmetry of the material.

In the transverse plane (x and z): $S_{11} = S_{33} = \frac{1}{E}$.

Along the fiber axis (y): $S_{22} = \frac{1}{E'}$.

Strain in x due to stress in z is governed by ν. $S_{13} = -\frac{\nu}{E}$. By symmetry (x-z isotropy), $S_{31} = -\frac{\nu}{E}$.

Strain in x due to stress in y (longitudinal) is governed by ν'. $S_{12} = -\frac{\nu'}{E'}$. By symmetry of the transverse plane (z behaves like x), $S_{32} = -\frac{\nu'}{E'}$.

Shear involving the fiber axis (y) and a transverse axis (x or z) is governed by the longitudinal shear modulus μ'. $S_{44} = S_{66} = \frac{1}{\mu'}$

Shear within the isotropic transverse plane is governed by μ. $S_{55} = \frac{1}{\mu}$

Thermodynamic consistency requires the matrix to be symmetric. $S_{21} = S_{12} = -\frac{\nu'}{E'}$, $S_{23} = S_{32} = -\frac{\nu'}{E'}$.

Substituting these coefficients back into the matrix equation:

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_z - \frac{\nu'}{E'}\sigma_y$$

$$\epsilon_y = -\frac{\nu'}{E'}(\sigma_x + \sigma_z) + \frac{1}{E'}\sigma_y$$

$$\epsilon_z = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_z - \frac{\nu'}{E'}\sigma_y$$

$$\gamma_{xy} = \frac{1}{\mu'}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{\mu'}\tau_{yz}$$

The transverse shear γ_{zx} is implicit in the plane strain formulation, but constitutive theory gives:

$$\gamma_{zx} = \frac{1}{\mu}\tau_{zx}$$

We are solving for displacements u(x,z) and w(x,z). While true "plane strain" usually implies $\epsilon_y = 0$, isolates the transverse stresses σ_x and σ_z by ignoring the coupling to σ_y in the in-plane stiffness matrix calculation, or assuming σ_y contributes to the "string force" handled separately.

We invert the relations for ϵ_x and ϵ_z. Solving the system for σ_x and σ_z:

$$E\epsilon_x = \sigma_x - \nu\sigma_z$$

$$E\epsilon_z = -\nu\sigma_x + \sigma_z$$

$$E\nu\epsilon_z = -\nu^2\sigma_x + \nu\sigma_z$$

$$E\epsilon_x + E\nu\epsilon_z = \sigma_x(1 - \nu^2)$$

$$\sigma_x = \frac{E}{1 - \nu^2}(\epsilon_x + \nu\epsilon_z)$$

Using the shear modulus relation $\mu = \frac{E}{2(1+\nu)}$, we can write $E = 2\mu(1+\nu)$.

$$\sigma_x = \frac{2\mu(1+\nu)}{1-\nu^2}(\epsilon_x + \nu\epsilon_z) = \frac{2\mu(1+\nu)}{(1-\nu)(1+\nu)}(\epsilon_x + \nu\epsilon_z) = \frac{2\mu}{1-\nu}(\epsilon_x + \nu\epsilon_z)$$

$$\sigma_x = c_1\mu\epsilon_x + c_2\mu\epsilon_z$$
$$\sigma_z = c_2\mu\epsilon_x + c_1\mu\epsilon_z$$

Where $\epsilon_x = \frac{\partial u}{\partial x}$ and $\epsilon_z = \frac{\partial w}{\partial z}$.

$$c_1 = \frac{2(\alpha - \nu'^2)}{\alpha(1-\nu) - 2\nu'^2}, \quad c_2 = \frac{2(\alpha\nu - \nu'^2)}{\alpha(1-\nu) - 2\nu'^2}$$

$$\alpha = E'/E$$

$$\sigma_x = c_1\mu\frac{\partial u}{\partial x} + c_2\mu\frac{\partial w}{\partial z}$$
$$\sigma_z = c_2\mu\frac{\partial u}{\partial x} + c_1\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \mu'\frac{\partial u}{\partial y}$$

$$\tau_{yz} = \mu'\frac{\partial w}{\partial y}$$

$$\tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\pi = U_{strain} - W_{external}$$

$$U_{total} = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_z \epsilon_z + \tau_{zx}\gamma_{zx}) + \frac{1}{2}(\tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz})$$

In-Plane Energy governs the 2D vibration within the cross-section:

$$U_0 = U_{in-plane} = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_z \epsilon_z + \tau_{zx}\gamma_{zx})$$

$$U_{in-plane} = \frac{\mu}{2}[c_1 \epsilon_x^2 + c_1 \epsilon_z^2 + 2c_2 \epsilon_x \epsilon_z + \gamma_{zx}^2]$$

$$U_{in-plane} = \frac{\mu}{2}\left[c_1 u_{\partial x}^2 + c_1 w_{\partial z}^2 + 2c_2 u_{\partial x} w_{\partial z} + (u_{\partial z} + w_{\partial x})^2\right]$$

The remaining terms are:

$$U_{shear} = \frac{1}{2}(\tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz})$$

$$U_{shear} = \frac{\mu'}{2}[\gamma_{xy}^2 + \gamma_{yz}^2] = \frac{\mu'}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

This term is "dropped temporarily here due to the assumption of a uniform displacement field across the layer." Physically, within a single 2D finite element slice, the thickness h is small, so the solver assumes $\partial u \,/\, \partial y \approx 0$ for the stiffness matrix construction of that element. They are reintroduced in later (see below) as the "String Force" or "Shear Force" acting between adjacent layers, effectively coupling the 2D slices into a 3D model.

The total strain energy is the integral over the volume V.

$$U_{strain} = \iiint_V U_0 dx\, dy dz$$

The model assumes the vocal fold is composed of layers with thickness h. Within a layer, the properties and deformation are assumed uniform in the y direction. Thus, $dV = h\, dA$, where $dA = dx\, dz$.

$$U_{strain} = h \iint_A U_0 dA$$

$$U_{strain} = \frac{\mu h}{2}\iint_A \left[c_1\left(\frac{\partial u}{\partial x}\right)^2 + c_1\left(\frac{\partial w}{\partial z}\right)^2 + 2c_2 \frac{\partial u}{\partial x}\frac{\partial w}{\partial z}\right] dA + \frac{\mu h}{2}\iint_A \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right]^2 dA$$

The work term $W_{external}$ consists of body forces and surface tractions.

Using D'Alembert's principle, dynamic effects are treated as inertial body forces opposing the acceleration:

$$B_x = -\rho \frac{\partial^2 u}{\partial t^2}, \quad B_z = -\rho \frac{\partial^2 w}{\partial t^2}$$

Substituting into the work integral:

$$W_{body} = \int_V (B_x u + B_z w) dV = \int_V \left(-\rho u \frac{\partial^2 u}{\partial t^2} - \rho w \frac{\partial^2 w}{\partial t^2} \right) dV$$

$$W_{body} = -h \iint_A \rho \left(u \frac{\partial^2 u}{\partial t^2} + w \frac{\partial^2 w}{\partial t^2} \right) dA$$

Surface Forces, boundary tractions T_x and T_z act on the boundary S.

$$W_{surface} = \int_S (T_x u + T_z w) dS$$

In the potential energy functional, this remains subtracted:

$$\pi \text{ term} = -h \oint_{boundary} (T_x u + T_z w) ds$$

Combining:

$$\pi = \frac{\mu h}{2} \iint_A \left[c_1 \left(\frac{\partial u}{\partial x} \right)^2 + c_1 \left(\frac{\partial w}{\partial z} \right)^2 + 2c_2 \frac{\partial u}{\partial x} \frac{\partial w}{\partial z} \right] dA + \frac{\mu h}{2} \iint_A \left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right]^2 dA$$
$$+ h \iint_A \rho \left(u \frac{\partial^2 u}{\partial t^2} + w \frac{\partial^2 w}{\partial t^2} \right) dA - h \oint_S (T_x u + T_z w) ds$$

$$\int_V \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} + B_x - \rho\ddot{u}\right)\delta u \ dV = 0$$

We use the product rule of differentiation: $\nabla \cdot (\sigma\delta u) = (\nabla \cdot \sigma)\delta u + \sigma : \nabla(\delta u)$. Rearranging and applying the Divergence Theorem ($\int_V \nabla \cdot (\ldots) dV = \int_S (\ldots) \cdot n dS$):

$$\underbrace{\int_S (t_x \delta u + t_z \delta w) dS}_{\text{External Work}} - \underbrace{\int_V (\sigma_x \delta\epsilon_x + \sigma_z \delta\epsilon_z + \tau_{zx}\delta\gamma_{zx}) dV}_{\text{Internal Strain Energy}} + \underbrace{\int_V (B_x \delta u + \cdots) dV}_{\text{Body Force}}$$
$$- \underbrace{\int_V \rho(\ddot{u}\delta u + \ddot{w}\delta w) dV}_{\text{Inertial Work}} = 0$$

Rearranging to match the Principle of Virtual Work:

$$\int_V \delta\boldsymbol{\epsilon}^{\boldsymbol{T}}\boldsymbol{\sigma} \ dV + \int_V \rho\delta\boldsymbol{u}^{\boldsymbol{T}}\ddot{\boldsymbol{u}} \ dV = \int_S \delta\boldsymbol{u}^{\boldsymbol{T}}\boldsymbol{t} \ dS$$

We approximate the continuous field using nodes.

Nodes: i, j, m (3 nodes per element). 2 DOFs per node (u, w). Total 6 DOFs per element.

$$d = [U_1, U_2, U_3, W_1, W_2, W_3]^T$$

Displacement u is interpolated from d using shape functions $[N]$:

$$\begin{matrix} u \\ w \end{matrix} = Nd = \begin{matrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{matrix} d$$

Linear Shape Function: $N_i = \frac{1}{2A}(\alpha_i + \beta_i x + \gamma_i z)$.

Strain-Displacement Matrix

$$\begin{matrix} \epsilon_x \\ \epsilon_z \\ \gamma_{zx} \end{matrix} = \begin{matrix} u_{\partial x} \\ w_{\partial z} \\ u_{\partial z} + w_{\partial x} \end{matrix} = Bd$$

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & \beta_1 & \beta_2 & \beta_3 \end{bmatrix}$$

$$\sigma = C\epsilon$$

$$\mathbf{C}=\mu\begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\int_V \rho\delta u^T \ddot{u}\ dV = \delta d^T \left[\int_V \rho N^T N\ dV\right] \ddot{d}$$

$$M = \rho h \iint_A N^T N\ dA$$

Since N decouples u and w (see Step 3.3 A), M is block diagonal.

$$\mathbf{M}=\begin{bmatrix} M_{uu} & 0 \\ 0 & M_{ww} \end{bmatrix}$$

The sub-block terms are $M_{ij} = \rho h \int_A N_i N_j dA$.

$$\int_A N_i^a N_j^b N_k^c dA = \frac{a!\,b!\,c!\,2A}{(a+b+c+2)!}$$

$$M_{ii} = \rho h \int N_i^2 dA = \rho h \frac{2!\,0!\,0!\,2A}{(2+0+0+2)!} = \rho h \frac{2(2A)}{24} = \frac{\rho h A}{6}\ (no\ sum)$$

$$M_{ij} = \rho h \int N_i N_j dA = \rho h \frac{1!\,1!\,0!\,2A}{(1+1+0+2)!} = \rho h \frac{1(2A)}{24} = \frac{\rho h A}{12}\ (i \neq j)$$

$$\int_V \delta\epsilon^T \sigma\ dV = \int_V (B\delta d)^T (CBd)\ dV = \delta d^T \left[\int_V B^T C\, B\, dV\right] d$$

$$K = hAB^T CB$$

We calculate the 3×3 sub-block K_{uu} (Top-Left, rows 1-3, cols 1-3). This block uses the first 3 columns of B, let's call this sub-matrix B_u.

$$\mathbf{B}_u=\frac{1}{2A}\begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

Calculate (CB_u):

$$\mu\begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{1}{2A}\begin{bmatrix} \beta_j \\ 0 \\ \gamma_j \end{bmatrix}=\frac{\mu}{2A}\begin{bmatrix} c_1\beta_j \\ c_2\beta_j \\ \gamma_j \end{bmatrix}$$

Calculate $B_u^T(CB_u)$

$$\left(\frac{1}{2A}[\beta_i, 0, \gamma_i]\right) \cdot \left(\frac{\mu}{2A}\begin{bmatrix} c_1\beta_j \\ c_2\beta_j \\ \gamma_j \end{bmatrix}\right)$$

$$= \frac{\mu}{(2A)^2}\left(\beta_i\left(c_1\beta_j\right) + 0 + \gamma_i\left(\gamma_j\right)\right)$$

$$K_{uu}(i,j) = hA \cdot \frac{\mu}{4A^2}\left(c_1\beta_i\beta_j + \gamma_i\gamma_j\right) = \frac{\mu h}{4A}\left(c_1\beta_i\beta_j + \gamma_i\gamma_j\right)$$

$$K_{ww} = \frac{\mu h}{4A}\left(c_1\gamma_i\gamma_j + \beta_i\beta_j\right)$$

$$K_{uw} = \frac{\mu h}{4A}\left(c_2\beta_i\gamma_j + \gamma_i\beta_j\right)$$

$$D = 0.001K + M$$

4. Forces (Section VI)

4.1 Pressure Force

Pressure acts on the surface (edge) of the element. Work done: $W_p = \int_S P_x u \, ds$. Nodal Force $F_i = \int N_i P \, ds$. For a linear element edge of length L, the integral of a linear shape function N_i times a constant pressure is PL/2. If Pressure varies linearly from P_1 to P_2: Force at node 1: $\frac{L}{6}(2P_1 + P_2)$.

The model assumes plane strain (x-z), which normally implies infinite length in y. However, the vocal fold has finite length and u(y) varies along the length. The "String Force" approximates the longitudinal shear resistance between layers (xy and yz shears) that was dropped from the 2D stiffness matrix formulation.

Without this force, the 2D tissue slices would vibrate independently, ignoring the physical continuity of the tissue fibers running anterior-to-posterior. This force couples the slices together, restoring the 3D integrity of the vocal fold.

Arises from the variation of the longitudinal shear energy:

$$\delta U_{shear} = \int_V \mu' \left(\frac{\partial u}{\partial y} \delta \frac{\partial u}{\partial y} + \cdots \right) dV$$

Discretized, this appears as a restoring force proportional to the curvature in the y-direction:

$$F_{string} \approx \mu' A \frac{\partial^2 u}{\partial y^2} \approx \mu' A \frac{u_{next} - 2u_{curr} + u_{prev}}{\Delta y^2}$$

This acts as an external force on the RHS of the equation of motion.