Contents

1 Lecture Note 17

1.1 how to classify groups with semi-directed products

1.1.1 Warm up exercise: $|G| = 6$

Say G_{13} of size 6. Let $N \lhd G, |N| = 3$ Let H be a group, $|H| = 2, H \subset G$. Notice $H \cap N = \{e\}$. So $G = NH$.

What is $Aut < N >$? automorphism group. $N \cong C_3$. $Aut < N > \cong$ TODO

There are 2 homomorphisms $\phi : H \to Aut < N >$, $x \to id_N$, these are ϕ_1. and the other one ϕ_2

$C_3 \times_{\phi_1} C_2$, muliplication in this exmaple is $< n_1, h_1 >< n_2, h_2 >= n_1\phi_1(h_1, n_2)h_1h_2 = n_1n_2 \cdot h_1h_2$ ie $C_3 \times C_2$ or C_6

$C_3 \times_{\phi_2} C_2 \cong S_3$

1.1.2 Classify all groups $|G| = 12$

$12 = 2^2 \cdot 3$, so we have 2-Sylow subgroup of size 4 and a 3-Sylow subgroup of size 3.

1. Example $C_12 \cong C_4 \times C_3$, $C_2 \times C_2 \times C_3$, A_4 Dihearal group of with 12 elements $\to D_6$.

   What about $S_3 \times C_2$? Is it actually $S_3 \times C_2 \cong D6$?

   There are 3 non-isomorphic non-abelian group of order 12 but the missing one is called the dicyclic group of order 12.

   G of size 12 and $n_3 \equiv 1 \ mod\ 3$ and $n_3 | 4$. so either $n_3 = 1$ or 4. If $n_3 = 1$ then the 3-Sylow subgroup is normal otherwise we have $n_3 = 4$.

   If we have 4 3-Sylow subgroups, then 8 elements of G have order 3 and we also have the identity 60 this only allows 3 other elements so $n_2 = 1$. In this case the 2-Sylow subgroups is normal.

- Case 1 Let $n_2 = 1$ with 2-Sylow subgroup N and some 3-Sylow subgroup H. $N \cap H = \{e\}$ and $G = NH$. So G is a semi-direct product. (Hint: We only get 1 non-abelian up to isomorphism.)
- Case 2 $n_3 = 1$ so we have normal 3-Sylow subgroup N and some 2-Sylow subgroup H. Again $G = NH$ and G is a semi-direct product. There are 2 non-abelian examples here up to isomorphism.

Notice if $G \cong N \times H$ then G is abelian.

1.2 Part 2 of Math 4GR3: Rings and Fields

1.2.1 What is a Ring?

A set R with 2 binary operations $+$ and $\cdot$ is called a ring if

- $(R, +)$ is an abelian group. Usually write 0 for the identity
- $\cdot$ is associative $x \cdot (y \cdot z) = (x \cdot y) \cdot z \forall x, y, z \in R$

  If there is a unit, we usually write it as 1, i.e. $1 \cdot x = x \cdot 1 = x \forall x \in R$.
- Distributivity $x \cdot (y + z) = x \cdot y + x \cdot z$ (left-distributivity) $(y + z) \cdot x = y \cdot x + z \cdot x$ (right-distributivity) $\forall x, y, z \in R$.

1. Example

   - Z iwth usual $+$ and $\cdot$ is a ring
   - $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ with usual $+$ and $\cdot$ are rings. In fact, every non-zero element has a multiplicative inverse, so there are fields. (Multiplication is commutative as well).
   - If we have multiplicative inverse but not commutative TODO
   - Polynomial rings: $\mathbb{Z}\{x\}, \mathbb{R}\{x\}, \mathbb{C}\{x\}$ with usual $+$ and $\cdot$
   - Non-commutative example of a ring: $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$, matrix rings ($n \times n$ with entries in $\mathbb{R}$ or $\mathbb{C}$)
   - Naturual example of a ring without 1 $R = \{F : \mathbb{R} \to \mathbb{R}\}$, such that $\displaystyle\int_{-\infty}^{+\infty} |f| \, dx < \infty$ with usual addition of functions and usual multiplication. But natural choice for 1 would be the constant function 1 but $1 \notin \mathbb{R}$