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1.1: $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

$\because$ $A \Rightarrow$ upper triangle $\Rightarrow$ it's eigenvalues are diagonal elements.

$\begin{vmatrix} 3-\lambda & 2 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3 = 0 \quad \lambda = 3$ with multiplicity 3

for $\lambda = 3$.

rank $(A-\lambda I) = 1$, rank$((A-\lambda I)^2) = 0$.

then for: ① $(A-\lambda I) \cdot x = 0$, $\Rightarrow$ solution set: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$

② $(A-\lambda I)^2 X = 0 \Rightarrow$ solution set: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

for $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $(A-\lambda I)^2 X_1 = 0 \Rightarrow$ not a generalized eigenvector

$X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $(A-\lambda I)^2 X_2 \neq 0 \Rightarrow u^1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $u^2 = (A-\lambda I) \cdot u^1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $(A-\lambda I)^2 X_3 \neq 0 \Rightarrow u^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $u^2 = (A-\lambda I) u^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Then $P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & 3 & \frac{3}{2} \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

1.2 $\frac{dx(t)}{dt} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} x(t)$, $\quad x(0) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$x(t) = [\, x_1(t) \quad x_2(t) \quad x_3(t) \,]^T$. $\Rightarrow \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} x(t)$

$x(t) = Py(t)$ when $y(t)$ is the solution of $\dot{y} = Jy$. $y(0) = P^{-1}x_0$

From 1.1 $\Rightarrow J = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow \frac{dy(t)}{dt} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} y(t)$,

$y(t) = [\, y_1(t) \quad y_2(t) \quad y_3(t) \,]$

$\Downarrow$

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}.$$

After inspection. $\Lambda_1 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad \Lambda_2 = [\,3\,]$.

$\Downarrow$

$y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \end{bmatrix}$ $\quad y^{(1)} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\quad y^{(2)} = [\, y_3 \,]$.

$\Downarrow$

$$\begin{cases} \dot{y}_1(t) = 3y_1(t) + y_2(t) \\ \dot{y}_2(t) = 3y_2(t) \end{cases} \text{(A)}$$

$\dot{y}_3(t) = 3y_3(t)$ (B)

with $y(0) = P^{-1}x_0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ -1 \end{bmatrix}$

$(A):\ \begin{cases} y_1(t) = (\frac{1}{2}+\frac{3}{2}t)\, e^{3t} \\ y_2(t) = \frac{3}{2}e^{3t} \end{cases}$

$(B):\ y_3(t) = -e^{3t}.$

$$X(t) = Py(t) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{3}{2}te^{3t} \\ \frac{3}{2}e^{3t} \\ -e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{3t} + 3te^{3t} \\ 2e^{3t} \\ -e^{3t} \end{bmatrix} = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}$$

$\therefore\ X_1(t) = (1+3t)e^{3t}$

$X_2(t) = 2e^{3t}$

$X_3(t) = -e^{3t}$

2、

For $n \times n$ matrix A, all n Gerschgorin circles of A are symmetric about the real axis. And since they are disjoint,

$\Downarrow$

A have n eigenvalues and all of them are distince.

And each Gerschgorin circle has one.

From the theorem $\Rightarrow$

If a union of k of these discs forms a connected region disjoint from the remaining $n-k$ discs, then there are exactly k eigenvalues in the region.

we may get the result because:

if we have one eigenvalue which is not real, then it must be a complex conjugate pair. and they are symmetric about the real axis in complex plane. Which means if their is a complex eigenvalue A on one Gersch $\cdot$rin circles there must be another complex eigenvalues B which is conjugate of A But it's against the theorem we mentioned before.

So every eigenvalue of A is real when A is real.

3.

Theorem:

For any matrix norm $\|\cdot\|$, $\rho(A) \le \|A\|$, with A is a square matrix.

Since $A = \begin{pmatrix} 1 & 1 \\ -1.5 & 2 \end{pmatrix} \Rightarrow A$ is square matrix

$D = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ $P_1 > 0,\ P_2 > 0. \Rightarrow |D| > 0$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Downarrow$

D is invertible matrix

Then we have

$$C = D^{-1}AD = \begin{pmatrix} \frac{1}{P_1} & 0 \\ 0 & \frac{1}{P_2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{3}{2} & 2 \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{P_1} & \frac{1}{P_1} \\ -\frac{3}{2P_2} & \frac{2}{P_2} \end{pmatrix} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{P_2}{P_1} \\ -\frac{3P_1}{2P_2} & 2 \end{pmatrix} \Rightarrow C \text{ is square matrix}$$

And for C is similar to A, means

$\Downarrow$

$D^{-1}AD$ and A have same eigenvalue.

And For C, $\rho(C) \leq \|C\|_\infty \rightarrow \rho(D^{-1}AD) \leq \|D^{-1}AD\|_\infty$

$\rho(D^{-1}AD) = \rho(A)$

$\rho(A) \leq \min \|D^{-1}AD\|_\infty$