Volatility-of-Volatility Risk

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Abstract

We show that market volatility of volatility is a significant risk factor which affects index and VIX option returns, beyond volatility itself. Volatility and volatility-of-volatility, identified model-free as the VIX and VVIX indices, respectively, are only weakly related to each other. Delta-hedged index and VIX option returns are negative on average, and are more negative for strategies which are more exposed to volatility and volatility-of-volatility risks. Further, volatility and volatility of volatility significantly negatively predict future delta-hedged option payoffs. The evidence suggests that volatility and volatility-of-volatility risks are jointly priced and have negative market prices of risk.

I. Introduction

Recent studies show that volatility risks significantly affect asset prices and the macroeconomy.[1] In the data, asset market volatility is often captured by the volatility index (VIX). Calculated in real time from the cross section of S&P 500 option prices, the VIX index provides a risk-neutral forecast of the index volatility over the next 30 days. The VIX index exhibits substantial fluctuations, which in the data and in many economic models drive the movements in asset prices and the risk premia. Interestingly, the volatility of the VIX index itself varies over time. Computed from VIX options in an analogous way to the VIX, the volatility-of-volatility index (VVIX) directly measures the risk-neutral expectations of the volatility of volatility in the financial markets. In the data, we find that the VVIX has separate dynamics from the VIX, so that fluctuations in volatility of volatility are not directly tied to movements in market volatility. We further show that volatility-of-volatility risks are a significant risk factor which affects the time series and the cross section of index and VIX option returns, above and beyond volatility risks. The evidence is consistent with a no-arbitrage model which features time-varying market volatility and volatility-of-volatility factors which are priced by the investors. In particular, volatility and volatility of volatility have negative market prices of risk, so that investors

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[1]For example, Bansal and Yaron (2004), Bloom (2009), Bansal, Kiku, Shaliastovich, and Yaron (2014), and Fernandez-Villaverde and Rubio-Ramírez (2013) discuss the implications of macroeconomic volatility risks, while Coval and Shumway (2001), Bakshi and Kapadia (2003), and Campbell, Giglio, Polk, and Turley (2018) consider market volatility risks.

dislike increases in volatility and volatility of volatility, and demand a risk compensation for the exposure to these risks.

Our no-arbitrage model extends the 1-factor stochastic volatility specification of equity returns in Bakshi and Kapadia (2003). Specifically, we introduce a separate time-varying volatility-of-volatility risk factor which drives the conditional variance of the variance of market returns.[2] We use the model to characterize the payoffs to delta-hedged equity and VIX options. The zero-cost, delta-hedged positions represent the gains on a long position in the option, continuously hedged by an offsetting short position in the underlying asset. As argued in Bakshi and Kapadia (2003), delta-hedged option payoffs most cleanly isolate the exposures to volatility risks, and thus are very useful to study the pricing of volatility-related risks.[3] Indeed, we show that the expected payoff on the delta-hedged index option positions capture risk compensations for both volatility and volatility-of-volatility risks, while for VIX options, the expected gains primarily reflect the compensation for volatility-of-volatility risk. Imposing a standard linear risk premium assumption, we can further decompose the risk premia components of the expected gains on the index options into the product of the market price of risk, the risk exposure, and the time-varying quantity of each source of risk. For VIX options, the expected gains do not

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[2]We use the terms "variance" and "volatility" interchangeably unless specified otherwise.

[3]For example, unlike delta-hedged positions, zero-beta straddles analyzed in Coval and Shumway (2001) are not dynamically rebalanced and may contain a significant time-decay option premium component.

4

generally admit a linear factor structure, so we resort to numerical calibration techniques to support our theoretical arguments.

The model delivers clear, testable predictions for expected option returns and their relation to volatility and volatility-of-volatility risks. In the model, if investors dislike volatility and volatility of volatility, so that the market prices of these risks are negative, delta-hedged equity and VIX option gains are negative on average. In the cross section, the average returns are more negative for option strategies which have higher exposure to the volatility and volatility-of-volatility risks. Finally, in the time series, higher volatility and volatility of volatility predict more negative delta-hedged option gains in the future.

We find that the model implications for volatility and volatility-of-volatility risks are strongly supported in the cross section of index and VIX option returns. First, we use the VIX and VVIX indices to validate our volatility measures, and compare and contrast their dynamics. In our monthly 2006–2016 sample, the VVIX behaves quite differently from the VIX, consistent with a setup of our model which separates market volatility from volatility of volatility. Using predictive regressions, we show that the VIX is a significant predictor of the future realized variance of market returns, while the VVIX significantly forecasts future realized variation in the VIX index itself.

On average, the risk-neutral volatilities of the market return and market volatility captured by the VIX and VVIX exceed the realized volatilities of returns and the VIX. The difference between the risk-neutral and physical volatilities of market returns is known as the variance premium (variance-of-variance premium for the VIX), and the findings of

5

positive variance and variance-of-variance premium suggest that investors dislike variance and variance-of-variance risks, and demand a premium for being exposed to these risks.

We next turn to the asset-price evidence from the equity index and VIX option markets. In line with our model, we consider discrete-time counterparts to the continuously-rebalanced delta-hedged gains; this approach is similar to Bakshi and Kapadia (2003)[4]. Consistent with the evidence in previous studies, the average delta-hedged returns on out-of-the-money equity index calls and puts are significantly negative in our sample. The novel evidence in our paper is that the average delta-hedged returns on VIX options are also negative at all strikes and statistically significant for all the contracts except for the out-of-the-money puts and in-the-money calls. Estimates of the loss for call options range from -0.25% of the index value for in-the-money VIX calls to -1.20% for out-of-the-money calls. The negative average returns on index and VIX options directly suggest that the market prices of volatility and volatility-of-volatility risks are negative.

We then show that the cross-sectional spreads in average option returns are significantly related to the volatility and volatility-of-volatility risks. In lieu of calculating exact model betas, we compute option exposures using the Black and Scholes (1973) vega and volga. The two quantities vary intuitively with the moneyness of the option in the cross section, and help us proxy for the betas of the options to the underlying risks. Empirically, we document that average option returns are significantly and negatively related to our proxies for volatility and volatility-of-volatility risks. Hence, using the cross

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[4]We consider several checks to show the robustness of our results to the measurements of delta.

section of equity index options and VIX options, we find strong evidence for a negative market price of volatility and volatility-of-volatility risks.

Finally, we consider a predictive role of our volatility measures for future option returns. In the model, expected delta-hedged gains are time-varying and are driven by the volatility and volatility of volatility (by volatility of volatility for VIX options). In particular, as option betas are all positive, when the market prices of volatility-related risks are negative, both volatility measures should forecast future returns with a negative sign. This model prediction is consistent with the data.

Related Literature. Our paper is most closely related to Bakshi and Kapadia (2003) who consider the implications of volatility risk for equity index option markets. We extend their approach to include volatility-of-volatility risk, and bring evidence from VIX options. To help us focus on the volatility-related risks, we consider dynamic delta-hedging strategies where a long position in an option is dynamically hedged by taking an offsetting position in the underlying. Delta-hedged strategies are also used in Bertsimas, Kogan, and Lo (2000), Cao and Han (2013), and Frazzini and Pedersen (2012), and are a standard risk management technique of option traders in the financial industry (Hull (2011)). In an earlier study, Coval and Shumway (2001) consider the returns on zero-beta straddles to identify volatility risk sensitive assets.

We also contribute to a growing literature which address the role of volatility-of-volatility risks in asset pricing. Zhang and Zhu (2006) and Lu and Zhu (2010) highlight the nature and importance of volatility risks by analyzing the pricing of VIX futures. Related to our findings, Park (2015) shows that the VVIX has predictive power for

7

VIX option returns. Hollstein and Prokopczuk (2017) study the cross section of stocks and find that stocks with a strongly negative return exposure to volatility of volatility have higher returns on average. This is consistent with our evidence for a negative market price of volatility-of-volatility risk. Branger, Hülsbusch, and Kraftschik (2017) analyze the information contained in the VVIX term structure. They find that it predicts S&P 500 and VIX straddle returns and is incremental to the information contained in the VIX term structure which has been analyzed by Johnson (2017). This is in line with our finding that variance dynamics are richer than that of the square-root process typically considered in the literature, and is consistent with the evidence in Christoffersen, Jacobs, and Mimouni (2010), Mencía and Sentana (2013), and Branger, Kraftschik, and Völkert (2016).

In a structural approach, Bollerslev, Tauchen, and Zhou (2009) consider a version of the Bansal and Yaron (2004) long-run risks model which features recursive utility and fluctuations in the volatility and volatility of volatility of the aggregate consumption process. They show that in equilibrium, investors require compensation for the exposure to volatility and volatility-of-volatility risks. With preference for early resolution of uncertainty, the market prices of the two risks are negative. As a result, the variance risk premium is positive on average, and can predict future equity returns. Bollerslev et al. (2009) and Drechsler and Yaron (2011) show that the calibrated version of such a model can account for the key features of equity markets and the variance premium in the data. Our empirical results are consistent with the economic intuition in these models and complement the empirical evidence in these studies.

8

Finally, it is worth noting that in our paper we abstract from jumps in equity returns, and focus on diffusive volatilities as the main drivers of asset prices and risk premia. For robustness, we confirm that our predictability results are robust to controlling for jump risk measures such as the slope of the implied volatility curve, realized jump intensity (Barndorff-Nielsen and Shephard (2006), Wright and Zhou (2009)), and risk-neutral skewness (Bakshi, Kapadia, and Madan (2003)). Hence, we argue that the VIX and VVIX have a significant impact on option returns even in the presence of stock market and volatility jumps; we leave a formal treatment of jumps for future research. Reduced-form models which highlight the role of jumps include Bates (2000), Pan (2002), and Duffie, Pan and Singleton (2000), among others. Muravyev (2016) documents that inventory risk faced by market-makers also affects option prices.

Our paper proceeds as follows: In Section II we discuss our model which links expected delta-hedged equity and volatility option gains to risk compensations for volatility and volatility-of-volatility risk. We also discuss an illustrative numerical example to show the relation between VIX option gains and the volatility-related risks. Details about this study can be found in the Supplementary Material for our paper. In Section III, we describe the construction of both the model-free implied variance measures and high-frequency realized variance measures, and summarize their dynamics in the time-series. Section IV provides the empirical evidence from option prices by empirically implementing the delta-hedged option strategies in our model. Section V presents robustness tests for alternative measures of variance, as well as robustness of the results in the presence of jump risks. Section VI concludes the paper.

9

II. Model

In this section we describe our model for stock index returns, as well as for equity and volatility option prices. Our model is an extension of Bakshi and Kapadia (2003) and features separate time-variation in the market volatility and volatility of volatility. Both volatility risks are priced, and affect the level and time-variation of the expected asset payoffs.

A. Dynamics of Equity and Equity Option Prices

Under the physical measure ($\mathbb{P}$) the stock index price S_t evolves according to:

(1)
$$\frac{dS_t}{S_t} = \mu(S_t, V_t, \eta_t)dt + \sqrt{V_t}dW_t^1,$$

$$dV_t = \theta(V_t)dt + \sqrt{\eta_t}dW_t^2,$$

$$d\eta_t = \gamma(\eta_t)dt + \phi\sqrt{\eta_t}dW_t^3,$$

where W^i ($i = 1, 2, 3$) are the Brownian motions which drive stock returns, the stock return variance, and the variance of variance, respectively. The Brownian components can be correlated (i.e., we assume $dW_t^i \, dW_t^j = \rho_{i,j} \, dt$ for all $i \neq j$). V_t is the variance of instantaneous returns, and η_t is the variance of innovations in V_t. Note that the drift of the variance V_t only depends on itself, and not on the stock price S_t or the volatility of volatility η_t. Similarly, the drift of the volatility of volatility η_t is a function of the volatility of volatility η_t.

10

Under the risk-neutral measure ($\mathbb{Q}$) the stock price S_t follows a similar process, but with the drifts adjusted by the risk compensations for the corresponding risks:

$$(2) \qquad \frac{dS_t}{S_t} = r_f dt + \sqrt{V_t}d\widetilde{W}_t^1,$$

$$dV_t = (\theta(V_t) - \lambda_t^V)dt + \sqrt{\eta_t}d\widetilde{W}_t^2,$$

$$d\eta_t = (\gamma(\eta_t) - \lambda_t^\eta)dt + \phi\sqrt{\eta_t}d\widetilde{W}_t^3.$$

In this representation, the $\widetilde{W}^i$ represent Brownian motions under the risk-neutral measure $\mathbb{Q}$. λ_t^V captures the compensation for variance risk, and λ_t^η reflects the compensation for innovations in the stochastic variance of variance. If investors dislike variance and variance-of-variance risks, the two risk compensations are negative. In this case, the variances have higher drifts under the risk-neutral than under the physical measure.

Let $C_t(K, \tau)$ denote the time t price of a call option on the stock with strike price K and time to maturity τ. Assume the risk-free rate r_f is constant. To simplify the presentation, we further abstract from dividends. While we focus our discussion on call options, the case of put options follows analogously. Given the specified dynamics of the stock price under the two probability measures, the option price is given by a twice-differentiable function C of the state variables: $C_t(K, \tau) = C(S_t, V_t, \eta_t, t)$. By Itô's Lemma,

$$(3) \qquad dC_t = \frac{\partial C}{\partial S}dS_t + \frac{\partial C}{\partial V}dV_t + \frac{\partial C}{\partial \eta}d\eta_t + b_t dt,$$

for a certain drift component b_t.

11

The discounted option price $e^{-r_f t} C_t$ is a martingale under $\mathbb{Q}$ and thus has zero drift. We use Itô's Lemma again to obtain that:

$$(4) \qquad \frac{\partial C}{\partial S} S_t r_f + \frac{\partial C}{\partial V}\left(\theta(V_t) - \lambda_t^V\right) + \frac{\partial C}{\partial \eta}\left(\gamma(\eta_t) - \lambda_t^\eta\right) + b_t - r_f C_t = 0.$$

This implies that:

$$(5) \qquad b_t = r_f\left(C_t - \frac{\partial C}{\partial S} S_t\right) - \frac{\partial C}{\partial V}(\theta(V_t) - \lambda_t^V) - \frac{\partial C}{\partial \eta}(\gamma(\eta_t) - \lambda_t^\eta).$$

Let $\Pi_{t,t+\tau}$ stand for the delta-hedged option gain for call options:

$$(6) \qquad \Pi_{t,t+\tau} \equiv C_{t+\tau} - C_t - \int_t^{t+\tau} \frac{\partial C}{\partial S} dS_u - \int_t^{t+\tau} r_f\left(C_u - \frac{\partial C}{\partial S} S_u\right) du.$$

The delta-hedged option gain represents the gain on a long position in the option, continuously hedged by an offsetting short position in the stock, with the net balance earning the risk-free rate.

Combining equations (3) and (5) and noting that $C_{t+\tau} = C_t + \int_t^{t+\tau} dC_u$, we see that the delta-hedged gain for call options is given as

$$
\begin{aligned}
(7) \quad \Pi_{t,t+\tau} &\equiv C_{t+\tau} - C_t - \int_t^{t+\tau} \frac{\partial C}{\partial S} dS_u - \int_t^{t+\tau} r_f\left(C_u - \frac{\partial C}{\partial S} S_u\right) du \\
&= \int_t^{t+\tau} \lambda_u^V \frac{\partial C}{\partial V} du + \int_t^{t+\tau} \lambda_u^\eta \frac{\partial C}{\partial \eta} du + \int_t^{t+\tau} \frac{\partial C}{\partial V} \sqrt{\eta_u} dW_u^2 + \int_t^{t+\tau} \frac{\partial C}{\partial \eta} \phi \sqrt{\eta_u} dW_u^3.
\end{aligned}
$$

12

Since the expectation of Itô integrals is 0, the expected delta-hedged equity option gains are given by

$$
(8) \qquad \mathbb{E}_t\left[\Pi_{t,t+\tau}\right] = \mathbb{E}_t\left[\int_t^{t+\tau} \lambda_u^V \frac{\partial C}{\partial V} du\right] + \mathbb{E}_t\left[\int_t^{t+\tau} \lambda_u^\eta \frac{\partial C}{\partial \eta} du\right].
$$

The expected option gains depend on the risk compensation components for the volatility and volatility-of-volatility risks (λ_t^V and λ_t^η), and the option price exposures to these two sources of risks ($\partial C/\partial V$ and $\partial C/\partial \eta$).

For tractability, following the literature let us further assume that the risk premium structure is linear:

$$
(9) \qquad \lambda_t^V = \lambda^V V_t, \qquad \lambda_t^\eta = \lambda^\eta \eta_t,
$$

where λ^V is the market price of the variance risk and λ^η is the market price of the variance-of-variance risk. We can further operationalize (8) by applying Itô-Taylor expansions (Milstein, 1995). This gives us a linear factor structure (see details in the Supplementary Material):

$$
(10) \qquad \frac{\mathbb{E}_t\left[\Pi_{t,t+\tau}\right]}{S_t} = \lambda^V \beta_t^V V_t + \lambda^\eta \beta_t^\eta \eta_t.
$$

The sensitivities to the risk factors are given by:

$$
(11) \qquad \beta_t^V = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^V > 0, \qquad \beta_t^\eta = \sum_{n=0}^{\infty} \frac{\tau^{1+n}}{(1+n)!} \Phi_{t,n}^\eta > 0,
$$

where $\Phi_{t,n}^V$ and $\Phi_{t,n}^\eta$ are positive functions which depend on the moneyness of the option and on the sensitivities $\partial C / \partial V$ and $\partial C / \partial \eta$, respectively. Hence, the expected payoff on the delta-hedged option position combines the risk compensations for the volatility and volatility-of-volatility risks. The two risk compensations are given by the product of the market price of risk, the exposure of the asset to the corresponding risk, and the quantity of risk. In particular, options are positive-beta assets with respect to both volatility and volatility-of-volatility risks. Hence, if investors dislike volatility and volatility-of-volatility risks so that their market prices of risks are negative, the expected option payoffs are negative as well.

B.   Dynamics of VIX Option Prices

The squared VIX index for a time to maturity τ is the annualized risk-neutral expectation of the quadratic variation of returns from time t to $t + \tau$, given by:

$$(12) \qquad \text{VIX}_t^2 = \frac{1}{\tau} \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{t+\tau} V_s \, ds \right].$$

Given our model assumptions, the VIX index is a function of the stock market variance: $\text{VIX}_t = \text{VIX}(V_t)$. For example, in a linear model where the variance drift $\theta(V_t)$ is linear in V_t, the squared VIX is a linear function of the stock market variance V_t.

Let F_t be the time t price of a VIX futures contract expiring at $t + \tau$. Under the assumptions of no arbitrage and continuous mark-to-market, F_t is a martingale under the

14

risk-neutral measure $\mathbb{Q}$:

$$
(13) \qquad F_t = \mathbb{E}_t^{\mathbb{Q}}\left[\text{VIX}_{t+\tau}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\text{VIX}(V_{t+\tau})\right].
$$

Under our model structure, the futures price F is a function of the market variance V_t and volatility of volatility η. Under economically plausible scenarios, the futures price is monotone in the two volatility processes.[5] Knowing F_t and η_t is sufficient for V_t, so we can re-write the economic states $\begin{bmatrix} V_t & \eta_t \end{bmatrix}$ in terms of $\begin{bmatrix} F_t & \eta_t \end{bmatrix}$.

Let C_t^* be the time t price of a VIX call option, whose underlying is a VIX forward contract. The option price is given by a twice differentiable function C^* of the state variables, so that $C_t^*(K, \tau) = C^*(F_t, \eta_t, t)$. By Itô's Lemma:

$$
(14) \qquad dC_t^* = \frac{\partial C^*}{\partial F} dF_t + \frac{\partial C^*}{\partial \eta} d\eta_t + b_t^* dt,
$$

for a drift component b_t^*.

Under the risk-neutral measure $\mathbb{Q}$, the discounted VIX option price process $e^{-r_f t} C_t^*$ is a martingale, so it must have zero drift:

$$
(15) \qquad \frac{\partial C^*}{\partial F} \mathcal{D}^{\mathbb{Q}}[F_t] + \frac{\partial C^*}{\partial \eta}(\gamma(\eta_t) - \lambda_t^\eta) - r_f C_t^* + b_t^* dt = 0,
$$

[5]See also Zhang and Zhu (2006), Lu and Zhu (2010), and Branger et al. (2016) for VIX futures pricing models.

where $\mathcal{D}^{\mathbb{Q}}[F_t]$ is the infinitesimal generator applied to F (i.e., the conditional expectation of dF_t under $\mathbb{Q}$).

This implies that

$$
\begin{aligned}
(16) \qquad b_t^* &= r_f C_t^* + \frac{\partial C^*}{\partial \eta} \lambda_t^\eta - \frac{\partial C^*}{\partial \eta} \gamma(\eta_t) - \frac{\partial C^*}{\partial F} \mathcal{D}^{\mathbb{Q}}[F_t] \\
&= r_f C_t^* + \frac{\partial C^*}{\partial \eta} \lambda_t^\eta - \frac{\partial C^*}{\partial \eta} \gamma(\eta_t),
\end{aligned}
$$

where the second line follows from the fact that F_t is a martingale under $\mathbb{Q}$, so that its drift is equal to 0.

Combining equation (16) with equation (14), we obtain the equation for the delta-hedged VIX option gain:

$$
\begin{aligned}
(17) \qquad \Pi_{t,t+\tau}^* &= C_{t+\tau}^* - C_t^* - \int_t^{t+\tau} \frac{\partial C^*}{\partial F} dF_s - \int_t^{t+\tau} r_f C_s^* ds \\
&= \int_t^{t+\tau} \frac{\partial C^*}{\partial \eta} \lambda_s^\eta ds + \int_t^{t+\tau} \frac{\partial C^*}{\partial \eta} \phi \sqrt{\eta_s} dW_s^3.
\end{aligned}
$$

The delta-hedged VIX option gain in equation (17) is the counterpart to the delta-hedged equity option gain in equation (7). The difference comes from the fact that the short stock position serving as the hedge in the case of equity options is funded at r_f, while for VIX futures the hedging position is zero cost.

Taking expectations, we can derive the corresponding expected gain on delta-hedged VIX options:

$$
(18) \qquad \frac{\mathbb{E}_t \left[\Pi^*_{t,t+\tau} \right]}{F_t} = \frac{1}{F_t} \int_t^{t+\tau} \mathbb{E}_t \left[\frac{\partial C^*_s}{\partial \eta_s} \lambda^\eta_s \right] ds.
$$

Similar to index options, the expected gain on delta-hedged VIX options depends on the market price of volatility-related risks and the option exposure to these risks. However, in contrast to index options, at least for short maturities τ, VIX options are mainly exposed to volatility-of-volatility risks. Their exposure to these risks, captured by $\partial C^*_s / \partial \eta_s$, is expected to be positive, so that the average level of the expected gains on VIX options and its dependence on the volatility-of-volatility factor is governed by the market price of the volatility risks λ^η. Unlike the expected gains for index options in (10), the expected gains for VIX options do not generally admit a linear factor structure because the option price is no longer homogeneous in the underlying asset value. We provide a numerical example below to support our arguments, and then consider the relative importance of the risks in the data.

C.   Illustrative Numerical Example

We calibrate the model to match key asset-pricing moments in the data. We use numerical methods to calculate model-implied delta-hedged VIX option gains in equation (18), and analyze how the expected VIX option gains depend on the market price

of the volatility-of-volatility risks. The Supplementary Material contains a detailed description of the calibration approach and the solution technique.

Using numerical solutions, we find that the sign of the expected option gain is pinned down by the sign of the market price of volatility-of-volatility risk. In particular, we find that the derivative $\partial C^*/\partial \eta$ of the VIX option price with respect to the volatility of volatility is positive for all levels of V and η. As a consequence, the expected gains are negative if and only if the market price of volatility-of-volatility risks λ^η is negative.

We also consider the relation between the expected gains and the level of the volatility of volatility. The option price sensitivity to volatility of volatility, $\partial C^*/\partial \eta$, is always positive. The derivative is mildly decreasing in volatility of volatility, in absolute value. In total, when the market price of volatility-of-volatility risks is negative, the expected gains are more negative at times of high volatility of volatility. In this case, volatility of volatility predicts future VIX option gains with a negative sign.

III.   Variance Measures

A.   Construction of Variance Measures

The VIX index is a model-free, forward-looking measure of implied volatility in the U.S. stock market, published by the Chicago Board Options Exchange (CBOE). The square of the VIX index is defined as in equation (12) where $\tau = 30/365$. Carr and Madan (1998), Britten-Jones and Neuberger (2000), and Jiang and Tian (2005) show that VIX^2

18

can be computed from the prices of call and put options with the same maturity at different strike prices:

$$(19) \qquad \text{VIX}_t^2 = \frac{2e^{r_f\tau}}{\tau} \left[\int_0^{S_t^*} \frac{1}{K^2} P_t(K) dK + \int_{S_t^*}^{\infty} \frac{1}{K^2} C_t(K) dK \right],$$

where K is the strike price, C_t and P_t are the put and call prices, S_t^* is the fair forward price of the S&P 500 index, and r_f is the risk-free rate. The VIX index published by the CBOE is discretized, truncated, and interpolated across the two nearest maturities to achieve a constant 30-day maturity.[6] Jiang and Tian (2005) show through simulation analysis that the approximations used in the VIX index calculation are quite accurate.

Since Feb. 2006, options on the VIX have been trading on the CBOE, which give investors a way to trade the volatility of volatility. As of 2012:Q3, the open interest in front-month VIX options was about 2.5 million contracts, which is similar to the open interest in front-month S&P 500 index option contracts.

We calculate our measure of the implied volatility of volatility using the same method as the VIX, applied to VIX options instead of S&P 500 options. The index, which has since been published by the CBOE as the "VVIX index" in 2012 and back-filled, is calculated as:

$$(20) \qquad \text{VVIX}_t^2 = \frac{2e^{r_f\tau}}{\tau} \left[\int_0^{F_t} \frac{1}{K^2} P_t^*(K) dK + \int_{F_t}^{\infty} \frac{1}{K^2} C_t^*(K) dK \right],$$

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[6]More details on the exact implementation of the VIX can be found in the white paper available on the CBOE Web site: http://www.cboe.com/micro/vix

where F_t is the VIX futures price, and C_t^*, P_t^* are the prices of call and put options on the VIX, respectively.[7] The squared VVIX is calculated from a portfolio of out-of-the-money call and put options on VIX futures contracts. It captures the implied volatility of VIX futures returns over the next 30 days, and is a model-free, forward-looking measure of the implied volatility of volatility.

In addition to the implied volatilities, we can also compute the realized volatilities for the stock market and the VIX. The construction here follows Barndorff-Nielsen and Shephard (2004) using high-frequency, intraday data.[8] Realized variance is defined as the sum of squared high-frequency log returns over the trading day:

$$(21) \qquad \mathrm{RV}_t = \sum_{j=1}^{N} r_{t,j}^2.$$

Barndorff-Nielsen and Shephard (2004) show that RV_t converges to the quadratic variation as $N \to \infty$. We follow the standard approach of considering 5-minute return intervals. A finer sampling frequency results in better asymptotic properties of the realized variance estimator, but also introduces more market microstructure noise such as the bid-ask bounce discussed in Heston, Korajczyk, and Sadka (2010). Liu, Patton, and

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[7]The official index is back-filled until 2007. We apply the same methodology and construct the index for an additional year back to 2006. The correlation between our measure of the VVIX and the published index is over 99% in the post-2007 sample. Our empirical results remain essentially unchanged if we restrict our sample to only the post-2007 period.

[8]The data are obtained from http://www.tickdata.com.

Sheppard (2015) show that the 5-minute realized variance is very accurate, difficult to beat in practice, and is typically the ideal sampling choice in most applications combining accuracy and parsimony.

We estimate two realized variance measures, one for the S&P 500 and one for the VIX. For the S&P 500, we use the S&P 500 futures contract and the resulting realized variance will be denoted RV^{SPX}. For the VIX, we use the spot VIX index and denote the resulting realized variance of the VIX by RV^{VIX}, which is our measure of the physical volatility of volatility. For robustness, we also entertain an alternative measure of the realized variance RV^{VX^*} which is computed using the 5-minute VIX futures returns. The sample for this measure is shorter, and starts in July 2012.

B.   Variance Dynamics

All of our variables are at the monthly frequency. The implied variance measures are given by the index values at the end of the month, and the realized variance measures are calculated over the past month and annualized. The sample for the benchmark measures runs from Feb. 2006 to Dec. 2016.

Table 1 presents summary statistics for the implied and realized variance measures. While the average level of the VIX is about 22%, the average level of the VVIX is much higher at about 88%, which captures the fact that VIX futures returns are much more volatile than market returns: volatility, itself, is very volatile. $VVIX^2$ is also much more volatile and less persistent than VIX^2, with a first-order autocorrelation coefficient of 0.33

compared to 0.81 for VIX[2]. The VVIX exhibits relatively low correlation with the VIX, with a correlation coefficient of about 0.27. The mean of realized variance for S&P 500 futures returns is 0.024, which corresponds to an annualized volatility of 15.5%. S&P 500 realized variance is persistent and quite strongly correlated to the VIX index (correlation coefficient = 0.88) and much more weakly correlated to the VVIX index (correlation coefficient ≈ 0.26). The realized variance of VIX is strongly related to the VVIX index (correlation = 0.61), and to a much lesser extent, the VIX index (correlation = 0.18).

In the last row of Table 1 we also consider the realized variance of VIX futures returns, since the VIX futures contract (not the spot VIX) is the underying asset for VIX options and there is no simple cost-of-carry relationship between VIX futures and spot VIX. Compared to spot VIX realized variance, VIX futures realized variance is lower on average, less volatile, more persistent, and more correlated with VIX. Recall, however, that our data for VIX futures is from 2012:M7 to 2016:M12 because of the availability of tick data.

Figure 1 shows the time series of the VIX and VVIX from Feb. 2006 to Dec. 2016. There are some common prominent moves in both series, such as the a peak during the financial crisis. Notably, however, the VVIX also peaks during other times of economic uncertainty, such as the summer of 2007 (quant meltdown, beginning of the subprime crisis), May 2010 (Eurozone debt crisis, flash crash), Aug. 2011 (U.S. debt ceiling crisis), and August 2015 (sell-off driven by the Chinese stock market crash). The movements in VIX during these events are far smaller than the spikes in the VVIX. The suggests that the VVIX captures important uncertainty-related risks in the aggregate market, distinct from the VIX itself.

In Figure 2 we present time series plots for both S&P 500 and VIX realized variances. As shown in Graph A, the realized and implied variances of the stock market follow a similar pattern, and S&P 500 realized variance is nearly always below the implied variance. There is a large spike in both series around the financial crisis in October 2008, at which point realized variance exceeded implied variance. The difference between the mean of VIX^2 and RV^{SPX} is typically interpreted as a variance premium, which is the difference between end-of-month model-free, forward-looking implied variance calculated from S&P 500 index options and the realized variance of S&P 500 futures returns over the past month. Unconditionally, the average level of the VIX (22%) is greater than the average level of the S&P 500 realized volatility (15.6%), so that the variance premium is positive, consistent with the evidence in Bollerslev et al. (2009) and Drechsler and Yaron (2011). This also implies a negative market price of volatility risk.

Graph B of the Figure 2 shows the time series of the realized and implied volatility of the VIX index. Generally, the implied volatility tends to increase at times of pronounced spikes in the realized volatility. The implied volatility is also high during other times of economic distress and uncertainty, such as May 2010 (Eurozone debt crisis and flash crash), and August 2011 (U.S. debt ceiling crisis). During normal times, the VVIX is above the VIX realized variance, although during times of extreme distress we see the realized variance of VIX can exceed the VVIX. The average level of the VVIX (87.8%) is greater than the average level of the VIX realized volatility (81.8%), so that the

23

volatility-of-volatility premium is also positive.[9] As we see in Table 1, VIX futures realized volatility is lower than VIX index realized volatility, so the actual magnitude of the volatility-of-volatility risk premium is even larger when compared to VIX futures. Similar to our discussion of the variance premium, this evidence suggests that investors dislike volatility-of-volatility risks, and the market price of these risks is negative.

In addition to unconditional moments, we can also analyze the conditional dependence of volatility on volatility of volatility. Specifically, we consider the predictability of future realized variances by the VIX and VVIX, in spirit of Canina and Figlewski (1993), Christensen and Prabhala (1998), and Jiang and Tian (2005) who use option implied volatilities to predict future realized volatilities. Following that approach, we consider both univariate and multivariate encompassing regressions to assess the predictability of future realized variances by the VIX and VVIX.

In our main specification, the dependent variable is the realized variance (RV) over the next month, for both the S&P 500 and the VIX. Univariate regressions test whether each implied volatility measure (the VIX or the VVIX) can forecast future realized variances; multivariate encompassing regressions compare the relative forecasting importance of the VIX and VVIX and whether one implied volatility measure subsumes

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[9]Song (2013) shows that the average level of his VVIX measure, computed using numerical integration rather than the model-free VIX construction, is lower than the average realized volatility of VIX. One of the key differences between his and our computations is the frequency of returns used in the realized variance computations. Consistent with the literature, we rely on 5-minute returns to compute the realized variances, while Song (2013) uses daily returns.

the information content of the other. The univariate regressions are restricted versions of the corresponding multivariate encompassing regression. For the S&P 500 the regression is given as

$$(22) \qquad \mathrm{RV}_{t+1}^{\mathrm{SPX}} = \beta_0 + \beta_1 \mathrm{VIX}_t^2 + \beta_2 \mathrm{VVIX}_t^2 + \beta_3 \mathrm{RV}_t^{\mathrm{SPX}} + \epsilon_{t+1}.$$

Similarly for the VIX, we have:

$$(23) \qquad \mathrm{RV}_{t+1}^{\mathrm{VIX}} = \beta_0 + \beta_1 \mathrm{VIX}_t^2 + \beta_2 \mathrm{VVIX}_t^2 + \beta_3 \mathrm{RV}_t^{\mathrm{VIX}} + \epsilon_{t+1}.$$

Our benchmark results are presented for all variables calculated in annualized variance units.

The first regression in Panel A of Table 2 shows that the VIX can forecast future realized variance of S&P 500 returns. This is consistent with the findings of Jiang and Tian (2005). The VVIX can also forecast future S&P 500 realized variance somewhat, although the statistical significance is weaker than that of the VIX and the magnitude of the regression coefficient is several times smaller. In the encompassing regression, we see that the VIX dominates the VVIX in forecasting future S&P 500 realized variance. A one standard deviation increase in VIX^2 is associated with a 0.6 standard deviation increase in the realized variance of S&P 500 returns next month. The coefficient on the VIX does not change much when we include the VVIX. Including lags of the realized variances themselves do not materially change the results. These results are consistent with our model specification.

Panel B of Table 2 shows our predictability results for VIX realized variance, which is our proxy for physical volatility of volatility. The VIX is not significantly related to future VIX realized variation; in fact, the point estimate is negative. The t-statistic is nearly 0, and the adjusted R^2 is below 0. In contrast, the VVIX is a significant predictor of future VIX realized variation. The regression coefficient for the VVIX is about 0.6 in a univariate regression, and is 0.7 in the multivariate regression. A 1-standard-deviation increase in the current value of VVIX^2 is associated with more than a 0.2-standard-deviation increase in next month realized variance of VIX. In Panel C, we use VIX futures realized return volatility as the proxy for physical volatility-of-volatility and also find that VVIX is a significant predictor of future realized variance which drives out the VIX in a multivariate regression.

The empirical evidence suggests that fluctuations in the volatility of volatility are not directly related to the level of the volatility itself. This is consistent with our 2-volatility model specification in Section II. In many reduced form and structural models, the volatility of volatility is directly linked to the level of the volatility. For example, Heston (1993) models volatility as following a Cox, Ingersoll, and Ross (1985) square-root process. In that case, the level of volatility itself should forecast future realized volatility of volatility. The evidence in the data does not support this assumption, and supports richer dynamics of the volatility process with separate movements in the volatility of volatility.

IV.   Evidence from Options

In this section we analyze the implications of equity and VIX option price dynamics for the pricing of volatility and volatility-of-volatility risks in the data. Our model suggests that the market prices of volatility and volatility-of-volatility risks determine the key properties of the cross section and time series of delta-hedged equity and VIX option gains. Specifically, if the market prices of volatility and volatility-of-volatility risks are negative, the average delta-hedged equity and VIX option gains are also negative. In the cross section, the average returns are more negative for option strategies which have higher exposure to the volatility and volatility-of-volatility risks. Finally, in the time series higher volatility and volatility of volatility predicts more negative gains in the future. We evaluate these model predictions in the data, and find a strong support that both volatility and volatility-of-volatility risks are priced in the option markets, and have negative market prices of risks.

A.   Delta-Hedged Option Gains

We consider discrete-time counterparts to the continuously-rebalanced delta-hedged gains in equations (7) and (17):

$$
(24) \qquad \Pi_{t,t+\tau} = \underbrace{C_{t+\tau} - C_t}_{\text{option gain/loss}} - \underbrace{\sum_{n=0}^{N-1} \Delta_{t_n} \left(S_{t_{n+1}} - S_{t_n} \right)}_{\text{delta hedging gain/loss}} + \underbrace{\sum_{n=0}^{N-1} r_f \left(\Delta_{t_n} S_{t_n} - C_t \right) \frac{\tau}{N}}_{\text{risk-free rate}},
$$

$$
\Pi_{t,t+\tau}^* = \underbrace{C_{t+\tau}^* - C_t^*}_{\text{option gain/loss}} - \underbrace{\sum_{n=0}^{N-1} \Delta_{t_n} \left(F_{t_{n+1}} - F_{t_n} \right)}_{\text{delta hedging gain/loss}} - \underbrace{\sum_{n=0}^{N-1} r_f C_t^* \frac{\tau}{N}}_{\text{risk-free rate}}.
$$

Δ_{t_n} indicates the option delta at time t_n, for example, for a call option, $\Delta_{t_n} = \partial C_{t_n}/\partial S_{t_n}$, and N is the number of trading days in the month. This discrete delta-hedging scheme is also used in Bakshi and Kapadia (2003) and Bertsimas et al. (2000).

At the close of each option expiration, we look at the prices of all options with nonzero open interest and nonzero trading volume. We take a long position in the option, and hedge each day using the Δ according to the Black–Scholes (1973) model, with the net investment earning the risk-free interest rate appropriately.[10] To minimize the effect of

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[10]This requires an estimate of the implied volatility of the option, which may require an option price. We use implied volatilities directly backed out from market prices of options whenever possible; if an option does not have a quoted price on any intermediate date, we fit a cubic polynomial to the implied volatility curve given by options with quoted prices, and back out the option's implied volatility. This is similar to typical option position risk management done by professional traders.

recording errors, we discard options that have implied volatilities below the 1st percentile or above the 99th percentile. All options have exactly one calendar month to maturity; S&P 500 options expire on the third Friday of every month, while VIX options expire on the Wednesday that is 30 days away from the third Friday of the following month.

Table 3 shows average index and VIX delta-hedged option gains in our sample. We separate options by call or put, and group each option into 4 bins by moneyness to obtain 8 bins for both S&P 500 and VIX options. The column Π/S gives the delta-hedged option gain scaled by the initial index level, and the column Π/C gives the delta-hedged option gain scaled by the initial option price, which can be interpreted more readily as a "return" in the traditional sense.

Panel A of Table 3 shows that on average out-of-the-money delta-hedged S&P 500 call options have significantly negative returns. Likewise, delta-hedged put options on the S&P 500 also have significantly negative returns at all levels of moneyness. To guard against potential outliers in option returns, we show that the results are similar using medians rather than average values; in percentages, delta-hedged index put options are negative about 80% of the time. This evidence is largely consistent with Bakshi and Kapadia (2003). In the model, negative average returns on delta-hedged index options imply that volatility and/or volatility-of-volatility risks have a negative market price of risk. S&P 500 option gains display mild positive serial correlation, which we will account for in our time series predictive regressions in the later sections.

Notably, the average gains for in-the-money index calls are positive, which is puzzling given the evidence for the negative average gains across all the other option

29

contracts.[11] Bakshi and Kapadia (2003) and Chen, Shu, and Zhang (2016) document a similar pattern in different samples. As argued in Bakshi and Kapadia (2003) and Bakshi, Cao, and Chen (2000), deep in-the-money index call options are quite illiquid, so that the market makers often do not update call prices in response to changes in index levels. In line with them, we also find that in-the-money calls are relatively illiquid, compared to out-of-the-money calls or especially out-of-the-money puts. The corresponding out-of-the-money puts are much more liquid and have significantly negative delta-hedged gains.

Panel B of Table 3 shows average returns for delta-hedged VIX options. The average delta-hedged VIX option returns are negative and statistically significant in all bins except for out-of-the-money puts and in-the-money calls, which are marginally significant. Call options lose more money as they become more out of the money, regardless of whether we are scaling by the index or by the option price. Estimates of the loss for call options ranges from −0.25% of the index value for in-the-money VIX calls to −1.20% of the index value for out-of-the-money VIX calls. When viewed as a percentage of the option price, delta-hedged VIX calls return about −1% per month at the money, and −20% out-of-the-money. The call option returns are negative 60% to 80% of the time. The results for VIX put options are similar. In the model, negative average returns for delta-hedged VIX options imply that investors dislike volatility-of-volatility risks, and are systematically paying a premium hedge against increases in the volatility of volatility. This suggests that the price of the

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[11]The median gains are actually 0 or below 0 for in-the-money calls, so that the distribution of gains is right-skewed.

volatility-of-volatility risk is negative. VIX option gains exhibit weak negative serial correlation, which we also account for in our time series predictive regressions in the later sections. For both the S&P 500 and VIX, the delta-hedged option gains are quite volatile.

In the next section, we provide further direct evidence by controlling for the exposures of the delta-hedged option positions to the underlying risks.

B.   Cross-Sectional Evidence

As shown in the previous section, the average delta-hedged option gains are negative for S&P 500 and VIX options. Our model further implies (see equations (10) and (18)) that options with higher sensitivity to volatility and volatility-of-volatility risks should have more negative gains. To compute the estimates of option exposures to the underlying risks, we follow the approach of Bakshi and Kapadia (2003) which relies on using the Black and Scholes (1973) model to proxy for the true option betas.

Specifically, to compute the proxy for the option beta to volatility risk, we consider the vega of the option:

(25)
$$\frac{\partial C}{\partial \sigma} = S\sqrt{\frac{\tau}{2\pi}}e^{-\frac{d_1^2}{2}} \propto e^{-\frac{d_1^2}{2}},$$

where $d_1 = \left[\ln(S/K) + \left(r_f - q + \sigma^2/2\right)\tau\right]/(\sigma\sqrt{\tau})$, q is the dividend yield, and σ is the implied volatility of the option. This approach allows us to compute proxies for the exposures of equity options to volatility risks, and of VIX options to the volatility-of-volatility risks.

31

To illustrate the relation between the moneyness of the option and the vega-measured exposure of options to volatility risks, we show the option vega as a function of option moneyness in Figure 3. Vega represents an increase in the value of the option as implied volatility increases by one percentage point. Higher volatility translates into higher future profits from delta-hedging due to the convexity effect; hence both call and put options have strictly positive vegas. Further, since the curvature of option value is the highest for at-the-money options, at-the-money options have the highest vega in the cross section, and thus the largest exposure to volatility risks. An alternative way to proxy for the option sensitivity to volatility risks is to use the "gamma" of the option, which represents the second derivative of the option price to the underlying stock price: $\Gamma = \partial^2 C / \partial S^2$. As shown in the upper graph in Figure 3, the shape of the vega and gamma functions are almost identical, hence, the implied cross-sectional dispersion in volatility betas by moneyness are very similar as well.

To capture the sensitivity of option prices to the volatility of volatility, we compute the Black and Scholes (1973) second partial derivative of the option price with respect to the volatility, which is known in "volga" for "volatility gamma." Volga is calculated as:

$$(26) \qquad \frac{\partial^2 C}{\partial \sigma^2} = S \sqrt{\frac{\tau}{2\pi}} e^{-\frac{d_1^2}{2}} \left(\frac{d_1 d_2}{\sigma} \right) = \frac{\partial C}{\partial \sigma} \left(\frac{d_1 d_2}{\sigma} \right),$$

where $d_2 = d_1 - \sigma\sqrt{\tau}$. The lower graph in Figure 3 shows the plot of volga as a function of the moneyness of the option. Volga is positive, and exhibits twin peaks with a valley around at-the-money. At-the-money options are essentially pure bets on volatility, and are

approximately linear in volatility (see Stein (1989)). Therefore, the volga is the lowest for at-the-money options. Deep-out-of-the-money options and deep-in-the-money options do not have much sensitivity to volatility of volatility either, since for the former it is a pure directional bet, and for the latter the option value is almost entirely comprised of intrinsic value. Options that are somewhat away from at-the-money are most exposed to volatility-of-volatility risks.

Table 4 shows our cross-sectional evidence from the regressions of average option returns on our proxies of options' volatility and volatility-of-volatility betas. Panel A shows univariate and multivariate regressions of delta-hedged S&P 500 option gains scaled by the index on the sensitivities of the options to volatility and volatility-of-volatility risks. Our encompassing regression for delta-hedged S&P 500 options is:

$$(27) \qquad \text{GAINS}_{t,t+\tau}^i = \frac{\Pi_{t,t+\tau}^i}{S_t} = \tilde{\lambda}_1 \text{VEGA}_t^i + \tilde{\lambda}_2 \text{VOLGA}_t^i + \gamma_t + \epsilon_{t,t+\tau}^i.$$

Since each date includes multiple options, as in Bakshi and Kapadia (2003) we allow for a date-specific component in $\Pi_{t,t+1}^i$ due to the option expirations (i.e., we include time-fixed effects represented by γ_t). Conceptually, our approach is related to Fama and MacBeth (1973) regressions. Instead of estimating risk betas in the first stage, due to the nonlinear structure of option returns, we measure the exposures from economically motivated proxies for the risk sensitivities.[12]

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[12]Song and Xiu (2016) demonstrate an alternative method of estimating risk sensitivities nonparametrically using local linear regression methods.

33

The results in Panel A of Table 4 show that both volatility and the volatility of volatility are priced in the cross section of delta-hedged S&P 500 option returns. While univariate estimates for vega are insignificantly different from 0, univariate estimates for volga are significantly negative at -0.005 with a t-statistic of -4.28. In multivariate regressions, once we control for the effect of volga, the loading on vega becomes signicantly negative as we expect from the volatility premium. This demonstrates that especially in a period where we observe large spikes in volatility, it is important to control for exposure to volatility-of-volatility to properly see the impact of exposure to volatility on delta-hedged S&P 500 option returns.

Panel B of Table 4 presents cross-sectional results for delta-hedged VIX options. As equation (17) shows, delta-hedged VIX options are mainly exposed to volatility-of-volatility risks, and the vega for VIX options captures the sensitivity of VIX options to innovations in the volatility of volatility. The coefficient on vega of -1.23 is negative and statistically significant. Thus, in the cross section of both S&P 500 options and VIX options, we find strong evidence of a negative price of volatility-of-volatility risk.

C.   Time-Series Evidence

In the model, time-variation in the expected delta-hedged option gains is driven by V and η, and the loadings are determined by the market prices of volatility and volatility-of-volatility risk. We group options into the same bins as we used for average returns in Table 3, and average the scaled gains within each bin, so that we have a time series of option returns for each moneyness bin. To examine the contribution of both risks

34

for the time-variation in expected index option payoffs, we consider the following regression:

$$(28) \qquad \text{GAINS}_{t,t+\tau}^i = \frac{\Pi_{t,t+\tau}^i}{S_t} = \beta_0 + \beta_1 \text{VIX}_t^2 + \beta_2 \text{VVIX}_t^2 + \gamma \text{GAINS}_{t-\tau}^i + u^i + \epsilon_{t+\tau}^i,$$

where we include fixed effects u^i to account for the heterogeneity in the sensitivity of options in different moneyness bins to the underlying risks. We regress the delta-hedged option gain scaled by the index from expiration to expiration on the value of the VIX and VVIX indices at the end of the earlier expiration; in other words, we run one-month ahead predictive regressions of delta-hedged option returns on the VIX and VVIX. We include lagged gains to adjust for serial correlation in the residuals, following Bakshi and Kapadia (2003).

Panel A of Table 5 shows the regression results for the index options. The univariate regression of delta-hedged S&P 500 option gains on VIX^2 yields a negative coefficient which is statistically significant, consistent with Bakshi and Kapadia (2003). VVIX^2 also predicts future S&P 500 option gains with a negative coefficient, and in a multivariate regression on both VIX and VVIX we find both loadings are negative and statistically significant.

It is important to point out that the true loadings on the volatility factors are time-varying, and generally depend on the underlying volatility and volatility-of-volatility states (see, e.g., equation (11)). As shown in Bakshi and Kapadia (2003), these loadings are insensitive to volatility for at-the-money options. However, away from the money, the loadings may depend on the variance risk factors, so that the relation between delta-hedged gains and the variance-related risk factors is nonlinear. To guard against finding a spurious

35

coefficient on the VVIX, we repeat the exercise using only at-the-money options, and find very similar results.

Panel B of Table 5 shows the corresponding evidence for VIX options. The VVIX negatively and significantly predicts future VIX option gains. This is consistent with a negative market price of volatility-of-volatility risk. The results remain very similar for only at-the-money options.

V. Robustness

A. Delta Computations

A key input into the computations of delta-hedged option returns is the delta of the option. The delta hedging insulates option returns from changes in the direction of the underlying, and allows us to focus on the volatility-related risks. The "true" deltas are not observable, and depend on the model specification. It is even more challenging to find the "true" delta for the VIX options since the VIX is a square root of a portfolio of options and the integrated volatility is not a traded asset. In our paper we have pursued a typical approach in the literature to approximate the unknown "true" deltas by the Black–Scholes implied ones (see, e.g., Bakshi and Kapadia (2003), Duarte and Jones (2007)). In addition, we consider a variety of checks to show the robustness of our key results to measurements of delta.

From the theoretical perspective, we use numerical simulations of our illustrative model in Section II to compute and compare the "true" and Black–Scholes deltas, and evaluate the adequacy of the approximation by the Black–Scholes deltas for the VIX options. In the Supplementary Material we show that in our model specification, the two variants of delta are quite similar, in particular for at-the-money options.

We also consider several robustness checks in the empirical application. First, we use empirically observed, rather than Black–Scholes implied, deltas to construct delta-hedged gains. Specifically, we regress changes in option prices on changes in the prices of the underlying assets, option characteristics, and the interactions between the two, and obtain the estimate of the empirical delta as a time-varying loading on the underlying price change. For example, for index calls the regression takes form:

$$(29) \qquad \Delta C_{t+1} = C_{t+1} - C_t = \text{CONST} + \alpha X_t + \beta_0 \Delta S_{t+1} + \beta_1 \Delta S_{t+1} X_t + \epsilon_{t+1}$$
$$= \text{CONST} + \alpha X_t + (\beta_0 + \beta_1 X_t) \Delta S_{t+1} + \epsilon_{t+1}$$

so that an empirical estimate of delta is given by $\beta_0 + \beta_1 X_t$. We run separate regressions for index and VIX calls and puts, and include in X_t option characteristics that are known to affect option sensitivity to the underlying, such as its moneyness, implied volatility, and its maturity.

Overall, we find that while empirical estimation is noisy, the main results are quite similar to the benchmark. The correlation between the empirical and the Black–Scholes deltas is about 90% across all the contracts. As shown in Table 6, the results for the

average delta-hedged gains are quite similar to the benchmark specification. Similar to the Black–Scholes delta hedging, all the gains with the exception of in-the-money calls are negative, nearly all are significant, and the magnitudes are quite similar across the two specifications.

To provide additional robustness checks, we also considered an exercise similar to Coval and Shumway (2001) by perturbing the Black–Scholes deltas used in the option hedging. We set the deltas to 0.95 and 1.05 times the value from the Black–Scholes model, respectively. Our main results are not sensitive to the choice of using Black–Scholes deltas for option hedging, and remain quite robust. Results are provided in the Supplementary Material.

B. Sensitivity to Jump Risk Measures

The evidence in our paper highlights the roles of the volatility and volatility-of-volatility factors, which are driven by smooth Brownian motion shocks. In principle, the losses on delta-hedged option portfolios can also be attributed to large, discontinuous movements (jumps) in the stock market and in the market volatility. In this section we verify that our empirical evidence for the importance of the volatility-related factors is robust to the inclusion of jump measures considered in the literature.

Specifically, we consider three measures of jump risks, which we construct for the S&P 500 and the VIX. Our first jump measure corresponds to the slope of the implied

volatility curve:

$$\text{SLOPE}^{\text{SPX}} = \sigma_{\text{OTM}}^{\text{SPX}} - \sigma_{\text{ATM}}^{\text{SPX}}, \tag{30}$$

$$\text{SLOPE}^{\text{VIX}} = \sigma_{\text{OTM}}^{\text{VIX}} - \sigma_{\text{ATM}}^{\text{VIX}}.$$

The OTM contract for the S&P 500 options is defined as a put option with a moneyness closest to 0.9, and for VIX options as a call option with a moneyness closest to 1.1. In both cases, the ATM option has moneyness of 1. These slopes are positive for both index and VIX options. Positive slope of the index volatility smile is consistent with the notion of negative jumps in market returns (see, e.g., Bates (2000), Pan (2002), and Eraker, Johannes, and Polson (2003)), while the fact that the implied volatility curve for VIX options slopes upwards is consistent with positive volatility jumps (Drechsler and Yaron (2011) and Eraker and Shaliastovich (2008), among others). In this sense, these slope measures help capture the variation in the market and volatility jumps in the economy.

Our second jump measure incorporates the whole cross section of option prices, beyond just the slope of the smile. It is based on the model-free risk-neutral skewness of Bakshi et al. (2003):

$$\text{SKEW}(t, t+\tau) = \frac{e^{r_f \tau} W_{t,t+\tau} - 3\mu_{t,t+\tau} e^{r_f \tau} V_{t,t+\tau} + 2\mu_{t,t+\tau}^3}{\left[e^{r_f \tau} V_{t,t+\tau} - \mu_{t,t+\tau}^2 \right]^{3/2}}, \tag{31}$$

where $V_{t,t+\tau}, W_{t,t+\tau}, X_{t,t+\tau}$ are given by the prices of the volatility, cubic, and quartic contracts, and $\mu_{t,t+\tau}$ is function of them. Importantly, these measures are computed

39

model-free using only observed option prices. The details for the computations are provided in the Supplementary Material.

Finally, our third measure of jump risks is based on high-frequency index and VIX data, rather than option prices. It corresponds to the realized jump intensity, and relies on the bipower variation methods in Barndorff-Nielsen and Shephard (2004), Huang and Tauchen (2005), and Wright and Zhou (2009). Specifically, while the realized variance defined in (21) captures both the continuous and jump variation, the bipower variation, defined as

$$
\text{(32)} \qquad \text{BV}_t = \frac{\pi}{2}\left(\frac{M}{M-1}\right)\sum_{j=2}^{M}|r_{t,j-1}||r_{t,j}|
$$

measures the amount of continuous variation returns. Hence, we can use the following test statistic to determine if there is a jump on any given day:

$$
\text{(33)} \qquad J_t = \frac{\dfrac{\text{RV}_t - \text{BV}_t}{\text{RV}_t}}{\sqrt{\dfrac{\theta}{M}\max\left(1,\dfrac{\text{QV}_t}{\text{BV}_t^2}\right)}},
$$

where $\theta = (\pi/2)^2 + \pi - 5$, and QV_t is the quad-power quarticity defined in Huang and Tauchen (2005) and Barndorff-Nielsen and Shephard (2004). The test statistic is standard normally distributed. We flag the day as having a jump if the probability exceeds 99.9% both for index returns and for the VIX. These cut-offs imply an average frequency of jumps of once every two months for the index, and about three jumps a month for the VIX. This is broadly consistent with the findings of Tauchen and Todorov (2011), who find that VIX

jumps tend to happen much more frequently than S&P 500 jumps. Over a month, we sum up all the days where we have a jump, and we define our jump intensity measure on a monthly level as:

$$(34) \qquad \mathrm{RJ}_t = \frac{1}{T} \sum_{i=0}^{T-1} J_{t+i},$$

where T is the number of trading days in the given month.

We use the jump statistics to document the robustness of the link between the volatility and volatility-of-volatility factors and options gains. We consider a regression of the form

$$(35) \qquad \mathrm{GAINS}_{t,t+\tau}^{i} = \beta_0 + \beta_1 \mathrm{VIX}_t^2 + \beta_2 \mathrm{VVIX}_t^2 + \beta_3 \mathrm{JUMP}_t + \gamma \mathrm{GAINS}_{t-\tau}^{i} + u^i + \epsilon_{t+\tau}^{i},$$

where JUMP_t is one of the above jump risk proxies. We use index jump measures for index gains, and VIX jump measures for VIX gains.

Table 7 displays our results. Both for S&P 500 and VIX options, controlling for SLOPE does not change the ability of VIX and VVIX to predict future delta-hedged option gains. Both factors are still significant, and the point estimates $\hat{\beta}_1$ and $\hat{\beta}_2$ remain largely unchanged. SLOPE itself is not significant at conventional levels for S&P 500 options but significantly positive for VIX options. The risk-neutral skewness also does not affect the predictive ability of VIX and VVIX, although, again, it is significant for VIX options. The two estimates of β_3 have the correct signs, since skewness is negative for S&P 500 options and positive for VIX options; this is broadly similar to the findings of Bakshi

41

and Kapadia (2003). Finally, we control for realized jump intensity RJ and we see a similar result where the statistical significance of VIX and VVIX as well as their point estimates are largely unchanged. RJ seems to marginally predict future S&P 500 option gains; for VIX options, however, RJ does not seem to predict future VIX option gains.

Hence, our evidence suggests that the VIX and VVIX have a significant impact on option returns even in the presence of stock market and volatility jumps. We leave a formal treatment of jumps for future research.

C.   Subsamples

To further investigate robustness of our results, we split our data into 2 subperiods of roughly equal length, a pre-2012 and a post-2012 period. Tables 8 and 9 show the results in the two subperiods. As we can see from the tables, the subperiod results are very similar to the full sample results in Table 3, and show large negative average S&P 500 and VIX option gains, consistent with investors pricing volatility and volatility-of-volatility risks with negative market prices of risk.

D.   Alternative Variance Specifications

Our results for the predictability of realized by implied variance are robust to alternative specifications of volatility. Specifically, we consider regressing in volatility units

or log-volatility units, rather than variance. The robustness regressions follow the form:

$$\sqrt{\mathrm{RV}_{t,t+1}^x} = \beta_0 + \beta_1 \mathrm{VIX}_t + \beta_2 \mathrm{VVIX}_t + \epsilon_{t+1},$$

(36)

$$\ln \sqrt{\mathrm{RV}_{t,t+1}^x} = \beta_0 + \beta_1 \ln \mathrm{VIX}_t + \beta_2 \ln \mathrm{VVIX}_t + \epsilon_{t+1},$$

where x refers to SPX or VIX.

In Table 10, we see that the point estimates and significance are very close to our baseline specification in variance units.

VI.   Conclusion

Using S&P 500 and VIX options data, we show that a time-varying volatility of volatility is a separate risk factor which affects option returns, above and beyond market volatility risks. We measure volatility risks using the VIX index, and volatility-of-volatility risk using the VVIX index. The VIX and VVIX have separate dynamics, and are only weakly related in the data: the correlation between the two series is under 30%. On average, risk-neutral volatilities identified by the VIX and VVIX exceed the realized physical volatilities of the corresponding variables in the data. Hence, the variance premium and variance-of-variance premium for VIX are positive, which suggests that investors dislike variance and variance-of-variance risks.

We show the pricing implications of volatility and volatility-of-volatility risks using options market data. Average delta-hedged option gains are negative, which suggests that

investors pay a premium to hedge against innovations in not only volatility but also the volatility of volatility. In the cross section of both delta-hedged S&P 500 options and VIX options, options with higher sensitivities to volatility-of-volatility risk earn more negative returns. In the time-series, higher values of the VVIX predict more negative delta-hedged option returns, for both S&P 500 and VIX options.

Our findings are consistent with a no-arbitrage model which features time-varying market volatility and volatility-of-volatility factors. The volatility factors are priced by the investors, and in particular, volatility and volatility of volatility have negative market prices of risks.

References

Bakshi, G.; C. Cao; and Z. Chen. "How Often Does the Call Move in the Opposite Direction to the Underlying?" *Review of Financial Studies*, 13 (2000), 549–584.

Bakshi, G., and N. Kapadia. "Delta-Hedged Gains and the Negative Market Volatility Risk Premium." *Review of Financial Studies*, 16 (2003), 527–566.

Bakshi, G.; N. Kapadia; and D. Madan. "Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options." *Review of Financial Studies*, 16 (2003), 101–143.

Bansal, R.; D. Kiku; I. Shaliastovich; and A. Yaron. "Volatility, the Macroeconomy, and Asset Prices." *Journal of Finance*, 69 (2014), 2471–2511.

Bansal, R., and A. Yaron. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles." *Journal of Finance*, 54 (2004), 1481–1509.

Barndorff-Nielsen, O., and N. Shephard. "Power and Bipower Variation with Stochastic Volatility and Jumps." *Journal of Financial Econometrics*, 2 (2004), 1–37.

Barndorff-Nielsen, O., and N. Shephard. "Econometrics of Testing for Jumps in Financial Economics Using Bipower Variation." *Journal of Financial Econometrics*, 4 (2006), 1–30.

Bates, D. "Post-'87 Crash Fears in the S&P 500 Futures Option Market." *Journal of Econometrics*, 94 (2000), 181–238.

Bertsimas, D.; L. Kogan; and A. Lo. "When is Time Continuous?" *Journal of Financial Economics*, 55 (2000), 173–204.

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81 (1973), 637–654.

Bloom, N. "The Impact of Uncertainty Shocks." *Econometrica*, 77 (2009), 623–685.

Bollerslev, T.; G. Tauchen; and H. Zhou. "Expected Stock Returns and Variance Risk Premia." *Review of Financial Studies*, 22 (2009), 4463–4492.

Branger, N.; H. Hülsbusch; and A. Kraftschik. "The Volatility-of-Volatility Term Structure." Working Paper, University of Münster (2017).

Branger, N.; A. Kraftschik; and C. Völkert. "The Fine Structure of Variance: Pricing VIX Derivatives in Consistent and Log-VIX Models." Working Paper, University of Münster (2016).

Britten-Jones, M., and A. Neuberger. "Option Prices, Implied Price Processes, and Stochastic Volatility." *Journal of Finance*, 51 (2000), 621–651.

Campbell, J. Y.; S. Giglio; C. Polk; and R. Turley. "An intertemporal CAPM with stochastic volatility." *Journal of Financial Economics*, 128 (2018), 207–233.

Canina, L., and S. Figlewski. "The Informational Content of Implied Volatility." *Review of Financial Studies*, 6 (1993), 659–681.

Cao, J., and B. Han. "Cross-Section of Option Returns and Idiosyncratic Stock Volatility." *Journal of Financial Economics*, 20 (2013), 1–10.

Carr, P., and D. Madan. "Towards a Theory of Volatility Trading." In *Advances in Mathematical Finance,* J. Cvitanic, E. Jouini, and M. Musiela, eds. Cambridge, UK: University Press (1998).

Chen, Y.; J. Shu; and J. E. Zhang. "Investor Sentiment, Variance Risk Premium and Delta-Hedged Gains." *Applied Economics*, 48 (2016), 2952–2964.

Christensen, B., and N. Prabhala. "The Relation Between Implied and Realized Volatility." *Journal of Financial Economics*, 50 (1998), 125–150.

Christoffersen, P.; K. Jacobs; and K. Mimouni. "Volatility Dynamics for the S&P500: Evidence from Realized Volatility, Daily Returns, and Option Prices." *Review of Financial Studies*, 23 (2010), 3141–3189.

Coval, J., and T. Shumway. "Expected Option Returns." *Journal of Finance*, 56 (2001), 983–1009.

Cox, J.; J. Ingersoll; and S. Ross. "A Theory of the Term Structure of Interest Rates." *Econometrica*, 53 (1985), 385–407.

Drechsler, I., and A. Yaron. "What's vol got to do with it." *Review of Financial Studies*, (2011), 1–45.

Duarte, J., and C. Jones. "The Price of Market Volatility Risk." Working Paper, Rice University (2007).

Duffie, D.; J. Pan; and K. Singleton. "Transform Analysis and Asset Pricing for Affine Jump-Diffusions." *Econometrica*, 68 (2000), 1343–1376.

47

Eraker, B.; M. Johannes; and N. Polson. "The Impact of Jumps in Returns and Volatility." *Journal of Finance*, 53 (2003), 1269–1300.

Eraker, B., and I. Shaliastovich. "An Equilibrium Guide to Designing Affine Pricing Models." *Mathematical Finance*, 18 (2008), 519–543.

Fama, E., and J. MacBeth. "Risk, Return, and Equilibrium: Empirical Tests." *Journal of Political Economy*, 81 (1973), 607–636.

Fernandez-Villaverde, J., and J. F. Rubio-Ramírez. "Macroeconomics and volatility: Data, models, and estimation." In *Advances in Economics and Econometrics: Theory and Applications,* D. Acemoglu, M. Arellano, and E. Dekel, eds. Cambridge, UK: University Press (2013).

Frazzini, A., and L. Pedersen. "Embedded Leverage." Working Paper, New York University (2012).

Heston, S. "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bonds and Currency Options." *Review of Financial Studies*, 6 (1993), 327–343.

Heston, S.; R. Korajczyk; and R. Sadka. "Intraday Patterns in the Cross-Section of Stock Returns." *Journal of Finance*, 65 (2010), 1369–1407.

Hollstein, F., and M. Prokopczuk. "How Aggregate Volatility-of-Volatility Affects Stock Returns." *Review of Asset Pricing Studies*, forthcoming (2017).

Huang, X., and G. Tauchen. "The Relative Contribution of Jumps to Total Price Variation." *Journal of Financial Econometrics*, 3 (2005), 456–499.

Hull, J. *Options, Futures, and Other Derivatives*, Prentice Hall, 8th ed. (2011).

Jiang, G., and Y. Tian. "The Model-Free Implied Volatility and its Information Content." *Review of Financial Studies*, 18 (2005), 1305–1342.

Johnson, T. "Risk Premia and the VIX Term Structure." *Journal of Financial and Quantitative Analysis*, 52 (2017), 2461–2490.

Liu, L.; A. Patton; and K. Sheppard. "Does Anything Beat 5-Minute RV? A Comparison of Realized Measures across Multiple Asset Classes." *Journal of Econometrics*, 187 (2015), 293–311.

Lu, Z., and Y. Zhu. "Volatility Components: The Term Structure Dynamics of VIX Futures." *Journal of Futures Markets*, 30 (2010), 230–256.

Mencía, J., and E. Sentana. "Valuation of VIX Derivatives." *Journal of Financial Economics*, 108 (2013), 367–391.

Milstein, G. *Numerical Integration of Stochastic Differential Equations*. Boston, MA: Kluwer Academic (1995).

Muravyev, D. "Order Flow and Expected Option Returns." *Journal of Finance*, 71 (2016), 673–708.

Pan, J. "The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study." *Journal of Financial Economics*, 63 (2002), 3–50.

Park, Y.-H. "Volatility-of-Volatility and Tail Risk Hedging Returns." *Journal of Financial Markets*, 26 (2015), 38–63.

Song, Z. "Expected VIX Option Returns." Working Paper, Johns Hopkins University (2013).

Song, Z., and D. Xiu. "A Tale of Two Option Markets: Pricing Kernels and Volatility Risk." *Journal of Econometrics*, 190 (2016), 176–196.

Stein, J. "Overreactions in the Options Market." *Journal of Finance*, 44 (1989), 1011–1023.

Tauchen, G., and V. Todorov. "Volatility Jumps." *Journal of Business and Economic Statistics*, 29 (2011), 235–371.

Wright, J., and H. Zhou. "Bond Risk Premia and Realized Jump Risk." *Journal of Banking and Finance*, 33 (2009), 2036–2049.

Zhang, J., and Y. Zhu. "VIX Futures." *Journal of Futures Markets*, 26 (2006), 521–531.

Table 1: **Summary Statistics**

Table 1 shows summary statistics for the implied and realized variances of S&P 500 and the VIX index. The implied variances are computed from the option data: VIX^2 corresponds to $(\text{VIX}/100)^2$ and VVIX^2 stands for $(\text{VVIX}/100)^2$. Realized variances are annualized and computed using 5-minute data on S&P 500 returns (RV^{SPX}); log VIX index innovations (RV^{VIX}); and VIX futures returns (RV^{VX^*}). The data are monthly from 2006:M2 to 2016:M12; VIX futures high-frequency data are from 2012:M7 to 2016:M12.

Variable	Mean	Std.	AR(1)	Corr. VIX^2	Corr. VVIX^2
VIX^2	0.048	0.053	0.812	1.000	0.269
VVIX^2	0.771	0.214	0.333	0.269	1.000
RV^{SPX}	0.024	0.050	0.631	0.880	0.258
RV^{VIX}	0.670	0.727	0.182	0.182	0.609
RV^{VX^*}	0.306	0.181	0.284	0.726	0.640

Table 2: **Predictability of Realized Variance Measures**

Table 2 shows the evidence from the projections of future realized variances of S&P 500 and the VIX index on the current implied variances. Numbers in square brackets indicate Newey–West t-statistics with 6 lags. The data are monthly from 2006:M2 to 2016:M12; VIX futures high-frequency data are from 2012:M7 to 2016:M12.

| | VIX^2 | | $VVIX^2$ | | |
	Slope	t-Stat.	Slope	t-Stat.	R^2_{adj}
Panel A. S&P 500 Index					
$RV^{SPX}_{t,t+1}$	0.604	[4.82]			39.42
			0.039	[1.48]	2.01
	0.605	[4.53]	−0.001	[−0.10]	38.95
Panel B. VIX Index					
$RV^{VIX}_{t,t+1}$	−0.607	[−0.57]			-0.35
			0.612	[3.56]	2.49
	−1.384	[−1.34]	0.704	[4.05]	2.67
Panel C. VIX Futures					
$RV^{VX^*}_{t,t+1}$	3.215	[3.07]			2.55
			0.243	[2.24]	7.19
	−0.935	[−0.46]	0.282	[2.34]	5.49

Table 3: **Delta-Hedged Option Gains**

Table 3 shows average delta-hedged option gains on the S&P 500 and VIX options across
their moneyness. Options have 1 month to maturity, are grouped into an equal-weighted
portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is
computed using the Black–Scholes formula, with daily rebalancing and the margin
difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by
the index or by the option price. The t-statistics are testing the null that the delta-hedged
option gain is equal to 0 and are shown in square brackets. The $\% < 0$ column shows the
fraction of observations with negative gains. The data are monthly from 2006:M2 to
2016:M12.

| Moneyness | $\Pi/S(\%)$ | | | | | | $\Pi/C(\%)$ | | |
	Mean	t-Stat.	Median	$\% < 0$	Std.	AR(1)	Mean	t-Stat.	Median
Panel A. S&P 500 Options									
Call									
0.950 to 0.975	0.07	[3.23]	0.00	50%	0.55	0.36	1.61	[3.71]	0.01
0.975 to 1.000	0.04	[2.63]	−0.02	52%	0.50	0.19	2.38	[3.72]	−0.79
1.000 to 1.025	−0.04	[−2.64]	−0.05	57%	0.44	0.09	−2.45	[−1.74]	−5.90
1.025 to 1.050	−0.08	[−6.51]	−0.05	62%	0.38	0.14	−32.98	[−9.00]	−22.50
Put									
0.950 to 0.975	−0.14	[−7.86]	−0.18	81%	0.54	0.35	−19.27	[−9.01]	−28.74
0.975 to 1.000	−0.16	[−9.89]	−0.19	75%	0.50	0.20	−10.24	[−7.93]	−16.89
1.000 to 1.025	−0.22	[−13.17]	−0.22	75%	0.48	0.04	−8.77	[−13.99]	−10.85
1.025 to 1.050	−0.25	[−12.11]	−0.23	78%	0.47	0.05	−5.84	[−12.72]	−5.97
Panel B. VIX Options									
Call									
0.800 to 0.900	−0.25	[−1.66]	−0.59	62%	2.26	−0.03	−1.38	[−1.54]	−3.12
0.900 to 1.000	−0.77	[−3.83]	−1.26	70%	2.95	−0.09	−6.76	[−3.80]	−12.40
1.000 to 1.100	−0.84	[−3.54]	−1.37	71%	3.24	−0.17	−10.03	[−3.34]	−16.29
1.100 to 1.200	−1.20	[−5.04]	−1.52	77%	3.16	−0.13	−19.01	[−4.81]	−25.49
Put									
0.800 to 0.900	−0.50	[−1.67]	−0.90	64%	2.69	−0.06	−12.76	[−1.55]	−25.18
0.900 to 1.000	−0.80	[−3.73]	−1.29	71%	3.05	−0.14	−10.91	[−3.08]	−21.59
1.000 to 1.100	−0.84	[−3.42]	−1.45	70%	3.31	−0.18	−6.13	[−3.17]	−11.02
1.100 to 1.200	−1.17	[−4.86]	−1.31	77%	3.16	−0.14	−5.53	[−4.77]	−6.45

Table 4: **Delta-Hedged Option Gains by Volatility Risk Sensitivities**

Table 4 shows the evidence from the cross-sectional regressions of the delta-hedged S&P 500 and VIX option gains on the vega and volga of the options. Regressions include time fixed effects, and option characteristics are computed using the Black–Scholes formula. Numbers is square brackets indicate Newey–West t-statistics. The data are monthly from 2006:M2 to 2016:M12.

Option Gains	Vega $= \dfrac{\partial C}{\partial \sigma}$		Volga $= \dfrac{\partial^2 C}{\partial \sigma^2}$	
	Slope	t-Stat.	Slope	t-Stat.
Panel A. S&P 500 Options				
$\Pi_{t,t+1}/S_t$	0.009	[0.61]		
			-0.005	$[-4.28]$
	-0.104	$[-4.40]$	-0.013	$[-6.21]$
Panel B. VIX Options				
$\Pi^*_{t,t+1}/F_t$	-1.23	$[-3.35]$		

54

Table 5: **Predictability of Delta-Hedged Option Gains**

Table 5 shows the evidence of predictability of future delta-hedged S&P 500 and VIX option gains by the implied volatility and volatility-of-volatility measures. The regressions across all monenyness bins include moneyness fixed effects; the regressions for the ATM options are for the moneyness ranges between 0.975 to 1.025 for the index and 0.9 to 1.1 for the VIX options. Lag gains are included to correct for serial correlation of the residuals. Numbers is square brackets indicate Newey–West t-statistics. The data are monthly from 2006:M2 to 2016:M12.

	VIX^2		$VVIX^2$		$GAINS_{t-1}$	
Option Gains	Slope	t-Stat.	Slope	t-Stat.	Slope	t-Stat.
Panel A. S&P 500 Options						
$\Pi_{t,t+\tau}/S_t$	-0.69	$[-3.42]$			0.24	$[4.08]$
			-0.26	$[-2.87]$	0.23	$[4.16]$
	-0.50	$[-3.41]$	-0.23	$[-2.75]$	0.26	$[4.20]$
Panel B. S&P 500 Options (ATM only)						
$\Pi_{t,t+\tau}/S_t$	-0.53	$[-2.81]$			0.16	$[3.43]$
			-0.25	$[-2.41]$	0.18	$[3.22]$
	-0.35	$[-2.28]$	-0.24	$[-2.29]$	0.19	$[3.38]$
Panel C. VIX Options						
$\Pi^*_{t,t+\tau}/F_t$			-1.01	$[-3.38]$	-0.10	$[-7.07]$
Panel D. VIX Options (ATM only)						
$\Pi^*_{t,t+\tau}/F_t$			-0.77	$[-4.43]$	-0.11	$[-6.89]$

Table 6: **Delta-Hedged Option Gains: Empirical Delta**

Table 6 shows average delta-hedged option gains on the S&P 500 and VIX options across their moneyness constructed using the empirical, regression-based estimates of the option deltas. The included factors that affect the sensitivity of option price to change in underlying and are given by the moneyness, implied volatility, and maturity. Delta-hedging is done with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to 0 and are shown in square brackets. The $\% < 0$ column shows the fraction of observations with negative gains. The data are monthly from 2006:M2 to 2016:M12.

Moneyness	Π/S (%)						Π/C (%)		
	Mean	t-Stat.	Median	$\% < 0$	Std.	AR(1)	Mean	t-Stat.	Median
Panel A. S&P 500 Options									
Call									
0.950 to 0.975	0.09	[1.89]	0.08	47%	1.24	0.10	1.99	[1.97]	1.61
0.975 to 1.000	0.02	[0.44]	−0.08	54%	1.17	0.13	0.47	[0.31]	−3.99
1.000 to 1.025	−0.11	[−2.48]	−0.40	64%	1.40	0.00	−3.73	[−0.50]	−39.12
1.025 to 1.050	−0.27	[−4.97]	−0.69	66%	1.63	−0.03	−129.30	[−3.38]	−142.13
Put									
0.950 to 0.975	−0.10	[−2.34]	−0.06	52%	1.31	0.13	−21.93	[−2.94]	−5.67
0.975 to 1.000	−0.16	[−4.11]	−0.27	59%	1.19	0.18	−12.36	[−4.24]	−21.07
1.000 to 1.025	−0.26	[−5.81]	−0.45	67%	1.33	0.03	−9.84	[−5.61]	−21.78
1.025 to 1.050	−0.43	[−6.28]	−0.74	72%	1.56	−0.03	−9.41	[−5.93]	−18.69
Panel B. VIX Options									
Call									
0.800 to 0.900	−0.44	[−1.72]	−1.03	62%	3.84	−0.12	−2.43	[−1.64]	−5.94
0.900 to 1.000	−0.70	[−2.33]	−1.33	60%	4.43	−0.14	−6.18	[−2.29]	−11.05
1.000 to 1.100	−0.24	[−0.66]	−0.15	51%	4.95	−0.10	−1.82	[−0.38]	−1.66
1.100 to 1.200	−0.55	[−1.44]	0.39	48%	5.09	−0.07	−7.85	[−1.15]	6.42
Put									
0.800 to 0.900	−2.44	[−3.24]	−3.58	77%	6.76	0.37	−67.33	[−3.07]	−93.83
0.900 to 1.000	−1.42	[−2.92]	−2.98	69%	6.91	−0.06	−19.25	[−2.16]	−45.96
1.000 to 1.100	−0.86	[−1.81]	−1.45	58%	6.44	−0.01	−6.76	[−1.72]	−11.84
1.100 to 1.200	−1.37	[−3.08]	−1.40	60%	5.82	−0.03	−6.62	[−3.05]	−7.10

Table 7: **Predictability of Delta-Hedged Option Gains: Robustness to Jump Measures**

Table 7 shows the evidence of predictability of future delta-hedged S&P 500 and VIX option gains by the implied volatility and volatility-of-volatility measures, controlling for the jump risk measures. The cross-sectional regressions include moneyness fixed effects, and lag gains are included to correct for serial correlation of the residuals. The jump measures are as follows. SLOPE is the slope of the implied volatility smile, calculated as Black–Scholes implied volatility of an out-of-the-money ($K/S = 0.9$) minus the implied volatility of an at-the-money put option ($K/S = 1$) for S&P 500 options, and between a $K/S = 1.1$ call option and an at-the-money call option for the VIX options. RJ is the realized jump variation calculated using high-frequency S&P 500 futures tick data for the S&P 500 and VIX tick data for the VIX. SKEW is the model-free measure of risk-neutral skewness. Number is square brackets indicate Newey–West t-statistics. The data are monthly from 2006:M2 to 2016:M12.

Option Gains	VIX^2		VVIX^2		SLOPE		SKEW		RJ		GAINS$_{t-1}$	
	Slope	t-Stat.	Slope	t-Stat.	Slope	t-Stat.	Slope	t-Stat.	Slope	t-Stat.	Slope	t-Stat.
Panel A. S&P 500 Options												
$\Pi_{t,t+\tau}/S_t$	−0.64	[−2.49]	−0.21	[−3.25]	−1.44	[−1.02]					0.27	[4.00]
	−0.49	[−2.58]	−0.23	[−2.92]			−0.01	[−0.15]			0.26	[4.06]
	−0.49	[−3.44]	−0.23	[−2.75]					0.01	[1.81]	0.26	[4.20]
Panel B. VIX Options												
$\Pi^*_{t,t+\tau}/F_t$			−0.98	[−3.19]	2.55	[2.60]					−0.10	[−7.14]
			−0.76	[−2.17]			0.34	[3.64]			−0.10	[−7.58]
			−1.39	[−3.59]					0.03	[1.12]	−0.08	[−3.55]

Table 8: **Delta-Hedged Option Gains: Pre-2012 Subsample**

Table 8 shows average delta-hedged option gains on the S&P 500 and VIX options across their moneyness, in the pre-2012 subsample. Options have one month to maturity, are grouped into an equal-weighted portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is computed using the Black–Scholes formula, with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to 0 and are shown in square brackets. The $\% < 0$ column shows the fraction of observations with negative gains. The data are monthly from 2006:M2 to 2011:M12.

| Moneyness | Π/S (%) | | | | | | Π/C (%) | | |
	Mean	t-Stat.	Median	$\% < 0$	Std.	AR(1)	Mean	t-Stat.	Median
Panel A. S&P 500 Options									
Call									
0.950 to 0.975	0.12	[2.67]	−0.04	54%	0.76	0.40	2.48	[2.94]	−0.85
0.975 to 1.000	0.04	[1.12]	−0.02	53%	0.64	0.26	1.61	[1.52]	−0.94
1.000 to 1.025	−0.08	[−2.92]	−0.11	61%	0.58	0.11	−5.83	[−3.00]	−11.16
1.025 to 1.050	−0.13	[−4.92]	−0.07	63%	0.51	0.12	−28.22	[−5.74]	−17.53
Put									
0.950 to 0.975	−0.12	[−3.31]	−0.25	81%	0.77	0.38	−12.24	[−3.23]	−28.10
0.975 to 1.000	−0.19	[−5.98]	−0.31	75%	0.66	0.24	−9.36	[−4.68]	−18.05
1.000 to 1.025	−0.27	[−8.64]	−0.34	72%	0.62	0.05	−9.64	[−9.24]	−12.92
1.025 to 1.050	−0.30	[−8.58]	−0.30	76%	0.57	0.06	−6.51	[−8.80]	−7.33
Panel B. VIX Options									
Call									
0.800 to 0.900	−0.62	[−2.96]	−0.65	68%	2.16	−0.10	−3.74	[−3.10]	−3.66
0.900 to 1.000	−1.49	[−5.63]	−2.01	75%	2.62	−0.21	−13.00	[−5.48]	−15.61
1.000 to 1.100	−1.02	[−3.16]	−1.46	73%	2.90	−0.10	−11.97	[−2.72]	−20.79
1.100 to 1.200	−1.38	[−4.25]	−1.60	80%	2.89	−0.08	−22.24	[−3.72]	−29.28
Put									
0.800 to 0.900	−0.65	[−1.75]	−1.37	64%	2.64	−0.10	−14.14	[−1.28]	−34.73
0.900 to 1.000	−1.40	[−4.94]	−1.96	74%	2.77	−0.21	−19.34	[−3.98]	−32.81
1.000 to 1.100	−0.97	[−2.82]	−1.56	74%	3.05	−0.13	−7.01	[−2.48]	−12.75
1.100 to 1.200	−1.29	[−3.88]	−1.38	79%	2.87	−0.07	−6.12	[−3.68]	−7.49

Table 9: **Delta-Hedged Option Gains: Post-2012 Subsample**

Table 9 shows average delta-hedged option gains on the S&P 500 and VIX options across their moneyness, in the post-2012 subsample. Options have one month to maturity, are grouped into an equal-weighted portfolio inside the moneyness bin, and are held till expiration. The delta-hedge is computed using the Black–Scholes formula, with daily rebalancing and the margin difference earning the risk-free rate. The delta-hedged option gains Π are scaled either by the index or by the option price. The t-statistics are testing the null that the delta-hedged option gain is equal to 0 and are shown in square brackets. The % < 0 column shows the fraction of observations with negative gains. The data are monthly from 2012:M1 to 2016:M12.

Moneyness	$\Pi/S(\%)$						$\Pi/C(\%)$		
	Mean	t-Stat.	Median	% < 0	Std.	AR(1)	Mean	t-Stat.	Median
Panel A. S&P 500 Options									
Call									
0.950 to 0.975	0.03	[2.05]	0.01	46%	0.27	−0.07	0.88	[2.40]	0.31
0.975 to 1.000	0.05	[3.21]	−0.01	51%	0.35	−0.14	3.00	[3.83]	−0.40
1.000 to 1.025	0.00	[0.21]	−0.03	54%	0.29	−0.08	0.22	[0.11]	−3.70
1.025 to 1.050	−0.05	[−5.08]	−0.04	62%	0.20	0.17	−36.85	[−6.95]	−32.35
Put									
0.950 to 0.975	−0.15	[−13.37]	−0.14	81%	0.26	0.02	−24.72	[−10.37]	−29.33
0.975 to 1.000	−0.14	[−9.75]	−0.15	75%	0.32	−0.03	−10.96	[−6.49]	−15.59
1.000 to 1.025	−0.17	[−11.76]	−0.18	77%	0.31	−0.11	−8.06	[−10.69]	−9.39
1.025 to 1.050	−0.20	[−9.91]	−0.21	82%	0.31	−0.01	−5.11	[−9.87]	−5.39
Panel B. VIX Options									
Call									
0.800 to 0.900	0.07	[0.36]	−0.25	57%	2.30	−0.05	0.64	[0.50]	−1.57
0.900 to 1.000	−0.17	[−0.61]	−0.87	66%	3.08	−0.08	−1.63	[−0.65]	−6.79
1.000 to 1.100	−0.71	[−2.07]	−1.21	69%	3.49	−0.23	−8.52	[−2.07]	−13.85
1.100 to 1.200	−1.05	[−3.06]	−1.32	75%	3.36	−0.19	−16.37	[−3.10]	−19.43
Put									
0.800 to 0.900	−0.25	[−0.50]	−0.86	65%	2.79	0.22	−10.54	[−0.85]	−20.76
0.900 to 1.000	−0.26	[−0.85]	−0.90	68%	3.19	−0.13	−3.43	[−0.68]	−12.19
1.000 to 1.100	−0.74	[−2.15]	−1.18	66%	3.50	−0.22	−5.46	[−2.06]	−8.49
1.100 to 1.200	−1.08	[−3.15]	−1.26	76%	3.38	−0.19	−5.08	[−3.15]	−5.49

Table 10: **Predictability of Realized Variance Measures: Alternative Specifications**

Table 10 shows the evidence from the alternative projections of future realized variances of S&P 500 and the VIX index on the current implied volatility and volatility of volatility measures. The realized and implied variances are modified to be expressed in standard deviation or log units. Numbers in square brackets indicate Newey–West t-statistics with 6 lags. The data are monthly from 2006:M2 to 2016:M12.

Variable	VIX		VVIX		ln VIX^2		ln VVIX^2	
	Slope	t-Stat.	Slope	t-Stat.	Slope	t-Stat.	Slope	t-Stat.
Panel A. S&P 500 Index								
$\sqrt{\mathrm{RV}^{\mathrm{SPX}}_{t,t+1}}$	0.723	[6.37]						
			0.128	[1.63]				
	0.732	[5.98]	−0.023	[−0.52]				
$\ln \mathrm{RV}^{\mathrm{SPX}}_{t,t+1}$					0.964	[8.70]		
							0.528	[1.38]
					0.986	[8.26]	−0.215	[−1.07]
Panel B. VIX Index								
$\sqrt{\mathrm{RV}^{\mathrm{VIX}}_{t,t+1}}$	0.200	[−0.57]						
			0.687	[5.10]				
	−0.495	[−1.42]	0.789	[5.15]				
$\ln \mathrm{RV}^{\mathrm{VIX}}_{t,t+1}$					−0.069	[−0.68]		
							0.752	[5.18]
					−0.161	[−1.57]	0.873	[4.80]

60

Figure 1: **Volatility and Volatility-of-Volatility Measures**

Figure 1 shows the time series of the volatility and volatility-of-volatility measures. The solid line is the VIX and the dashed line is the VVIX. Data are on a monthly frequency from 2006:M2 to 2016:M12.
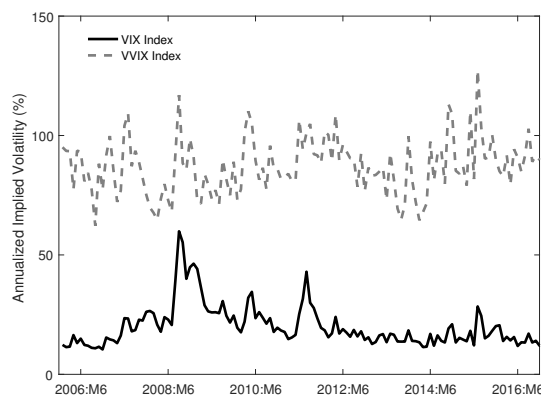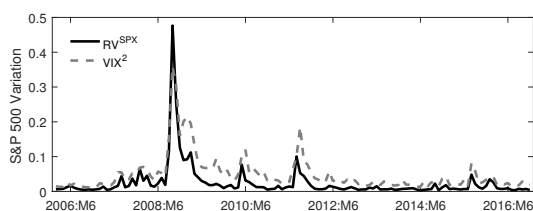
Figure 2: **Realized and Implied Variance Measures**

Figure 2 shows the time series of the realized and implied variances for the S&P 500 (Graph A) and the VIX (Graph B). The solid lines are the realized variances, and dashed lines are the implied variances. All measures are in annualized variance units. Monthly data from 2006:M2 to 2016:M12.

Graph A. Realized and Implied Variance of S&P 500

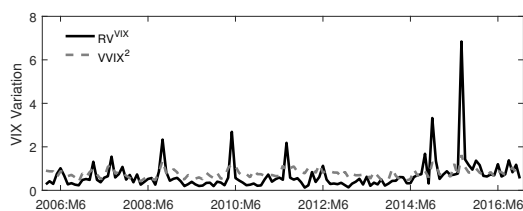*Graph B. Realized and Implied Variance of VIX*

Figure 3: **Gamma, Vega, and Volga by Moneyness**

Graph A of Figure 3 shows the Black–Scholes gamma and vega of the option by option moneyness for average levels of volatility. Graph B shows the Black–Scholes volga of the option by option moneyness for average levels of volatility.

Graph A. Gamma and Vega

Graph B. Volga