Linear Algebra

高等代数作业

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时间： January 30, 2020

版本： 1.00

Even though the truth is not a happy time, we have to be honest ,because more effort needed to cover the truth. — Bertr Russell

第 1 章 高等代数

1.1 作业

✍ **练习 1.1** 用正交变换化二次型$f(x_1,x_2,x_3)=x_1^2-2x_2^2-2x_3^2-4x_1x_2+4x_1x_3+8x_2x_3$为标准型,并给出所用的正交变换.

解 这个实二次型的矩阵$\boldsymbol{A}$为

$$\boldsymbol{A}=\begin{bmatrix}1 & -2 & 2\\ -2 & -2 & 4\\ 2 & 4 & -2\end{bmatrix}. \tag{1.1}$$

$$|\lambda\boldsymbol{I}-\boldsymbol{A}|=\begin{vmatrix}\lambda-1 & 2 & -2\\ 2 & \lambda+2 & -4\\ -2 & 2-4 & \lambda+2\end{vmatrix}=(\lambda-2)^2(\lambda+7). \tag{1.2}$$

$\boldsymbol{A}$的全部特征值是2(二重),-7.

对特征值2,求出$(2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{X}=0$的基础解系:

$$\boldsymbol{\alpha}_1=\begin{bmatrix}2\\0\\1\end{bmatrix},\boldsymbol{\alpha}_2=\begin{bmatrix}-2\\-1\\0\end{bmatrix}. \tag{1.3}$$

对特征值-7,求出$(-7\boldsymbol{I}-\boldsymbol{A})\boldsymbol{X}=0$的基础解系:

$$\boldsymbol{\alpha}_3=\begin{bmatrix}-1\\-2\\2\end{bmatrix}. \tag{1.4}$$

得到三个线性无关的特征向量

$$\boldsymbol{\alpha}_1=\begin{bmatrix}2\\0\\1\end{bmatrix},\boldsymbol{\alpha}_2=\begin{bmatrix}-2\\-1\\0\end{bmatrix},\boldsymbol{\alpha}_3=\begin{bmatrix}-1\\-2\\2\end{bmatrix}. \tag{1.5}$$

将$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$正交化

$$\beta_1 = \alpha_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{8}{5} \end{bmatrix} \tag{1.6}$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

再将$\beta_1, \beta_2, \beta_3$单位化得$\mu_1, \mu_2, \mu_3$ 即

$$T = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2\sqrt{5}}{5} & 0 \end{bmatrix} \tag{1.7}$$

则T是正交矩阵,且$T^{-1}AT = diag\{2, 2, -7\}$.

令

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{1.8}$$

则

$$f(y_1, y_2, y_3) = 2y_1^2 + 2y_2^2 - 7y_3^2. \tag{1.9}$$

练习 1.2 设$f(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + ax_3^2 = 2x_1x_2 - 8x_1x_3 + 2x_2x_3$在正交变换$x = Qy$下的标准型为$\lambda_1 y_1^2 + \lambda_2 y_2^2$,求$a$的值及一个正交矩阵$Q$.

解 因为标准型为$\lambda_1 y_1^2 + \lambda_2 y_2^2$所以$det(A) = 0$得$a = 2$:

$$A = \begin{bmatrix} 2 & 1 & -4 \\ 1 & -1 & 1 \\ -4 & 1 & 2 \end{bmatrix} |\lambda I - A| = (\lambda - 6)(\lambda + 3)\lambda = 0 \tag{1.10}$$

$\lambda = 6, \lambda = -3, \lambda = 0$的特征向量分别为:

$$\alpha_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \alpha_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \tag{1.11}$$

将$\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3$单位化,即得:

$$\boldsymbol{T} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \tag{1.12}$$

则$\boldsymbol{T}$是正交矩阵,且$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = diag\{6, -3, 0\}$.

令

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{1.13}$$

则

$$f(y_1, y_2, y_3) = 6y_1^2 - 3y_2^2. \tag{1.14}$$

练习 1.3 用正交变换化二次型

$$f(x_1, \cdots, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_3x_4) \tag{1.15}$$

为标准型,并给出所用的正交线性变化.

解 二次型方程的矩阵$\boldsymbol{A}$为

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, |\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda - 2 & -1 & 0 & 0 \\ -1 & \lambda - 2 & 0 & 0 \\ 0 & 0 & \lambda - 2 & -1 \\ 0 & 0 & -1 & \lambda - 2 \end{vmatrix} = (\lambda - 3)^2(\lambda - 1)^2. \tag{1.16}$$

$\boldsymbol{A}$的全部特征值是3,1.

对特征值3,求出$(3\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X} = 0$的基础解系:

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{\alpha}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.17}$$

把$\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$单位化,得

对特征值1,求出$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X} = 0$的基础解系:

$$\boldsymbol{\alpha}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{\alpha}_4 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.18}$$

因$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_4$已经为正交向量组,即正交化得:

$$\boldsymbol{T}=\begin{bmatrix}0&\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}&0\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\end{bmatrix}\qquad \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}=\boldsymbol{D}=diag(3,3,1,1); \tag{1.19}$$

令

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\boldsymbol{T}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}. \tag{1.20}$$

则

$$f(y_1,y_2,y_3,y_4)=3y_1^2+3y_2^2+y_3^2+y_4^2. \tag{1.21}$$

练习 1.4 求一个正交变换将二次型

$$f=2x_1^2+3x_2^2+3x_3^2+4x_2x_3 \tag{1.22}$$

化成标准形.

解 二次型矩阵$\boldsymbol{A}$为

$$\boldsymbol{A}=\begin{bmatrix}2&0&0\\0&3&2\\0&2&3\end{bmatrix} \tag{1.23}$$

解得$\lambda_1=5,\lambda_2=2,\lambda_3=1$以及对应的特征向量$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$为:

$$\boldsymbol{\alpha}_1=\begin{bmatrix}0\\1\\1\end{bmatrix},\boldsymbol{\alpha}_2=\begin{bmatrix}1\\0\\0\end{bmatrix},\boldsymbol{\alpha}_3=\begin{bmatrix}0\\-1\\1\end{bmatrix}. \tag{1.24}$$

对$\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3$单位化得:

$$\boldsymbol{T}=\begin{bmatrix}0&1&0\\\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{bmatrix}\qquad \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}=\boldsymbol{D}=diag(5,2,1); \tag{1.25}$$

令

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\boldsymbol{T}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}. \tag{1.26}$$

则

$$f(y_1,y_2,y_3)=5y_1^2+2y_2^2+y_3^2. \tag{1.27}$$

练习 1.5 设$\boldsymbol{A}$是一个n阶正定矩阵，则$|\boldsymbol{A}+\boldsymbol{I}|>1$。

证明 设$\boldsymbol{A}$的n个实特征值为$\lambda_1, \cdots, \lambda_n$
有正交矩阵$\boldsymbol{C}$使得$\boldsymbol{A} = (\boldsymbol{C}')^{-1}\boldsymbol{D}\boldsymbol{C}^{-1}$

$$|\boldsymbol{A}+\boldsymbol{I}| = |\boldsymbol{CDC}'+\boldsymbol{I}| = |\boldsymbol{CDC}'+\boldsymbol{CC}'| = |\boldsymbol{C}||\boldsymbol{DC}'+\boldsymbol{C}'| = |\boldsymbol{D}+\boldsymbol{I}||\boldsymbol{C}|^2 = (\lambda_1+1)\cdots(\lambda_n+1) > 1 \quad (1.28)$$

且$\lambda_1, \cdots, \lambda_n > 0$即

$$|\boldsymbol{A}+\boldsymbol{I}| > 1. \quad (1.29)$$

练习 1.6 设n阶实对称矩阵$\boldsymbol{A}$正定，证明$\boldsymbol{A}^*$也是正定矩阵，其中$\boldsymbol{A}^*$表示$\boldsymbol{A}$的伴随矩阵.

证明 存在实矩阵$\boldsymbol{C}$,使得

$$\boldsymbol{A} = \boldsymbol{C}'\boldsymbol{C}, \qquad \boldsymbol{A}^{-1} = \boldsymbol{C}^{-1}\boldsymbol{I}(\boldsymbol{C}^{-1})' \quad (1.30)$$

即$\boldsymbol{A}^{-1} \backsimeq \boldsymbol{I}$得$\boldsymbol{A}^{-1}$是正定的.

$$\begin{aligned}\boldsymbol{A}^* &= |\boldsymbol{A}|\boldsymbol{A}^{-1}\\ &= |\boldsymbol{A}|\boldsymbol{C}^{-1}\boldsymbol{I}(\boldsymbol{C}^{-1})'\\ &= (\sqrt{|\boldsymbol{A}|}\boldsymbol{C}^{-1})'\boldsymbol{I}(\sqrt{|\boldsymbol{A}|}\boldsymbol{C}^{-1})'\end{aligned} \quad (1.31)$$

即$\boldsymbol{A} \backsimeq \boldsymbol{I}$

练习 1.7 设n阶方阵$\boldsymbol{A}$满足条件$\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{I}$,其中$\boldsymbol{A}'$是$\boldsymbol{A}$的转置矩阵，$\boldsymbol{I}$为单位矩阵，证明$\boldsymbol{A}$的实特征向量所对应的特征值的绝对值等于1.

证明 已知$\boldsymbol{A}' = \boldsymbol{A}^{-1}$
由于$|\lambda\boldsymbol{I}| = |(\lambda\boldsymbol{I}-\boldsymbol{A})'| = |\lambda\boldsymbol{I}-\boldsymbol{A}'| = |\lambda\boldsymbol{I}-\boldsymbol{A}^{-1}| = |\boldsymbol{A}^{-1}||\lambda\boldsymbol{I}-\boldsymbol{A}| =$

$$\frac{1}{|\boldsymbol{A}|}\frac{1}{\lambda}(-1)^n|\lambda\boldsymbol{I}-\boldsymbol{A}| \quad (1.32)$$

所以$\frac{1}{|\boldsymbol{A}|}\frac{1}{\lambda}(-1)^n = 1$,而$|\boldsymbol{AA}'| = |\boldsymbol{A}|^2 = 1$,所以$|\lambda| = 1$

练习 1.8 设$f(x_1, \cdots, x_n) = \boldsymbol{X}^T\boldsymbol{A}\boldsymbol{X}$是一实二次型,若存在n维实向量

$$\boldsymbol{X}_1, \boldsymbol{X}_2, s.t., \boldsymbol{X}_1^T\boldsymbol{A}\boldsymbol{X}_1 > 0, \boldsymbol{X}_2^T\boldsymbol{A}\boldsymbol{X}_2 < 0, \quad (1.33)$$

证明:存在n维实向量$\boldsymbol{A}, s.t., \boldsymbol{A}'\boldsymbol{A}\boldsymbol{A} = 0.$

证明 设$\boldsymbol{A}$的秩为r,作实非退化线性替换$\boldsymbol{X} = \boldsymbol{CY}$,将$f$化为规范形

$$f(x_1, \cdots, x_n) = \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} = y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_{p+q}^2 (r = p+q) \quad (1.34)$$

由于存在两个向量$\boldsymbol{X}_1, \boldsymbol{X}_2$,使$\boldsymbol{X}_1'\boldsymbol{A}\boldsymbol{X}_1 > 0, \boldsymbol{X}_2'\boldsymbol{A}\boldsymbol{X}_2 < 0$,,从而可得$p > 0, q > 0$.
令

$$y_1 = y_{p+1} = 1, y_2 = \cdots = y_p = y_{p+2} = \cdots = y_{p+q} = 0, \quad (1.35)$$

取n维列向量

$$X_0 = C\begin{bmatrix}1\\0\\\vdots\\0\\1\\0\\\vdots\\0\end{bmatrix} \tag{1.36}$$

中间的1位于第$p+1$行.

由于$\boldsymbol{X}=\boldsymbol{C}\boldsymbol{Y}$非退化,故$\boldsymbol{X}_0\neq\boldsymbol{0}$,且有

$$X_0'AX_0 = 0 \tag{1.37}$$

练习 1.9 设$\boldsymbol{A}$是n阶实对称矩阵,证明:

$$\exists c>0, s.t., \forall \boldsymbol{X}\in \boldsymbol{R}^n, |\boldsymbol{X}^T\boldsymbol{A}\boldsymbol{X}|\leq c\boldsymbol{X}^T\boldsymbol{X}. \tag{1.38}$$

证明 设$\boldsymbol{A}$的全部特征值是$\lambda_1,\cdots,\lambda_n$.则$t\boldsymbol{I}+\boldsymbol{A}$的全部特征值为$t+\lambda_1,\cdots,t+\lambda_n$,当$t>S_r(\boldsymbol{A})$时.

$$t+\lambda_i\geq t-|\lambda_i|\geq t-S_r(\boldsymbol{A})>0,\ i=1,\cdots,n. \tag{1.39}$$

即$t\boldsymbol{I}+\boldsymbol{A}$为正定矩阵.同理$t\boldsymbol{I}-\boldsymbol{A}$也为正交矩阵.

任意一个实n阶向量$\boldsymbol{X}$,都有.

$$\boldsymbol{X}'(c\boldsymbol{I}+\boldsymbol{A})\boldsymbol{X}\geqslant 0,\ \boldsymbol{X}'(c\boldsymbol{I}-\boldsymbol{A})\geqslant 0 \tag{1.40}$$

$$-c\boldsymbol{X}'\boldsymbol{X}\leqslant \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}\leqslant c\boldsymbol{X}'\boldsymbol{X} \tag{1.41}$$

即

$$|\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}|\leqslant c\boldsymbol{X}'\boldsymbol{X} \tag{1.42}$$

练习 1.10 (1)证明:如果$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j(a_{ij}=a_{ji})$是正定二次型.那么

$$f(y_1,\dots,y_n)=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}&y_1\\a_{21}&a_{22}&\cdots&a_{2n}&y_2\\\vdots&\vdots&\ddots&\vdots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}&y_n\\y_1&y_2&\cdots&y_n&0\end{bmatrix} \tag{1.43}$$

是负定二次型;

(2)如果$\boldsymbol{A}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{bmatrix}$是正定矩阵,那么

$$|\boldsymbol{A}|\leq a_{nn}\boldsymbol{P}_{n-1}, \tag{1.44}$$

其中$\boldsymbol{P}_{n-1}$是$\boldsymbol{A}$的n-1阶顺序主子式;

(3)$|\boldsymbol{A}| \leq a_{11}a_{22}\cdots a_{nn}$.

证明 (1)作变换$\boldsymbol{Y}=\boldsymbol{A}\boldsymbol{Z}$即

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \tag{1.45}$$

则

$$\begin{aligned} f(y_1,\cdots,y_n) &= \begin{vmatrix} a_{11} & \cdots & a_{1n} & a_{11}z_1+\cdots+a_{1n}z_n \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n1}z_1+\cdots+a_{nn}z_n \\ y_1 & \cdots & y_n & 0 \end{vmatrix} \\ &\xRightarrow[(i=1,\cdots,n-1)]{c_n - z_n c_i} \begin{vmatrix} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \\ y_1 & \cdots & y_n & -(y_1z_1+\cdots y_nz_n) \end{vmatrix} \\ &= -|\boldsymbol{A}|(y_1z_1+\cdots y_nz_n) \\ &= -|\boldsymbol{A}|\boldsymbol{Y}'\boldsymbol{Z} = -|\boldsymbol{A}|\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} \end{aligned} \tag{1.46}$$

由$\boldsymbol{A}$为正定矩阵知,$f(y_1,y_2,\cdots,y_{n-1})$为负定二次型.

证明 (2)记

$$g'(y_1,y_2,\cdots,y_n) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & y_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & y_n \\ y_1 & y_2 & \cdots & y_{n-1} & 0 \end{vmatrix} \tag{1.47}$$

由(1)可知,$g'(y_1,y_2,\cdots,y_{n-1})$负定

$$|\boldsymbol{A}| = g'(a_{n1},\cdots,a_{n,n-1}) + a_{nn}\boldsymbol{P}_{n-1} \tag{1.48}$$

由于$g'(a_{n1},\cdots,a_{n,n-1}) < 0$,所以$|\boldsymbol{A}| \leq a_{nn}\cdot\boldsymbol{P}_{n-1}$.

证明 (3)由(2),得

$$|\boldsymbol{A}| \leq a_{nn}, |\boldsymbol{P}_{n-1}| \leq a_{nn}a_{n-1,n-1}, |\boldsymbol{P}_{n-2}| \leq \cdots \leq a_{nn}a_{n-1,n-1}\cdots a_{11} \tag{1.49}$$

练习 1.11 设$\boldsymbol{A},\boldsymbol{B}$都是$n\times n$实对称矩阵，且$\boldsymbol{A}$正定,

(1)证明:存在实可逆矩阵$\boldsymbol{T}$，使得$\boldsymbol{T}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{T}$为对角矩阵;

(2)假设$\boldsymbol{B}$也正定.证明:$|\boldsymbol{A}+\boldsymbol{B}| \geq |\boldsymbol{A}|+|\boldsymbol{B}|$.

(3)假设$\boldsymbol{B}$也正定,且$\boldsymbol{A}\boldsymbol{B}=\boldsymbol{B}\boldsymbol{A}$,证明$\boldsymbol{A}\boldsymbol{B}$也是正定矩阵.

证明 (1)

$\boldsymbol{A}$正定,所以存在正交矩阵$\boldsymbol{P}^{-1}=\boldsymbol{P}'$使得,

$$\boldsymbol{P}'\boldsymbol{A}\boldsymbol{P}=diag\left(a_1,\cdots,a_n\right), \tag{1.50}$$

$a_1,\cdots,a_n>0$为$\boldsymbol{A}$的特征值.
$\boldsymbol{P}'\boldsymbol{B}\boldsymbol{P}=\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$也是对称矩阵,且与$\boldsymbol{B}$有相同的特征值.所以存在正交矩阵$\boldsymbol{Q}^{-1}=\boldsymbol{Q}'$使得,

$$\boldsymbol{Q}'(\boldsymbol{P}'\boldsymbol{B}\boldsymbol{P})\boldsymbol{Q}=\boldsymbol{Q}^{-1}(\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P})\boldsymbol{Q}=diag\left(b_1,\cdots,b_n\right), \tag{1.51}$$

其中$b_1,\cdots,b_n$为$\boldsymbol{B}$的特征值.所以:

$$\boldsymbol{Q}'(\boldsymbol{P}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{P})\boldsymbol{Q}=\boldsymbol{Q}^{-1}(\boldsymbol{P}^{-1}(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{P})\boldsymbol{Q}=diag\left(a_1+b_1,\cdots,a_n+b_n\right). \tag{1.52}$$

所以存在实可逆矩阵$\boldsymbol{T}=\boldsymbol{P}\boldsymbol{Q}$,使得$\boldsymbol{T}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{T}$为对角矩阵.
证明 (2)
$\boldsymbol{B}$也正定,因此$b_1,\cdots,b_n>0$.

$$\begin{aligned}|\boldsymbol{Q}'(\boldsymbol{P}'(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{P})\boldsymbol{Q}| &= |diag(a_1+b_1,\cdots,a_n+b_n)| \\ &= \prod_{i=1}^{n}(a_i+b_i) \\ &> \prod_{i=1}^{n}a_i+\prod_{i=1}^{n}b_i \\ &= |\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}|+|\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}|=|\boldsymbol{A}|+|\boldsymbol{B}|.\end{aligned}$$

$$\Rightarrow |\boldsymbol{A}+\boldsymbol{B}|\geq|\boldsymbol{A}|+|\boldsymbol{B}|. \tag{1.53}$$

证明 (3)存在一个n级正交矩阵$\boldsymbol{T}$, $s.t.$

$$\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}=diag\{\lambda_1,\cdots,\lambda_n\}\quad \boldsymbol{T}'\boldsymbol{B}\boldsymbol{T}=diag\{\mu_1,\cdots,\mu_n\}. \tag{1.54}$$

其中$\lambda_1,\cdots,\lambda_n$是$\boldsymbol{A}$的全部特征值,$\mu_1,\cdots,\mu_n$是$\boldsymbol{B}$的全部特征值.$\boldsymbol{A},\boldsymbol{B}$都是正定矩阵,由于:

$$\boldsymbol{T}'(\boldsymbol{A}\boldsymbol{B})\boldsymbol{T}=(\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T})(\boldsymbol{T}'\boldsymbol{B}\boldsymbol{T})=diag\{\lambda_1\mu_1,\cdots,\lambda_n\mu_n\} \tag{1.55}$$

即$\boldsymbol{A}\boldsymbol{B}$合同与$diag\{\lambda_1\mu_1,\cdots,\lambda_n\mu_n\}$,所以$\boldsymbol{A}\boldsymbol{B}$是正定矩阵.

✍ **练习 1.12** 化二次型

$$f(x_1,x_2,x_3)=2x_1^2+4x_1x_2-4x_1x_3+5x_2^2-8x_2x_3+5x_3^2 \tag{1.56}$$

为标准型并写出线性变换.

证明 二次型f的矩阵$\boldsymbol{A}$,以及$\boldsymbol{A}$的全部特征值以及特征向量为:

$$\boldsymbol{A}=\begin{bmatrix}2&2&-2\\2&5&-4\\-2&-4&5\end{bmatrix}; \tag{1.57}$$

$\lambda_1, \lambda_2, \lambda_3$的特征值分别为10,1,1特征向量分别为

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \ \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \ \boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \tag{1.58}$$

将$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$正交化,单位化得$\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$:

$$\boldsymbol{\beta}_1 = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix} \boldsymbol{\beta}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix} \boldsymbol{\beta}_3 = \begin{bmatrix} -\frac{2}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \end{bmatrix} \tag{1.59}$$

即

$$\boldsymbol{T} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \end{bmatrix} \qquad \boldsymbol{T}'\boldsymbol{A}\boldsymbol{T} = \boldsymbol{D} = diag\,(10, 1, 1)\,; \tag{1.60}$$

令

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{T} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}. \tag{1.61}$$

则

$$f(y_1, y_2, y_3) = 10y_1^2 + y_2^2 + y_3^2. \tag{1.62}$$

练习 1.13 设二次型

$$f(x_1, x_2, x_3) = ax_1^2 + ax_2^2 + (a-1)x_3^2 + 2x_1x_3 - 2x_2x_3. \tag{1.63}$$

(1)求二次型f的矩阵的所有特征值;

(2)若二次型的规范性为$y_1^2 + y_2^2$,求a的值和各特征值的一个特征向量.

证明 (1)二次型的矩阵$\boldsymbol{A}$为:

$$\boldsymbol{A} = \begin{bmatrix} a & 0 & 1 \\ 0 & a & -1 \\ 1 & -1 & a-1 \end{bmatrix} \ |\lambda \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \lambda - a & 0 & -1 \\ 0 & \lambda - a & 1 \\ -1 & 1 & \lambda - (a-1) \end{vmatrix} \tag{1.64}$$

解得$\lambda_1, \lambda_2, \lambda_3$的特征值分别为$a-2, a, a+1$,特征向量分别为

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix},\ \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}. \tag{1.65}$$

证明 (2)若规范性为$y_1^2 + y_2^2$, a只能为2,即:

$\lambda_1,\lambda_2,\lambda_3$的特征值分别为0, 2, 3,特征向量分别为

$$\boldsymbol{v}_1=\begin{bmatrix}-1\\1\\2\end{bmatrix},\ \boldsymbol{v}_2=\begin{bmatrix}1\\1\\0\end{bmatrix},\ \boldsymbol{v}_3=\begin{bmatrix}1\\-1\\1\end{bmatrix}. \tag{1.66}$$

练习 1.14 若$n\times n$矩阵$\boldsymbol{A}$可逆,

(1)证明$\boldsymbol{A}'\boldsymbol{A}$正定.

(2)证明:存在正交矩阵$\boldsymbol{P},\boldsymbol{Q}$，使得

$$\boldsymbol{P}'\boldsymbol{A}\boldsymbol{Q}=diag\left(a_1,\cdots,a_n\right),where\ a_i>0,i=1,\cdots,n. \tag{1.67}$$

证明 (1)

$$\boldsymbol{A}'\boldsymbol{A}=\boldsymbol{A}'\boldsymbol{I}\boldsymbol{A} \tag{1.68}$$

因此$\boldsymbol{A}'\boldsymbol{A}$合同与单位矩阵,即$\boldsymbol{A}'\boldsymbol{A}$正定.

证明 (2) $\boldsymbol{A}'\boldsymbol{A}$正定,因此存在正交矩阵$\boldsymbol{P}$,使得

$$\boldsymbol{P}'\boldsymbol{A}'\boldsymbol{A}\boldsymbol{P}=diag(b_1,\cdots,b_n),where\ b_i>0,i=0,\cdots,n. \tag{1.69}$$

令

$$\boldsymbol{D}=diag\left(\sqrt{b_1},\cdots,\sqrt{b_n}\right),\ \boldsymbol{Q}=\boldsymbol{A}'\boldsymbol{P}\boldsymbol{D}, \tag{1.70}$$

容易求得

$$\boldsymbol{Q}'\boldsymbol{Q}=(\boldsymbol{A}'\boldsymbol{P}\boldsymbol{D})'\boldsymbol{A}'\boldsymbol{P}\boldsymbol{D}=\boldsymbol{I}_n, \tag{1.71}$$

因此矩阵$\boldsymbol{Q}$为正交矩阵,令$a_i=\sqrt{b_i},i=1\cdots,n$.则

$$\boldsymbol{P}'\boldsymbol{A}\boldsymbol{Q}=\boldsymbol{D}=diag\left(a_1,\cdots,a_n\right). \tag{1.72}$$

练习 1.15 求线性方程组 $\begin{cases}2x_1+x_2+3_3x_3-x_4=0\\3x_1+2x_2+0x_3-2x_4=0\\3x_1+x_2+9x_3-x_4=0\end{cases}$ 的解空间W及其正交补空间$W^{\perp}$.

证明 系数矩阵作初等行变换得:

$$\boldsymbol{A}=\begin{bmatrix}2&1&3&-1\\3&2&0&-2\\3&1&9&-1\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&6&0\\0&1&-9&-1\\0&0&0&0\end{bmatrix} \tag{1.73}$$

因此,方程组的两个线性无关的向量为:

$$\boldsymbol{\alpha}_1=\begin{bmatrix}0\\1\\0\\1\end{bmatrix},\ \boldsymbol{\alpha}_2=\begin{bmatrix}-6\\9\\1\\0\end{bmatrix}. \tag{1.74}$$

$W = L(\alpha_1, \alpha_2)$. 设正交补空间$W^\perp$的任一向量为$\beta = (y_1, y_2, y_3, y_4)'$.则$\alpha_1'\beta = \alpha_2'\beta = 0$.即

$$\begin{cases} y_2 + y_4 = 0 \\ -6y_1 + 9y_2 + y_3 = 0 \end{cases} \tag{1.75}$$

因此,上面的方程组的两个线性无关的向量为:

$$\beta_1 = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix},\ \beta_2 = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 2 \end{bmatrix}. \tag{1.76}$$

所以,正交补空间$W^\perp = L(\beta_1, \beta_2)$.

练习 1.16 设σ是有限维线性空间V的可逆线性变换.设W是V中$\sigma-$不变子空间,证明:W是V中$\sigma^{-1}-$不变子空间.

证明 已知$\alpha \in W, \sigma(\alpha) \in W$.
设$\sigma(\alpha) = \beta$,因σ可逆,即若:

$$\sigma^{-1}(\beta) = \alpha \qquad \beta\ and\ \sigma^{-1}(\beta) \in W \tag{1.77}$$

所以W是V中$\sigma^{-1}-$不变子空间

下证σ为双射:

证明 设σ为可逆变换,它的逆变换为σ^{-1}.
任取$\xi, \eta \in W$,且$\xi \neq \eta$,则必有$\sigma(\xi) \neq \sigma(\eta)$.不然,设$\sigma(\xi) = \sigma(\eta)$,两边左乘$\sigma^{-1}$,有$\xi = \eta$,这与条件矛盾,所以$\sigma$是单射.
其次,对任一向量$\xi \in W$,必有η使$\sigma(\xi) = \eta$,事实上,令$\sigma^{-1}(\eta) = \xi$即可,所以σ是满射,故σ是双射,即$\sigma^{-1}(\beta) = \alpha$.

练习 1.17 设σ是正交变换,证明:σ的不变子空间的正交补也是σ的不变子空间.

证明 设W是σ的任意一个不变子空间,取W的一组标准正交基$\varepsilon_1, \cdots, \varepsilon_n$,把它扩成$V$的一组标准正交基$\varepsilon_1, \cdots, \varepsilon_m, \varepsilon_{m+1}, \cdots, \varepsilon_n$,由

$$dim(W) + dim(W^\perp) = n \tag{1.78}$$

$$W = L(\varepsilon_1, \cdots, \varepsilon_m),\ W^\perp = L(\varepsilon_{m+1}, \cdots, \varepsilon_n). \tag{1.79}$$

又σ是正交变换,则$\sigma(\varepsilon_1), \cdots, \sigma(\varepsilon_n)$也是标准正交基,由于$W$是$\sigma$的不变子空间,所以$\sigma(\varepsilon_1), \cdots, \sigma(\varepsilon_m) \in W$.且为$W$的一组标准正交基,从而$\sigma(\varepsilon_{m+1}, \cdots, \sigma(\varepsilon_n)) \in W^\perp$
任取$\alpha \in W^\perp$则

$$\alpha = k_{m+1}\varepsilon_{m+1} + \cdots + k_n\varepsilon_n \in W^\perp, \tag{1.80}$$

所以

$$\sigma(\alpha) = k_{m+1}\varepsilon_{m+1} + \cdots + k_n\sigma(\varepsilon_n) \in W^\perp. \tag{1.81}$$

$W^\perp$是σ的不变子空间.

练习 1.18 证明:反对称矩阵的特征值为0或者纯虚数。

证明 设$\boldsymbol{A}$是反对称实矩阵,λ是$\boldsymbol{A}$的一个特征值,$\boldsymbol{\xi}$为相应的特征向量,即$\boldsymbol{A}\boldsymbol{\xi}=\lambda\boldsymbol{\xi}$,则

$$\bar{\xi}'A\xi=\bar{\xi}'(-A')\xi=-\bar{\xi}'A'\xi=-(A\bar{\xi})'\xi=-(\overline{A\xi})'\xi \tag{1.82}$$

即有$\lambda\bar{\xi}\xi=-\bar{\lambda}\bar{\xi}\xi$,从而$\lambda=-\bar{\lambda}$.

令$\lambda=a+ib$,代入上式得$a+ib=-(a-ib)$,即有$a=0$.故λ是0或纯虚数.

练习 1.19 如果λ是正交矩阵$\boldsymbol{A}$的特征值,则$\frac{1}{\lambda}$也是正交矩阵$\boldsymbol{A}$的特征值.

证明 设λ是$\boldsymbol{A}$的特征值,已知λ^{-1}是$\boldsymbol{A}^{-1}$的特征值,由于$\boldsymbol{A}$是正交矩阵,$\boldsymbol{A}'=\boldsymbol{A}^{-1}$,所以$\lambda^{-1}$是$\boldsymbol{A}'$的特征值.因为$\boldsymbol{A}$与$\boldsymbol{A}'$有相同的特征值,所以$\lambda^{-1}$也是$\boldsymbol{A}$的特征值.

练习 1.20 设$\boldsymbol{A},\boldsymbol{B}$都是实对称矩阵,证明:存在正交矩阵$\boldsymbol{Q}, s.t., \boldsymbol{Q}^{-1}\boldsymbol{A}\boldsymbol{Q}=\boldsymbol{B}$的充要条件为$\boldsymbol{A},\boldsymbol{B}$的特征多项式的根全部相同.

证明 必要性.若$\boldsymbol{T}'\boldsymbol{A}\boldsymbol{T}=\boldsymbol{B}$,即$\boldsymbol{A},\boldsymbol{B}$相似,则$\boldsymbol{A},\boldsymbol{B}$的特征多项式相同,所以它们的特征根相同.

充分性.设$\lambda_1,\lambda_2,\cdots,\lambda_n$为$\boldsymbol{A}$和$\boldsymbol{B}$的特征值,由$\boldsymbol{A},\boldsymbol{B}$为实对称矩阵,则存在正交矩阵$\boldsymbol{X}$和$\boldsymbol{Y}$,使

$$X^{-1}AX=\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}=Y^{-1}BY \tag{1.83}$$

于是

$$YX^{-1}AXY^{-1}=B \tag{1.84}$$

令$\boldsymbol{T}=\boldsymbol{X}\boldsymbol{Y}^{-1}$,由$\boldsymbol{X},\boldsymbol{Y}$是正交矩阵知,$\boldsymbol{T}$也是正交矩阵,且有$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}=\boldsymbol{B}$.

练习 1.21 设$\boldsymbol{A}$是n阶实对称矩阵,且$\boldsymbol{A}^2=\boldsymbol{A}$.证明:存在正交矩阵

$$Q, s.t., Q^{-1}AQ=diag\,(1,\cdots,1,0,\cdots,0)\,. \tag{1.85}$$

证明 设λ是$\boldsymbol{A}$的任意特征值,$\boldsymbol{\xi}$是属于λ的特征向量,则$\boldsymbol{A}\boldsymbol{\xi}=\lambda\boldsymbol{\xi}$,从而

$$A^2\xi=A(A\xi)=A(\lambda\xi)=\lambda A\xi=\lambda^2\xi. \tag{1.86}$$

又由于$\boldsymbol{A}^2=\boldsymbol{A}$,所以有$\lambda^2\boldsymbol{\xi}=\boldsymbol{A}^2\boldsymbol{\xi}=\boldsymbol{A}\boldsymbol{\xi}=\lambda\boldsymbol{\xi}$,即$(\lambda^2-\lambda)\boldsymbol{\xi}=0$,因为$\boldsymbol{\xi}\neq\boldsymbol{0}$,故$\lambda^2-\lambda=0$,从而$\lambda=0$或1,再由定理7知,存在正交矩阵$\boldsymbol{Q}$,使

$$Q^{-1}AQ=diag\,(1,\cdots,1,0,\cdots,0)\,. \tag{1.87}$$

在上式中的对角线元素中,1的个数为$\boldsymbol{A}$的特征值1的个数,0的个数为$\boldsymbol{A}$的特征值0的个数.