Introduction in Machine Learning
Perceptron

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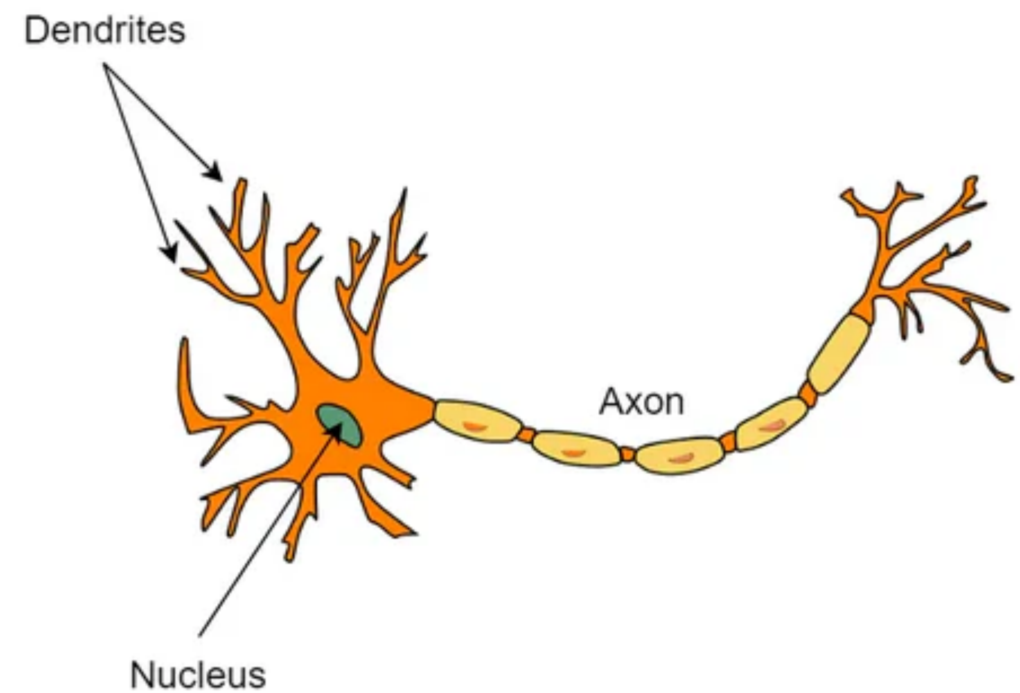
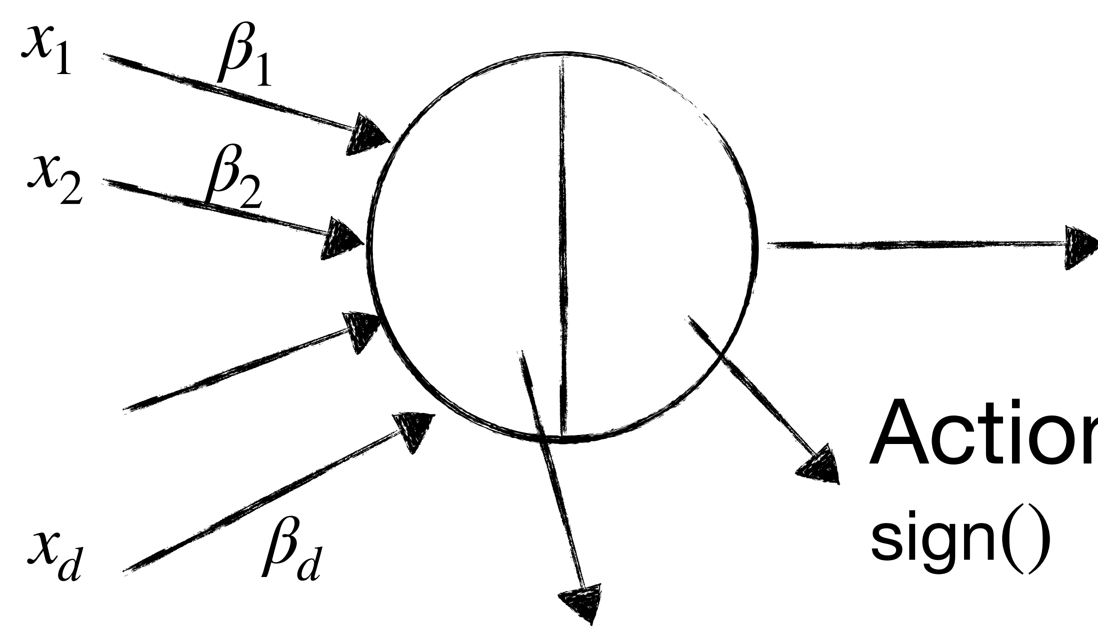
Perceptron



- Simple Linear Classifier
- Input $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_d \end{bmatrix} \in \mathbb{R}^d$,
- Weight $\underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_d \end{bmatrix} \in \mathbb{R}^d$
- bias $\beta_0 \in \mathbb{R}$
- Output $y \in \{-1,1\}$



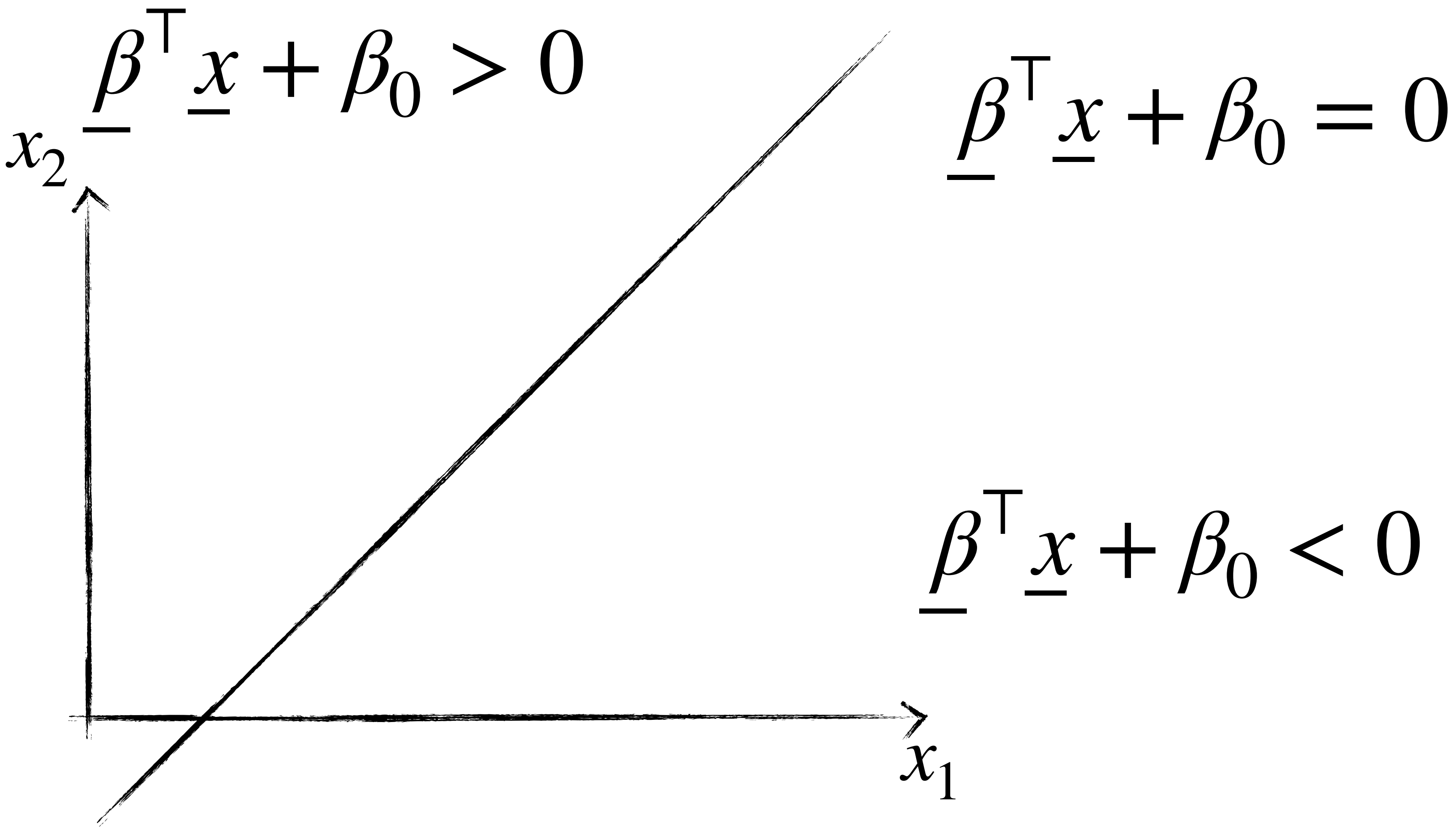
Action Function:
sign()

Linear Function $\sum_{i=1}^{d} \beta_i x_i + \beta_0$

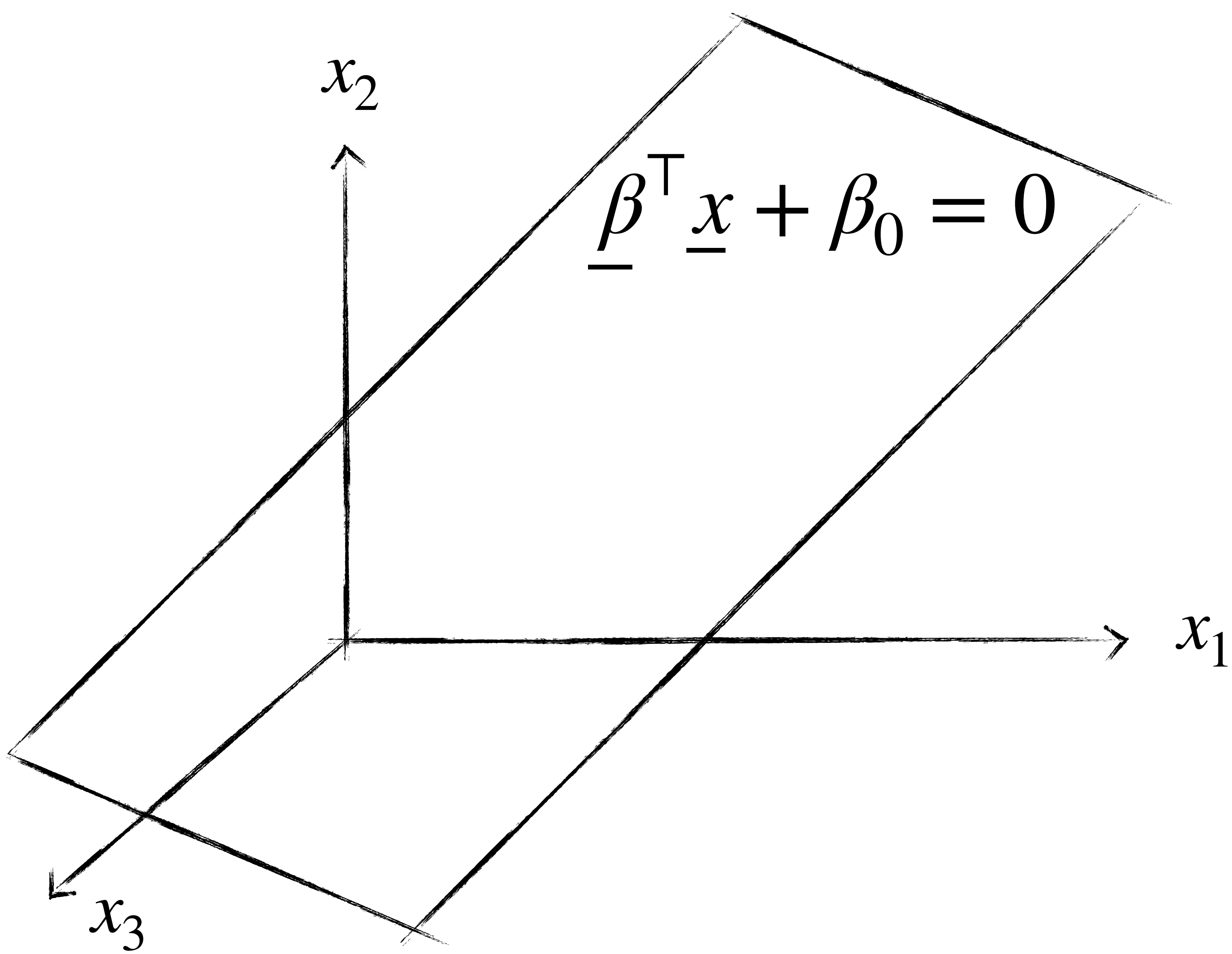
Vector inner product notation

- $$\underline{\beta}^\top \underline{x}_1 = \begin{bmatrix} \beta_1 & \beta_2 & \dots & \beta_d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_d \end{bmatrix} = \sum_{i=1}^{d} \beta_i x_i \in \mathbb{R}$$

The decision boundary (when d=2)

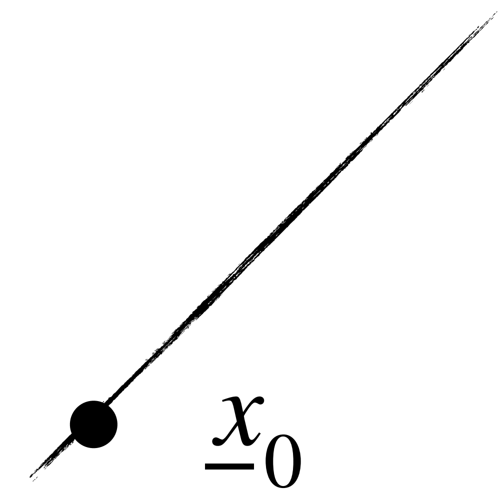


The decision boundary (when d=3)



- For any $\underline{x}_0$ on this line (or hyper plane)
- $\underline{\beta}^\top \underline{x}_0 + \beta_0 = 0$
- $\rightarrow \beta_0 = -\underline{\beta}^\top \underline{x}_0$

$$\underline{\beta}^\top \underline{x} + \beta_0 = 0$$

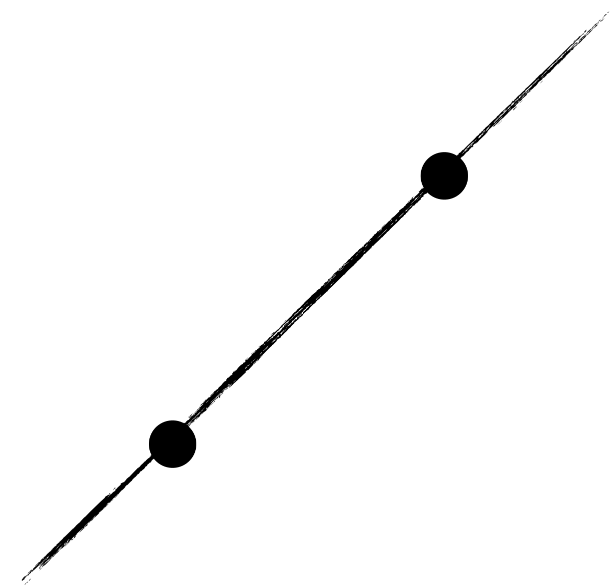


$\underline{x}_0$

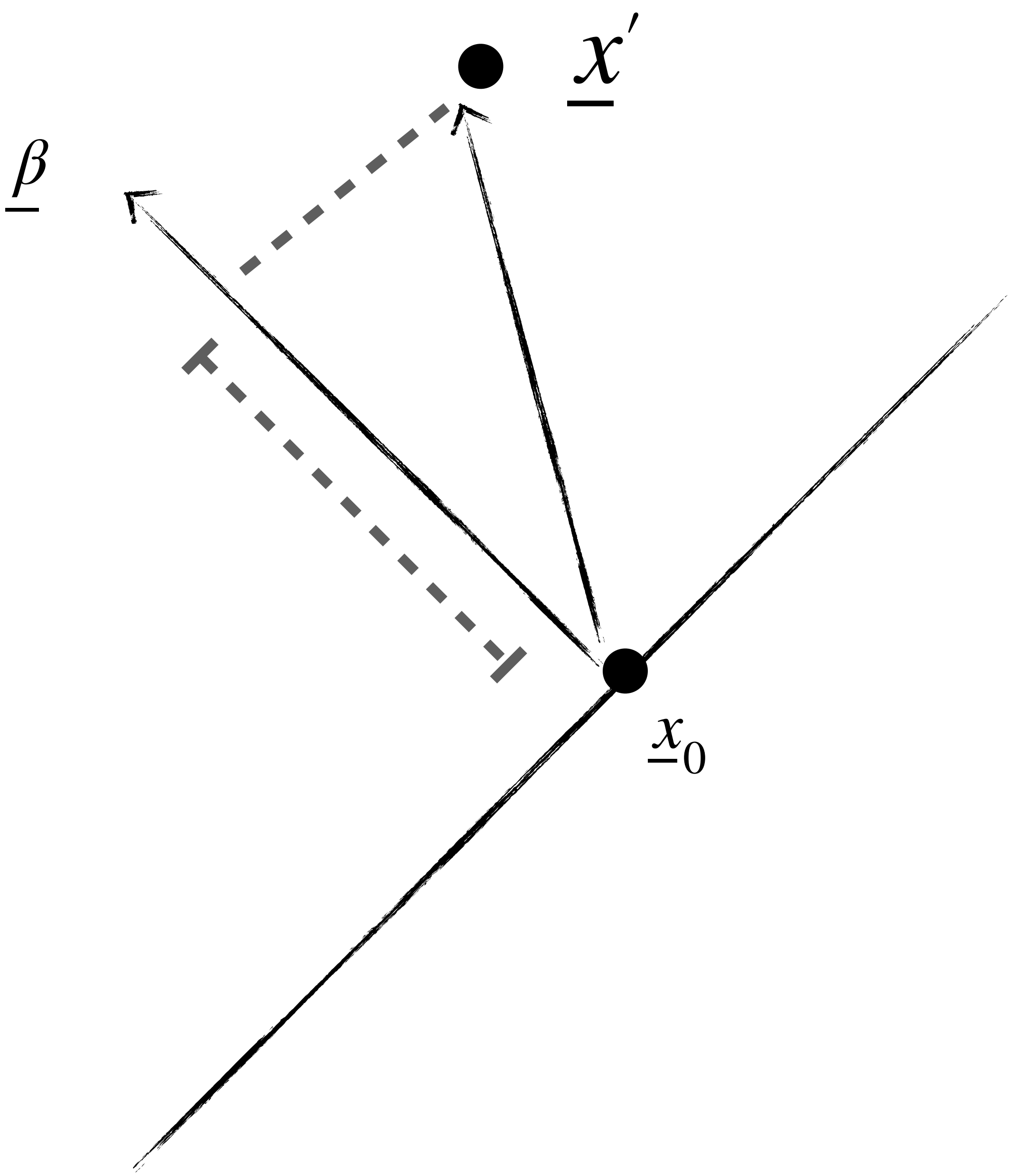
- For any $\underline{x}'$ not on this line (or hyper plane)

- For any $\underline{x}_1$ and $\underline{x}_2$ on this line (or hyper plane):
- $\underline{\beta}^\top \underline{x}_1 + \beta_0 = \underline{\beta}^\top \underline{x}_2 + \beta_0 = 0$
- $\underline{\beta}^\top \underline{x}_1 - \underline{\beta}^\top \underline{x}_2 = 0$
- $\underline{\beta}^\top \left(\underline{x}_1 - \underline{x}_2 \right) = 0$
- vector $\underline{\beta}^\top \perp$ vector $\left(\underline{x}_1 - \underline{x}_2 \right)$

$$\underline{\beta}^\top \underline{x} + \beta_0 = 0$$



The distance from the point to the line



$$\underline{\beta}^\top \underline{x} + \beta_0 = 0$$

The distance from
the point to the line

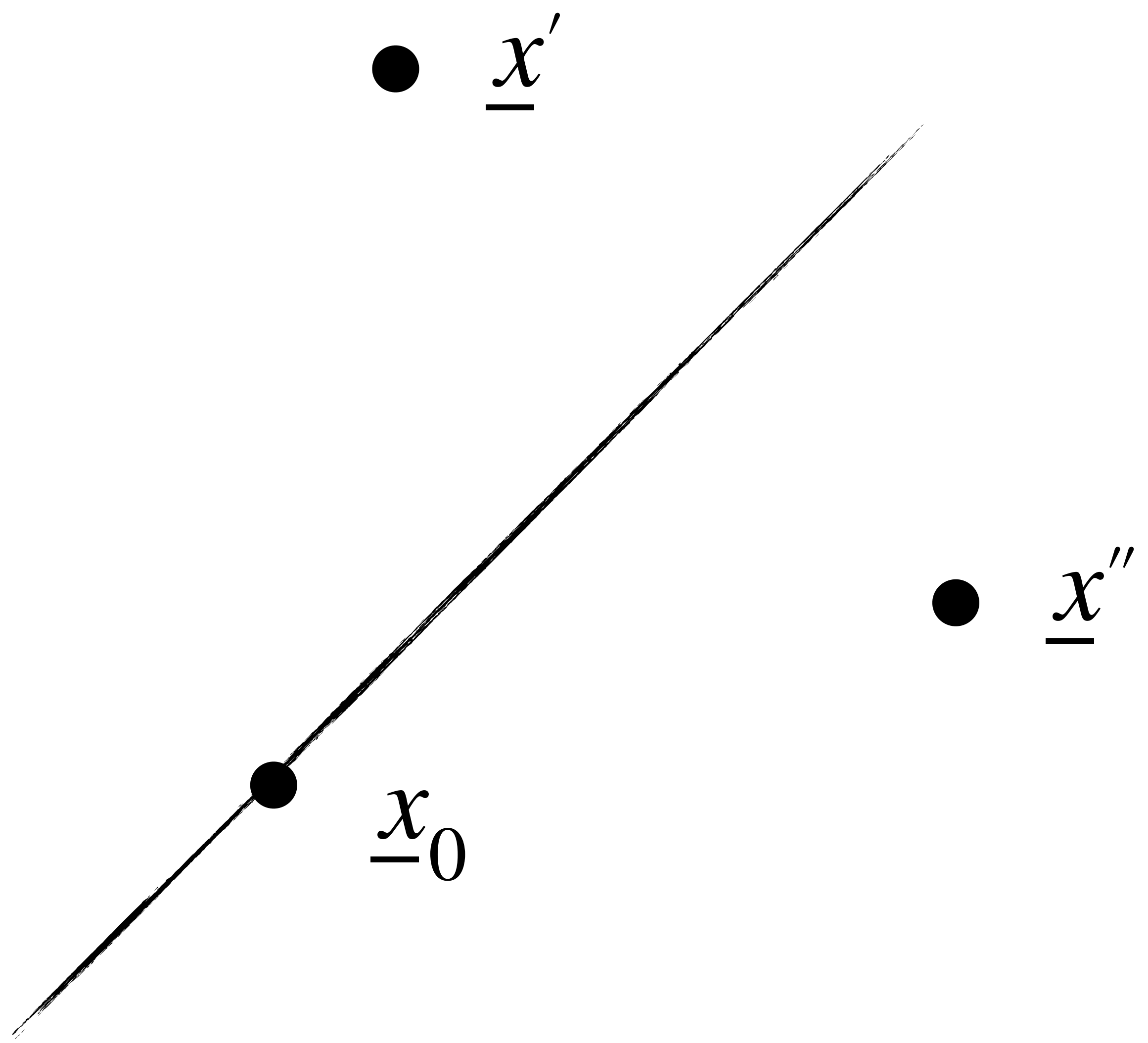
$$\underline{\beta}^\top \left(\underline{x}' - \underline{x}_0 \right)$$

$$= \underline{\beta}^\top \underline{x}' - \underline{\beta}^\top \underline{x}_0$$

$$= \underline{\beta}^\top \underline{x}' + \beta_0$$

The distance from the point to the line

The distance $\underline{\beta}^\top \underline{x}_i + \beta_0$, The output $y_i \in \{+1, -1\}$

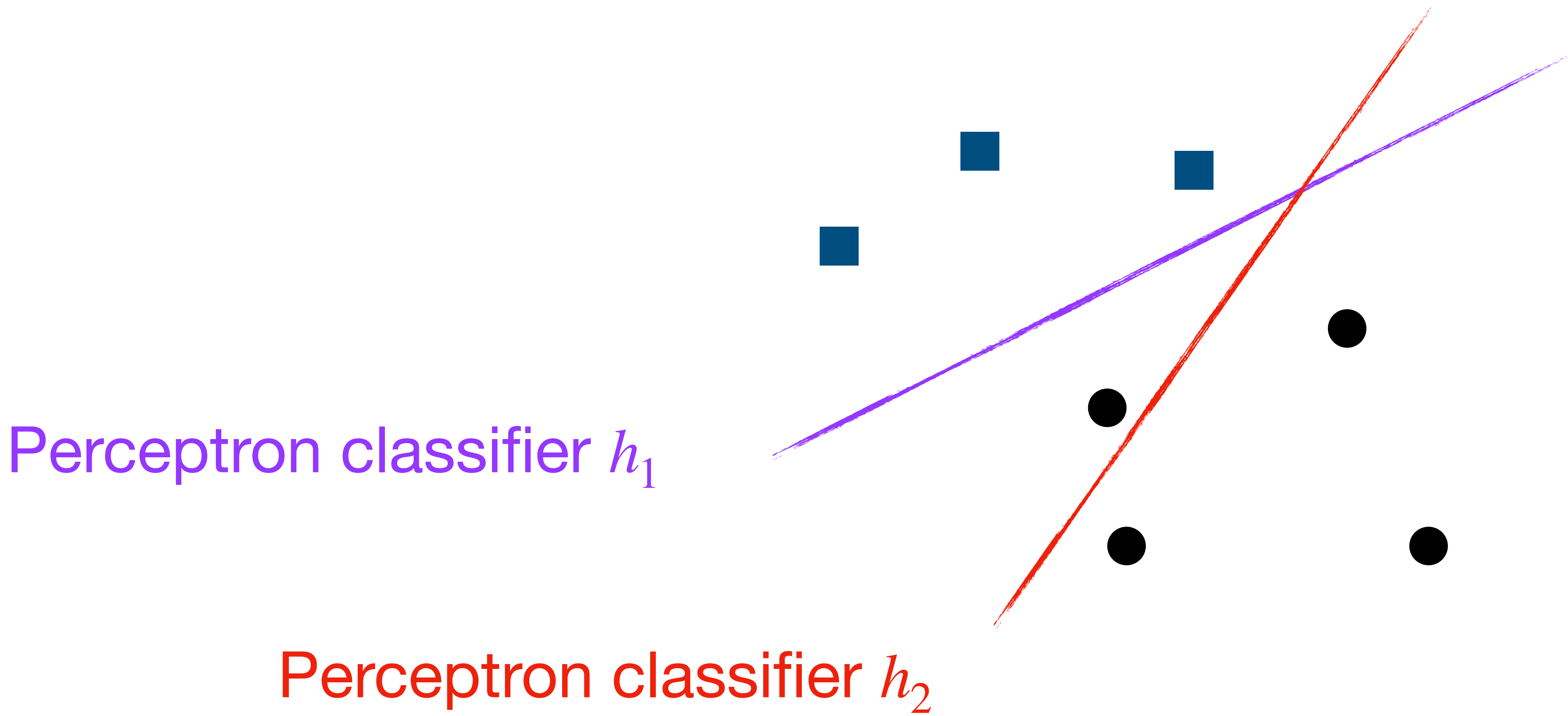


The distance

$$y_i \left(\underline{\beta}^\top \underline{x}_i + \beta_0 \right)$$

Which classifier is better?

- Given a dataset, quantify which classifier is better?

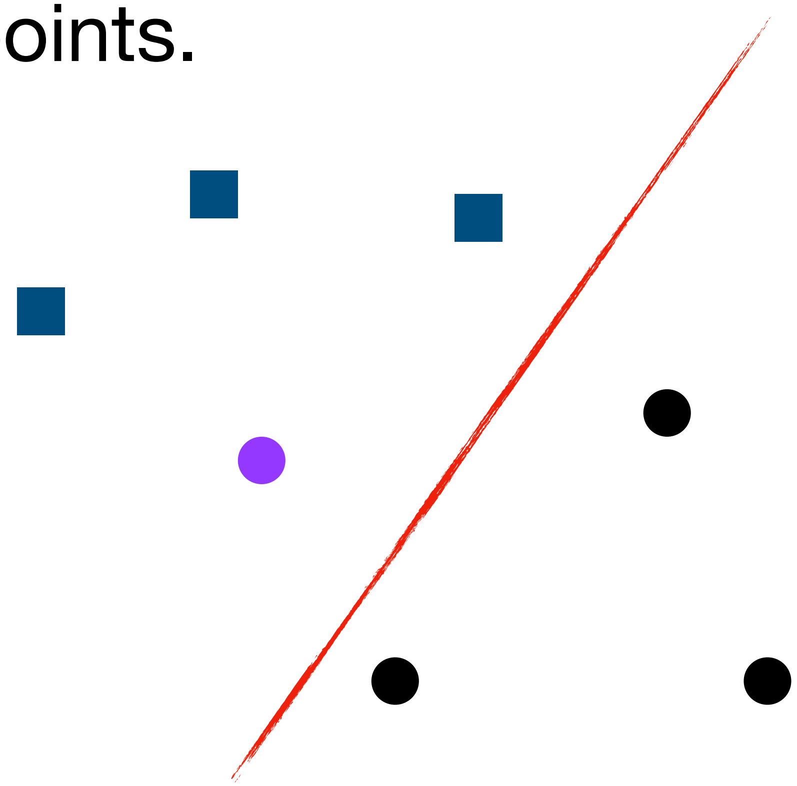


Cost Function

$$\boxed{\text{The distance } y_i\left(\underline{\beta}^\top \underline{x}_i + \beta_0\right)}$$

Let M be the number of misclassified points.

$$\ell(\underline{\beta}, \beta_0) = \sum_{i=1}^{M} y_i\left(\underline{\beta}^\top \underline{x}_i + \beta_0\right)$$

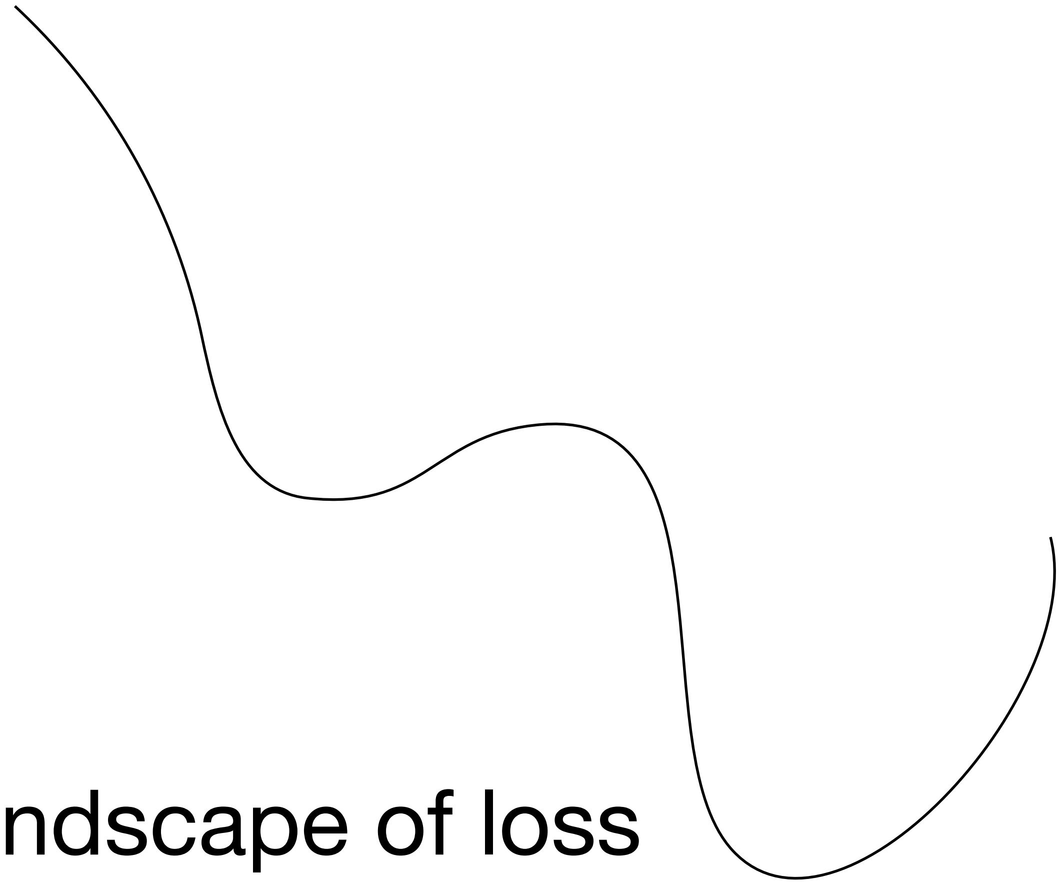


Update Rule of Perceptron

Gradient Descent:

- Let β be the parameters of the model and $0 < \gamma \leq 1$ be the learning rate, we compute:

$$\beta_{new} \leftarrow \beta_{old} - \gamma \frac{\partial \ell}{\partial W}$$



The landscape of loss

Compute the gradient

Let M be the number of misclassified points.

The loss function

$$\ell(\underline{\beta}, \beta_0) = \sum_{i=1}^{M} y_i \left(\underline{\beta}^\top \underline{x}_i + \beta_0 \right)$$

The gradient is:

$$\frac{\partial \ell}{\partial \beta} = -\sum_{i=1}^{M} y_i \underline{x}_i$$

$$\frac{\partial \ell}{\partial \beta_0} = -\sum_{i=1}^{M} y_i$$

Update Rule of Perceptron

- In the Perceptron algorithm, we update parameters as follow:

$$\boxed{\begin{aligned} \frac{\partial \ell}{\partial \beta} &= -\sum_{i=1}^{M} y_i \underline{x}_i \\ \frac{\partial \ell}{\partial \beta_0} &= -\sum_{i=1}^{M} y_i \end{aligned}}$$

$$\boxed{\beta_{new} \leftarrow \beta_{old} - \gamma \frac{\partial \ell}{\partial W}}$$

$$\underline{\beta}_{new} \leftarrow \underline{\beta}_{old} + \gamma y_i \underline{x}_i$$

$$\underline{\beta}_{new} \leftarrow \underline{\beta}_{old} + \gamma y_i$$

Cost Function

- Quantify how of good of the classifier.

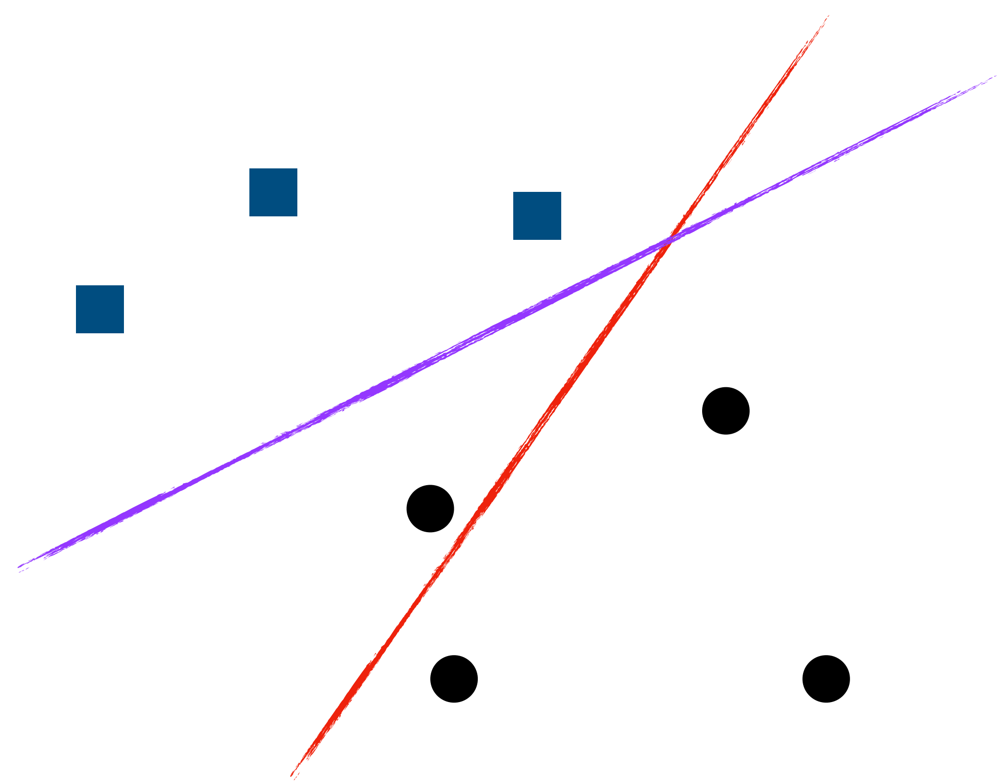
$$\ell(\underline{\beta}, \beta_0) = -\sum_{i=1}^{M} y_i \left(\underline{\beta}^\top \underline{x}_i + \beta_0\right)$$

M is the number of misclassified points.

$$\frac{\partial \ell}{\partial \beta} = -\sum_{i=1}^{M} y_i \underline{x}_i$$

$$\frac{\partial \ell}{\partial \beta_0} = -\sum_{i=1}^{M} y_i$$

Which line is better?



The Whole Pipeline of Perceptron

- For t=1 to T
  - For I=1 to N
    - If $y_i\left(\underline{\beta}^\top \underline{x} + \beta_0\right) \leq 0,$
      - $\underline{\beta} \leftarrow \underline{\beta} + \rho y_i \underline{x}_i$
      - $\underline{\beta} \leftarrow \underline{\beta} + \rho y_i$

The Whole Pipeline of Perceptron

```
For t=1 to T
  For i=1 to N
    # incorrect prediction
```

If ________ $y_i\left(\underline{\beta}^{\top}\underline{x}+\beta_0\right)\leq 0$ ________,

```
      # Update parameters of perceptron
```

$$\underline{\beta} \leftarrow \underline{\beta} + \rho y_i \underline{x}_i$$

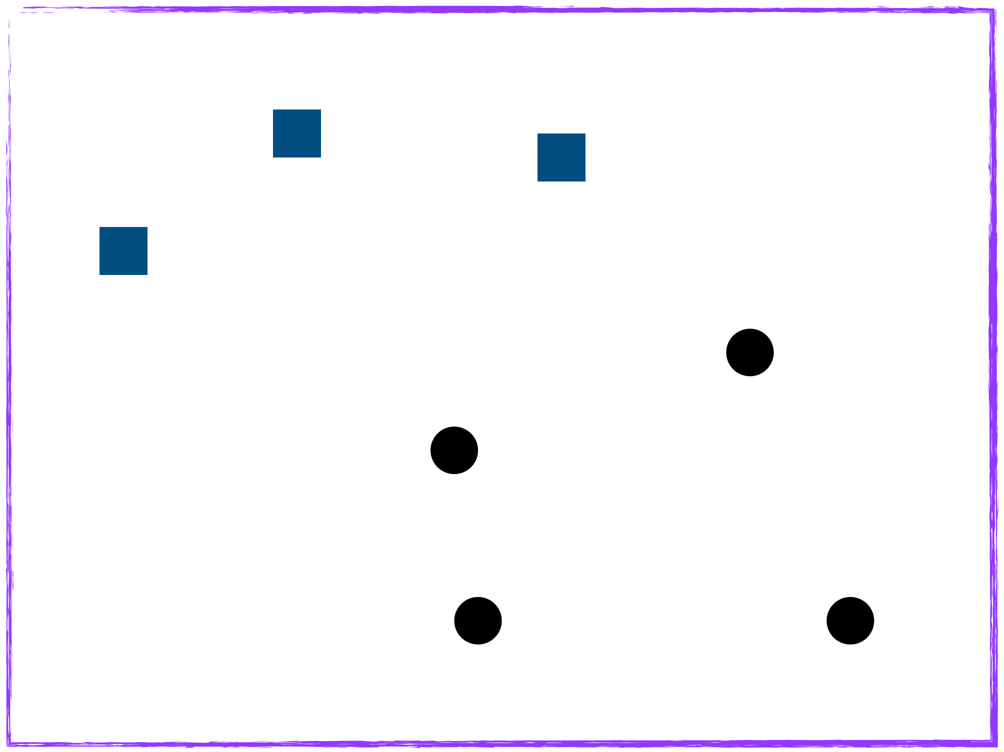
$$\underline{\beta} \leftarrow \underline{\beta} + \rho y_i$$

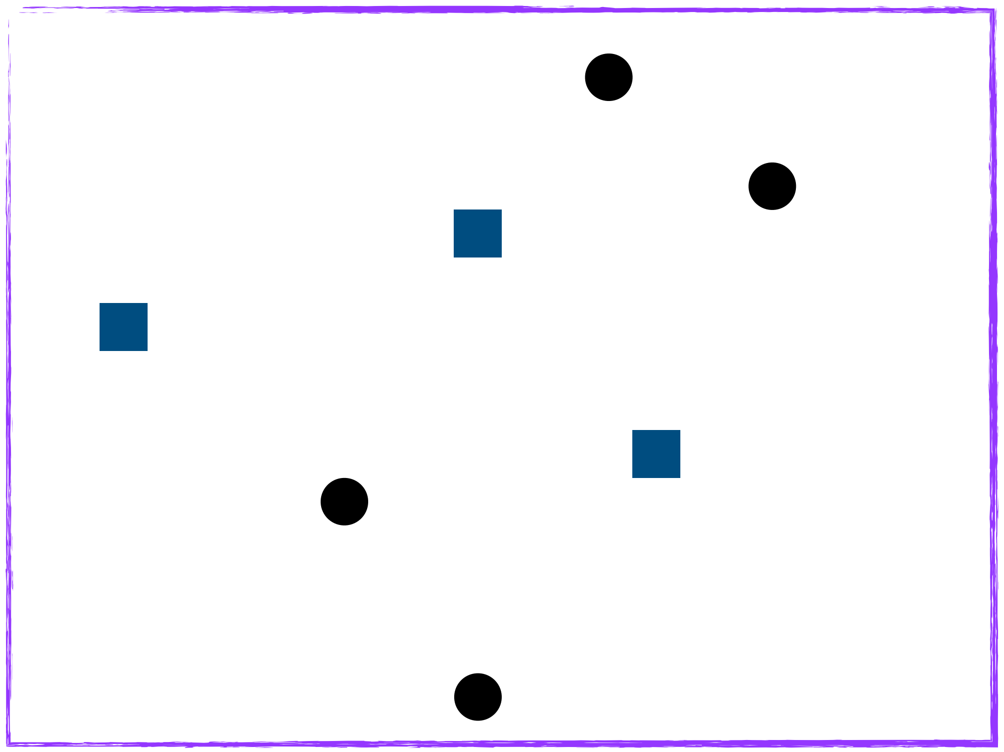
- total learning iterations
- total data points

Linear separable assumption

- There exists $\delta > 0, \underline{\beta}, \beta_0$, such that:

$$y_i \left(\underline{\beta}^\top \underline{x}_i + \beta_0 \right) \geq \delta, \qquad \text{for all } y_i, \underline{x}_i$$





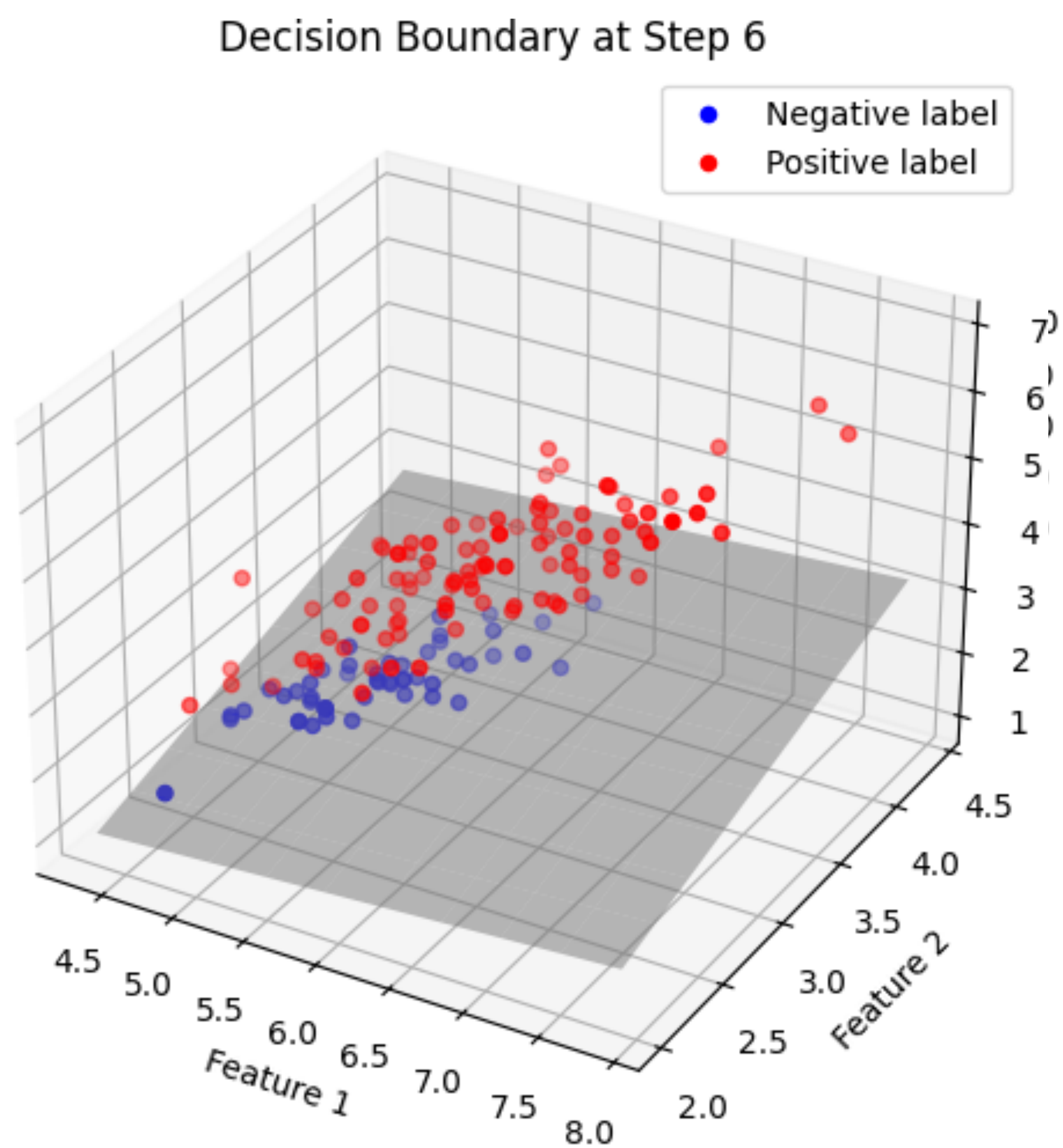
Code Implementation Demo

```
def Perceptron(X, y, gamma=0.1, T=50):
    # N = num_samples, d = num_features
    N, d = X.shape

    # Randomly initialize weights and bias
    beta = np.random.uniform(-1.0, 1.0, size=d)
    beta0 = np.random.uniform(-1.0, 1.0)

    # Iterate over the dataset for T epochs
    for t in range(T):
      for xi, yi in zip(X, y):
          if yi * (np.dot(xi, beta) + beta0) <= 0:
              # Update weights
              beta += gamma * yi * xi
              beta0 += gamma * yi
  return beta, beta0
```

Visualize the process



https://github.com/jiangnanhugo/intro-to-ML/

Thank you!