MATH 113 FINAL EXAM (PRACTICE 1)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

This exam consists of 7 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Carefully define what it means for a set R to be a ring. State all the axioms precisely.

Solution:

Ring is a set R with 2 binary operations.

1) $(R,+)$ be an abelian group { ① associativity ② identity ③ inverse

2) $(R,\times)$ associativity.

3) $(R,+,\times)$ distributivity

4) identity & unit.

(b) Define the units $R^* \subset R$.

Solution:

A unit $a \in R$ is an element s.t. $\exists b \in R$, $ab = 1_R$

R^* contains all units.

(c) Prove, using only the axioms, that $R^* = R$ implies that $|R| = 1$.

Solution:

$\because 0_R a = (0_R + 0_R) a$

$\therefore 0_R a + 0_R = (0_R + 0_R) a$

$\therefore 0_R = 0_R a$

$\because R^* = R$

$\therefore \exists b \in R$ s.t. $0_R = 0_R \cdot b = 1_R$.

$\therefore 0_R = 1_R$.

$\therefore 0_R = 0_R \cdot a = 1_R \cdot a = a$

$\therefore \forall a \in R$, $a = 0_R \Rightarrow |R| = 1$

2. (25 points) Let R be a ring.

(a) Define what it means for a subset $I \subset R$ to be an ideal.

Solution:

1) additive subgroup

2) $\forall x \in I,\ r \in R.\ r \cdot x \in m$

(b) Prove that the binary operation

$$\begin{aligned} \phi : R/I \times R/I &\longrightarrow R/I \\ (x+I, y+I) &\longrightarrow (xy)+I \end{aligned}$$

is well-defined, i.e. independent of coset representative choices.

Solution:

$\forall\ x_1+I,\ x_2+I \in R/I,\ x_1+I = x_2+I.$

$\forall\ y_1+I,\ y_2+I \in R/I.\ y_1+I = y_2+I$

$x_1 - x_2 \in I$

$y_1 - y_2 \in I.$

$\therefore x_1y_1 - x_2y_2 = \underbrace{x_1(y_1-y_2)}_{I} + \underbrace{(x_1-x_2)y_2}_{I} \in I.$

$\therefore x_1y_1 - x_2y_2 \in I.$ ∎

(c) If R/I is the quotient ring, is the following true: $x + I \in (R/I)^* \Rightarrow x \in R^*$. Be sure to justify your answer.

Solution:

not exactly

counterexample: $R = \mathbb{Z}$. $I = 5\mathbb{Z}$.

$[3] \in (\mathbb{Z}/5\mathbb{Z})^*$. $[3]*[2] = [1]$

However. there are no inverse of 3 in $\mathbb{Z}$.

$R = \mathbb{C}[x]$

$I = (x+1)$.

$\mathbb{C}[x]/(x+1)$ is a field and

$x + (x+1) \neq 0 + (x+1)$

$\therefore x + (x+1) \in (\mathbb{C}[x]/(x+1))^*$ but $x \notin (\mathbb{C}[x])^*$.

3. (25 points) Let R be an integral domain.

   (a) Define the characteristic of R.

   Solution:

   (b) Prove that if the characteristic of R is p, then there is an injective homomorphism $\phi : \mathbb{F}_p \to R$. Be sure to carefully justify your answer.

4. (25 points) Let R be a commutative ring.

(a) Define what it means for two elements $a, b \in R$ to be associated.

Solution:

x, y are associate if $a = ub$ for a unit u of R.

(b) Prove that if R is an integral domain then a and b are associated if and only if there exists $u \in R^*$ such that $a = ub$.

$u = ab^{-1}$

Solution:

"$\Rightarrow$"

$\because$ R is an integral domain

$\therefore$ $\forall b \in R$, $\exists c \in R$. s.t. $bc = 1_R$.

$\therefore$ let $u = ac$

$\therefore$ $RHS = ub = acb = a = LHS$.

"$\Leftarrow$" $\forall a, b \in R$, $\exists u \in R^*$ s.t. $a = ub$

$\therefore$ $\forall a \in R$. let $b = 1_R$.

$\because$ $u \in R^*$

$\therefore$ $\exists u^{-1}$. s.t. $uu^{-1} = 1_R$

$\therefore$ $au^{-1} = 1_R$. ∎

(c) Using this, prove that $2\sqrt{2}+1$ and $5+3\sqrt{2}$ are associated in $\mathbb{Z}[\sqrt{2}]$.

$(2\sqrt{2}+1)(a+b\sqrt{2}) = 5+3\sqrt{2}$

$\therefore \begin{cases} 4b+a=5 \\ 2a+b=3 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=1 \end{cases}$

$\therefore$ $u = 1+\sqrt{2}$

$\therefore$ $2\sqrt{2}+1$ and $5+3\sqrt{2}$ are associated.

5. (25 points) Prove that if R is a PID then $a \in R$ is irreducible $\iff$ $(a) \subset R$ is maximal.

Solution:

$\because$ R is a PID

$\therefore$ R is an integral domain and every ideal in R is principal ideal.

"$\Rightarrow$" if $a \in R$ is irreducible

$\therefore$ whenever $a = uv$ in R either u or v is a unit

assume (a) is not maximal. (b) is

$\therefore (a) \subseteq (b) \Rightarrow b < a$. $b \in R$.

$\forall a^k \in (a)$, $b^p = a^k$

$\because$ a is irreducible

$\therefore$ b is a unit

"$\Rightarrow$" a irreducible $\Rightarrow a \notin R^*$

6. (25 points) Let R be an integral domain.

(a) Define what it means for an ideal $I \subset R$ to be maximal.

Solution:

I is a maximal ideal in R. there are no $J \trianglelefteq R$. s.t. $I \subsetneq J \subsetneq R$

or The ideal I is maximal iff any ideal J containing I either equals I or R.

(b) Is the ideal $(x^4 - 1, x^5 - x^3) \subset \mathbb{Q}[X]$ maximal? Be sure to carefully justify your answer. If you use any results from lecture be sure to state them clearly.

Solution:

it is not maximal

$\gcd(x^4-1, x^5-x^3) = (x^2-1)$

$\langle (x^2-1) \rangle \subseteq \langle x-1 \rangle$

7. (25 points) (a) Let E/F be a field extension and let $\alpha \in E$ be algebraic over F. Define the minimal polynomial of α over F.

Solution:

there is a non-zero polynomial $f \in F[x]$. such that $f(\alpha)=0$, f has the least degree, then this f is the minimal polynomial of α over F.

(b) Prove the minimal polynomial is irreducible.

Solution:

assume this minimal polynomial is reducible,

$\therefore f(\alpha) = p(\alpha) q(x)$

where $\deg(q(x)) \geq 1$

However. it shows that

$$\deg(p(\alpha)) = \deg(f(\alpha)) - \deg(q(x))$$
$$< \deg(f(\alpha))$$

which caused a contradiction.

$\therefore$ minimal polynomial is irreducible.

(c) Determine the degree of the extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. You may use any results from lectures as long as they are clearly stated.

$$\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}[1, \sqrt[3]{2}, \sqrt[3]{4}]$$

$\therefore$ degree is 3.

END OF EXAM