Shanghai Jiao Tong University
Faculty of Department of Automation

EE353 Network Optimization

Fall 2014

Homework Solution 1

Assigned on Fri. Oct. 17th, Due on Fri. Oct. 31st

Problem 1: Which of the following sets are convex? And explain the reason.

(a) A *slab*, *i.e.*, a set of the form $\left\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\right\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \ldots, n\}$.A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\left\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1, a_2^T x \leq b_2\right\}$.

(d) The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

Solution.

(a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

(d) This set is convex. $x + S_2 \subseteq S_1$ if $x + y \in S_1$ for all $y \in S_2$. Therefore

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y),$$

the intersection of convex sets $S_1 - y$.

Problem 2: *Running average of a convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, with $\mathbf{R}_+ \subseteq \mathbf{dom} f$. Show that its *running average* F, defined as

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt, \quad \mathbf{dom}\, F = \mathbf{R}_{++}$$

is convex. You can assume f is differentiable.

Solution.

F is differentiable with

$$\begin{aligned}
F'(x) &= -\left(1/x^2\right)\int_0^x f(t)\,dt + f(x)/x \\
F''(x) &= \left(2/x^3\right)\int_0^x f(t)\,dt - 2f(x)/x^2 + f'(x)/x \\
&= \left(2/x^3\right)\int_0^x \left(f(t) - f(x) - f'(x)(t-x)\right)dt
\end{aligned}$$

Convexity now follows from the fact that

$$f(t) \geq f(x) + f'(x)(t-x)$$

for all $x, t \in \mathbf{dom} f$, which implies $F''(x) \geq 0$.

Problem 3: *Dual of general LP*. Find the dual function of LP

$$\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Gx \preceq h \\
& Ax = b.
\end{array}$$

Give the dual problem, and make the implicit equality constraints explicit.

Solution.

(a)The Lagrangian is

$$\begin{aligned}
L(x,\lambda,\nu) &= c^T x + \lambda^T(Gx-h) + \nu^T(Ax-b) \\
&= (c^T + \lambda^T G + \nu^T A)x - h\lambda^T - \nu^T b
\end{aligned}$$

which is an affine function of x. It follows that the dual function is given by

$$g(\lambda,\nu) = \inf_x L(x,\lambda,\nu) = \begin{cases} -\lambda^T h - \nu^T b & c + G^T\lambda + A^T\nu = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

(b)The dual problem is

$$\begin{array}{ll}
\text{maximize} & g(\lambda,\nu) \\
\text{subject to} & \lambda \succeq 0
\end{array}$$

After making the implicit constraints explicit, we obtain

$$\begin{array}{ll}
\text{maximize} & -\lambda^T h - \nu^T b \\
\text{subject to} & c + G^T\lambda + A^T\nu = 0 \\
& \lambda \succeq 0.
\end{array}$$

Problem 4:

(a) Show that the *log-sum-exp* function $f(x) = \frac{1}{\beta}\log\left(\sum_{i=1}^{n} e^{\beta x_i}\right)$ is the optimal value of the following optimization problem

$$\begin{aligned} \textbf{P1:} \quad \max_{y\geq 0} \quad & \sum_{i=1}^{n} y_i x_i - \frac{1}{\beta}\sum_{i=1}^{n} y_i \log y_i \\ s.t. \quad & \sum_{i=1}^{n} y_i = 1. \end{aligned}$$

(b) Solve the KKT conditions to obtain the optimal solution for Problem **P1**.
(Note: justify to yourself why the solution is optimal.)

Solution.

(a) For the log-sum-exp function $f(x) = \frac{1}{\beta}\log\left(\sum_{i=1}^{n} e^{\beta x_i}\right)$, we have its conjugate function given by

$$f^*(y) = \sup_{x\in\mathbf{dom}f} (y^T x - f(x))$$

By setting the gradient w.r.t. x equal to zero, we obtain the condition

$$y_i = \frac{e^{x_i}}{\sum_{j=1}^{n} e^{x_j}}, \quad i = 1, \ldots, n$$

Note that these equations are solvable for x if and only if $\sum_{i=1}^{n} y_i = 1$ and $y \succ 0$. Let $\sum_{j=1}^{n} e^{x_j} = 1$. We obtain $x_i = \frac{1}{\beta}\text{log}y_i$. By substituting the expression for y_i into $y^T x - f(x)$ we obtain

$$f^*(y) = \begin{cases} \dfrac{1}{\beta}\displaystyle\sum_{i=1}^{n} y_i \log y_i & \text{if } y \succeq 0 \text{ and } \mathbf{1}^T y = 1 \\ -\infty & \text{otherwise.} \end{cases}$$

As f^* is still correct if some components of y are zero, where we interpret $0\text{log}0 = 0$. f is a convex and closed function; hence, the conjugate of its conjugate f^* is itself, i.e. $f = f^{**}$. In other words, $f(x)$ is the same as the optimal value of **P1**.

(b) Let λ be the Lagrange multiplier associated with the inequality constraint and ν be the Lagrange multiplier associated with the equality constraint. The KKT conditions are:

$$y \succeq 0, \;\; \mathbf{1}^T y = 1 \tag{1}$$

$$\lambda \succeq 0 \tag{2}$$

$$\lambda^T y = 0 \tag{3}$$

$$\begin{aligned} \frac{\partial}{\partial y_i}\Big(\frac{1}{\beta}\sum_{i=1}^{n} y_i \log y_i - \sum_{i=1}^{n} y_i x_i \\ - \sum_{i=1}^{n} \lambda_i x_i + \nu\big(\sum_{i=1}^{n} y_i - 1\big)\Big) = 0, \quad i = 1, \ldots, n \end{aligned} \tag{4}$$

According to the fourth KKT condition (4), we have

$$\frac{1}{\beta}\left(\log y_i + 1\right) - x_i - \lambda_i + \nu = 0, \quad i = 1, \ldots n$$

Let us take the third KKT condition (3) into account. When $\lambda_i = 0$, x_i are bounded, thus

$$y_i = e^{\beta(x_i - \nu) - 1}, \quad i = 1, \ldots, n$$

When $y_i = 0$, x_i are unbounded.
Considering the first of KKT conditions (1), we can obtain

$$\nu = \frac{1}{\beta}\log\left(\sum_{i=1}^{n} e^{\beta x_i}\right) - \frac{1}{\beta}$$

Plugging ν into the expression of y_i, we can obtain the optimal solution y_i

$$y_i^* = \frac{e^{\beta x_i}}{\sum_{i=1}^{n} e^{\beta x_i}}, \quad i = 1, \ldots, n$$

With the optimal solution y^*, it can be verified that the optimal value of **P1** equals to $\frac{1}{\beta}\log\left(\sum_{i=1}^{n} e^{\beta x_i}\right)$.