# Lazy Subtyping

| | | | |
|---|---|---|---|
| Type variable | $a, b$ | | |
| Existential variable | $\widehat{\alpha}, \widehat{\beta}$ | | |
| Type | $A, B, C$ | $::=$ | $1 \mid a \mid \forall x.A \mid A \to B \mid \widehat{\alpha}$ |
| Declarative Monotype | $\tau$ | $::=$ | $1 \mid a \mid \tau_1 \to \tau_2$ |
| Algorithmic Monotype | $l, u$ | $::=$ | $1 \mid a \mid \tau_1 \to \tau_2 \mid \widehat{\alpha}$ |
| Algorithmic Monotype List | $\mathcal{L}, \mathcal{U}$ | $::=$ | $[l] \mid [u]$ |
| Declarative Context | $\Psi$ | $::=$ | $\cdot \mid \Psi, a$ |
| Algorithmic Worklist | $\Gamma$ | $::=$ | $\cdot \mid \Gamma, a \mid \Gamma, \mathcal{L} \le \widehat{\alpha} \le \mathcal{U} \mid \Gamma \Vdash A \le B$ |

$\boxed{\Psi \vdash A \le B}$ Declarative Subtyping

$$\frac{a \in \Psi}{\Psi \vdash a \le a} \le \texttt{Var} \qquad \frac{}{\Psi \vdash 1 \le 1} \le \texttt{Unit}$$

$$\frac{\Psi \vdash B_1 \le A_1 \quad \Psi \vdash A_2 \le B_2}{\Psi \vdash A_1 \to A_2 \le B_1 \to B_2} \le\to \qquad \frac{\Psi \vdash \tau \quad \Psi \vdash [\tau/a]A \le B}{\Psi \vdash \forall a.A \le B} \le \forall\texttt{L} \qquad \frac{\Psi, b \vdash A \le B}{\Psi \vdash A \le \forall b.B} \le \forall\texttt{R}$$

$\boxed{\Gamma \Longrightarrow \Gamma'}$ $\Gamma$ reduces to $\Gamma'$ (Algorithmic Subtyping)

$$\Gamma, a \Longrightarrow \Gamma$$
$$\Gamma, \widehat{\alpha} \Longrightarrow \Gamma$$
$$\Gamma, \cdot \le \widehat{\alpha} \le \cdot \Longrightarrow \Gamma$$
$$\Gamma, \mathcal{L} \le \widehat{\alpha} \le \cdot \Longrightarrow \Gamma \Vdash l_1 \le l_2 \Vdash \ldots \Vdash l_1 \le l_n \qquad \mathcal{L} = \{l\}_n, n > 0$$
$$\Gamma, \cdot \le \widehat{\alpha} \le \mathcal{U} \Longrightarrow \Gamma \Vdash u_1 \le u_2 \Vdash \ldots \Vdash u_1 \le u_m \qquad \mathcal{U} = \{u\}_m, m > 0$$
$$\Gamma, \mathcal{L} \le \widehat{\alpha} \le \mathcal{U} \Longrightarrow \Gamma \Vdash l_1 \le u_1 \Vdash \ldots \Vdash l_n \le u_m \qquad \mathcal{L} = \{l\}_n, \mathcal{U} = \{u\}_m, m, n > 0$$
$$\Gamma \Vdash 1 \le 1 \Longrightarrow \Gamma$$
$$\Gamma[a] \Vdash a \le a \Longrightarrow \Gamma$$
$$\Gamma[\widehat{\alpha}] \Vdash \widehat{\alpha} \le \widehat{\alpha} \Longrightarrow \Gamma$$
$$\Gamma \Vdash A_1 \to A_2 \le B_1 \to B_2 \Longrightarrow \Gamma \Vdash A_2 \le B_2 \Vdash B_1 \le A_1$$
$$\Gamma \Vdash \forall a.A \le B \Longrightarrow \Gamma, \widehat{\alpha} \Vdash [\widehat{\alpha}/a]A \le B \qquad B \ne \forall * .*$$
$$\Gamma \Vdash A \le \forall b.B \Longrightarrow \Gamma, b \Vdash A \le B$$
$$\Gamma[\widehat{\alpha}][\widehat{\beta}] \Vdash \widehat{\alpha} \le \widehat{\beta} \Longrightarrow \{\widehat{\alpha}/\widehat{\beta}\}^{>}_{\cdot}\Gamma$$
$$\Gamma[\widehat{\alpha}][\widehat{\beta}] \Vdash \widehat{\beta} \le \widehat{\alpha} \Longrightarrow \{\widehat{\alpha}/\widehat{\beta}\}^{<}_{\cdot}\Gamma$$
$$\Gamma[\widehat{\alpha}] \Vdash \widehat{\alpha} \le u \Longrightarrow \{u/\widehat{\alpha}\}^{<}_{\cdot}\Gamma \qquad \widehat{\alpha} \notin FV(u)$$
$$\Gamma[\widehat{\alpha}] \Vdash l \le \widehat{\alpha} \Longrightarrow \{l/\widehat{\alpha}\}^{>}_{\cdot}\Gamma \qquad \widehat{\alpha} \notin FV(l)$$
$$\Gamma[\widehat{\alpha}] \Vdash \widehat{\alpha} \le A \to B \Longrightarrow \{\widehat{\alpha}_1 \to \widehat{\alpha}_2/\widehat{\alpha}\}^{<}_{\cdot}(\Gamma, \cdot \le \widehat{\alpha}_1 \le \cdot, \cdot \le \widehat{\alpha}_2 \le \cdot) \Vdash \widehat{\alpha}_1 \to \widehat{\alpha}_2 \le A \to B$$
$$\text{not monotype } (A \to B) \wedge \widehat{\alpha} \notin FV(A \to B)$$
$$\Gamma[\widehat{\alpha}] \Vdash A \to B \le \widehat{\alpha} \Longrightarrow \{\widehat{\alpha}_1 \to \widehat{\alpha}_2/\widehat{\alpha}\}^{>}_{\cdot}(\Gamma, \cdot \le \widehat{\alpha}_1 \le \cdot, \cdot \le \widehat{\alpha}_2 \le \cdot) \Vdash \widehat{\alpha}_1 \to \widehat{\alpha}_2 \le A \to B$$
$$\text{not monotype } (A \to B) \wedge \widehat{\alpha} \notin FV(A \to B)$$

$$\boxed{\{A/\widehat{\alpha}\}^*_{\tilde{\Gamma}}\,\Gamma \longmapsto \Gamma'}$$ ReorderWL

$$
\begin{aligned}
\{A/\widehat{\alpha}\}^{>}_{\tilde{\Gamma}}(\Gamma, \mathcal{L} \le \widehat{\alpha} \le \mathcal{U}) &\longmapsto \Gamma,,\texttt{rev}(\tilde{\Gamma}), \mathcal{L} \le \widehat{\alpha} \le \mathcal{U} \cup \{A\} \\
\{A/\widehat{\alpha}\}^{<}_{\tilde{\Gamma}}(\Gamma, \mathcal{L} \le \widehat{\alpha} \le \mathcal{U}) &\longmapsto \Gamma,,\texttt{rev}(\tilde{\Gamma}), L \cup \{A\} \le \widehat{\alpha} \le \mathcal{U} \\
\{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}}(\Gamma, b) &\longmapsto \{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}}\Gamma, b && b \notin \mathrm{FV}(\tilde{\Gamma}) \cup \mathrm{FV}(A) \\
\{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}}(\Gamma, \mathcal{L} \le \widehat{\beta} \le \mathcal{U}) &\longmapsto \{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}}\Gamma, \mathcal{L} \le \widehat{\beta} \le \mathcal{U} && \widehat{\beta} \notin \mathrm{FV}(\tilde{\Gamma}) \cup \mathrm{FV}(A) \\
\{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}}(\Gamma, \mathcal{L} \le \widehat{\beta} \le \mathcal{U}) &\longmapsto \{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}, \mathcal{L} \le \widehat{\beta} \le \mathcal{U}}\Gamma && \widehat{\beta} \in \mathrm{FV}(\tilde{\Gamma}) \cup \mathrm{FV}(A) \wedge \widehat{\alpha} \notin \mathrm{FV}(L) \cup \mathrm{FV}(U) \\
\{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}}(\Gamma \Vdash B \le C) &\longmapsto \{A/\widehat{\alpha}\}^{*}_{\tilde{\Gamma}}\Gamma \Vdash B \le C
\end{aligned}
$$

where `rev` reverses a worklist and $,,$ concatenates two worklists.