Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

ICSI 210 Discrete Structures

Basic Structures

- Sets
  - The Language of Sets, Set Operations
- Functions
  - Types of Functions, Operations on Functions
- Sequences and Summations
  - Types of Sequences, Summation Formulae
- Matrices
  - Matrix Arithmetic

Sets

Section 2.1

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.

Sets

- A *set* is an **unordered** collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
  - Notice that an uppercase letter is used for a set name.
- If a is not a member of A, write $a \notin A$.

Describing a Set: Roster Method

- $S = \{a, b, c, d\}$
- Order is not important.

$$S = \{a, b, c, d\} = \{b, c, a, d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

$$S = \{a, b, c, d\} = \{a, b, c, b, c, d\}$$

- Ellipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

$$S = \{a, b, c, d, \ldots\ldots, z\}$$

Roster Method

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,........,99\}$

- Set of all integers less than 0:

  $S = \{...., -3,-2,-1\}$

Some Important Sets

$\mathbf{N}$ = *natural numbers* = $\{0,1,2,3\ldots.\}$

$\mathbf{Z}$ = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z^+}$ = *positive integers* = $\{1,2,3,\ldots..\}$

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers*.

$\mathbf{Q}$ = set of rational numbers

Describing a Set: Set-Builder Notation

- Specify the property or properties that all members must satisfy:

$$S = \{x \mid x \text{ is a positive integer less than } 100\}$$

$$O = \{x \mid x \text{ is an odd positive integer less than } 10\}$$

$$O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$$

Positive rational numbers:

$$\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p,q\}$$

- A predicate may be used:

$$S = \{x \mid \mathrm{P}(x)\}$$

  - Example: $S = \{x \mid \mathrm{Prime}(x)\}$

Interval Notation

$$[a,b] = \{x \mid a \leq x \leq b\}$$
$$[a,b) = \{x \mid a \leq x < b\}$$
$$(a,b] = \{x \mid a < x \leq b\}$$
$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

Sets containing other sets

- **Example:** The set {**N**,**Z**,**Q**,**R**} is a set containing **four** elements, each of which is a set. The four elements of this set are

  N, the set of natural numbers;

  Z, the set of integers;

  Q, the set of rational numbers; and

  R, the set of real numbers.

Datatype in Computer Science

- The concept of a datatype, or type, in computer science is built upon the concept of a set.
- A **datatype** or **type** is the name of a set, together with a set of operations that can be performed on objects from that set.
  - Name + Operations
  - *boolean* is the name of the set {0, 1} together with operators on one or more elements of this set, such as NOT, AND, and OR.
    - boolean + NOT, AND, and OR

Universal Set and Empty Set

- The *universal set U* is the set containing **everything** currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set (**null** set) is the set with **no elements**. Symbolized ∅, but **{}** also used.

More About Sets

- Sets can be elements of sets.

$$\{\{1,2,3\}, a, \{b, c\}\}$$

$$\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$$

- The empty set is different from a set containing the empty set.

$$\emptyset \neq \{\emptyset\}$$

Set Equality

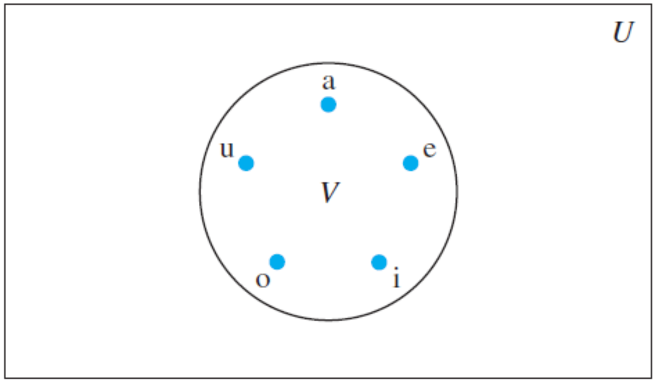
- **Definition**: Two sets are *equal* if and only if they have the same elements.
- If A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.
- Write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Venn Diagrams

- Sets can be represented graphically using Venn diagrams.
  - **A rectangle:** the **universal set** *U* containing all the objects under consideration
  - A **circle** (or **oval**): a set A
  - A **dot (sometimes)**: an element
- Venn diagram for the set of vowels:



- Often used to indicate the relationships between sets.

Subsets

- **Definition**: The set A is **a *subset* of** B, if and only if every element of A is also an element of B.
  - $A \subseteq B$: A is a subset of B.
  - $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.
  - **Special cases: $\emptyset \subseteq S$ and $S \subseteq S$, for every set S.**

    Proof (optional):

    Because $a \in \emptyset$ is always false, $a \in \emptyset \to a \in S$ is true,

    $\emptyset \subseteq S$, for every set S.

    Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.

Show $A \subseteq B$ or $A \nsubseteq B$

- ***A*** $\subseteq$ ***B***: show that if x belongs to A, then x also belongs to B.
  - The set of all computer science majors at your school is a subset of all students at your school.
- ***A*** $\nsubseteq$ ***B***: find an element $x \in A$ with $x \notin B$. (x is a **counterexample** .)
  - The set of integers with squares less than 100 is not a subset of the set of nonnegative integers. Is -3 a counterexample?

Equality of Sets

- $A = B$ iff $\forall x(x \in A \leftrightarrow x \in B)$
- $A = B$ iff

  $$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

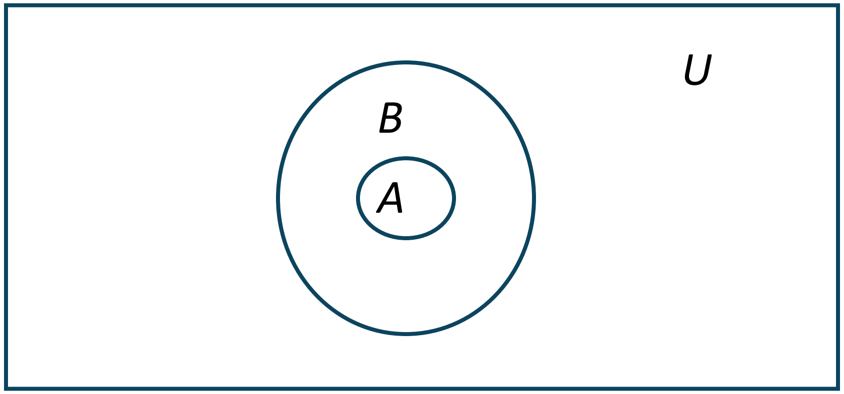
- This is equivalent to **$A \subseteq B$ and $B \subseteq A$.**

Proper Subsets

- **Definition**: If $A \subseteq B$, but $A \neq B$, then we say A is ***a proper subset* of** B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.



Proper Subsets vs. Subsets

- **Example**: List all subsets and all proper subsets of A = {1,2,3}.
- **Solution:**

  Subsets: There are $\mathbf{2^n}$ subsets.
  - Ø, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, **{1,2,3}**

  Proper subsets: There are $\mathbf{2^n - 1}$ proper subsets.
  - Ø, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}

Set Cardinality

- **Definition**: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise, it is *infinite*.

- **Definition**: The *cardinality* of a **finite** set A, denoted by $|A|$, is the number of (distinct) elements of A.

Set Cardinality

- **Examples**:

  $|\emptyset| = 0$

  S: the letters of the English alphabet, $|S| = 26$.

  $|\{1,2,3\}| = 3$

  $|\{\emptyset\}| = 1$

  The set of integers is infinite.

Power Sets

- **Definition**: The set of all **subsets** of a set A, denoted $\mathcal{P}(A)$, is called the *power set* of A.
- If a set has n elements, then the cardinality of the power set is 2^n.

- **Example**:

  $A = \{a, b\}$
  $\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a, b\}\}$.

  $2^2 = 4$ subsets in total

Power Sets

- **Alternative solution** for the same example:
  - Use binary digit 1 to indicate an element is contained in a subset.
  - Use binary digit 0 to indicate an element is not contained in a subset.

a	***b***	
0	0	neither *a* nor *b* is contained in a subset. ø
0	1	*b*, not *a*, is contained in a subset. $\{b\}$
1	0	*a*, not *b*, is contained in a subset. $\{a\}$
1	1	*a*, not *b*, is contained in a subset. $\{a, b\}$

Power Sets

- **Example**: How many ways are there to purchase a hammer and a screwdriver?

Solution:

A = {a hammer, a screwdriver} then

$\mathcal{P}$(A) = {ø,

{a hammer},

{a screwdriver},

{a hammer, a screwdriver}}

Power Sets

- **Example**: What is the power set of the empty set $\emptyset$? What is the power set of the set $\{\emptyset\}$?

Solution:

The empty set has 0 element. $2^0 = 1$. It has exactly one subset, namely, itself. Consequently, $\mathcal{P}(\emptyset) = \{\emptyset\}$.

The set $\{\emptyset\}$ has 1 element. $2^1 = 2$. It has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore, $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

Ordered n-Tuples

- The ordered n-tuple $(a_1, a_2, \ldots.., a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
  - Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuple (a_1, a_2) is called an ordered pair.
  - The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.
  - For example, (1,2) is equal to (1,2).

Cartesian Product

- **Definition**: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a, b\}$ $B = \{1, 2, 3\}$

$A \times B = \{(a, 1),(a, 2),(a, 3), (b, 1),(b, 2),(b, 3)\}$

Cartesian Product

- **Definition**: The cartesian products of the sets A_1, A_2, ……, A_n, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered *n*-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Cartesian Product

- **Example**: What is $A \times B \times$ C where A = {0,1}, B = {1,2} and C = {0,1,2}?

Solution:

$$
\begin{aligned}
A \times B \times C = \{&(0,1,0), (0,1,1), (0,1,2), \\
&(0,2,0), (0,2,1), (0,2,2), \\
&(1,1,0), (1,1,1), (1,1,2), \\
&(1,2,0), (1,2,1), (1,2,2)\}
\end{aligned}
$$

Set Operations

Section 2.2

Set Operators

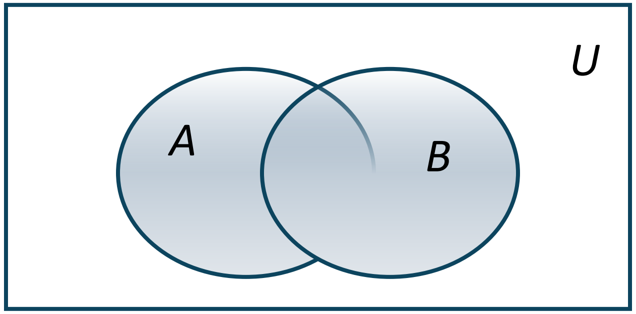
- The operators in set theory are analogous to the corresponding logical connectives in propositional logic.

Set Operators		Logical connectives
Union	∪	∨ (Disjunction, OR)
Intersection	∩	∧ (Conjunction, AND)
Complement	-	¬ (Negation, NOT)

Union

- **Definition**: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x | x \in A \lor x \in B\}$$

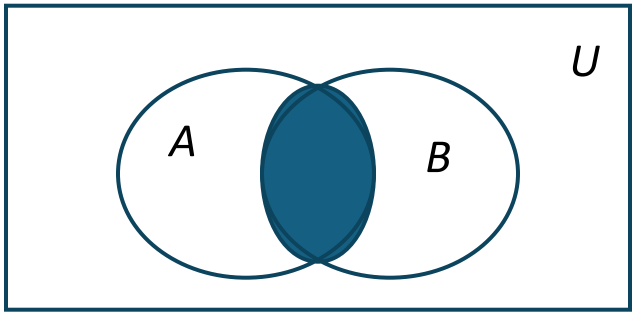


- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?
  Solution: $\{1,2,3,4,5\}$

Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is the set:

$$\{x | x \in A \wedge x \in B\}$$



- If the intersection is empty, then *A* and *B* are said to be *disjoint*.

Intersection

- **Example**: What is $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$

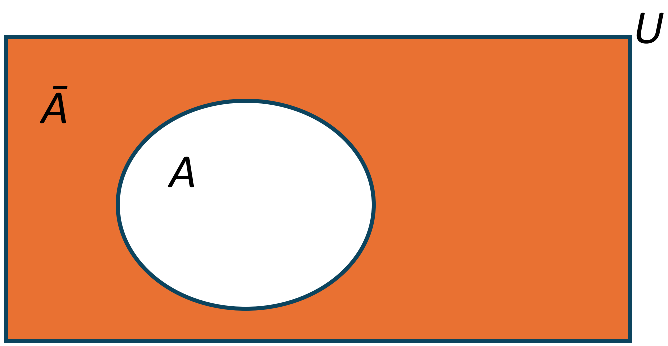
- **Example:** What is $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Complement

- **Definition**: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$.

$$\bar{A} = \{x \in U \mid x \notin A\}$$



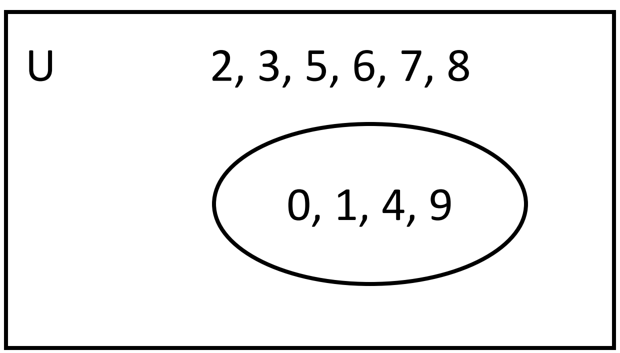
- **Example**: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

Complement

- **Example**: If U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}, and Sq = {0, 1, 4, 9}, what is the complement of Sq.

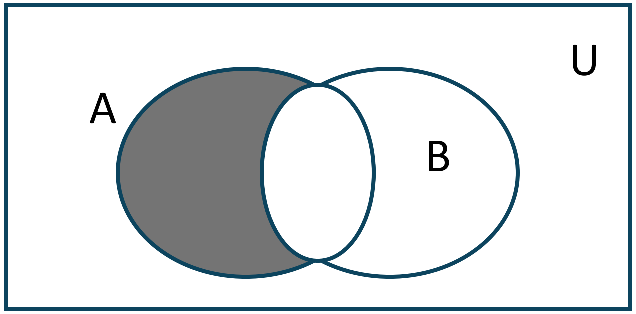
Solution: The complement of Sq is {2, 3, 5, 6, 7, 8}



Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $\boldsymbol{A - B}$, is **the set containing the elements of $\boldsymbol{A}$ that are not in $\boldsymbol{B}$**.
  - $\boldsymbol{A - B}$: called **the complement of $\boldsymbol{B}$ with respect to $\boldsymbol{A}$**.

$$A - B = \{x \mid x \in \mathrm{A} \wedge x \notin B\} = A \cap \overline{B}$$



Difference

- **Example**: If U = $\{1, 2, 3, 4, 5\}$ A = $\{1, 2, 4\}$, and B =$\{4, 5\}$, what is $A - B$ and $B - A$?

  Solution:

  A – B = {1, 2}

  B – A = {5}

Review Questions

- **Example**: $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$.

1. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
2. $A \cap B = \{4, 5\}$
3. $\bar{A} = \{0, 6, 7, 8, 9, 10\}$
4. $\bar{B} = \{0, 1, 2, 3, 9, 10\}$
5. $A - B = \{1, 2, 3\}$
6. $B - A = \{6, 7, 8\}$

The Union of Two Sets

- **Example**:

  A - the math majors in your class

  B - the CS majors in your class

  $|A \cap B|$ - the number of the double-majored students

  $|A \cup B|$ - the number of students who are either math majors or CS majors

  The total number of students:

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- The generalization of this result to unions of an arbitrary number of sets is called the *principle of inclusion–exclusion.*

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \quad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. **Membership Tables:** We consider each combination of sets that an element can belong to and verify that elements in the same combinations of sets belong to both the sets in the identity.

Proving Set Identities

Membership Tables:

- Analog to truth tables in propositional logic.
- Columns for different set expressions.
- Rows for all combinations of memberships in constituent sets.
- Use “1” to indicate membership in the derived set, “0” for non-membership.
- Prove equivalence with identical columns.

Membership Table

- **Example**: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	**1**	1	1	**1**
1	1	0	0	**1**	1	1	**1**
1	0	1	0	**1**	1	1	**1**
1	0	0	0	**1**	1	1	**1**
0	1	1	1	**1**	1	1	**1**
0	1	0	0	**0**	1	0	**0**
0	0	1	0	**0**	0	1	**0**
0	0	0	0	**0**	0	0	**0**

Computer Representation of Sets

- First, specify an arbitrary ordering of the elements of *U,* for instance $a_1, a_2, \ldots, a_n$.
- Represent a subset *A* of *U* with the bit string of length *n,* where the *i*th bit in this string is 1 if a_i belongs to *A* and is 0 if a_i does not belong to *A*.

Computer Representation of Sets

- **Example:** Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order; that is, $a_i = i$. what are the bit strings of the following sets?

  Solution:

  The subset containing all odd integers in *U:*

  10 1010 1010

  The subset containing all even integers in *U:*

  01 0101 0101

  The subset containing the integers not exceeding 5 in *U:*

  11 1110 0000

Computer Representation of Sets

- What is the relation between the first two subsets?
  - The subset containing all odd integers in *U:* 10 1010 1010
  - The subset containing all even integers in *U:* 01 0101 0101

- The complement of a set - simply change each 1 to a 0 and each 0 to 1.
- What about the union or the intersection of two sets?
  - Union: The entire set (1111111111)
  - Intersection: Ø (0000000000)

Computer Representation of Sets

- The **union** of two sets - perform **bitwise OR** Boolean operations.
  - The bit in the *i*th position of the bit string of the union is 1 if either of the bits in the *i*th position in the two strings is 1 or both are 1 and is 0 when both bits are 0.

- The **intersection** of two sets - perform **bitwise AND** Boolean operations.
  - The bit in the *i*th position of the bit string of the intersection is 1 when the bits in the corresponding position in the two strings are both 1 and is 0 when either of the two bits is 0 (or both are).

Computer Representation of Sets

- **Example**: The bit strings for the sets {1, 2, 3, 4, 5} and {1, 3, 5, 7, 9} are 11 1110 0000 and 10 1010 1010, respectively. Use bit strings to find the union and intersection of these sets.

  Solution:

  The union of these sets:

  $$11\ 1110\ 0000 \lor 10\ 1010\ 1010 = 11\ 1110\ 1010$$

  {1, 2, 3, 4, 5, 7, 9}

  The intersection of these sets:

  $$11\ 1110\ 0000 \land 10\ 1010\ 1010 = 10\ 1010\ 0000$$

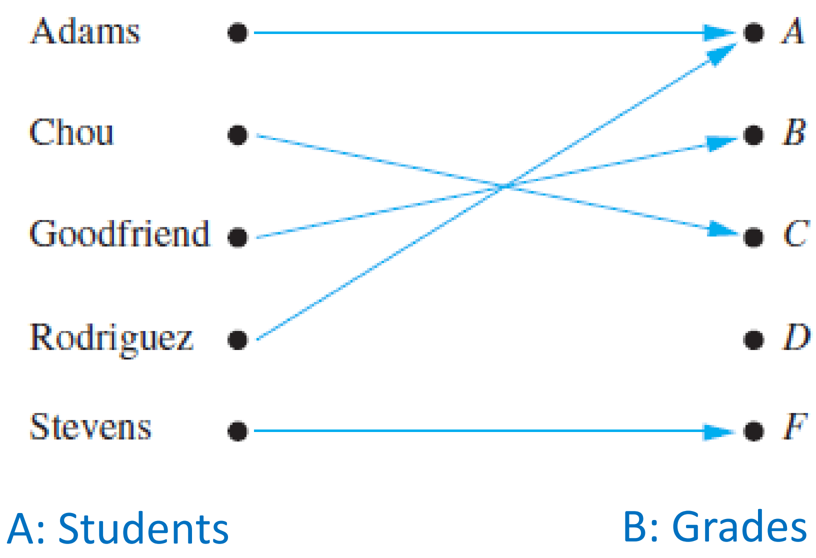
  {1, 3, 5}

Functions

Section 2.3

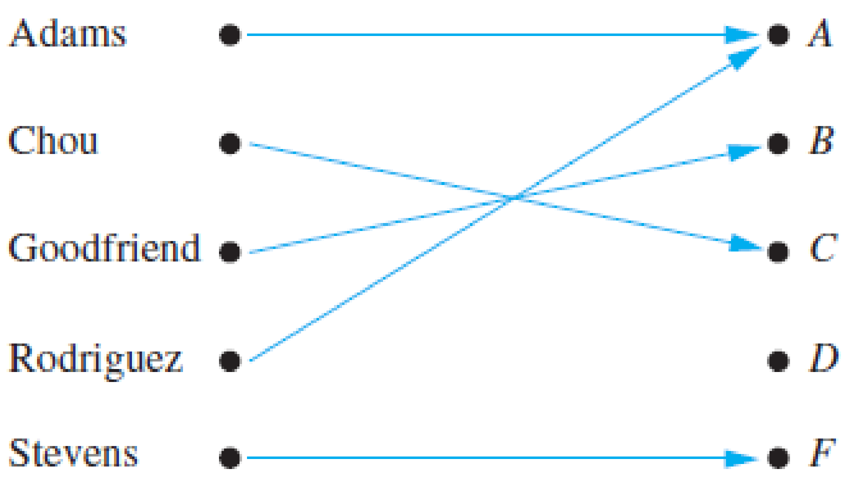
Functions

- **Definition**: Let A and B be nonempty sets. A *function f* from A to B, denoted $f: A \rightarrow B$, is an assignment of each element of A to **exactly one** element of B. We write $f(a) = b$ if b is the **unique** element of B assigned by the function f to the element a of A.
  - For example, $f(\text{Adams}) = \text{A}$.
  - Each student must be assigned with a grade.
  - A student can only be assigned with one grade.
- Functions are sometimes called *mappings* or *transformations*.



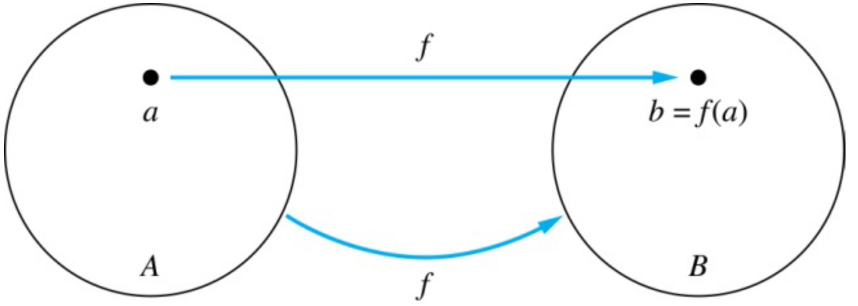
Functions

- **The ordered pairs** (*a, b*) for every element $a \in A$:
  - (Adams, *A*)
  - (Chou, *C*)
  - (Goodfriend, *B*)
  - (Rodriguez, *A*)
  - (Stevens, *F*)



Functions

- Function f: $A \rightarrow B$ *maps* A to B or f is a *mapping* from A to B.
  - A*:* the *domain* of f*.*
  - B*:* the *codomain* of f*.*
- If $f(a) = b$, then
  - b*:* the *image* of a under f.
  - a*:* the *preimage* of b*.*



- The *range* of f , denoted by $f(\boldsymbol{A})$*:*
  - the set of all *images* of points in $\boldsymbol{A}$ under f.

Functions

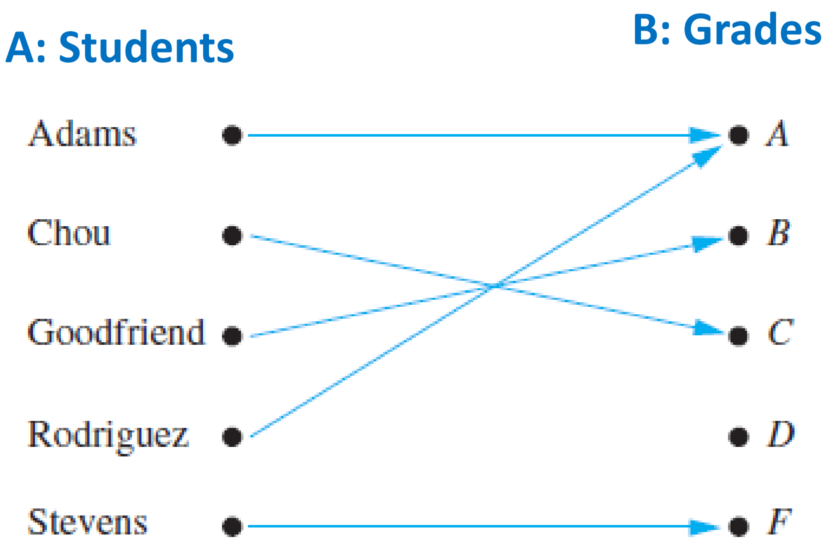
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.

Representing Functions

- Functions may be specified in different ways:
  - An explicit statement of the assignment.

    The grades are A for Adams, C for Chou, B for Goodfriend, A for Rodriguez, and F for Stevens.
  - A formula.

    $f(x) = x + 1$
  - A computer program.
    - A Java program that when given an integer n, produces the nth Fibonacci Number.



Representing Functions

- The domain and codomain of functions are often specified in programming languages.
  - For instance, the Java statement

public static **int floor**(**float** real){. . .}

Codomain: integers

Domain: real numbers

Questions

The domain of *f*: A

The codomain of *f*: B

f(a): z

The image of d : z

The preimage of y: b

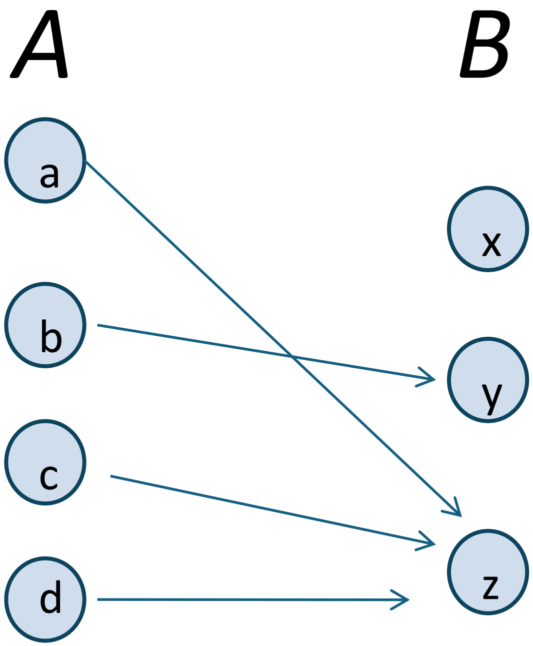
The preimage(s) of z: {a,c,d}

f(A): {y,z}

f ({a,b,c,}): {y,z}

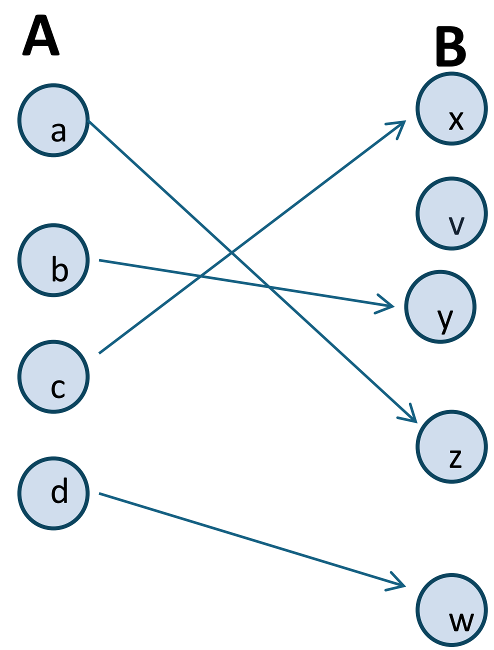
f ({c,d}): {z}

Function *f*: *A* → *B*



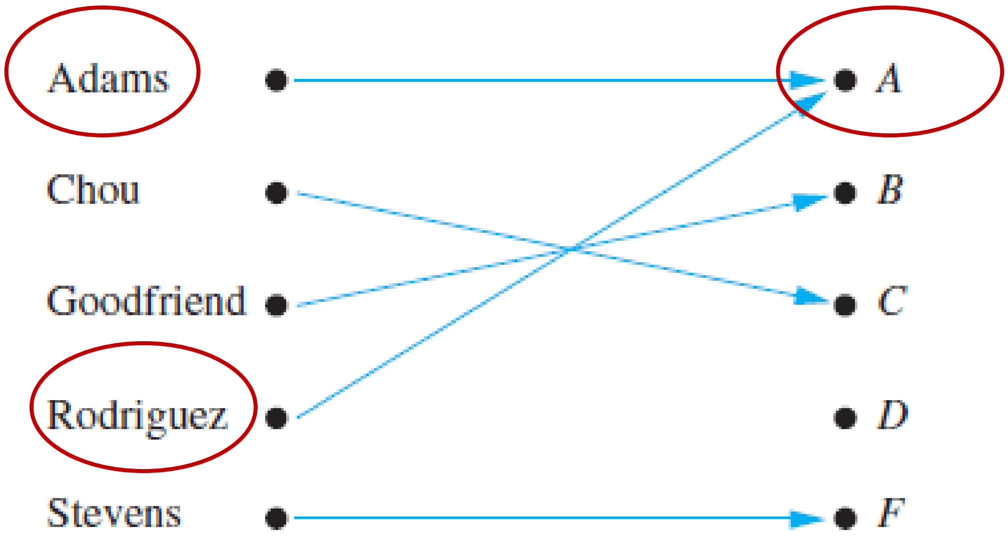
Injections

- A function that never assigns the same value to two different domain elements is said to be **one-to-one or an injection**.
- **Definition**: A function f is said to be *one-to-one* , or *injective*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.



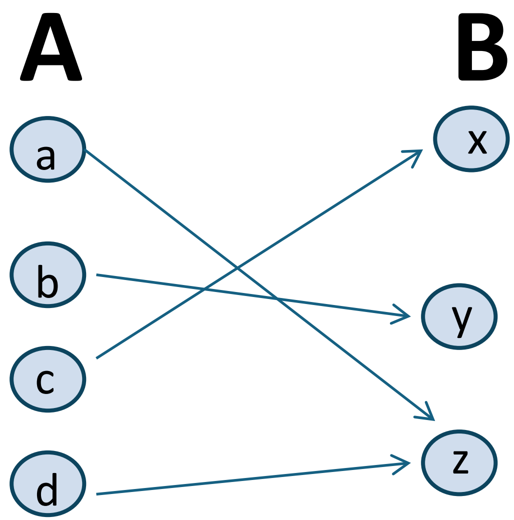
Injections

- In an injection, each student must be assigned with a different letter grade.
- The following is not an injection.



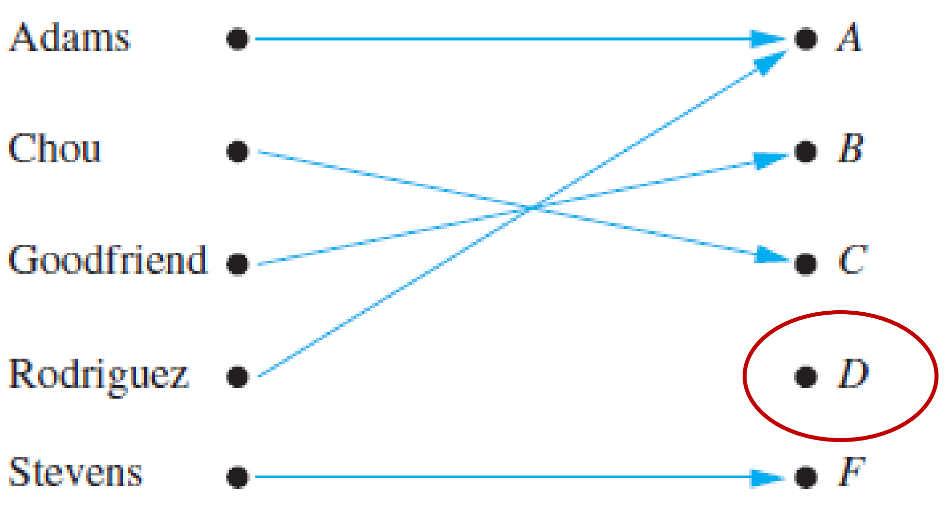
Surjections

- **Definition**: A function *f* from *A* to *B* is called *onto* or *surjective* if and only if **for every element** $b \in B$ **there is an element** $a \in A$ **with** $f(a) = b$. A function *f* is called a *surjection* if it is *onto*.
  - Each element in the codomain B must be used.



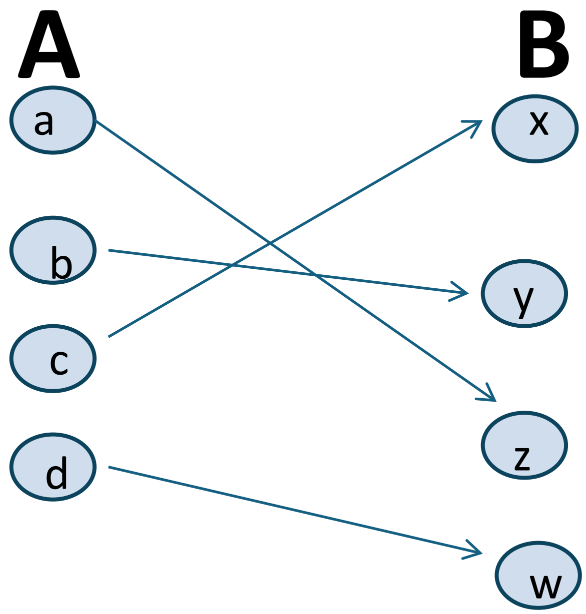
Surjections

- In a surjection, each grade must be assigned.
- The following is not a surjection.



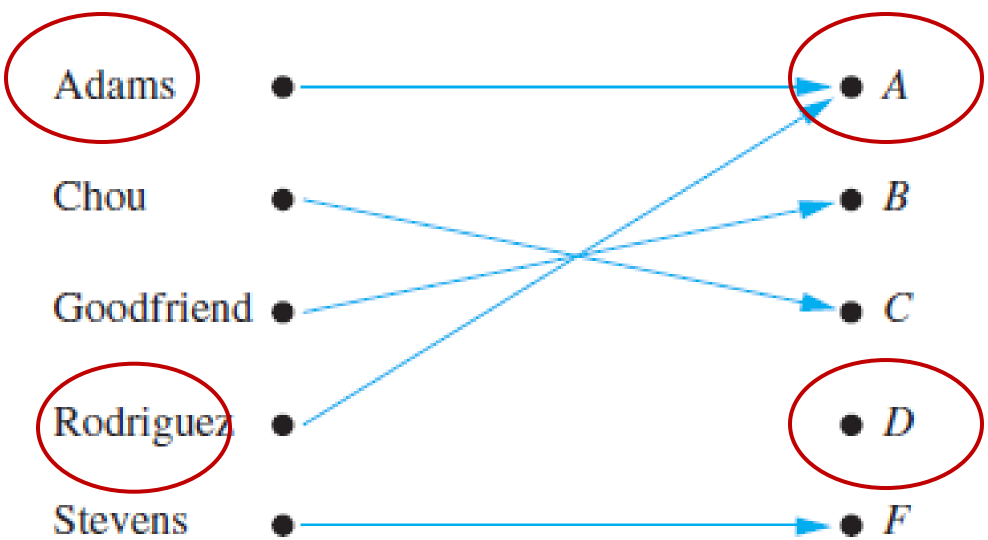
Bijections

- **Definition**: A function f is a *one-to-one correspondence*, or a *bijection*, if it is both one-to-one and onto (surjective and injective).

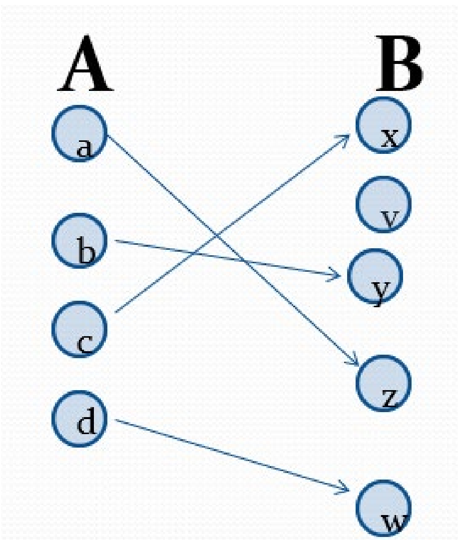


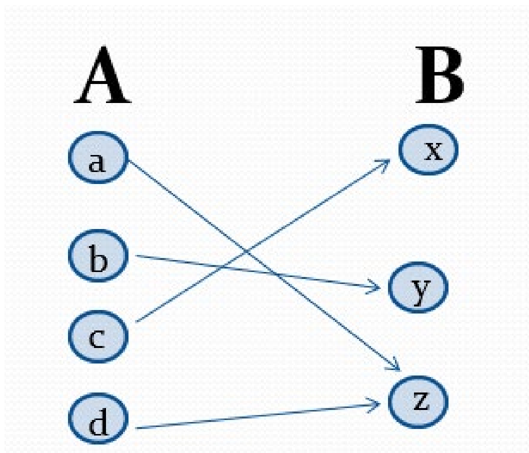
Bijections

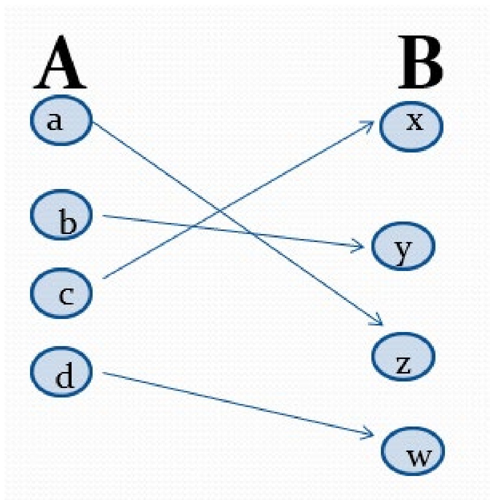
- To be an bijection (injection + surjection), each student must be assigned with a different grade and each grade must be assigned.
- The following is not a bijection.



Injections (One-to-one)	**Surjections (Onto)**	**Bijections** (One-to-one and Onto**)**
Never assigns the same value to two different domain elements.	For every element $b \in$ B, there is an element $a \in$ A with $f(a) = b$.	surjective and injective



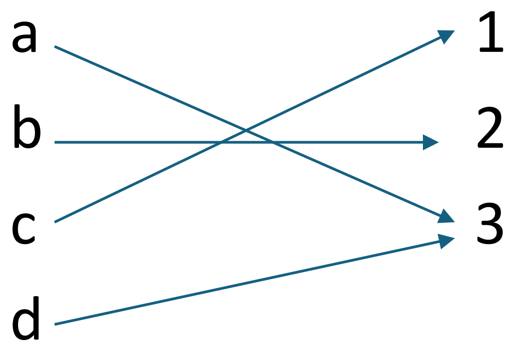




One-to-one or onto

- **Example** : Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

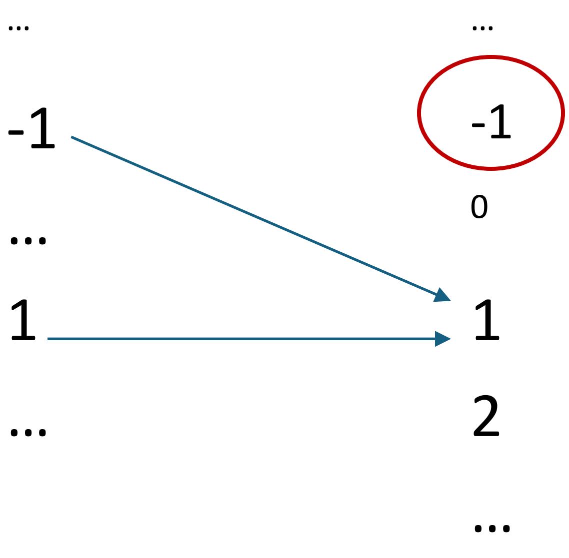
  Solution: Yes, f is onto since all three elements of the codomain are images of elements in the domain.



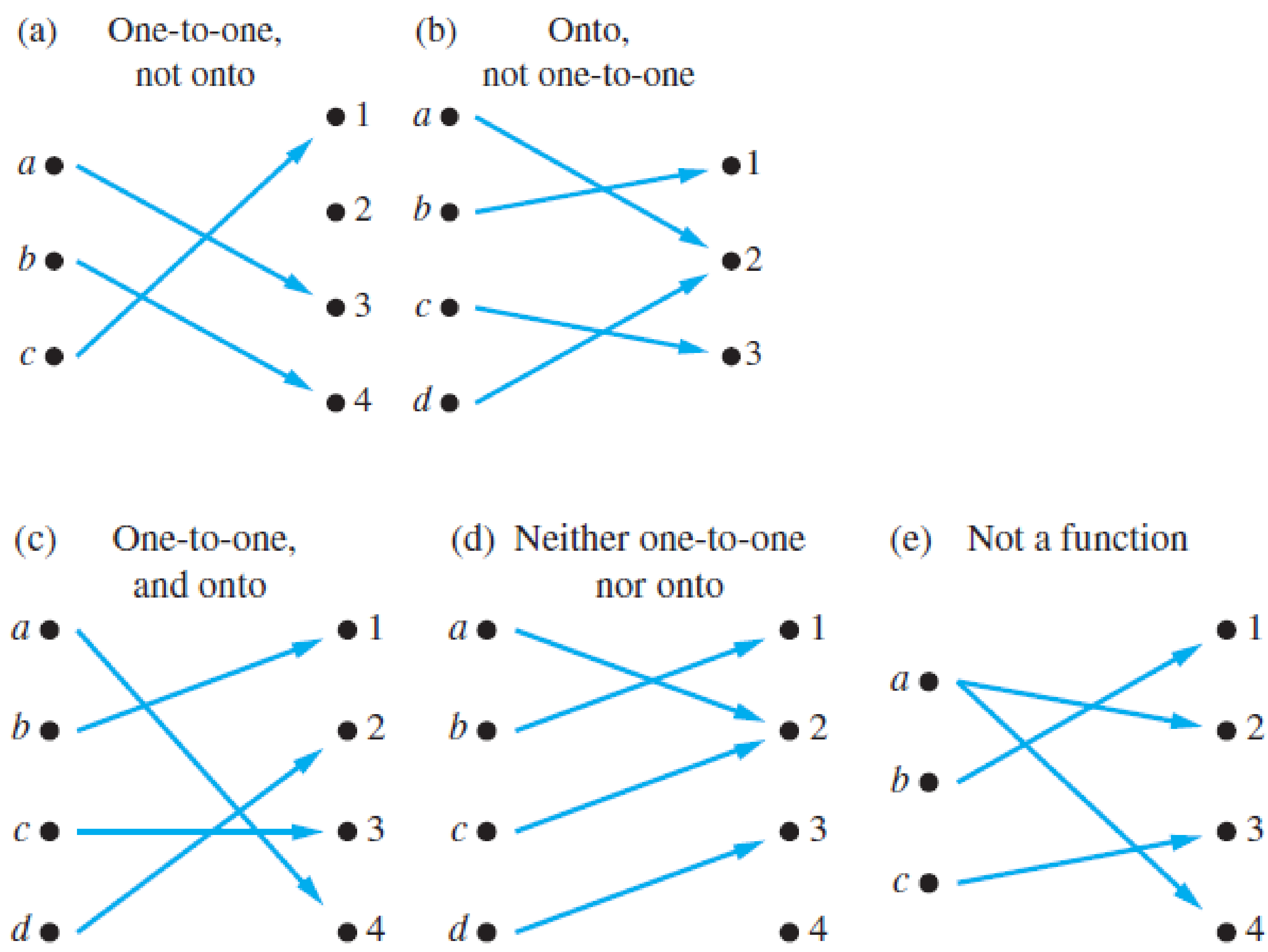
One-to-one or onto

- **Example**: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

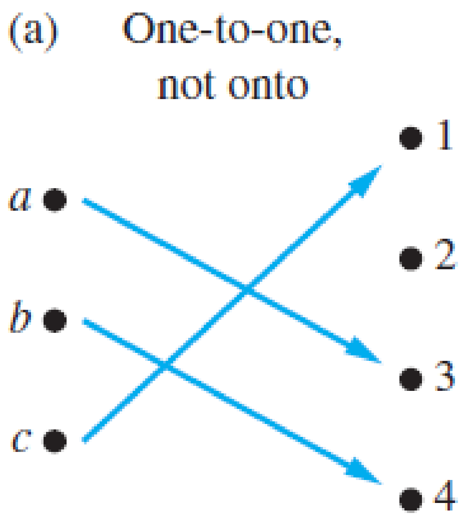
Solution: No, f is not onto because there is no integer x with $x^2 = -1$.



Different Types of Correspondences

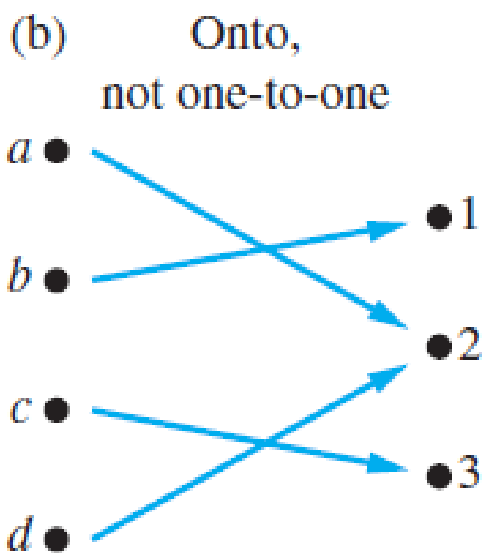


Different Types of Correspondences



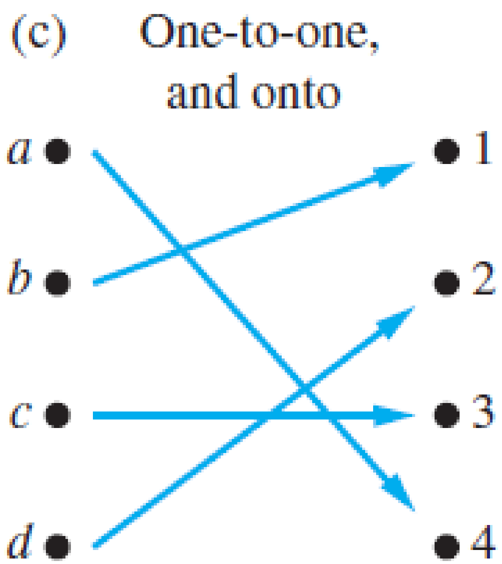
- It is One-to-One since each domain element is assigned with a different codomain element.
- It is not Onto since codomain element 2 is not assigned to any domain element.

Different Types of Correspondences



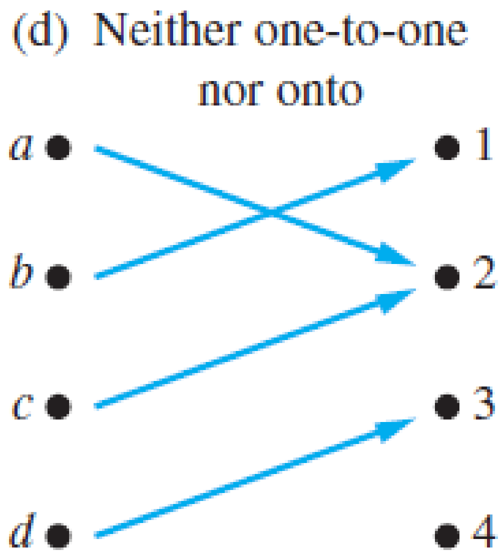
- It is Onto since each of codomain element is assigned to a domain element.
- It is Not One-to-One since codomain element 2 is assigned to two domain elements (*a* and *d*).

Different Types of Correspondences



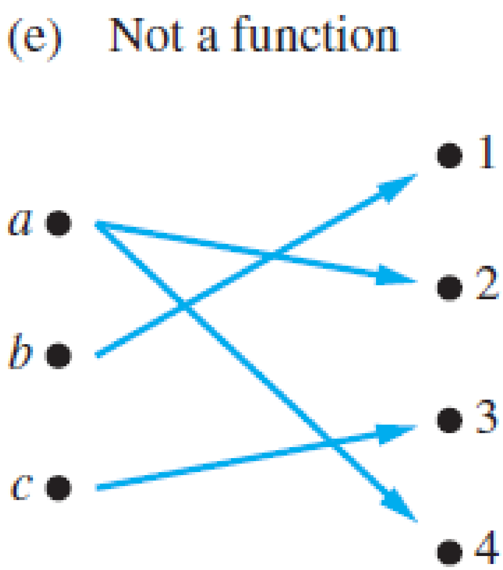
- It is One-to-One since each domain element is assigned with a different codomain element.
- It is Onto since each of codomain element is assigned to a domain element.

Different Types of Correspondences



- It is not One-to-One since codomain element 2 is assigned to two domain elements (*a* and *c*).
- It is not Onto since codomain element 4 is not assigned to any domain element.

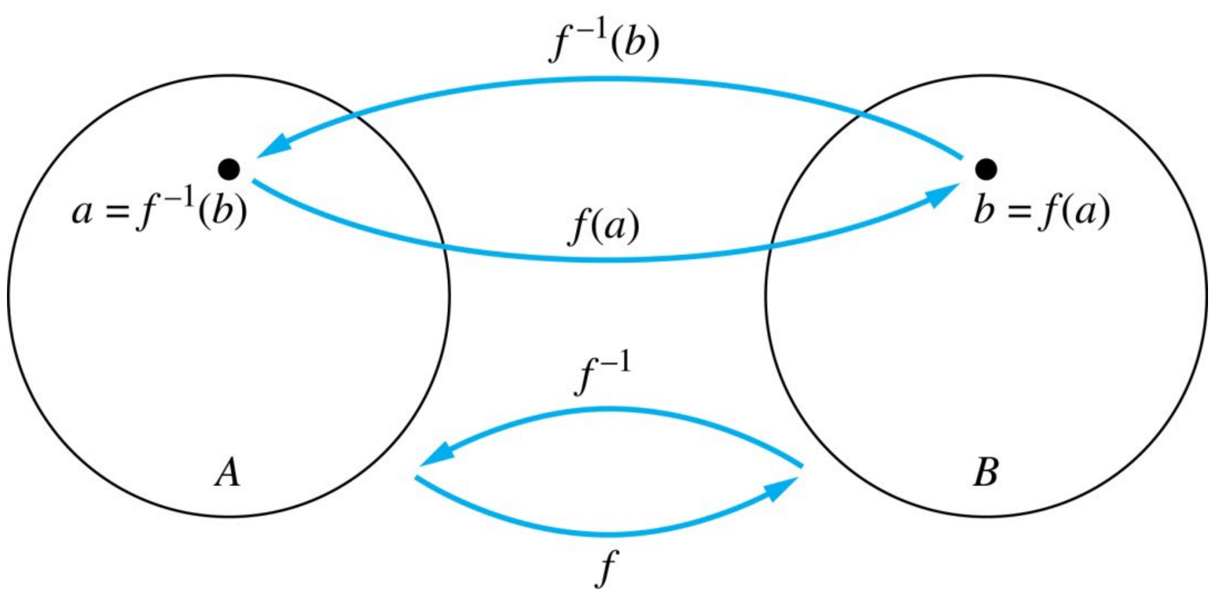
Different Types of Correspondences



- It is not function. In a function, each domain element must be assigned to **exactly one** unique codomain element. In this case, domain element *a* is assigned to two codomain element (2 and 4).

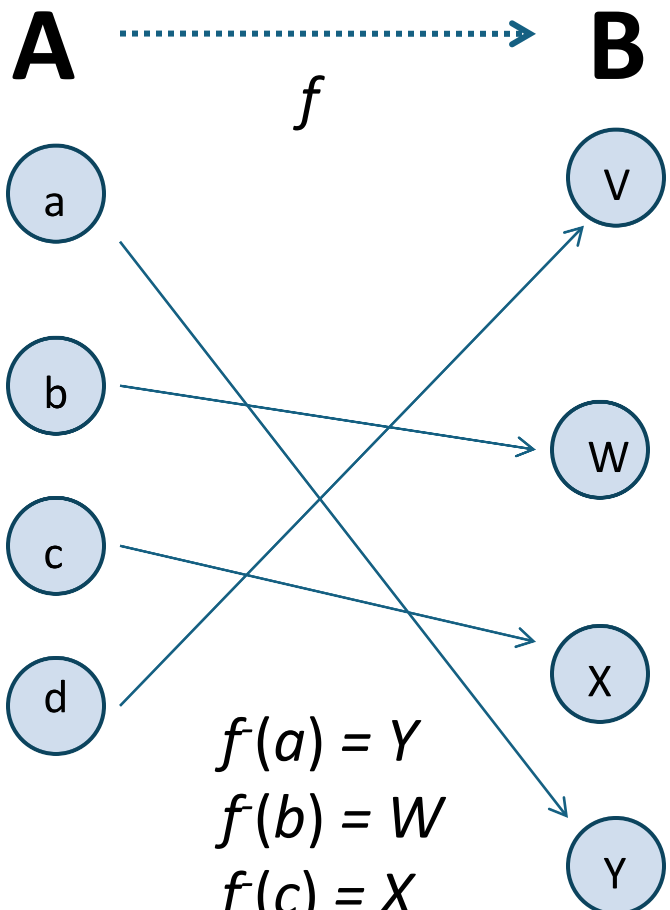
Inverse Functions

- **Definition**: Let *f* be a **bijection** from *A* to *B*. Then the *inverse* of *f*, denoted f^{-1} , is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$.
  - Interchange the domain and the codomain.
  - A bijection is also called an **invertible** function.

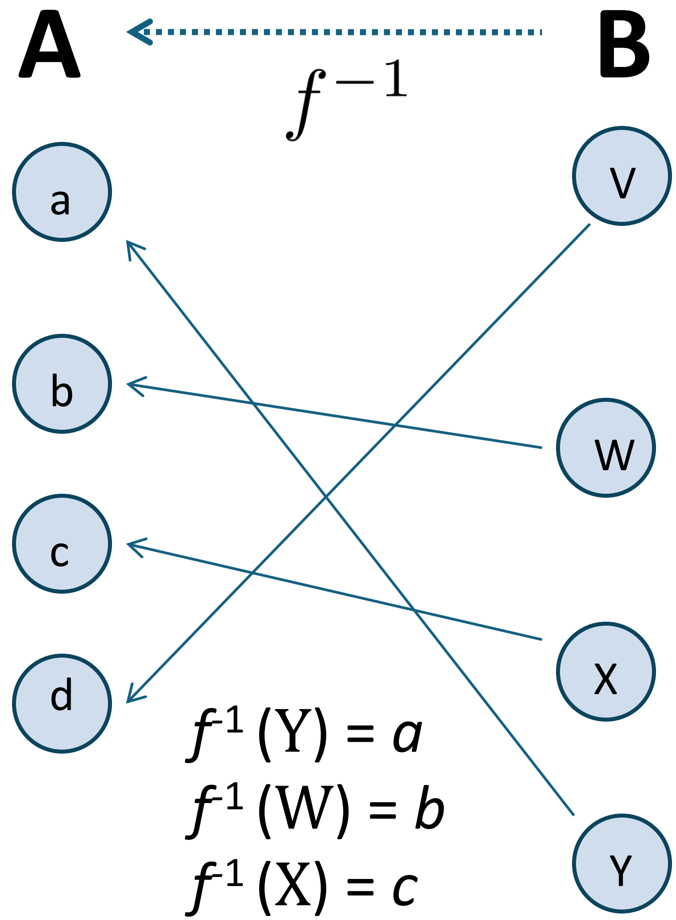


Inverse Functions

- f is a bijection.



$f(a) = Y$
$f(b) = W$
$f(c) = X$
$f(d) = V$



$f^{-1}(Y) = a$
$f^{-1}(W) = b$
$f^{-1}(X) = c$
$f^{-1}(V) = d$

Inverse Functions

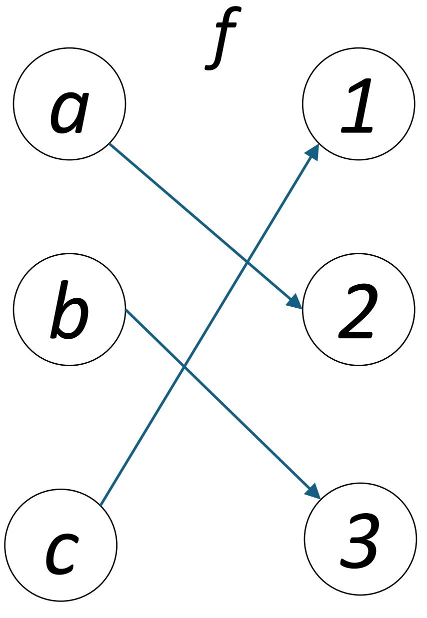
- **Example**: Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible and if so, what is its inverse?

Solution: f is a bijection, therefore, it is invertible.

$f^{-1}(1) = c$

$f^{-1}(2) = a$

$f^{-1}(3) = b$



Inverse Functions

- The inverse function of a function f (also called the inverse of f) is a function that **undoes** the operations of f.
- For a function *f*: A → B, its inverse f^{-1}: B → A sends each element *b* ∈ B to the **unique** element *a* ∈ A such as *f*(*a*) = *b*.
- **How to undo the operations of a function *f* ?**
  - Addition undoes subtraction
  - Division undoes multiplication

Inverse Functions

- **Example**: Let $f: \mathbf{Z} \rightarrow \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if so, what is its inverse?

Solution:

Step 1: f is a bijection, therefore, it is invertible.

Step 2: Undo the operation(s) of function f .

Undo the addition in $f(x) = x + 1$.

Let $y = f(x)$. Therefore, $y = x + 1$.

Undo the addition with subtraction: $y - 1$.

The inverse function of f is $f^{-1}(y) = y - 1$ or $f^{-1}(x) = x - 1$.

Note: the choice of variable is arbitrary, we choose y something different than x as this is already involved in the function. $f^{-1}(y) = y - 1$ can be written as $f^{-1}(x) = x - 1$.

Inverse Functions

- **Example**: Let f: **Z** → **Z** be such that $f(x) = 5x - 7$. Is f invertible, and if so, what is its inverse?

Solution:

Step 1: f is a bijection, therefore, it is invertible.

Step 2: Undo the operation(s) of function f .

Undo the subtraction and the multiplication in $f(x) = 5x – 7$.

Let $y = f(x)$. Therefore, $y = 5x - 7$.

Undo the subtraction with addition: $y + 7$.

Undo the multiplication with division: $\frac{y+7}{5}$

The inverse function of f is $f^{-1}(y) = \frac{y+7}{5}$ or $f^{-1}(x) = \frac{x+7}{5}$.

Inverse Functions

- Different way of thinking about "Undoing operations in *f*: Solving for *x*.

Let $y = f(x)$. Therefore, $y = 5x - 7$.

To determine $f^{-1}(y)$ for y, we must find the unique x such that $y = 5x - 7$. Let's **<u>solve $y = 5x - 7$ for x.</u>**

Add 7 to both sides: $y + 7 = 5x - 7 + 7$ becomes $y + 7 = 5x$.

Divide both sides by 5: $\frac{y+7}{5} = \frac{5x}{5}$ becomes $\frac{y+7}{5} = x$.

The inverse function of f is $f^{-1}(y) = \frac{y+7}{5}$ or $f^{-1}(x) = \frac{x+7}{5}$.

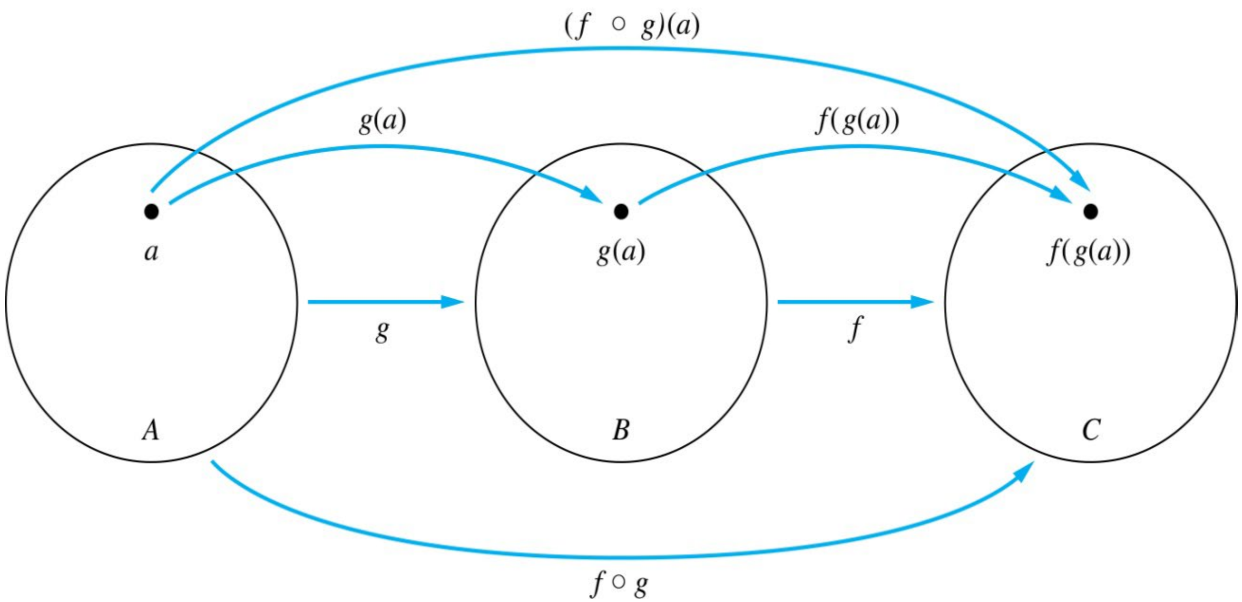
Inverse Functions

- **Example**: Let f: **R** $\rightarrow$ **R** be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is not one-to-one correspondence or bijective.

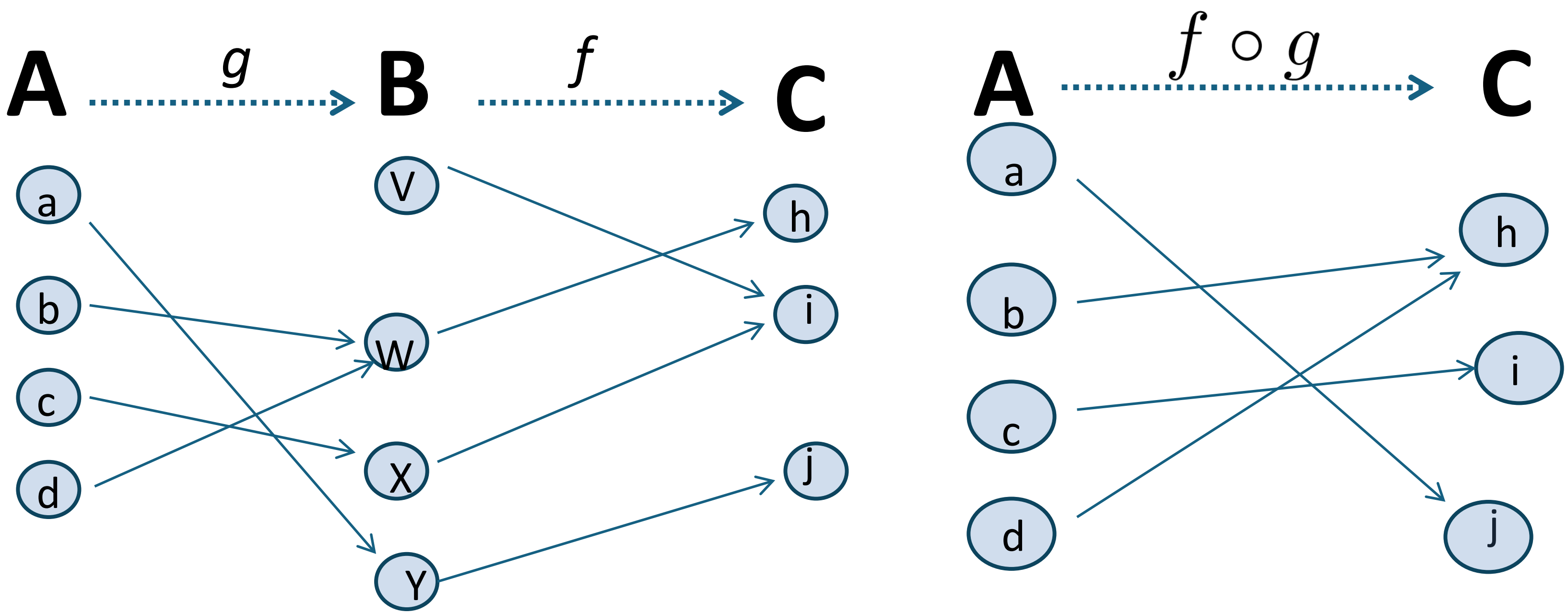
Composition

- **Definition**: Let g: $A \to B$, f: $B \to C$. The *composition* of f with g, denoted $f \circ g$ is the function from A to C defined by $f \circ g(x) = f(g(x))$.
  - Read as f of g of x.
  - The new function is composed of the two given functions f and g, where one function is substituted into the other.



Composition

- The composition $f \circ g$ cannot be defined **unless the *range* of *g* is a subset of the domain of *f***.
  - The *range* of *g*: {W, X, Y}
  - The domain of *f*: {V, W, X, Y}
  - $\{W, X, Y\} \subseteq \{V, W, X, Y\}$



Composition

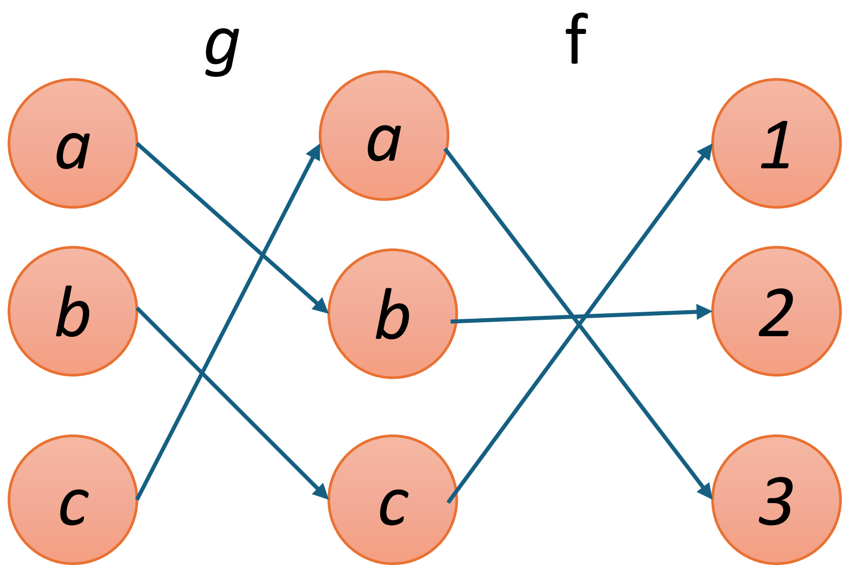
- **Example**: $f(x) = x^2$ and $g(x) = 2x + 1$. what is $f \circ g = f(g(x))$? What is $g \circ f = g(f(x))$?

  Solution:
  The *range* of g is a subset of the domain of f .
  $f \circ g$ is defined. $f(g(x)) = (2x + 1)^2$.

  The *range* of f is a subset of the domain of g .
  $g \circ f$ is defined. $g(f(x)) = 2x^2 + 1$.

Composition

- **Example**: Let $g : \{a, b, c\} \rightarrow \{a, b, c\}$ be such that $g(a) = b$, $g(b) = c$, and $g(c) = a$. Let $f: \{a, b, c\} \rightarrow \{1, 2, 3\}$ such that $f(a) = 3$, $f(b) = 2$, and $f(c) = 1$. What is the composition of f with g, and what is the composition of g with f?



Composition

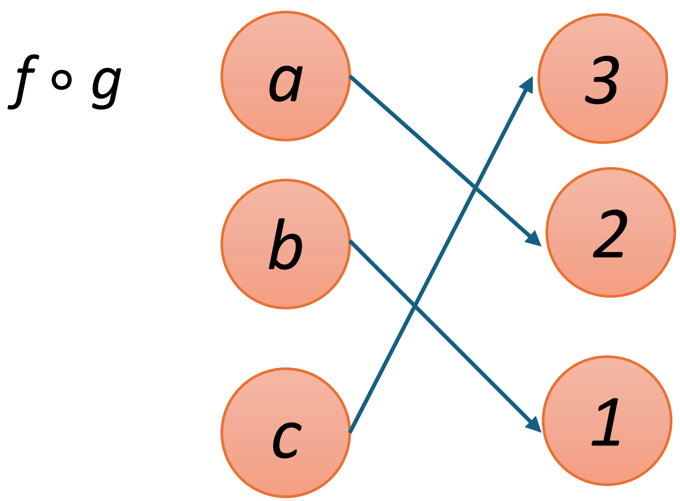
Solution:

The *range* of g is a subset of the domain of f. $f \circ g$ is defined.

$$f \circ g(a) = f(g(a)) = f(b) = 2.$$

$$f \circ g(b) = f(g(b)) = f(c) = 1.$$

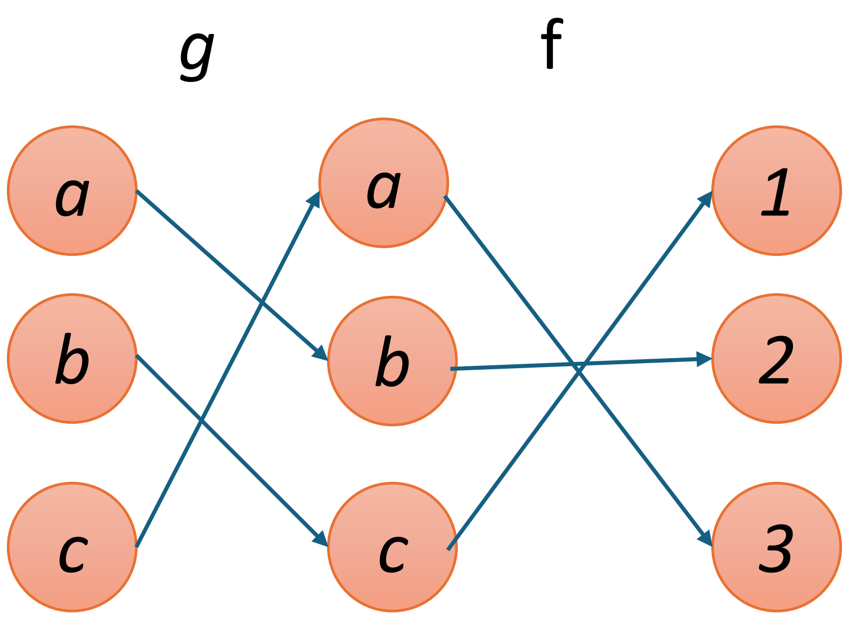
$$f \circ g(c) = f(g(c)) = f(a) = 3.$$



Composition

Solution cont. :

The *range* of f is a subset of the domain of g . $g \circ f$ is not defined.



Composition

- **Example**: Let f and g be functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f with g and the composition of g with f ?

Solution:

The *range* of g is a subset of the domain of f . $f \circ g$ is defined.

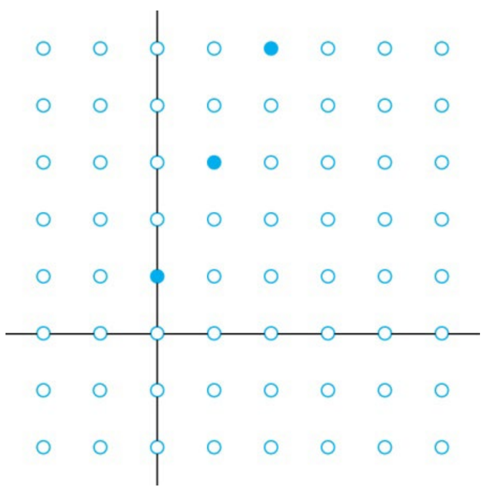
$$f \circ g\ (x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

The *range* of f is a subset of the domain of g . $g \circ f$ is defined.

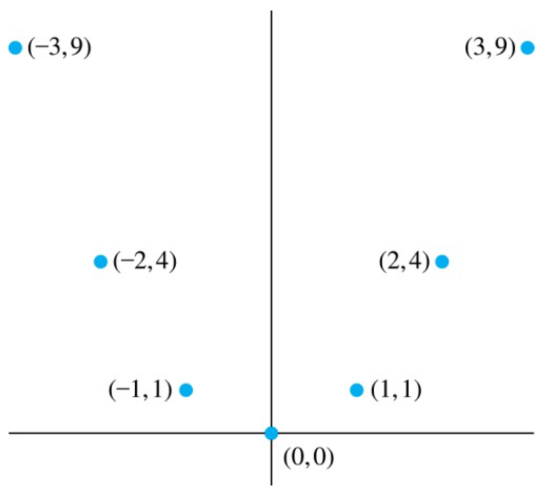
$$g \circ f(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$$

Graphs of Functions

- The graph of the function $f{:}A \rightarrow B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } f(a) = b\}$.



$f(n) = 2n + 1$
from Z to Z



$f(x) = x^2$
from Z to Z

floor & ceiling

- The *floor* function, $f(x) = \lfloor x \rfloor$, returns the largest integer less than or equal to *x*.

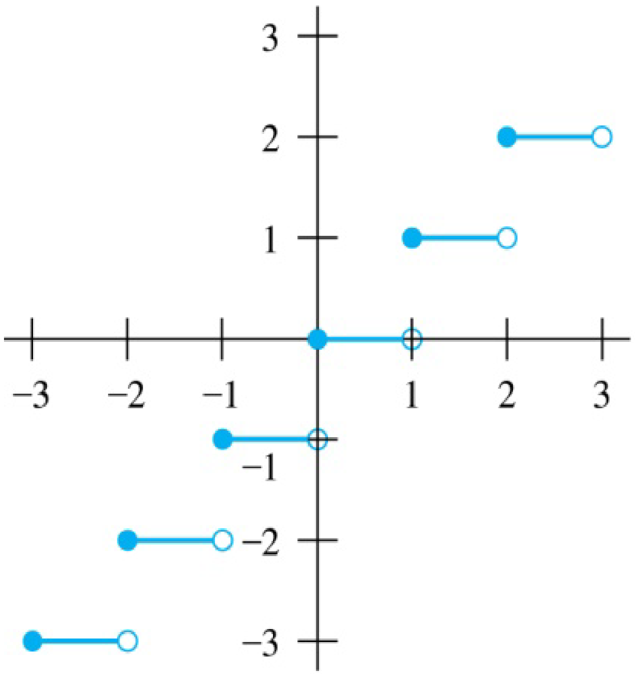
$$\lfloor 3.5 \rfloor = 3 \quad \lfloor -1.5 \rfloor = -2$$

- The *ceiling* function, $f(x) = \lceil x \rceil$ returns the smallest integer greater than or equal to *x.*

$$\lceil 3.5 \rceil = 4 \quad \lceil -1.5 \rceil = -1$$

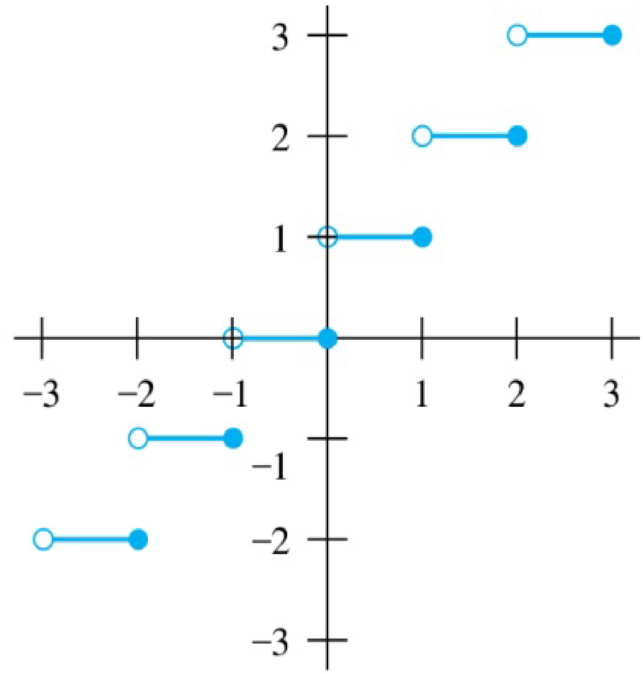
- What is the floor of 3? What is the ceiling of 3?

Floor and Ceiling Functions



(a) $y = [x]$

Floor



(b) $y = [x]$

Ceiling

Factorial Function

- **Definition:** f: $\mathbf{N} \rightarrow \mathbf{Z}^+$ **,** denoted by $f(n) = n!$, is the product of the first n **positive** integers when n is a **nonnegative** integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

- **Examples:**

$$f(1) = 1! = 1$$

$$f(2) = 2! = 1 \cdot 2 = 2$$

$$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$$

$$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$$

Stirling's Formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

$$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$$

$\sim$ "is asymptotic to"

Sequences and Summations

Section 2.4

Introduction

- Sequences are **ordered** lists of elements.
  - 1,2,3,5,8
  - 1,3,9,27,81,….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from **b**otany to music.

Sequences

- **Definition**: A *sequence* is a **function** from a subset of the integers A to a set S.

  f**:** $A \rightarrow S$

  - A: {0, 1, 2, 3, 4, …..} - indexes of the terms
  - S: $\{a_{0,} a_{1,} a_{3,} a_{4, \ldots..}\ \}$ - the terms

Sequences

- **Examples**: Consider the following sequences:

0 1 2 3 4 Indexes

$\{a_n\} = 1, 2, 3, 5, 8$

0 1 2 3 4 5 Indexes

$\{a_n\} = 1, 3, 9, 27, 81, 243....$

1 2 3 4 Indexes starting at 1

$\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

Geometric Progression

- **Definition**: A *geometric progression* is a sequence of the form: $a, ar, ar^2, \ldots, ar^n, \ldots$ where the *initial term a* and the *common **ratio** r* are real numbers.
- Grows exponentially

Geometric Progression

- **Examples**:
  - $a = 1$ and $r = -1$:

    $$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

  - $a = 2$ and $r = 5$:

    $$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

  - $a = 6$ and $r = 1/3$:

    $$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Arithmetic Progression

- **Definition**: An *arithmetic progression* is a sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$ where the *initial term a* and the *common* ***difference*** *d* are real numbers.

Arithmetic Progression

- **Examples**:
  - Let *a* = −1 and *d* = 4:

    $$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

  - Let *a* = 7 and *d* = −3:

    $$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

  - Let *a* = 1 and d = 2:

    $$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

- **Definition:** A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a ***solution*** of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Recurrence Relations

- **Example** : Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that the initial condition $a_0 = 2$. What are a_1, a_2 and a_3?

Solution:

$a_n = a_{n-1} + 3$ means a term is the sum of the previous term and 3.

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = a_1 + 3 = 5 + 3 = 8$$
$$a_3 = a_2 + 3 = 8 + 3 = 11$$

{5, 8, 11, 14, …}

Recurrence Relations

- **Example** : Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that the initial conditions are $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

Solution:

$a_n = a_{n-1} - a_{n-2}$ means a term is its predecessor minus the predecessor of the predecessor.

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

$\{2, -3, -5, -2, \ldots\}$

Fibonacci Sequence

- **Example:** The *Fibonacci* sequence, $f_0, f_1, f_2, \ldots,$ is defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$. The Initial Conditions are $f_0 = 0$, $f_1 = 1$. Find f_2, f_3, f_4, f_5 and f_6.

Solution:

$f_n = f_{n-1} + f_{n-2}$ means a term is the sum of the previous two terms.

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

$\{1,2,3,5,8,13, \ldots\}$

Solving Recurrence Relations

- Finding a formula (*closed formula*) for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Among many methods, a straightforward method is known as *iteration*.
- An iteration method can be executed using
  - Forward substitution (Working upward)
    - Start with the lowest term and work toward to the *n*th term.

    OR
  - Backward substitution(Working downward)
    - Start with the *n*th term and work toward to lowest term.

Solving Recurrence Relations

- **Example**: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$. Solve the recurrence relation by finding a closed formula for the *n*th term of the sequence.

Solving Recurrence Relations

- **Solution 1**: Use forward substitution (Working upward).

$$a_2 = 2 + 3 = 2 + 3 \cdot 1$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$.$$

$$.$$

$$.$$

$$a_n = a_{n\text{-}1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

$$= 3n - 1$$

Solving Recurrence Relations

- **Solution 2**: Use backward substitution (Working downward).

$$a_n = a_{n\text{-}1} + 3 = a_{n\text{-}1} + 3 \cdot 1$$
$$= (a_{n\text{-}2} + 3) + 3 = a_{n\text{-}2} + 3 \cdot 2$$
$$= (a_{n\text{-}3} + 3\,) + 3 \cdot 2 \;= a_{n\text{-}3} + 3 \cdot 3$$
$$.$$
$$.$$
$$.$$
$$= a_2 \;+ 3(n-2)$$
$$= (a_1 \;+ 3) + 3(n-2) \;= a_1 \;+ 3(n-1)$$
$$= 2 + 3(n-1) = 3n \text{ -}1$$

Financial Application

- **Example**: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after 30 years. P_n satisfies the recurrence relation $P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$ with the initial condition $P_0 = 10{,}000$.

Financial Application

Solution cont. : Use Forward Substitution to solve.

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2 P_0$

$P_3 = (1.11)P_2 = (1.11)^3 P_0$

$\vdots$

$P_n = (1.11)P_{n-1} = (1.11)^n P_0 = (1.11)^n\, 10{,}000$

$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

Summations

- Sum of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$. The indexes run from *m* to *n*.
- The notation:

$$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{m \leq j \leq n} a_j$$

- The variable *j* is called the *index of summation*.
  - *m*: the *lower limit*
  - *n*: *the upper limit*

Summations

- **Examples**: Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$.

  Solution:

  The lower limit for the index of summation is 1, and the upper limit is 100.

$$\sum_{j=1}^{100} \frac{1}{j}$$

Summations

- **Examples**: What is the value of $\sum_{j=1}^{5} j^2$?

Solution: Add the terms from index 1 to 5.

$$\begin{aligned}\sum_{j=1}^{5} j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55.\end{aligned}$$

Summations

- The indexes of the terms that will be summed can be stored as a set *S. j* is an element of the set *S.*

$$\sum_{j \in S} a_j$$

- **Example**:

$$\text{If } S = \{2, 5, 7, 10\} \text{ then } \sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$$

Add the terms at the indexes: 2, 5, 7, and 10.

Summations

- When two summations' indexes do not match,
  - shift the index of one summation in a sum (usually the one whose indexes don't start with "0".
  - And then, add the two sums.

- **Example:**

$$\sum_{i=0}^{4} \boldsymbol{i^2} + \sum_{j=1}^{5} \boldsymbol{j^2}$$

Summations

Solution:

Shift the indexes of $\sum_{j=1}^{5} j^2$ from 1 to 5 to o to 4.

let $k = j - 1$.

$k = 1 - 1 = 0$ when $j = 1$.

$k = 5 - 1 = 4$ when $j = 5$.

j^2 becomes $(k + 1)^2$.

Then the new summation index runs from 0 to 4.

$$\sum_{j=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2.$$

Summation Formulae

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$ Geometric Series

$$\sum_{k=1}^{n} k \qquad \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 \qquad \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 \qquad \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=0}^{\infty} x^k, |x| < 1 \qquad \frac{1}{1-x}$$

$$\sum_{k=1}^{\infty} kx^{k-1}, |x| < 1 \qquad \frac{1}{(1-x)^2}$$

Summation Formulae

- **Example**: Find $\sum_{k=50}^{100} k^2$?

  Solution: Use subtraction technique.

  Since $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$,

  therefore, $\displaystyle\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$.

  Use the formula: $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$.

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925.$$

Double Summations

- To evaluate the double sum, $\sum_{i=1}^{4}\sum_{j=1}^{3} ij$, first expand the inner summation and then continue by computing the outer summation.
    - like a nested loop in computer program
- For example,

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{3} ij &= \sum_{i=1}^{4}(i + 2i + 3i)\\ &= \sum_{i=1}^{4} 6i\\ &= 6 + 12 + 18 + 24 = 60.\end{aligned}$$

Matrices

Section 2.6

Matrices

- Matrices, useful discrete structures, can be used to:
  - describe certain types of functions known as linear transformations.
  - Express which vertices of a graph are connected by edge.
- Matrices are used to build models of transportation systems and communication networks.

Matrix

- **Definition**: A *matrix* is a rectangular array of numbers. A matrix with m rows and n columns is called a m x n matrix.
  - A matrix with the same number of rows as columns is called *square*.
  - Two matrices are *equal* if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \qquad 3 \text{ x } 2 \text{ matrix}$$

Matrix

- Let *m* and *n* be positive integers and let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- The (i,j)th *element* or *entry* of **A** is the element a_{ij}.
- We can use $\mathbf{A} = [a_{ij}]$ to denote the matrix with its (i,j)th element equal to a_{ij}.

Matrix

- The *i*th row of **A** is the $1 \times n$ matrix.

$$[a_{i1}, a_{i2}, \ldots, a_{in}]$$

- The *j*th column of **A** is the $m \times 1$ *matrix:*

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$$

Matrix Arithmetic: Addition

- **Definition**: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be m x n matrices. **A** + **B**, the sum of **A** and **B**, is the m x n matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element.
  - $A + B = [a_{ij} + b_{ij}]$
  - A and B must be matrices of same size.
- For example,

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Matrix Arithmetic: Multiplication

- **Definition**: Let **A** be an m x k matrix and **B** be a k x n matrix. **AB** , the *product* of **A** and **B**, is the m x n matrix that has its (i,j)th element equal to the sum of the products of the corresponding elements from **the** ith row of **A** and the jth column of **B**.
  - if **AB** = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1\mathrm{j}} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.
- **The number of columns in the first matrix A must be the same as the number of rows in the second matrix B.**

Matrix Arithmetic: Multiplication

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ . & . & & . \\ . & . & & . \\ a_{i1} & a_{i2} & \ldots & a_{ik} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \ldots & a_{mk} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & a_{12} & \ldots & b_{1j} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2n} \\ . & . & & . & & \\ . & . & & . & & \\ b_{k1} & b_{k2} & \ldots & b_{kj} & \ldots & b_{kn} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ . & . & & . \\ . & . & c_{ij} & . \\ . & . & & . \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix}$$

Matrix Arithmetic: Multiplication

- **Example:**

$$\underset{\mathbf{A}}{\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix}} \underset{\mathrm{B}}{\begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$$

1 x 2 + 0 x 1 + 4 x 3 = 14
1 x 4 + 0 x 1 + 4 x 0 = 4

2 x 2 + 1 x 1 + 1 x 3 = 8
2 x 4 + 1 x 1 + 1 x 0 = 9

3 x 2 + 1 x 1 + 0 x 3 = 7
3 x 4 + 1 x 1 + 0 x 0 = 13

0 x 2 + 2 x 1 + 2 x 3 = 8
0 x 4 + 2 x 1 + 2 x 0 = 2

AB ≠ BA (Not Commutative)

- **Example**: Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, is **AB = BA**?

Solution:

AB **=** $\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ and BA **=** $\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$.

AB $\neq$ BA

Identity Matrix

- **Definition**: The *identity matrix* of order n is the m x n matrix $\mathbf{I}_n$ = $[\delta_{ij}]$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

$$\mathbf{I_n} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

$AI_n = I_m A = A$ when $\mathbf{A}$ is an m x n matrix.

Powers of Matrices

- Powers of square matrices can be defined. When A is an $n \times n$ matrix, we have:

$$\mathbf{A}^0 = \mathbf{I}_n \qquad \mathbf{A}^r = \underbrace{\mathbf{AAA}\cdots\mathbf{A}}_{\text{r times}}$$

Transposes of Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an m x n matrix. The *transpose* of **A**, denoted by $\mathbf{A}^t$,is the n x m matrix obtained by interchanging the rows and columns of **A**.
  - The ith row in a matrix becomes ith column in its transpose of the matrix.
  - If $\mathbf{A}^t = [b_{ij}]$, then $b_{ij} = a_{ji}$ for $i = 1,2,\ldots,n$ and $j = 1,2, \ldots,m$.
  - For example, the transpose of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$.

Transposes of Matrices

- **Definition**: A square matrix **A** is called symmetric if $\mathbf{A} = \mathbf{A}^t$. Thus **A** = $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$.
  - Square matrices do not change when their rows and columns are interchanged.
  - For example, the following matrix is a square.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Zero-One Matrices

- **Definition**: A matrix all of whose entries are either 0 or 1 is called a *zero-one matrix*.
- Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

Zero-One Matrices

- **Definition**: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be an $m \times n$ zero-one matrices.
  - The *join* of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. The *join* of **A** and **B** is denoted by $\mathbf{A} \vee \mathbf{B}$.
  - The meet of of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. The *meet* of **A** and **B** is denoted by $\mathbf{A} \wedge \mathbf{B}$.

Set Operators		**Logical connectives**	**Boolean Algebra**
Union	∪	∨ (Disjunction, OR)	Join, + (Sum)
Intersection	∩	∧ (Conjunction, AND)	Meet, • (Product)
Complement	-	¬ (Negation, NOT)	

Join and Meet

- **Example**: Find the join and meet of the zero-one matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: The join of **A** and **B** is

$$\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The meet of **A** and **B** is

$$\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Boolean Product of Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an m x k zero-one matrix and **B** = $[b_{ij}]$ be a k x n zero-one matrix. The *Boolean product* of **A** and **B**, denoted by **A** ⊙ **B**, is the m x n zero-one matrix with(i,j)th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

Boolean Product of Matrices

- **Example**: The Boolean product $\mathbf{A} \odot \mathbf{B}$ is given by

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Boolean Powers of Matrices

Definition: Let **A** be a square zero-one matrix and let *r* be a positive integer. The *r*th Boolean power of **A** is the Boolean product of *r* factors of **A**, denoted by $\mathbf{A}^{[r]}$. Hence,

$$\mathbf{A}^{[r]} = \underbrace{\mathbf{A} \odot \mathbf{A} \odot \ldots \odot \mathbf{A}}_{r \text{ times}}.$$

We define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.

Boolean Powers of Zero-One Matrices

- **Example**: Find **A**^{n} for all positive integers *n with n* ≤ 5.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution:

$$\mathbf{A}^{[2]} = \mathbf{A} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{A}^{[3]} = \mathbf{A}^{[2]} \odot \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{A}^{[4]} = \mathbf{A}^{[3]} \odot \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \mathbf{A}^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$