CHAPTER 18

Binomial Trees in Practice

As we have seen in Chapters 13, 15, and 16, the Black–Scholes–Merton model and its extensions can be used to value European call and put options on stocks, stock indices, currencies, and futures contracts. For American options we rely on binomial trees. In this chapter we cover, more completely than in Chapter 12, how binomial trees are used in practice. In particular, we explain how the binomial tree methodology can be used to value American options on a range of different underlying assets, including dividend-paying stocks, and how it can be used to calculate the Greek letters that were introduced in Chapter 17. The DerivaGem software that accompanies this book can be used to carry out the calculations described in the chapter and to display the binomial trees that are used.

18.1 THE BINOMIAL MODEL FOR A NON-DIVIDEND-PAYING STOCK

The binomial tree methodology for handling American-style options was proposed by Cox, Ross, and Rubinstein in 1979.[1] Consider the evaluation of an option on a non-dividend-paying stock. We start by dividing the life of the option into a large number of small time intervals of length Δt. We assume that in each time interval the stock price moves from its initial value of S to one of two new values, Su and Sd. This model is illustrated in Figure 18.1. In general, $u > 1$ and $d < 1$. The movement from S to Su is, therefore, an "up" movement and the movement from S to Sd is a "down" movement. The probability of an up movement will be denoted by p. The probability of a down movement is $1 - p$.

Risk-Neutral Valuation

The risk-neutral valuation principle, discussed in Chapters 12 and 13, states that any security dependent on a stock price can be valued on the assumption that the world is

[1] See J. C. Cox, S.A. Ross, and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, 7 (October 1979), 229–64.

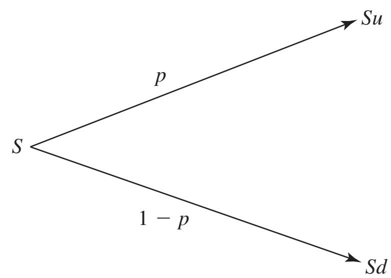


Figure 18.1 Stock price movements in time Δt under the binomial model

risk neutral. This means that, for the purposes of valuing an option (or any other derivative), we can assume the following:

1. The expected return from all traded securities is the risk-free interest rate.
2. Future cash flows can be valued by discounting their expected values at the risk-free interest rate.

We will make use of this result when using a binomial tree.

Determination of *p, u,* and *d*

We design the tree to represent the behavior of a stock price in a risk-neutral world. The parameters p, u, and d must give correct values for the mean and variance of the stock price return during a time interval Δt in this world. The expected return from a stock is the risk-free interest rate, r. Hence, the expected value of the stock price at the end of a time interval Δt is $Se^{r\Delta t}$, where S is the stock price at the beginning of the time interval. To match the mean stock price return with the tree, we therefore need

$$Se^{r\Delta t} = pSu + (1-p)Sd \tag{18.1}$$

or

$$e^{r\Delta t} = pu + (1-p)d \tag{18.2}$$

The variance of a variable Q is defined as $E(Q^2) - E(Q)^2$, where E denotes expected value. Defining R as the proportional change in the asset price in time Δt, there is a probability p that $1 + R$ is u and a probability $1 - p$ that $1 + R$ is d. It follows that the variance of $1 + R$ is $pu^2 + (1 - p)d^2 - [pu + (1 - p)d]^2$. Because adding a constant to a variable makes no difference to its variance, the variance of $1 + R$ is the same as the variance of R. As explained in Section 13.1, this is $\sigma^2\Delta t$. Hence,

$$\sigma^2\Delta t = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 \tag{18.3}$$

Equations (18.2) and (18.3) impose two conditions on p, u, and d. A third condition used by Cox, Ross, and Rubinstein is

$$u = \frac{1}{d}$$

When Δt is small, equations (18.2), (18.3), and this equation are satisfied by

$$p = \frac{a-d}{u-d} \tag{18.4}$$

$$u = e^{\sigma\sqrt{\Delta t}} \tag{18.5}$$

$$d = e^{-\sigma\sqrt{\Delta t}} \tag{18.6}$$

where

$$a = e^{r\Delta t} \tag{18.7}$$

The variable a is sometimes referred to as the *growth factor*. Equations (18.4) to (18.7) are the same as equations (12.11) to (12.14) in Chapter 12.

The Tree of Stock Prices

Figure 18.2 shows the complete tree of stock prices that is considered when the binomial model is used and there are four time steps. At time zero, the stock price, S_0, is known. At time Δt, there are two possible stock prices, S_0u and S_0d; at time $2\Delta t$, there are three possible stock prices, S_0u^2, S_0, and S_0d^2; and so on. In general, at time $i\,\Delta t$, we consider $i+1$ stock prices. These are

$$S_0u^j d^{i-j} \quad (j = 0, 1, \dots, i)$$

Note that the relationship $u = 1/d$ is used in computing the stock price at each node of the tree in Figure 18.2. For example, the asset price when $j = 2$ and $i = 3$ is $S_0u^2d = S_0u$. Note also that the tree recombines in the sense that an up movement followed by a down movement leads to the same stock price as a down movement followed by an up movement.

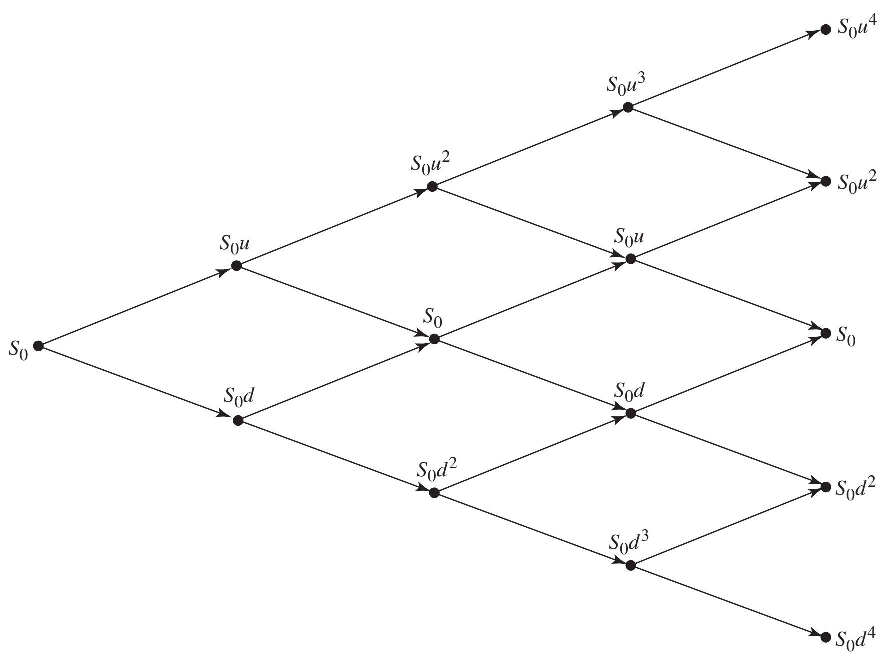


Figure 18.2 Tree used to value a stock option

Working Backward through the Tree

Options are evaluated by starting at the end of the tree (time T) and working backward, a procedure known as *backward induction*. The value of the option is known at time T. For example, a put option is worth $\max(K - S_T, 0)$ and a call option is worth $\max(S_T - K, 0)$, where S_T is the stock price at time T and K is the strike price. Because a risk-neutral world is assumed, the value at each node at time $T - \Delta t$ can be calculated as the expected value at time T discounted at rate r for a time period Δt. Similarly, the value at each node at time $T - 2\Delta t$ can be calculated as the expected value at time $T - \Delta t$ discounted for a time period Δt at rate r, and so on. If the option is American, it is necessary to check at each node to see whether early exercise is preferable to holding the option for a further time period Δt. Eventually, by working back through all the nodes, we obtain the value of the option at time zero.

Illustration

We now illustrate the procedure with a five-step tree. Consider an American put option on a non-dividend-paying stock when the stock price is \$50, the strike price is \$50, the risk-free interest rate is 10% per annum, the life is five months, and the volatility is 40% per annum. With our usual notation, this means that $S_0 = 50$, $K = 50$, $r = 0.10$, $\sigma = 0.40$, and $T = 0.4167$. We divide the life of the option into five intervals of length one month for the purposes of constructing a binomial tree. Then $\Delta t = 1/12$ and, using equations (18.4) to (18.7),

$$u = e^{\sigma\sqrt{\Delta t}} = 1.1224, \qquad d = e^{-\sigma\sqrt{\Delta t}} = 0.8909, \qquad a = e^{r\Delta t} = 1.0084$$

$$p = \frac{a-d}{u-d} = 0.5073, \qquad 1 - p = 0.4927$$

Figure 18.3 shows the binomial tree (produced using DerivaGem). At each node there are two numbers. The top one shows the stock price at the node; the lower one shows the value of the option at the node. The probability of an up movement is always 0.5073; the probability of a down movement is always 0.4927.

The stock price at the jth ($j = 0, 1, \ldots, i$) node at time $i\,\Delta t$ ($i = 0, 1, \ldots, 5$) is calculated as $S_0 u^j d^{i-j}$. For example, the stock price at node A ($i = 4, j = 1$) (i.e., the second node up at the end of the fourth time step) is $50 \times 1.1224 \times 0.8909^3 = \$39.69$.

The option prices at the final nodes are calculated as $\max(K - S_T, 0)$. For example, the option price at node G is $50.00 - 35.36 = 14.64$. The option prices at the penultimate nodes are calculated from the option prices at the final nodes. First, we assume no exercise of the option at a node. This means that the option price is calculated as the present value of the expected option price one time step later. For example, at node E, the option price is calculated as

$$(0.5073 \times 0 + 0.4927 \times 5.45)e^{-0.10 \times 1/12} = 2.66$$

whereas at node A it is calculated as

$$(0.5073 \times 5.45 + 0.4927 \times 14.64)e^{-0.10 \times 1/12} = 9.90$$

We then check to see if early exercise is preferable to waiting. At node E, early exercise

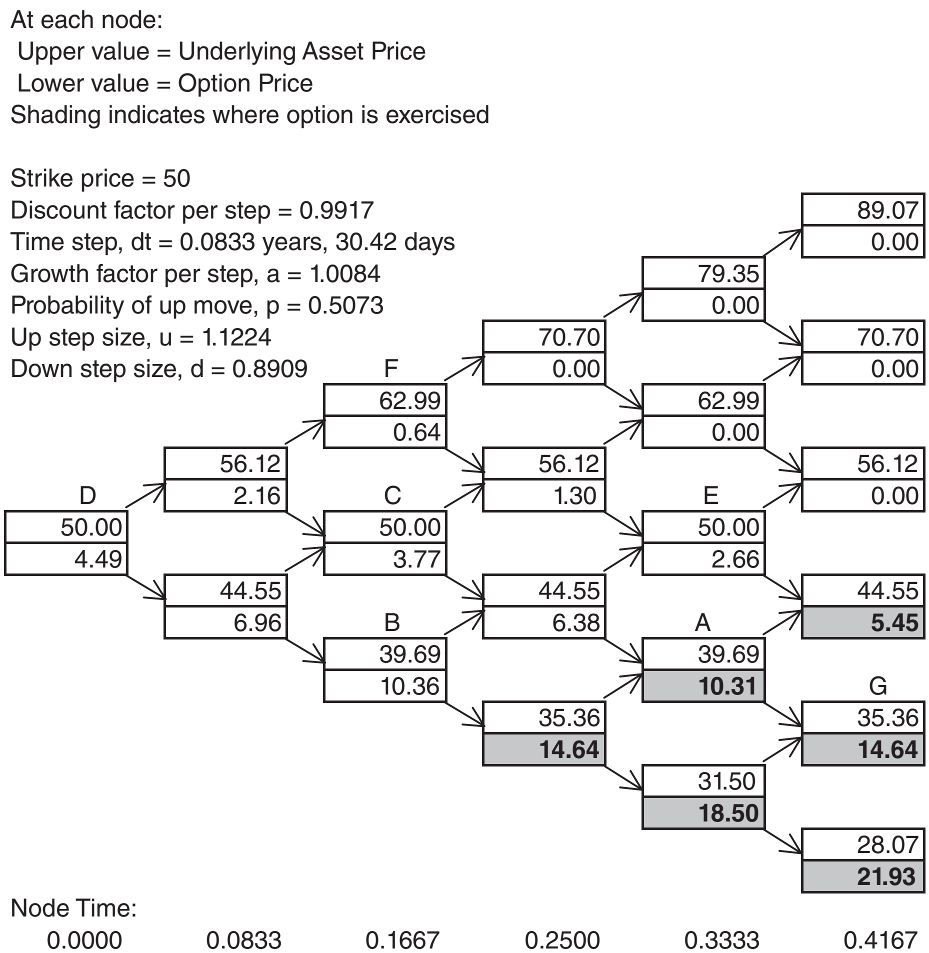


Figure 18.3 Binomial tree from DerivaGem for American put on non-dividend-paying stock

would give a value for the option of zero because both the stock price and strike price are \$50. Clearly, it is best to wait. The correct value for the option at node E is, therefore, \$2.66. At node A, it is a different story. If the option is exercised, it is worth \$50.00 – \$39.69, or \$10.31. This is more than \$9.90. If node A is reached, the option should, therefore, be exercised and the correct value for the option at node A is \$10.31.

Option prices at earlier nodes are calculated in a similar way. Note that it is not always best to exercise an option early when it is in the money. Consider node B. If the option is exercised, it is worth \$50.00 – \$39.69, or \$10.31. However, if it is not exercised, it is worth

$$(0.5073 \times 6.38 + 0.4927 \times 14.64)e^{-0.10 \times 1/12} = 10.36$$

The option should, therefore, not be exercised at this node, and the correct option value at the node is \$10.36.

Working back through the tree, we find the value of the option at the initial node to be \$4.49. This is our numerical estimate for the option's current value. In practice, a smaller value of Δt, and many more nodes, would be used. DerivaGem shows that with 30, 50, and 100 time steps we get values for the option of 4.263, 4.272, and 4.278, respectively.

Expressing the Approach Algebraically

Suppose that the life of an American put option on a non-dividend-paying stock is divided into N subintervals of length Δt. We will refer to the jth node at time $i\,\Delta t$ as the (i, j) node, where $0 \le i \le N$ and $0 \le j \le i$. This means that the lowest node at time $i\,\Delta t$ is $(i, 0)$, the next lowest is $(i, 1)$, and so on. Define $f_{i,j}$ as the value of the option at the (i, j) node. The stock price at the (i, j) node is $S_0 u^j d^{i-j}$. Because the value of an American put at its expiration date is $\max(K - S_T, 0)$, we know that

$$f_{N,j} = \max(K - S_0 u^j d^{N-j}, 0) \quad (j = 0, 1, \ldots, N)$$

There is a probability, p, of moving from the (i, j) node at time $i\,\Delta t$ to the $(i+1, j+1)$ node at time $(i+1)\Delta t$, and a probability $1-p$ of moving from the (i, j) node at time $i\,\Delta t$ to the $(i+1, j)$ node at time $(i+1)\,\Delta t$. Assuming no early exercise, risk-neutral valuation gives

$$f_{i,j} = e^{-r\Delta t}[pf_{i+1,j+1} + (1-p)f_{i+1,j}]$$

for $0 \le i \le N-1$ and $0 \le j \le i$. When early exercise is possible, this value for $f_{i,j}$ must be compared with the option's intrinsic value, and we obtain

$$f_{i,j} = \max\{K - S_0 u^j d^{i-j},\, e^{-r\Delta t}[pf_{i+1,j+1} + (1-p)f_{i+1,j}]\}$$

Note that, because the calculations start at time T and work backward, the value at time $i\,\Delta t$ captures not only the effect of early exercise possibilities at time $i\,\Delta t$, but also the effect of early exercise at subsequent times.

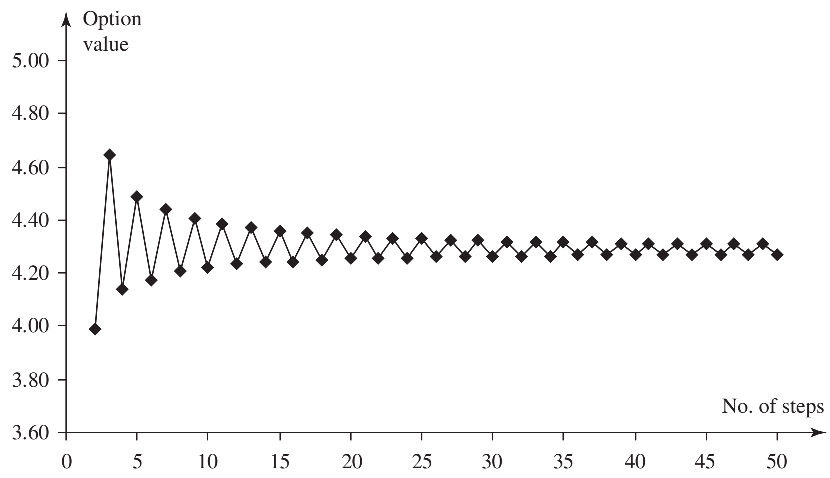


Figure 18.4 Convergence of option price calculated from a binomial tree

In the limit as Δt tends to zero, an exact value for the American put is obtained. In practice, $N = 30$ usually gives reasonable results. Figure 18.4 shows the convergence of the option price in the example we have been considering. (This figure was calculated using the Application Builder functions provided with the DerivaGem software. See Sample Application A.)

If a call rather than a put is being valued, the value at the final node is $\max(S_T - K, 0)$, so that

$$f_{N,j} = \max(S_0 u^j d^{N-j} - K, 0) \quad (j = 0, 1, \ldots, N)$$

Also, when early exercise is possible, the expression for $f_{i,j}$ must reflect the payoff from a call rather than a put, so that

$$f_{i,j} = \max\{S_0 u^j d^{i-j} - K, e^{-r\Delta t}[pf_{i+1,j+1} + (1+p)f_{i+1,j}]\}$$

Estimating Delta and Other Greek Letters

It will be recalled that the delta, Δ, of an option is the rate of change of its price with respect to the underlying stock price. It can be calculated as

$$\frac{\Delta f}{\Delta S}$$

where ΔS is a small change in the stock price and Δf is the corresponding small change in the option price. At time Δt, we have an estimate $f_{1,1}$ for the option price when the stock price is $S_0 u$ and an estimate $f_{1,0}$ for the option price when the stock price is $S_0 d$. This means that, when $\Delta S = S_0 u - S_0 d$, we have $\Delta f = f_{1,1} - f_{1,0}$. An estimate of Δ at time Δt is therefore

$$\Delta = \frac{f_{1,1} - f_{1,0}}{S_0 u - S_0 d} \tag{18.8}$$

To determine gamma, Γ, we note that there are two estimates of Δ at time $2\Delta t$. When the stock price is $(S_0 u^2 + S_0)/2$ (halfway between the second and third node at time $2\Delta t$), delta is $(f_{2,2} - f_{2,1})/(S_0 u^2 - S_0)$; when the stock price is $(S_0 + S_0 d^2)/2$ (halfway between the first and second node at time $2\Delta t$), delta is $(f_{2,1} - f_{2,0})/(S_0 - S_0 d^2)$. The difference between the two stock prices is h, where

$$h = 0.5(S_0 u^2 - S_0 d^2)$$

Gamma is the change in delta divided by h, or

$$\Gamma = \frac{[(f_{2,2} - f_{2,1})/(S_0 u^2 - S_0)] - [(f_{2,1} - f_{2,0})/(S_0 - S_0 d^2)]}{h} \tag{18.9}$$

These procedures provide estimates of delta at time Δt and of gamma at time $2\Delta t$. In practice, they are usually used as estimates of delta and gamma at time zero as well.[2]

[2] If slightly more accuracy is required for delta and gamma, we can start the binomial tree at time $-2\Delta t$ and assume that the stock price is S_0 at this time. This leads to the option price being calculated for three different stock prices at time zero.

A further hedge parameter that can be obtained directly from the tree is theta, Θ. This is the rate of change of the option price with time when all else is kept constant. The value of the option at time zero, when the stock price is S_0, is $f_{0,0}$. The value of the option at time $2\Delta t$, when the stock price is S_0, is $f_{2,1}$. An estimate of theta is therefore

$$\Theta = \frac{f_{2,1} - f_{0,0}}{2\Delta t} \tag{18.10}$$

Vega can be calculated by making a small change, $\Delta\sigma$, in the volatility and constructing a new tree to obtain a new value of the option (the number of time steps should be kept the same). The estimate of vega is

$$\mathcal{V} = \frac{f^* - f}{\Delta\sigma}$$

where f and f^* are the estimates of the option price from the original and the new tree, respectively. Rho can be calculated similarly.

As an illustration, consider the tree in Figure 18.3. In this case, $f_{1,0} = 6.96$ and $f_{1,1} = 2.16$. Equation (18.8) gives an estimate of delta of

$$\frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$

From equation (18.9), an estimate of the gamma of the option can be obtained from the values at nodes B, C, and F as

$$\frac{[(0.64 - 3.77)/(62.99 - 50.00)] - [(3.77 - 10.36)/(50.00 - 39.69)]}{11.65} = 0.03$$

From equation (18.10), an estimate of the theta of the option can be obtained from the values at nodes D and C as

$$\frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}$$

or –0.012 per calendar day. These are, of course, only rough estimates. They become progressively better as the number of time steps on the tree is increased. Using 50 time steps, DerivaGem provides estimates of –0.414, 0.033, and –0.0117 for delta, gamma, and theta, respectively.

18.2 USING THE BINOMIAL TREE FOR OPTIONS ON INDICES, CURRENCIES, AND FUTURES CONTRACTS

As shown in Section 12.10, the binomial tree approach to valuing options on non-dividend-paying stocks can easily be adapted to valuing American calls and puts on a stock paying a continuous dividend yield at rate q.

Because the dividends provide a return of q, the stock price itself must, on average, in a risk-neutral world provide a return of $r - q$. Hence, equation (18.1) becomes

$$Se^{(r-q)\Delta t} = pSu + (1-p)Sd$$

so that

$$e^{(r-q)\Delta t} = pu + (1-p)d$$

Example 18.1 Tree for option on index futures

Consider a four-month American call option on index futures. The current futures price is 300, the exercise price is 300, the risk-free interest rate is 8% per annum, and the volatility of the index is 30% per annum. We divide the life of the option into four one-month intervals for the purposes of constructing the tree. In this case, $F_0 = 300$, $K = 300$, $r = 0.08$, $\sigma = 0.3$, $T = 4/12$, and $\Delta t = 1/12$. Because a futures contract is analogous to a stock paying dividends at a continuous rate r, q should be set equal to r in equation (18.11). This gives $a = 1$. The other parameters necessary to construct the tree are

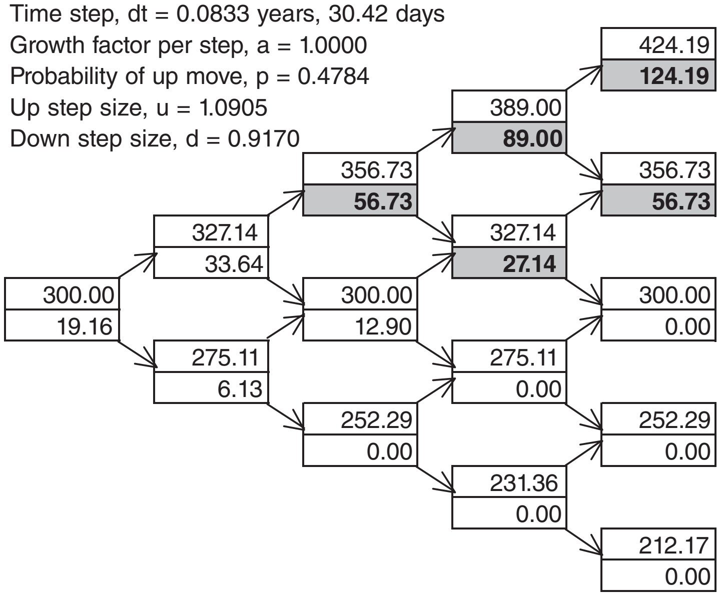
$$u = e^{\sigma\sqrt{\Delta t}} = 1.0905, \quad d = \frac{1}{u} = 0.9170, \quad p = \frac{a-d}{u-d} = 0.4784, \quad 1-p = 0.5216$$

The tree is shown in the figure below (the upper number is the futures price; the lower number is the option price). The estimated value of the option is 19.16. More accuracy is obtained with more steps. With 50 time steps DerivaGem gives a value of 20.18; with 100 time steps it gives a value of 20.22.

DerivaGem Output:

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 300
Discount factor per step = 0.9934
Time step, dt = 0.0833 years, 30.42 days
Growth factor per step, a = 1.0000
Probability of up move, p = 0.4784
Up step size, u = 1.0905
Down step size, d = 0.9170



Node Time:
0.0000 0.0833 0.1667 0.2500 0.3333

Example 18.2 Tree for option on currency

Consider a one-year American put option on the British pound (GBP). The current exchange rate (USD per GBP) is 1.6100, the strike price is 1.6000, the U.S. risk-free interest rate is 8% per annum, the sterling risk-free interest rate is 9% per annum, and the volatility of the sterling exchange rate is 12% per annum. In this case, $S_0 = 1.61$, $K = 1.60$, $r = 0.08$, $r_f = 0.09$, $\sigma = 0.12$, and $T = 1.0$. We divide the life of the option into four three-month periods for the purposes of constructing the tree, so that $\Delta t = 0.25$. In this case, $q = r_f$ and equation (18.11) gives

$$a = e^{(0.08-0.09)\times 0.25} = 0.9975$$

The other parameters necessary to construct the tree are:

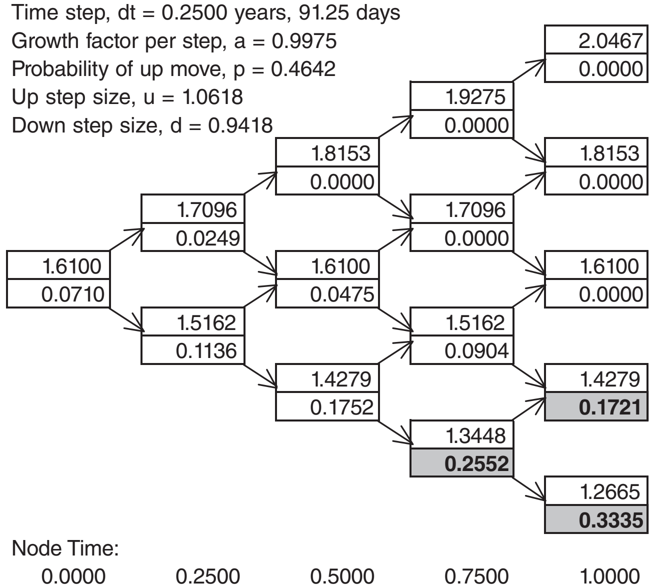
$$u = e^{\sigma\sqrt{\Delta t}} = 1.0618, \quad d = \frac{1}{u} = 0.9418, \quad p = \frac{a-d}{u-d} = 0.4642, \quad 1-p = 0.5358$$

The tree is shown in the figure below (the upper number is the exchange rate; the lower number is the option price). The estimated value of the option is \$0.0710. Using 50 time steps, DerivaGem gives the value of the option as 0.0738; with 100 time steps, it also gives the value 0.0738.

DerivaGem Output:

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 1.6
Discount factor per step = 0.9802
Time step, dt = 0.2500 years, 91.25 days
Growth factor per step, a = 0.9975
Probability of up move, p = 0.4642
Up step size, u = 1.0618
Down step size, d = 0.9418



The parameters p, u, and d must satisfy this equation and equation (18.3). Equations (18.4), (18.5), and (18.6) are still correct but with

$$a = e^{(r-q)\Delta t} \tag{18.11}$$

The binomial tree numerical procedure can therefore be used exactly as before with this new value of a.

We showed in Chapters 15 and 16 that stock indices, currencies, and futures contracts can for the purposes of option evaluation be considered as stocks paying continuous dividend yields. In the case of a stock index, the relevant dividend yield is the dividend yield on the stock portfolio underlying the index; in the case of a currency, it is the foreign risk-free interest rate; in the case of a futures contract, it is the domestic risk-free interest rate. Examples 18.1 and 18.2 illustrate this.

18.3 THE BINOMIAL MODEL FOR A DIVIDEND-PAYING STOCK

We now move on to the more tricky issue of how the binomial model can be used for a stock paying discrete dividends. As in Chapter 13, the word "dividend" will for the purposes of our discussion be used to refer to the reduction in the stock price on the ex-dividend date as a result of the dividend.

Known Dividend Yield

For long-life stock options, it is sometimes assumed for convenience that there is a known continuous dividend yield of q on the stock. The options can then be valued in the same way as options on a stock index. For more accuracy, known dividend yields can be assumed to be paid discretely. Suppose first that a single dividend will be paid at a certain time and that it will be a proportion δ of the stock price at that time. The parameters u, d, and p can be calculated as though no dividends are expected. The tree takes the form shown in Figure 18.5 and can be analyzed in a way that is analogous to that just described. If the time $i\,\Delta t$ is prior to the stock going ex-dividend, the nodes on the tree correspond to stock prices

$$S_0 u^j d^{i-j} \quad (j = 0, 1, \ldots, i)$$

If the time $i\,\Delta t$ is after the stock goes ex-dividend, the nodes correspond to stock prices

$$S_0(1-\delta)u^j d^{i-j} \quad (j = 0, 1, \ldots, i)$$

Several known dividends during the life of an option can be dealt with similarly. If δ_i is the total dividend yield associated with all ex-dividend dates between time zero and time $i\,\Delta t$, the nodes at time $i\,\Delta t$ correspond to stock prices

$$S_0(1-\delta_i)u^j d^{i-j}$$

Known Dollar Dividend

In some situations, particularly when the life of the option is short, it is more realistic to assume that the dollar amount of the dividend rather than the dividend yield is known in advance. If the volatility of the stock, σ, is assumed constant, the tree takes the form shown in Figure 18.6. It does not recombine, which means that the number of nodes that have to be evaluated is liable to become very large. Suppose that there is only one dividend, that the ex-dividend date, τ, is between $k\Delta t$ and $(k+1)\Delta t$, and that the dollar amount of the dividend is D. When $i \le k$, the nodes on the tree at time $i\,\Delta t$ correspond to stock prices

$$S_0 u^j d^{i-j} \quad (j = 0, 1, 2, \ldots, i)$$

as before. When $i = k+1$, the nodes on the tree correspond to stock prices

$$S_0 u^j d^{i-j} - D \quad (j = 0, 1, 2, \ldots, i)$$

When $i = k+2$, the nodes on the tree correspond to stock prices

$$(S_0 u^j d^{i-1-j} - D)u \quad \text{and} \quad (S_0 u^j d^{i-1-j} - D)d$$

for $j = 0, 1, 2, \ldots, i-1$, so that there are $2i$ rather than $i+1$ nodes. At time $(k+m)\Delta t$, there are $m(k+2)$ rather than $k+m+1$ nodes. The number of nodes expands even faster when there are several ex-dividend dates during the option's life.

The node-proliferation problem can be solved by assuming, as in the analysis of European options in Chapter 13, that the stock price has two components: a part that is uncertain and a part that is the present value of all future dividends during the life of the option. Suppose that there is only one ex-dividend date, τ, during the life of the option and that $k\Delta t \le \tau \le (k+1)\Delta t$. The value S^* of the uncertain component (i.e., the

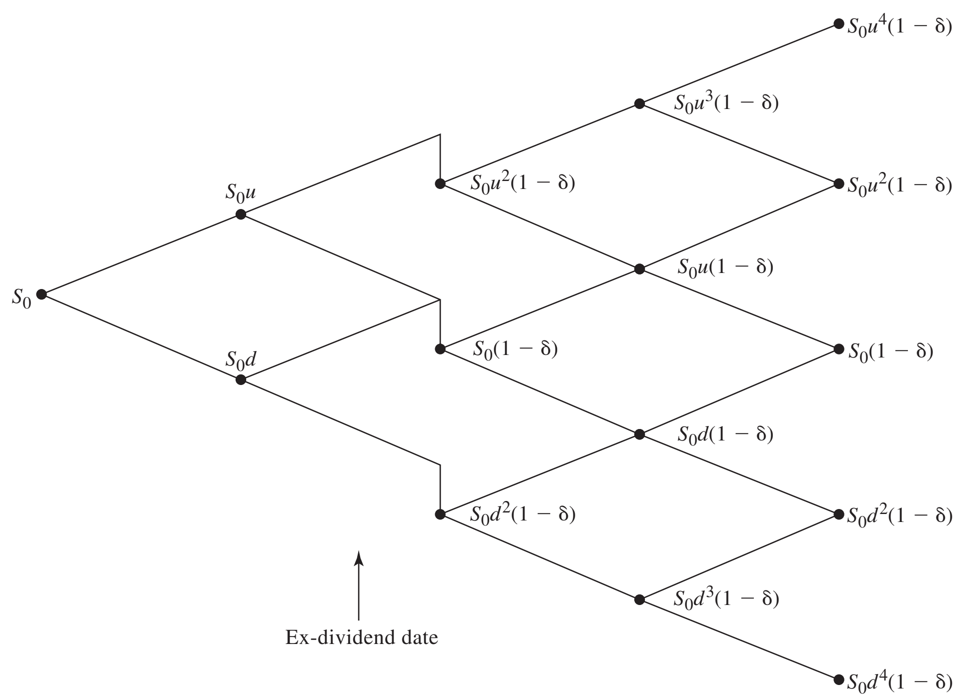


Figure 18.5 Tree when stock pays a known dividend yield at one particular time

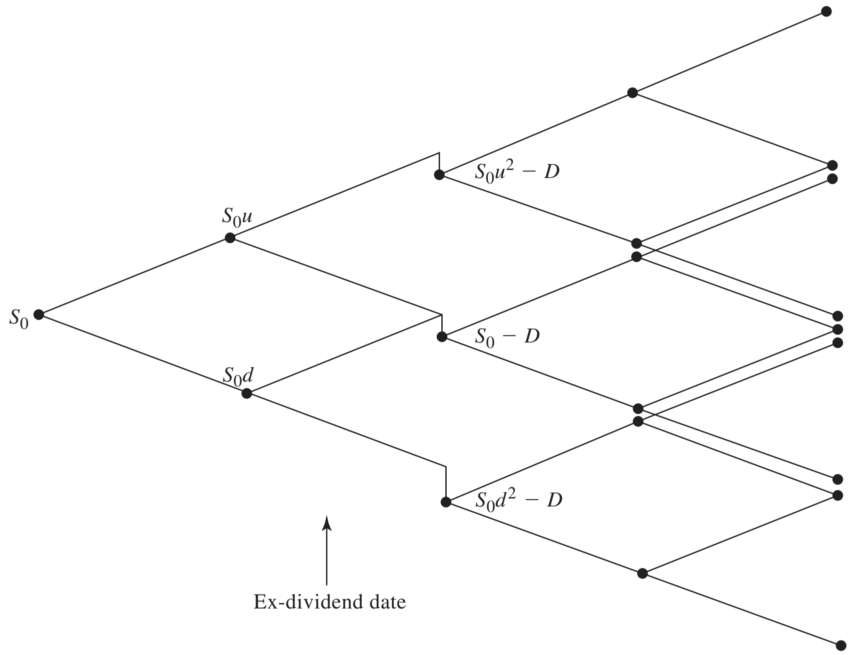


Figure 18.6 Tree when dollar amount of dividend is assumed known and volatility is assumed constant

component not used to pay dividends) at time $i\,\Delta t$ is given by

$$S^* = S \quad \text{when } i\,\Delta t > \tau$$

and

$$S^* = S - De^{-r(\tau - i\Delta t)} \quad \text{when } i\,\Delta t \le \tau$$

where D is the dividend. Define σ^* as the volatility of S^* and assume that σ^* is constant.[3] The parameters p, u, and d can be calculated from equations (18.4), (18.5), (18.6), and (18.7) with σ replaced by σ^*, and a tree can be constructed in the usual way to model S^*. By adding to the stock price at each node, the present value of future dividends (if any), the tree can be converted into another tree that models S. Suppose that S_0^* is the value of S^* at time zero. At time $i\,\Delta t$, the nodes on this tree correspond to the stock prices

$$S_0^* u^j d^{i-j} + De^{-r(\tau - i\,\Delta t)} \quad (j = 0, 1, \ldots, i)$$

when $i\,\Delta t < \tau$ and

$$S_0^* u^j d^{i-j} \quad (j = 0, 1, \ldots, i)$$

when $i\,\Delta t > \tau$. This approach, which has the advantage of being consistent with the approach for European options in Section 13.10, succeeds in achieving a situation where the tree recombines so that there are $i + 1$ nodes at time $i\,\Delta t$. It can be generalized in a straightforward way to deal with the situation where there are several dividends. The

[3] In theory, σ^* is slightly greater than σ, the volatility of S. In practice, implied volatilities are used, so that distinguishing between σ and σ^* is not necessary.

Example 18.3 Tree for option on a dividend-paying stock

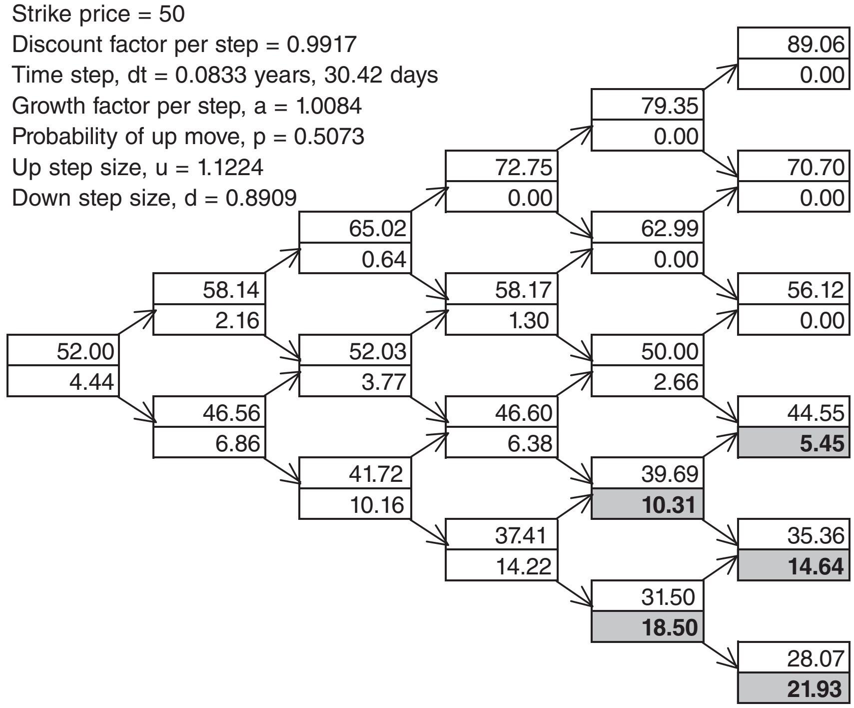
Consider a five-month American put option on a stock that is expected to pay a single dividend of \$2.06 during the life of the option. The initial stock price is \$52, the strike price is \$50, the risk-free interest rate is 10% per annum, the volatility is 40% per annum, and the ex-dividend date is in 3.5 months.

We first construct a tree to model S^*, the stock price less the present value of future dividends during the life of the option. Initially, the present value of the dividend is $2.06 \times e^{-0.1\times 3.5/12} = 2.00$. The initial value of S^* is therefore 50. Assuming that the 40% per annum volatility refers to S^*, Figure 18.3 provides a binomial tree for S^*. (S^* has the same initial value and volatility as the stock price on which Figure 18.3 was based.) Adding the present value of the dividend at each node leads to the figure below, which is a binomial tree for S. The probabilities at each node are, as in Figure 18.3, 0.5073 for an up movement and 0.4927 for a down movement. Working back through the tree in the usual way gives the option price as \$4.44 (50 steps give 4.208; 100 steps give 4.214).

DerivaGem Output:

At each node:
Upper value = Underlying Asset Price
Lower value = Option Price
Shading indicates where option is exercised

Strike price = 50
Discount factor per step = 0.9917
Time step, dt = 0.0833 years, 30.42 days
Growth factor per step, a = 1.0084
Probability of up move, p = 0.5073
Up step size, u = 1.1224
Down step size, d = 0.8909



Node Time:
0.0000 0.0833 0.1667 0.2500 0.3333 0.4167

approach is illustrated in Example 18.3. As indicated earlier, for long-life options a known dividend yield rather than a known dollar dividend is often assumed.

18.4 EXTENSIONS OF THE BASIC TREE APPROACH

We now explain two ways in which the binomial tree approach can be extended.

Time-Dependent Interest Rates and Volatilities

Up to now we have been assuming that interest rates are constant. When the term structure is steeply upward or downward sloping and American options are being valued, this may not be a satisfactory assumption. It is more appropriate to assume that the interest rate for a period of length Δt in the future equals the current forward interest rate for that period. We can accommodate this assumption by setting

$$a = e^{f(t)\Delta t} \tag{18.12}$$

for nodes at time t where $f(t)$ is the forward rate between times t and $t + \Delta t$. This does not change the geometry of the tree because u and d do not depend on a. The probabilities on the branches emanating from nodes at time t are as before:[4]

$$p = \frac{a-d}{u-d} \quad \text{and} \quad 1-p = \frac{u-a}{u-d}$$

The rest of the way in which we use the tree is the same as before, except that when discounting from time $t + \Delta t$ to time t we use $f(t)$. A similar modification of the basic tree can be used to value index options, foreign exchange options, and futures options. In these applications the dividend yield on an index or a foreign risk-free rate can be made a function of time by following a similar approach to that just described.

Making the volatility σ a function of time in a binomial tree is more difficult. Suppose $\sigma(t)$ is the volatility used to price an option with a life of t. One approach is to make the length of each time step inversely proportional to the average variance rate during the time step. The values of u and d are then the same everywhere and the tree recombines. Define the $V = \sigma(T)^2 T$, where T is the life of the tree and define t_i as the end of the ith time step. For N time steps, we choose t_i to satisfy $\sigma(t_i)^2 t_i = iV/N$ and set $u = e^{\sqrt{V/N}}$ with $d = 1/u$. The parameter p is defined as for a constant volatility. This procedure can be combined with the procedure just mentioned for dealing with non-constant interest rates, so that both interest rates and volatilities are time-dependent.

The Control Variate Technique

A technique known as the *control variate technique* can be used for the evaluation of an American option.[5] This involves using the same tree to calculate both the value of the American option, f_A, and the value of the corresponding European option, f_E. The Black–Scholes–Merton price of the European option, f_{BSM}, is also calculated. The error given by the tree in the pricing of the European option, $f_{BSM} - f_E$, is assumed

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[4] For a sufficiently large number of time steps, these probabilities are always positive.

[5] See J. C. Hull and A. White, "The Use of the Control Variate Technique in Option Pricing," *Journal of Financial and Quantitative Analysis* 23 (September 1988): 237–51.

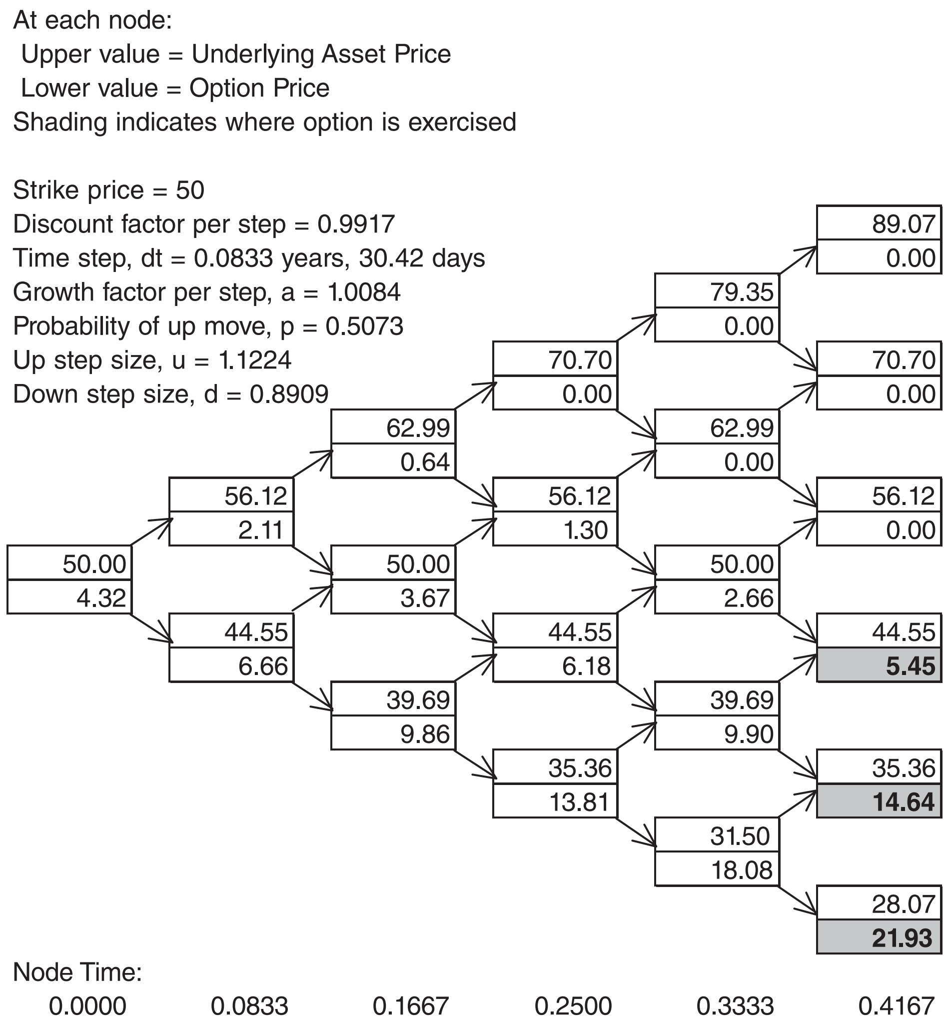

Figure 18.7 Tree produced by DerivaGem for European version of option in Figure 18.3. At each node, the upper number is the stock price and the lower number is the option price

equal to that given by the tree in the pricing of the American option. This gives the estimate of the price of the American option to be

$$f_{\mathrm{A}} + (f_{\mathrm{BSM}} - f_{\mathrm{E}})$$

To illustrate this approach, Figure 18.7 values the option in Figure 18.3 on the assumption that it is European. The price obtained, f_{E}, is \$4.32. From the Black–Scholes–Merton formula, the true European price of the option, f_{BSM}, is \$4.08. The estimate of the American price, f_{A}, in Figure 18.3 is \$4.49. The control variate estimate of the American price is therefore

$$4.49 + (4.08 - 4.32) = 4.25$$

A good estimate of the American price, calculated using 100 steps, is 4.278. The control variate approach does, therefore, produce a considerable improvement over the basic tree estimate of 4.49 in this case. In effect, it uses the tree to calculate the difference between the European and the American price rather than the American price itself.

18.5 ALTERNATIVE PROCEDURE FOR CONSTRUCTING TREES

The Cox, Ross, and Rubinstein approach is not the only way of building a binomial tree. Instead of imposing the assumption $u = 1/d$ on equations (18.2) and (18.3), we can set $p = 0.5$. A solution to the equations for small Δt is then

$$u = e^{(r-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}}, \quad d = e^{(r-\sigma^2/2)\Delta t-\sigma\sqrt{\Delta t}}$$

When the stock provides a continuous dividend yield at rate q, the variable r becomes $r - q$ in these formulas. This allows trees with $p = 0.5$ to be built for options on indices, foreign exchange, and futures. The procedure is illustrated in Example 18.4.

This alternative tree-building procedure has the advantage over the Cox, Ross, and Rubinstein approach that the probabilities are always 0.5 regardless of the value of σ or the number of time steps.[6] A small disadvantage is that the calculation of delta, gamma, and theta from the tree is not quite as accurate because the values of the underlying asset at times Δt and $2\Delta t$ are no longer centered at S_0.

18.6 MONTE CARLO SIMULATION

Binomial trees can be used in conjunction with a procedure known as *Monte Carlo simulation* for valuing derivatives. Once the tree has been constructed, we randomly sample paths through it. Instead of working backward from the end of the tree to the beginning, we work forward through the tree. The basic procedure is as follows. At the first node we sample a random number between 0 and 1. If the number lies between 0 and p, we take the upper branch; if it lies between p and 1, we take the lower branch. We repeat this procedure at the node that is then reached and at all subsequent nodes that are reached until we get to the end of the tree. We then calculate the payoff on the option for the particular path sampled. This completes the first trial. We carry out many more trials by repeating the whole procedure. Our estimate of the value of the option is the arithmetic average of the payoffs from all the trials discounted at the risk-free interest rate. Example 18.5 provides an illustration.

Monte Carlo simulation, as just described, cannot easily be used for American options, because we have no way of knowing whether early exercise is optimal when a certain node is reached. It can be used to value European options so that a check is provided on the pricing formulas for these options. It can also be used to price some of the exotic options we will discuss in Chapter 22 (e.g. Asian options and lookback options).

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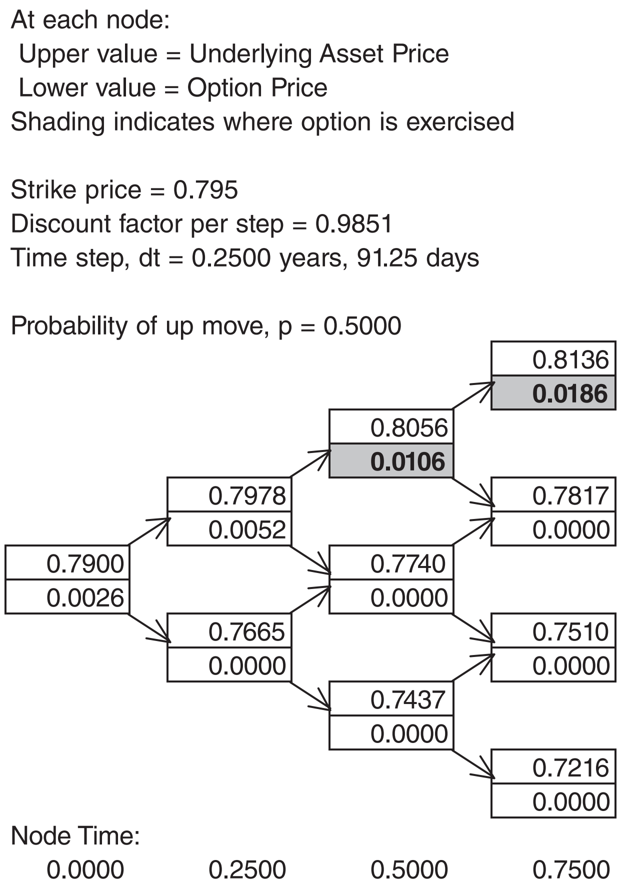
[6] In the unusual situation that time steps are so large that $\sigma < \left|(r-q)\sqrt{\Delta t}\right|$, the Cox, Ross, and Rubinstein tree gives negative probabilities. The alternative procedure described here does not have that drawback.

Example 18.4 Alternative tree construction

A nine-month American call option on a foreign currency has a strike price of 0.7950 (USD per unit of foreign currency). The current exchange rate is 0.7900, the domestic risk-free interest rate is 6% per annum, the foreign risk-free interest rate is 10% per annum, and the volatility of the exchange rate is 4% per annum. In this case, $S_0 = 0.79$, $K = 0.795$, $r = 0.06$, $r_f = 0.10$, $\sigma = 0.04$, and $T = 0.75$. We divide the life of the option into three-month periods for the purposes of constructing the tree so that $\Delta t = 0.25$. We set the probabilities on each branch to 0.5 and

$$u = e^{(0.06-0.10-0.0016/2)0.25+0.04\sqrt{0.25}} = 1.0098$$
$$d = e^{(0.06-0.10-0.0016/2)0.25-0.04\sqrt{0.25}} = 0.9703$$

The tree for the exchange rate is shown in the figure below. The tree gives the value of the option as 0.0026.



Example 18.5 Using Monte Carlo simulation with a tree

Suppose that the tree in Figure 18.3 is used to value an option that pays off $\max(S_{ave} - 50, 0)$ where S_{ave} is the average stock price during the five months (with the first and last stock price being included in the average). This is known as an Asian option. When ten simulation trials are used, one possible result is shown in the following table (U = up movement; D = down movement):

Trial	*Path*	*Average stock price*	*Option payoff*
1	UUUUD	64.98	14.98
2	UUUDD	59.82	9.82
3	DDDUU	42.31	0.00
4	UUUUU	68.04	18.04
5	UUDDU	55.22	5.22
6	UDUUD	55.22	5.22
7	DDUDD	42.31	0.00
8	UUDDU	55.22	5.22
9	UUUDU	62.25	12.25
10	DDUUD	45.56	0.00
Average			7.08

The option payoff is the amount by which the average stock price exceeds \$50. The value of the option is calculated as the average payoff discounted at the risk-free rate. In this case, the average payoff is \$7.08 and the risk-free rate is 10%, and so the calculated value is $7.08e^{-0.1\times 5/12} = 6.79$. (This illustrates the method. In practice we would have to use more time steps on the tree and many more simulation trials to get an accurate answer.)

SUMMARY

This chapter has described how options can be valued using the binomial tree approach. This approach involves dividing the life of the option into a number of small intervals of length Δt and assuming that an asset price at the beginning of an interval can lead to only one of two alternative asset prices at the end of the interval. One of these alternative asset prices involves an up movement; the other involves a down movement.

The sizes of the up movements and down movements, and their associated probabilities, are chosen so that the change in the asset price has the correct mean and standard deviation for a risk-neutral world. Option prices are calculated by starting at the end of the tree and working backward. At the end of the tree, the price of an option is its intrinsic value. At earlier nodes on the tree, the value of an option, if it is American, must be calculated as the greater of

1. The value it has if exercised immediately
2. The value it has if held for a further period of time of length Δt.

If it is exercised at a node, the value of the option is its intrinsic value. If it is held for a further period of length Δt, the value of the option is its expected value at the end of the time period Δt discounted at the risk-free rate.

Delta, gamma, and theta can be estimated directly from the values of the option at the various nodes of the tree. Vega can be estimated by making a small change to the volatility and recomputing the value of the option using a similar tree. Rho can similarly be estimated by making a small change to the interest rate and recomputing the tree.

The binomial tree approach can handle options on stocks paying continuous dividend yields. Because stock indices, currencies, and most futures contracts can be regarded as analogous to stocks paying continuous yields, binomial trees can handle options on these assets as well.

When the binomial tree approach is used to value options on a stock paying known dollar dividends, it is convenient to use the tree to model the stock price less the present value of all future dividends during the life of the option. This keeps the number of nodes on the tree from becoming unmanageable and is consistent with the way European options on dividend-paying stocks are valued.

The computational efficiency of the binomial model can be improved by using the control variate technique. This involves valuing both the American option that is of interest and the corresponding European option using the same tree. The error in the price of the European option is used as an estimate of the error in the price of the American option.

FURTHER READING

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Rendleman, R., and B. Bartter. "Two State Option Pricing," *Journal of Finance*, 34 (1979): 1092–1110.

Quiz (Answers at End of Book)

18.1. Which of the following can be estimated for an American option by constructing a single binomial tree: delta, gamma, vega, theta, rho?

18.2. The probability for an up-movement on a binomial tree is $(a - d)/(u - d)$. Explain how the growth factor a is calculated for (a) a non-dividend-paying stock, (b) a stock index, (c) a foreign currency, and (d) a futures contract.

18.3. Calculate the price of a three-month American put option on a non-dividend-paying stock when the stock price is \$60, the strike price is \$60, the risk-free interest rate is 10% per annum, and the volatility is 45% per annum. Use a binomial tree with a time step of one month.

18.4. Explain how the control variate technique is implemented.

18.5. Calculate the price of a nine-month American call option on corn futures when the current futures price is 198 cents, the strike price is 200 cents, the risk-free interest rate is 8% per annum, and the volatility is 30% per annum. Use a binomial tree with a time step of three months.

18.6. "For a dividend-paying stock the tree for the stock price does not recombine, but the tree for the stock price less the present value of future dividends does recombine." Explain.

18.7. Explain the problem in using Monte Carlo simulation to value an American option.

Practice Questions (Answers in Solutions Manual/Study Guide)

18.8. Consider an option that pays off the amount by which the final stock price exceeds the average stock price achieved during the life of the option. Can this be valued from a binomial tree using backwards induction?

18.9. A nine-month American put option on a non-dividend-paying stock has a strike price of \$49. The stock price is \$50, the risk-free rate is 5% per annum, and the volatility is 30% per annum. Use a three-step binomial tree to calculate the option price.

18.10. Use a three-time-step tree to value a nine-month American call option on wheat futures. The current futures price is 400 cents, the strike price is 420 cents, the risk-free rate is 6%, and the volatility is 35% per annum. Estimate the delta of the option from your tree.

18.11. A three-month American call option on a stock has a strike price of \$20. The stock price is \$20, the risk-free rate is 3% per annum, and the volatility is 25% per annum. A dividend of \$2 is expected in 1.5 months. Use a three-step binomial tree to calculate the option price.

18.12. A one-year American put option on a non-dividend-paying stock has an exercise price of \$18. The current stock price is \$20, the risk-free interest rate is 15% per annum, and the volatility of the stock is 40% per annum. Use the DerivaGem software with four three-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

18.13. A two-month American put option on a stock index has an exercise price of 480. The current level of the index is 484, the risk-free interest rate is 10% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum. Divide the life of the option into four half-month periods and use the binomial tree approach to estimate the value of the option.

18.14. How would you use the control variate approach to improve the estimate of the delta of an American option when the binomial tree approach is used?

18.15. How would you use the binomial tree approach to value an American option on a stock index when the dividend yield on the index is a function of time?

Further Questions

18.16. An American put option to sell a Swiss franc for dollars has a strike price of \$0.80 and a time to maturity of one year. The volatility of the Swiss franc is 10%, the dollar interest rate is 6%, the Swiss franc interest rate is 3%, and the current exchange rate is 0.81. Use a tree with three time steps to value the option. Estimate the delta of the option from your tree.

18.17. A one-year American call option on silver futures has an exercise price of \$9.00. The current futures price is \$8.50, the risk-free rate of interest is 12% per annum, and the volatility of the futures price is 25% per annum. Use the DerivaGem software with four three-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.

18.18. A six-month American call option on a stock is expected to pay dividends of \$1 per share at the end of the second month and the fifth month. The current stock price is \$30, the exercise price is \$34, the risk-free interest rate is 10% per annum, and the volatility of the part of the stock price that will not be used to pay the dividends is 30% per annum. Use the DerivaGem software with the life of the option divided into 100 time steps to estimate the value of the option. Compare your answer with that given by Black's approximation (see Section 13.10).

18.19. The DerivaGem Application Builder functions enable you to investigate how the prices of options calculated from a binomial tree converge to the correct value as the number of time steps increases (see Figure 18.4 and Sample Application A in DerivaGem). Consider a put option on a stock index where the index level is 900, the strike price is 900, the risk-free rate is 5%, the dividend yield is 2%, and the time to maturity is 2 years:
(a) Produce results similar to Sample Application A on convergence for the situation where the option is European and the volatility of the index is 20%.
(b) Produce results similar to Sample Application A on convergence for the situation where the option is American and the volatility of the index is 20%.
(c) Produce a chart showing the pricing of the American option when the volatility is 20% as a function of the number of time steps when the control variate technique is used.
(d) Suppose that the price of the American option in the market is 85.0. Produce a chart showing the implied volatility estimate as a function of the number of time steps.

18.20. Estimate delta, gamma, and theta from the tree in Example 18.1. Explain how each can be interpreted.

18.21. How much is gained from exercising early at the lowest node at the nine-month point in Example 18.2?

18.22. A four-step Cox–Ross–Rubinstein binomial tree is used to price a one-year American put option on an index when the index level is 500, the strike price is 500, the dividend yield is 2%, the risk-free rate is 5%, and the volatility is 25% per annum. What is the option price, delta, gamma, and theta? Explain how you would calculate vega and rho.