NP-complete Problems: Reductions

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Outline

1. Reductions
2. Showing **NP**-completeness
3. Independent Set $\rightarrow$ Vertex Cover
4. 3-SAT $\rightarrow$ Independent Set
5. SAT $\rightarrow$ 3-SAT
6. All of **NP** $\rightarrow$ SAT
7. Using SAT-solvers

Informally

We say that a search problem A is reduced to a search problem B and write $A \to B$, if a polynomial time algorithm for B can be used (as a black box) to solve A in polynomial time.

Reduction: $A \to B$

instance I of A

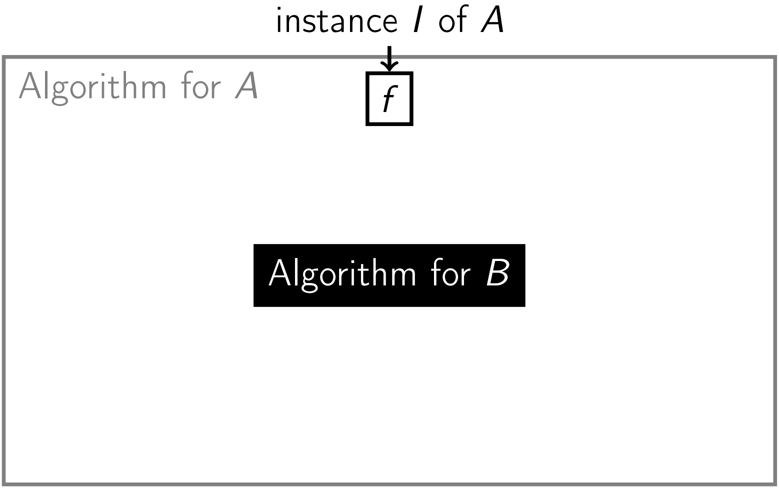
Reduction: $A \to B$

instance I of A

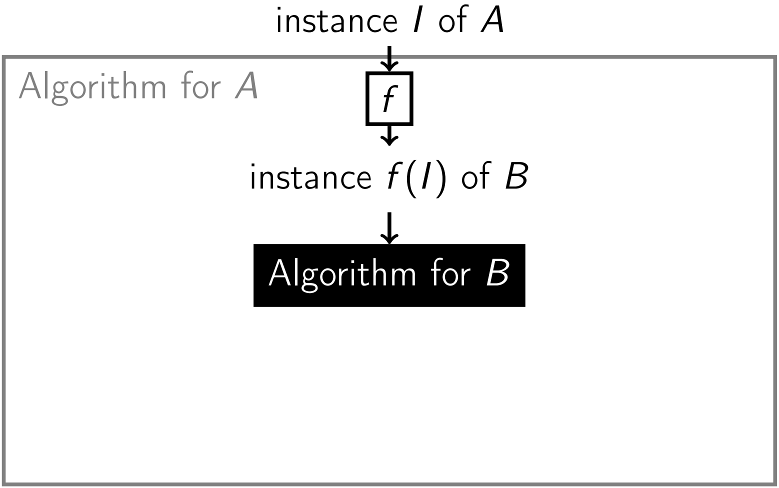
Algorithm for A

Algorithm for B

Reduction: $A \to B$



Reduction: $A \rightarrow B$



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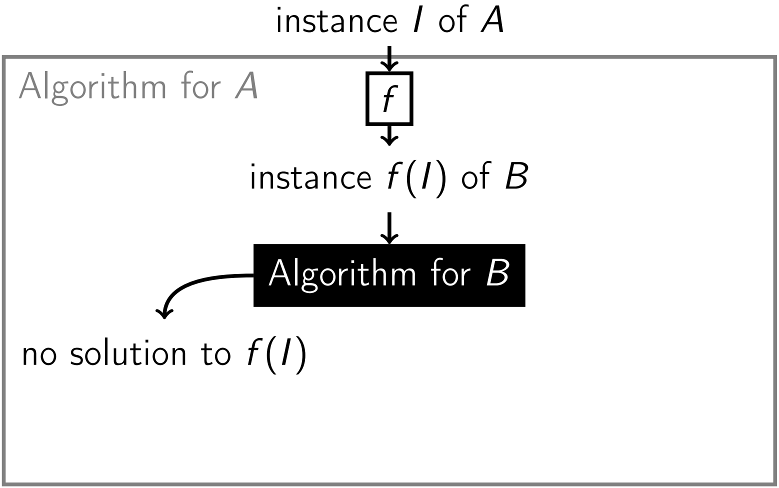

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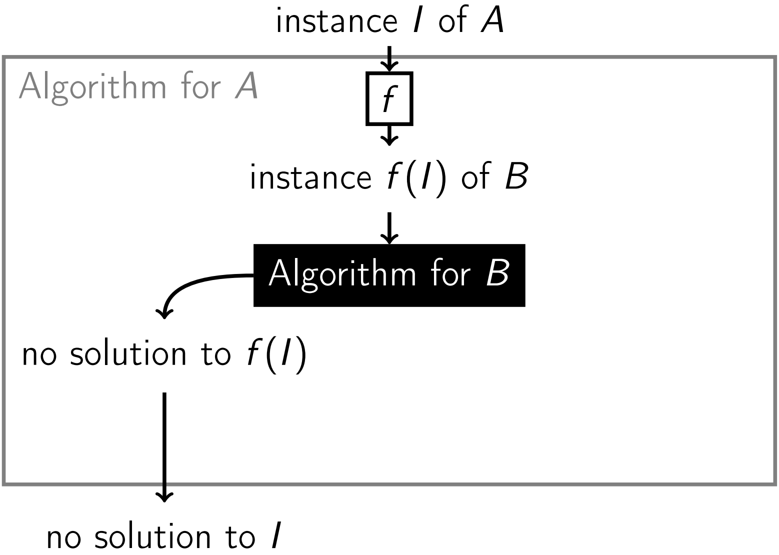

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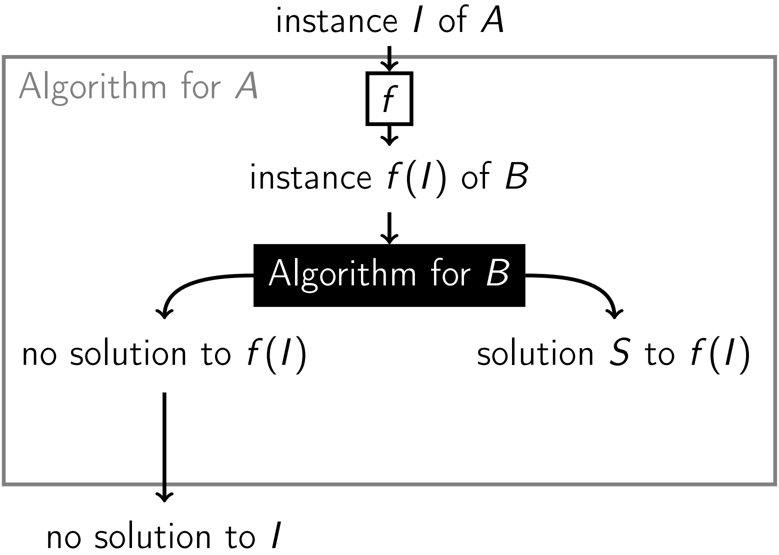

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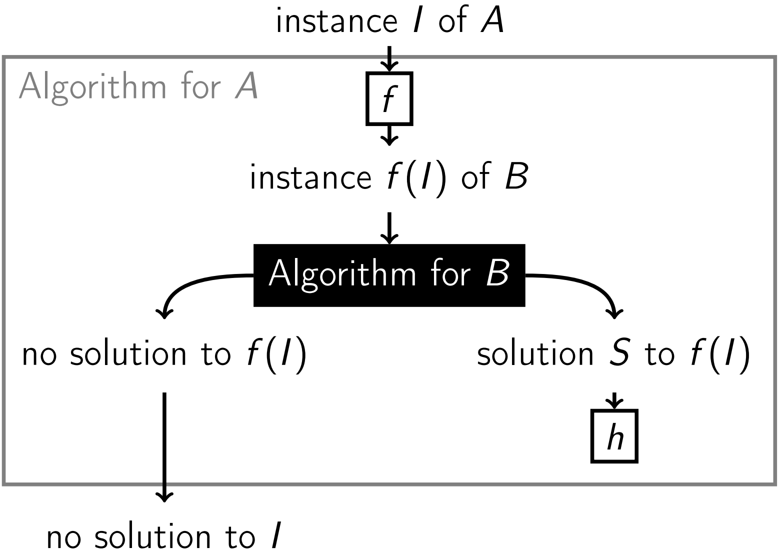

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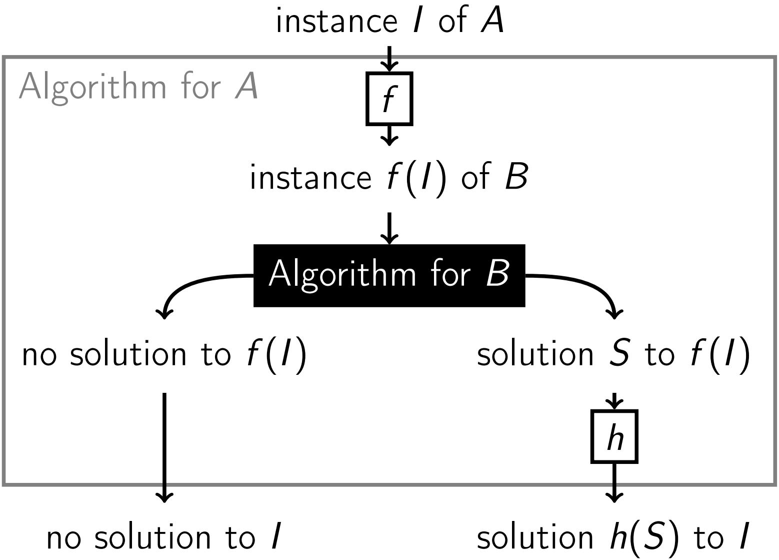
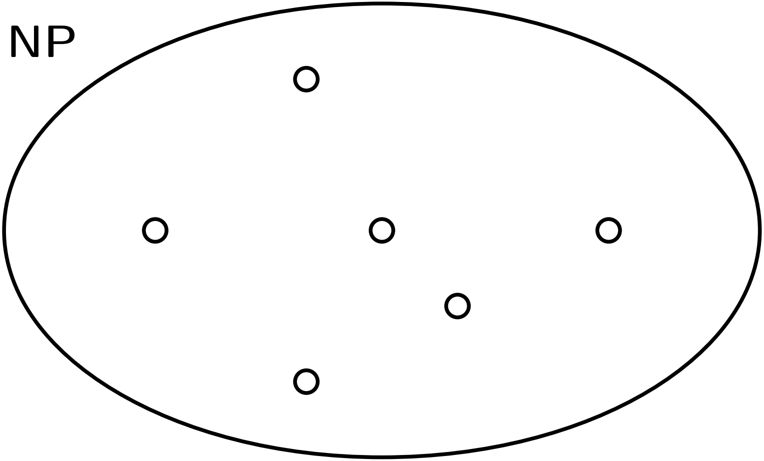


Formally

Definition

We say that a search problem A is reduced to a search problem B and write $A \to B$, if there exists a polynomial time algorithm f that converts any instance I of A into an instance $f(I)$ of B, together with a polynomial time algorithm h that converts any solution S to $f(I)$ back to a solution $h(S)$ to A. If there is no solution to $f(I)$, then there is no solution to I.

Graph of Search Problems



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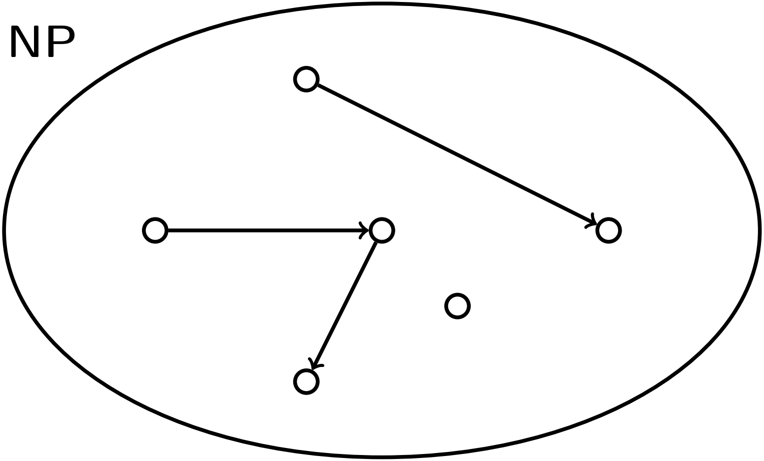

NP-complete Problems

Definition

A search problem is called **NP-complete** if all other search problems reduce to it.

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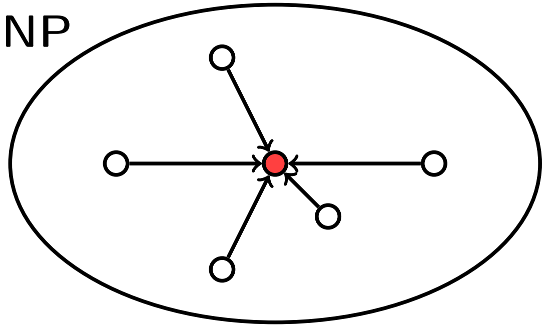

Do they exist?

It's not at all immediate that **NP**-complete problems even exist. We'll see later that all hard problems that we've seen in the previous part are in fact **NP**-complete!

Outline

Two ways of using $A \rightarrow B$:

1. if B is easy (can be solved in polynomial time), then so is A
2. if A is hard (cannot be solved in polynomial time), then so is B

Reductions Compose

Lemma

If $A \to B$ and $B \to C$, then $A \to C$.

Proof

- The reductions $A \to B$ and $B \to C$ are given by pairs of polytime algorithms (f_{AB}, h_{AB}) and (f_{BC}, h_{BC}).

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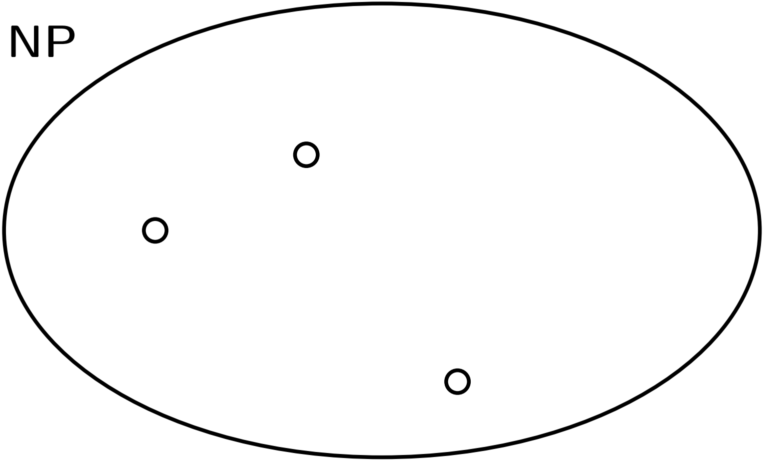
- To transform an instance I_A of A to an instance I_C of C we apply a polytime algorithm $f_{BC} \circ f_{AB}$: $I_C = f_{BC}(f_{AB}(I_A))$.
- To transform a solution S_C to I_C to a solution S_A to I_A we apply a polytime algorithm $h_{AB} \circ h_{BC}$: $S_A = h_{AB}(h_{BC}(S_C))$. $\square$

Pictorially



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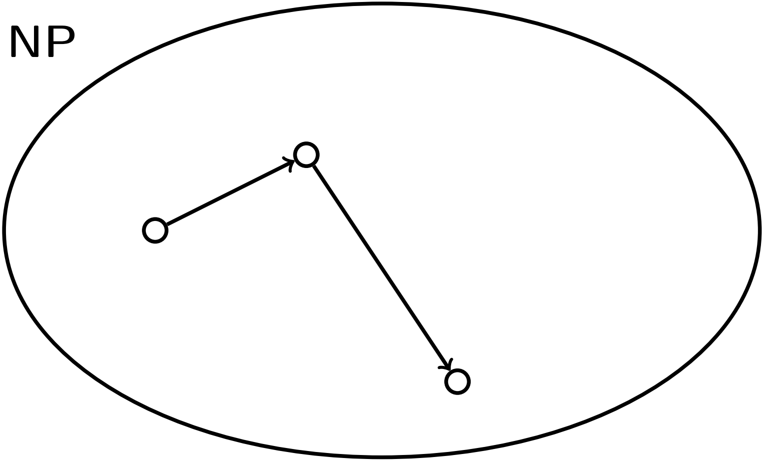

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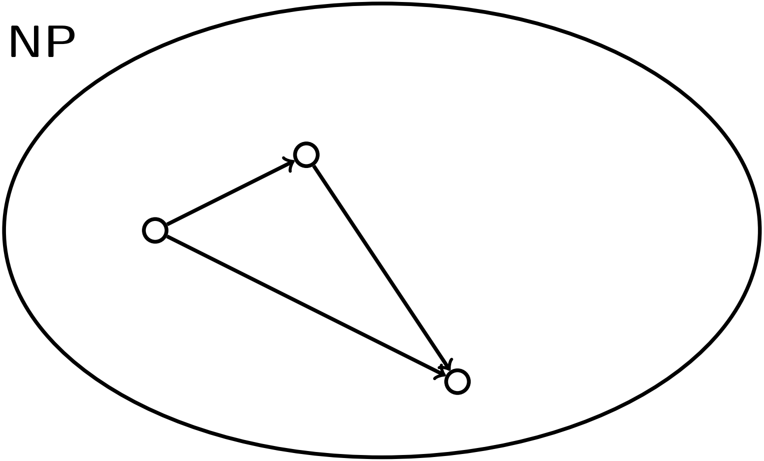

Showing **NP**-completeness

Corollary

If $A \rightarrow B$ and A is **NP**-complete, then so is B.

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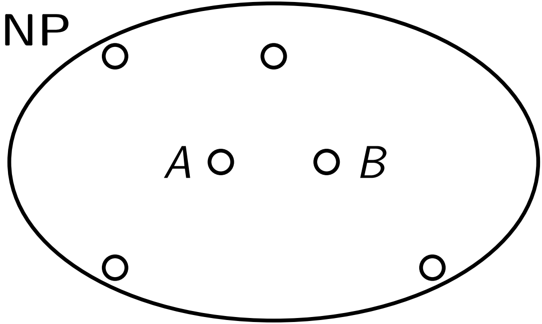

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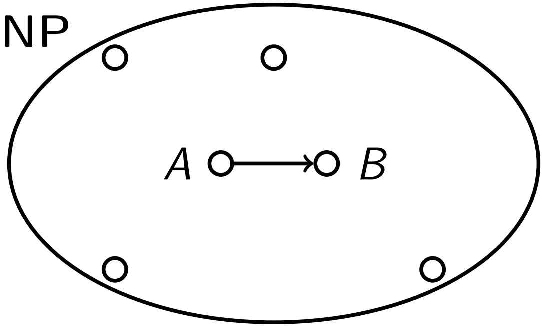

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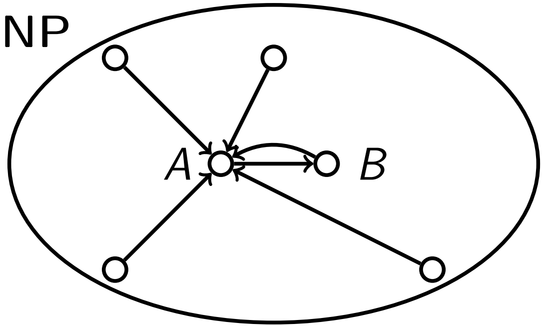

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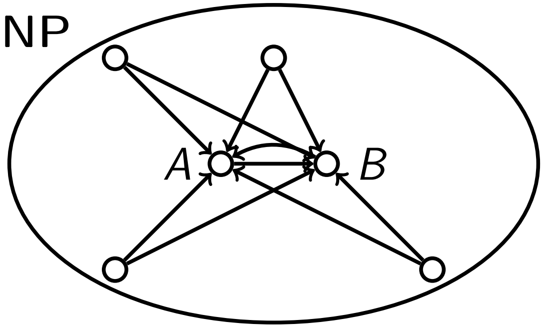
If $A \to B$ and A is **NP**-complete, then so is B.



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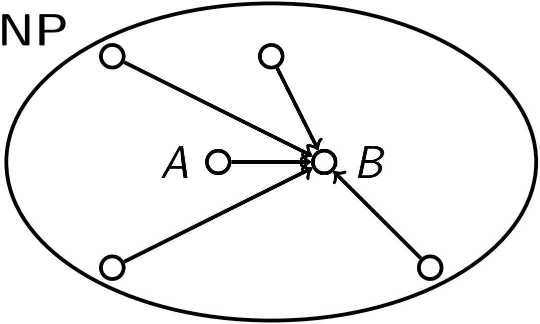
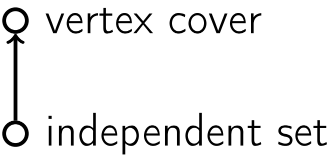


Plan



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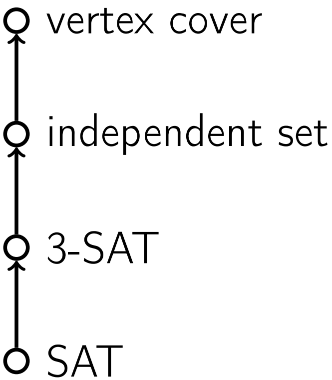

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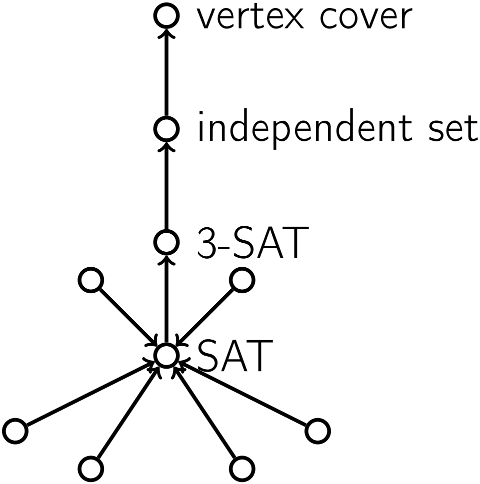

Outline

Independent set

Input: A graph and a budget b.

Output: A subset of at least b vertices such that no two of them are adjacent.

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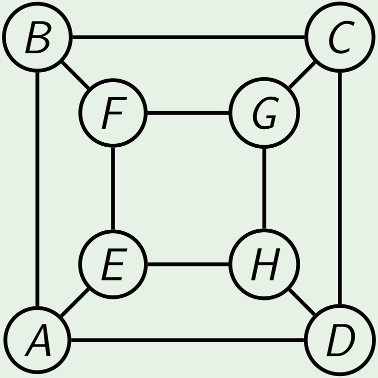
Vertex cover

Input: A graph and a budget b.

Output: A subset of at most b vertices that touches every edge.

Example



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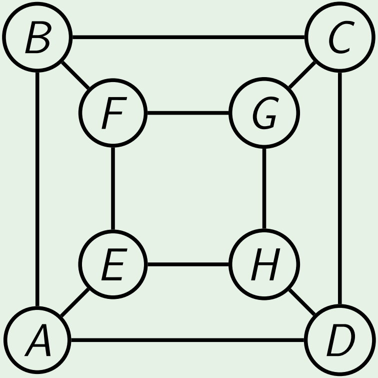
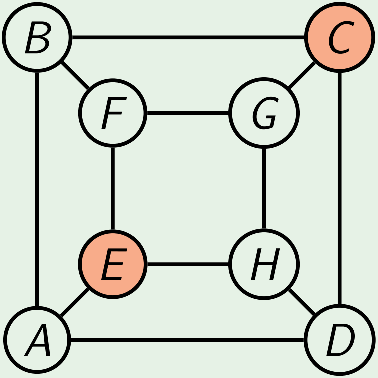


Independent sets:

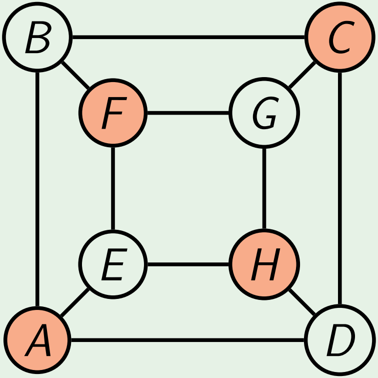
Example



Independent sets:

$\{E, C\}$

Example



Independent sets:

$\{E, C\}$ $\{A, C, F, H\}$

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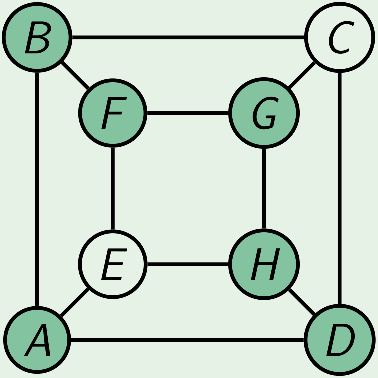

Independent sets:

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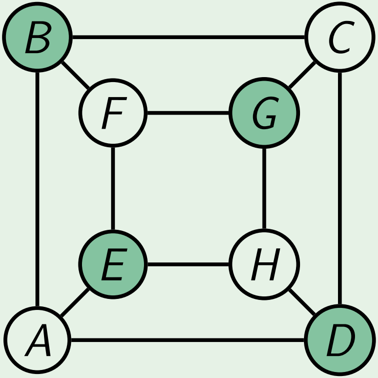
Vertex covers:

$\{A, B, D, F, G, H\}$

Example



Independent sets:

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$\{B, D, E, G\}$

Lemma

I is an independent set of $G(V, E)$, if and only if $V - I$ is a vertex cover of G.

Proof

- $\Rightarrow$ If I is an independent set, then there is no edge with both endpoints in I.
  Hence $V - I$ touches every edge.
- $\Leftarrow$ If $V - I$ touches every edge, then each edge has at least one endpoint in $V - I$.
  Hence I is an independent set. $\square$

Reduction

Independent set $\rightarrow$ vertex cover: to check whether $G(V, E)$ has an independent set of size at least b, check whether G has a vertex cover of size at most $|V| - b$:

- $f(G(V, E), b) = (G(V, E), |V| - b)$
- $h(S) = V - S$

Outline

3-SAT

Input: Formula F in 3-CNF (a collection of clauses each having at most three literals).

Output: An assignment of Boolean values to the variables of F satisfying all clauses, if exists.

Goal

Design a polynomial time algorithm that, given a 3-CNF formula F, outputs a graph G and an integer b, such that:

> *F is satisfiable, if and only if G has an independent set of size at least b.*

We need to find an assignment of Boolean values to variables, such that each clause contains at least one satisfied literal.

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Example

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Example

- Setting $x = 1, y = 1, z = 1$ satisfies a formula $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.
- Setting $x = 1, y = 0, z = 0$ also satisfies it: $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.

Alternatively, we need to select at least one literal from each clause, such that the set of selected literals is consistent: it does not contain a literal ℓ together with its negation $\overline{\ell}$.

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- Consistent: $\{x, x, \overline{z}\}$, $\{x, x, y\}$, $\{x, \overline{y}, \overline{z}\}$.
- Inconsistent: $\{y, \overline{y}, \overline{z}\}$, $\{z, x, \overline{z}\}$.

Using Alternative Statement

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$$

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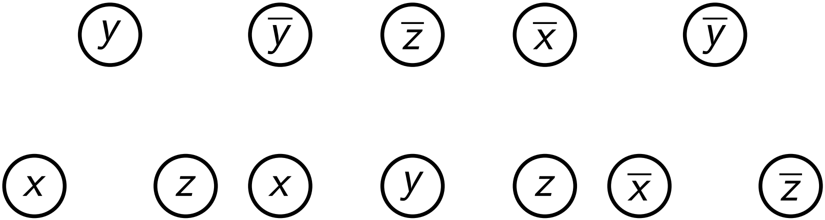

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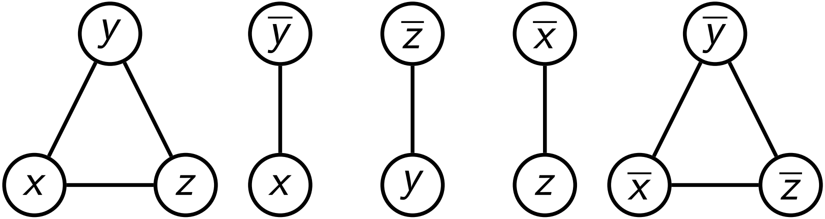

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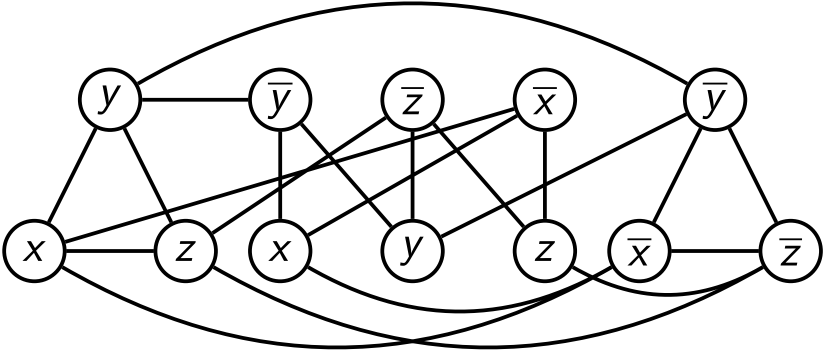

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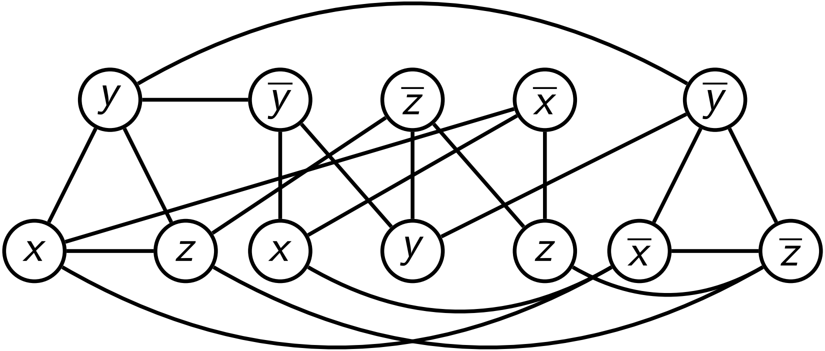

the formula is satisfiable iff the resulting graph has independent set of size 5

Transforming an Instance

- For each clause of the input formula F, introduce three (or two, or one) vertices in G labeled with the literals of this clause. Join every two of them.

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- F is satisfiable if and only if G has independent set of size equal to the number of clauses in F.
- Transformation takes polynomial time.

Transforming a Solution

- Given a solution S for G, just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).

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- Given a solution S for G, just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).
- If there is no solution for G, then F is unsatisfiable: indeed, a satisfying assignment for F would give a required independent set in G.

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Goal

Transform a CNF formula into an equisatisfiable 3-CNF formula. That is, reduce a problem to its special case.

Transforming an Instance

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- The second clause is shorter than C
- Repeat while there is a long clause

Running time

The running time of the transformation is polynomial: at each iteration we replace a clause with a shorter clause and a 3-clause. Hence the total number of iterations is at most the total number of literals of the initial formula.

Correctness

Lemma

The formulas $F = (\ell_1 \vee \ell_2 \vee A) \ldots$ and $F' = (\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A) \ldots$ are equisatisfiable.

Proof

$F = (\ell_1 \vee \ell_2 \vee A) \ldots$

$F' = (\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A) \ldots$

$\Rightarrow$ If either ℓ_1 or ℓ_2 is satisfied, set $y = 0$. Otherwise A must be satisfied. Then set $y = 1$.

$\Leftarrow$ If $(\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A)$ are satisfied, then so is $(\ell_1 \vee \ell_2 \vee A)$. $\square$

Transforming a Solution

Given a satisfying assignment for F', just throw away the values of all new variables (y's) to get a satisfying assignment of the initial formula.

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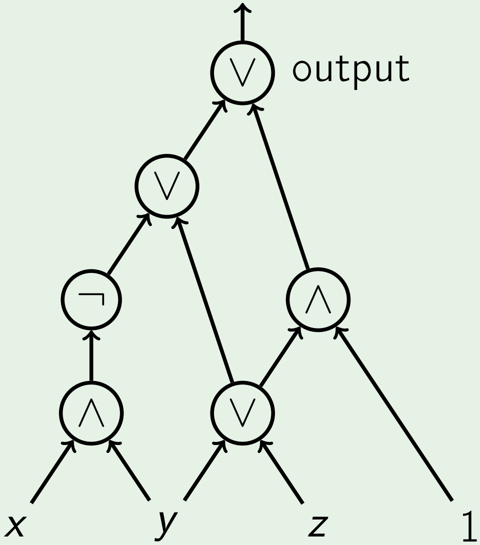
Instead, we show that any problem reduces to Circuit SAT problem, which, in turn, reduces to SAT.

Circuit



Definition

A circuit is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called inputs and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called gates: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as output.

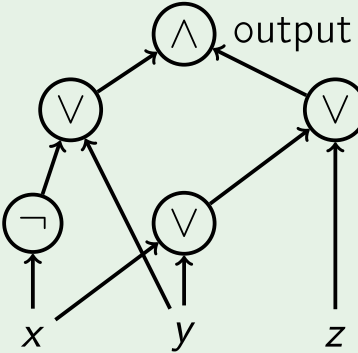
Circuit-SAT

Input: A circuit.

Output: An assignment of Boolean values to the input variables of the circuit that makes the output true.

SAT is a special case of Circuit-SAT as a CNF formula can be represented as a circuit:

Example: $(x \vee y \vee z)(y \vee \overline{x})$



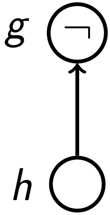
Circuit-SAT $\to$ SAT

To reduce Circuit-SAT to SAT, we need to design a polynomial time algorithm that for a given circuit outputs a CNF formula which is satisfiable, if and only if the circuit is satisfiable

Idea

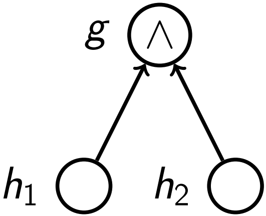
- Introduce a Boolean variable for each gate
- For each gate, write down a few clauses that describe the relationship between this gate and its direct predecessors

NOT Gates



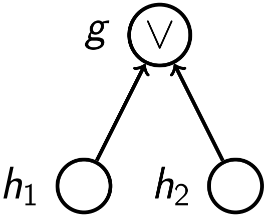
$$(h \vee g)(\overline{h} \vee \overline{g})$$

AND Gates



$$(h_1 \vee \overline{g})(h_2 \vee \overline{g})(\overline{h}_1 \vee \overline{h}_2 \vee g)$$

OR Gates



$$(\overline{h}_1 \vee g)(\overline{h}_2 \vee g)(h_1 \vee h_2 \vee \overline{g})$$

Output Gate

$$g \bigcirc \text{output} \tag{g}$$

- The resulting CNF formula is consistent with the initial circuit: in any satisfying assignment of the formula, the value of g is equal to the value of the gate labeled with g in the circuit

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- Therefore, the CNF formula is equisatisfiable to the circuit
- The reduction takes polynomial time

Goal

Reduce every search problem to Circuit-SAT.

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Goal

Reduce every search problem to Circuit-SAT.

- Let A be a search problem
- We know that there exists an algorithm $\mathcal{C}$ that takes an instance I of A and a candidate solution S and checks whether S is a solution for I in time polynomial in $|I|$
- In particular, $|S|$ is polynomial in $|I|$

Turn an Algorithm into a Circuit

- Note that a computer is in fact a circuit (of constant size!) implemented on a chip

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- Each step of the algorithm $\mathcal{C}(I, S)$ is performed by this computer's circuit
- This gives a circuit of size polynomial in $|I|$ that has input bits for I and S and outputs whether S is a solution for I (a separate circuit for each input length)

Reduction

To solve an instance I of the problem A:

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$

Reduction

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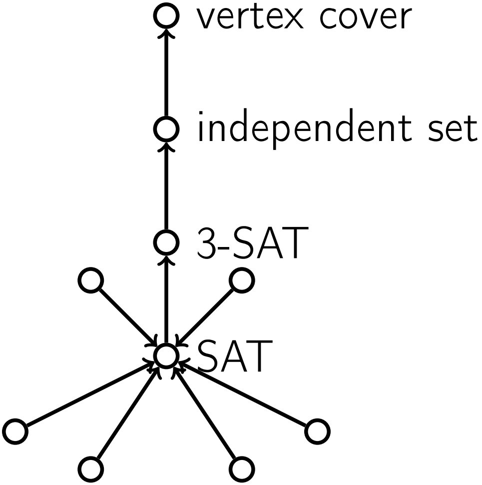
To solve an instance I of the problem A:

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$
- the inputs to this circuit encode candidate solutions
- use a Circuit-SAT algorithm for this circuit to find a solution (if exists)

Summary



Outline

Sudoku Puzzle

This part

A simple and efficient Sudoku solver

SAT: Theory and Practice

Theory: we have no algorithm checking the satisfiability of a CNF formula F with n variables in time $\text{poly}(|F|) \cdot 1.99^n$

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Practice: SAT-solvers routinely solve instances with thousands of variables

Solving Hard Problems in Practice

An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)

Solving Hard Problems in Practice

An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)
- Use a SAT solver

Sudoku Puzzle

Goal: fill in with digits the partially completed 9×9 grid so that each row, each column, and each of the nine 3×3 subgrids contains all the digits from 1 to 9.

Example

Variables

There will be $9 \times 9 \times 9 = 729$ Boolean variables: for $1 \leq i, j, k \leq 9$, $x_{ijk} = 1$, if and only if the cell $[i, j]$ contains the digit k

Exactly One Is True

Clauses expressing the fact that exactly one of the literals ℓ_1, ℓ_2, ℓ_3 is equal to 1:

$$(\ell_1 \lor \ell_2 \lor \ell_3)(\overline{\ell}_1 \lor \overline{\ell}_2)(\overline{\ell}_1 \lor \overline{\ell}_3)(\overline{\ell}_2 \lor \overline{\ell}_3)$$

Constraints

- Cell $[i, j]$ contains exactly one digit: $\texttt{ExactlyOneOf}(x_{ij1}, x_{ij2}, \ldots, x_{ij9})$

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- k appears exactly once in a 3×3 block:
  $\texttt{ExactlyOneOf}(x_{11k}, x_{12k}, \ldots, x_{33k})$
- $[i, j]$ already contains k: (x_{ijk})

Resulting Formula

State-of-the-art SAT-solvers find a satisfying assignment for the resulting formula in blink of an eye, though the corresponding search space has size about $2^{729} \approx 10^{220}$

NP-complete Problems: Search Problems

Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Advanced Algorithms and Complexity
Data Structures and Algorithms

Outline

Polynomial vs Exponential

running time:	n	n^2	n^3	2^n
less than 10^9:	10^9	$10^{4.5}$	10^3	29

Improving Brute Force Search

Usually, an efficient (polynomial) algorithm searches for a solution among an exponential number of candidates:

- there are $n!$ permutations of n objects

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Usually, an efficient (polynomial) algorithm searches for a solution among an exponential number of candidates:

- there are $n!$ permutations of n objects
- there are 2^n ways to partition n objects into two sets
- there are n^{n-2} spanning trees in a complete graph on n vertices

This module

- For thousands of practically important problems we don't have an efficient algorithm yet

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- For thousands of practically important problems we don't have an efficient algorithm yet
- An efficient algorithm for one such problem automatically gives efficient algorithms for all these problems!
- $1M prize for constructing such an algorithm or proving that this is impossible!

Outline

Boolean Formulas

Formula in conjunctive normal form

$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$

- x, y, z are Boolean variables (values: `true`/`false` or 1/0)
- literals are variables (x, y, z) and their negations ($\overline{x}, \overline{y}, \overline{z}$)
- clauses are disjunctions (logical or) of literals

Satisfiability (SAT)

Input: Formula F in conjunctive normal form (CNF).

Output: An assignment of Boolean values to the variables of F satisfying all clauses, if exists.

Examples

- The formula $(x \vee \overline{y})(\overline{x} \vee \overline{y})(x \vee y)$ is satisfiable: set $x = 1$, $y = 0$.
- The formula $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$ is satisfiable: set $x = 1, y = 1, z = 1$ or $x = 1, y = 0, z = 0$.
- The formula $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$ is unsatisfiable.

Satisfiability

- Classical hard problem
- Many applications: e.g., hardware/software verification, planning, scheduling
- Many hard problems are stated in terms of SAT naturally
- SAT solvers (will see later), SAT competition

- SAT is a typical search problem
- Search problem: given an instance I, find a solution S or report that none exists
- Main property: one must be able to check quickly whether S is indeed a solution for I
- By saying quickly, we mean in time polynomial in the length of I. In particular, the length of S should be polynomial in the length of I

Definition

A search problem is defined by an algorithm $\mathcal{C}$ that takes an instance I and a candidate solution S, and runs in time polynomial in the length of I. We say that S is a solution to I iff $\mathcal{C}(S, I) = \texttt{true}$.

Example

For SAT, I is a Boolean formula, S is an assignment of Boolean constants to its variables. The corresponding algorithm $\mathcal{C}$ checks whether S satisfies all clauses of I.

Next part

A few practical search problems for which polynomial algorithms remain unknown

Outline

Outline

Traveling salesman problem (TSP)

Input: Pairwise distances between n cities and a budget b.

Output: A cycle that visits each vertex exactly once and has total length at most b.

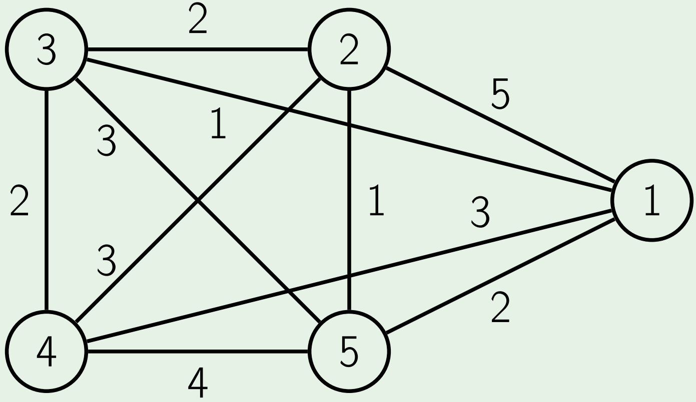
Delivery Company

`https://simple.wikipedia.org/wiki/Travelling_salesman_problem`

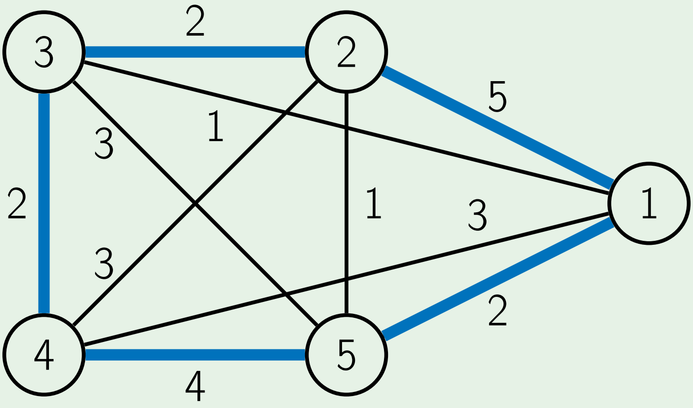
Drilling Holes in a Circuit Board

https://developers.google.com/optimization/routing/tsp

Example

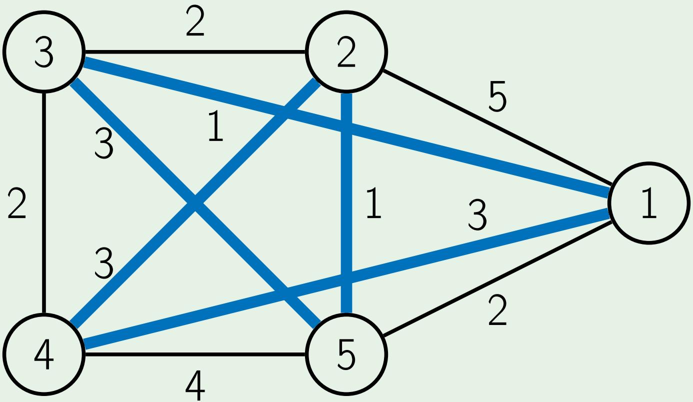


Example



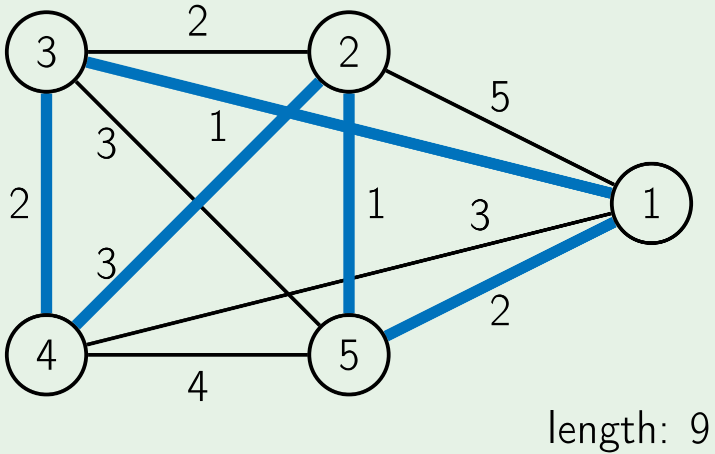
length: 15

Example



length: 13

Example



Search Problem

- TSP is a search problem: given a sequence of vertices, it is easy to check whether it is a cycle visiting all the vertices of total length at most b
- TSP is usually stated as an optimization problem; we stated its decision version to guarantee that a candidate solution can be efficiently checked for correctness

Algorithms

- Check all permutations: about $O(n!)$, extremely slow
- Dynamic programming: $O(n^2 2^n)$
- No significantly better upper bound is known
- There are heuristic algorithms and approximation algorithms

Comparing to MST

MST

Decision version:
given n cities, connect them by $(n-1)$ roads of minimal total length

Comparing to MST

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Can be solved efficiently

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TSP

Decision version: given n cities, connect them in a path by $(n-1)$ roads of minimal total length

Can be solved efficiently

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TSP

Decision version:
given n cities, connect them in a path by $(n-1)$ roads of minimal total length

No polynomial algorithm known!

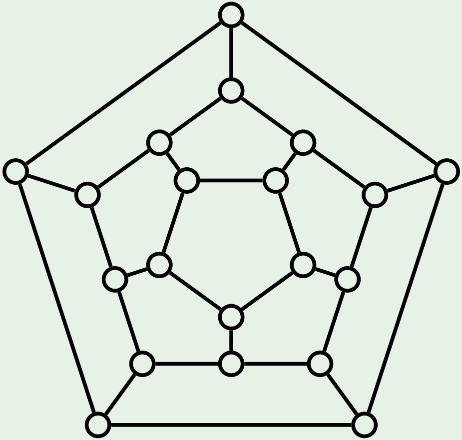
Outline

Hamiltonian cycle

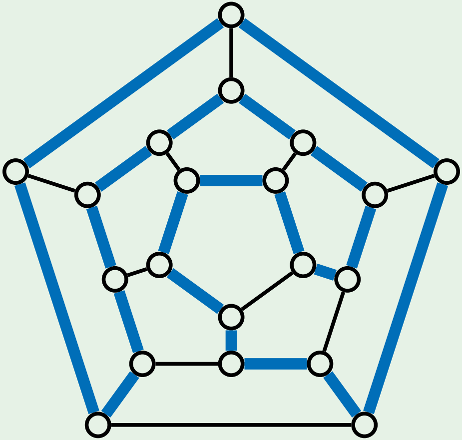
Input: A graph.

Output: A cycle that visits each vertex of the graph exactly once.

Example



Example



Eulerian cycle

Input: A graph.

Output: A cycle that visits each edge of the graph exactly once.

Eulerian cycle

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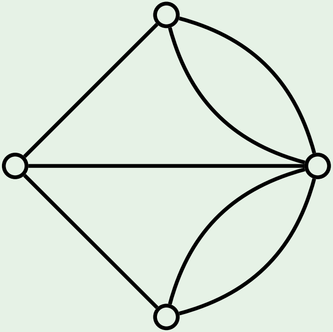
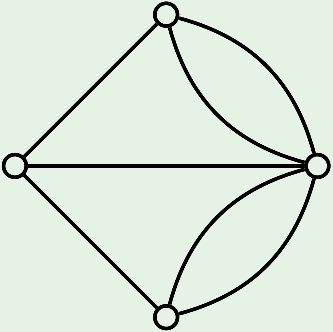
Theorem

A graph has an Eulerian cycle if and only if it is connected and the degree of each vertex is even.

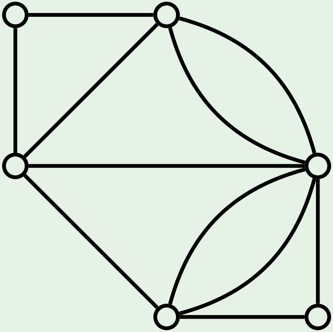
Non-Eulerian graph



Non-Eulerian graph



Eulerian graph



Eulerian cycle

Find a cycle visiting each edge exactly once

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Can be solved efficiently

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Hamiltonian cycle

Find a cycle visiting each vertex exactly once

Can be solved efficiently

Eulerian cycle

Find a cycle visiting each edge exactly once

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Hamiltonian cycle

Find a cycle visiting each vertex exactly once

No polynomial algorithm known!

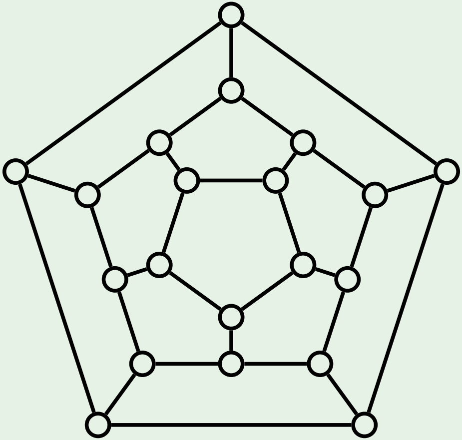
Outline

Longest path

Input: A weighted graph, two vertices s, t, and a budget b.

Output: A simple path (containing no repeated vertices) of total length at least b.

Example



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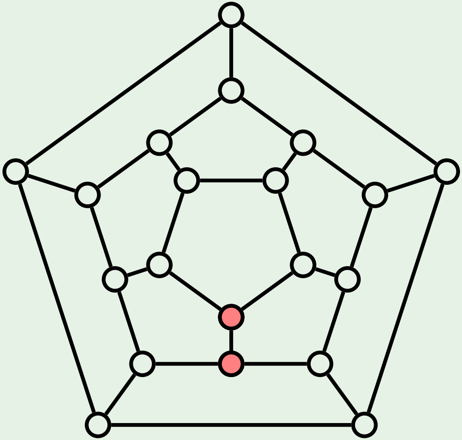

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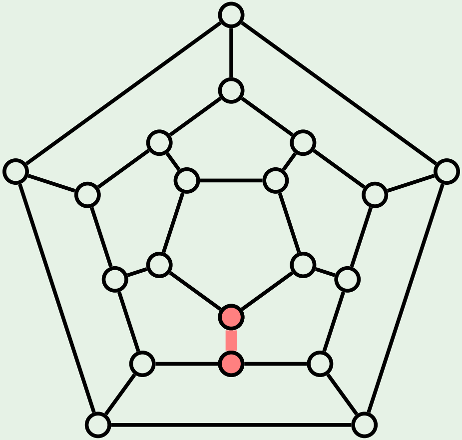

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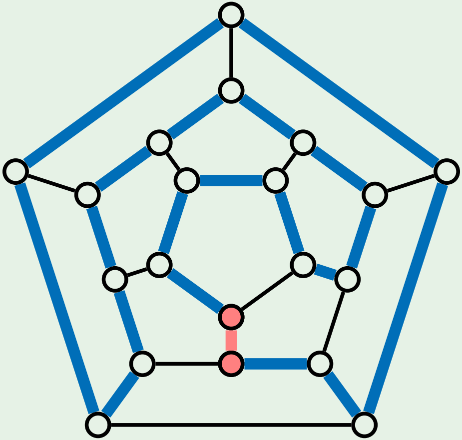

Shortest path

Find a simple path from s to t of total length at most b

Shortest path

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Can be solved efficiently

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Longest path

Find a simple path from s to t of total length at least b

No polynomial algorithm known!

Outline

Integer linear programming

Input: A set of linear inequalities $\mathbf{A}\mathbf{x} \leq \mathbf{b}$.

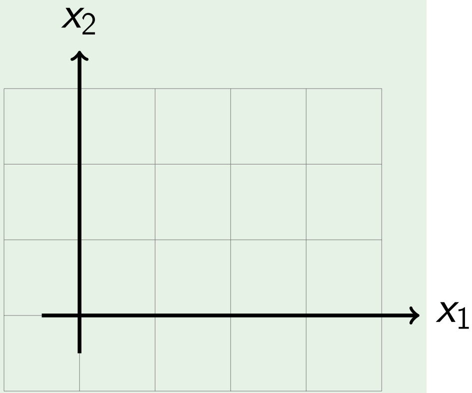
Output: Integer solution.

Example

$$
\begin{aligned}
x_1 &\geq 0.5 \\
-x_1 + 8x_2 &\geq 0 \\
-x_1 - 8x_2 &\geq -8
\end{aligned}
$$

Example

$$x_1 \geq 0.5$$
$$-x_1 + 8x_2 \geq 0$$
$$-x_1 - 8x_2 \geq -8$$



Example

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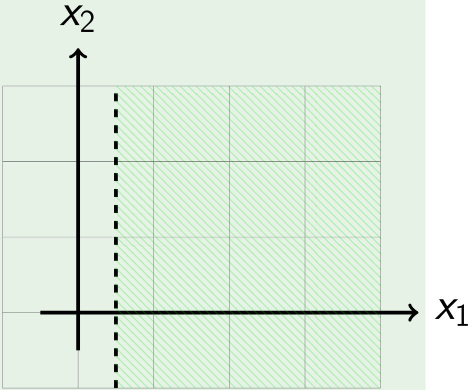

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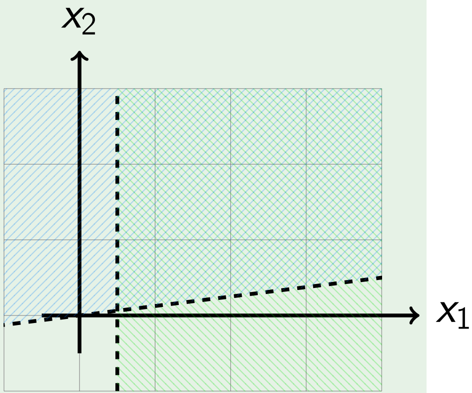

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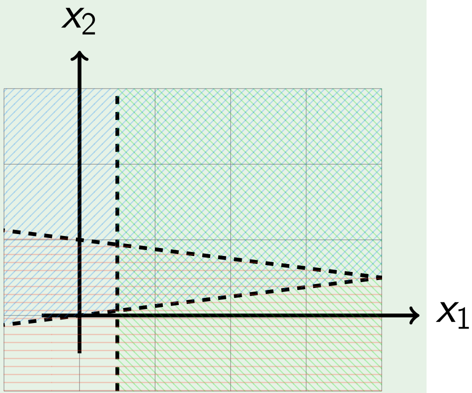


LP (decision version)

Find a real solution of a system of linear inequalities

LP
(decision version)

Find a real solution of a system of linear inequalities

Can be solved efficiently

LP (decision version)

Find a real solution of a system of linear inequalities

ILP

Find an integer solution of a system of linear inequalities

Can be solved efficiently

LP (decision version)

Find a real solution of a system of linear inequalities

Can be solved efficiently

ILP

Find an integer solution of a system of linear inequalities

No polynomial algorithm known!

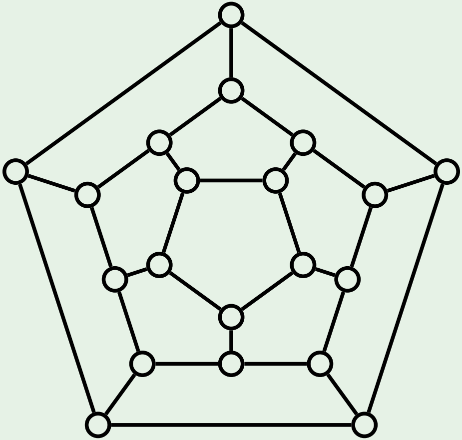
Outline

Independent set

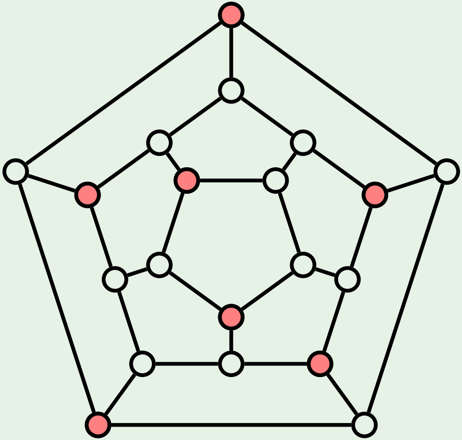
Input: A graph and a budget b.

Output: A subset of vertices of size at least b such that no two of them are adjacent.

Example

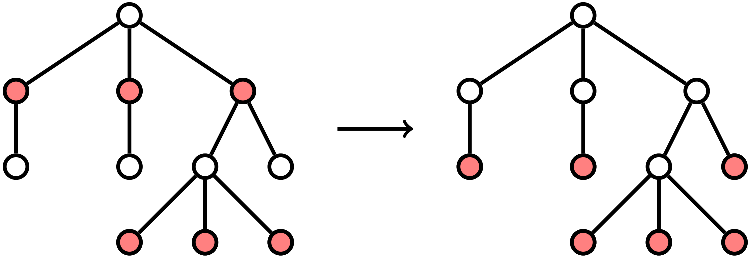


Example



Independent Sets in a Tree

A maximal independent set in a tree can be found by a simple greedy algorithm: it is safe to take into a solution all the leaves.



Independent set in a tree

Find an independent set of size at least b in a given tree

Independent set in a tree

Find an independent set of size at least b in a given tree

Can be solved efficiently

Independent set in a tree

Find an independent set of size at least b in a given tree

Independent set in a graph

Find an independent set of size at least b in a given graph

Can be solved efficiently

Independent set in a tree	Independent set in a graph
Find an independent set of size at least b in a given tree	Find an independent set of size at least b in a given graph

Can be solved efficiently	No polynomial algorithm known!

Next part

It turns out that all these hard problems are in a sense a single hard problem:
a polynomial time algorithm for any of these problems can be used to solve all of them in polynomial time!

Outline

Class NP

Definition

A search problem is defined by an algorithm $\mathcal{C}$ that takes an instance I and a candidate solution S, and runs in time polynomial in the length of I. We say that S is a solution to I iff $\mathcal{C}(S, I) = \texttt{true}$.

Class NP

Definition

A search problem is defined by an algorithm $\mathcal{C}$ that takes an instance I and a candidate solution S, and runs in time polynomial in the length of I. We say that S is a solution to I iff $\mathcal{C}(S, I) = \texttt{true}$.

Definition

NP is the class of all search problems.

- **NP** stands for “non-deterministic polynomial time”: one can guess a solution, and then verify its correctness in polynomial time
- In other words, the class **NP** contains all problems whose solutions can be efficiently verified

Class **P**

Definition

P is the class of all search problems that can be solved in polynomial time.

Class P

Problems whose solution can be found efficiently

Class P

Problems whose solution can be found efficiently

- MST
- Shortest path
- LP
- IS on trees

Class **P**

Problems whose solution can be found efficiently

Class **NP**

Problems whose solution can be verified efficiently

- MST
- Shortest path
- LP
- IS on trees

Class P

Problems whose solution can be found efficiently

- MST
- Shortest path
- LP
- IS on trees

Class NP

Problems whose solution can be verified efficiently

- TSP
- Longest path
- ILP
- IS on graphs

The main open problem in Computer Science

Is P equal to NP?

The main open problem in Computer Science

Is P equal to NP?

Millenium Prize Problem

Clay Mathematics Institute: $1M prize for solving the problem

- If **P**=**NP**, then all search problems can be solved in polynomial time.

- If **P**$=$**NP**, then all search problems can be solved in polynomial time.
- If **P**$\neq$**NP**, then there exist search problems that cannot be solved in polynomial time.

Next part

We'll show that the satisfiability problem, the traveling salesman problem, the independent set problem, the integer linear programming are the hardest problems in **NP**.