Convex Optimization: Fall 2019

Based on lectures by Ryan Tibshirani

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These notes are scribed by me, and it is difficult to avoids errors.

Course Information

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Schedule

This course is divided into five parts:

(i) **Theory I: Fundamentals**

- Introduction
- Convexity I: Sets and Functions
- Convexity II: Optimization Basics
- Canonical Problem Forms

(ii) **Algorithms I: First-order methods**

(iii) **Theory II: Duality and optimality**

(iv) **Algorithms II: Second-order methods**

(v) **Advanced topics**

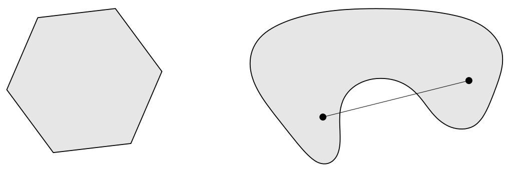
Contents

1 Convexity I: Sets and Functions

1.1 Convex sets

Definition (Convex set)**.** $C \subseteq \mathbb{R}^n$ such that

$$x, y \in C \Longrightarrow tx + (1-t)y \in C, \text{ for all } 0 \leq t \leq 1$$



Definition (Convex combination)**.** For $x_1, \cdots, x_k \in \mathbb{R}^n$, any linear combination

$$\theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_i \geq 0, i = 1, \cdots, k$, and $\sum_{i=1}^k \theta_i = 1$.

Definition (Convex hull)**.** The convex hull of C, $\text{conv}(C)$, is all convex combinations of elements, and is always convex.

1.1.1 Example of convex sets:

(i) **Trivial ones:** empty set, point, line

(ii) **Norm ball:** $\{x : \|x\| \leq r\}$, for given norm $\|\cdot\|$, radius r

(iii) **Hyperplane:** $\{x : a^T x = b\}$, for given a, b

(iv) **Halfspace:** $\{x : a^T x \leq b\}$, for given a, b

(v) **Affine space:** $\{x : Ax = b\}$, for given A, b

(vi) **Polyhedron:** $\{x : Ax \leq b\}$, while inequality $\leq$ is interpreted componentwise. Note: the set $\{x : Ax \leq b, Cx = d\}$ is also a Polyhedron

(vii) **Simplex:** special case of polyhedra, given by $\text{conc}\{x_0, \cdots, x_k\}$, where these points are affinely independent. The canonical example is the **probability simplex**,

$$\text{conv}\{e_1, \cdots, e_n\} = \{w : w \geq 0, 1^T w = 1\}$$

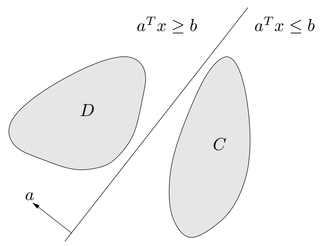
1.1.2 Key properties of convex sets:

(i) **Separating hyperplane theorem:** two disjoint convex sets have a separating between hyperplane them Formally, if C, D are nonempty convex sets with $C \cap D = \emptyset$, then there exists a, b such that

$$C \subseteq \{x : a^T x \leq b\}$$
$$D \subseteq \{x : a^T x \geq b\}$$

(ii) **Supporting hyperplane theorem:** a boundary point of a convex set has a supporting hyperplane passing through it. Formally, if C is a nonempty convex set, and $x_0 \in \text{bd}(C)$, then there exists a such that

$$C \subseteq \{x : a^T x \leq a^T x_0\}$$



1.1.3 Operations preserving convexity

(i) **Intersection:** the intersection of convex sets is convex

(ii) **Scaling and translation:** if C is convex, then

$$aC + b = \{ax + b : x \in C\}$$

is convex for any a, b

(iii) **Affine images and preimages:** if $f(x) = Ax + b$ and C is convex then

$$f(C) = \{f(x) : x \in C\}$$

is convex, and if D is convex then

$$f^{-1}(D) = \{x : f(x) \in D\}$$

is convex

(iv) **Perspective images and preimages:** the perspective function is $P : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ where $\mathbb{R}_{++}$ denotes positive reals,

$$P(x, z) = x/z$$

for $z > 0$. If $C \subseteq \text{dom}(P)$ is convex then so is $P(C)$, and if D is convex then so is $P^{-1}(D)$

(v) **Linear-fractional images and preimages:** the perspective map composed with an affine function,

$$f(x) = \frac{Ax + b}{c^T x + d}$$

is called a linear-fractional function, defined on $c^T x + d > 0$. If $C \subseteq \text{dom}(f)$ is convex then so if $f(C)$, and if D is convex then so is $f^{-1}(D)$

1.2 Cones

Definition (Cone)**.** $C \subseteq \mathbb{R}^n$ such that

$$x \in C \Longrightarrow tx \in C, \text{ for all } t \geq 0$$

Definition (Convex cone)**.** Cone that is also convex, i.e.,

$$x_1, x_2 \in C \Longrightarrow t_1 x_1 + t_2 x_2 \in C, \text{ for all } t_1, t_2 \geq 0$$

Definition (Conic combination)**.** For $x_1, \cdots, x_k \in \mathbb{R}^n$, any linear combination

$$\theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_i \geq 0, i = 1, \cdots, k$.

Definition (Conic hull)**.** The conic hull of C, cone(C), is all conic combinations of elements.

1.2.1 Example of convex cones:

(i) **Norm cone:** $\{(x,t) : \|x\| \le r\}$, for a norm $\|\cdot\|$. Under the ℓ_2 norm $\|\cdot\|_2$, called second-order cone

(ii) **Normal cone:** given any set C and point $x \in C$, we can define

$$\mathcal{N}_C(x) = \{g : g^T x \ge g^T y, \text{ for all } y \in C\}$$

This is always a convex cone, regardless of C

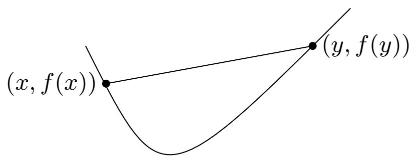
(iii) **Positive semidefinite cone:** $\mathbb{S}^n_+ = \{X \in \mathbb{S}^n : X \succeq 0\}$, where $X \succeq 0$ means that X is positive semidefinite (and $\mathbb{S}^n$ is the set of $n \times n$ symmetric matrices)

1.3 Convex function

Definition (Convex function)**.** $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{dom}(f) \subseteq \mathbb{R}^n$ convex, and

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \text{ for all } 0 \le t \le 1$$

and all $x, y \in \mathrm{dom}(f)$



Definition (Concave function)**.** $f : \mathbb{R}^n \to \mathbb{R}$ such that $\mathrm{dom}(f) \subseteq \mathbb{R}^n$ convex, and

$$f(tx + (1-t)y) \ge tf(x) + (1-t)f(y), \text{ for all } 0 \le t \le 1$$

and all $x, y \in \mathrm{dom}(f)$, so that

$$f \text{ concave} \Leftrightarrow -f \text{ convex}$$

Important modifiers:

(i) **Strictly convex:** $f(tx+(1-t)y) < tf(x)+(1-t)f(y)$, for $x \ne y$ and $0 < t < 1$. In words, f is convex and has greater curvature than a linear function

(ii) **Strongly convex:** with parameter $m > 0 : f - \frac{m}{2}\|x\|_2^2$ is convex. In words, f is at least as convex as a quadratic function

(iii) **Note:** strongly convex $\Rightarrow$ strictly convex $\Rightarrow$ convex (Analogously for concave functions)

1.3.1 Example of convex functions

(i) **Univariate functions:**

(a) Exponential function: e^{ax} is convex for any a over $\mathbb{R}$

(b) Power function: x^a is convex for $a \ge 1$ or $a \le 0$ over $\mathbb{R}_+$ (nonnegative reals)

(c) Power function: x^a is concave for $0 \le a \le 1$ over $\mathbb{R}_+$

(d) Logarithmic function: $\log(x)$ is concave over $\mathbb{R}_{++}$

(ii) **Affine function:** $a^T x + b$ is both convex and concave

(iii) **Quadratic function:** $\frac{1}{2}x^TQx + b^Tx + c$ is convex provided that $Q \succeq 0$ (positive semidefinite)

(iv) **Least squares loss:** $\|y - Ax\|_2^2$ is always convex (since A^TA is always positive semidefinite)

(v) **Norm:** $\|x\|$ is convex for any norm; e.g. ℓ_p norms,

$$\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p} \text{ for } p \geq 1, \quad \|x\|_\infty = \max_{i=1,\cdots .n} |x_i|$$

and also operator (spectral) and trace (nuclear) norms,

$$\|X\|_{op} = \sigma_1(X), \|X\|_{tr} = \sum_{i=1}^r \sigma_r(X)$$

where $\sigma_1(X) \geq \cdots \geq \sigma_r(X) \geq 0$ are the singular values of the matrix X

(vi) **Indicator function:** if C is convex, then its indicator function

$$I_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

(vii) **Support function:** for any set C(convex or not), its indicator function

$$I_C^*(x) = \max_{y \in C} x^Ty$$

is convex

(viii) **Max function:** $f(x) = \max\{x_1, \cdots, x_n\}$ is convex

1.3.2 Key properties of convex functions

(i) A function is convex if and only if its restriction to any line is convex

(ii) **Epigraph characterization:** a function f is convex if and only if its epigraph

$$\text{epi}(f) = \{(x,t) \in \text{dom}(f) \times \mathbb{R} : f(x) \leq t\}$$

is a convex set

(iii) **Convex sublevel sets:** if f is convex, then its sublevel sets

$$\{x \in \text{dom}(f) : f(x) \leq t\}$$

are convex, for all $t \in \mathbb{R}$. The converse is not true

(iv) **First order characterization:** if f is differentiable, then f is convex if and only if dom(f) is convex, and

$$f(y) \geq f(x) + \nabla f(x)^T(y - x)$$

for all $x, y \in \text{dom}(f)$. Therefore for a differentiable convex function $\nabla f(x) = 0 \Leftrightarrow x$ minimizes f

(v) **Second order characterization:** if f is twice differentiable, then f is convex if and only if dom(f) is convex, and $\nabla^2 f(x) \succeq 0$ for all $x \in \text{dom}(f)$

(vi) **Jensen's inequality:** if f is convex, and X is a random variable supported on dom(f), then $f(\mathbb{E}[X]) \leq \mathbb{E}[f(x)]$

1.3.3 Operations preserving convexity

(i) **Nonnegative linear combination:** $f_1, \cdots, f_m$ convex implies $a_1 f_1 + \cdots + a_m f_m$ convex for any $a_1, \cdots, a_m \geq 0$

(ii) **Pointwise maximization:** if f_s is convex for any $s \in S$,then $f(x) = \max_{s \in S} f_s(x)$ is convex. Note that the set S here (number of functions f_s) can be infinite

(iii) **Partial minimization:** if $g(x, y)$ is convex in x, y, and C is convex, then $f(x) = \min_{y \in C} g(x, y)$ is convex

(iv) **Affine composition:** if f is convex, then $g(x) = f(Ax + b)$ is convex

(v) **General composition:** suppose $f = h \circ g$, where $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$. Then:

    f is convex if h is convex and nondecreasing, g is convex
    f is convex if h is convex and nonincreasing, g is concave
    f is concave if h is concave and nondecreasing, g is concave
    f is concave if h is concave and nonincreasing, g is convex

(vi) **Vector composition:** suppose that

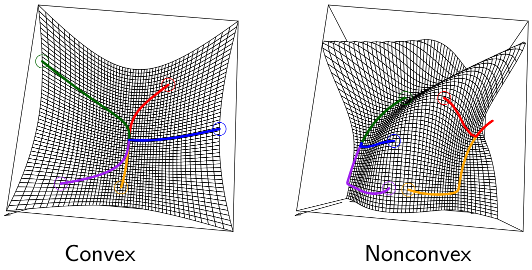
$$f(x) = h(g(x)) = h(g_1(x), \cdots, g_k(x))$$

where $g : \mathbb{R}^n \to \mathbb{R}^k$, $h : \mathbb{R}^k \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$. Then:

    f is convex if h is convex and nondecreasing in each argument, g is convex
    f is convex if h is convex and nonincreasing in each argument, g is concave
    f is concave if h is concave and nondecreasing in each argument, g is concave
    f is concave if h is concave and nonincreasing in each argument, g is convex

2 Convexity II: Optimization Basics

2.1 Optimization terminology



Definition (Optimization problem).

$$\begin{aligned} \min_{x \in D} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0,\ i = 1, \ldots, m \\ & h_j(x) = 0,\ j = 1, \ldots, r \end{aligned}$$

here $D = \text{dom}(f) \cap \bigcap_{i=1}^{m} \text{dom}(g_i) \cap \bigcap_{j=1}^{r} \text{dom}(h_j)$, common domain of all functions.

Definition (Convex optimization problem).

$$\begin{aligned} \min_{x \in D} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0,\ i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

where f and $g_i, i = 1, \cdots, m$ are all convex, and the optimization domain is $D = \text{dom}(f) \cap \bigcap_{i=1}^{m} \text{dom}(g_i)$.

For convex optimization problem, **local minima are global minima**. Formally, if x is feasible—$x \in D$, and satisfies all constraints and minimizes f in a local neighborhood,

$$f(x) \leq f(y) \text{ for all feasible } y,\ \|x - y\|_2 \leq \rho$$

then

$$f(x) \leq f(y) \text{ for all feasible } y$$

Terminologies:

(i) f is called criterion or objective function

(ii) g_i is called inequality constraint function

(iii) If $x \in D, g_i(x) \leq 0, i = 1, \cdots, m$, and $Ax = b$ then x is called a feasible point

(iv) The minimum of $f(x)$ over all feasible points x is called the optimal value, written f^*

(v) If x is feasible and $f(x) = f^*$, then x is called optimal; also called a solution, or a minimizer

(vi) If x is feasible and $f(x) \leq f^* + \epsilon$, then x is called ϵ-suboptimal

(vii) If x is feasible and $g_i(x) = 0$, then we say g_i is active at x

(viii) Convex minimization can be reposed as concave maximization, i.e. $\min f(x) \Leftrightarrow \max -f(x)$, both are called convex optimization problems

(ix) the optimization problem can be rewritten as

$$\min_x f(x) \quad \text{subject to } x \in C$$

where C is the feasible set. Hence the formulation is complete general. With I_C the indicator of C, it can also be written as the unconstrained form

$$\min_x \; f(x) + I_C(x)$$

Definition (Solution set)**.** Let X_{opt} be the set of all solutions of convex problem, written

$$\begin{aligned} X_{opt} = \arg\min \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \; i = 1, \ldots, m \\ & Ax = b \end{aligned}$$

Two key properties:

(i) X_{opt} is a convex set, and it can be proofed using definitions. If x, y are solutions, then for $0 \le t \le 1$, $t(x + (1-t)y)$ is also a solution

(ii) If f is strictly convex, then solution is unique, i.e., X_{opt} contains one element

2.2 First-order optimality condition

Definition (First-order optimality condition)**.** For a convex problem

$$\min_x f(x) \quad \text{subject to } x \in C$$

and differentiable f, a feasible point x is optimal if and only if

$$\nabla f(x)^T (y - x) \ge 0 \quad \text{for all } y \in C$$

this means all feasible directions from x are aligned with gradient $\nabla f(x)$

Important case: if $C \in \mathbb{R}^n$ (unconstrained optimization), then optimality condition reduces to familiar $\nabla f(x) = 0$.

2.2.1 Example: quadratic minimization

Consider minimizing the quadratic function

$$f(x) = \frac{1}{2} x^T Q x + b^T x + c$$

where $Q \succeq 0$. The first-order condition says that solution satisfies

$$\nabla f(x) = Qx + b = 0$$

(i) if $Q \succeq 0$, then there is a unique solution $x = -Q^{-1} b$

(ii) if Q is singular and $b \notin \text{col}(Q)$, then there is no solution (i.e., $\min_x f(x) = -\infty$)

(iii) if Q is singular and $b \in \text{col}(Q)$, then there are infinitely many solutions

$$x = -Q^+ b + z, \quad z \in \text{null}(Q)$$

where Q^+ is the pseudoinverse of Q. Note: $\text{null}(Q) = \{z \in C^n : Qz = 0\}$

2.2.2 Example: equality-constrained minimization

Consider the equality-constrained convex problem

$$\min_x f(x) \quad \text{subject to } Ax = b$$

with f differentiable. Let's prove Lagrange multiplier optimality condition

$$\nabla f(x) + A^T u = 0 \quad \text{for some } u$$

According to first-order optimality, solution x satisfies $Ax = b$ and

$$\nabla f(x)^T (y - x) \geq 0 \quad \text{for all } y \text{ such that } Ay = b$$

This is equivalent to

$$\nabla f(x)^T v = 0 \text{ for all } v \in \text{null}(A)$$

Result follows because $\text{null}(A)^\perp = \text{row}(A)$.

2.2.3 Example: projection onto a convex set

Consider projection onto convex set C

$$\min_x \|a - x\|_2^2 \quad \text{subject to } x \in C$$

First-order optimality condition says that the solution x satisfies

$$\nabla f(x)^T (y - x) = (x - a)^T (y - x) \geq 0 \quad \text{for all } y \in C$$

Equivalent, this says that

$$a - x \in \mathcal{N}_C(x)$$

where recall $\mathcal{N}_C(x)$ is the normal cone to C at x.

2.3 Partial optimization

We can always partial optimize a convex problem and retain convexity, e.g.

$$\begin{array}{lll} \min\limits_{x_1, x_2} & f(x_1, x_2) \\ \text{subject to} & g_1(x_1) \leq 0 \\ & g_2(x_2) \leq 0 \end{array} \Longleftrightarrow \begin{array}{ll} \min\limits_{x_1} & \tilde{f}(x_1) \\ \text{subject to} & g_1(x_1) \leq 0 \end{array}$$

where $\tilde{f}(x_1) = \min\{f(x_1, x_2) : g_2(x_2) \leq 0\}$. The right problem is convex if the left problem is.

2.3.1 Excample: hinge form of SVMs

Recall the SVM problem

$$\begin{array}{ll} \min\limits_{\beta, \beta_0, \xi} & \frac{1}{2}\|\beta\|_2^2 + C \sum\limits_{i=1}^n \xi_i \\ \text{subject to} & \xi_i \geq 0, y_i \left(x_i^T \beta + \beta_0\right) \geq 1 - \xi_i, i = 1, \ldots, n \end{array}$$

Rewrite the constraints as $\xi_i \geq \max\{0, 1 - y_i \left(x_i^T \beta + \beta_0\right)\}$. Indeed we can argue that we have $=$ at solution. Therefor plugging in for optimal ξ gives the hinge form of SVMs:

$$\min_{\beta, \beta_0} \frac{1}{2}\|\beta\|_2^2 + C \sum_{i=1}^n [1 - y_i \left(x_i^T \beta + \beta_0\right)]_+$$

where $a_+ = \max\{0, a\}$ is called the hinge function

2.4 Transformations and change of variables

If $h : \mathbb{R} \to \mathbb{R}$ is a monotone increasing translation, then

$$\min_x f(x) \quad \text{subject to } x \in C \iff \min_x h(f(x)) \quad \text{subject to } x \in C$$

Similarly, inequality or equality constraints can be transformed and yield equivalent optimization problem. This means we can use this to reveal the 'hidden convexity' of a problem.

If $\phi : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, and its image covers feasible set C, then we can change variables in an optimization problem:

$$\min_x f(x) \quad \text{subject to } x \in C \iff \min_y f(\phi(y)) \quad \text{subject to } \phi(y) \in C$$

2.4.1 Example: geometric programming

A monomial is a function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ of the form

$$f(x) = \gamma x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

for $\gamma > 0, a_1, \cdots, a_n \in \mathbb{R}$. A posynomial is a sum of monomials,

$$f(x) = \sum_{k=1}^{p} \gamma_k x_1^{a_k 1} x_2^{a_k 2} \cdots x_n^{a_k n}$$

A geometric program is of the form

$$\begin{array}{ll} \min\limits_x & f(x) \\ \text{subject to} & g_i(x) \leq 0, \ i = 1, \dots, m \\ & h_j(x) = 1, \ j = 1, \cdots, r \end{array}$$

where $f, g_i, i = 1, \cdots, m$ are posynomials and $h_j, j = 1, \cdots, r$ are monomials. This is nonconvex.

Given $f(x) = \gamma x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, let $y_i = \log x_i$ and rewrite this as

$$\gamma (e^{y_1})^{a_1} (e^{y_2})^{a_2} \cdots (e^{y_n})^{a_n} = e^{a^T y + b}$$

for $b = \log \gamma$. Also, a posynomial can be written as $\sum_{k=1}^{p} e^{a_k^T y + b_k}$. With this variable substitution, and after taking logs, a geometric program is equivalent to

$$\begin{array}{ll} \min\limits_x & \log(\sum\limits_{k=1}^{p_0} e^{a_{0k}^T y + b_{0k}}) \\ \text{subject to} & \log(\sum\limits_{k=1}^{p_i} e^{a_{ik}^T y + b_{ik}}) \leq 0, \ i = 1, \cdots, m \\ & c_j^T y + d_j = 0, \ j = 1, \cdots, r \end{array}$$

This is convex, recalling the convexity of soft max functions.

2.5 Eliminating equality constraints

Important special case of change of variables: eliminating equality constraints. Given the problem

$$\begin{array}{ll} \min\limits_x & f(x) \\ \text{subject to} & g_i(x) \leq 0, \ i = 1, \dots, m \\ & Ax = b \end{array}$$

we can always express any feasible point as $x = My + x_0$, where $Ax_0 = b$ and $\text{col}(M) = \text{null}(A)$. Hence the above is equivalent to

$$\begin{array}{ll} \min\limits_{y} & f(My + x_0) \\ \text{subject to} & g_i(My + x_0) \leq 0,\ i = 1, \dots, m \end{array}$$

Note: this is fully general but not always a good idea (practically).

2.6 Introducing slack variables

Essentially opposite to eliminating equality constraints: introducing slack variables. Given the problem

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{subject to} & g_i(x) \leq 0,\ i = 1, \dots, m \\ & Ax = b \end{array}$$

we can transform the inequality constraints via

$$\begin{array}{ll} \min\limits_{x,s} & f(x) \\ \text{subject to} & s_i \geq 0,\ i = 1, \dots, m \\ & g_i(x) + s_i = 0,\ i = 1, \dots, m \\ & Ax = b \end{array}$$

Note: this is no longer convex unless $g_i, i = 1, \cdots, n$ are affine.

2.7 Relaxing nonaffine equalities

Given an optimization problem

$$\min_x f(x) \quad \text{subject to } x \in C$$

we can always take an enlarged constraint set $\tilde{C} \supseteq C$ and consider

$$\min_x f(x) \quad \text{subject to } x \in \tilde{C}$$

This is called a relaxation and its optimal value is always smaller or equal to that of the original problem. An important special case: relaxing nonaffine equality constraints, i.e.,

$$h_j(x) = 0,\ j = 1, \cdots, r$$

where $h_j, j = 1, \cdots, r$ are convex but nonaffine, are replaced with

$$h_j(x) \leq 0,\ j = 1, \cdots, r$$

2.7.1 Example: maximum utility problem

The maximum utility problem models investment/consumption:

$$\begin{array}{ll} \min\limits_{x,b} & \sum_{t=1}^{T} \alpha_t u(x_t) \\ \text{subject to} & b_{t+1} = b_t + f(b_t) - x_t,\ t = 1, \dots, T \\ & 0 \leq x_t \leq b_t, t = 1, \cdots, T \end{array}$$

Here b_t is the budget and x_t is the amount consumed at time t; f is an investment return function, u utility function, both concave and increasing (Here, $f(0) = 0$, and $b_0 > 0$ is given).

2.7.2 Example: principal components analysis

Given $X \in \mathbb{R}^{n \times p}$, consider the low rank approximation problem:

$$\min_R \|X - R\|_F^2 \quad \text{subject to } \operatorname{rank}(R) = k$$

Here $\|A\|_F^2 = \sum_{i=1}^n \sum_{j=1}^p A_{ij}^2$, the entrywise squared ℓ_2 norm, and $\operatorname{rank}(A)$ denotes the rank of A. The problem is also called principal components analysis or PCA problem. Given $X = UDV^T$, singular value decomposition or SVD, the solution is

$$R = U_k D_k V_k^T$$

where U_k, V_k are the first k columns of U, V and D_k is the first k diagonal elements of D. That is, R is reconstruction of X from its first k principal components.

The PCA problem is not convex, but we can recast it. First rewrite as

$$\begin{aligned} &\min_{Z \in \mathbb{S}^p} \|X - XZ\|_F^2 \quad \text{subject to } \operatorname{rank}(Z) = k, Z \text{ is a projection} \\ \iff &\max_{Z \in \mathbb{S}^p} tr(SZ) \quad \text{subject to } \operatorname{rank}(Z) = k, Z \text{ is a projection} \end{aligned}$$

where $S = X^T X$. Hence constraint set is the nonconvex set

$$C = \{Z \in \mathbb{S}^p : \lambda_i(Z) \in \{0, 1\}, i = 1, \cdots, p, \operatorname{tr}(Z) = k\}$$

where $\lambda_i(Z), i = 1, \cdots, n$ are the eigenvalues of Z. Solution in this formulation is

$$Z = V_k V_k^T$$

where V_k gives first k columns of V.

Now consider relaxing constraint set to $\mathcal{F}_k = \operatorname{conv}(V)$, its convex hull. Note

$$\begin{aligned} \mathcal{F}_k &= \{Z \in \mathbb{S}^p : \lambda_i(Z) \in [0, 1], i = 1, \cdots, p, \operatorname{tr}(Z) = k\} \\ &= \{Z \in \mathbb{S}^p : 0 \preceq Z \preceq I, \operatorname{tr}(Z) = k\} \end{aligned}$$

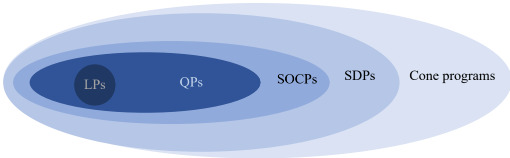
This set is called the Fantope of order k and it is convex. Hence, the linear maximization over the Fantope, namely

$$\max_{Z \in \mathcal{F}_k} \quad \operatorname{tr}(SZ)$$

is a convex problem. Remarkably, this is equivalent to the original nonconvex PCA problem (admit the same solution).

3 Canonical Problem Forms

The relationship among linear programs (LPs), quadratic programs (QPs), semidefinite programs (SDPs), second-order cone program (SOCPs) and cone programs is shown in the following figure.



3.1 Linear program

Definition (Linear program)**.** A linear program is an optimization problem of the form

$$\begin{aligned} \min_x \quad & c^T x \\ \text{subject to} \quad & Dx \leq d \\ & Ax = b \end{aligned}$$

and this is always a convex optimization problem.

The standard form of LP is

$$\begin{aligned} \min_x \quad & c^T x \\ \text{subject to} \quad & x \geq 0 \\ & Ax = b \end{aligned}$$

any LP can be rewritten in standard form.

3.1.1 Excample: basis pursuit

Given $y \in \mathbb{R}^n$ and $X \in \mathbb{R}^{n \times p}$, where $p > n$. Suppose that we seek the sparest solution to underdetermined linear system $X\beta = y$. If using ℓ_0 norm, it will be a nonconvex formulation. The ℓ_1 approximation, often called basis pursuit

$$\begin{aligned} \min_\beta \quad & \|\beta\|_1 \\ \text{subject to} \quad & X\beta = y \end{aligned}$$

This problem can be reformulated as the LP form

$$\begin{aligned} \min_{\beta, z} \quad & 1^T z \\ \text{subject to} \quad & z \geq \beta \\ & z \geq -\beta \\ & X\beta = y \end{aligned}$$

3.1.2 Excample: Dantzig selector

Modification of previous problem, where we allow for $X\beta \approx y$, the Dantzig selector

$$\begin{aligned} \min_\beta \quad & \|\beta\|_1 \\ \text{subject to} \quad & |X^T(y - X\beta)\|_\infty \leq \lambda \end{aligned}$$

where $\lambda \geq 0$ is a tuning parameter. It can also be reformulated as a LP.

3.2 Convex quadratic program

Definition (Convex quadratic program)**.** A convex quadratic program is an optimization problem of the form

$$\begin{array}{ll} \min\limits_{x} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Dx \leq d \\ & Ax = b \end{array}$$

where $Q \succeq 0$. Note that this problem is not convex when $Q \not\succeq 0$. From now on, when we say QP, we implicitly assume that $Q \succeq 0$.

The standard form of QP is

$$\begin{array}{ll} \min\limits_{x} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & x \geq 0 \\ & Ax = b \end{array}$$

any QP can be rewritten in standard form.

3.2.1 Example: support vector machines

Given $y \in \{-1, 1\}^n$, $X \in \mathbb{R}^{n \times p}$ having rows $x_1, \cdots, x_n$, recall the SVM problem

$$\begin{array}{ll} \min\limits_{\beta, \beta_0, \xi} & \frac{1}{2} \|\beta\|_2^2 + C \sum\limits_{i=1}^{n} \xi_i \\ \text{subject to} & \xi_i \geq 0, y_i \left(x_i^T \beta + \beta_0 \right) \geq 1 - \xi_i, i = 1, \ldots, n \end{array}$$

3.3 Semidefinite program

Definition (Semidefinite program)**.** A semidefinite program is an optimization problem of the form

$$\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots x_n F_n \succeq F_0 \\ & Ax = b \end{array}$$

where $F_j \in \mathbb{S}^d$, for $j = 0, 1, \cdots, n$, and $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n, b \in \mathbb{R}^m$.

The standard form of SDP is

$$\begin{array}{ll} \min\limits_{X} & C \bullet X \\ \text{subject to} & A_i \bullet X = b_i, i = 1, \cdots, m \\ & X \succeq 0 \end{array}$$

where $X \bullet Y = \text{tr}(XY)$ and any SDP can be rewritten in standard form.

Note: $\mathbb{S}^n$ is space of $n \times n$ symmetric matrices, $\mathbb{S}_+^n$ is the space of positive semidefinite matrices, i.e.,

$$\mathbb{S}_+^n = \{X \in \mathbb{S}^n : u^T X u \geq 0 \text{ for all } u \in \mathbb{R}^n\}$$

$\mathbb{S}_{++}^n$ is the space of positive define matrices, i.e.,

$$\mathbb{S}_{++}^n = \{X \in \mathbb{S}^n : u^T X u > 0 \text{ for all } u \in \mathbb{R}^n \setminus \{0\}\}$$

3.3.1 Example: theta function

Let $G = (N, E)$ be an undirected graph, $N = \{1, \cdots, n\}$, $\omega(G)$ and $\chi(G)$ is the clique number and chromatic number of G, respectively. The Lovasz theta function is

$$\begin{array}{rl} \vartheta(G) = \max\limits_{X} & 11^T \bullet X \\ \text{subject to} & I \bullet X = 1 \\ & X_{ij} = 0, (i,j) \notin E \\ & X \succeq 0 \end{array}$$

The Lovasz sandwich theorem: $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$, where $\overline{G}$ is the complement graph of G.

3.3.2 Example: trace norm minimization

Let $A : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ be a linear map

$$A(X) = \begin{pmatrix} A_1 \bullet X \\ \cdots \\ A_p \bullet X \end{pmatrix}$$

for $A_1, \cdots, A_p \in \mathbb{R}^{m\times n}$ and $A_i \bullet X = \text{tr}(A_i^T X)$. Finding lowest-rank solution to an underdetermined system, nonconvex

$$\begin{array}{ll} \min\limits_{X} & \text{rank}(X) \\ \text{subject to} & A(X) = b \end{array}$$

Trace norm approximation:

$$\begin{array}{ll} \min\limits_{X} & \|X\|_{tr} \\ \text{subject to} & A(X) = b \end{array}$$

3.4 Conic program

Definition (Conic program)**.** A conic program is an optimization problem of the form

$$\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{subject to} & D(x) + d \in K \\ & Ax = b \end{array}$$

where $c, x \in \mathbb{R}^n$, and $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. $D : \mathbb{R}^n \to Y$ is a linear map, $d \in Y$, for Euclidean space Y, $K \subseteq Y$ is a closed convex cone.

Both LPs and SDPs are special cases of conic programming. For LPs, $K = \mathbb{R}^n_+$; for SDPs, $K = \mathbb{S}^n_+$.

Definition (Second-order cone program)**.** A second-order cone program or SOCP is an optimization problem of the form

$$\begin{array}{ll} \min\limits_{x} & c^T x \\ \text{subject to} & \|D_i(x) + d_i\|_2 \leq e_i^T x + f_i, i = 1, \cdots, p \\ & Ax = b \end{array}$$

This is indeed a cone program. Recall the second-order cone $Q = \{(x,t) : \|x\|_2 \leq t\}$, we have

$$\|D_i(x) + d_i\|_2 \leq e_i^T x + f_i \Longleftrightarrow (D_i(x) + d_i, e_i^T x + f_i) \in Q_i$$

for second-order cone Q_i of appropriate dimensions. Now take $K = Q_1 \times \cdots \times Q_p$.

Observe that every LP is an SOCP and every SOCP is an SDP. Turns out that

$$\|x\|_2 \leq t \Longleftrightarrow \begin{bmatrix} tI & x \\ x^T & t \end{bmatrix} \succeq 0$$

hence any SOCP constraint can be written as an SDP constraint. The above is a special case of the Schur complement theorem:

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \Longleftrightarrow A - BC^{-1}B^T \succeq 0$$

for A, C symmetric and $C \succ 0$.

In addition, QPs are SOCPs, which can be seen by rewriting a QP as

$$\begin{array}{ll} \min\limits_{x,t} & c^T x + t \\ \text{subject to} & Dx \leq d, \frac{1}{2}x^T Qx \leq t \\ & Ax = b \end{array}$$

now write $\frac{1}{2}x^T Qx \leq t \Leftrightarrow \left\|\left(\frac{1}{\sqrt{2}}Q^{1/2}x, \frac{1}{2}(1-t)\right)\right\|_2 \leq \frac{1}{2}(1+t)$. Thus we have established the hierachy.

$$\text{LPs} \subseteq \text{QPs} \subseteq \text{SOCPs} \subseteq \text{SDPs} \subseteq \text{Conic programs}$$