Polynomial Representation of Transfer Functions

Rather than specifying the functions *G* and *H* in (3-10) in terms of functions of the frequency variable ω, you can describe them as rational functions of q^{-1} and specify the numerator and denominator coefficients in some way.

A commonly used parametric model is the ARX model that corresponds to

$$G(q) = q^{-nk} \cdot \frac{B(q)}{A(q)}; \qquad H(q) = \frac{1}{A(q)} \tag{3-12}$$

where *B* and *A* are polynomials in the delay operator q^{-1}:

$$\begin{aligned} A(q) &= 1 + a_1 q^{-1} + \dots\dots + a_{na} q^{-na} \\ B(q) &= b_1 + b_2 q^{-1} + \dots\dots + b_{nb} q^{-nb+1} \end{aligned} \tag{3-13}$$

Here, the numbers *na* and *nb* are the orders of the respective polynomials. The number *nk* is the number of delays from input to output. The model is usually written

$$A(q)y(t) = B(q)u(t-nk) + e(t) \tag{3-14}$$

or explicitly

$$\begin{aligned} &y(t) + a_1 y(t-1) + \dots\dots + a_{na} y(t-na) = \\ &b_1 u(t-nk) + b_2 u(t-nk-1) + \dots\dots + b_{nb} u(t-nk-nb+1) + e(t) \end{aligned} \tag{3-15}$$

Note that (3-14) - (3-15) apply also to the multivariable case, with *ny* output channels and *nu* input channels. Then $A(q)$ and the coefficients a_i become *ny*-by-*ny* matrices, $B(q)$ and the coefficients b_i become *ny*-by-*nu* matrices.

Another very common, and more general, model structure is the ARMAX structure

$$A(q)y(t) = B(q)u(t-nk) + C(q)e(t) \tag{3-16}$$

Here, $A(q)$ and $B(q)$ are as in (3-13), while

$$C(q) = 1 + c_1 q^{-1} + \dots + c_{nc} q^{-nc}$$

An *Output-Error* (OE) structure is obtained as

(3-17)

$$y(t) = \frac{B(q)}{F(q)}u(t-nk)+e(t)$$

with

$$F(q) = 1+f_1q^{-1}+\ldots+f_{nf}q^{-nf}$$

The so-called *Box-Jenkins* (BJ) model structure is given by

$$y(t) = \frac{B(q)}{F(q)}u(t-nk)+\frac{C(q)}{D(q)}e(t) \qquad \textbf{(3-18)}$$

with

$$D(q) = 1+d_1q^{-1}+\ldots+d_{nd}q^{-nd}$$

All these models are special cases of the general parametric model structure.

$$A(q)y(t) = \frac{B(q)}{F(q)}u(t-nk)+\frac{C(q)}{D(q)}e(t) \qquad \textbf{(3-19)}$$

The variance of the white noise $\{e(t)\}$ is assumed to be λ.

Within the structure of (3-19), virtually all of the usual linear black-box model structures are obtained as special cases. The ARX structure is obviously obtained for $nc = nd = nf = 0$. The ARMAX structure corresponds to $nf = nd = 0$. The ARARX structure (or the "generalized least squares model") is obtained for $nc = nf = 0$, while the ARARMAX structure (or "extended matrix model") corresponds to $nf = 0$. The Output-Error model is obtained with $na = nc = nd = 0$, while the Box-Jenkins model corresponds to $na = 0$. (See Section 4.2 in Ljung (1999) for a detailed discussion.)

The same type of models can be defined for systems with an arbitrary number of inputs. They have the form

$$A(q)y(t) = \frac{B_1(q)}{F_1(q)}u_1(t-nk_1)+\ldots+\frac{B_{nu}(q)}{F_{nu}(q)}u_{nu}(t-nk_{nu})+\frac{C(q)}{D(q)}e(t) \qquad \textbf{(3-20)}$$