Estimating Spectra and Frequency Functions

This section describes methods that estimate the frequency functions and spectra (3-11) directly. The cross-covariance function $R_{yu}(\tau)$ between $y(t)$ and $u(t)$ is defined as $Ey(t+\tau)u(t)$ analogously to (3-7). Its Fourier transform, the cross spectrum, $\Phi_{yu}(\omega)$ is defined analogously to (3-6). Provided that the input $u(t)$ is independent of $v(t)$, the relationship (3-1) implies the following relationships between the spectra.

$$\begin{aligned} \Phi_y(\omega) &= \left|G(e^{i\omega})\right|^2\Phi_u(\omega) + \Phi_v(\omega) \\ \Phi_{yu}(\omega) &= G(e^{i\omega})\Phi_u(\omega) \end{aligned} \quad \textbf{(3-32)}$$

By estimating the various spectra involved, the frequency function and the disturbance spectrum can be estimated as follows.

Form estimates of the covariance functions (as defined in (3-7)) $\hat{R}_y(\tau)$, $\hat{R}_{yu}(\tau)$, and $\hat{R}_u(\tau)$, using

$$\hat{R}_{yu}(\tau) = \frac{1}{N}\sum_{t=1}^{N} y(t+\tau)u(t) \quad \textbf{(3-33)}$$

and analog expressions for the others. Then, form estimates of the corresponding spectra

$$\hat{\Phi}_y(\omega) = \sum_{\tau=-M}^{M} \hat{R}_y(\tau)W_M(\tau)e^{-i\omega\tau} \quad \textbf{(3-34)}$$

and analogously for $\hat{\Phi}_u$ and $\hat{\Phi}_{yu}$. Here $W_M(\tau)$ is the so-called *lag window* and *M* is the width of the lag window. The estimates are then formed as

$$\hat{G}_N(e^{i\omega}) = \frac{\hat{\Phi}_{yu}(\omega)}{\hat{\Phi}_u(\omega)}; \qquad \hat{\Phi}_v(\omega) = \hat{\Phi}_y(\omega) - \frac{\left|\hat{\Phi}_{yu}(\omega)\right|^2}{\hat{\Phi}_u(\omega)} \quad \textbf{(3-35)}$$

This procedure is known as *spectral analysis*. (See Chapter 6 in Ljung (1999).)