Tools and Techniques for ML Homework 2

Zian Jiang (zj444)

March 22, 2021

1 Complete case mean - unbiased or what

1.1

2 Regression imputation with $E[Y|X = x]$

2.1

$$\begin{aligned}
E(\hat{\mu}_{\hat{f}\text{-full}}) &= E(\frac{1}{n}\sum_{i=1}^{n}\hat{f}(X_i)) \\
&= \frac{1}{n}\sum_{i=1}^{n}E(\hat{f}(X_i))) \\
&= \frac{1}{n}\sum_{i=1}^{n}E(E(Y|X = X_i)) \\
&= \frac{1}{n}\sum_{i=1}^{n}E(Y) \text{ (by Adam's law)} \\
&= E(Y).
\end{aligned}$$

$\blacksquare$

2.2

$$
\begin{aligned}
E[\frac{1}{n}\sum_{i=1}^{n}[R_iY_i+(1+R_i)\hat{f}(X_i)]] &= E[E[\frac{1}{n}\sum_{i=1}^{n}[R_iY_i+(1+R_i)\hat{f}(X_i)]]|X_1,...,X_n]|X_1,...,X_n] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}E[R_i|X_i]E[Y_i|X_i]+(1-E[R_i|X_i])E[\hat{f}(X_i)|X_i]|X_1,...,X_n] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}E[R_i|X_i]E[Y_i|X_i]+(1-E[R_i|X_i])E[Y_i|X_i]|X_1,...,X_n] \\
&= E[\frac{1}{n}\sum_{i=1}^{n}E[Y_i|X_i]|X_1,...,X_n] \\
&= E[E[Y]|X] \\
&= E[Y].
\end{aligned}
$$

■

3 A family of simple AIPW estimators

3.1

$$
\hat{\mu}_{\mathrm{ipw}-a} = \hat{\mu}_{\mathrm{ipw}}(\mathcal{D}+a) - a = \frac{1}{n}\sum_{i=1}^{n}\frac{R_iY_i}{\pi(X_i)} + \frac{a}{n}\sum_{i=1}^{n}\frac{R_i}{\pi(X_i)} - a,
$$

so the control variate is $\frac{a}{n}\sum_{i=1}^{n}\frac{R_i}{\pi(X_i)}$ with expectation a.

3.2

We want to minimize the variance of $\hat{\mu}_{\mathrm{ipw}-a}$.

$$
\begin{aligned}
Var(\hat{\mu}_{\mathrm{ipw}-a}) &= Var(\frac{RY}{\pi(X)} + \frac{aR}{\pi(X)} - a) \\
&= Var(\frac{RY}{\pi(X)} + a^2Var(\frac{R}{\pi(X)}) + 2aCov(\frac{RY}{\pi(X)}, \frac{R}{\pi(X)})).
\end{aligned}
$$

Thus to minimize $Var(\hat{\mu}_{\mathrm{ipw}-a})$, we can set

$$
a < -\frac{2Cov(\frac{RY}{\pi(X)}, \frac{R}{\pi(X)}))}{Var(\frac{R}{\pi(X)})}.
$$

4 Election forecasting

4.1 Fitting the regression

4.1.1

4.1.2

$$\frac{1}{n}\sum_{i=1}^{n}\frac{P(Y_i, X_i|T=1)}{P(Y_i, X_i|R=1)}l(f(X_i), Y_i)$$

4.2 Using our regression to forecast the election

4.2.1

$$\begin{aligned}
E[f(x)|T=1] &= E[f(x)|T=1, X=x] \\
&= 1 \times P(Y=1|T=1, X=x) + 0 \times P(Y=0|T=1, X=x) \\
&= P(Y=1|T=1, X=x) \\
&= P(Y=1|T=1).
\end{aligned}$$

$\blacksquare$

$$\begin{aligned}
E[Tf(X)] &= E[E[Tf(X)]|X] \\
&= E[E[T|X]E[f(X)|X]] \\
&= E[P(T=1|X)E[f(X)|T=1, X]] \\
&= P(T=1)E[f(X)|T=1].
\end{aligned}$$

$\blacksquare$

$$\begin{aligned}
\frac{E[\pi_t(X)f(X)]}{P(T=1)} &= \frac{E[E[T|X]f(X)]}{P(T=1)} \\
&= \frac{E[f(X)]E[T]}{P(T=1)} \\
&= E[f(X)] \\
&= E[E[Y|X,T]] \\
&= E[Y|T].
\end{aligned}$$

$\blacksquare$

$$\begin{aligned}
E[\pi_t(X)] &= E[E[T|X]] \\
&= E[T] \\
&= P(T=1).
\end{aligned}$$

$\blacksquare$