

# GMT Notes

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My two primary sources are [Mag12] and [Sim14].

## 1 Lebesgue equals Hausdorff (3.9.23)

We follow (Thm 2.9, [Sim14]). Recall the definitions of Lebesgue and Hausdorff measures.

$$\mathcal{L}^n(A) = |A| := \inf \left\{ \sum_{j=1}^{\infty} |R_j| : R_j \text{ is a open rectangle, } A \subset \bigcup_{j=1}^{\infty} R_j \right\},$$

$$\mathcal{H}_{\delta}^n(A) := \inf \left\{ \sum_{j=1}^{\infty} \omega_n \left( \frac{\text{diam} C_j}{2} \right)^n : \text{diam}(C_j) < \delta, A \subset \bigcup_{j=1}^{\infty} C_j \right\},$$

and  $\mathcal{H}^n(A) := \lim_{\delta \searrow 0} \mathcal{H}_{\delta}^n(A)$ .

**Theorem 1.1.** *For every  $\delta > 0$  and  $A \subset \mathbb{R}^n$ ,*

$$\mathcal{H}^n(A) = \mathcal{H}_{\delta}^n(A) = |A|.$$

*Proof.* Taking  $\delta \searrow 0$ , it suffices to show  $\mathcal{H}_{\delta}^n = \mathcal{L}^n$ . Since taking closure does not affect diameter, we may assume  $C_k$  are closed. Note also  $|E| = 0 \implies \mathcal{H}_{\delta}^n(E) = 0$  since we may inscribe a ball  $B_j$  with diameter  $< \delta$  in each rectangle  $R_j$ .

( $\mathcal{H}_{\delta}^n \leq \mathcal{L}^n$ ) The idea here is to “uniformly eat up” all the measure  $A$  with finitely many pairwise disjoint balls, then iterate this algorithm *ad infinitum*.<sup>1</sup> Fix an arbitrary cover  $\{R_j\}$  of  $A$  by open rectangles.

Step  $k = 1$ . For each  $j$ , consider a disjoint family of cubes  $\{I_k\}$ <sup>2</sup> with  $\text{diam}(I_k) < \delta$  such that the interiors of  $I_k$  are pairwise disjoint and  $\bigcup I_k = R_j$ . This is possible since  $R_j$  is an open set (Thm 1.4, [SS05]). For each  $k$ , inscribe a *closed* ball  $B_k$  into  $I_k$  such that  $\text{diam}(B_k) > \frac{1}{2}s_k$  where  $s_k$  is the side length of  $I_k$  [Figure 1]. This is obviously not sharp, but it doesn't matter. What does matter is that the most measure we've left off out is

$$|I_k - B_k| \leq s_k^n - \omega_n \left( \frac{s_k}{4} \right)^n = \left( 1 - \frac{\omega_n}{4^n} \right) s_k^n.$$

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<sup>1</sup>i.e. Step 1 leaves  $\frac{1}{2}$  the measure, step 2 leaves  $\frac{1}{4}$  the measure, etc (these are not the correct constants, but the idea is right).

<sup>2</sup> $I_k$  depends on  $j$  also, which we suppress from the notation. Note that  $I_k^j$  may intersect  $I_k^{j'}$  for  $j \neq j'$  - the pairwise disjoint condition is with respect to a fixed rectangle.

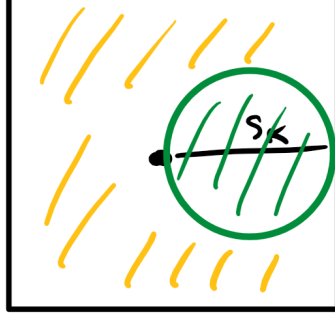


Figure 1:  $B_k$  must be bigger than the green ball.

Note the constant in front of  $s_k$  is uniform in  $k$  and strictly smaller than 1. Therefore,

$$|R_j - \cup_k B_k| = |\cup_k I_k - B_k| \leq (1 - \frac{\omega_n}{4^n}) \sum_k s^k = (1 - \frac{\omega_n}{4^n}) |R_j|.$$

Thus, for each  $j$ , we may choose finitely many ball  $B_j^1, \dots, B_j^N$  such that the same inequality holds.


Step  $k \geq 2$ . For  $k = 2$ , we repeat the above argument but for  $R_j - \cup_{k=1}^N B_j^k$  instead of  $R_j$ , which is *again an open set*. Repeat this construction for  $k > 2$ .

In total, for each  $j$  we get a countable disjoint collection of balls  $\{B_j^k\}$  of  $R_j$  with radius  $< \delta$  such that  $|R_j - \cup_k B_j^k| = 0$ , so  $\mathcal{H}_\delta^n(R_j - \cup_k B_j^k) = 0$ . Therefore,

$$\mathcal{H}_\delta^n(R_j) = \mathcal{H}_\delta^n(\cup_k B_j^k) \leq \sum_{k=1}^{\infty} \omega_n \left( \frac{\text{diam} B_j^k}{2} \right)^n = \sum_{k=1}^{\infty} |B_j^k| = |R_j|.$$

Thus

$$\mathcal{H}_\delta^n(A) \leq \sum_{j=1}^{\infty} \mathcal{H}_\delta^n(R_j) \leq \sum_{j=1}^{\infty} |R_j|,$$

and since  $R_j$  was an arbitrary cover, the result is shown. 

( $\mathcal{H}_\delta^n \geq \mathcal{L}^n$ ) The idea here is to use *Steiner symmetrization* to prove the isodiametric inequality.

**Lemma 1.2.** *For any  $A \subset \mathbb{R}^n$ ,*

$$|A| \leq \omega_n \left( \frac{\text{diam} A}{2} \right)^n.$$

Assuming the lemma, take  $C_j$  an arbitrary collection of sets which cover  $A$ , with  $\text{diam} C_j < \delta$ . Then,

$$|A| \leq |\cup_j C_j| \leq \sum_j |C_j| \leq \sum_j \omega_n \left( \frac{\text{diam} C_j}{2} \right)^n.$$

(of Lemma 1.2). It suffices to prove  $A$  compact since taking closure does not increase diameter. We sketch a symmetrization process as follows. For the hyperplane  $H_i := \{x^i = 0\}$  for  $i = 1, \dots, n$ , let  $\xi_i \in H_i$  parameterize  $A$  in the sense that the fibers (under orthogonal projection to  $H_i$ ) of  $A$  are at most 1-dimensional. We record the Lebesgue measure (length) of the fibers, the symmetrically distribute the length across  $\xi^i$ . This symmetrizes  $A$  across the hyperplane  $H_i$ , is diameter non-increasing, and the Lebesgue measure of  $A$  is constant. Furthermore, this preserves symmetrizations across other hyperplanes. Thus, symmetrizing  $A$  across each hyperplane  $H_i$  produces a subset of the ball of radius  $\frac{\text{diam} A}{2}$ , proving the lemma. ■

Both inequalities are proven, and so the result follows. ■

## 2 Riesz Representation Theorem II (4.6.23)

We state and prove RRT.

**Theorem 2.1.** *If  $L$  is a bounded linear functional on  $C_c(\mathbb{R}^n, \mathbb{R}^m)$ , then its variation  $|L|$  is a (scaler-valued) Radon measure on  $\mathbb{R}^n$  and there exists a  $|L|$ -measurable function  $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that  $g = 1$   $|L|$ -a.e., and for all  $\phi \in C_c(\mathbb{R}^n, \mathbb{R}^m)$*

$$\langle L, \phi \rangle = \int_{\mathbb{R}^n} (\phi \cdot g) d|L|.$$

Moreover,

$$|L|(A) = \sup \left\{ \int_{\mathbb{R}^n} \phi \cdot g d|L| : \phi \in C_c(A, \mathbb{R}^m), |\phi| \leq 1 \right\}.$$

*Proof.* Recall the definition of the total variation (measure) of  $L$ .

$$|L| :=$$

Zack had previously shown that  $|L|$  is Radon, and that a version of RRT holds for  $L^1$ . ■

## 3 Compactness and Regularization (4.27.23)

*Maggi proves this by intertwining the functional analysis with the geometric measure theory. I think it's more clear to separate them out.* ☹ ————— 🎉🎉🎉🎉

Let  $(V, \|\cdot\|)$  be a normed linear space, and  $B \subset V$  the unit ball defined by  $\|\cdot\|$ . Recall the following basic fact about the strong topology

**Proposition 3.1.** *A normed linear space  $V$  is finite dimensional iff the unit ball is compact.*


The proof is easy but irrelevant. What matters is the moral of the story - if we're out looking for compact sets in infinite dimensional space, the strong topology is not the correct one to look in. Even the most basic candidate for a compact set is not compact, so there are way too many open sets. We need a coarser topology.

Look not in  $V$ , but in its dual  $V^*$  together with its unit ball  $B^*$  defined by the operator norm. We see

$$B^* := \{\mu : V \rightarrow \mathbb{R} : |\mu(\phi)| \leq \|\phi\|\} = \{L : V \rightarrow \mathbb{R} : |\mu(\phi)| \leq 1 \text{ for } \|\phi\| \leq 1\}.$$

In other words,  $\mu : B \rightarrow [-1, 1]$  and so  $\mu \in [-1, 1]^B$ ; in other words,  $B^*$  canonically sits inside  $[-1, 1]^B$ , so it is compact in the subspace topology by Tychonoff's. However, the subspace topology agrees does *not* agree with the topology induced by the operator norm. The subspace topology oversees only that the evaluation pairing between  $V, V^*$  is continuous, and so it agrees with the weak star topology. That is, a sequence  $\{\mu_i\}$  in  $V^*$  converges weak-\* to  $\mu$  iff for every  $\phi \in V$ ,

$$\mu_i \xrightarrow{*} \mu \iff \langle \phi, \mu_i \rangle \rightarrow \langle \phi, \mu \rangle,$$

where  $\langle -, - \rangle$  denotes the evaluation pairing. This is known as *Banach-Alaoglu* and will serve as the backbone functional analysis tool for the compactness result for the space of Radon measures (which by RRT is dual to  $V = C_c$  with a mildly strange topology). Truthfully, it's important that we have the sequential version of Banach-Alaoglu - this is true if  $V$  is *separable* (counterexamples exist otherwise) which is true in our case. 

**Theorem 3.2.** *Given a sequence  $\{\mu_i\}$  of Radon measures which are locally uniformly bounded, i.e. for  $K \subset \mathbb{R}^n$  compact*

$$\sup_i \mu_i(K) < \infty,$$

*then there exists a subsequence, upon reindexing,  $\{\mu_k\}$  which weak-\* converge to a Radon measure. That is, by RRT,*

$$\int_K \phi d\mu_i \rightarrow \int_K \phi d\mu,$$

*for each  $\phi \in C_c(\mathbb{R}^n)$ .*

*Proof.* The idea is basically sequential Banach-Alaoglu and RRT. We first give a construction of  $\{\mu_k\}$  via a diagonalization procedure. Consider an exhaustion of  $\mathbb{R}^n$  by balls  $\{B_j\}$  centered at 0. Define functionals

$$F_{i,j}(\phi) := \int_{B_j} \phi d\mu_i,$$

for  $\phi \in C_c(B_j)$ . Linearity of  $F_{i,j}$  is clear, and boundedness follows from the local uniformity assumption as

$$|F_{i,j}(\phi)| \leq \sup \phi \cdot \mu_i(B_j) \leq C \sup \phi,$$

where  $C = C(j)$ . By sequential Banach-Alaoglu, there is a functional  $F_j : C_c(B_j) \rightarrow \mathbb{R}$  and a subsequence  $\{F_{i',j}\}$  such that  $F_{i',j} \xrightarrow{*} F_j$ . By extracting subsequences subsequently, then selecting the diagonal subsequence, we extract the desired  $\{\mu_k\}$  upon relabeling. We claim that  $F = \lim_k \mu_k$ . By construction, this limit is well-defined. Furthermore, by RRT we have

$$F(\phi) := \int_{\mathbb{R}^n} \phi d\mu,$$

for some Radon measure  $\mu$ . For  $\phi \in C_c(\mathbb{R}^n)$ , take  $\text{spt} \phi \subset B_j$ . Linearity follows since  $F(\phi) = F_j(\phi)$ , and  $F_j$  is linear. Boundedness follows from

$$|F(\phi)| \leq \sup \phi \cdot \mu(B_j) \leq C \sup \phi.$$



There are two obvious requirements that need to hold for the above formula to be true. First, if  $E := \{x \in \mathbb{R}^n : Jf(x) = 0\}$ , then  $\mathcal{H}^n(f(E)) = 0$  (injectivity is dropped for this statement). This is the content of Proposition 8.7. The second is that under the above assumptions,  $f(E)$  had better be  $\mathcal{H}^n$  measurable.

**Proposition 4.2.** *If  $E$  is Lebesgue measurable in  $\mathbb{R}^n$  and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is an injective Lipschitz function, then  $f(E)$  is  $\mathcal{H}^n$  measurable in  $\mathbb{R}^m$ .*

*Proof.* Assume  $E$  is bounded, and exhaust  $E$  by a sequence of compact sets  $\{K_i\}$  wrt  $\mathcal{L}^n$ , so  $|E - \cup K_i| = 0$ . Since  $f$  is Lipschitz, it's continuous and so  $f(K_i)$  is compact and  $\cup f(K_i)$  is Borel. Then,

$$\mathcal{H}^n(f(E) - \cup f(K_i)) = \mathcal{H}^n(f(E - \cup K_i)) \leq \text{Lip} f^n \cdot \mathcal{H}^n(E - \cup K_i) = \text{Lip} f^n \cdot |E - \cup K_i| = 0.$$

Note that injectivity is used in the first equality. ■

Here, we used that Lipschitz functions play nicely with the Hausdorff measure in the sense that  $\mathcal{H}^s(f(E)) \leq \text{Lip} f^s \mathcal{H}^s(E)$ , and is equal to the Lebesgue measure. For future reference, recall also that Hausdorff measure plays nicely with scaling in that  $\mathcal{H}^s(rE) = r^s \mathcal{H}^s(E)$ .

The next statement is to prove that area formula holds on arbitrary (not necessarily injective) linear functions  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

**Proposition 4.3.** *If  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear, then for every  $E \subset \mathbb{R}^n$ ,*

$$\mathcal{H}^n(T(E)) = JT \cdot |E|.$$

For this, we must first recall some linear algebra. Set  $\text{Lin}(n, m) := \{T : \mathbb{R}^n \rightarrow \mathbb{R}^m : T \text{ linear}\}$  and  $\text{Isom}(n, m) := \{P \in \text{Lin}(n, m) : \langle Px, Py \rangle = \langle x, y \rangle\}$ . We think of  $\text{Isom}(n, m)$  as the set of linear isometric embeddings - clearly all must be injective.

**Lemma 4.4.** *For  $P \in \text{Isom}(\mathbb{R}^n, \mathbb{R}^m)$ , we have  $P^*P = id_n$ .*

*Proof.* Recall  $P^* := (\mathbb{R}^m)^* \rightarrow (\mathbb{R}^n)^*$  is precomposition with  $P$ . The composition  $P^*P$  is understood as the following isomorphism:

$$\mathbb{R}^n \xrightarrow{P} \mathbb{R}^m \rightarrow (\mathbb{R}^m)^* \xrightarrow{P^*} (\mathbb{R}^n)^* \rightarrow \mathbb{R}^n$$

$$x \mapsto Px \mapsto \langle Px, - \rangle \mapsto \langle Px, P- \rangle \mapsto x$$

where the last map is justified since the covector  $y \mapsto \langle Px, Py \rangle = \langle x, y \rangle$  is metrically equivalent to the vector  $y \in \mathbb{R}^n$ . ■

Since  $P \in \text{Isom}(n, m)$  is an isometry, both  $P$  and its adjoint  $P^*$  (orthogonal projections) have Lipschitz constant 1. Recall also the spectral theorem - every symmetric real-valued matrix diagonalizes with real eigenvalues (and orthogonal eigenspaces) and the polar decomposition - every linear map  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  can be realized as

$$T = PS$$

with  $S \in \text{Lin}(n, n)$  symmetric and  $P \in \text{Isom}(n, m)$  (both  $S, T$  are given explicitly in terms of the diagonalization of  $T$ ). It turns out linear isometric embeddings do not change the fine structure of the geometry, and in a sense we'll make precise, all the change in the geometry happens with  $S$ .

(of 4.3.) Set  $\kappa := D_{\mathcal{L}^n} \nu = \frac{\mathcal{H}^n(T(B))}{|B|}$  for  $\nu(E) := \mathcal{H}^n(T(E))$  to be the density. The last equality is justified by scaling since for all  $r > 0$ ,

$$\begin{aligned}\mathcal{H}^n(T(rB)) &= r^n \mathcal{H}^n(T(B)) \\ |rB| &= r^n |B|.\end{aligned}$$

It suffices to show the following two statements:


1.  $\mathcal{H}^n(T(E)) = \kappa |E|$ ,
2.  $JT = \kappa$ .

First assume  $\kappa = 0$  ( $T$  is non-injective). Then,  $\mathcal{H}^n(T(B)) = 0 \implies \mathcal{H}^n(T(rB)) = 0$  for every  $r > 0$  by linearity of  $T$  and so in the limit,  $\mathcal{H}^n(\mathbb{R}^n) = 0$ . By monotonicity of the measure,  $\mathcal{H}^n(T(E)) = 0$  for every  $E$ .

Next, assume  $\kappa > 0$  ( $T$  is injective). Since  $E \mapsto \mathcal{H}^n(T(E))$  is Radon, by the decomposition theorem for measures, it suffices to show the equality is true on balls  $B_r(x)$  for  $r > 0$  and  $x \in \mathbb{R}^n$ . This follows by translation invariance and scaling, as

$$\begin{aligned}\mathcal{H}^n(T(B_r(x))) &= \mathcal{H}^n(T(x + rB)) \\ &= r^n \mathcal{H}^n(T(B)) \\ &= r^n \kappa |B| \\ &= \kappa |B_r(x)|,\end{aligned}$$

which shows  $\mathcal{H}^n(T(-)) \ll \mathcal{L}^n$  and identifies  $\kappa$  as the density.

*Remark 4.5.* For linear functions, non-injectivity is very easy to deal with as both sides evaluates to 0. We don't need to consider multiplicity for this case. 

To show  $JT = \kappa$ , we first show for  $T = PS$  and  $S(Q) = E$  for a cube  $Q$ , we have  $\frac{\mathcal{H}^n(P(E))}{|E|} = 1$ , since

$$|E| = |P^*P(E)| \leq \text{Lip}(P^*)^n |P(E)| = |P(E)| = \mathcal{H}^n(P(E)) \leq \text{Lip}(P)^n |E| = |E|.$$

Therefore,

$$\kappa = \frac{\mathcal{H}^n(PS(Q))}{|Q|} = \frac{\mathcal{H}^n(P(E))}{|E|} \frac{|S(Q)|}{|Q|} = \frac{|S(Q)|}{|Q|}.$$

Note we can switch to other sets when defining density (Thm 3.22, [Fol13]). By homogeneity, we take  $Q$  to be the unit cube, so we must prove  $JT = |S(Q)|$ . Consider now the spectral decomposition of  $Sv_i = \lambda_i v_i$ . Using the fact that  $\nabla T = T$  as  $T$  is linear, pullback is contravariant, and

$$\langle S^*Sv_i, v_j \rangle = \langle Sv_i, Sv_j \rangle = \begin{cases} \lambda_i^2 & i = j \\ 0 & i \neq j, \end{cases}$$

we compute

$$JT = \sqrt{\det T^*T} = \sqrt{\det(S^*P^*PS)} = \sqrt{\det(S^*S)} = \sqrt{\prod_i |\lambda_i|^2} = |\det(S)| = |S(Q)|.$$

■

The next proof shows that the area formula does not see the *singular set*  $\mathcal{S} := \{Jf = 0\}$ . For notation, balls without mention to the dimension will be assumed to be dimension  $n$ , and without mention to the center will be assumed to be centered at the origin.

**Proposition 4.6.** *If  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is Lipschitz, then on the singular set  $\mathcal{S}$ ,*

$$\mathcal{H}^n(f(\mathcal{S})) = 0.$$

*Proof.* We will deduce  $\mathcal{H}_\infty^n(f(\mathcal{S} \cap B_R)) = 0$  for arbitrary  $R > 0$  (recall  $\mathcal{H}^n \ll \mathcal{H}_\infty^n$ ). For  $x \in \mathcal{S}$  and  $1 > \epsilon > 0$ ,<sup>3</sup> by definition  $\nabla f(x)$  is the linear map which satisfies the inequality

$$|f(x+v) - f(x) - \nabla f(x)v| < \epsilon|v|,$$

for  $|v| < r(\epsilon, x)$ . We can reinterpret the above (think of  $f(x) = 0$ .) in terms of small balls<sup>4</sup>: for  $r = |v| < r(\epsilon, x)$ , we have

$$f(B_r(x)) \subset f(x) + B_{\epsilon r}(\nabla f(x)(B_r)).$$

It would serve us well, therefore, to find a bound on the (translation-invariant) size of  $B_{\epsilon r}(\nabla f(x)(B_r))$

**Lemma 4.7.** *For  $0 < \epsilon < 1$  and  $C = C(n, \text{Lip} f)$ ,*

$$\mathcal{H}_\infty^n(B_{\epsilon r}(\nabla f(x)(B_r))) \leq Cr^n \epsilon.$$

The proof will use that  $\mathcal{S}$  is singular by using a dimension drop argument, which produces the  $\epsilon$ . Assuming this lemma, we get for  $x \in \mathcal{S}$ ,

$$\mathcal{H}_\infty^n(f(B_r(x))) \leq Cr^n \epsilon. \tag{2}$$

We take the family  $\mathcal{F}$  of balls with centers in  $\mathcal{S} \cap B_R$ ,

$$\mathcal{F} := \{B_r(x) \subset \mathbb{R}^n : x \in \mathcal{S} \cap B_R, 0 < r < r(\epsilon, x)\}.$$

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<sup>3</sup>As far as I can tell, the restriction to  $1 > \epsilon$  is for polish.

<sup>4</sup>Compare this to the definition of continuity in terms of balls.  $f$  is continuous at  $x$  if for  $r < r(\epsilon, x)$ ,

$$f(B_r(x)) \subset f(x) + B_\epsilon.$$

So if you can take a derivative, the information you gain is having a modulus of control over the error tolerance  $\epsilon r$  in terms on your parameter  $r$ .



We cutoff  $\mathcal{S}$  by  $B_R$  to have bounded centers; we may therefore apply Besicovitch covering theorem to extract subfamilies  $\mathcal{F}_1, \dots, \mathcal{F}_{\xi(n)}$  such that each  $\mathcal{F}_i$  is countable, pairwise disjoint and  $\mathcal{S} \cap B_R \subset \cup_{\xi(n)} \mathcal{F}_i$ . By inequality 2,

$$\begin{aligned}
\mathcal{H}_\infty^n(f(\mathcal{S} \cap B_R)) &\leq \sum_{i=1}^{\xi(n)} \sum_{B_r(x) \in \mathcal{F}_i} \mathcal{H}_\infty^n(f(B_r(x))) \\
&\leq C\omega_n \epsilon \sum_{i=1}^{\xi(n)} \sum_{B_r(x) \in \mathcal{F}_i} r^n \\
&\leq \frac{C\epsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \sum_{B_r(x) \in \mathcal{F}_i} |B_r(x)| \\
&= \frac{C\epsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \left| \bigcup_{B_r(x) \in \mathcal{F}_i} B_r(x) \right| \\
&\leq \frac{C\epsilon \xi(n)}{\omega_n} |B_1(\mathcal{S} \cap B_R)|.
\end{aligned}$$

Therefore, it suffices to prove the lemma.

(of Lemma 4.7). We first identify that  $\nabla f(x)(B_r) \subset B_{r\text{Lip}f}^m \cap \nabla f(x)(\mathbb{R}^n)$ , where the right hand side is a disk  $D_{r\text{Lip}f}$  of dimension  $k < n$  since  $x \in \mathcal{S}$ . This follows from homogeneity and  $\|\nabla f\| \leq \text{Lip}f$ , since we may characterize  $\text{Lip}f$  as

$$\text{Lip}f = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Therefore, we must obtain a bound of the form

$$\mathcal{H}_\infty^n(B_\epsilon(D_s)) \leq C(n, s)\epsilon, \quad (3)$$

since taking a neighborhood of a disk is linear,

$$\begin{aligned}
\mathcal{H}_\infty^n(B_{\epsilon r}(\nabla f(x)(B_r))) &\leq \mathcal{H}_\infty^n(B_{\epsilon r}(B_{r\text{Lip}f}^m \cap \nabla f(x)(\mathbb{R}^n))) \\
&\leq r^n \mathcal{H}_\infty^n(B_\epsilon(B_{\text{Lip}f}^m \cap \nabla f(x)(\mathbb{R}^n))) \\
&\leq r^n C(n, \text{Lip}f)\epsilon.
\end{aligned}$$

To obtain inequality 3, we produce a covering of  $B_\epsilon(D_s) \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$  by translates of

$$B_{\epsilon s}^k \times B_\epsilon^{m-k} \subset \mathbb{R}^k \times \mathbb{R}^{m-k}.$$

Note that we need  $C\epsilon^{-k}$  number of translates to cover  $B_\epsilon(D_s) \approx B_s^k \times B_\epsilon^{m-k}$  for small  $\epsilon$ , since area of  $B_s^k$  grows like  $s^k$ .<sup>5</sup> By Pythagorean theorem,

$$\text{diam}(B_{\epsilon s}^k \times B_\epsilon^{m-k})^2 = \text{diam}(B_{\epsilon s}^k)^2 + \text{diam}(B_\epsilon^{m-k})^2 = 4\epsilon^2(1 + s^2).$$

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<sup>5</sup>e.g. For 2-dimensional disks, need (up to a dimensional constant) one-hundred small disks  $D_{1/10}$  to cover one big disk  $D_1$ .

Since  $k < n$ , we have

$$\begin{aligned}
\mathcal{H}_\infty^n(B_\epsilon(D_s)) &\leq \omega_n \sum_{\# \text{ translates}} \left( \frac{\text{diam}(B_{\epsilon s}^k \times B_\epsilon^{m-k})}{2} \right)^n \\
&= \omega_n C \epsilon^{-k} (2\epsilon^2(1+s^2))^{\frac{n}{2}} \\
&= C(n, s) \epsilon^{n-k} \\
&\leq C\epsilon.
\end{aligned}$$

■

By the lemma,  $\mathcal{H}^n(\mathcal{S}) = 0$ .

■

## 5 Approximate Tangent Spaces (9.20.23)

We follow [Mag12] Chapters 10.1-10.2.

Set  $k \leq n$ . We say a set  $M \subset \mathbb{R}^n$  is  **$k$ -rectifiable** if there are countably many Lipschitz maps  $f_i : \mathbb{R}^k \rightarrow \mathbb{R}^n$  such that

$$\mathcal{H}^k(M \setminus \cup_{i=1}^\infty f_i(\mathbb{R}^k)) = 0.$$

The point is that a  $k$ -rectifiable set is a measure-theoretic version of a  $k$ -dimensional manifold. We say  $M$  is **locally  $k$ -rectifiable** if for every compact  $K$ ,  $\mathcal{H}^k(K \cap M) < \infty$ . The point here is that  $\mathcal{H}^k \llcorner M$  is Radon (not just Borel) iff  $M$  is locally  $k$ -rectifiable. By Kirezbraun theorem and regularity properties of Hausdorff measure,  $M$  is  $k$ -rectifiable iff there exists Borel  $M_0$  which is  $\mathcal{H}^k$ -null such that

$$M = M_0 \cup \bigcup_{i=1}^\infty f_i(E_i),$$

where  $E_i$  are bounded, Borel sets, and  $f_i$  is Lipschitz. This decomposition is highly non-unique, and we will exploit this in the following lemma by asking for a decomposition with nice regularity properties. Namely, for each  $i \in \mathbb{N}$ , the pair  $(f_i, E_i)$  will form a regular Lipschitz image. We say  $(f, E)$  form a **regular Lipschitz image** if

1.  $f$  is injective (so  $k \leq n$ ) and differentiable on  $E$ , and  $Jf > 0$  (i.e.  $Jf \neq 0$  anywhere) on  $E$  ( $\implies f$  is an injective immersion on  $E$ ).
2. All points in  $E$  form a point of density 1 for  $E$ .
3. All point in  $E$  are Lebesgue points of  $\nabla f$  ( $\implies$  all points are Lebesgue points of  $Jf$ ).
4. There exists a lower bound for  $\text{Lip}(f)$  on  $E$ .

Morally speaking, this is saying  $(f, E)$  is a Lipschitz injective immersion with good  $C^1$  control. Such a decomposition of a rectifiable set  $M$  always exists (Theorem 10.1) by a number of previous theorems. Here are the quoted ones from [Mag12].

1. **Theorem 2.10:** Borel sets admit inner/outer approximations, if the measure is a locally finite Borel measure.
2. **Theorem 8.7:** Singular values of a Lipschitz function are  $\mathcal{H}^k$ -null.
3. **Theorem 8.8:** The regular points of a Lipschitz function admit a countable, almost flat partition, on the function is injective on each leaf.
4. **Rademacher's:** Lipschitz functions are differentiable almost everywhere.
5. **Theorem 5.16:** Almost every point of a  $L^1_{\text{loc}}(\mu)$  function satisfies MVP, if  $\mu$  is Radon.

The point of the above four axioms is that they are sufficient conditions to make the following lemma true. This lemma classifies the approximate tangent space for the image of a regular Lipschitz function.

**Lemma 5.1.** *Suppose  $M^k = f(E)$  with  $(f, E)$  a regular Lipschitz image. Then, then for any  $x \in M$ , the approximate tangent space is given by*

$$T_x M = (\nabla f|_z)(\mathbb{R}^k),$$

where  $x = f(z)$ .

*Proof.* Recall that the approximate tangent plane to  $M$  at  $x$  is the  $k$ -plane  $T_x M \subset \mathbb{R}^n$  such that

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int_M \phi \left( \frac{y - x}{r} \right) d\mathcal{H}^k(y) = \int_{T_x M} \phi(y) d\mathcal{H}^k(y),$$

for all  $\phi \in C_c(\mathbb{R}^n)$ . This is the measure-theoretic version of a tangent space. By pulling back along  $f$ , and using change of variables,

$$\begin{aligned} \frac{1}{r^k} \int_M \phi \left( \frac{y - x}{r} \right) d\mathcal{H}^k(y) &= \frac{1}{r^k} \int_E \phi \left( \frac{f(\tilde{w}) - f(z)}{r} \right) Jf(\tilde{w}) d\tilde{w} \\ &= \underbrace{\int_{\mathbb{R}^k} \mathbb{1}_E(z + rw) \cdot \phi \left( \frac{f(z + rw) - f(z)}{r} \right) Jf(z + rw) dw}_{:= u_r(w)} \end{aligned}$$

Since  $z \in E$  is a point of density 1 for  $E$ ,

$$\lim_{r \downarrow 0} \int_{\mathbb{R}^k} \mathbb{1}_E(z + rw) dw = \mathbb{1}_E(z) = 1.$$

Since  $z \in E$  is Lebesgue point for  $Jf$ ,

$$\lim_{r \downarrow 0} \int_{\mathbb{R}^k} Jf(z + rw) dw = J(z).$$

Take the limit as  $r \downarrow 0$  on both sides. For the moment, take on faith that DCT can be justified, and switch  $\lim_{r \downarrow 0}$  with  $\int$ . Since  $\phi$  is continuous and  $f$  is differentiable at  $z \in E$ ,

$$\phi \left( \frac{f(z + rw) - f(z)}{r} \right) \xrightarrow{r \downarrow 0} \phi(\nabla f|_z w).^6$$

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<sup>6</sup> *Type check:* note that  $\nabla f|_z$  is a linear map from  $\mathbb{R}^k \rightarrow \mathbb{R}^n$  (the Jacobian), and  $w$  is a vector in  $\mathbb{R}^k$ .

Altogether,<sup>7</sup>

$$\begin{aligned}
\lim_{r \downarrow 0} \int_{\mathbb{R}^k} u_r(w) dw &= \int_{\mathbb{R}^k} \phi(\nabla f|_z w) Jf(z) dw \\
&= \int_{\mathbb{R}^k} \phi(\nabla f|_z w) J(\nabla f|_z)(w) dw \\
&= \int_{\nabla f|_z(\mathbb{R}^k)} \phi d\mathcal{H}^k,
\end{aligned}$$

where the last step follows from area formula of injective, Lipschitz maps. Now to justify DCT. Observe the  $L^\infty$  (not pointwise, as values of  $Jf$  can take  $\infty$  on a null set) upper bound

$$\|u_r\|_\infty \leq \left( \sup_{\mathbb{R}^n} \phi \right) \cdot \text{Lip}(f)^k,$$

and notice the RHS is a constant function. It suffices then to show that for all  $r > 0$ ,  $\text{spt} u_r \subset B_R$ , for some  $R$  which is independent of  $r$ . Since  $\text{spt} u_r := \overline{\{u_r > 0\}}$ , and a set is bounded iff its closure is, it suffices to show  $\{u_r > 0\} \subset B_R$ . Take  $w \in \{u_r > 0\}$ , so that

$$0 < \mathbb{1}_E(z + rw) \cdot \phi \left( \frac{f(z + rw) - f(z)}{r} \right) Jf(z + rw);$$

in particular,  $z + rw \in E$ . Thus, last condition of a regular Lipschitz image immediately gives for some  $\lambda = \lambda(E) > 0$ ,

$$|f(z + rw) - f(z)| \geq \lambda r |w|. \quad (4)$$

On the other hand, since

$$0 < \mathbb{1}_E(z + rw) \cdot \phi \left( \frac{f(z + rw) - f(z)}{r} \right) Jf(z + rw),$$

it follows that

$$\frac{f(z + rw) - f(z)}{r} \in \text{spt} \phi.$$

Since  $\text{spt} \phi \subset B_\Lambda$  for some  $\Lambda = \Lambda(\phi)$ ,

$$\left| \frac{f(z + rw) - f(z)}{r} \right| \leq \Lambda. \quad (5)$$

Together, inequalities 4 and 5 together give the conclusion upon setting  $R := \frac{\Lambda}{\lambda}$ , as

$$\lambda r |w| \leq r \Lambda \implies w \in B_{\frac{\Lambda}{\lambda}}.$$

■

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<sup>7</sup>The gradient of the linear map  $\nabla f|_z$  is  $\nabla f|_z$ . Thus

$$J(\nabla f|_z)(w) = \sqrt{\det((\nabla(\nabla f|_z)|_w)^* \cdot (\nabla(\nabla f|_z)|_w))} = \sqrt{\det(\nabla f|_z^* \cdot \nabla f|_z)} = Jf(z).$$

**Theorem 5.2.** Suppose  $M \subset \mathbb{R}^n$  is a locally  $k$ -rectifiable set. Then for  $\mathcal{H}^k$ -a.e. point in  $M$ , there exists a unique  $k$ -dimensional plane  $T_x M$  such that:

1. The blow-up (defined below) of the measure  $\mathcal{H}^k \llcorner M$  weak-star converges to  $\mathcal{H}^k \llcorner T_x M$ . That is, as  $r \downarrow 0$ ,

$$\mathcal{H}^k \llcorner \left( \frac{M - x}{r} \right) \xrightarrow{*} \mathcal{H}^k \llcorner T_x M.$$

2. For  $\mathcal{H}^k$ -a.e.  $x \in M$ ,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^k(M \cap B_r(x))}{\omega_k r^k} = 1.$$

More precisely, this happens whenever  $x$  admits an approximate tangent space.

*Proof.* In the proof, we will need the notion of a blow-up. For fixed  $x \in M$  and  $r > 0$ , we set

$$\eta_{x,r}(y) = \eta(y) := \frac{y - x}{r},$$

which centers a chosen point  $x \in M$  at the origin, then rescales (in fact, magnifies as  $r \downarrow 0$ ) a point of distance  $r$  to unit distance.

1. This would immediately follow from Lemma 5.1 upon setting  $T_x M = \nabla f|_z(\mathbb{R}^k)$ , if not for the  $\mathcal{H}^k$ -null set  $M_0$ . This will be taken care of by the *upper density theorem* which tells us for each  $i$ , as  $M_i$  is locally  $k$ -rectifiable, for a.e.  $x \notin M_i$ ,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^k(M_i \cap B_r(x))}{\omega_k r^k} = 0.$$

Note that since the RHS is 0, in fact the numerator and denominator need not decay at the same rate. So more generally, we have for  $r, R > 0$ ,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^k(M_i \cap B_{rR}(x))}{\omega_k r^k} = 0.$$

We use this as follows. Let  $\phi \in C_c(\mathbb{R}^n)$  such that  $\text{spt} \phi \subset B_R(0)$ . Then, for  $x \in M_i$ ,

$$\begin{aligned} \left| \frac{1}{r^k} \int_{M \setminus M_i} \phi \circ \eta_{x,r} d\mathcal{H}^k \right| &= \left| \int_{\eta_{x,r}(M \setminus M_i)} \phi d\mathcal{H}^k \right| \\ &\leq \mathcal{H}^k(B_R(0) \cap \eta_{x,r}(M \setminus M_i)) \cdot \sup_{\mathbb{R}^n} \phi \\ &= \omega_k \sup_{\mathbb{R}^n} \phi \cdot \frac{\mathcal{H}^k(B_{rR}(0) \cap (M \setminus M_i) - x)}{\omega_k r^k} \\ &= \omega_k \sup_{\mathbb{R}^n} \phi \cdot \frac{\mathcal{H}^k(B_{rR}(x) \cap (M \setminus M_i))}{\omega_k r^k} \\ &\xrightarrow{r \downarrow 0} 0. \end{aligned}$$

2. This follows from algebraic set-theoretic interactions of the blow-up with Hausdorff measure. Choose a sequence  $\{\phi_i\} \subset C_c(\mathbb{R}^n)$  such that  $\phi_i \xrightarrow{i \rightarrow \infty} \mathbb{1}_{B_1^n(0)}$ . On one hand,

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int_M \phi_i \circ \eta(y) d\mathcal{H}^k(y) = \lim_{r \downarrow 0} \int_{\eta M} \phi_i(y) d\mathcal{H}^k(y) \xrightarrow{i \rightarrow \infty} \lim_{r \downarrow 0} \mathcal{H}^k(\eta M \cap B_1^n(0)).$$

On the other hand,

$$\int_{T_x M} \phi_i(y) d\mathcal{H}^k(y) \xrightarrow{i \rightarrow \infty} \mathcal{H}^k(T_x M \cap B_1(0)) = \mathcal{H}^k(B_1^k(0)) = \omega_k.$$

As the two expressions are equal, we calculate

$$\begin{aligned} 1 &= \lim_{r \downarrow 0} \frac{\mathcal{H}^k(\eta M \cap B_1^n(0))}{\omega_k} \\ &= \lim_{r \downarrow 0} \frac{r^k \mathcal{H}^k(\eta M \cap B_1^n(0))}{\omega_k r^k} \\ &= \lim_{r \downarrow 0} \frac{\mathcal{H}^k((M - x) \cap B_r^n(0))}{\omega_k r^k} \\ &= \lim_{r \downarrow 0} \frac{\mathcal{H}^k(M \cap B_r^n(x))}{\omega_k r^k}. \end{aligned}$$

■

*Remark 5.3.* This completes the forwards (easy) direction of the heuristic “measure-theoretic manifold iff admits measure-theoretic tangent planes”.

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