

Notes on Schauder Estimates

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The following are the *standard assumptions*. On bounded domains $\Omega \subset \mathbb{R}^n$, will consider 2nd order (uniformly) elliptic operators

$$Lu := \sum_{i,j} a^{ij} u_{ij} + \sum_i b^i u_i + cu = f,$$

where a^{ij}, b^i, c are functions with regularity to be specified later. We assume symmetry on the leading order coefficient terms $a^{ij} = a^{ji}$, and for all uniform ellipticity says $\xi \in \mathbb{R}^n$, for some positive constant $\lambda > 0$,

$$\sum_{i,j} a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2.$$

The regularity of this approach will require our functions u, f to lie in appropriate Hölder spaces. In this note, we will introduce a zoo of seminorms and norms on the way to proving Schauder estimates, each of which will slightly differently exploit the fact that Ω is bounded.



To start, we introduce the standard Hölder spaces and their respective norms and seminorms. This approach will be pointwise in nature, compared to the integral-based approach of Sobolev spaces last semester. Let $\alpha \in (0, 1)$. With respect to Ω , we have the pointwise (and global) Hölder seminorms which detect pointwise (and uniform) α -Hölder continuity. We may also define local α -Hölder continuity by asking for every compact $K \subset \Omega$, that $[u]_{\alpha;K}$ is finite. In such a case, we say u is locally uniformly continuous.

1. **(Pointwise)** For $x_o \in \Omega$,

$$[u]_{0,\alpha;x_o} := \sup_{x \in \Omega} \frac{|u(x) - u(x_o)|}{|x - x_o|^\alpha}.$$

2. **(Local)** For all compact $K \subset \Omega$,

$$\sup_{x \neq y \in K} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$$

3. **(Global)**

$$[u]_{0,\alpha;\Omega} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

As usual, global implies local implies pointwise. The above gives rise to $C^{k,\alpha}(\bar{\Omega})$ (and $C^{k,\alpha}(\Omega)$) consisting of functions whose k -th order derivatives are uniformly (and locally uniformly) continuous. In the global case, we get a seminorm as defined above. We can formally extend this to $\alpha = 0, 1$ which gives continuity and Lipschitz continuity respectively.

Hölder regularity gives a quantitative measurement of continuity. In fact, greater the α , the more regularity we have.¹ One can see this in the following illustrative example:

Example 0.0.1. *The function $f(x) = |x|^\beta$ is α -Hölder continuous at $x = 0$ iff $\alpha \leq \beta$.*

Proof. Let Ω be a domain containing 0. We compute

$$\sup_{x \neq 0 \in \Omega} \frac{|f(x) - f(0)|}{|x - 0|^\alpha} = \sup_{x \neq 0 \in \Omega} |x|^{\beta-\alpha} = \begin{cases} \text{finite} & \alpha \leq \beta \\ \infty & \alpha > \beta. \end{cases}$$

■

Recall the $C^k(\Omega)$ seminorm is defined by measuring how large the k th derivative of u on Ω is,

$$[u]_{k,0;\Omega} := |D^k u|_{0;\Omega} := \sup_{x \in \Omega, |\beta|=k} |D^\beta u|,$$

where β is a multi-index of length k . Similarly, we have the $C^k(\Omega)$ norm by adding the largest of all the derivatives to order k ,

$$|u|_{k,0;\Omega} := \sum_{j=1}^k |D^j u|_{0,\Omega}.$$

These definitions admit easy extensions to include α -Hölder regularity, namely we have the *standard (semi)norm* of $C^{k,\alpha}(\bar{\Omega})$.

1. $[u]_{k,\alpha;\Omega} := [D^k u]_{0,\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u]$.
2. $|u|_{k,\alpha;\Omega} := |u|_{C^k(\bar{\Omega})} + [u]_{k,\alpha;\Omega}$.

One may check that $C^{k,\alpha}(\bar{\Omega})$ for $0 \leq \alpha < 1$ are Banach spaces with the standard norms. One should think of seminorms as taking the “highest degree” portion of the norm.

A short word on notation on our (semi)norms. From here on out, we drop the α from the notation when $\alpha = 0$, but never drop how many derivatives we take.

¹We briefly mention that for $k = 0$, we have a Rellich-type result for Hölder spaces. Namely for $0 < \alpha < \beta \leq 1$, we have the compact inclusion $C^{0,\beta}(\Omega) \subset C^{0,\alpha}(\Omega)$.

1 The Classical Proof

We unabashedly follow Chapters 2, 4, 6 of [2].

1.1 Reduction to the Laplacian

Roughly speaking, the Schauder interior estimates tell us that we can bound the $C^{2,\alpha}$ norm of u in terms of the C^0 norm of u and the $C^{0,\alpha}$ norm of f . Explicitly, we have the following theorem.

Theorem 1.1.1. (*Interior Estimates*)² *Given the standard assumptions, if*

$$|a^{ij}|_{0,\alpha;\Omega}^{(0)}, |b^i|_{0,\alpha;\Omega}^{(1)}, |c|_{0,\alpha;\Omega}^{(2)} \leq \Lambda,$$

then the following estimate holds

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

We now introduce the relevant norms so the above statement is legible. The following Hölder (semi)norms are weighted by their distances to the boundary. Namely for $x \neq y \in \Omega$, consider

$$d_x := d(x, \partial\Omega) \quad d_{x,y} := \min\{d_x, d_y\}.$$

We use the above two quantities to define the following (semi)norms. For $\alpha \in [0, 1]$ and $u \in C^{k,\alpha}(\Omega)$, we first define the θ -homogeneous (semi)norm of $u \in C^{k,\alpha}(\Omega)$.

1. **0-homogeneous seminorm.** For $\alpha = 0$,

$$[u]_{k;\Omega}^* := \sup_{x \in \Omega, |\beta|=k} d_x^k |D^\beta u(x)|.$$

For $\alpha > 0$,

$$[u]_{k,\alpha;\Omega}^* := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

2. **0-homogeneous norm.** For $\alpha = 0$,

$$|u|_{k;\Omega}^* := \sum_{j=0}^k [u]_{j,\Omega}^*.$$

For $\alpha > 0$,

$$|u|_{k,\alpha;\Omega}^* := |u|_{k;\Omega}^* + [u]_{k,\alpha;\Omega}^*.$$

²Compare Theorem 1.1.1 to the regularity statement of ([1], pg 329, “Interior H^2 -regularity”).

Next, we introduce the σ -homogeneous (semi)norm of $u \in C^{k,\alpha}(\Omega)$ for $\sigma \in \mathbb{R}$, generalizing the above. The 0-homogeneous (semi)norms can obviously be recovered by setting $\sigma = 0$.

1. **σ -homogeneous seminorm.** For $\alpha = 0$,

$$[u]_{k;\Omega}^{(\sigma)} := \sup_{x \in \Omega, |\beta|=k} d_x^{k+\sigma} |D^\beta u(x)|.$$

For $\alpha > 0$,

$$[u]_{k,\alpha;\Omega}^{(\sigma)} := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

2. **σ -homogeneous norm.** For $\alpha = 0$,

$$|u|_{k;\Omega}^{(\sigma)} := \sum_{j=0}^k [u]_{j,\Omega}^{(\sigma)}.$$

For $\alpha > 0$,

$$|u|_{k,\alpha;\Omega}^{(\sigma)} := |u|_{k,\Omega}^{(\sigma)} + [u]_{k,\alpha;\Omega}^{(\sigma)}.$$

It's worth pointing out that in the $k = 0$ case, the σ -homogeneous (semi)norms agree with the standard ones. We show that the σ -inhomogeneous norms enjoy a submultiplicative property.³

Lemma 1.1.2. *For $\sigma + \tau \geq 0$, we have*

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} \leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}.$$

Proof. This is a Baby Rudin proof. Recall

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} := \underbrace{\sup_{x \neq y \in \Omega} \left(d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x - y|^\alpha} \right)}_{(*)} + \underbrace{\sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|)}_{(**)}.$$

We estimate the first term by

$$\begin{aligned} (*) &= \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x - y|^\alpha} \right) \\ &\leq \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\sigma+\alpha+\tau} \cdot d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)|}{|x - y|^\alpha} \right) \\ &\leq \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)|}{|x - y|^\alpha} \right) + \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\tau+\alpha+\sigma} \frac{|g(y)| \cdot |f(x) - f(y)|}{|x - y|^\alpha} \right) \\ &\leq |f|_{0;\Omega}^{(\sigma)} \cdot [g]_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot [f]_{0,\alpha;\Omega}^{(\sigma)}. \end{aligned}$$

³(6.11) of [2].

We estimate the second term by

$$\begin{aligned}
(**) &= \sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|) \\
&\leq \sup_{x \in \Omega} (d_x^\sigma |f(x)|) \cdot \sup_{x \in \Omega} (d_x^\tau |g(x)|) \\
&= |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)}
\end{aligned}$$

Combining the two estimates,

$$\begin{aligned}
(*) + (**) &= |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot |f|_{0,\alpha;\Omega}^{(\sigma)} + |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)} \\
&\leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}
\end{aligned}$$

■

Throughout the course of the proof, we will frequently employ the interpolation inequalities.⁴ Unfortunately, we must omit the proof.

Lemma 1.1.3. *Let $u \in C^{2,\alpha}(\Omega)$, then for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that*

$$[u]_{j,\beta;\Omega}^* \leq |u|_{j,\beta;\Omega}^* \leq C|u|_{0;\Omega} + \varepsilon[u]_{2,\alpha;\Omega}^*,$$

where $j = 0, 1, 2$ and $0 \leq \alpha, \beta \leq 1$ and $j + \beta < 2 + \alpha$.



We first reduce showing Schauder estimates from the general case to the case of constant coefficients with no lower order terms.⁵

Proof. (of 1.1.1.) By using a proper, compact exhaustion, it suffices to show the equality holds for every compact subset of Ω . By the interpolation inequalities 1.1.3, it suffices to show the result where the LHS is $[u]_{2,\alpha;\Omega}^*$. Thus, the relevant quantity to consider is

$$d_{x_o, y_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha},$$

for arbitrary $x_o, y_o \in \Omega$. The idea here is to “freeze” one variable to reduce to the case of constant coefficients with no lower order terms. Suppose $d_{x_o} = d_{x_o, y_o}$, and consider

$$\sum_{i,j} a^{ij}(x_o) u_{ij}(x) = F(x),$$

where $F := \sum_{i,j} a^{ij}(x_o) u_{ij} - Lu + f$. Assuming local Schauder estimates on the ball

$$|u|_{2,\alpha;B}^* \leq C(|u|_{0;B} + |F|_{0,\alpha;B}^{(2)}),$$

⁴(6.8) and (6.9) of [2].

⁵Theorem 6.2 of [2].

where $B := B_{x_o}(\frac{d}{2})$ where $d := \mu d_{x_o}$ for some $\mu \leq \frac{1}{2}$ we will choose later.

We now perform a near-and-far estimate on the frozen case, and we estimate the “junk” term F in the sequel. For the near estimate, if $y_o \in B$, we have the estimate

$$\left(\frac{d}{2}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq C(|u|_{0;B} + |F|_{0,\alpha;B}^{(2)}).$$

Since $(\frac{d}{2})^{2+\alpha} = \left(\frac{d_{x_o}\mu}{2}\right)^{2+\alpha}$, we enlarge our domain to obtain the estimate

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right).$$

For the far estimate, if $y_o \notin B \iff 2|x_o - y_o| \geq d$, and so $(2|x_o - y_o|)^\alpha \geq d^\alpha$. Thus,

$$\begin{aligned} d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} &= \left(\frac{d}{\mu}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \\ &\leq \frac{d^2 \cdot 2^\alpha}{\mu^{2+\alpha}} |D^2 u(x_o) - D^2 u(y_o)| \\ &= \frac{(d_{x_o}\mu)^2 \cdot 2^\alpha}{\mu^2 \cdot \mu^\alpha} |D^2 u(x_o) - D^2 u(y_o)| \\ &\leq \left(\frac{2}{\mu}\right)^\alpha (d_{x_o}^2 |D^2 u(x_o)| + d_{y_o}^2 |D^2 u(y_o)|) \\ &\leq \frac{2^{\alpha+1}}{\mu^\alpha} [u]_{2;\Omega}^* \\ &\leq \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*. \end{aligned}$$

Combining the two estimates, we have

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right) + \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*$$



The next step is to estimate the “junk” term $|F|_{0,\alpha;B}^{(2)}$ in terms of $|u|_{0;\Omega}$ and $[u]_{2,\alpha;\Omega}^*$. This is a mildly annoying process, but straightforward conceptually. Specifically, we will estimate the terms on the RHS of

$$|F|_{0,\alpha;B}^{(2)} \leq \sum_{i,j} |(a^{ij}(x_o) - a^{ij}(x))u_{ij}|_{0,\alpha;B}^{(2)} + \sum_i |b^i u_i|_{0,\alpha;B}^{(2)} + |cu|_{0,\alpha;B}^{(2)} + |f|_{0,\alpha;B}^{(2)}.$$

We first provide a generic estimate local-to-global estimate for arbitrary $g \in C^{0,\alpha}(\Omega)$. For $x \in B$, define $d_x^{\partial B} := \text{dist}(x, \partial B)$ and $d_{x,y}^{\partial B} := \min\{d_x^{\partial B}, d_y^{\partial B}\}$. Note that upon localizing to

B , we need to consider the distance to ∂B not to $\partial\Omega$.⁶

$$\begin{aligned}
|g|_{0,\alpha;B}^{(2)} &\leq \sup_{x \in B} (d_x^{\partial B})^2 \cdot \sup_{x \in B} |g(x)| + \sup_{x \neq y \in B} (d_{x,y}^{\partial B})^{2+\alpha} \cdot \sup_{x \neq y \in B} \frac{|g(x) - g(y)|}{|x - y|^\alpha} \\
&= d^2 |g|_{0;B} + d^{2+\alpha} [g]_{\alpha;B} \\
&\leq |g|_{0;\Omega}^{(2)} + [g]_{\alpha;\Omega}^{(2)} \\
&\leq \left(\frac{\mu}{1-\mu} \right)^2 |g|_{0;\Omega}^{(2)} + \left(\frac{\mu}{1-\mu} \right)^{2+\alpha} [g]_{\alpha;\Omega}^{(2)} \\
&\leq 4\mu^2 |g|_{0;\Omega}^{(2)} + 8\mu^{2+\alpha} [g]_{0,\alpha;\Omega}^{(2)}
\end{aligned}$$

The third line follows since for any $x \in B$ which comes from the computation

$$d = \mu d_{x_o} \leq (1 - \mu) d_{x_o} \leq d_x.$$

Replacing d with d_x is nondecreasing on B , so the result follows. The fifth line follows from $\mu \leq \frac{1}{2} \implies \left(\frac{1}{1-\mu} \right)^2 \leq 4$ and $\left(\frac{1}{1-\mu} \right)^{2+\alpha} \leq 8$.

We now apply estimates in detail to the principal term; the lower order terms are treated similarly. By Lemma 1.1.2, we compute

$$\begin{aligned}
|(a^{ij}(x_o) - a^{ij})u_{ij}|_{0,\alpha;B}^{(2)} &\leq |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)} \cdot |u_{ij}|_{0,\alpha;B}^{(2)} \\
&\leq \underbrace{|a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)}}_{(*)} \cdot (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*).
\end{aligned}$$

We also estimate the local term $(*)$ by

$$\begin{aligned}
(*) &= \sup_{x \in B} (|a^{ij}(x_o) - a^{ij}(x)|) + \sup_{x \neq y \in B} \left((d_{x,y}^{\partial B})^\alpha \frac{|a^{ij}(x_o) - a^{ij}(x) - (a^{ij}(x_o) - a^{ij}(y))|}{|x - y|^\alpha} \right) \\
&\leq \sup_{x \neq x_o \in B} \left(\frac{|a^{ij}(x_o) - a^{ij}(x)|}{|x - x_o|^\alpha} \cdot |x - x_o|^\alpha \right) + d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2(2\mu)^\alpha [a^{ij}]_{0,\alpha;\Omega}^* \\
&\leq 4\Lambda\mu^\alpha,
\end{aligned}$$

where the second to last inequality follows since $d \leq d_x$ for all $x \in B$, and $\mu \leq \frac{1}{2}$. Altogether, we find

$$\begin{aligned}
\sum_{i,j} |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(2)} &\leq 4n^2 \Lambda \mu^\alpha (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} ([u]_{2;\Omega}^* + \mu^\alpha [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} (C(\mu) |\mu|_{0;\Omega} + 2\mu^\alpha [u]_{2,\alpha;\Omega}^*),
\end{aligned}$$

⁶Caution, [2] uses d_x for our $d_x^{\partial B}$, which may be a cause of confusion.

where the last step follows from the interpolation inequalities 1.1.3. Similarly, we have the estimates

$$\begin{aligned}\sum_i |b^i u_i|_{0,\alpha;B}^{(2)} &\leq 8n\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]_{2,\alpha;\Omega}^*), \\ |cu|_{0,\alpha;B}^{(2)} &\leq 8\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]_{2,\alpha;\Omega}^*), \\ |f|_{0,\alpha;\Omega}^{(2)} &\leq 8\mu^2|f|_{0,\alpha;\Omega}^{(2)}.\end{aligned}$$

Altogether, we estimate F by

$$|F|_{0,\alpha;B}^{(2)} \leq C\mu^{2+2\alpha}[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$



Combining the two steps and using the interpolation inequalities 1.1.3, we get

$$[u]_{2,\alpha;\Omega}^* \leq C\mu^\alpha[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

Fixing μ so that $C\mu^\alpha \leq \frac{1}{2}$ yields the desired result. ■

We quickly sketch a proof of Schauder estimates for the case of constant coefficients with no lower order terms.⁷ Consider

$$L_0 u := \sum_{i,j} A^{ij} u_{ij},$$

where $A := (A^{ij}) \in \text{GL}_n(\mathbb{R})$ is a constant matrix. In this case, we diagonalize A by an orthogonal matrix⁸ to a diagonal one, turning $L_0 u(x) = f(x)$ to $\Delta \tilde{u}(y) = \tilde{f}(y)$. Since our transformations are all linear, the norms are equivalent, and so it suffices to prove Schauder estimates for the Laplacian.

1.2 The Laplacian

We first introduce the *fundamental solution* and some basic ideas surrounding it. The upshot is that we obtain some estimates for future use, and the following *Green's representation formula*,⁹ to prove the following theorem.

Theorem 1.2.1. *Let $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$ such that $\Delta u = f$. Then,*

$$u = h + w,$$

where h is some harmonic function, and w is the Newtonian potential of f .

⁷Lemma 6.1 of [2].

⁸This follows by spectral theorem, since A is a symmetric, real-valued, nonsingular matrix.

⁹(2.16) of [2].

Fix $x \in \Omega \subset \mathbb{R}^n$ for $n \geq 2$.¹⁰ Let ω_n be the volume of the n -ball. Define the fundamental solution on $\Omega - x$ as the following radial function about x :

$$\Gamma(x - y) := \begin{cases} \frac{1}{n(2-n)\omega_n} |x - y|^{2-n} & n > 2 \\ \frac{1}{2\pi} \log |x - y| & n = 2. \end{cases}$$

We first obtain some basic estimates on Γ . The following derivatives will be taken with respect to the variable x . For convenience, define

$$S(x) := \sum_{k=1}^n (x_k - y_k)^2,$$

so that $|x - y| = S^{\frac{1}{2}}$, and $S_i = 2(x_i - y_i)$. We calculate for $n > 2$,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{n(2-n)\omega_n} \left(\frac{2-n}{2} S^{\frac{2-n}{2}-1} \cdot 2(x_i - y_i) \right) \\ &= \frac{1}{n\omega_n} |x - y|^{-n} (x_i - y_i). \end{aligned}$$

We calculate for $n = 2$,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{2\pi} \frac{\frac{1}{2} S^{-\frac{1}{2}} \cdot S_i}{S^{\frac{1}{2}}} \\ &= \frac{1}{2\pi} S^{-1} (x_i - y_i) \\ &= \frac{1}{2\pi} |x - y|^{-2} (x_i - y_i). \end{aligned}$$

Continuing, we calculate the second derivatives for $n \geq 2$,

$$\begin{aligned} \Gamma_{ij}(x - y) &= \frac{1}{n\omega_n} \left(\delta_{ij} S^{-\frac{n}{2}} + (x_i - y_i) \left(\frac{-n}{2} S^{-\frac{n}{2}-1} \cdot 2(x_j - y_j) \right) \right) \\ &= \frac{S^{-\frac{n}{2}-1}}{n\omega_n} (\delta_{ij} S^1 - n(x_i - y_i)(x_j - y_j)) \\ &= \frac{|x - y|^{-n-2}}{n\omega_n} (\delta_{ij} |x - y|^2 - n(x_i - y_i)(x_j - y_j)). \end{aligned}$$

Γ is indeed harmonic, since

$$\Delta \Gamma = \sum_i \Gamma_{ii} = \frac{|x - y|^{-n-2}}{n\omega_n} \sum_{i=1}^n (\delta_{ii} |x - y|^2 - n(x_i - y_i)^2) = 0.$$

For near future use, we record estimates on the fundamental solutions.

¹⁰Careful! [2] uses “y” to denote the singularity in Chapter 2, but switches to “x” to denote the singularity in Chapter 4.

Proposition 1.2.2. *Let $\rho = |x - y|$ and for any multi-index β , we estimate*

$$|D^\beta \Gamma(\rho)| \leq C \rho^{2-n-|\beta|},$$

where $C = C(n, |\beta|)$. In fact, we calculate directly

$$\Gamma'(\rho) = \frac{1}{n\omega_n} \rho^{1-n}.$$

In Cartesian coordinates, we make the constant explicit to get

$$|\Gamma_i(x - y)| \leq \frac{1}{n\omega_n} |x - y|^{1-n},$$

$$|\Gamma_{ij}(x - y)| \leq \frac{1}{\omega_n} |x - y|^{-n}.$$

Next, we recall *Green's identities*. First, recall for $g \in C^\infty(\Omega)$ and $X \in \mathfrak{X}(\Omega)$,

$$\operatorname{div}(gX) = \nabla g \cdot X + g \operatorname{div} X.$$

In the case $X = \nabla u$ for $u \in C^2(\Omega)$, we get

$$\operatorname{div}(g\nabla u) = \nabla g \cdot \nabla u + g \Delta u.$$

Integrating and using *divergence theorem* yields *Green's first identity*

$$\int_{\Omega} \nabla g \cdot \nabla u + \int_{\Omega} g \Delta u = \int_{\partial\Omega} (g \nabla u) \cdot \nu = \int_{\partial\Omega} g \frac{\partial u}{\partial \nu}.$$

Switching the roles of g, u and subtracting yields *Green's second identity*

$$\int_{\Omega} g \Delta u - u \Delta g = \int_{\partial\Omega} g \frac{\partial u}{\partial \nu} - u \frac{\partial g}{\partial \nu}.$$



We are in a position to give a proof of Theorem 1.2.1. In general, we call

$$w(x) := \Gamma * f(x) = \int_{\Omega} \Gamma(x - y) f(y) dy$$

the Newtonian potential of f .

Proof. (of 1.2.1.) We would like to use Green's second identity for $g = \Gamma$. But due to the singularity at x , we must use it instead for $\Omega - B_x(\rho)$, for small ρ . We compute

$$\int_{\Omega - B_x(\rho)} \Gamma \Delta u = \int_{\partial\Omega} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} + \underbrace{\int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu}}_{(*)}.$$

We show $(*) \rightarrow u(x)$ as $\rho \rightarrow 0$. Keeping in mind that Γ is radial, we proceed with estimating the first term of $(*)$,

$$\begin{aligned} \int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} &= \Gamma(\rho) \cdot \int_{\partial B_x(\rho)} \frac{\partial u}{\partial \nu} \\ &\leq C(n) \rho^{2-n} \cdot n \omega_n \rho^{n-1} \sup_{\partial B_x(\rho)} |Du|. \end{aligned}$$

Since $u \in C^2(\Omega)$, the exponents work out so that the last quantity vanishes as $\rho \rightarrow 0$.¹¹ For the second term, we calculate by 1.2.2,¹²

$$\begin{aligned} - \int_{\partial B_x(\rho)} u \frac{\partial \Gamma}{\partial \nu} &= \Gamma'(\rho) \int_{\partial B_x(\rho)} u \\ &= \frac{1}{n \omega_n \rho^{n-1}} \int_{\partial B_x(\rho)} u. \end{aligned}$$

Since u is continuous, the last quantity goes to $u(x)$ as $\rho \rightarrow 0$. Altogether, sending $\rho \rightarrow 0$ yields Green's representation formula¹³

$$u(x) = \underbrace{\int_{\partial \Omega} u(y) \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu}(y) ds}_{h(x)} + \underbrace{\int_{\Omega} \Gamma(x-y) \Delta u(x) dx}_{w(x)}.$$

We argue h is harmonic. Since $\partial \Omega$ is compact, we can commute ∂_i past the integral. By IBP we can put both ∂_i derivatives on Γ , which is harmonic; by Clairaut's theorem, we can put ∂_i past any ∂_ν terms. IBP leaves no boundary term since $\partial(\partial \Omega) = \partial^2 \Omega = 0$. ■

We give explicit formulas for the first two derivatives of the Newtonian potential.¹⁴

Proposition 1.2.3. *Suppose f is bounded, integrable in Ω with Newtonian potential w . Then $w \in C^1(\mathbb{R}^n)$ and its derivative given by the formula*

$$w_i(x) = \int_{\Omega} \Gamma_i(x-y) f(y) dy.$$

Proof. Set $v(x) := \int_{\Omega} \Gamma_i(x-y) f(y) dy$. By the asymptotic estimates of 1.2.2 and the fact that f is bounded, we find that v is well-defined. We consider the sequence $w^\varepsilon := (\Gamma \eta^\varepsilon) * f$, where $\eta^\varepsilon = \eta\left(\frac{|x-y|}{\varepsilon}\right)$. Here, $\eta \in C^1(\mathbb{R})$ is a function such that

$$\eta(t) = \begin{cases} 1 & t \geq 2 \\ 0 & t \leq 1, \end{cases} \text{ and } 0 \leq \eta' \leq 2.$$

¹¹At least for $n > 2$. But the $n = 2$ case follows similarly.

¹²Note since ν is outward pointing on $\Omega - B$, then ν into B . However, ρ is defined so that it is positive as you go away from x . Therefore, the sign changes.

¹³Observe for compactly supported u , the second term vanishes, while for harmonic u , the first term vanishes.

¹⁴Lemmas 4.1 and 4.2 of [2].

Therefore, $w^\varepsilon(x)$ has the effect of doing nothing far from the singularity of Γ (at distances $|x - y| \geq 2\varepsilon$), and killing off the function near the singularity of Γ (at distances $|x - y| \leq \varepsilon$). Therefore, it is clear that $w^\varepsilon(x)$ converges pointwise to $w(x)$. Let $K \subset \Omega$ be compact, and for any $x \in K$ we compute for $n > 2$,¹⁵

$$\begin{aligned}
|v(x) - w_i^\varepsilon(x)| &= \left| \int_{\Omega} \frac{\partial}{\partial x^i} \Gamma(x - y) f(y) dy - \frac{\partial}{\partial x^i} \int_{\Omega} \Gamma(x - y) \eta^\varepsilon(x, y) f(y) dy \right| \\
&= \left| \int_{B_e(2\varepsilon)} \left(\frac{\partial \Gamma(x - y)}{\partial x^i} - \frac{\partial \Gamma(x - y) \cdot \eta^\varepsilon}{\partial x^i} \right) f(y) dy \right| \\
&\leq \sup_{B_x(2\varepsilon)} |f| \int_{B_e(2\varepsilon)} |D\Gamma| + \left(|D\Gamma| + \frac{2}{\varepsilon^n} \cdot \varepsilon^{n-1} |\Gamma| \right) dy \\
&\leq \frac{\sup |f|}{n\omega_n} \int_{B_x(2\varepsilon)} 2|x - y|^{1-n} + \frac{2}{\varepsilon(2-n)} |x - y|^{2-n} dy \\
&= \frac{\sup |f|}{n\omega_n} \left(\frac{2n\omega_n(2\varepsilon)^{n-1}}{(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{1-n} (\rho^{n-1} d\rho) + \frac{2n\omega_n(2\varepsilon)^{n-1}}{\varepsilon(2-n)(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{2-n} (\rho^{n-1} d\rho) \right) \\
&= \sup |f| \left(2\varepsilon + \frac{2}{\varepsilon(n-2)} \cdot \frac{(2\varepsilon)^2}{2} \right) \\
&= \sup |f| \frac{2n\varepsilon}{n-2}.
\end{aligned}$$

The third to last equality is from co-area formula, along with change of variables. Since the upper bound is independent of x , the uniformity follows. For $n = 2$, we leave the proof to the reader. ■

Proposition 1.2.4. *Suppose f is bounded and locally Hölder continuous (with exponent $\alpha \leq 1$) in Ω and let w be the Newtonian potential of f . Then w is $C^2(\Omega)$, $\Delta w = f$ in Ω and for any $x \in \Omega$,*

$$w_{ij}(x) = \int_{\Omega_0} \Gamma_{ij}(x - y) (f(y) - f(x)) dy - f(x) \int_{\partial\Omega_0} \Gamma_i(x - y) \nu_j(y) ds_y.$$

Here Ω_0 is domain containing Ω where the divergence theorem holds, and f is extended to vanish outside Ω .

Proof. Similar to above. Come back to this if there's time. Proof techniques combine those in Theorems 1.2.3 and 1.2.5. ■

We provide the fundamental estimate for the Schauder estimate for the Laplacian.¹⁶

Lemma 1.2.5. *Let $B_1 := B_R(x_o)$ and $B_2 := B_{2R}(x_o)$ be concentric balls in \mathbb{R}^n . If $f \in C^{0,\alpha}(\bar{B}_2)$ with w the Newtonian potential of f in B_2 , then $w \in C^{2,\alpha}(\bar{B}_1)$. Furthermore,*

$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

¹⁵Check this computation...

¹⁶Lemma 4.4 of [2].

Proof. As a preliminary, we remark that $|B_2| = \omega_n(2R)^n$ and $|\partial B_2| = n\omega_n(2R)^{n-1}$. For $x \in B_1$, we obtain from 1.2.4 and 1.2.2, the first estimate

$$\begin{aligned}
|w_{ij}(x)| &\leq |f(x)| \int_{\partial B_2} |\Gamma_i(x-y)\nu_j(y)| ds_y + \int_{B_2} |\Gamma_{ij}(x-y)(f(y) - f(x))| dy \\
&\leq \frac{|f(x)|}{n\omega_n} \int_{\partial B_2} \underbrace{|x-y|^{1-n}}_{(*)} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \int_{B_2} |x-y|^{\alpha-n} dy \\
&\leq \frac{|f(x)|}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \underbrace{\int_{B_{3R}(x)} |x-y|^{\alpha-n} dy}_{(**)} \\
&\leq 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha [f]_{x;\alpha}, \\
&\leq C(|f(x)| + R^\alpha [f]_{\alpha;x}),
\end{aligned}$$

where $C = C(n, \alpha)$. Here, $(*)$ follows since $R \leq |x-y|$, and so $|x-y|^{1-n} \leq R^{1-n}$ since $n \geq 2$. We calculate $(**)$ as

$$\begin{aligned}
(**) &= \frac{n\omega_n(3R)^{n-1}}{(3R)^{n-1}} \int_0^{3R} \rho^{\alpha-n} (\rho^{n-1} d\rho) \\
&= (n\omega_n) \frac{\rho^\alpha}{\alpha} \Big|_0^{3R} \\
&= \frac{n\omega_n}{\alpha} (3R)^\alpha.
\end{aligned}$$

Notice the idea here to shift the center of the ball from x_o to x ; in doing so, we must enlarge the radius to $3R$ so that we capture B_2 . The first estimate gives

$$|D^2 w|_{0;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

For our second estimate, for $x, \bar{x} \in B_1$, $\xi = \frac{1}{2}(x + \bar{x})$ and $\delta = |x - \bar{x}|$,

$$\begin{aligned}
w_{ij}(\bar{x}) - w_{ij}(x) &= -f(\bar{x}) \int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y + f(x) \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \\
&\quad + \int_{B_2} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy - \int_{B_2} \Gamma_{ij}(x - y)(f(y) - f(x)) dy \\
&= f(x) \underbrace{\int_{\partial B_2} \Gamma_i(x - y) - \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_1} + (f(x) - f(\bar{x})) \underbrace{\int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_2} \\
&\quad + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(x - y)(f(x) - f(y)) dy}_{I_3} + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy}_{I_4} \\
&\quad + (f(x) - f(\bar{x})) \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) dy}_{I_5} \\
&\quad + \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) - \Gamma_{ij}(\bar{x} - y)(f(\bar{x}) - f(y)) dy}_{I_6},
\end{aligned}$$

where the second equality follows from (detailed) inspection. We estimate I_1, \dots, I_6 .¹⁷

1. We use mean value inequality for some \hat{x} between x, \bar{x} , then follow with the same trick as the first term in the first estimate to compute

$$\begin{aligned}
|I_1| &\leq |x - \bar{x}| \int_{\partial B_2} |D\Gamma_i(\hat{x} - y)| ds_y \\
&\leq |x - \bar{x}| \int_{\partial B_2} |\hat{x} - y|^{-n} ds_y \\
&\leq \frac{|x - \bar{x}|}{\omega_n} \int_{\partial B_2} R^{-n} ds_y \\
&= \frac{|x - \bar{x}| n 2^{n-1}}{R} \\
&\leq n 2^{n-\alpha} \left(\frac{\delta}{R} \right)^\alpha,
\end{aligned}$$

where the last estimate follows by simple computation from $\delta = |x - \bar{x}| < 2R$.¹⁸

2. The estimate follows the ideas when estimating the first term of $w_{ij}(x)$. We record our estimates for ease of replication in I_5 . Note that the final answer is completely

¹⁷My labelling of the six integrals follows that in [2].

¹⁸There is an extra factor of n in [2], which I don't see where it comes from. Similarly, this extra factor appears when estimating $|I_6|$.

independent of the ball.

$$\begin{aligned} |I_2| &\leq \frac{1}{n\omega_n} \int_{\partial B_2} |x - y|^{1-n} ds_y \\ &\leq \frac{1}{n\omega_n} \int_{\partial B_2} R^{1-n} ds_y = 2^{n-1}. \end{aligned}$$

3. $|I_3| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;\bar{x}}$. The estimate follows the ideas when estimating the second term of $w_{ij}(x)$ with $B_\delta(x)$ in the place of B_2 and $B_{\frac{3\delta}{2}}$ in the place of $B_{3R}(x)$.
4. $|I_4| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;\bar{x}}$. Ditto, but swap x for \bar{x} .
5. Since the estimate in I_2 is independent of the ball, the desired estimate follows immediately.

$$\begin{aligned} |I_5| &\leq \left| \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \right| + \left| \int_{\partial B_\delta(\xi)} \Gamma_i(x - y) \nu_j(y) ds_y \right| \\ &\leq 2(2^{n-1}) = 2^n \end{aligned}$$

6. Using ideas as before, we calculate for $C = C(n)$ and \hat{x} some point between \bar{x} and x ,

$$\begin{aligned} |I_6| &\leq |x - \bar{x}| \int_{B_2 - B_\delta(\xi)} |D\Gamma_{ij}(\hat{x} - y)| |f(\bar{x}) - f(y)| dy \\ &\leq C\delta \int_{B_2 - B_\delta(\xi)} \frac{|f(\bar{x}) - f(y)|}{|\hat{x} - y|^{n+1}} dy \\ &\leq C\delta [f]_{\alpha,x} \left(\frac{3}{2}\right)^\alpha \int_{B_\delta(\xi)} |\xi - y|^{\alpha-n-1} dy \\ &\leq \frac{C}{1-\alpha} \delta^\alpha [f]_{\alpha,x} \left(\frac{3}{2}\right)^\alpha \end{aligned}$$

Altogether, the second estimate gives

$$R^\alpha [D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

■

Lemma 1.2.5 proves a Schauder estimate for concentric balls. We introduce our final norm. Let $\eta = \text{diameter of } \Omega$, then

$$|u|'_{k,\alpha;\Omega} := \sum_{j=0}^k \sup_{x \in \Omega} \eta^j |D^j u(x)| + \sup_{x \neq y \in \Omega} \eta^{k+\alpha} \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\alpha}.$$

Again, observe in the $k = 0$ case, this norm agree with the standard one. We now state the fundamental estimate to show Schauder estimates for the Laplacian.¹⁹

¹⁹Theorem 4.6 of [2].

Lemma 1.2.6. *Let $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$ satisfy $\Delta u = f$. Then $u \in C^{2,\alpha}(\Omega)$ and for any two concentric balls $B_1 := B_R(x_o)$, $B_2 := B_{2R}(x_o)$ compactly contained in Ω ,*

$$|u|'_{2,\alpha;B_1} \leq C(|u|_{0;B_2} + |f|'_{0,\alpha;B_2}).$$

Proof. By Theorem 1.2.1, we may write

$$u = v + w,$$

for harmonic h and w is the Newtonian potential of f . The estimates 1.2.3 and 1.2.5 together give the result. \blacksquare

Finally, we have arrived at the promised land.²⁰

Theorem 1.2.7. *For $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$, with $\Delta u = f$,*

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

Proof. We first obtain an estimate on the $|u|_{2,\Omega}^*$ by the RHS. Since $|u|_{0;\Omega}$ appears on the RHS, we only need to worry about bounding $[u]_{j;\Omega}^*$ for $j = 1, 2$. We compute for $3R = d_x$ and $B_1 := B_R(x)$ and $B_2 := B_{2R}(x)$,

$$\begin{aligned} d_x |Du(x)| + d_x^2 |D^2 u(x)| &\leq (3R) |Du|_{0;B_1} + (3R)^2 |D^2 u|_{0;B_1} \\ &\leq C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}) \\ &\leq C(|u|_{0;\Omega} + R^2 |f|_{0,\alpha;B_2}) \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}), \end{aligned}$$

where the second inequality follows from Lemma 1.2.6. Therefore, we have

$$|u|_{2;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}). \quad (*)$$

Next, we obtain an estimate on $[u]_{2,\alpha;\Omega}^*$ by the RHS. We compute for $x \neq y \in \Omega$, with $d_x = d_{x,y}$.

$$\begin{aligned} d_x^{2+\alpha} \frac{|D^2 u(x) - D^2 u(y)|}{|x - y|^\alpha} &\leq (3R)^{2+\alpha} [D^2 u]_{0,\alpha;B_1} + 3^{\alpha+1} (3R)^2 \sup_{x \in \Omega} |D^2 u(x)| \\ &\leq C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}) + 6[u]_{2;\Omega}^* \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}) \end{aligned}$$

where the first inequality follows from a near-and-far estimate on B_1 , as in the proof of 1.1.1. The second inequality follows from Lemma 1.2.6. The last inequality follows from (*). \blacksquare

²⁰Theorem 4.8 of [2].

References

- [1] L. Evans. *Partial differential equations*. 2nd ed. Providence, R.I.: American Mathematical Society, 2010.
- [2] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer Berlin Heidelberg, 2001.