# Bernstein Theorem Notes

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The goal of this note is to provide a proof of *Bernstein's theorem* in $\mathbb{R}^3$ following 1.4 - 1.5 of [CM11].

**Theorem 0.1** (Bernstein)**.** *If $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire solution to the minimal surface equation, then its graph must be an affine plane.*

*Remark* 0.2. It's interesting to compare this statement to *Liouville theorem*, namely bounded (or sublinear) entire harmonic functions are constant.

## 1 Preliminaries

For $\Sigma^2 \subset \mathbb{R}^3$ be orientable and $\nu$ be a choice of unit normal on $\Sigma$. We will define two maps, which will be identified with each other [Figure 1]. The first map is the *Weingarten map*,

$$\begin{aligned} T\Sigma &\to T\Sigma \\ X &\mapsto \nabla_X \nu. \end{aligned}$$

Note that changing this to the $(0,2)$ version gives, up to sign, the second fundamental form

$$\mathrm{II}(X,Y) := \langle \nabla_X \nu, Y \rangle.$$

In particular, the Weingarten map is symmetric, real-valued so it diagonalizes with eigenvalues $\kappa_1, \kappa_2$. The second map is the differential of the *Gauss map*,

$$\begin{aligned} d\nu : T\Sigma &\to T\mathbb{S}^2 \\ X &\mapsto d\nu(X). \end{aligned}$$

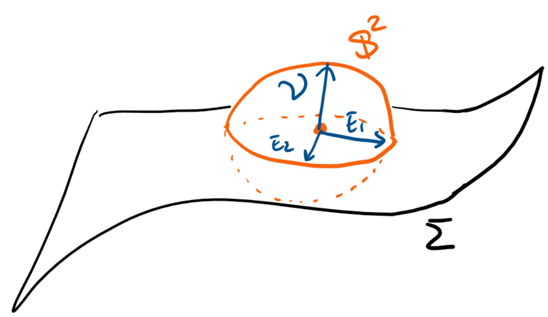


Figure 1: Weingarten map is the differential of Gauss map

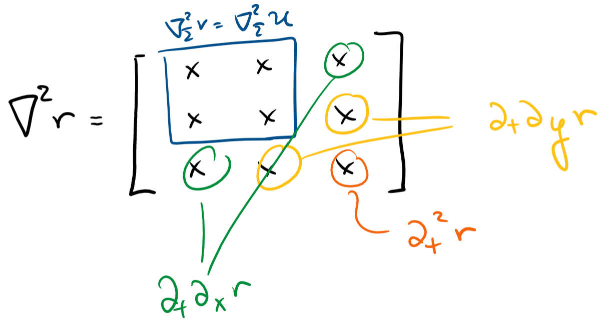


Figure 2: The full Hessian of $r$

The two maps can be identified since any orthonormal frame $E_1, E_2$ on $\Sigma$ can be carried to one on $\mathbb{S}^2$, so there is no point in distinguishing the codomain between $T\Sigma$ or $T\mathbb{S}^2$. Furthermore assuming $\Sigma$ is minimal forces $\kappa_2 = -\kappa_1$ and gives anti-conformality of the Gauss map,

$$|d\nu|^2 = |\mathrm{II}|^2 = \kappa_1^2 + \kappa_2^2 = -2\kappa_1\kappa_2 = -2\det(d\nu). \tag{1}$$

We briefly remark that $\det(d\nu)$ is a common definition of *Gauss curvature.*

We observe if $\Sigma$ is given as the regular level set of a function $r : \mathbb{R}^3 \to \mathbb{R}$, then its second fundamental form is proportional the surface Hessian. Recall for $X, Y \in \mathfrak{X}(\Sigma)$,

$$\nabla^2_\Sigma r(X, Y) := \langle \nabla_X \nabla r, Y \rangle.$$

Since the gradient is perpendicular to level sets, we see that

$$\mathrm{II} = \frac{\nabla^2_\Sigma r}{|\nabla r|}.$$

We will apply this observation to Bernstein's theorem via the following procedure which turns any graph into a level set. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a solution to the minimal surface equation, and consider $\Sigma = \operatorname{graph} u \subset \mathbb{R}^3$ with the induced metric. Consider the signed distance function $r$ in a neighborhood of $\Sigma$,

$$\begin{aligned} r : \Sigma \times (-\epsilon, \epsilon) &\to \mathbb{R} \\ (x, t) &\mapsto u(x) - t. \end{aligned}$$

We compute the norm of the gradient of $r$ and the Hessian of $r$ as

$$|\nabla r| = \sqrt{(\partial_t r)^2 + |\nabla_\Sigma r|^2} = \sqrt{1 + |\nabla u|^2},$$

$$\nabla^2_\Sigma r = \nabla^2_\Sigma u.$$

The Hessian identity follows since all derivatives in the direction of $\Sigma$ fall onto $u$ [Figure 2]. Therefore, the second fundamental form can be expressed purely in terms of $u$,

$$\Pi = \frac{\nabla^2_\Sigma r}{|\nabla r|} = \frac{\nabla^2_\Sigma u}{\sqrt{1 + |\nabla u|^2}}.$$

In particular, if the second fundamental form vanishes, then so must the Hessian, so $u$ will graph an affine plane.

## 2 Bernstein's theorem

We begin our quest of showing $\mathrm{II} \equiv 0$ on $\Sigma := \operatorname{graph} u$ by showing that the total curvature is bounded by the energy of any cutoff function.

**Lemma 2.1.** *Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a solution to the minimal surface equation. For any non-negative, Lipschitz*[1] *function $\eta$ with support contained in $\Omega \times \mathbb{R}$,*

$$\int_\Sigma \eta^2 |II|^2 \leq C \int_\Sigma |\nabla_\Sigma \eta|^2.$$

*Proof.* Let $\omega$ be the area form on $\mathbb{S}^2$ and consider the upper hemisphere. Consider the 1-form $\alpha$ such that $d\alpha = \omega$ on the upper hemisphere. Equation 1 implies

$$|\mathrm{II}|^2 d\Sigma = -2\det(d\nu)d\Sigma = 2\nu^*\omega = 2d\nu^*\alpha.$$

Furthermore, in local coordinates we have $(\nu^*\alpha)_i = (d\nu)_i^j \alpha_j$, so Cauchy-Schwarz implies

$$|\nu^*\alpha| \leq C|\mathrm{II}|,$$

with $C = C(\alpha)$. In total,

$$\begin{aligned}\int_\Sigma \eta^2 |\mathrm{II}|^2 d\Sigma = 2\int_\Sigma \eta^2 d\nu^*\alpha = &\underbrace{-4\int_\Sigma \eta d\eta \wedge \nu^*\alpha}_{\text{Stokes \& } \eta^2 \text{ vanishes on } \partial\Sigma} \\ &\leq 4C\int_\Sigma \eta |\nabla_\Sigma \eta||\mathrm{II}| d\Sigma \leq 4C\left(\int_\Sigma \eta^2 |\mathrm{II}|^2 d\Sigma\right)^{\frac{1}{2}} \left(\int_\Sigma |\nabla_\Sigma \eta|^2 d\Sigma\right)^{\frac{1}{2}},\end{aligned}$$

and so we reincorporate to get

$$\int_\Sigma \eta^2 |\mathrm{II}|^2 d\Sigma \leq 16C^2 \int_\Sigma |\nabla_\Sigma \eta|^2 d\Sigma.$$

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*Remark* 2.2. I'm not convinced that non-negativity of $\eta$ is used in any meaningful way in the above proof. This assumption can probably be dropped.

In light of Lemma 2.1, game now is to find a sequence of non-negative Lipschitz cutoff functions $\eta_N$ tending to 1, with energy tending to 0. Let us first work heuristically. Define radial cutoff functions [Figure 3] for $r = |x|$,

$$\eta_N(r) := \begin{cases} 1 & r \leq e^N \\ 2 - \frac{\log(r)}{N} & e^N < r \leq e^{2N} \\ 0 & e^{2N} < r. \end{cases}$$

[1] It is helpful to recall Rademacher's theorem, which tells us that Lipschitz functions are differentiable almost everywhere.

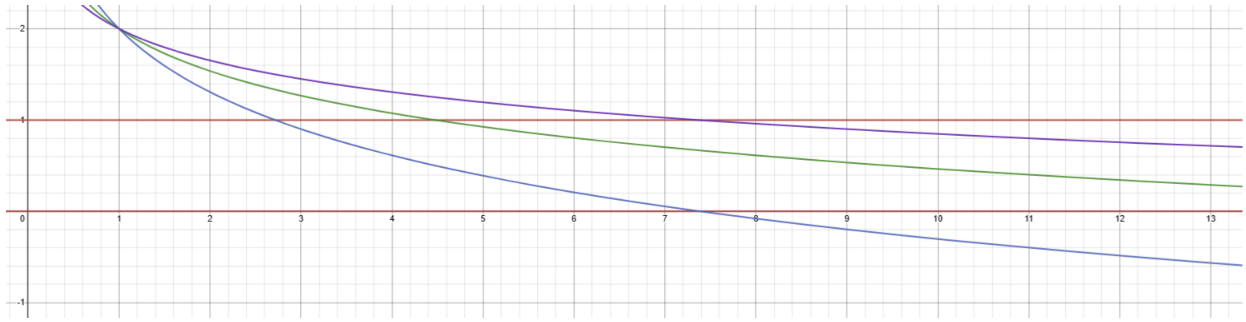


Figure 3: $\eta_N$ for $N = 1, 1.5, 2$

Observe $\eta_N \to 1$ as $N \to \infty$. We compute $|\nabla \eta_N| = \frac{1}{Nr}$,[2] and by co-area formula we compute

$$\int_{\mathbb{R}^2} |\nabla \eta_N|^2 = \int_0^\infty \int_{\partial B_r} |\nabla \eta_N|^2 d\sigma dr = \int_{e^N}^{e^{2N}} \frac{2\pi r}{(Nr)^2} dr = \frac{2\pi}{N}.$$

In particular, the energy of $\eta_N$ vanishes as $N \to \infty$. The same computation holds with *linear perimeter growth*

$$\text{Length}(\partial B_r) \leq Cr,$$

as we may basically repeat the above argument

$$\int_\Sigma |\nabla \eta_N|^2 = \int_0^\infty \int_{\partial B_r} |\nabla \eta_N|^2 d\sigma dr = \int_{e^N}^{e^{2N}} \frac{\text{Length}(\partial B_r)}{(Nr)^2} dr \leq \frac{C}{N}.$$

However, what's important is that the same computation holds under the sharper assumption of *quadratic area growth*

$$\text{Area}(B_r) \leq Cr^2.$$

This is important since by a calibration argument (Corollary 1.2 of [CM11]), a minimal surface $\Sigma^2 \subset \mathbb{R}^3$ will always satisfy a quadratic area growth (with $C = 2\pi$). To prove the energy bound under this assumption, first observe $|\nabla \eta_N|$ is monotonically decreasing, so

$$\sup_{B_{e^k} \setminus B_{e^{k-1}}} |\nabla_N \eta|^2 = |\nabla \eta_N|^2 \Big|_{\partial B_{e^{k-1}}} = N^{-2} e^{2-2k}.$$

We break up the "middle section" into concentric annuli and compute

$$\begin{aligned}
\int_\Sigma |\nabla \eta_N|^2 &\leq \sum_{k=N+1}^{2N} \int_{B_{e^k} \setminus B_{e^{k-1}}} N^{-2} e^{2-2k} d\sigma \\
&\leq \sum_{k=N+1}^{2N} N^{-2} e^{2-2k} \text{Area}(B_{e^k} \setminus B_{e^{k-1}}) \leq \sum_{k=N+1}^{2N} C N^{-2} e^2 = \frac{Ce^2}{N}.
\end{aligned}$$

*Remark* 2.3. Let's take a second to summarize what happened. The energy integrand decays like $r^{-2}$, while the domain grows like $r^2$. To get the desired decay rate, you associate a constant $N$ with the energy integrand, which pops out as $N^{-2}$. This combats the linear $N$ that pops out of the sum over the annuli, leaving a final rate of $N^{-1}$.

[2] In fact, you probably start at this and define $\eta$ from here.

*Remark* 2.4. This whole heuristic can obviously be sharpened. Two immediate directions are replacing Cauchy-Schwarz with Hölder in Lemma 2.1, and requiring a decay on the energy of $\eta_N$ of $N^{-\alpha}$ for any $\alpha > 0$. Generalizing in these directions is the content of Chapter 2.

Inspired by these heuristic computations, we perform a similar logarithmic cutoff trick to conclude Bernstein's theorem.

**Corollary 2.5.** *If $u : \Omega : \mathbb{R}^2 \to \mathbb{R}$ is a solution to the minimal surface equation, $\kappa > 1$ and $\Omega$ contains a ball of radius $\kappa R$ centered at the origin, then*

$$\int_{B_{\sqrt{\kappa}R}\cap\Sigma} |II|^2 \leq \frac{C}{\log\kappa}.$$

*Remark* 2.6. Note that the parameter $R$ is needed as we require $\kappa > 1$ for taking log. But for Bernstein purposes, we can think of fixing $R = 1$ and $\kappa \to \infty$.

*Proof.* Define $\eta : \mathbb{R}^3 \to \mathbb{R}$ with support contained in $B_{\kappa R}$. Again for $r = |x|$, define

$$\eta(r) := \begin{cases} 1 & r \leq \sqrt{\kappa}R \\ 2 - \frac{2\log(\frac{r}{R})}{\log\kappa} & \sqrt{\kappa}R < r \leq \kappa R \\ 0 & \kappa R < r. \end{cases}$$

We compute $|\nabla_\Sigma \eta| \leq \frac{2}{r\log\kappa}$, and assuming for simplicity $\log\sqrt{\kappa} = \log\kappa/2$ is an integer,

$$\begin{aligned} \int_{B_{\sqrt{\kappa}R}\cap\Sigma} |\mathrm{II}|^2 \leq \int_\Sigma \eta^2|\mathrm{II}|^2 \leq C\int_\Sigma |\nabla_\Sigma\eta|^2 &\leq \frac{4C}{(\log\kappa)^2}\underbrace{\int_{B_{\kappa R}\cap\Sigma} r^{-2}dr}_{\text{quadratic decay}} \\ &\leq \frac{4C}{(\log\kappa)^2}\sum_{k=\log\sqrt{\kappa}}^{\log\kappa}\int_{B_{e^kR}\setminus B_{e^{k-1}R}\cap\Sigma} r^{-2}dr \\ &\leq \frac{4C}{(\log\kappa)^2}\sum_{k=\log\sqrt{\kappa}}^{\log\kappa}(e^{k-1}R)^{-2}\cdot\underbrace{2\pi(e^kR)^2}_{\text{quadratic area growth}} \\ &= \frac{4C}{(\log\kappa)^2}\sum_{k=\log\sqrt{\kappa}}^{\log\kappa} 2e^2\pi = \frac{4\pi e^2C}{\log\kappa}. \end{aligned}$$

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# References


[CM11] T.H. Colding and W.P. Minicozzi. *A Course in Minimal Surfaces*. Graduate studies in mathematics. American Mathematical Society, 2011. ISBN: 9780821853238.