Jacobi's Formula

At the heart of everything are the following two definitions for determinant and trace. For $M \in M_n(\mathbb{R})$ and $\{e_i\}$ a basis for $\mathbb{R}^n$, we define $\det M$ and $\mathrm{tr} M$ as the following scalars

$$Me_1 \wedge \ldots \wedge Me_n = (\det M) e_1 \wedge \ldots \wedge e_n, \tag{1}$$

$$\sum_{i=1}^{n} e_1 \wedge \ldots Me_i \ldots \wedge e_n = (\mathrm{tr} M) e_1 \wedge \ldots \wedge e_n. \tag{2}$$

Definition (1) is standard, but (2) may be unfamiliar. Definition (2) recovers the trace as the sum of the diagonals by the following computation.

$$\begin{aligned}
\sum_{i=1}^{n} e_1 \wedge \ldots Me_i \ldots \wedge e_n &= \sum_{i=1}^{n} e_1 \wedge \ldots \left(\sum_{j=1}^{n} M_i^j e_j \right) \ldots \wedge e_n \\
&= \sum_{i=1}^{n} e_1 \wedge \ldots M_i^i e_i \ldots \wedge e_n \\
&= (\mathrm{tr} M) e_1 \wedge \ldots \wedge e_n.
\end{aligned}$$

where the second equality follows since all terms $j \neq i$ die in the wedge.

From these definitions, some basic formulas follow easily. The first is Jacobi's formula. For geometers, this enters when talking about first variation of surface area.

Proposition 0.1 (Jacobi's formula)**.** *For M_t a curve in $GL_{n\times n}(\mathbb{R})$,*

$$\frac{d}{dt} \det(M_t) = \det(M_t) tr \left(M_t^{-1} \cdot \frac{d}{dt} M_t \right) \tag{3}$$

Proof. Definitions (1) and (2) suggest that the correct setting to show equation (3) is in the top exterior power.

$$
\begin{aligned}
\frac{d}{dt}\left(\det(M_t)\right) e_1 \wedge \ldots \wedge e_n &= \frac{d}{dt}\left(\det(M_t) e_1 \wedge \ldots \wedge e_n\right) \\
&= \frac{d}{dt}(M_t e_1 \wedge \ldots \wedge M_t e_n) \\
&= \sum_{i=1}^{n} \left(M_t e_1 \wedge \ldots \frac{d}{dt} M_t e_i, \ldots \wedge M_t e_n \right) \\
&= \sum_{i=1}^{n} \left(M_t e_1 \wedge \ldots M_t M_t^{-1} \frac{d}{dt} M_t e_i, \ldots \wedge M_t e_n \right) \\
&= \det(M_t) \sum_{i=1}^{n} e_1 \wedge \ldots M_t^{-1} \frac{d}{dt} M_t e_i, \ldots \wedge e_n \\
&= \det(M_t) \mathrm{tr} \left(M_t^{-1} \cdot \frac{d}{dt} M_t \right).
\end{aligned}
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Using the same method, one also easily sees that the "derivative of determinant is the trace".

Proposition 0.2. *For any matrix $M \in M_{n\times n}(\mathbb{R})$,*

$$
\left.\frac{d}{dt}\right|_{t=0} \det(I + tM) = trM. \tag{4}
$$

Proof. Again this equality happens in the top exterior power. We compute using definitions (1) and (2),

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \det(I + tM) e_1 \wedge \ldots \wedge e_n &= \left.\frac{d}{dt}\right|_{t=0} (I + tM) e_1 \wedge \ldots \wedge (I + tM) e_n \\
&= \sum_{i=1}^{n} e_1 \wedge \ldots M e_i \ldots \wedge e_n \\
&= (\mathrm{tr} M) e_1 \wedge \ldots \wedge e_n.
\end{aligned}
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