

Notes on Schauder Estimates

Paul Tee

The following are the *standard assumptions*. On bounded domains $\Omega \subset \mathbb{R}^n$, will consider 2nd order (uniformly) elliptic operators

$$Lu := \sum_{i,j} a^{ij} u_{ij} + \sum_i b^i u_i + cu = f,$$

where a^{ij}, b^i, c are functions with regularity to be specified later. We assume symmetry on the leading order coefficient terms $a^{ij} = a^{ji}$, and for all uniform ellipticity says $\xi \in \mathbb{R}^n$, for some positive constant $\lambda > 0$,

$$\sum_{i,j} a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2.$$

The regularity of this approach will require our functions u, f to lie in appropriate Hölder spaces. In this note, we will introduce a zoo of seminorms and norms on the way to proving Schauder estimates, each of which will slightly differently exploit the fact that Ω is bounded.



To start, we introduce the standard Hölder spaces and their respective norms and seminorms. This approach will be pointwise in nature, compared to the integral-based approach of Sobolev spaces last semester. Let $\alpha \in (0, 1)$. With respect to Ω , we have the pointwise (and global) Hölder seminorms which detect pointwise (and uniform) α -Hölder continuity. We may also define local α -Hölder continuity by asking for every compact $K \subset \Omega$, that $[u]_{\alpha;K}$ is finite. In such a case, we say u is locally uniformly continuous.

1. **(Pointwise)** For $x_o \in \Omega$,

$$[u]_{0,\alpha;x_o} := \sup_{x \in \Omega} \frac{|u(x) - u(x_o)|}{|x - x_o|^\alpha}.$$

2. **(Local)** For all compact $K \subset \Omega$,

$$\sup_{x \neq y \in K} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$$

3. **(Global)**

$$[u]_{0,\alpha;\Omega} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

As usual, global implies local implies pointwise. The above gives rise to $C^{k,\alpha}(\bar{\Omega})$ (and $C^{k,\alpha}(\Omega)$) consisting of functions whose k -th order derivatives are uniformly (and locally uniformly) continuous. In the global case, we get a seminorm as defined above. We can formally extend this to $\alpha = 0, 1$ which gives continuity and Lipschitz continuity respectively.

Hölder regularity gives a quantitative measurement of continuity. In fact, greater the α , the more regularity we have.¹ One can see this in the following illustrative example:

Example 0.0.1. *The function $f(x) = |x|^\beta$ is α -Hölder continuous at $x = 0$ iff $\alpha \leq \beta$.*

Proof. Let Ω be a domain containing 0. We compute

$$\sup_{x \neq 0 \in \Omega} \frac{|f(x) - f(0)|}{|x - 0|^\alpha} = \sup_{x \neq 0 \in \Omega} |x|^{\beta-\alpha} = \begin{cases} \text{finite} & \alpha \leq \beta \\ \infty & \alpha > \beta. \end{cases}$$

■

Recall the $C^k(\Omega)$ seminorm is defined by measuring how large the k th derivative of u on Ω is,

$$[u]_{k,0;\Omega} := |D^k u|_{0;\Omega} := \sup_{x \in \Omega, |\beta|=k} |D^\beta u|,$$

where β is a multi-index of length k . Similarly, we have the $C^k(\Omega)$ norm by adding the largest of all the derivatives to order k ,

$$|u|_{k,0;\Omega} := \sum_{j=1}^k |D^j u|_{0,\Omega}.$$

These definitions admit easy extensions to include α -Hölder regularity, namely we have the *standard (semi)norm* of $C^{k,\alpha}(\bar{\Omega})$.

1. $[u]_{k,\alpha;\Omega} := [D^k u]_{0,\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u]$.
2. $|u|_{k,\alpha;\Omega} := |u|_{C^k(\bar{\Omega})} + [u]_{k,\alpha;\Omega}$.

One may check that $C^{k,\alpha}(\bar{\Omega})$ for $0 \leq \alpha < 1$ are Banach spaces with the standard norms. One should think of seminorms as taking the “highest degree” portion of the norm.

A short word on notation on our (semi)norms. From here on out, we drop the α from the notation when $\alpha = 0$, but never drop how many derivatives we take.

¹We briefly mention that for $k = 0$, we have a Rellich-type result for Hölder spaces. Namely for $0 < \alpha < \beta \leq 1$, we have the compact inclusion $C^{0,\beta}(\Omega) \subset C^{0,\alpha}(\Omega)$.

1 The Classical Proof

We unabashedly follow Chapters 2, 4, 6 of [2].

1.1 Reduction to the Laplacian

Roughly speaking, the Schauder interior estimates tell us that we can bound the $C^{2,\alpha}$ norm of u in terms of the C^0 norm of u and the $C^{0,\alpha}$ norm of f . Explicitly, we have the following theorem.

Theorem 1.1.1. (*Interior Estimates*)² *Given the standard assumptions, if*

$$|a^{ij}|_{0,\alpha;\Omega}^{(0)}, |b^i|_{0,\alpha;\Omega}^{(1)}, |c|_{0,\alpha;\Omega}^{(2)} \leq \Lambda,$$

then the following estimate holds

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

We now introduce the relevant norms so the above statement is legible. The following Hölder (semi)norms are weighted by their distances to the boundary. Namely for $x \neq y \in \Omega$, consider

$$d_x := d(x, \partial\Omega) \quad d_{x,y} := \min\{d_x, d_y\}.$$

We use the above two quantities to define the following (semi)norms. For $\alpha \in [0, 1]$ and $u \in C^{k,\alpha}(\Omega)$, we first define the θ -homogeneous (semi)norm of $u \in C^{k,\alpha}(\Omega)$.

1. **0-homogeneous seminorm.** For $\alpha = 0$,

$$[u]_{k;\Omega}^* := \sup_{x \in \Omega, |\beta|=k} d_x^k |D^\beta u(x)|.$$

For $\alpha > 0$,

$$[u]_{k,\alpha;\Omega}^* := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

2. **0-homogeneous norm.** For $\alpha = 0$,

$$|u|_{k;\Omega}^* := \sum_{j=0}^k [u]_{j,\Omega}^*.$$

For $\alpha > 0$,

$$|u|_{k,\alpha;\Omega}^* := |u|_{k;\Omega}^* + [u]_{k,\alpha;\Omega}^*.$$

²Compare Theorem 1.1.1 to the regularity statement of ([1], pg 329, “Interior H^2 -regularity”).

Next, we introduce the σ -homogeneous (semi)norm of $u \in C^{k,\alpha}(\Omega)$ for $\sigma \in \mathbb{R}$, generalizing the above. The 0-homogeneous (semi)norms can obviously be recovered by setting $\sigma = 0$.

1. **σ -homogeneous seminorm.** For $\alpha = 0$,

$$[u]_{k;\Omega}^{(\sigma)} := \sup_{x \in \Omega, |\beta|=k} d_x^{k+\sigma} |D^\beta u(x)|.$$

For $\alpha > 0$,

$$[u]_{k,\alpha;\Omega}^{(\sigma)} := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

2. **σ -homogeneous norm.** For $\alpha = 0$,

$$|u|_{k;\Omega}^{(\sigma)} := \sum_{j=0}^k [u]_{j,\Omega}^{(\sigma)}.$$

For $\alpha > 0$,

$$|u|_{k,\alpha;\Omega}^{(\sigma)} := |u|_{k,\Omega}^{(\sigma)} + [u]_{k,\alpha;\Omega}^{(\sigma)}.$$

It's worth pointing out that in the $k = 0$ case, the σ -homogeneous (semi)norms agree with the standard ones. We show that the σ -inhomogeneous norms enjoy a submultiplicative property.³

Lemma 1.1.2. *For $\sigma + \tau \geq 0$, we have*

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} \leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}.$$

Proof. This is a Baby Rudin proof. Recall

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} := \underbrace{\sup_{x \neq y \in \Omega} \left(d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x - y|^\alpha} \right)}_{(*)} + \underbrace{\sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|)}_{(**)}.$$

We estimate the first term by

$$\begin{aligned} (*) &= \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x - y|^\alpha} \right) \\ &\leq \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\sigma+\alpha+\tau} \cdot d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)|}{|x - y|^\alpha} \right) \\ &\leq \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)|}{|x - y|^\alpha} \right) + \sup_{x \neq y \in \Omega} \left(d_{x,y}^{\tau+\alpha+\sigma} \frac{|g(y)| \cdot |f(x) - f(y)|}{|x - y|^\alpha} \right) \\ &\leq |f|_{0;\Omega}^{(\sigma)} \cdot [g]_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot [f]_{0,\alpha;\Omega}^{(\sigma)}. \end{aligned}$$

³(6.11) of [2].

We estimate the second term by

$$\begin{aligned}
(**) &= \sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|) \\
&\leq \sup_{x \in \Omega} (d_x^\sigma |f(x)|) \cdot \sup_{x \in \Omega} (d_x^\tau |g(x)|) \\
&= |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)}
\end{aligned}$$

Combining the two estimates,

$$\begin{aligned}
(*) + (**) &= |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot |f|_{0,\alpha;\Omega}^{(\sigma)} + |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)} \\
&\leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}
\end{aligned}$$

■

Throughout the course of the proof, we will frequently employ the interpolation inequalities.⁴ Unfortunately, we must omit the proof.

Lemma 1.1.3. *Let $u \in C^{2,\alpha}(\Omega)$, then for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that*

$$[u]_{j,\beta;\Omega}^* \leq |u|_{j,\beta;\Omega}^* \leq C|u|_{0;\Omega} + \varepsilon[u]_{2,\alpha;\Omega}^*,$$

where $j = 0, 1, 2$ and $0 \leq \alpha, \beta \leq 1$ and $j + \beta < 2 + \alpha$.



We first reduce showing Schauder estimates from the general case to the case of constant coefficients with no lower order terms.⁵

Proof. (of 1.1.1.) By using a proper, compact exhaustion, it suffices to show the equality holds for every compact subset of Ω . By the interpolation inequalities 1.1.3, it suffices to show the result where the LHS is $[u]_{2,\alpha;\Omega}^*$. Thus, the relevant quantity to consider is

$$d_{x_o, y_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha},$$

for arbitrary $x_o, y_o \in \Omega$. The idea here is to “freeze” one variable to reduce to the case of constant coefficients with no lower order terms. Suppose $d_{x_o} = d_{x_o, y_o}$, and consider

$$\sum_{i,j} a^{ij}(x_o) u_{ij}(x) = F(x),$$

where $F := \sum_{i,j} a^{ij}(x_o) u_{ij} - Lu + f$. Assuming local Schauder estimates on the ball

$$|u|_{2,\alpha;B}^* \leq C(|u|_{0;B} + |F|_{0,\alpha;B}^{(2)}),$$

⁴(6.8) and (6.9) of [2].

⁵Theorem 6.2 of [2].

where $B := B_{x_o}(\frac{d}{2})$ where $d := \mu d_{x_o}$ for some $\mu \leq \frac{1}{2}$ we will choose later.

We now perform a near-and-far estimate on the frozen case, and we estimate the “junk” term F in the sequel. For the near estimate, if $y_o \in B$, we have the estimate

$$\left(\frac{d}{2}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq C(|u|_{0;B} + |F|_{0,\alpha;B}^{(2)}).$$

Since $(\frac{d}{2})^{2+\alpha} = \left(\frac{d_{x_o}\mu}{2}\right)^{2+\alpha}$, we enlarge our domain to yield the estimate

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right).$$

For the far estimate, if $y_o \notin B \iff 2|x_o - y_o| \geq d$, and so $(2|x_o - y_o|)^\alpha \geq d^\alpha$. Thus,

$$\begin{aligned} d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} &= \left(\frac{d}{\mu}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \\ &\leq \frac{d^2 \cdot 2^\alpha}{\mu^{2+\alpha}} |D^2 u(x_o) - D^2 u(y_o)| \\ &= \frac{(d_{x_o}\mu)^2 \cdot 2^\alpha}{\mu^2 \cdot \mu^\alpha} |D^2 u(x_o) - D^2 u(y_o)| \\ &\leq \left(\frac{2}{\mu}\right)^\alpha (d_{x_o}^2 |D^2 u(x_o)| + d_{y_o}^2 |D^2 u(y_o)|) \\ &\leq \frac{2^{\alpha+1}}{\mu^\alpha} [u]_{2;\Omega}^* \\ &\leq \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*. \end{aligned}$$

Combining the two estimates, we have

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right) + \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*$$



The next step is to estimate the “junk” term $|F|_{0,\alpha;B}^{(2)}$ in terms of $|u|_{0;\Omega}$ and $[u]_{2,\alpha;\Omega}^*$. This is a mildly annoying process, but straightforward conceptually. Specifically, we will estimate the terms on the RHS of

$$|F|_{0,\alpha;B}^{(2)} \leq \sum_{i,j} |(a^{ij}(x_o) - a^{ij}(x))u_{ij}|_{0,\alpha;B}^{(2)} + \sum_i |b^i u_i|_{0,\alpha;B}^{(2)} + |cu|_{0,\alpha;B}^{(2)} + |f|_{0,\alpha;B}^{(2)}.$$

We first provide a generic estimate local-to-global estimate for arbitrary $g \in C^{0,\alpha}(\Omega)$. For $x \in B$, define $d_x^{\partial B} := \text{dist}(x, \partial B)$ and $d_{x,y}^{\partial B} := \min\{d_x^{\partial B}, d_y^{\partial B}\}$. Note that upon localizing to

B , we need to consider the distance to ∂B not to $\partial\Omega$.⁶

$$\begin{aligned}
|g|_{0,\alpha;B}^{(2)} &\leq \sup_{x \in B} (d_x^{\partial B})^2 \cdot \sup_{x \in B} |g(x)| + \sup_{x \neq y \in B} (d_{x,y}^{\partial B})^{2+\alpha} \cdot \sup_{x \neq y \in B} \frac{|g(x) - g(y)|}{|x - y|^\alpha} \\
&= d^2 |g|_{0;B} + d^{2+\alpha} [g]_{\alpha;B} \\
&\leq |g|_{0;\Omega}^{(2)} + [g]_{\alpha;\Omega}^{(2)} \\
&\leq \left(\frac{\mu}{1-\mu} \right)^2 |g|_{0;\Omega}^{(2)} + \left(\frac{\mu}{1-\mu} \right)^{2+\alpha} [g]_{\alpha;\Omega}^{(2)} \\
&\leq 4\mu^2 |g|_{0;\Omega}^{(2)} + 8\mu^{2+\alpha} [g]_{0,\alpha;\Omega}^{(2)}
\end{aligned}$$

The third line follows since for any $x \in B$ which comes from the computation

$$d = \mu d_{x_o} \leq (1 - \mu) d_{x_o} \leq d_x.$$

Replacing d with d_x is nondecreasing on B , so the result follows. The fifth line follows from $\mu \leq \frac{1}{2} \implies \left(\frac{1}{1-\mu} \right)^2 \leq 4$ and $\left(\frac{1}{1-\mu} \right)^{2+\alpha} \leq 8$.

We now apply estimates in detail to the principal term; the lower order terms are treated similarly. By Lemma 1.1.2, we compute

$$\begin{aligned}
|(a^{ij}(x_o) - a^{ij})u_{ij}|_{0,\alpha;B}^{(2)} &\leq |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)} \cdot |u_{ij}|_{0,\alpha;B}^{(2)} \\
&\leq \underbrace{|a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)}}_{(*)} \cdot (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*).
\end{aligned}$$

We also estimate the local term $(*)$ by

$$\begin{aligned}
(*) &= \sup_{x \in B} (|a^{ij}(x_o) - a^{ij}(x)|) + \sup_{x \neq y \in B} \left((d_{x,y}^{\partial B})^\alpha \frac{|a^{ij}(x_o) - a^{ij}(x) - (a^{ij}(x_o) - a^{ij}(y))|}{|x - y|^\alpha} \right) \\
&\leq \sup_{x \neq x_o \in B} \left(\frac{|a^{ij}(x_o) - a^{ij}(x)|}{|x - x_o|^\alpha} \cdot |x - x_o|^\alpha \right) + d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2(2\mu)^\alpha [a^{ij}]_{0,\alpha;\Omega}^* \\
&\leq 4\Lambda\mu^\alpha,
\end{aligned}$$

where the second to last inequality follows since $d \leq d_x$ for all $x \in B$, and $\mu \leq \frac{1}{2}$. Altogether, we find

$$\begin{aligned}
\sum_{i,j} |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(2)} &\leq 4n^2 \Lambda \mu^\alpha (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} ([u]_{2;\Omega}^* + \mu^\alpha [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} (C(\mu) |\mu|_{0;\Omega} + 2\mu^\alpha [u]_{2,\alpha;\Omega}^*),
\end{aligned}$$

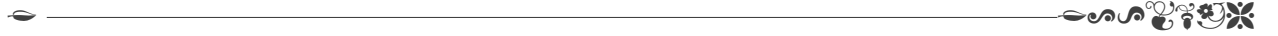
⁶Caution, [2] uses d_x for our $d_x^{\partial B}$, which may be a cause of confusion.

where the last step follows from the interpolation inequalities 1.1.3. Similarly, we have the estimates

$$\begin{aligned}\sum_i |b^i u_i|_{0,\alpha;B}^{(2)} &\leq 8n\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]_{2,\alpha;\Omega}^*), \\ |cu|_{0,\alpha;B}^{(2)} &\leq 8\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]_{2,\alpha;\Omega}^*), \\ |f|_{0,\alpha;\Omega}^{(2)} &\leq 8\mu^2|f|_{0,\alpha;\Omega}^{(2)}.\end{aligned}$$

Altogether, we estimate F by

$$|F|_{0,\alpha;B}^{(2)} \leq C\mu^{2+2\alpha}[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$



Combining the two steps and using the interpolation inequalities 1.1.3, we get

$$[u]_{2,\alpha;\Omega}^* \leq C\mu^\alpha[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

Fixing μ so that $C\mu^\alpha \leq \frac{1}{2}$ yields the desired result. ■

We quickly sketch a proof of Schauder estimates for the case of constant coefficients with no lower order terms.⁷ Consider

$$L_0 u := \sum_{i,j} A^{ij} u_{ij},$$

where $A := (A^{ij}) \in \text{GL}_n(\mathbb{R})$ is a constant matrix. In this case, we diagonalize A by an orthogonal matrix⁸ to a diagonal one, turning $L_0 u(x) = f(x)$ to $\Delta \tilde{u}(y) = \tilde{f}(y)$. Since our transformations are all linear, the norms are equivalent, and so it suffices to prove Schauder estimates for the Laplacian.

1.2 The Laplacian

We first introduce the *fundamental solution* and some basic ideas surrounding it. The upshot is that we obtain some estimates for future use, and the following *Green's representation formula*,⁹ to prove the following theorem.

Theorem 1.2.1. *Let $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$ such that $\Delta u = f$. Then,*

$$u = h + w,$$

where h is some harmonic function, and w is the Newtonian potential of f .

⁷Lemma 6.1 of [2].

⁸This follows by spectral theorem, since A is a symmetric, real-valued, nonsingular matrix.

⁹(2.16) of [2].

Fix $x \in \Omega \subset \mathbb{R}^n$ for $n \geq 2$.¹⁰ Let ω_n be the volume of the n -ball. Define the fundamental solution on $\Omega - x$ as the following radial function about x :

$$\Gamma(x - y) := \begin{cases} \frac{1}{n(2-n)\omega_n} |x - y|^{2-n} & n > 2 \\ \frac{1}{2\pi} \log |x - y| & n = 2. \end{cases}$$

We first obtain some basic estimates on Γ . The following derivatives will be taken with respect to the variable x . For convenience, define

$$S(x) := \sum_{k=1}^n (x_k - y_k)^2,$$

so that $|x - y| = S^{\frac{1}{2}}$, and $S_i = 2(x_i - y_i)$. We calculate for $n > 2$,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{n(2-n)\omega_n} \left(\frac{2-n}{2} S^{\frac{2-n}{2}-1} \cdot 2(x_i - y_i) \right) \\ &= \frac{1}{n\omega_n} |x - y|^{-n} (x_i - y_i). \end{aligned}$$

We calculate for $n = 2$,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{2\pi} \frac{\frac{1}{2} S^{-\frac{1}{2}} \cdot S_i}{S^{\frac{1}{2}}} \\ &= \frac{1}{2\pi} S^{-1} (x_i - y_i) \\ &= \frac{1}{2\pi} |x - y|^{-2} (x_i - y_i). \end{aligned}$$

Continuing, we calculate the second derivatives for $n \geq 2$,

$$\begin{aligned} \Gamma_{ij}(x - y) &= \frac{1}{n\omega_n} \left(\delta_{ij} S^{-\frac{n}{2}} + (x_i - y_i) \left(\frac{-n}{2} S^{-\frac{n}{2}-1} \cdot 2(x_j - y_j) \right) \right) \\ &= \frac{S^{-\frac{n}{2}-1}}{n\omega_n} (\delta_{ij} S^1 - n(x_i - y_i)(x_j - y_j)) \\ &= \frac{|x - y|^{-n-2}}{n\omega_n} (\delta_{ij} |x - y|^2 - n(x_i - y_i)(x_j - y_j)). \end{aligned}$$

Γ is indeed harmonic, since

$$\Delta \Gamma = \sum_i \Gamma_{ii} = \frac{|x - y|^{-n-2}}{n\omega_n} \sum_{i=1}^n (\delta_{ii} |x - y|^2 - n(x_i - y_i)^2) = 0.$$

For near future use, we record estimates on the fundamental solutions.

¹⁰Careful! [2] uses “y” to denote the singularity in Chapter 2, but switches to “x” to denote the singularity in Chapter 4.

Proposition 1.2.2. *Let $\rho = |x - y|$ and for any multi-index β , we estimate*

$$|D^\beta \Gamma(\rho)| \leq C \rho^{2-n-|\beta|},$$

where $C = C(n, |\beta|)$. In fact, we calculate directly

$$\Gamma'(\rho) = \frac{1}{n\omega_n} \rho^{1-n}.$$

In Cartesian coordinates, we make the constant explicit to get

$$|\Gamma_i(x - y)| \leq \frac{1}{n\omega_n} |x - y|^{1-n},$$

$$|\Gamma_{ij}(x - y)| \leq \frac{1}{\omega_n} |x - y|^{-n}.$$

Next, we recall *Green's identities*. First, recall for $g \in C^\infty(\Omega)$ and $X \in \mathfrak{X}(\Omega)$,

$$\operatorname{div}(gX) = \nabla g \cdot X + g \operatorname{div} X.$$

In the case $X = \nabla u$ for $u \in C^2(\Omega)$, we get

$$\operatorname{div}(g\nabla u) = \nabla g \cdot \nabla u + g \Delta u.$$

Integrating and using *divergence theorem* yields *Green's first identity*

$$\int_{\Omega} \nabla g \cdot \nabla u + \int_{\Omega} g \Delta u = \int_{\partial\Omega} (g \nabla u) \cdot \nu = \int_{\partial\Omega} g \frac{\partial u}{\partial \nu}.$$

Switching the roles of g, u and subtracting yields *Green's second identity*

$$\int_{\Omega} g \Delta u - u \Delta g = \int_{\partial\Omega} g \frac{\partial u}{\partial \nu} - u \frac{\partial g}{\partial \nu}.$$



We are in a position to give a proof of Theorem 1.2.1. In general, we call

$$w(x) := \Gamma * f(x) = \int_{\Omega} \Gamma(x - y) f(y) dy$$

the Newtonian potential of f .

Proof. (of 1.2.1.) We would like to use Green's second identity for $g = \Gamma$, but because of the singularity at x , we use it instead for $\Omega - B_x(\rho)$ for small ρ . We compute

$$\int_{\Omega - B_x(\rho)} \Gamma \Delta u = \int_{\partial\Omega} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} + \underbrace{\int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu}}_{(*)}.$$

We show $(*) \rightarrow u(x)$ as $\rho \rightarrow 0$. Keeping in mind that Γ is radial, we proceed with estimating the first term of $(*)$,

$$\begin{aligned} \int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} &= \Gamma(\rho) \cdot \int_{\partial B_x(\rho)} \frac{\partial u}{\partial \nu} \\ &\leq C(n) \rho^{2-n} \cdot n \omega_n \rho^{n-1} \sup_{\partial B_x(\rho)} |Du|. \end{aligned}$$

Since $u \in C^2(\Omega)$, the exponents work out so that the last quantity vanishes as $\rho \rightarrow 0$.¹¹ For the second term, we calculate by 1.2.2,¹²

$$\begin{aligned} - \int_{\partial B_x(\rho)} u \frac{\partial \Gamma}{\partial \nu} &= \Gamma'(\rho) \int_{\partial B_x(\rho)} u \\ &= \frac{1}{n \omega_n \rho^{n-1}} \int_{\partial B_x(\rho)} u. \end{aligned}$$

Since u is continuous, the last quantity goes to $u(x)$ as $\rho \rightarrow 0$. Altogether, sending $\rho \rightarrow 0$ yields Green's representation formula

$$u(x) = \underbrace{\int_{\partial \Omega} u(y) \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu}(y) ds}_{h(x)} + \underbrace{\int_{\Omega} \Gamma(x-y) \Delta u(x) dx}_{w(x)}.$$

Observe for compactly supported u , the second term vanishes, while for harmonic u , the first term vanishes. ■

We give explicit formulas for the first two derivatives of the Newtonian potential.¹³

Proposition 1.2.3. *Suppose f is bounded, integrable in Ω with Newtonian potential w . Then $w \in C^1(\mathbb{R}^n)$ and its derivative given by the formula*

$$w_i(x) = \int_{\Omega} \Gamma_i(x-y) f(y) dy.$$

Proof. Set $v(x) := \int_{\Omega} \Gamma_i(x-y) f(y) dy$. By the asymptotic estimates of 1.2.2 and the fact that f is bounded, we find that v is well-defined. We consider the sequence $w^\varepsilon := (\Gamma \eta^\varepsilon) * f$, where $\eta^\varepsilon = \eta\left(\frac{|x-y|}{\varepsilon}\right)$. Here, $\eta \in C^1(\mathbb{R})$ is a function such that

$$\eta(t) = \begin{cases} 1 & t \geq 2 \\ 0 & t \leq 1, \end{cases} \quad \text{and } 0 \leq \eta' \leq 2.$$

¹¹At least for $n > 2$. But the $n = 2$ case follows similarly.

¹²Note since ν is outward pointing on $\Omega - B$, then ν into B . However, ρ is defined so that it is positive as you go away from y . Therefore, the sign changes.

¹³Lemmas 4.1 and 4.2 of [2].

Therefore, $w^\varepsilon(x)$ has the effect of doing nothing far from the singularity of Γ (at distances $|x - y| \geq 2\varepsilon$), and killing off the function near the singularity of Γ (at distances $|x - y| \leq \varepsilon$). Therefore, it is clear that $w^\varepsilon(x)$ converges pointwise to $w(x)$. Let $K \subset \Omega$ be compact, and for any $x \in K$ we compute for $n > 2$,¹⁴

$$\begin{aligned}
|v(x) - w_i^\varepsilon(x)| &= \left| \int_{\Omega} \frac{\partial}{\partial x^i} \Gamma(x - y) f(y) dy - \frac{\partial}{\partial x^i} \int_{\Omega} \Gamma(x - y) \eta^\varepsilon(x, y) f(y) dy \right| \\
&= \left| \int_{B_e(2\varepsilon)} \left(\frac{\partial \Gamma(x - y)}{\partial x^i} - \frac{\partial \Gamma(x - y) \cdot \eta^\varepsilon}{\partial x^i} \right) f(y) dy \right| \\
&\leq \sup_{B_x(2\varepsilon)} |f| \int_{B_e(2\varepsilon)} |D\Gamma| + \left(|D\Gamma| + \frac{2}{\varepsilon^n} \cdot \varepsilon^{n-1} |\Gamma| \right) dy \\
&\leq \frac{\sup |f|}{n\omega_n} \int_{B_x(2\varepsilon)} 2|x - y|^{1-n} + \frac{2}{\varepsilon(2-n)} |x - y|^{2-n} dy \\
&= \frac{\sup |f|}{n\omega_n} \left(\frac{2n\omega_n(2\varepsilon)^{n-1}}{(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{1-n} (\rho^{n-1} d\rho) + \frac{2n\omega_n(2\varepsilon)^{n-1}}{\varepsilon(2-n)(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{2-n} (\rho^{n-1} d\rho) \right) \\
&= \sup |f| \left(2\varepsilon + \frac{2}{\varepsilon(n-2)} \cdot \frac{(2\varepsilon)^2}{2} \right) \\
&= \sup |f| \frac{2n\varepsilon}{n-2}.
\end{aligned}$$

The third to last equality is from co-area formula, along with change of variables. Since the upper bound is independent of x , the uniformity follows. For $n = 2$, we leave the proof to the reader. \blacksquare

Proposition 1.2.4. *Suppose f is bounded and locally Hölder continuous (with exponent $\alpha \leq 1$) in Ω and let w be the Newtonian potential of f . Then w is $C^2(\Omega)$, $\Delta w = f$ in Ω and for any $x \in \Omega$,*

$$w_{ij}(x) = \int_{\Omega_0} \Gamma_{ij}(x - y) (f(y) - f(x)) dy - f(x) \int_{\partial\Omega_0} \Gamma_i(x - y) \nu_j(y) dz_y.$$

Here Ω_0 is domain containing Ω where the divergence theorem holds, and f is extended to vanish outside Ω .

Proof. Similar to above. Come back to this if there's time. Proof techniques combine those in Theorems 1.2.3 and 1.2.5. \blacksquare

We provide the fundamental estimate for the Schauder estimate for the Laplacian.¹⁵

Lemma 1.2.5. *Let $B_1 := B_R(x_o)$ and $B_2 := B_{2R}(x_o)$ be concentric balls in \mathbb{R}^n . If $f \in C^{0,\alpha}(\bar{B}_2)$ with w the Newtonian potential of f in B_2 , then $w \in C^{2,\alpha}(\bar{B}_1)$. Furthermore,*

$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

¹⁴Check this computation...

¹⁵Lemma 4.4 of [2].

Proof. As a preliminary, we remark that $|B_2| = \omega_n(2R)^n$ and $|\partial B_2| = n\omega_n(2R)^{n-1}$. For $x \in B_1$, we obtain from 1.2.4 and 1.2.2, the first estimate

$$\begin{aligned}
|w_{ij}(x)| &= |f(x)| \int_{\partial B_2} |\Gamma_i(x-y)\nu_j(y)| ds_y + \int_{B_2} |\Gamma_{ij}(x-y)(f(y) - f(x))| dy \\
&\leq \frac{|f(x)|}{n\omega_n} \int_{\partial B_2} \underbrace{|x-y|^{1-n}}_{(*)} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \int_{B_2} |x-y|^{\alpha-n} dy \\
&\leq \frac{|f(x)|}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \underbrace{\int_{B_{3R}(x)} |x-y|^{\alpha-n} dy}_{(**)} \\
&\leq 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha [f]_{x;\alpha}, \\
&\leq C_1(|f(x)| + R^\alpha [f]_{\alpha;x}),
\end{aligned}$$

where $C_1 = C_1(n, \alpha)$. Here, $(*)$ follows since $R \leq |x-y|$, and so $|x-y|^{1-n} \leq R^{1-n}$ since $n \geq 2$. We calculate $(**)$ as

$$\begin{aligned}
(**) &= \frac{n\omega_n(3R)^{n-1}}{(3R)^{n-1}} \int_0^{3R} \rho^{\alpha-n} (\rho^{n-1} d\rho) \\
&= (n\omega_n) \frac{\rho^\alpha}{\alpha} \Big|_0^{3R} \\
&= \frac{n\omega_n}{\alpha} (3R)^\alpha.
\end{aligned}$$

Notice the idea here to shift the center of the ball from x_o to x ; in doing so, we must enlarge the radius to $3R$ so that we capture B_2 . The first estimate gives

$$|D^2 w|_{0;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

For our second estimate, for $x, \bar{x} \in B_1$, $\xi = \frac{1}{2}(x + \bar{x})$ and $\delta = |x - \bar{x}|$,

$$\begin{aligned}
w_{ij}(\bar{x}) - w_{ij}(x) &= -f(\bar{x}) \int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y + f(x) \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \\
&\quad + \int_{B_2} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy - \int_{B_2} \Gamma_{ij}(x - y)(f(y) - f(x)) dy \\
&= f(x) \underbrace{\int_{\partial B_2} \Gamma_i(x - y) - \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_1} + (f(x) - f(\bar{x})) \underbrace{\int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_2} \\
&\quad + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(x - y)(f(x) - f(y)) dy}_{I_3} + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy}_{I_4} \\
&\quad + (f(x) - f(\bar{x})) \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) dy}_{I_5} \\
&\quad + \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) - \Gamma_{ij}(\bar{x} - y)(f(\bar{x}) - f(y)) dy}_{I_6},
\end{aligned}$$

where the second equality follows from (detailed) inspection. We estimate I_1, \dots, I_6 .¹⁶

1. We use mean value inequality for some \hat{x} between x, \bar{x} , then follow with the same trick as the first term in the first estimate to compute

$$\begin{aligned}
|I_1| &\leq |x - \bar{x}| \int_{\partial B_2} |D\Gamma_i(\hat{x} - y)| ds_y \\
&\leq |x - \bar{x}| \int_{\partial B_2} |\hat{x} - y|^{-n} ds_y \\
&\leq \frac{|x - \bar{x}|}{\omega_n} \int_{\partial B_2} R^{-n} ds_y \\
&= \frac{|x - \bar{x}| n 2^{n-1}}{R} \\
&\leq n 2^{n-\alpha} \left(\frac{\delta}{R} \right)^\alpha,
\end{aligned}$$

where the last estimate follows by simple computation from $\delta = |x - \bar{x}| < 2R$.¹⁷

2. The estimate follows the ideas when estimating the first term of $w_{ij}(x)$. We record our estimates for ease of replication in I_5 . Note that the final answer is completely

¹⁶My labelling of the six integrals follows that in [2].

¹⁷There is an extra factor of n in [2], which I don't see where it comes from. Similarly, this extra factor appears when estimating $|I_6|$.

independent of the ball.

$$\begin{aligned} |I_2| &\leq \frac{1}{n\omega_n} \int_{\partial B_2} |x - y|^{1-n} ds_y \\ &\leq \frac{1}{n\omega_n} \int_{\partial B_2} R^{1-n} ds_y = 2^{n-1}. \end{aligned}$$

3. $|I_3| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha; \bar{x}}$. The estimate follows the ideas when estimating the second term of $w_{ij}(x)$ with $B_\delta(x)$ in the place of B_2 and $B_{\frac{3\delta}{2}}$ in the place of $B_{3R}(x)$.
4. $|I_4| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha; \bar{x}}$. Ditto, but swap x for \bar{x} .
5. Since the estimate in I_2 is independent of the ball, the desired estimate follows immediately.

$$\begin{aligned} |I_5| &\leq \left| \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \right| + \left| \int_{\partial B_\delta(\xi)} \Gamma_i(x - y) \nu_j(y) ds_y \right| \\ &\leq 2(2^{n-1}) = 2^n \end{aligned}$$

6. Using ideas as before, we calculate for $C = C(n)$ and \hat{x} some point between \bar{x} and x ,

$$\begin{aligned} |I_6| &\leq |x - \bar{x}| \int_{B_2 - B_\delta(\xi)} |D\Gamma_{ij}(\hat{x} - y)| |f(\bar{x}) - f(y)| dy \\ &\leq C\delta \int_{B_2 - B_\delta(\xi)} \frac{|f(\bar{x}) - f(y)|}{|\hat{x} - y|^{n+1}} dy \\ &\leq C\delta [f]_{\alpha, x} \left(\frac{3}{2}\right)^\alpha \int_{B_\delta(\xi)} |\xi - y|^{\alpha-n-1} dy \\ &\leq \frac{C}{1-\alpha} \delta^\alpha [f]_{\alpha, x} \left(\frac{3}{2}\right)^\alpha \end{aligned}$$

Altogether, the second estimate gives

$$R^\alpha [D^2 w]_{\alpha; B_1} \leq C(|f|_{0; B_2} + R^\alpha [f]_{\alpha; B_2}).$$

■

Lemma 1.2.5 proves a Schauder estimate for concentric balls. We introduce our final norm. Let $\eta = \text{diameter of } \Omega$, then

$$|u|'_{k, \alpha; \Omega} := \sum_{j=0}^k \sup_{x \in \Omega} \eta^j |D^j u(x)| + \sup_{x \neq y \in \Omega} \eta^{k+\alpha} \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\alpha}.$$

Again, observe in the $k = 0$ case, this norm agree with the standard one. We now state the fundamental estimate to show Schauder estimates for the Laplacian.¹⁸

¹⁸Theorem 4.6 of [2].

Lemma 1.2.6. *Let $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$ satisfy $\Delta u = f$. Then $u \in C^{2,\alpha}(\Omega)$ and for any two concentric balls $B_1 := B_R(x_o)$, $B_2 := B_{2R}(x_o)$ compactly contained in Ω ,*

$$|u|'_{2,\alpha;B_1} \leq C(|u|_{0;B_2} + |f|'_{0,\alpha;B_2}).$$

Proof. By Theorem 1.2.1, we may write

$$u = v + w,$$

for harmonic h and w is the Newtonian potential of f . The estimates 1.2.3 and 1.2.5 together give the result. \blacksquare

Finally, we have arrived at the promised land.¹⁹

Theorem 1.2.7. *For $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$, with $\Delta u = f$,*

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

Proof. We first obtain an estimate on the $|u|_{2,\Omega}^*$ by the RHS. Since $|u|_{0;\Omega}$ appears on the RHS, we only need to worry about bounding $[u]_{j;\Omega}^*$ for $j = 1, 2$. We compute for $3R = d_x$ and $B_1 := B_R(x)$ and $B_2 := B_{2R}(x)$,

$$\begin{aligned} d_x |Du(x)| + d_x^2 |D^2 u(x)| &\leq (3R) |Du|_{0;B_1} + (3R)^2 |D^2 u|_{0;B_1} \\ &\leq C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}) \\ &\leq C(|u|_{0;\Omega} + R^2 |f|_{0,\alpha;B_2}) \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}), \end{aligned}$$

where the second inequality follows from Lemma 1.2.6. Therefore, we have

$$|u|_{2;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}). \quad (*)$$

Next, we obtain an estimate on $[u]_{2,\alpha;\Omega}^*$ by the RHS. We compute for $x \neq y \in \Omega$, with $d_x = d_{x,y}$.

$$\begin{aligned} d_x^{2+\alpha} \frac{|D^2 u(x) - D^2 u(y)|}{|x - y|^\alpha} &\leq (3R)^{2+\alpha} [D^2 u]_{0,\alpha;B_1} + 3^{\alpha+1} (3R)^2 \sup_{x \in \Omega} |D^2 u(x)| \\ &\leq C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}) + 6[u]_{2;\Omega}^* \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}) \end{aligned}$$

where the first inequality follows from a near-and-far estimate on B_1 , as in the proof of 1.1.1. The second inequality follows from Lemma 1.2.6. The last inequality follows from (*). \blacksquare

¹⁹Theorem 4.8 of [2].

References

- [1] L. Evans. *Partial differential equations*. 2nd ed. Providence, R.I.: American Mathematical Society, 2010.
- [2] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer Berlin Heidelberg, 2001.