

# Notes on Schauder Estimates

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The following are the *standard assumptions*. On bounded domains  $\Omega \subset \mathbb{R}^n$ , will consider 2nd order (uniformly) elliptic operators


$$Lu := \sum_{i,j} a^{ij} u_{ij} + \sum_i b^i u_i + cu = f,$$

where  $a^{ij}, b^i, c$  are functions with regularity to be specified later. We assume symmetry on the leading order coefficient terms  $a^{ij} = a^{ji}$ , and for all uniform ellipticity says  $\xi \in \mathbb{R}^n$ , for some positive constant  $\lambda > 0$ ,

$$\sum_{i,j} a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2.$$

The regularity of this approach will require our functions  $u, f$  to lie in appropriate Hölder spaces. In this note, we will introduce a zoo of seminorms and norms on the way to proving Schauder estimates, each of which will slightly differently exploit the fact that  $\Omega$  is bounded.

In these notes, we will explore *a priori bounds*. It's worth paying attention to how much regularity we assume on  $u$  in the beginning. One way to produce such a solution is to first find a *fundamental solution*  $\Gamma$  for  $L$ , then a solution will be given by  $\Gamma * f$ . We will take this route for  $L = \Delta$  equation.

In addition, notice the regularity on may be unexpected at first, namely we require  $f \in C^{0,\alpha}$ . Counterexamples exists for  $f$  merely continuous even in the simplest case of  $L = \Delta$ , as we will discuss in Example 1.2.1. 

To start, we introduce the standard Hölder spaces and their respective norms and seminorms. This approach will be pointwise in nature, compared to the integral-based approach of Sobolev spaces last semester. Let  $\alpha \in (0, 1)$ . With respect to  $\Omega$ , we have the pointwise (and global) Hölder seminorms which detect pointwise (and uniform)  $\alpha$ -Hölder continuity. We may also define local  $\alpha$ -Hölder continuity by asking for every compact  $K \subset \Omega$ , that  $[u]_{\alpha;K}$  is finite. In such a case, we say  $u$  is locally uniformly continuous.

1. **(Pointwise)** For  $x_o \in \Omega$ ,

$$[u]_{0,\alpha;x_o} := \sup_{x \in \Omega} \frac{|u(x) - u(x_o)|}{|x - x_o|^\alpha}.$$

2. **(Local)** For all compact  $K \subset \Omega$ ,

$$\sup_{x \neq y \in K} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$$

3. **(Global)**

$$[u]_{0,\alpha;\Omega} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

As usual, global implies local implies pointwise. The above gives rise to  $C^{k,\alpha}(\bar{\Omega})$  (and  $C^{k,\alpha}(\Omega)$ ) consisting of functions whose  $k$ -th order derivatives are uniformly (and locally uniformly) continuous. In the global case, we get a seminorm as defined above. We can formally extend this to  $\alpha = 0, 1$  which gives continuity and Lipschitz continuity respectively.

Hölder regularity gives a quantitative measurement of continuity. In fact, greater the  $\alpha$ , the more regularity we have.<sup>1</sup> One can see this in the following illustrative example:

**Example 0.0.1.** *The function  $f(x) = |x|^\beta$  is  $\alpha$ -Hölder continuous at  $x = 0$  iff  $\alpha \leq \beta$ .*

*Proof.* Let  $\Omega$  be a domain containing 0. We compute

$$\sup_{x \neq 0 \in \Omega} \frac{|f(x) - f(0)|}{|x - 0|^\alpha} = \sup_{x \neq 0 \in \Omega} |x|^{\beta-\alpha} = \begin{cases} \text{finite} & \alpha \leq \beta \\ \infty & \alpha > \beta. \end{cases}$$

■

Recall the  $C^k(\Omega)$  seminorm is defined by measuring how large the  $k$ th derivative of  $u$  on  $\Omega$  is,

$$[u]_{k,0;\Omega} := |D^k u|_{0;\Omega} := \sup_{x \in \Omega, |\beta|=k} |D^\beta u|,$$

where  $\beta$  is a multi-index of length  $k$ . Similarly, we have the  $C^k(\Omega)$  norm by adding the largest of all the derivatives to order  $k$ ,

$$|u|_{k,0;\Omega} := \sum_{j=1}^k |D^j u|_{0,\Omega}.$$

These definitions admit easy extensions to include  $\alpha$ -Hölder regularity, namely we have the *standard (semi)norm* of  $C^{k,\alpha}(\bar{\Omega})$ .

$$1. [u]_{k,\alpha;\Omega} := [D^k u]_{0,\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u].$$

$$2. |u|_{k,\alpha;\Omega} := |u|_{C^k(\bar{\Omega})} + [u]_{k,\alpha;\Omega}.$$

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<sup>1</sup>We briefly mention that for  $k = 0$ , we have a Rellich-type result for Hölder spaces. Namely for  $0 < \alpha < \beta \leq 1$ , we have the compact inclusion  $C^{0,\beta}(\Omega) \subset C^{0,\alpha}(\Omega)$ .

One may check that  $C^{k,\alpha}(\bar{\Omega})$  for  $0 \leq \alpha < 1$  are Banach spaces with the standard norms. One should think of seminorms as taking the “highest degree” portion of the norm.

A short word on notation on our (semi)norms. From here on out, we drop the  $\alpha$  from the notation when  $\alpha = 0$ , but never drop how many derivatives we take.

# 1 The Classical Proof

We unabashedly follow Chapters 2, 4, 6 of [2].

## 1.1 Reduction by Perturbation

Roughly speaking, the Schauder interior estimates tell us that we can bound the  $C^{2,\alpha}$  norm of  $u$  in terms of the  $C^0$  norm of  $u$  and the  $C^{0,\alpha}$  norm of  $f$ . Explicitly, we have the following theorem.

**Theorem 1.1.1. (*Interior Estimates*)**<sup>2</sup> *Given the standard assumptions, if  $u \in C^{2,\alpha}(\Omega)$  is a bounded solution to Poisson’s equation with  $f \in C^{0,\alpha}(\Omega)$ , and*

$$|a^{ij}|_{0,\alpha;\Omega}^{(0)}, |b^i|_{0,\alpha;\Omega}^{(1)}, |c|_{0,\alpha;\Omega}^{(2)} \leq \Lambda,$$

*then the following estimate holds*

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

We now introduce the relevant norms so the above statement is legible. The following Hölder (semi)norms are weighted by their distances to the boundary. Namely for  $x \neq y \in \Omega$ , consider

$$d_x := d(x, \partial\Omega) \quad d_{x,y} := \min\{d_x, d_y\}.$$

We use the above two quantities to define the following (semi)norms. For  $\alpha \in [0, 1]$  and  $u \in C^{k,\alpha}(\Omega)$ , we first define the  $0$ -homogeneous (semi)norm of  $u \in C^{k,\alpha}(\Omega)$ .

1. **0-homogeneous seminorm.** For  $\alpha = 0$ ,

$$[u]_{k;\Omega}^* := \sup_{x \in \Omega, |\beta|=k} d_x^k |D^\beta u(x)|.$$

For  $\alpha > 0$ ,

$$[u]_{k,\alpha;\Omega}^* := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

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<sup>2</sup>Compare Theorem 1.1.1 to the regularity statement of ([1], pg 329, “Interior  $H^2$ -regularity”).

2. **0-homogeneous norm.** For  $\alpha = 0$ ,

$$|u|_{k;\Omega}^* := \sum_{j=0}^k [u]_{j,\Omega}^*.$$

For  $\alpha > 0$ ,

$$|u|_{k,\alpha;\Omega}^* := |u|_{k,\Omega}^* + [u]_{k,\alpha;\Omega}^*.$$

Next, we introduce the  $\sigma$ -homogeneous (semi)norm of  $u \in C^{k,\alpha}(\Omega)$  for  $\sigma \in \mathbb{R}$ , generalizing the above. The 0-homogeneous (semi)norms can obviously be recovered by setting  $\sigma = 0$ .

1.  **$\sigma$ -homogeneous seminorm.** For  $\alpha = 0$ ,

$$[u]_{k;\Omega}^{(\sigma)} := \sup_{x \in \Omega, |\beta|=k} d_x^{k+\sigma} |D^\beta u(x)|.$$

For  $\alpha > 0$ ,

$$[u]_{k,\alpha;\Omega}^{(\sigma)} := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

2.  **$\sigma$ -homogeneous norm.** For  $\alpha = 0$ ,

$$|u|_{k;\Omega}^{(\sigma)} := \sum_{j=0}^k [u]_{j,\Omega}^{(\sigma)}.$$

For  $\alpha > 0$ ,

$$|u|_{k,\alpha;\Omega}^{(\sigma)} := |u|_{k,\Omega}^{(\sigma)} + [u]_{k,\alpha;\Omega}^{(\sigma)}.$$

It's worth pointing out that in the  $k = 0$  case, the  $\sigma$ -homogeneous (semi)norms agree with the standard ones. We show that the  $\sigma$ -inhomogeneous norms enjoy a submultiplicative property.<sup>3</sup>

**Lemma 1.1.2.** *For  $\sigma + \tau \geq 0$ , we have*

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} \leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}.$$

*Proof.* This is a Baby Rudin proof. Recall

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} := \underbrace{\sup_{x \neq y \in \Omega} \left( d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x - y|^\alpha} \right)}_{(*)} + \underbrace{\sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|)}_{(**)}.$$

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<sup>3</sup>(6.11) of [2].

We estimate the first term by

$$\begin{aligned}
(*) &= \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x-y|^\alpha} \right) \\
&\leq \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\sigma+\alpha+\tau} \cdot d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)|}{|x-y|^\alpha} \right) \\
&\leq \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)|}{|x-y|^\alpha} \right) + \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\tau+\alpha+\sigma} \frac{|g(y)| \cdot |f(x) - f(y)|}{|x-y|^\alpha} \right) \\
&\leq |f|_{0;\Omega}^{(\sigma)} \cdot [g]_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot [f]_{0,\alpha;\Omega}^{(\sigma)}.
\end{aligned}$$

We estimate the second term by

$$\begin{aligned}
(**) &= \sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|) \\
&\leq \sup_{x \in \Omega} (d_x^\sigma |f(x)|) \cdot \sup_{x \in \Omega} (d_x^\tau |g(x)|) \\
&= |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)}
\end{aligned}$$

Combining the two estimates,

$$\begin{aligned}
(*) + (**) &= |f|_{0;\Omega}^{(\sigma)} \cdot [g]_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot [f]_{0,\alpha;\Omega}^{(\sigma)} + |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)} \\
&\leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}
\end{aligned}$$

■

Throughout the course of the proof, we will frequently employ the interpolation inequalities.<sup>4</sup> Unfortunately, we must omit the proof.

**Lemma 1.1.3.** *Let  $u \in C^{2,\alpha}(\Omega)$ , then for any  $\varepsilon > 0$ , there exists a constant  $C = C(\varepsilon)$  such that*

$$[u]_{j,\beta;\Omega}^* \leq |u|_{j,\beta;\Omega}^* \leq C|u|_{0;\Omega} + \varepsilon[u]_{2,\alpha;\Omega}^*,$$

where  $j = 0, 1, 2$  and  $0 \leq \alpha, \beta \leq 1$  and  $j + \beta < 2 + \alpha$ .

We first reduce showing Schauder estimates from the general case to the case of constant coefficients with no lower order terms.<sup>5</sup>

*Proof.* (of 1.1.1.) By using a proper, compact exhaustion, it suffices to show the equality holds for every compact subset of  $\Omega$ . By the interpolation inequalities 1.1.3, it suffices to show the result where the LHS is  $[u]_{2,\alpha;\Omega}^*$ . Thus, the relevant quantity to consider is

$$d_{x_o,y_o}^{2+\alpha} \frac{|D^2u(x_o) - D^2u(y_o)|}{|x_o - y_o|^\alpha},$$

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<sup>4</sup>(6.8) and (6.9) of [2].

<sup>5</sup>Theorem 6.2 of [2].

for arbitrary  $x_o, y_o \in \Omega$ . The idea here is to “freeze” one variable to reduce to the case of constant coefficients with no lower order terms. Suppose  $d_{x_o} = d_{x_o, y_o}$ , and consider

$$\sum_{i,j} a^{ij}(x_o) u_{ij}(x) = F(x),$$

where  $F := \sum_{i,j} a^{ij}(x_o) u_{ij} - Lu + f$ . Assuming local Schauder estimates on the ball

$$|u|_{2,\alpha;B}^* \leq C(|u|_{0;B} + |F|_{0,\alpha;B}^{(2)}),$$

where  $B := B_{x_o}(\frac{d}{2})$  where  $d := \mu d_{x_o}$  for some  $\mu \leq \frac{1}{2}$  we will choose later.

We now perform a near-and-far estimate on the frozen case, and we estimate the “junk” term  $F$  in the sequel. For the near estimate, if  $y_o \in B$ , we have the estimate

$$\left(\frac{d}{2}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq C(|u|_{0;B} + |F|_{0,\alpha;B}^{(2)}).$$

Since  $(\frac{d}{2})^{2+\alpha} = \left(\frac{d_{x_o}\mu}{2}\right)^{2+\alpha}$ , we enlarge our domain to obtain the estimate

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right).$$

For the far estimate, if  $y_o \notin B \iff 2|x_o - y_o| \geq d$ , and so  $(2|x_o - y_o|)^\alpha \geq d^\alpha$ . Thus,

$$\begin{aligned} d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} &= \left(\frac{d}{\mu}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \\ &\leq \frac{d^2 \cdot 2^\alpha}{\mu^{2+\alpha}} |D^2 u(x_o) - D^2 u(y_o)| \\ &= \frac{(d_{x_o}\mu)^2 \cdot 2^\alpha}{\mu^2 \cdot \mu^\alpha} |D^2 u(x_o) - D^2 u(y_o)| \\ &\leq \left(\frac{2}{\mu}\right)^\alpha (d_{x_o}^2 |D^2 u(x_o)| + d_{y_o}^2 |D^2 u(y_o)|) \\ &\leq \frac{2^{\alpha+1}}{\mu^\alpha} [u]_{2;\Omega}^* \\ &\leq \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*. \end{aligned}$$

Combining the two estimates, we have

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right) + \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*$$

The next step is to estimate the “junk” term  $|F|_{0,\alpha;B}^{(2)}$  in terms of  $|u|_{0;\Omega}$  and  $[u]_{2,\alpha;\Omega}^*$ . This is a mildly annoying process, but straightforward conceptually. Specifically, we will estimate the terms on the RHS of

$$|F|_{0,\alpha;B}^{(2)} \leq \sum_{i,j} |(a^{ij}(x_o) - a^{ij}(x)) u_{ij}|_{0,\alpha;B}^{(2)} + \sum_i |b^i u_i|_{0,\alpha;B}^{(2)} + |cu|_{0,\alpha;B}^{(2)} + |f|_{0,\alpha;B}^{(2)}.$$

We first provide a generic estimate local-to-global estimate for arbitrary  $g \in C^{0,\alpha}(\Omega)$ . For  $x \in B$ , define  $d_x^{\partial B} := \text{dist}(x, \partial B)$  and  $d_{x,y}^{\partial B} := \min\{d_x^{\partial B}, d_y^{\partial B}\}$ . Note that upon localizing to  $B$ , we need to consider the distance to  $\partial B$  not to  $\partial\Omega$ .<sup>6</sup>

$$\begin{aligned}
|g|_{0,\alpha;B}^{(2)} &\leq \sup_{x \in B} (d_x^{\partial B})^2 \cdot \sup_{x \in B} |g(x)| + \sup_{x \neq y \in B} (d_{x,y}^{\partial B})^{2+\alpha} \cdot \sup_{x \neq y \in B} \frac{|g(x) - g(y)|}{|x - y|^\alpha} \\
&= d^2 |g|_{0;B} + d^{2+\alpha} [g]_{\alpha;B} \\
&\leq |g|_{0;\Omega}^{(2)} + [g]_{\alpha;\Omega}^{(2)} \\
&\leq \left( \frac{\mu}{1-\mu} \right)^2 |g|_{0;\Omega}^{(2)} + \left( \frac{\mu}{1-\mu} \right)^{2+\alpha} [g]_{\alpha;\Omega}^{(2)} \\
&\leq 4\mu^2 |g|_{0;\Omega}^{(2)} + 8\mu^{2+\alpha} [g]_{0,\alpha;\Omega}^{(2)}
\end{aligned}$$

The third line follows since for any  $x \in B$  which comes from the computation

$$d = \mu d_{x_o} \leq (1 - \mu) d_{x_o} \leq d_x.$$

Replacing  $d$  with  $d_x$  is nondecreasing on  $B$ , so the result follows. The fifth line follows from  $\mu \leq \frac{1}{2} \implies \left( \frac{1}{1-\mu} \right)^2 \leq 4$  and  $\left( \frac{1}{1-\mu} \right)^{2+\alpha} \leq 8$ .

We now apply estimates in detail to the principal term; the lower order terms are treated similarly. By Lemma 1.1.2, we compute

$$\begin{aligned}
|(a^{ij}(x_o) - a^{ij})u_{ij}|_{0,\alpha;B}^{(2)} &\leq |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)} \cdot |u_{ij}|_{0,\alpha;B}^{(2)} \\
&\leq \underbrace{|a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)}}_{(*)} \cdot (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*).
\end{aligned}$$

We also estimate the local term  $(*)$  by

$$\begin{aligned}
(*) &= \sup_{x \in B} (|a^{ij}(x_o) - a^{ij}(x)|) + \sup_{x \neq y \in B} \left( (d_{x,y}^{\partial B})^\alpha \frac{|a^{ij}(x_o) - a^{ij}(x) - (a^{ij}(x_o) - a^{ij}(y))|}{|x - y|^\alpha} \right) \\
&\leq \sup_{x \neq x_o \in B} \left( \frac{|a^{ij}(x_o) - a^{ij}(x)|}{|x - x_o|^\alpha} \cdot |x - x_o|^\alpha \right) + d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2(2\mu)^\alpha [a^{ij}]_{0,\alpha;\Omega}^* \\
&\leq 4\Lambda\mu^\alpha,
\end{aligned}$$

where the second to last inequality follows since  $d \leq d_x$  for all  $x \in B$ , and  $\mu \leq \frac{1}{2}$ . Altogether, we find

$$\begin{aligned}
\sum_{i,j} |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(2)} &\leq 4n^2 \Lambda \mu^\alpha (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} ([u]_{2;\Omega}^* + \mu^\alpha [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} (C(\mu) |\mu|_{0;\Omega} + 2\mu^\alpha [u]_{2,\alpha;\Omega}^*),
\end{aligned}$$

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<sup>6</sup>Caution, [2] uses  $d_x$  for our  $d_x^{\partial B}$ , which may be a cause of confusion.

where the last step follows from the interpolation inequalities 1.1.3. Similarly, we have the estimates

$$\begin{aligned}\sum_i |b^i u_i|_{0,\alpha;B}^{(2)} &\leq 8n\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]_{2,\alpha;\Omega}^*), \\ |cu|_{0,\alpha;B}^{(2)} &\leq 8\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]_{2,\alpha;\Omega}^*), \\ |f|_{0,\alpha;\Omega}^{(2)} &\leq 8\mu^2|f|_{0,\alpha;\Omega}^{(2)}.\end{aligned}$$

Altogether, we estimate  $F$  by

$$|F|_{0,\alpha;B}^{(2)} \leq C\mu^{2+2\alpha}[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

Combining the two steps and using the interpolation inequalities 1.1.3, we get

$$[u]_{2,\alpha;\Omega}^* \leq C\mu^\alpha[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

Fixing  $\mu$  so that  $C\mu^\alpha \leq \frac{1}{2}$  yields the desired result. ■

We quickly sketch a proof of Schauder estimates for the case of constant coefficients with no lower order terms.<sup>7</sup> Consider

$$L_0 u := \sum_{i,j} A^{ij} u_{ij},$$

where  $A := (A^{ij}) \in \text{GL}_n(\mathbb{R})$  is a constant matrix. In this case, we diagonalize  $A$  by an orthogonal matrix<sup>8</sup> to a diagonal one, turning  $L_0 u(x) = f(x)$  to  $\Delta \tilde{u}(y) = \tilde{f}(y)$ . Since our transformations are all linear, the norms are equivalent, and so it suffices to prove Schauder estimates for the Laplacian.

## 1.2 The Laplacian

We want to study the regularity problem Poisson equation of

$$\Delta u = f,$$

where  $f \in C^{0,\alpha}$  and  $u \in C^2$ . Intuitively we expect that  $u$  should have two more derivatives than  $f$ . However, counterexamples exist as shown in Example 1.2.1, for  $f$  merely bounded, continuous.<sup>9</sup> The exposition follows the following MSE answer [3].

**Example 1.2.1.** Set  $P(x_1, x_2) := x_1 x_2$ ,  $c_k := \frac{1}{k}$ ,  $t_k := 2^k$  and  $\eta$  a smooth cutoff function with respect to  $B_0(1) \subset B_0(2)$ . We claim

$$f(x) := \sum_{k=1}^{\infty} c_k \Delta(\eta P)(t_k x)$$

is a bounded, continuous function, with no  $C^2$  solution to the Poisson equation  $\Delta u = f$ .

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<sup>7</sup>Lemma 6.1 of [2].

<sup>8</sup>This follows by spectral theorem, since  $A$  is a symmetric, real-valued, nonsingular matrix.

<sup>9</sup>Adapted from Exercise 4.9 of [2].



*Proof.* If we can show  $f$  is continuous, since  $f$  is compactly supported, then  $f$  will be bounded. Even though  $f$  is defined as an infinite sum, note that for any  $x \neq 0$ , there exists  $K = K(|x|)$  such that

$$f(x) = c_K \Delta(\eta P)(t_K x),$$

and so  $f$  is continuous away from the origin. This follows since  $\Delta P = 0$  when  $\eta \equiv 1$  since  $P$  is harmonic, and  $\eta P = 0$  for  $|t_k x| \geq 2$ . Since  $\Delta(\eta P)$  is continuous,

$$|f(x)| \leq C c_k,$$

where  $C$  is a global constant. One easily checks  $f(0) = 0$ , so  $f$  is continuous at 0.

Define  $v$  on  $B_0(2)$  as

$$v(x) := \sum_{k=1}^{\infty} \frac{c_k}{t_k^2} (\eta P)(t_k x),$$

which we claim is a solution to the Poisson equation. It suffices to show  $v$  is not  $C^2$  at the origin. Assuming this is shown, suppose for contradiction  $w$  is a  $C^2$  solution around 0 to the Poisson equation. Since  $v - w$  solves, say, weakly, the Dirichlet problem on  $B_0(2)$ , it is  $C^\infty$  by, say, Weyl's lemma. Therefore,  $v = (v - w) + w$  is  $C^2$  around 0.<sup>10</sup>

We first argue  $v$  is continuous at the origin. This follows by the Weierstrass test after recognizing the uniform bound

$$|\eta P| \leq C,$$

and that  $\sum_k \frac{c_k}{t_k^2} < \infty$ . However, we show  $\partial_2 \partial_1 v$  blows up at  $x \rightarrow 0$ . For any  $x \neq 0$ , for similar reasons as above, the sum becomes finite. Therefore, for some  $K < \infty$ ,

$$\begin{aligned} v_{12}(x) &= \sum_{k=1}^K c_k \partial_1 \partial_2 (\eta P)(t_k x) \\ &= \sum_K c_k (\eta_{12} P + \eta_1 P_2 + \eta_2 P_1 + \eta P_{12}) \\ &= \sum_K c_k (\eta_{12} x_1 x_2 + \eta_1 x_1 + \eta_2 x_2) + \sum_K c_k. \end{aligned}$$

As  $x \rightarrow 0$ , then  $K \rightarrow \infty$  and the last term blows up. ■

We first introduce the *fundamental solution* and some basic ideas surrounding it. The upshot is that we obtain some estimates for future use, and the following *Green's representation formula*,<sup>11</sup> to prove the following theorem.

**Theorem 1.2.2.** *Let  $u \in C^2(\Omega)$  and  $f \in C^{0,\alpha}(\Omega)$  such that  $\Delta u = f$ . Then,*

$$u = h + w,$$

*where  $h$  is some harmonic function, and  $w$  is the Newtonian potential of  $f$ .*

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<sup>10</sup>Every argument I've found online uses some asymptotic analysis at the origin. However, since  $v$  is well-defined at the origin, I wonder if I can proceed as such.

<sup>11</sup>(2.16) of [2].

Fix  $x \in \Omega \subset \mathbb{R}^n$  for  $n \geq 2$ .<sup>12</sup> Let  $\omega_n$  be the volume of the  $n$ -ball. Define the fundamental solution on  $\Omega - x$  as the following radial function about  $x$ :

$$\Gamma(x - y) := \begin{cases} \frac{1}{n(2-n)\omega_n} |x - y|^{2-n} & n > 2 \\ \frac{1}{2\pi} \log |x - y| & n = 2. \end{cases}$$

We first obtain some basic estimates on  $\Gamma$ . The following derivatives will be taken with respect to the variable  $x$ . For convenience, define

$$S(x) := \sum_{k=1}^n (x_k - y_k)^2,$$

so that  $|x - y| = S^{\frac{1}{2}}$ , and  $S_i = 2(x_i - y_i)$ . We calculate for  $n > 2$ ,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{n(2-n)\omega_n} \left( \frac{2-n}{2} S^{\frac{2-n}{2}-1} \cdot 2(x_i - y_i) \right) \\ &= \frac{1}{n\omega_n} |x - y|^{-n} (x_i - y_i). \end{aligned}$$

We calculate for  $n = 2$ ,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{2\pi} \frac{\frac{1}{2} S^{-\frac{1}{2}} \cdot S_i}{S^{\frac{1}{2}}} \\ &= \frac{1}{2\pi} S^{-1} (x_i - y_i) \\ &= \frac{1}{2\pi} |x - y|^{-2} (x_i - y_i). \end{aligned}$$

Continuing, we calculate the second derivatives for  $n \geq 2$ ,

$$\begin{aligned} \Gamma_{ij}(x - y) &= \frac{1}{n\omega_n} \left( \delta_{ij} S^{-\frac{n}{2}} + (x_i - y_i) \left( \frac{-n}{2} S^{-\frac{n}{2}-1} \cdot 2(x_j - y_j) \right) \right) \\ &= \frac{S^{-\frac{n}{2}-1}}{n\omega_n} (\delta_{ij} S^1 - n(x_i - y_i)(x_j - y_j)) \\ &= \frac{|x - y|^{-n-2}}{n\omega_n} (\delta_{ij} |x - y|^2 - n(x_i - y_i)(x_j - y_j)). \end{aligned}$$

$\Gamma$  is indeed harmonic, since

$$\Delta \Gamma = \sum_i \Gamma_{ii} = \frac{|x - y|^{-n-2}}{n\omega_n} \sum_{i=1}^n (\delta_{ii} |x - y|^2 - n(x_i - y_i)^2) = 0.$$

For near future use, we record estimates on the fundamental solutions. Keep in mind that the radial function  $|x - y|^k$  is integrable in a neighborhood of  $x \in \mathbb{R}^n$  iff  $k > -n$ .

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<sup>12</sup>Careful! [2] uses “y” to denote the singularity in Chapter 2, but switches to “x” to denote the singularity in Chapter 4.

**Proposition 1.2.3.** *Let  $\rho = |x - y|$  and for any multi-index  $\beta$ , we estimate*

$$|D^\beta \Gamma(\rho)| \leq C \rho^{2-n-|\beta|},$$

where  $C = C(n, |\beta|)$ . In fact, we calculate directly

$$\Gamma'(\rho) = \frac{1}{n\omega_n} \rho^{1-n}.$$

In Cartesian coordinates, we make the constant explicit to get

$$|\Gamma_i(x - y)| \leq \frac{1}{n\omega_n} |x - y|^{1-n},$$

$$|\Gamma_{ij}(x - y)| \leq \frac{1}{\omega_n} |x - y|^{-n}.$$

Next, we recall *Green's identities*. First, recall for  $g \in C^\infty(\Omega)$  and  $X \in \mathfrak{X}(\Omega)$ ,

$$\operatorname{div}(gX) = \nabla g \cdot X + g \operatorname{div} X.$$

In the case  $X = \nabla u$  for  $u \in C^2(\Omega)$ , we get

$$\operatorname{div}(g\nabla u) = \nabla g \cdot \nabla u + g \Delta u.$$

Integrating and using *divergence theorem* yields *Green's first identity*

$$\int_{\Omega} \nabla g \cdot \nabla u + \int_{\Omega} g \Delta u = \int_{\partial\Omega} (g \nabla u) \cdot \nu = \int_{\partial\Omega} g \frac{\partial u}{\partial \nu}.$$

Switching the roles of  $g, u$  and subtracting,

$$\int_{\Omega} g \Delta u - u \Delta g = \int_{\partial\Omega} g \frac{\partial u}{\partial \nu} - u \frac{\partial g}{\partial \nu}$$

known as *Green's second identity*. 

We are in a position to give a proof of Theorem 1.2.2. In general, we call

$$w(x) := \Gamma * f(x) = \int_{\Omega} \Gamma(x - y) f(y) dy$$

the Newtonian potential of  $f$ .

*Proof.* (of 1.2.2.) We would like to use Green's second identity for  $g = \Gamma$ . But due to the singularity at  $x$ , we must use it instead for  $\Omega - B_x(\rho)$ , for small  $\rho$ . We compute

$$\int_{\Omega - B_x(\rho)} \Gamma \Delta u = \int_{\partial\Omega} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} + \underbrace{\int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu}}_{(*)}.$$

We show  $(*) \rightarrow u(x)$  as  $\rho \rightarrow 0$ . Keeping in mind that  $\Gamma$  is radial, we proceed with estimating the first term of  $(*)$ ,

$$\begin{aligned} \int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} &= \Gamma(\rho) \cdot \int_{\partial B_x(\rho)} \frac{\partial u}{\partial \nu} \\ &\leq C(n) \rho^{2-n} \cdot n \omega_n \rho^{n-1} \sup_{\partial B_x(\rho)} |Du|. \end{aligned}$$

Since  $u \in C^2(\Omega)$ , the exponents work out so that the last quantity vanishes as  $\rho \rightarrow 0$ .<sup>13</sup> For the second term, we calculate by 1.2.3,<sup>14</sup>

$$\begin{aligned} - \int_{\partial B_x(\rho)} u \frac{\partial \Gamma}{\partial \nu} &= \Gamma'(\rho) \int_{\partial B_x(\rho)} u \\ &= \frac{1}{n \omega_n \rho^{n-1}} \int_{\partial B_x(\rho)} u. \end{aligned}$$

Since  $u$  is continuous, the last quantity goes to  $u(x)$  as  $\rho \rightarrow 0$ . Altogether, sending  $\rho \rightarrow 0$  yields Green's representation formula<sup>15</sup>

$$u(x) = \underbrace{\int_{\partial \Omega} u(y) \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu}(y) ds}_{h(x)} + \underbrace{\int_{\Omega} \Gamma(x-y) \Delta u(x) dx}_{w(x)}.$$

We argue  $h$  is harmonic. Since  $\partial \Omega$  is compact, we can commute  $\partial_i$  past the integral. By IBP we can put both  $\partial_i$  derivatives on  $\Gamma$ , which is harmonic; by Clairaut's theorem, we can put  $\partial_i$  past any  $\partial_\nu$  terms. IBP leaves no boundary term since  $\partial(\partial \Omega) = \partial^2 \Omega = 0$ . ■

We give explicit formulas for the first two derivatives of the Newtonian potential.<sup>16</sup>

**Proposition 1.2.4.** *Suppose  $f$  is bounded, integrable in  $\Omega$  with Newtonian potential  $w$ . Then  $w \in C^1(\mathbb{R}^n)$  and its derivative given by the formula*

$$w_i(x) = \int_{\Omega} \Gamma_i(x-y) f(y) dy.$$

*Proof.* Set  $v(x) := \int_{\Omega} \Gamma_i(x-y) f(y) dy$ . By the asymptotic estimates of 1.2.3 and the fact that  $f$  is bounded, we find that  $v$  is well-defined. We consider the sequence  $w^\varepsilon := (\Gamma \eta^\varepsilon) * f$ , where  $\eta^\varepsilon = \eta\left(\frac{|x-y|}{\varepsilon}\right)$ . Here,  $\eta \in C^1(\mathbb{R})$  is a function such that

$$\eta(t) = \begin{cases} 1 & t \geq 2 \\ 0 & t \leq 1, \end{cases} \text{ and } 0 \leq \eta' \leq 2.$$

<sup>13</sup>At least for  $n > 2$ . But the  $n = 2$  case follows similarly.

<sup>14</sup>Note since  $\nu$  is outward pointing on  $\Omega - B$ , then  $\nu$  into  $B$ . However,  $\rho$  is defined so that it is positive as you go away from  $x$ . Therefore, the sign changes.

<sup>15</sup>Observe for compactly supported  $u$ , the second term vanishes, while for harmonic  $u$ , the first term vanishes.

<sup>16</sup>Lemmas 4.1 and 4.2 of [2].

Therefore,  $w^\varepsilon(x)$  has the effect of doing nothing far from the singularity of  $\Gamma$  (at distances  $|x - y| \geq 2\varepsilon$ ), and killing off the function near the singularity of  $\Gamma$  (at distances  $|x - y| \leq \varepsilon$ ). Therefore, it is clear that  $w^\varepsilon(x)$  converges pointwise to  $w(x)$ . Let  $K \subset \Omega$  be compact, and for any  $x \in K$  we compute for  $n > 2$ ,<sup>17</sup>

$$\begin{aligned}
|v(x) - w_i^\varepsilon(x)| &= \left| \int_{\Omega} \frac{\partial}{\partial x^i} \Gamma(x - y) f(y) dy - \frac{\partial}{\partial x^i} \int_{\Omega} \Gamma(x - y) \eta^\varepsilon(x, y) f(y) dy \right| \\
&= \left| \int_{B_e(2\varepsilon)} \left( \frac{\partial \Gamma(x - y)}{\partial x^i} - \frac{\partial \Gamma(x - y) \cdot \eta^\varepsilon}{\partial x^i} \right) f(y) dy \right| \\
&\leq \sup_{B_x(2\varepsilon)} |f| \int_{B_e(2\varepsilon)} |D\Gamma| + \left( |D\Gamma| + \frac{2}{\varepsilon^n} \cdot \varepsilon^{n-1} |\Gamma| \right) dy \\
&\leq \frac{\sup |f|}{n\omega_n} \int_{B_x(2\varepsilon)} 2|x - y|^{1-n} + \frac{2}{\varepsilon(2-n)} |x - y|^{2-n} dy \\
&= \frac{\sup |f|}{n\omega_n} \left( \frac{2n\omega_n(2\varepsilon)^{n-1}}{(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{1-n} (\rho^{n-1} d\rho) + \frac{2n\omega_n(2\varepsilon)^{n-1}}{\varepsilon(2-n)(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{2-n} (\rho^{n-1} d\rho) \right) \\
&= \sup |f| \left( 2\varepsilon + \frac{2}{\varepsilon(n-2)} \cdot \frac{(2\varepsilon)^2}{2} \right) \\
&= \sup |f| \frac{2n\varepsilon}{n-2}.
\end{aligned}$$

The third to last equality is from co-area formula, along with change of variables. Since the upper bound is independent of  $x$ , the uniformity follows. For  $n = 2$ , we leave the proof to the reader.  $\blacksquare$

**Proposition 1.2.5.** *Suppose  $f$  is bounded and locally Hölder continuous (with exponent  $\alpha \leq 1$ ) in  $\Omega$  and let  $w$  be the Newtonian potential of  $f$ . Then  $w$  is  $C^2(\Omega)$ ,  $\Delta w = f$  in  $\Omega$  and for any  $x \in \Omega$ ,*

$$w_{ij}(x) = \int_{\Omega_0} \Gamma_{ij}(x - y) (f(y) - f(x)) dy - f(x) \int_{\partial\Omega_0} \Gamma_i(x - y) \nu_j(y) ds_y.$$

Here  $\Omega_0$  is domain containing  $\Omega$  where the divergence theorem holds, and  $f$  is extended to vanish outside  $\Omega$ .

*Proof.* Similar to above. Come back to this if there's time. Proof techniques combine those in Theorems 1.2.4 and 1.2.6.  $\blacksquare$

We provide the fundamental estimate for the Schauder estimate for the Laplacian.<sup>18</sup>

**Lemma 1.2.6.** *Let  $B_1 := B_R(x_o)$  and  $B_2 := B_{2R}(x_o)$  be concentric balls in  $\mathbb{R}^n$ . If  $f \in C^{0,\alpha}(\bar{B}_2)$  with  $w$  the Newtonian potential of  $f$  in  $B_2$ , then  $w \in C^{2,\alpha}(\bar{B}_1)$ . Furthermore,*

$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

<sup>17</sup>Check this computation...

<sup>18</sup>Lemma 4.4 of [2].

*Proof.* As a preliminary, we remark that  $|B_2| = \omega_n(2R)^n$  and  $|\partial B_2| = n\omega_n(2R)^{n-1}$ . For  $x \in B_1$ , we obtain from 1.2.5 and 1.2.3, the first estimate

$$\begin{aligned}
|w_{ij}(x)| &\leq |f(x)| \int_{\partial B_2} |\Gamma_i(x-y)\nu_j(y)| ds_y + \int_{B_2} |\Gamma_{ij}(x-y)(f(y) - f(x))| dy \\
&\leq \frac{|f(x)|}{n\omega_n} \int_{\partial B_2} \underbrace{|x-y|^{1-n}}_{(*)} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \int_{B_2} |x-y|^{\alpha-n} dy \\
&\leq \frac{|f(x)|}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \underbrace{\int_{B_{3R}(x)} |x-y|^{\alpha-n} dy}_{(**)} \\
&\leq 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha [f]_{x;\alpha}, \\
&\leq C(|f(x)| + R^\alpha [f]_{\alpha;x}),
\end{aligned}$$

where  $C = C(n, \alpha)$ . Here,  $(*)$  follows since  $R \leq |x-y|$ , and so  $|x-y|^{1-n} \leq R^{1-n}$  since  $n \geq 2$ . We calculate  $(**)$  as

$$\begin{aligned}
(**) &= \frac{n\omega_n(3R)^{n-1}}{(3R)^{n-1}} \int_0^{3R} \rho^{\alpha-n} (\rho^{n-1} d\rho) \\
&= (n\omega_n) \frac{\rho^\alpha}{\alpha} \Big|_0^{3R} \\
&= \frac{n\omega_n}{\alpha} (3R)^\alpha.
\end{aligned}$$

Notice the idea here to shift the center of the ball from  $x_o$  to  $x$ ; in doing so, we must enlarge the radius to  $3R$  so that we capture  $B_2$ . The first estimate gives

$$|D^2 w|_{0;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

For our second estimate, for  $x, \bar{x} \in B_1$ ,  $\xi = \frac{1}{2}(x + \bar{x})$  and  $\delta = |x - \bar{x}|$ ,

$$\begin{aligned}
w_{ij}(\bar{x}) - w_{ij}(x) &= -f(\bar{x}) \int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y + f(x) \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \\
&\quad + \int_{B_2} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy - \int_{B_2} \Gamma_{ij}(x - y)(f(y) - f(x)) dy \\
&= f(x) \underbrace{\int_{\partial B_2} \Gamma_i(x - y) - \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_1} + (f(x) - f(\bar{x})) \underbrace{\int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_2} \\
&\quad + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(x - y)(f(x) - f(y)) dy}_{I_3} + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy}_{I_4} \\
&\quad + (f(x) - f(\bar{x})) \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) dy}_{I_5} \\
&\quad + \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) - \Gamma_{ij}(\bar{x} - y)(f(\bar{x}) - f(y)) dy}_{I_6},
\end{aligned}$$

where the second equality follows from (detailed) inspection. We estimate  $I_1, \dots, I_6$ .<sup>19</sup>

1. We use mean value inequality for some  $\hat{x}$  between  $x, \bar{x}$ , then follow with the same trick as the first term in the first estimate to compute

$$\begin{aligned}
|I_1| &\leq |x - \bar{x}| \int_{\partial B_2} |D\Gamma_i(\hat{x} - y)| ds_y \\
&\leq |x - \bar{x}| \int_{\partial B_2} |\hat{x} - y|^{-n} ds_y \\
&\leq \frac{|x - \bar{x}|}{\omega_n} \int_{\partial B_2} R^{-n} ds_y \\
&= \frac{|x - \bar{x}| n 2^{n-1}}{R} \\
&\leq n 2^{n-\alpha} \left( \frac{\delta}{R} \right)^\alpha,
\end{aligned}$$

where the last estimate follows by simple computation from  $\delta = |x - \bar{x}| < 2R$ .<sup>20</sup>

2. The estimate follows the ideas when estimating the first term of  $w_{ij}(x)$ . We record our estimates for ease of replication in  $I_5$ . Note that the final answer is completely

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<sup>19</sup>My labelling of the six integrals follows that in [2].

<sup>20</sup>There is an extra factor of  $n$  in [2], which I don't see where it comes from. Similarly, this extra factor appears when estimating  $|I_6|$ .

independent of the ball.

$$\begin{aligned} |I_2| &\leq \frac{1}{n\omega_n} \int_{\partial B_2} |x - y|^{1-n} ds_y \\ &\leq \frac{1}{n\omega_n} \int_{\partial B_2} R^{1-n} ds_y = 2^{n-1}. \end{aligned}$$

3.  $|I_3| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;\bar{x}}$ . The estimate follows the ideas when estimating the second term of  $w_{ij}(x)$  with  $B_\delta(x)$  in the place of  $B_2$  and  $B_{\frac{3\delta}{2}}$  in the place of  $B_{3R}(x)$ .
4.  $|I_4| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;\bar{x}}$ . Ditto, but swap  $x$  for  $\bar{x}$ .
5. Since the estimate in  $I_2$  is independent of the ball, the desired estimate follows immediately.

$$\begin{aligned} |I_5| &\leq \left| \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \right| + \left| \int_{\partial B_\delta(\xi)} \Gamma_i(x - y) \nu_j(y) ds_y \right| \\ &\leq 2(2^{n-1}) = 2^n \end{aligned}$$

6. Using ideas as before, we calculate for  $C = C(n)$  and  $\hat{x}$  some point between  $\bar{x}$  and  $x$ ,

$$\begin{aligned} |I_6| &\leq |x - \bar{x}| \int_{B_2 - B_\delta(\xi)} |D\Gamma_{ij}(\hat{x} - y)| |f(\bar{x}) - f(y)| dy \\ &\leq C\delta \int_{B_2 - B_\delta(\xi)} \frac{|f(\bar{x}) - f(y)|}{|\hat{x} - y|^{n+1}} dy \\ &\leq C\delta [f]_{\alpha,x} \left(\frac{3}{2}\right)^\alpha \int_{B_\delta(\xi)} |\xi - y|^{\alpha-n-1} dy \\ &\leq \frac{C}{1-\alpha} \delta^\alpha [f]_{\alpha,x} \left(\frac{3}{2}\right)^\alpha \end{aligned}$$

Altogether, the second estimate gives

$$R^\alpha [D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

■

Lemma 1.2.6 proves a Schauder estimate for concentric balls. We introduce our final norm. Let  $\eta = \text{diameter of } \Omega$ , then

$$|u|'_{k,\alpha;\Omega} := \sum_{j=0}^k \sup_{x \in \Omega} \eta^j |D^j u(x)| + \sup_{x \neq y \in \Omega} \eta^{k+\alpha} \frac{|D^k u(x) - D^k u(y)|}{|x - y|^\alpha}.$$

Again, observe in the  $k = 0$  case, this norm agree with the standard one. We now state the fundamental estimate to show Schauder estimates for the Laplacian.<sup>21</sup>

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<sup>21</sup>Theorem 4.6 of [2].



**Lemma 1.2.7.** *Let  $u \in C^2(\Omega)$  and  $f \in C^{0,\alpha}(\Omega)$  satisfy  $\Delta u = f$ . Then  $u \in C^{2,\alpha}(\Omega)$  and for any two concentric balls  $B_1 := B_R(x_o)$ ,  $B_2 := B_{2R}(x_o)$  compactly contained in  $\Omega$ ,*

$$|u|'_{2,\alpha;B_1} \leq C(|u|_{0;B_2} + |f|'_{0,\alpha;B_2}).$$

*Proof.* By Theorem 1.2.2, we may write

$$u = v + w,$$

for harmonic  $h$  and  $w$  is the Newtonian potential of  $f$ . The estimates 1.2.4 and 1.2.6 together give the result.  $\blacksquare$

Finally, we have arrived at the promised land.<sup>22</sup>

**Theorem 1.2.8.** *If  $u \in C^2(\Omega)$  and  $f \in C^{0,\alpha}(\Omega)$ , with  $\Delta u = f$ , then  $u \in C^{2,\alpha}(\Omega)$  with*

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

*Proof.* We first obtain an estimate on the  $|u|_{2,\Omega}^*$  by the RHS. Since  $|u|_{0;\Omega}$  appears on the RHS, we only need to worry about bounding  $[u]_{j;\Omega}^*$  for  $j = 1, 2$ . We compute for  $3R = d_x$  and  $B_1 := B_R(x)$  and  $B_2 := B_{2R}(x)$ ,

$$\begin{aligned} d_x |Du(x)| + d_x^2 |D^2 u(x)| &\leq (3R) |Du|_{0;B_1} + (3R)^2 |D^2 u|_{0;B_1} \\ &\leq C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}) \\ &\leq C(|u|_{0;\Omega} + R^2 |f|_{0,\alpha;B_2}) \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}), \end{aligned}$$

where the second inequality follows from Lemma 1.2.7. Therefore, we have

$$|u|_{2;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}). \quad (*)$$

Next, we obtain an estimate on  $[u]_{2,\alpha;\Omega}^*$  by the RHS. We compute for  $x \neq y \in \Omega$ , with  $d_x = d_{x,y}$ .

$$\begin{aligned} d_x^{2+\alpha} \frac{|D^2 u(x) - D^2 u(y)|}{|x - y|^\alpha} &\leq (3R)^{2+\alpha} [D^2 u]_{0,\alpha;B_1} + 3^{\alpha+1} (3R)^2 \sup_{x \in \Omega} |D^2 u(x)| \\ &\leq C(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}) + 6[u]_{2;\Omega}^* \\ &\leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}) \end{aligned}$$

where the first inequality follows from a near-and-far estimate on  $B_1$ , as in the proof of 1.1.1. The second inequality follows from Lemma 1.2.7. The last inequality follows from (\*).  $\blacksquare$

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<sup>22</sup>Theorem 4.8 of [2].

## References

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