Chapter 0 Notes

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Spring 2022

We follow Griffiths and Harris [1]. This will be our main, and unless otherwise stated, main source.

1 Pre-Basics

1.1 SCV Analysis

Definition 1.1 (Cauchy-Riemann Equations).

Definition 1.2 (Holomorphic Functions).

Definition 1.3 (Analytic Functions).

Proposition 1.4 (Cauchy Integral Formula). *Let Δ be a disc in $\mathbb{C}$, $f \in \mathbb{C}^\infty(\Delta)$ and $z \in \Delta$, then*

$$f(z) = \frac{1}{2\pi i}\left(\int_\Delta \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z} + \int_{\partial\Delta} \frac{f(w)}{w - z} dw\right)$$

Proof. Let Δ_ϵ be a small disk contained in Δ centered at z. Consider smooth one form η defined by

$$\eta = \frac{1}{2\pi i} \frac{f(w)dw}{w - z} \in \Omega^1(\Delta - \Delta_\epsilon).$$

We compute its differential. Since $dw \wedge dw = 0$, the differential contributes only $d\bar{w}$ terms, so

$$\begin{aligned} d\eta &= \frac{1}{2\pi i} d\left(\frac{f(w)dw}{w - z}\right) && (1) \\ &= \frac{1}{2\pi i}\left(\frac{\partial f(w) d\bar{w}}{\partial \bar{w}} \frac{1}{w - z} dw + \frac{\partial}{\partial \bar{w}}\left(\frac{1}{w - z}\right) f(w) dw\right) && (2) \\ &= \frac{-1}{2\pi i}\left(\frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}\right) && (3) \end{aligned}$$

Where the last equality follows from holomorphicity of $\frac{1}{w-z}$.[1] We now use Stokes on $\Delta - \Delta_\epsilon$.

$$\int_{\Delta-\Delta_\epsilon} d\eta = \int_{\partial\Delta} \eta - \int_{\partial\Delta_\epsilon} \eta \tag{4}$$

$$\frac{-1}{2\pi i}\int_{\Delta-\Delta_\epsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(w)dw}{w-z} - \frac{1}{2\pi i}\int_{\partial\Delta_\epsilon} \frac{f(w)dw}{w-z} \tag{5}$$

$$\frac{1}{2\pi i}\int_{\Delta-\Delta_\epsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} = \frac{-1}{2\pi i}\int_{\partial\Delta} \frac{f(w)dw}{w-z} + \frac{1}{2\pi i}\int_{\partial\Delta_\epsilon} \frac{f(w)dw}{w-z} \tag{6}$$

Moving around terms, we get

$$\frac{1}{2\pi i}\int_{\partial\Delta_\epsilon} \frac{f(w)dw}{w-z} = \frac{1}{2\pi i}\left(\int_{\Delta-\Delta_\epsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} + \int_{\partial\Delta} \frac{f(w)dw}{w-z}\right)$$

As we take $\epsilon \to 0$, we want to show the LHS approaches $f(z)$ while the first term in the RHS approaches

$$\int_\Delta \frac{\partial f}{\partial \bar{w}}(w) \frac{dw \wedge d\bar{w}}{w-z}.$$

We focus on the LHS. We make the change of variables

$$w - z = re^{i\theta}$$

$$dw = d(w-z) = d(re^{i\theta}) = dre^{i\theta} + rie^{i\theta}d\theta.$$

However, notice that this is a contour integral around $\partial\Delta_\epsilon$ which fixes $r = \epsilon$, so $d(w-z) = rie^{i\theta}d\theta$. Therefore,

$$\frac{1}{2\pi i}\int_{\partial\Delta_\epsilon} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z+\epsilon e^{i\theta})}{\epsilon e^{i\theta}}(\epsilon i e^{i\theta} d\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta.$$

By continuity of f, as $\epsilon \to 0$, then $f(z+\epsilon e^{i\theta}) \to f(z)$. Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} f(z+\epsilon e^{i\theta})d\theta \to \frac{1}{2\pi}\int_0^{2\pi} f(z)d\theta = f(z).$$

We conclude the proof by showing

$$\lim_{\epsilon\to 0}\int_{\Delta-\Delta_\epsilon} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w-z} = \int_\Delta \frac{\partial f}{\partial \bar{w}}(w) \frac{dw \wedge d\bar{w}}{w-z}.$$

For this, let us prove a lemma from measure theory:

Lemma 1.5. *Let $S \subset \mathbb{R}^n$ and a chain of subsets of S, $S_\epsilon \to z \in \mathbb{R}^n$ as $\epsilon \to 0$. If $f \in L^1(S)$, then*

$$\lim_{\epsilon\to 0}\int_{S-S_\epsilon} f = \int_S f.$$

[1] Holomorphic functions compose, and we can realize $\frac{1}{w-z}$ and a translation followed by an inversion. This is holomorphic away from z.

Proof. We first assume $f \geq 0$, and define the monotonically increasing sequence f_n by

$$0 \leq f_n := |f - f \cdot \chi_{S_{1/n}}|$$

and observe $f_n \to f$ almost everywhere (in fact, everywhere except at z). By Monotone Convergence Theorem,

$$\int_S f = \lim_{n\to\infty} |f - f\chi_{S_{1/n}}| = \lim_{n\to\infty} \int_{S-S_\epsilon} |f| = \lim_{n\to\infty} \int_{S-S_\epsilon} f.$$

For a general function f, we do the usual trick of defining $f = f^+ + f^-$ where $f^+ := \max\{0, f\}$ and $f^- := \max\{0, -f\}$. Note that these two are both positive functions, and by the previous argument,

$$\int_S f^\pm = \lim_{n\to\infty} \int_{S-S_\epsilon} f^\pm.$$

Therefore,

$$\begin{aligned}
\int_S f &= \int_S f^+ + \int_S f^- &(7)\\
&= \lim_{n\to\infty} \int_{S-S_\epsilon} f^+ + \lim_{n\to\infty} \int_{S-S_\epsilon} f^- &(8)\\
&= \lim_{n\to\infty} \int_{S-S_\epsilon} f^+ + f^- &(9)\\
&= \lim_{n\to\infty} \int_{S-S_\epsilon} f &(10)
\end{aligned}$$

□

Going back to our problem, we see that it suffices to show our integrand is in $L^1(\Delta)$. To do this, we perform the change of variables $w = x + iy$, so $\bar{w} = x - iy$. Therefore,

$$dw \wedge d\bar{w} = d(x + iy) \wedge d(x - iy) = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy$$

Now performing a polar transform, we get

$$dx \wedge dy = rdr \wedge d\theta \implies dw \wedge d\bar{w} = -2irdr \wedge d\theta.$$

Therefore,

$$\left|\frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z}\right| = 2\left|\frac{\partial f}{\partial \bar{w}} dr \wedge d\theta\right| \leq M|dr \wedge d\theta|$$

where the last inequality follows as $f \in C^\infty(\Delta)$. Thus, our integrand is in $L^1(\Delta)$. □

Proposition 1.6. *For U an open set in $\mathbb{C}$ and $f \in C^\infty(U)$, f is holomorphic if and only if f is analytic.*

Proof. ($\implies$) Suppose f is holomorphic, i.e. $\frac{\partial f}{\partial \overline{z}} = 0$. Let $z \in U$; since U is open we know there exists $\varepsilon > 0$ such that $\Delta = \Delta(z_0, \varepsilon) \subseteq U$. From the Cauchy Integral Formula, we can write for any $z \in \Delta$

$$f(z) = \frac{1}{2\pi i}\left(\int_{\partial\Delta} \frac{f(w)dw}{w-z} + \int_{\Delta} \frac{\partial f(w)}{\partial \overline{w}} \frac{dw \wedge d\overline{w}}{w-z}\right) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f(w)dw}{w-z}$$

where the second term is 0 by hypothesis. To introduce a summation, note that we can rewrite the integrand $\frac{f(w)}{w-z} = \frac{f(w)}{(w-z_0)\left(1-\frac{z-z_0}{w-z_0}\right)}$. Then since $z \in \Delta$ and $w \in \partial\Delta$, we know $\left|\frac{z-z_0}{w-z_0}\right| < 1$ so by the formula for geometric series, $\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$. As a result,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}dw.$$

In order to write this in the form $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, we want to reverse the order of integration and summation (pull the summation outside the integral). It then suffices to show $\sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}$ is uniformly convergent. To that end, note because f is continuous on $\partial\Delta = S^1$ which is compact, $f(w)$ is bounded, i.e. there exists $M_1 > 0$ such that $|f(w)| \leq M_1$ for $w \in \partial\Delta$. We also know $\left|\frac{z-z_0}{w-z_0}\right| < 1$ so there exists some $M_2 > 0$ such that $\left|\frac{z-z_0}{w-z_0}\right| \leq M_2 < 1$. Lastly, $\frac{1}{w-z_0} = \frac{1}{\varepsilon}$. Thus $\left|f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\right| \leq \frac{1}{\varepsilon}M_1M_2^n$, and $\sum_{n=0}^{\infty}\frac{1}{\varepsilon}M_1M_2^n = \frac{M_1}{\varepsilon}\sum_{n=0}^{\infty} M_2^n$ converges (geometric series with $|r| < 1$). By the Weierstrass M-test we have the original series converges uniformly as desired, therefore

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}dw = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where $a_n = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)}{(w-z_0)^{n+1}}dw$. Thus f is analytic on U, which was to be shown.

($\impliedby$) Conversely suppose f is analytic on U, i.e. for any $z_0 \in U$ there exists $\varepsilon > 0$ such that for any $z \in \Delta$ we have $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, $a_n \in \mathbb{C}$, where the convergence of the series is uniform and absolute. Let $S_m(z) = \sum_{n=0}^{m} a_n(z-z_0)^n$ be the m^{th} partial sum of the series representation of f. Since each S_m is a polynomial we know it is holomorphic for each $n \in \mathbb{Z}^{\geq 0}$, so as before we can use the simplified Cauchy integral formula to say $S_m(z) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{S_m(w)dw}{w-z}$. Then,

$$f(z) = \lim_{m\to\infty} S_m(z) = \lim_{m\to\infty}\frac{1}{2\pi i}\int_{\partial\Delta}\frac{S_m(w)dw}{w-z}.$$

Since the convergence of the partial sums is uniform, we can pass the limit into the integral and find

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{f(w)dw}{w-z}.$$

Now we compute

$$\frac{\partial f(z)}{\partial \bar{z}} = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{\partial}{\partial \bar{z}} \left(\frac{f(w)}{w - z} \right) dw.$$

(**Subtle point**) Since $\frac{\partial}{\partial \bar{z}} \left(\frac{1}{w-z} \right) = 0$ for $w \neq z$, the integrand is 0 and hence $\frac{\partial f(z)}{\partial \bar{z}} = 0$. This is the definition of f being holomorphic, which was to be shown. □

Theorem 1.7 ($\bar{\partial}$-Poincaré Lemma in One Variable)**.** *Given $g(z) \in C^\infty(\bar{\Delta})$, the function*

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{g(w)}{w - z} dw \wedge d\bar{w}$$

is defined and C^∞ in Δ and satisfies

$$\frac{\partial f}{\partial \bar{z}} = g.$$

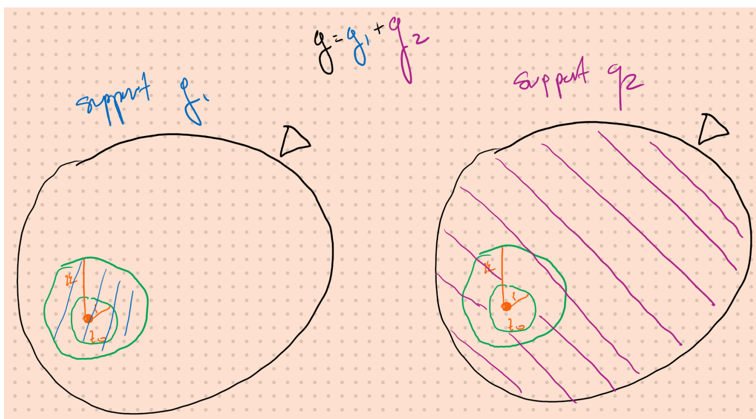

Figure 1: Shaded regions denote where g_i's support lie in.

Proof. Given $z_o \in \Delta$, we choose a small disk centered at z_o which we denote D_ϵ such that $D_{2\epsilon} \subset \Delta$.For $i = 1, 2$ we define g_i to be such that $g_1(z) + g_2(z) = g(z)$ and g_i vanish outside $D_{2\epsilon}$ and inside D_ϵ respectively. Therefore, by linearity of the integral, we have

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} \frac{g_1(w)}{w - z} dw \wedge d\bar{w} + \frac{1}{2\pi i} \int_{\Delta} \frac{g_2(w)}{w - z} dw \wedge d\bar{w}$$

so we define the two terms to be f_i respectively. We first aim to show $\frac{\partial f_2}{\partial \bar{z}} = 0$ and so it suffices to show

$$\frac{\partial f_1}{\partial \bar{z}} = g_1(z).$$

We begin by investigating $\partial_{\bar{z}} f_2$ with the caveat (IMPORTANT!) that we restrict attention $z \in D_\epsilon$. We differentiate under the integral to get

$$\partial_{\bar{z}} f_2(z) = \frac{1}{2\pi i} \int_{\Delta} \partial_{\bar{z}} \left(\frac{g_2(w)}{w - z} \right) dw \wedge d\bar{w}$$

We now apply Leibniz rule to the integrand yielding

$$\partial_{\bar{z}}((w-z)^{-1})g_2(w) + (w-z)^{-1}\partial_{\bar{z}}(g_2(w)).$$

Since $(w-z)^{-1}$ is holomorphic, the first term vanishes. Since $g_2(w) \equiv 0$ on D_ϵ, the second term vanishes. Note that we cannot make this argument for $z \in \Delta - D_\epsilon$.

We now turn our attention to $f_1(z)$. Let me note once again that we're really restricting $z \in D_\epsilon$. For values of $z \in \Delta - D_{2\epsilon}$, we see $g_1(z) \equiv 0$, so we can repeat our previous argument on this outer annulus. Since g_1 is contained in Δ, it has compact support, which gives the first equality below. The rest follows from various change of variables on $\mathbb{C}$.

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Delta \frac{g_1(z)}{w-z} dw \wedge d\bar{w} &= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(z)}{w-z} dw \wedge d\bar{w} && (11)\\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(z+u)}{u} dw \wedge d\bar{w} && (12)\\
&= \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{g_1(z+re^{i\theta})}{re^{i\theta}} (-2ir dr \wedge d\theta) && (13)\\
&= \frac{-1}{\pi}\int_{\mathbb{C}} g_1(z+re^{i\theta})e^{-i\theta} dr \wedge d\theta && (14)
\end{aligned}$$

Now computing $\partial_{\bar{z}} f_1(z)$.

$$\begin{aligned}
\partial_{\bar{z}} f_1 &= \frac{-1}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}} \left(\frac{g_1(z+re^{i\theta})}{e^{i\theta}}\right) dr \wedge d\theta && (15)\\
&= \frac{-1}{\pi}\int_{\mathbb{C}} \partial_{\bar{z}}(g_1)e^{-i\theta} + g_1 \partial_{\bar{z}} e^{-i\theta} dr \wedge d\theta && (16)
\end{aligned}$$

Since $e^{-i\theta}$ is a rotation by angle $-\theta$, it is a linear so it is holomorphic. To compute $\partial_{\bar{z}} g_1$, we use chain rule.

$$\frac{\partial g_1}{\partial \bar{z}}(z+re^{i\theta}) = \frac{\partial g_1}{\partial w}\frac{\partial w}{d\bar{z}} + \frac{\partial g_1}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial \bar{z}} = \frac{\partial g_1}{\partial w}\frac{\partial(z+re^{i\theta})}{d\bar{z}} + \frac{\partial g_1}{\partial \bar{w}}\frac{\partial(\bar{z}+re^{-i\theta})}{\partial \bar{z}} = \frac{\partial g_1}{\partial \bar{w}}$$

Thus, we change back to our original variables to get

$$\partial_{\bar{z}} f_1 = \frac{-1}{\pi}\int_{\mathbb{C}} \frac{\partial g_1(z+re^{i\theta})}{\partial \bar{w}} e^{-i\theta} dr \wedge d\theta = \frac{1}{2\pi i}\int_D \frac{\partial g_1(w)}{\partial \bar{w}}\frac{dw \wedge d\bar{w}}{w-z}$$

Since the support of g_1 is contained in the small disk, it vanishes near the boundary $\partial\Delta$. Thus Cauchy integral formula tells us

$$g_1(z) = \frac{1}{2\pi i}\int_\Delta \frac{\partial g_1(w)}{\partial \bar{w}}\frac{dw \wedge d\bar{w}}{w-z} = \frac{\partial f_1}{\partial \bar{z}}$$

Therefore, we have solved the desired equation on a small disk, namely on D_ϵ. □

Theorem 1.8 ($\bar{\partial}$-Poincaré Lemma in Several Variables)**.**

Theorem 1.9 (Hartog's Theorem)**.** *Any holomorphic function f in a neighborhood of $U-V$ extends to a holomorphic function on U.*

Proof. We extend f in each slice where z_1 =constant. These slices of $U-V$ are all disks or annuli (intuitively, slices of a larger ball minus a smaller one), so we are in a setting where the Cauchy integral formula is applicable. A natural choice for the extension F is then given by

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|w_2|=r} \frac{f(z_1, w_2)}{w_2 - z_2} dw_2.$$

Note then that F is defined throughout all of U ($z_2 \neq w_2$ since we are integrating over $|w_2| = r$, the outer boundary of each disk/annulus, so no division by 0; z_1 is fine in each slice, hence overall). Moreover,

$$\frac{\partial F}{\partial \overline{z}_2} = \frac{1}{2\pi i} \int_{|w_2|=r} \underbrace{\frac{\partial}{\partial \overline{z}_2} \left(\frac{f(z_1, w_z)}{w_2 - z_2} \right)}_{=0,\text{ subtle point}} dw_2 = 0,$$

$$\frac{\partial F}{\partial \overline{z}_1} = \frac{1}{2\pi i} \int_{|w_2|=r} \frac{\frac{\partial f(z_1, w_z)}{\partial \overline{z}_1}}{w_2 - z_2} dw_2 = 0$$

where we used the fact that f is holomorphic, hence holomorphic in z_1, to say $\frac{\partial f}{\partial \overline{z}_1} = 0$. Thus F is holomorphic, as desired. Indeed, for $(z_1, z_2) \in U - V$ the Cauchy integral formula directly states that $F(z_1, z_2) = \frac{1}{2\pi i} \int_{|w_2|=r} \frac{f(z_1, w_2)}{w_2 - z_2} dw_2 = f(z_1, z_2)$, i.e. $F\big|_{U-V} = f$ is a true extension of f. □

Definition 1.10 (Weierstrass polynomial)**.**

Theorem 1.11 (Weierstrass Preparation Theorem)**.** *If f is holomorphic around the origin in $\mathbb{C}^n$ and is not identically zero on the w-axis, then in some neighborhood of the origin f can be written uniquely as*

$$f = g \cdot h,$$

where g is a Weierstrass polynomial of degree d in w and $h(0) \neq 0$.

Theorem 1.12 (Riemann Extension Theorem)**.** *Suppose $f(z, w)$ is holomorphic in a disc $\Delta \subset \mathbb{C}^n$ and $g(z, w)$ is holomorphic in $\bar{\Delta} - \{f = 0\}$ and bounded. Then g extends to a holomorphic function on Δ.*

Proposition 1.13. *$\mathcal{O}_n$ is a UFD.*

Proposition 1.14. *If f and g are relatively prime in $\mathcal{O}_{n,0}$, then for $\|z\| < \varepsilon, f$ and g are relatively prime in $\mathcal{O}_{n,z}$.*

Theorem 1.15 (Weierstrass Division Theorem)**.** *Let $g(z, w) \in \theta_{n-I}[w]$ be a Weierstrass polynomial of degree k in w. Then for any $f \in \mathcal{O}_n$, we can write*

$$f = g \cdot h + r$$

with $r(z,w)$ a polynomial of degree $< k$ in w.

Corollary 1.16 (Weak Nullstellensatz)**.** *If $f(z,w) \in \mathcal{O}_n$ is irreducible and $h \in \mathcal{O}_n$ vanishes on the set $f(z,w) = 0$, then f divides h in $\mathcal{O}_n$.*

1.2 Extras

1. One thing that throws me off is the use of $z, \bar{z}$ as "independent" variables since they are "linked" by complex conjugation. For example, near the end of the proof of $\bar{\partial}-$Poincare Lemma, when computing the following derivative, where $w = z + re^{i\theta}$,

$$\frac{\partial g}{\partial \bar{z}}(z + re^{i\theta}) = \frac{\partial g}{\partial w}\frac{\partial w}{d\bar{z}} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{w}}{\partial \bar{z}} = \frac{\partial g}{\partial w}\frac{\partial(z + re^{i\theta})}{d\bar{z}} + \frac{\partial g}{\partial \bar{w}}\frac{\partial(\bar{z} + re^{-i\theta})}{\partial \bar{z}}$$

we treat $z, \bar{z}$ as "independent variables" so that

$$\frac{\partial(z + re^{i\theta})}{\partial \bar{z}} = 0$$

and

$$\frac{\partial(\bar{z} + re^{-i\theta})}{\partial \bar{z}} = \frac{\partial \bar{z}}{\partial \bar{z}} + \frac{\partial(re^{-i\theta})}{\partial \bar{z}} = 1.$$

For an illustration of my confusion, consider the identity function on $\mathbb{C}$,

$$f(z) = |z|^2.$$

If I write it like this, it looks like a function of only z. However, by the identity

$$|z|^2 = z\bar{z},$$

then actually this has dependency on $\bar{z}$ and thus is not holomorphic. Therefore, the question is when can I tell if my function is holomorphic or not?

2 Basics

2.1 Complex Manifolds Basics

In this section, we want to introduce the complex structure on manifolds. For this, let us recall what a differentiable manifold is.

Definition 2.1. *Let M be a topological space. We say M is an n-dimensional topological manifold if it has the following properties:*

- *M is Hausdorff,*
- *M is second countable,*
- *and M is locally Euclidean of dimension n, i.e., for every point $p \in M$, there exists a neighborhood U of p and a map $\varphi : U \to \tilde{U}$ such that $\tilde{U} \in \mathbb{R}^n$ is an open set and φ is homeomorphism.*

Roughly speaking, a differentiable or a smooth manifold is a topological manifold with a smooth structure, which is actually a maximal smooth atlas $\mathcal{A}$.

Definition 2.2 (Complex Manifold)**.** *A complex manifold M is a differentiable manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$. is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{C}^n$ for all α, β.*

Here we'll discuss some specific cases of complex manifolds and their properties:

1. A one-dimensional complex manifold is called a *Riemann surface*. Here are a few examples:

   - The complex plane $\mathbb{C}$ is the most basic Riemann surface and the identity map $f(z) = z$ defines a chart for it. Also, if we consider the conjugate map $g(z) = \bar{z}$, then we again have a Riemann surface structure on $\mathbb{C}$ which is not compatible with the one obtained by f and hence, we have two different Riemann surface structures on $\mathbb{C}$.
   - Every non-empty open set $U \subset \mathbb{C}$ can be viewed as a Riemann surface.
   - Let $S = \mathbb{C} \cup \{\infty\}$ and define $f(z) = z$ for $z \in S \setminus \{\infty\}$ and $g(z) = \frac{1}{z}$ for $z \in S \setminus \{0\}$. Then, f and g are charts for S and $\{f, g\}$ defines an atlas on S, making it to be a Riemann surface which is actually called a *Riemann sphere*. Note that this space is basically $\mathbb{CP}^1$ that we'll study in the next example.

2. The complex projective space $\mathbb{CP}^n$ of the lines passing through the origin in $\mathbb{C}^{n+1}$ has the structure of a complex manifold. On the subset $U_i = \{[Z] : Z_i \neq 0\} \subset \mathbb{P}^n$ of lines not contained in the hyperplane $(Z_i = 0)$, there is a bijective map φ_i to $\mathbb{C}^n$ given by

$$\varphi_i([Z_0, \ldots, Z_n]) = \left(\frac{Z_0}{Z_i}, \ldots, \frac{\hat{Z}_i}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right)$$

   On $(z_j \neq 0) = \varphi_i(U_j \cap U_i) \subset \mathbb{C}^n$,

$$\varphi_j \circ \varphi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{\hat{z}_j}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_n}{z_j}\right)$$

   is clearly holomorphic.

   - The coordinates $Z = [Z_0, \ldots, Z_n]$ are called *homogeneous coordinates* on $\mathbb{CP}^n$.
   - The coordinates given by the maps φ_i are called *Euclidean coordinates.*
   - Consider the map $\pi : \mathbb{S}^{2n+1} = \left\{(z_0, \ldots, z_n) \,|\, |z_0|^2 + \cdots + |z_n|^2 = 1\right\} \to \mathbb{CP}^n$, given by
$$(z_0, z_1, \ldots, z_n) \sim [z_0, z_1, \ldots, z_n].$$
     The map π is surjective, $\pi(\mathbb{S}^{2n+1}) = \mathbb{CP}^n$, and $\mathbb{S}^{2n+1}$ is a compact space, so $\mathbb{CP}^n$ is compact

3. Let $\Lambda = \mathbb{Z}^k \subset \mathbb{C}^n$. Then the quotient group $\mathbb{C}^n/\Lambda$ has the structure of a complex manifold induced by the projection map $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$.

- This complex manifold is compact if and only if $k = 2n$; in this case $\mathbb{C}^n/\Lambda$ is called a *complex torus*.
- If $\pi : M \to N$ is a covering space map for the complex manifold N, then M admits a complex manifold structure obtained from π.
- If M is a complex manifold and the deck transformations of M are holomorphic, then N inherits a complex manifold structure from M.
- As a specific example of this type of complex manifolds, let Λ be the group of automorphisms generated by $z \mapsto 2z$. Then, the qoutient space $\mathbb{C}^2 \setminus \{0\}/\Lambda$ has a complex manifold structure and it's called a *Hopf surface*. Note that a Hopf surface is the simplest example of a complex manifold that cannot be imbedded in projective spaces of any dimension. Hopf surfaces are from Kodaira dimension $-\infty$!

HW: cp1 is biholomorphic to riemann sphere

Theorem 2.3 (Inverse Function Theorem)**.** *Let U, V be open sets in $\mathbb{C}^n$ with $0 \in U$ and $f : U \to V$ a holomorphic map with $\mathscr{J}(f) = (\partial f_i/\partial z_j)$ nonsingular at 0. Then f is one-to-one in a neighborhood of 0 , and f^{-1} is holomorphic at $f(0)$.*

Proof. We know that $\det|\mathscr{J}_{\mathbb{R}}(f)| = |\det(\mathscr{J}(f))|^2 \geq 0$, and by assumption, $\mathscr{J}(f)$ is nonsingular at 0. So $\det|\mathscr{J}_{\mathbb{R}}(f)| \neq 0$ at 0, and using the ordinary inverse function theorem, f has a continuously differentiable inverse f^{-1} near 0 and we have $f^{-1}(f(z)) = z$. The only thing that remains to check is the holomorphicity of f^{-1}. For this, it suffices to show that $\frac{\partial}{\partial \bar{z}_i} f^{-1} = 0$. We have:

$$\begin{aligned} 0 = \frac{\partial}{\partial \bar{z}_i}(z) &= \frac{\partial}{\partial \bar{z}_i}(f^{-1}(f(z))) \\ &= \sum_k \frac{\partial f_j^{-1}}{\partial z_k} \cdot \frac{\partial f_k}{\partial \bar{z}_i} + \sum_k \frac{\partial f_j^{-1}}{\partial \bar{z}_k} \cdot \frac{\partial \bar{f}_k}{\partial \bar{z}_i} \\ &= \sum_k \frac{\partial f_j^{-1}}{\partial \bar{z}_k} \cdot \left(\overline{\frac{\partial f_k}{\partial z_i}} \right) \qquad \text{for all } i, j \end{aligned} \tag{17}$$

where the last equation holds because f is holomorphic and hence, $\sum_k \frac{\partial f_j^{-1}}{\partial z_k} \cdot \frac{\partial f_k}{\partial \bar{z}_i} = 0$. And we know that $\frac{\partial f_k}{\partial z_i}$ is nonsingular, so $\frac{\partial f_j^{-1}}{\partial \bar{z}_k}$ has to be 0 for all j, k which means that f^{-1} is holomorphic. □

2.2 Dolbeault Cohomology

We begin a discussion on Dolbeault cohomology. First, recall we have a decomposition of the complexified tangent space

$$T_p(M) \otimes \mathbb{C} \cong T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M = \mathbb{C}\{\frac{\partial}{\partial z}\}$ denotes the holomorphic tangent space, i.e. $\mathbb{C}$ linear derivations vanishing on f such that $\bar{f}$ is holomorphic. Since direct sums commute with dualization, we get a decomposition on the cotangent space

$$T_p^*(M) \otimes \mathbb{C} \cong (T^{1,0}M)^* \oplus (T^{0,1}M)^*.$$

This in turns yields a decomposition on exterior powers.

$$\bigwedge^k T_p^* M \otimes \mathbb{C} = \bigwedge^k (T^{1,0}M)^* \oplus (T^{0,1}M)^* \cong \bigoplus_{q=1}^{k} \bigwedge^q (T^{1,0}M)^* \otimes \bigwedge^{k-q} (T^{0,1}M)^*.$$

To see this isomorphism, we prove it for arbitrary finite dimensional vector spaces V, W. We show

Proposition 2.4. *For* $0 \leq i, j \leq k$

$$\bigoplus_{i+j=k} \left(\bigwedge^i V \otimes \bigwedge^j W \right) \cong \bigwedge^k V \oplus W$$

Proof. We first compute dimensions. Say $\dim V = n$ and $\dim W = m$, then

$$\begin{aligned} \dim \mathrm{LHS} &= \sum_{i=0}^{k} \dim(\wedge^i V \otimes \wedge^{k-i} W) \\ &= \sum_{i=0}^{k} \dim(\wedge^i V) \dim(\wedge^{k-i} W) \\ &= \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} \end{aligned}$$

while

$$\dim \mathrm{RHS} = \binom{n+m}{k}.$$

and this equality of dimensions is a combinatorial fact (ask Chinmay to prove this). Now, let us first define a bilinear map

$$\Phi_i : \bigwedge^i V \times \bigwedge^j W \to \bigwedge^k V \oplus W$$

$$\Phi_i(v_1 \wedge ... \wedge v_i, w_1 \wedge ... \wedge w_j) = (v_1, 0) \wedge .... \wedge (v_i, 0) \wedge (0, w_1) \wedge ... \wedge (0, w_j)$$

which by the universal property of tensors, factors as

$$\Phi_i : \bigwedge^i V \otimes \bigwedge^j W \to \bigwedge^k V \oplus W$$

$$\Phi_i(v_1 \wedge ... \wedge v_i \otimes w_1 \wedge ... \wedge w_j) = (v_1, 0) \wedge .... \wedge (v_i, 0) \wedge (0, w_1) \wedge ... \wedge (0, w_j).$$

By universal property of direct sum, have

$$\Phi : \bigoplus_{i+j=k} \left(\bigwedge^i V \otimes \bigwedge^j W \right) \to \bigwedge^k V \oplus W$$

$$\begin{aligned}\Phi(w_1 \wedge ... \wedge w_k, ..., v_1 \wedge ... \wedge v_k) &= \Phi_1(w_1 \wedge ... \wedge w_k) + ... + \Phi_k(v_1 \wedge ... \wedge v_k)\\ &= (0, w_1) \wedge ... \wedge (0, w_n) + ... + (v_1, 0) \wedge ... \wedge (v_k, 0)\end{aligned}$$

and we are finished if we can show Φ is either injective or surjective. We show surjectivity, so consider an arbitrary $(v_1, w_1) \wedge ... \wedge (v_k, w_k) \in \bigwedge^k V \oplus W$. Then, ?????????????????????? $\square$

Since this has been shown, we see that this decomposition descends down to forms, so

$$\Omega^k(M) \cong \bigoplus_{i+j=k} \Omega^{i,j}(M)$$

where $\Omega^{i,j}(M) := \wedge^i(T^{1,0}(M))^* \otimes \wedge^j(T^{0,1}(M))^*$. We say elements in $\Omega^{i,j}$ have form (i,j). Let $\pi^{i,j} : \Omega^*(M) \to \Omega^{i,j}M$ denote the natural projection map, and denote $\omega^{i,j} := \pi^{i,j}(\omega)$, then

$$\omega = \sum_{i,j} \omega^{i,j}.$$

We now want to argue the complexified exterior derivative respects this decomposition. Recall for $f \in C^\infty(M)$, we define $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$, and let us define

$$\partial f := \frac{\partial f}{\partial z}dz$$

$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

Notice also that $\partial f \in \Omega^{1,0}(M)$ and $\bar{\partial} f \in \Omega^{0,1}(M)$. Therefore, this decomposition of d preserves the direct sum from 0 to 1 forms, i.e. $d = \partial + \bar{\partial}$ and

$$\partial : \Omega^{0,0} \to \Omega^{1,1}$$

$$\bar{\partial} : \Omega^{0,0} \to \Omega^{0,1}$$

For higher degree forms, we define $\partial = \pi^{i+1,j} \circ d$ and $\bar{\partial} = \pi^{i,j+1} \circ d$, so restricting our domain to $\Omega^{i,j}$, we get

$$\partial : \Omega^{i,j} \to \Omega^{i+1,j}$$
$$\bar{\partial} : \Omega^{i,j} \to \Omega^{i,j+1}$$

defined locally by

$$\partial(\omega_{IJ} dz^I d\bar{z}^J) = \frac{\partial \omega_{IJ}}{\partial z^k} dz^k dz^I d\bar{z}^J.$$
$$\bar{\partial}(\omega_{IJ} dz^I \partial\bar{z}^J) = \frac{\partial \omega_{IJ}}{\partial \bar{z}^k} d\bar{z}^k dz^I d\bar{z}^J.$$

Let us see why $0 = \partial^2 = \bar{\partial}^2$. We have the following diagram representing

$$d^2 = \partial^2 + \bar{\partial}^2 + (\partial\bar{\partial} + \bar{\partial}\partial)$$

$$\begin{tikzcd}
 & & \Omega^{i+2,j} \\
 & \Omega^{i+1,j} \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"] & \\
\Omega^{i,j} \arrow[ur, "\partial"] \arrow[dr, "\bar{\partial}"'] & & \Omega^{i+1,j+1} \\
 & \Omega^{i,j+1} \arrow[ur, "\partial"'] \arrow[dr, "\bar{\partial}"'] & \\
 & & \Omega^{i,j+2}
\end{tikzcd}$$

Since $d^2 = 0$ and actually our decomposition of d^2 is a direct sum decomposition, this immediately gives the equations

$$0 = \partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial$$

Alternatively, we just compute. Then, we have

$$\partial^2\omega = \sum_{I,J}(\frac{\partial\omega_{IJ}}{\partial z^k dz^l}dz^l)dz^k dz^I dz^J$$

$$\bar{\partial}^2\omega = \sum_{I,J}(\frac{\partial\omega_{IJ}}{\partial\bar{z}^k d\bar{z}^l}d\bar{z}^l)d\bar{z}^k dz^I dz^J$$

For $k = l$, we see that the terms disappear. For $k \neq l$, this seems to suggest commutativity of second derivatives, i.e.

$$\frac{\partial\omega_{IJ}}{\partial z^k dz^l} = \frac{\partial\omega_{IJ}}{\partial z^l dz^k}$$
$$\frac{\partial\omega_{IJ}}{\partial\bar{z}^k d\bar{z}^l} = \frac{\partial\omega_{IJ}}{\partial\bar{z}^l d\bar{z}^k}.$$

If we exand $\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial y^i}\right)$ then we can easily see the desired commutativity.

Say a $(i, 0)$ for ω is *holomorphic* if all its coordinate functions are holomorphic. Equivalently, this is $0 = \bar{\partial}\omega$. Since holomorphic maps between manifolds preserve the direct sum decomposition, $\bar{\partial}$ commute with pullback. Finally, we define Dolbeault cohomology as

$$H^{p,q}_{\bar{\partial}} := \frac{\ker(\bar{\partial} : \Omega^{p,q} \to \Omega^{p,q+1})}{\mathrm{im}(\bar{\partial} : \Omega^{p,q-1} \to \Omega^{p,q})}.$$

2.3 Calculus

Examples of Hermitian Metrics:

1. The hermitian metric on $\mathbb{C}^n$ is given by

$$ds^2 = \sum_{i=1}^{n} dz_i \otimes d\bar{z}_i$$

and we call it the Euclidean or standard metric. The induced Riemannian metric is the standard metric on $\mathbb{C}^n = \mathbb{R}^{2n}$.

2. Let $\Lambda = \mathbb{Z}^k$. Then, the metric given on the complex torus $\mathbb{C}^n/\Lambda$ is

$$ds = \sum dz_i \otimes d\bar{z}_i$$

which is again called the Euclidean metric.

3. The hermitian metric on $\mathbb{CP}^n$ is an interesting one called the **Fubini-Study metric** and to define it we use a $(1,1)$-form to obtain the coframe of the space and then, see how's the hermitian metric structured.
For this, first, let $Z_0, \cdots, Z_n$ be coordinates on $\mathbb{C}^{n+1}$, and let $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ be the standard projection map. Now, let $U \subset \mathbb{CP}^n$ be an open set and $Z : U \to \mathbb{C}^{n+1} \setminus \{0\}$ be a lifting of U. By *lifting*, we mean a holomorphic map with $\pi \circ Z = id$. Now, consider the following $(1,1)$ differential form:

$$\omega = \frac{i}{2\pi} \partial\overline{\partial} \log \|Z\|^2.$$

First, we want to show that ω is independent of choice of the lifting, i.e., if $Z' : U \to \mathbb{C}^{n+1} \setminus \{0\}$ is another lifting, then we have $Z' = f.Z$ where f is a nonzero holomorphic function and we have:

$$\begin{aligned}
\omega' &= \frac{i}{2\pi} \partial\overline{\partial} \log \|Z'\|^2 \\
&= \frac{i}{2\pi} \partial\overline{\partial} \log \|f.Z\|^2 \\
&= \frac{i}{2\pi} \partial\overline{\partial} \log \left(|f.Z| \cdot |\overline{f.Z}| \right) \\
&= \frac{i}{2\pi} \partial\overline{\partial} \log \left(\|Z\|^2 \cdot f \cdot \overline{f} \right) \\
&= \frac{i}{2\pi} \partial\overline{\partial} \left(\log \|Z\|^2 + \log f + \log \overline{f} \right) \\
&= \frac{i}{2\pi} \left(\partial\overline{\partial} \log \|Z\|^2 + \partial\overline{\partial} \log f + \partial\overline{\partial} \log \overline{f} \right) \\
&= \omega + \frac{i}{2\pi} \left(\partial\overline{\partial} \log f - \overline{\partial}\partial \log \overline{f} \right) \\
&= \omega.
\end{aligned}$$

So, ω is independent of the chosen lifting and since liftings always exist locally, ω is defined globally on $\mathbb{CP}^{n+1}$.
Now, we have to see that ω is positive, note that the unitary group $U(n+1)$ acts transitively on $\mathbb{CP}^n$ and leaves ω invariant. So, if ω is positive at one point, it is positive

everywhere. Now, let $\{w_i = Z_i/Z_0\}$ be coordinates on the open set $U_0 = (Z_0 \neq 0)$ in $\mathbb{CP}^n$ and use the lifting $Z = (1, w_1, \cdots, w_n)$ on U_0. We have:

$$\begin{aligned}
\omega &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + \sum w_i\bar{w}_i\right) \\
&= \frac{i}{2\pi}\partial\left[\frac{\sum w_i d\bar{w}_i}{1 + \sum w_i\bar{w}_i}\right] \\
&= \frac{i}{2\pi}\left[\frac{\sum dw_i \wedge d\bar{w}_i}{1 + \sum w_i\bar{w}_i} - \frac{(\sum \bar{w}_i dw_i) \wedge (\sum w_i d\bar{w}_i)}{(1 + \sum w_i\bar{w}_i)^2}\right]
\end{aligned}$$

So, at point $[1, 0, \cdots, 0]$, we have:

$$\omega = \frac{i}{2\pi}\sum dw_i \wedge d\overline{w_i} > 0$$

which finally defines the hermitian metric on $\mathbb{CP}^n$.

2.4 Sheaf Basics

Much of this section comes from the Vakil's notes [3] along with some help from the Stacks project.

Fix a manifold M. Sheaves are meant to generalize DG structures on M such as -insert adj- functions or sections of -insert adj- various bundles.

Fix a topological space $(X, \mathcal{T})$. A **presheaf** is a contravariant functor $\mathcal{F} : \mathcal{T} \to$ Ab Grp. This definition is meant to (for example) generalize the fact that if $V \subset U$ and we have a (say) smooth function on U, we can restrict to V to get a (say) smooth function on V. This gives a map $C^\infty(U) \to C^\infty(V)$.

A **sheaf** is a presheaf which satisfies the additional gluing axiom: If $\{U_i\}$ is a collection of open sets, and $s_i \in \mathcal{F}(U)$ is a section defined such that all overlaps agree, then $\exists!$ extension s on the union

$$s \in \mathcal{F}\left(\bigcup_i U_i\right).$$

This is a local-to-global extension property. Observe if we instead take closed sets for the definition, this axiom fails in the most basic examples: consider $|x| : (-\infty, 0] \cup [0, \infty)$ as the union of two C^1 functions. We must glue on open sets, not closed ones.

At this point, you should be well convinced these are natural objects to study since they generalize so many DG objects one cares about. On the other hand, since the basic object already is a functor, it's not hard to see how this shit gets abstract fast.

Given sheaves $\mathcal{F}, \mathcal{G}$, a **map of sheaves** $\alpha : \mathcal{F} \to \mathcal{G}$ is a natural transformation, that is a collection of group homomorphisms α_U which commute with the sheaves. Explicitly the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "res_{\mathcal{F}}"'] & \mathcal{G}(U) \arrow[d, "res_{\mathcal{G}}"] \\
\mathcal{F}(V) \arrow[r, "\alpha_V"'] & \mathcal{G}(V)
\end{tikzcd}$$

We define the kernel of the map $\ker(\alpha)$ as the sheaf $U \to \ker(\alpha_U)$. Let us check this is a sheaf.

1. **Presheaf.** This is a diagram chase. The only thing to check is that given $V \subset U$, we have
$$res_{\mathcal{F}}|_{\ker(U_\alpha)} : \ker(\alpha_U) \to \ker(\alpha_V).$$
So suppose $s \in \ker(\alpha_U)$, so that $\alpha_U(s) = 0 \implies res_{\mathcal{G}}(\alpha_U(s)) = 0$ by group homomorphism. By commutativity of the diagram, this means $\alpha_V(res_{\mathcal{F}}(s)) = 0$. Therefore, by definition, $res_{\mathcal{F}}(s) \in \ker(\alpha_V)$. Thus the claim is shown.

   No more diagram chases will be explicitly written out.

2. **Gluing.** Given a collection of open sets $\{U_i\}$ and sections $s_i \in \ker(\alpha_{U_i})$ such that the overlaps agree, we want to verify that $\exists! s \in \ker(\alpha_{\cup_i U_i})$ which extends each s_i.

   Since for each i, we have $\ker(a_{U_i}) \subset \mathcal{F}(U_i)$, we can glue using $\mathcal{F}$, so that we guarantee that $s \in \mathcal{F}(\cup_i U_i)$, and it suffices to verify that actually s lies inside $\ker(\alpha_{\cup_i U_i})$. Consider the following diagram.
$$\begin{tikzcd}
\mathcal{F}(\cup_i U_i) \arrow[r, "\alpha_{\cup_i U_i}"] \arrow[d, "res_{\mathcal{F}}"'] & \mathcal{G}(\cup_i U_i) \arrow[d, "res_{\mathcal{G}}"] \\
\mathcal{F}(U_i) \arrow[r, "\alpha_{U_i}"'] & \mathcal{G}(U_i)
\end{tikzcd}$$
If we go down then go right, we have $s \mapsto s_i \mapsto 0$. Therefore, we have $s \mapsto \alpha_{\cup_i U_i}(s) \mapsto 0$. If we can show $res_{\mathcal{G}}$ is injective, then the result is shown. But this follows directly from the sheaf axioms: since $\alpha_{U_i}(s_i) = 0$ for each U_i, then we glue these sections to show 0 is the unique sheaf in $\mathcal{G}(\cup_i U_i)$ which restricts to each $\alpha_{U_i}(s_i)$. Th

Let us mention some more examples of sheaves. Let $\mathcal{F}$ be a sheaf and $V \subset U \subset X$.

1. **Constant sheaf.** Contrary to initial belief, we do *not* define this as the constant map taking any open set to a fixed abelian group A (and restriction maps are identities). This is a presheaf as we will check now.

   -FIND EXAMPLE OF WHAT FAILS-

Instead, let us define this as the

$$\underline{A}(U) = \{f : U \to A | \text{ f is locally constant}\}$$

where A is topologized with the discrete topology.

2. **Sheaf of continuous maps**. Fix a topological space Y. Generalizing our previous example, continuous maps $X \to Y$ form a sheaf. (This is a point-set proof).

3. **Pushforward sheaf.** Consider a continuous $\pi : X \to Y$ and we can pushforward $\mathcal{F}$ to a sheaf on Y, which we define now.

$$\pi_*(\mathcal{F})(V) := \mathcal{F}(\pi^-(V)).f$$

4. **Skyscraper sheaf.** This is the sheaf version of Kronecker delta. Fix $x \in X$, then we define the sheaf

$$(i_x)_*^S(U) := \begin{cases} S & x \in U \\ e & x \notin U \end{cases}$$

WORK OUT WHY THIS IS THE PUSHFORWARD SHEAF OF A POINT.

5. **Restriction sheaf.** We define this sheaf as

$$\mathcal{F}|_U(V) = \mathcal{F}(V).$$

We would also like to have a notion of an image sheaf and a cokernel sheaf (so we can employ tools from homological algebra to compute), but the naive definition fails to work. G&H gives the *exponential sequence* as an exact sequence which fails this. We will elaborate with more details later.

But let us fix this by introducing the notion of **sheafification** which turns any presheaf into a sheaf. The upshot is that we will define image and cokernels through this process. Sheafification is the free functor from the category of presheaves. We expand on this viewpoint at the very end.

Fix a point u in an open set U, then the **stalk** of a sheaf $\mathcal{F}$ is the direct limit

$$\mathcal{F}_u := \operatorname{colim}_{u \in U} \mathcal{F}(U).$$

Functionally, we can think of this as germs of sections at u:

$$\mathcal{F}_u = \{(U_i, s_i) \in \mathcal{T} \times \mathcal{F}(U) : u \in U_i\}/\{(U_i, s_i) \sim (U_j, s_j) \iff \exists U_k \subset U_i \cap U_j : s_i|_{U_k} = s_j|_k\}$$

but immediately we will abuse notation and think of (U_i, s_i) as s_i. Notice for any $u \in U$, we have a map

$$\mathcal{F}(U) \to \mathcal{F}_u$$
$$s \mapsto s_u$$

which is thought to be the process given a section on U, take germ of section of that section at u. Varying over u, we get the map

$$\mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}_u$$

$$s \mapsto \prod_{u \in U} s_u.$$

This map is injective. To see this, take $s, t \in \mathcal{F}(U)$ and suppose $(U_i, s_i) = (V_i, t_i) \in \prod_{u \in U} \mathcal{F}_u$. By definition of equality for germs, we have some neighborhood of u, $W_i \subset U_i \cap V_i$ and $r_i \in \mathcal{F}(W_i)$ such that $s_i|_{W_i} = r_i = t_i|_{W_i}$. Since this holds for every $u \in U$, we have equality of sections in an open cover, so we have equality throughout.

What is the image of this map? Notice first that it's impossible for this map to be surjective; there's too much free will involved. Somehow, there should be a compatibility condition dictated by restriction maps between germs $s_i = (U_i, s_i)$ and germs of points u_j lying in side U_i. Specifically, given (U_i, s_i), we want to take the germ at u_j

1. have its open set contained in U_i
2. have its section be induced from the restriction map from U_i to U_j. In other words, we demand the following equality:

$$s|_{U_j} = (s|_{U_i})\,|_{U_j}$$

Let us define the **sheafification** of $\mathcal{F}$ (here we think of $\mathcal{F}$ as just a presheaf) as

$$\mathcal{F}^{\#}(U) := \{(s_u) \in \prod_{u \in U} \mathcal{F}_u : \text{ the above compatibility holds}\} = \mathrm{im}\left(\mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}_u\right)$$

Remark: this bothers me a bit since $\mathcal{F}_u$ is a quotient, and I'm picking out a specific representative and using it, and it's not obvious is holds when changing representatives. So let us check this.

-INSERT CHECKING-.

We now argue that taking stalk is functorial. Namely, given a map of sheave $\alpha : \mathcal{F} \to \mathcal{G}$, this induces a map of stalks $\mathcal{F}_u \to \mathcal{G}_u$ for any $u \in X$. By definition of stalk, every element s_i must be a section over some open set U_i. Therefore, we can consider the diagram

$$\begin{tikzcd}
\mathcal{F}(U_i) \arrow[rrr, "\alpha"] \arrow[dd] & & & \mathcal{G}(U_i) \arrow[dd] \\
& s_i \arrow[r, mapsto, "\alpha"] \arrow[d, mapsto] & t_i \arrow[d, mapsto] & \\
\mathcal{F}_u & (U_i, s_i) & (U_i, t_i) & \mathcal{G}_u
\end{tikzcd}$$

and therefore, the map $\mathcal{F}_u \to \mathcal{G}_u$ is clear, although one needs to check this is independent of representative.

-INSERT CHECKING-.

Furthermore, and we won't prove this, we have

Theorem 2.5. *A map of sheaves is an isomorphism iff the induced map on stalks is an isomorphism.*

Let us now switch gears, back to sheafification. Notice since $V \subset U$, we have the following canonical map (using projections + universal property of product)

$$\prod_{u \in U} \mathcal{F}_u \to \prod_{v \in V} \mathcal{F}_v.$$

This gives the restriction maps for $\mathcal{F}^{\#}$.

2.5 Sheaf Cohomology

We didn't quite get to this last time, but the following is true and is the basis for our discussion.

Proposition 2.6. *Let $U \subset X$ be open and*

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0.$$

an exact sequence of sheaves.[2] *Name the first nontrivial map α and second β.*

1. *$\alpha : \mathcal{E}(U) \to \mathcal{F}(U)$ is injective.*
2. *"Up to a cover, $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective". Explicitly, for every $s \in \mathcal{G}(U)$, there exists a cover $\{U_i\}$ of U and sections $s_i \in \mathcal{F}(U_i)$ such that*

   $$\beta(s_i) = s|_{U_i} \in \mathcal{G}(U_i).$$

   The corresponding diagram here is (where elements are in pink)

$$\begin{tikzcd}
 & \mathcal{F}(U) \arrow[r, "\beta"] \arrow[d, "r_{\mathcal{F}}"'] & \mathcal{G}(U) \arrow[d, "r_{\mathcal{G}}"] & s \\
s_i & \mathcal{F}(U_i) \arrow[r, "\beta"'] & \mathcal{G}(U_i) & \beta(s_i) = s|_{U_i}
\end{tikzcd}$$

All this fussing about surjectivity but not injectivity was really the same reason we needed to go through sheafification. Phrased more abstractly - fixing an open set U, the functor from the category of sheaves to sections is only left exact. Now injectivity of α and surjectivity up to a covering of β on sections will similarly pass when we take cochains. Therefore, we can motivate **Cech cohomology** from the failure of right exactness of sheaves. Let us now setup our machinery.

[2]This means $\mathcal{E}$ is the kernel subsheaf of α and $\mathcal{F}$ is the sheafification of the cokernel presheaf for β.

Let M be a manifold, $\mathcal{F}$ be a sheaf on M, and $\underline{U} = \{U_i\}$ be a locally finite cover for M.[3] We define cochains on $\mathcal{F}$ as follows:

$$C^0(\underline{U}, \mathcal{F}) = \prod_i \mathcal{F}(U_i)$$
$$C^1(\underline{U}, \mathcal{F}) = \prod_{i \neq j} \mathcal{F}(U_i \cap U_j)$$
$$\vdots$$
$$C^p(\underline{U}, \mathcal{F}) = \prod_{i_0 \neq i_1 \neq \cdots \neq i_p} \mathcal{F}\left(U_{i_0} \cap \cdots \cap U_{i_p}\right)$$

An element $\sigma = \{\sigma_I \in \mathcal{F}(\cap U_{i_K})\}_{\#I = p+1}$ of $C^p(\underline{U}, \mathcal{F})$ is called a *p-cochain* of $\mathcal{F}$. We define a *coboundary* operator

$$\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F})$$

by the formula

$$(\delta\sigma)_{i_0 \ldots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0, \ldots, \widehat{i_j}, \ldots, i_{p+1}} \bigg|_{\cap U_I}$$

Note that is incredibly similar to singular (co)homology! This similarity will be reflected in facts such as $\delta^2 = 0$ and cochain maps (i.e. maps between cochain groups which commute with δ) will induce maps on cohomology. This argument shows that

$$C^0(\underline{U}, \mathcal{F}) \to C^1(\underline{U}, \mathcal{F}) \to \ldots$$

forms a complex. Therefore, we can define the pth-cohomology of the cover $\underline{U}$ as

$$H^p(\underline{U}, \mathcal{F}) := \frac{\ker(\delta : C^p(\underline{U}, \mathcal{F}) \to C^{p+1}(\underline{U}, \mathcal{F}))}{\operatorname{im}(\delta : C^{p-1}(\underline{U}, \mathcal{F}) \to C^p(\underline{U}, \mathcal{F}))}.$$

Observe now for fixed p, there are two variables for $C^p(-,-)$, the first taking in a cover of M and the second taking in a sheaf. Let us first focus our attention to the second variable. A map between sheaves $\beta : \mathcal{F} \to \mathcal{G}$ induces a map on cochains by the following. Fix a covering $\underline{U} = \{U_i\}$, then we define the induced map $\beta^\#$ on cochains by extending the maps β in the following way. For finite multi-index I with distinct entries and $|I| = k$,[4]

$$\beta : \mathcal{F}\left(\bigcap U_{i_I}\right) \to \mathcal{G}\left(\bigcap U_{i_I}\right)$$

[3] There is a bit of a subtle point here which is do we fix a well ordering on the cover? Let $i \in L$ (L for label) be a well ordered index for the cover. Since we are considering multi-indices with non-repeating values, after we quotient by the action of S_L on the set of length p multi-indices $\mathcal{I}$. This representative can be canonically identified with the well-ordered subset $I \subset \mathcal{P}(L)$ where the well-ordering is induced from L. Explicitly, we the representative I is chosen so that the labels are increasing, and this approach is taken in Hatcher. This choice really comes in when defining the homotopy operator. However, when talking about refinements of covers, the language becomes considerably more annoying, but equivalent, if we choose this well-ordering. For another more familiar of this instance, considered an ordered basis for a vector space $e_1, \ldots, e_n$ and then in the ordered view, we have a basis for $\wedge^p V$ as given by e_I where $|I| = p$ and the indices are strictly increasing. The unordered view is to consider $e_i \wedge e_j = -e_j \wedge e_i$, a relation that $G\&H$ explicitly mentions in the book when defining the coboundary operator.

[4] This language is used to avoid some notation G&H uses. It's also quite important to note that I depends on $\underline{U}$. Suppressing this dependence from the notation will slightly bite us in the ass later when we look at refinements, but not too hard.

we set

$$\beta^{\#} : \prod_{I \in \mathcal{I}} \mathcal{F}\left(\bigcap U_{i_I}\right) \to \prod_{I \in \mathcal{I}} \mathcal{G}\left(\bigcap U_{i_I}\right)$$

$$\beta^{\#}\Big((\sigma_I)_{I \in \mathcal{I}}\Big) := (\beta \sigma_I)_{I \in \mathcal{I}}.$$

where I varies over all multi-indices $\mathcal{I}$ satisfying the above adjectives. Notice the domain of $\beta^{\#}$ is $C^p(\underline{U}, \mathcal{F})$ and the codomain is $C^p(\underline{U}, \mathcal{G})$.

For any open set U, since our maps β are homomorphisms of abelian groups, they are linear by default. Therefore, so are $\beta^{\#}$ and so our maps will automatically commute with δ. By the same argument as in Hatcher, we get an induced map β^* on cohomology.

Now let us focus our attention to the first component. Fix a sheaf $\mathcal{F}$. What is a map of covers? Since from the very genesis of presheaves, we care about restrictions from large open sets to small open sets, then the correct notion of a map between covers is captured in the notion of a **refinement**. Given two covers $\underline{U}, \underline{U}'$, we say $\underline{U}'$ is a refinement of $\underline{U}$ if for any $U' \in \underline{U}'$, there exists $U \in \underline{U}$ such that $U' \subset U$. We denote this as $\underline{U} > \underline{U}'$. This gives us maps

$$\rho : C^p(\underline{U}, \mathcal{F}) \to C^p(\underline{U}', \mathcal{F})$$

compatible with the restriction maps from sets in the cover $\underline{U}$ to sets in the refined cover $\underline{U}'$. Let us spell this out in detail. Let I, I' index over sets in $\underline{U}, \underline{U}'$. $U > U'$ implies we can choose a $\phi : I' \to I$ such that for $U'_{i'} \in \underline{U}'$, we have $U'_{i'} \subset U_{\phi(i')}$.[5] Therefore, we define ρ by

$$(\rho_\phi \sigma)_{I'} := \sigma_{\phi(I')}|_{\cap U_{i_I}}.$$

It isn't hard to see ρ_ϕ commutes with δ so it also descends down to a map on cohomology

$$\rho : H^p(\underline{U}, \mathcal{F}) \to H^p(\underline{U}', \mathcal{F})$$

At this stage, fixing an open cover $\underline{U}$ and a refinement $\underline{U}'$, we check two maps $\phi, \psi : I' \to I$ which satisfy the above condition will be chain homotopic. Therefore, these induce equal maps on cohomology and on the level of cohomology, we can eliminate the dependence of ϕ completely.

-INSERT CHECKING-

We can finally define **pth Cech cohomology**, $H^p(M, \mathcal{F})$, namely as the direct limit over refinements of pth cohomology of covers. Computationally, we need a sufficient condition to actually compute anything. This is provided by Leray's theorem which says that if we take a cover which does not give any additional cohomology, then the cohomology of the cover computes Cech cohomology.

Theorem 2.7. *If $\underline{U}$ such that for all I*

$$H^q(\cap U_{i_I}, \mathcal{F}) = 0,$$

then $H^*(M, \mathcal{F}) = H^*(\underline{U}, \mathcal{F})$.

[5] We must later show whatever structure we care about must be independent of this choice of ϕ. This is encapsulated in the definition of a direct limit. Morally speaking, this choice shouldn't matter by uniqueness of gluing from sheaf axioms.

Consider an exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0.$$

where we name the first map α and second map β. In the following diagram, α is in blue, β is in green, δ is in orange, and ρ is in black. For ease, highlighted in pink are where σ, τ and μ live. This is really just the standard diagram chase, but you go down a level (i.e. refine more) each time you want to solve an equation. For refinements $\underline{U} > \underline{U}' > \underline{U}''$, chase the following diagram to define the coboundary map

$$\delta^* : H^p(M, \mathcal{G}) \to H^{p+1}(M, \mathcal{E}).$$

The three dimensional nature of this diagram reflects the fact that we can only require surjectivity up to taking a cover. This means every time we want to solve an equation, we must take a further refinement, which gives us a "z-direction". This gives another comparision to the singular cohomology case.

-INSERT TALA NOTES-

Theorem 2.8. *(de Rham) For any smooth manifold M, de Rham cohomology is isomorphic to singular cohomology. That is, $H^*_{dR}(M) = H^*_{sing}(M)$.*

Proof. We know any smooth manifold has an underlying simplicial structure; denote the corresponding cohomology groups by $H^*(M)$ (coefficients in $\mathbb{R}$). We always have $H^*(M) = H^*_{sing}(M)$, and by above $H^*(M) = \check{H}^*(M)$, so it suffices to show that the de Rham cohomology of M is isomorphic to the Čech cohomology of M. The Poincaré Lemma tells us that

$$0 \to \mathbb{R} \to A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots$$

is a long exactsequence of sheaves, so it can be split into a collection of short exact sequences:

$$0 \to \mathbb{R} \to A^0 \xrightarrow{d} Z^1 \to 0$$

$$0 \to Z^1 \xrightarrow{i} A^1 \xrightarrow{d} Z^2 \to 0$$

$$\vdots$$

$$0 \to Z^p \xrightarrow{i} A^p \xrightarrow{d} Z^{p+1} \to 0.$$

Each of the above short exact sequences induces a long exact sequence in Čech cohomology. From previous work we know $H^q(M, A^p) = 0$ for all $q > 0$, so the sequences look like

$$\cdots \to H^{p-1}(M, \mathbb{R}) \to 0 \to H^{p-1}(M, Z^1) \xrightarrow{\partial} H^p(M, \mathbb{R}) \to 0 \to \cdots$$

$$\cdots \to H^{p-2}(M, \mathbb{R}) \to 0 \to H^{p-2}(M, Z^1) \xrightarrow{\partial} H^{p-1}(M, \mathbb{R}) \to 0 \to \cdots$$

$$\vdots$$

etc. In each sequence we see a segment of the form $0 \to A \to B \to 0$, which by exactness implies $A = B$. This gives a cascade of isomorphisms $H^p(M, \mathbb{R}) = H^{p-1}(M, Z^1) = \cdots = H^1(M, Z^{p-1})$. If we can show $H^1(M, Z^{p-1}) = H^p_{dR}(M)$ we will be done. We have seen $H^0(M, Z^q) = Z^q$ and similarly $H^0(M, A^q) = A^q$ for any $q > 0$, so there is a long exact sequence

$$0 \to Z^q \to A^q \to Z^{q+1} \xrightarrow{\partial} H^1(M, Z^q) \to 0.$$

From exactness and the First Isomorphism Theorem, $H^1(M, Z^q) = Z^{q+1}/ker(\partial) = Z^{q+1}/im(d) =: H^{q+1}_{dR}(M)$. The proof is complete. □

Theorem 2.9. *(Dolbeault) Let M be a complex manifold, Ω^p holomorphic p-forms, $A^{p,q}$ smooth forms of type (p, q). Then $H^q(M, \Omega^p) = H^{p,q}_{\overline{\partial}}(M)$.*

Proof. Analogous to the proof of de Rham's theorem, we begin with a long exact sequence of sheaves:

$$0 \to \Omega^p \to A^{p,0} \xrightarrow{\overline{\partial}} A^{p,1} \xrightarrow{\overline{\partial}} A^{p,2} \cdots .$$

From this, we obtain short exact sequences

$$0 \to \Omega^p \to A^{p,0} \xrightarrow{\overline{\partial}} Z^{p,1} \to 0$$
$$0 \to Z^{p,1} \to A^{p,1} \xrightarrow{\overline{\partial}} Z^{p,2} \to 0$$
$$\vdots$$
$$0 \to Z^{p,q} \to A^{p,q} \xrightarrow{\overline{\partial}} Z^{p,q+1} \to 0.$$

Each short exact sequence of sheaves induces a long exact sequence in cohomology which is nearly identical to the corresponding one in the previous proof. Consequently, we find another cascade of isomorphisms $H^q(M, \Omega^p) = H^{q-1}(M, Z^{p,1}) = \cdots = H^1(M, Z^{p,q-1})$. From similar reasoning once more, we find $H^1(M, Z^{p,q-1}) = Z^{p,q}/im(\overline{\partial}) =: H^{p,q}_{\overline{\partial}}(M)$, which gives the result. □

3 Vector Bundles

Fix a complex manifold M. Intuitively, vector bundle is a collection of vector spaces parameterized by M such that there is no local twisting. One way of thinking of a rank k vector bundle is an open cover $\underline{U} = \{U_i\}$ together with maps $g_{ij} : U_i \cap U_j \to \mathrm{GL}_k(\mathbb{C})$ satisfying the **cocycle condition**:

$$id = g_{ki} \cdot g_{jk} \cdot g_{ij}.$$

3.1 Connections

3.1.1 Coordinate Free

A connection for a vector bundle $\pi : E \to M$ is a way to differentiate sections $\sigma : M \to E$ along vector fields $X \in \Gamma(E)$. This generalizes the notion of a directional derivative, or given a local frame, partial derivatives. We define a **connection** to be a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

$$(X, \sigma) \mapsto \nabla_X \sigma$$

satisfying the following: given $f, f' \in C^\infty(M)$, $X, X' \in \Gamma(TM)$, and $\sigma, \sigma' \in \Gamma(E)$

1. ∇ is tensorial (i.e. $C^\infty(M)$ linear) in the first component:

   $$\nabla_{fX+gX'}\sigma = f\nabla_X\sigma + f'\nabla_{X'}\sigma'.$$

   Tensoriality corresponds to the idea that taking the directional derivative of a section σ at $p \in M$ should be determined only by $X \in T_pM$ as opposed to local data. Namely if $X = 0 \in T_pM$, then choosing a coordinate neighborhood $(x^1, ..., x^n)$ around p, we have in a neighborhood of p,

   $$\nabla_X\sigma = \nabla_{X^i\partial_i}\sigma = X^i\nabla_{\partial_i}\sigma,$$

   We know that since $X^i(p) = 0$ by linear independence of ∂_i, then

   $$\nabla_X\sigma|_p = 0.$$

   As an obvious corollary, if $X = X' \in T_pM$, then

   $$\nabla_X\sigma|_p = \nabla_{X'}\sigma|_p.$$

2. ∇ satisfies Leibniz rule in the second component:

   $$\nabla_X(f\sigma) = (Xf)\sigma + f(\nabla_X\sigma).$$

   In particular, for constant functions $f \equiv c$, we have $Xf \equiv 0$, so we also have $\mathbb{R}$-linearity in this component. The Leibniz condition corresponds to *locality*. Namely if $\sigma \equiv 0$ in a neighborhood N_p of $p \in M$, then taking f to be a bump function supported in N_p. We first observe that outside of N_p we have no information on $\nabla_X\sigma$ as $f \equiv 0$ means

   $$\nabla_X(f\sigma) = \nabla_X(0 \cdot \sigma) = \nabla_X 0 = 0.$$

   Apply Leibniz, we have

   $$\nabla_X(f\sigma) = (Xf)\sigma + f\nabla_X\sigma = 0 \cdot \sigma + 0 \cdot \nabla_X\sigma.$$

   This should come as no surprise as the direction derivative itself is local.

However, around p you have good information. Consider a neighborhood $q \in N'_p \subset \{f \equiv 1\}$, then

$$0 = \nabla_X 0|_q = \nabla_X \sigma|_q = \nabla_X (f\sigma)|_q = (Xf)\sigma|_q + f(q)\nabla_X \sigma|_q = 0 \cdot \sigma|_q + 1 \cdot \nabla_X \sigma|_q = \nabla_X \sigma|_q.$$

Therefore in the neighborhood any N'_p, we have

$$\nabla_X \sigma = 0.$$

As an obvious corollary, we get if around p we have $\sigma = \sigma'$, then around p

$$\nabla_X \sigma = \nabla_X \sigma'.$$

We summarize the above discussion as the slogan: "tensorial is pointwise, Leibniz is local". By the Leibniz condition, we see that connections are not tensors. Therefore, if we hone in on a point $p \in M$, we may as well consider a connection to be a map

$$\nabla : T_pM \times \mathrm{Germ}_p \to \mathrm{Germ}_p$$

where ∇ satisfies the two conditions above.

There is an equivalent way of viewing the connection by "letting the X go". This defines the connection taking 0-forms to 1-forms. Namely, we can think of a connection on a vector bundle as a map

$$\Gamma(E) \to \Gamma(E \otimes \bigwedge^1 T^*M)$$

such that

$$\nabla(f\sigma) = df \otimes \sigma + f \cdot \nabla\sigma.$$

It's clear the first term is in the space, but to see why the second term is, we say without proof that

$$\Gamma(E \otimes \bigwedge^1 T^*M) \cong \Gamma(E) \otimes \Omega^1 M.$$

Then simply note that ∇ is $\mathbb{R}$-linear in both components. One common notation is to write

$$\Omega^k(M; E) := \Gamma(E \otimes \bigwedge^k T^*M)$$

where once again we have $\Gamma(E \otimes \bigwedge^k T^*M) \cong \Gamma(E) \otimes \Omega^k M$. Note in particular for $k = 0$, we have

$$\Omega^0(M; E) = \Gamma(E \otimes C^\infty M) = \Gamma(E).$$

One thinks of this as "E-valued k-forms", in the sense that if I enter k vector fields, I will end up with a section of $E \to M$. We'll see shortly that ∇ has a natural extension to these spaces. Then, we extend ∇ to higher tensor bundles by following the graded product rule:

$$\nabla : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$$

$$\nabla(\eta \otimes \sigma) = d\eta \otimes \sigma + (-1)^k \eta \otimes \nabla\sigma.$$

We wish to now extend our connection to the dual bundle and therefore also to all its tensor powers. To do so, we begin by focusing on the special case $E = TM$[6], we can extend ∇ to all (r, s) tensors by requiring

1. $\nabla = d$ on C^∞ functions.
2. ∇ treats the natural pairing as a product.

Let us begin by seeing how the first condition is a natural one. Consider the constant function $\sigma : M \to 1 \in \mathbb{R}$. Leibniz tells us

$$\nabla_X(f\sigma) = \nabla_X f,$$

and since σ is constant, it has 0 derivative, so

$$(Xf)\sigma + f(\nabla_X \sigma) = Xf + f(X\sigma) = Xf.$$

establishing the desired equality

$$\nabla = d : \Omega^0(M) \to \Omega^1(M).$$

Now choose $\eta \in \Gamma(T^*M)$ and $Y \in \Gamma(TM)$, so that $\eta(Y) \in C^\infty(M)$. Requiring product rule to hold and be compatible with evaluation, we have the following equality of C^∞ functions:

$$X(\eta(Y)) = \nabla_X(\eta(Y)) = (\nabla_X \eta)Y + \eta(\nabla_X Y).$$

This give us natural extension of ∇ to T^*M, namely

$$(\nabla_X \eta)(Y) := X(\eta(Y)) - \eta(\nabla_X Y).$$

The same procedure induces a connection on all tangent bundles. Namely given a (p, q) tensor T, with $Y_1, ..., Y_p \in \Gamma(TM)$ and $\eta_1, ..., \eta_q \in \Gamma(T^*M)$, we define $\nabla_X T$ to be the (p, q) tensor satisfying the equation

$$\nabla_X(T(Y_1, ..., Y_p, \eta_1, ..., \eta_q)) = (\nabla_X T)(...) + \sum_{i=1}^{p} T(..., \nabla_X Y_i, ...) + \sum_{j=1}^{q} T(..., \nabla_X \eta_j, ...).$$

In the general case of E, we similarly induce a connection on the dual bundle and therefore all tensors. This uses the fact that that for $\eta \in \Gamma(E^*)$ and $Y \in \Gamma(E)$, we have $\eta(Y) \in C^\infty M$ by defining ∇ on $\Omega^1(M; E^*)$ to satisfy the following equation in $C^\infty(M)$:

$$X(\eta(Y)) = (\nabla_X \eta)Y + \eta(\nabla_X Y).$$

Interestingly enough, $\nabla^2 : \Omega^0(M; E) \to \Omega^2(M; E)$ is tensorial, even though ∇ is not. To see this, we explicitly compute

$$\begin{aligned}
\nabla^2(f\sigma) &= \nabla(df \otimes \sigma + f\nabla\sigma) \\
&= d^2 f \otimes \sigma - df \otimes \nabla\sigma + df \otimes \nabla\sigma + f\nabla^2\sigma \\
&= f\nabla^2\sigma
\end{aligned}$$

[6]This is due to the initial worry that d cannot be extended to $\Omega^k(M; E)$, but we will be relieved to see this is not needed.

where the negative sign comes from the extension of the connection to $\nabla : \Omega^1(M;E) \to \Omega^2(M;E)$. By the correspondence tensorial maps on sections of vector bundles and bundle maps between their underlying vector bundles (see 7.7 in [2]), ∇^2 corresponds to a bundle map

$$\nabla^2 : E \to \bigwedge^2 TM \otimes E.$$

One has the fact that a map of bundles $E \to F$ is equivalent to a global section into $\mathrm{Hom}(E,F)$. Under these identifications, we can think of ∇^2 as a section Ω of the bundle

$$\begin{aligned}\Omega \in \Gamma(\mathrm{Hom}(E, \bigwedge^2 TM \otimes E)) &\cong \Gamma(\bigwedge^2 TM \otimes E^* \otimes E) \\ &= \Omega^2(M; E^* \otimes E)\end{aligned}$$

3.1.2 Local Coordinates

What does (the induced) ∇ (on its tensor bundles) look like in local coordinates? Since we have discussed the non-tensorality of ∇, it must obey a different transformation law. Pick a local frame $\{e_1, ..., e_n\}$ for $U \subset M$, so any element in $\Omega^k(M;E)$ can be written as

$$\sum_i \eta_i \otimes e_i$$

where each $\eta_i \in \Omega^k(M)$. Applying ∇, we have

$$\begin{aligned}\nabla\left(\sum_i \eta_i \otimes e_i\right) &= \sum_i d\eta_i \otimes e_i + (-1)^k \eta_i \otimes \nabla e_i \\ &= \sum_i d\eta_i \otimes e_i + (-1)^k \eta_i \otimes e_j \omega_i^j.\end{aligned}$$

We call the matrix of 1-forms ω the **connection form**[7]. The last equality may throw you off because we write the e_j's on the left of the matrix ω leading to the functional equation

$$\nabla e_i = \sum_j e_j \omega_i^j,$$

or if you prefer,

$$\nabla e = e\omega$$

is the index-invariant form. The reason we put the e on the left instead of the right is because when we change frames, we want to think of this as a **right action** instead of a left one. Let's now discuss the transformation behavior of a metric compatible ∇. Take another orthonormal frame $e' = (e'_1, ..., e'_k)$, so $e' = eg$ for some change of basis matrix

[7] To see this is a matrix, simply note that ∇ is $\mathbb{R}$-linear in its second component. To see it's entries are 1-forms, ???

$g : U \to \mathrm{GL}_k(\mathbb{C})$. Recall a basis is a linear isomorphism $e : \mathbb{C}^k \to E_k$, and $g \in \mathrm{GL}_k(\mathbb{C})$ acts from the right on e which corresponding to precomposition:

$$\begin{tikzcd} \mathbb{C}^k \arrow[r, "e'"] \arrow[d, "g"'] & E_k \\ \mathbb{C}^k \arrow[ur, "e"'] & \end{tikzcd}$$

Then, we recall $\nabla g = dg$ since g is a transition *function*, so that dg becomes a matrix of 1-forms. Therefore,

$$\nabla e' = \nabla(eg) = (\nabla e)g + edg = (e\omega)g + edg = e'g^{-}\omega + e'g^{-}dg = e'(g^{-}\omega + g^{-}dg).$$

Let us remark that if we chose $\nabla e = \omega e$ along with g a right action, then we get a cancellation that shouldn't be there, and the fact that e' is left on one term and right on another.

$$\nabla e' = \nabla(eg) = (\omega e)g + edg = \omega(e'g^{-})g + (e'g^{-})dg = \omega e' + (e'g^{-})dg$$

Therefore, you can think of $\nabla e = e\omega$ as a compatiblity condition forced by taking a right action of g on transition maps.

Let's see what $\Omega \in \Omega^2(M; E^* \otimes E)$ looks like in local coordinates. Since ∇^2 is tensorial, we expect it to transform accordingly. Choosing an orthonormal frame $e_1, ..., e_k$ for E induces an orthonormal frame $e^i \otimes e_j$ for $E^* \otimes E$. In this basis for $E^* \otimes E$, we can identify Ω with the matrix of 2-forms by the equation

$$\nabla^2 e_i := e_j \Omega_i^j.$$

This transforms as you'd expect. For another frame $e' = e \cdot g$ for some matrix g,

$$\begin{aligned} \nabla^2(e_i') &= \nabla^2(e_j g_i^j) \\ &= \nabla^2(e_j) g_i^j \\ &= e_k \Omega_j^k g_i^j \\ &= e_l'(g^{-})_k^l \Omega_j^k g_i^j \end{aligned}$$

which is expected by tensorality of ∇^2. If we instead apply ∇ one-by-one, we get the famous **Cartan's structure equation**.

$$\nabla^2 e_i = \nabla(e_j \omega_i^j) = -\nabla e_j \omega_i^j + e_j d\omega_i^j = -e_k \omega_j^k \omega_i^j + e_k d\omega_i^k.$$

In the above equation, we understand the product $\omega_j^k \omega_i^j$ to be wedge product (remember the entries of ω are 1-forms!). Rewriting, we have

$$d\omega = \Omega + \omega \wedge \omega.$$

Suppose we have a metric h, and so we enforce ∇ is metric compatible. Explicitly this means one of the two equivalent conditions:

1. $\nabla h = 0$ where we think of h as a $(0,2)$ form and ∇ as in the induced connection.
2. For all $Z \in \Gamma(TM)$ and $X, Y \in \Gamma(E)$ we have

$$\nabla_Z h(X,Y) = h(\nabla_Z X, Y) + h(X, \nabla_Z Y).$$

If we choose a unitary/orthonormal frame for E (always possible by GS), then coefficients of the connection matrix is determined by this data. Explicitly, let $e_1, ..., e_k$ be a frame over U for E. Then,

$$\delta^i_j = e^i(e_j).$$

Differentiating, i.e. taking a metric compatible connection,

$$0 = (\nabla e^i)(e_j) + e^i(\nabla e_j) = (e^k \alpha^i_k)(e_j) + e^i(e_k \omega^k_j) = \alpha^i_k(e^k(e_j)) + \omega^k_j(e^i(e_k)) = \alpha^i_j + \omega^i_j$$

3.2 Extending d

Suppose for a moment we are in one of the two scenarios:

1. $E = TM$ carries a Riemannian metric
2. E is an arbitrary bundle carrying a Hermitian metric

The requirement that ∇ be metric compatible does not single out unique choice of connection. However, in both scenarios, we can make an additional requirement which will define a unique connection.

1. **Riemannian.** In the case $E = TM$, then there is a unique metric compatible, *torsion free* connection. Torsion free means for all vector fields X, Y

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

   This is known as the **Levi-Civita connection.**

2. **Hermitian.** For arbitrary Hermitian bundles, there is a unique metric compatible, $\bar{\partial}$ *compatible* connection. $\bar{\partial}$ compatibility means we can decompose $\nabla = \nabla^{1,0} + \nabla^{0,1}$, and we require

$$\nabla^{0,1} = \bar{\partial} : \Omega^0(M; E) \to \Omega^1(M; E).$$

   This is known as the **Chern connection.** This section is dedicated to elucidating this condition.

Let us first discuss the holomorphic case. For any vector bundle E, we can extend $\bar{\partial}$ to any E-valued k-form, namely

$$\bar{\partial} : \Omega^k(M; E) \to \Omega^{k+1}(M; E)$$

$$\bar{\partial}(\eta \otimes \sigma) := \bar{\partial}(\eta) \otimes \sigma.$$

To show this definition is legitimate, we must show this is true in any local frame. Therefore, pick a local frame $e_1, ..., e_k$ over U.

4 Hodge Theorem

4.1 Setup

A good refernece for the Riemannian case is Jost's book. This is a complex version of this where we work with $\Omega^{p,q}$ instead of Ω^n.

Choose a local coframe, so for any $\eta \in \Omega^{p,q}$, $\eta = \eta_{I\bar{J}}\phi_I \wedge \bar{\phi}_{\bar{J}}$. In these coordaintes, the Hodge star is given by

$$\star\eta = 2^{p+q-n} \sum \pm \bar{\eta}_{I\bar{J}}\phi_{I^o} \wedge \bar{\phi}_{J^o},$$

where I^o denotes all the indices (looking at the indices as an ordered set) which do not appear in I.

We have the useful formula that the formal adjoint $\bar{\partial}^*$ is given by the formula

$$- \star \bar{\partial} \star = \bar{\partial}^*.$$

The reason we consider Hodge star in the first place is to consider turn integrands of inner products of forms into forms themselves so we can use things like Stokes. See pg 82 bottom for an example.

We now work out a computation in detail of δf where Δ denotes the Hodge Laplacian and $f \in \Omega^{0,0}$. Since $\bar{\partial}^* f = 0$, we have $\Delta f = \bar{\partial}^* \bar{\partial} f$.. Before beginning the computation, let us type check all our steps:

1. $\bar{\partial} f \in \Omega^{0,1}$
2. $\star\bar{\partial} f \in \Omega^{n,n-1}$
3. $\bar{\partial} \star \bar{\partial} f \in \Omega^{n,n}$
4. $\star\bar{\partial} \star \bar{\partial} f \in \Omega^{0,0}$.

This will help guide our local coordinates.

$$\begin{aligned}
\Delta f &= \bar{\partial}^* \bar{\partial} f && (18)\\
&= - \star \bar{\partial} \star \left(\sum_i \frac{\partial f}{\partial \bar{z}^i} d\bar{z}^i \right) && (19)\\
&= - \star \bar{\partial} \left(2^{1-n} \sum_i \pm \overline{\frac{\partial f}{\partial \bar{z}^i}} dz \wedge (d\bar{z}^i)^o \right) && (20)\\
&= - \star \left(2^{1-n} \sum_i \pm \sum_j \frac{\partial}{\partial \bar{z}^j} \overline{\frac{\partial f}{\partial \bar{z}^i}} d\bar{z}^j \wedge dz \wedge (d\bar{z}^i)^o \right) && (21)\\
& && (22)
\end{aligned}$$

At this point, let us stop and remark that actually $i = j$ since otherwise, $j \in i^o$, so $d\bar{z}^j \wedge (d\bar{z}^i)^o = 0$. Furthermore, the signs on the Hodge star are chosen so that all the signs are positive. Continuing,

$$= - * \left(2^{1-n} \sum_i \frac{\partial}{\partial \bar{z}^i} \overline{\frac{\partial f}{\partial \bar{z}^i}} dz \wedge d\bar{z} \right) \tag{23}$$

$$= * \left(2^{1-n} \sum_i - \frac{\partial}{\partial \bar{z}^i} \frac{\partial \bar{f}}{\partial z^i} dz \wedge d\bar{z} \right) \tag{24}$$

$$= 2 \left(- \sum_i \frac{\partial}{\partial z^i} \frac{\partial f}{\partial \bar{z}^i} \right) \tag{25}$$

We now state the Hodge theorem and its consequence the Hodge decomposition. Recall $\mathcal{H}^{p,q} \subset \Omega^{p,q}$ is the space of harmonic forms, i.e. solutions to the Laplace equation.

Theorem 4.1. $\dim \mathcal{H}^{p,q} < \infty$. *As such, we can define $\mathcal{H} : \Omega^{p,q} \to \mathcal{H}^{p,q}$ the orthogonal projection, then there exists unique $G : \Omega^{p,q} \to \Omega^{p,q}$ such that $\mathcal{H} \perp \mathcal{G}$ and $\mathcal{G}$ commutes with $\bar{\partial}$ and $\bar{\partial}^*$. Therefore, we can decompose*

$$id = \mathcal{H} \oplus \Delta \mathcal{G}$$

which implies we get the Hodge decomposition

$$\Omega^{p,q} = \mathcal{H}^{p,q} \oplus \bar{\partial} \Omega^{p,q-1} \oplus \bar{\partial}^* \Omega^{p,q+1}.$$

Thus we essentially need to solve the Laplace equation on a compact manifold. We do the particular case of the torus as an illuminating example.

4.2 Torus

Consider the torus $T := (\mathbb{R}/2\pi\mathbb{Z})^n$ where $\mathbb{R}^n$ is equipped with its usual coordinates. We define the abstract vector space of Fourier series

$$\mathcal{F} := \{u = \sum_{\xi \in \mathbb{Z}^n} u_\xi e^{i\langle \xi, x \rangle}\}$$

with Sobolev norm

$$\|u\|_s^2 := \sum_\xi (1 + \|\xi\|^2)^s |u_\xi|^2.$$

We remark a particularly frequent case will be the $s = 0$ case, where the Sobolev 0-norm is just the ℓ^2 norm

$$\|u\|_0^2 = \sum_\xi |u_\xi|^2.$$

We define the Sobolev space $H_s := \{u : \|u\|_s < \infty\}$ and we note that for $s \leq t$ then

$$H_t \subset H_s$$

so define

$$H_{-\infty} := \bigcup_s H_s$$

$$H_\infty := \bigcap_s H_s.$$

The first section is dedicated to investigating the relation between H_s and $C^s = C^z(T)$. We show $C^s \subset H_s$ and then show a partial converse which is known as the Sobolev lemma.

We begin with the $s = 0$ case. Let $\phi \in C^0(T)$, then it has a Fourier series

$$\phi = \sum_\xi \phi_\xi e^{i\langle \xi, x\rangle},$$

where the coefficient is given by the integral

$$\phi_\xi = \int_T \phi(x) e^{i\langle \xi, x\rangle} dx.$$

The volume form $dx = \frac{dx^1 \wedge \ldots \wedge dx^n}{(2\pi)^n}$, so the volume of the torus is 1. We now show what's known as Parseval's (sic?) identity. We note that by compactness of T and EVT, the first integral is finite

$$\begin{aligned}
\int_T |\phi|^2 &= \int_T \phi\bar{\phi} \\
&= \int_T \left(\sum_\xi \phi_\xi e^{i\langle \xi, x\rangle}\right)\left(\sum_{\xi'} \bar{\phi}_{\xi'} e^{-i\langle \xi', x\rangle}\right) dx \\
&= \int_T \sum_{\xi,\xi'} \phi_\xi \bar{\phi}_{\xi'} e^{i\langle \xi-\xi', x\rangle} dx \\
&= \sum_{\xi,\xi'} \phi_\xi \bar{\phi}_{\xi'} \int_T e^{i\langle \xi-\xi', x\rangle} dx
\end{aligned}$$

where this interchange is justified by... ??? At this point, let us remark that for $\xi = \xi'$, the term obviously reduces to $|\phi_\xi|^2$. For terms $\xi \neq \xi'$ if we parameterize the torus we see that by periodicity of the exponential function, this term evaluates to 0. Continuing,

$$\begin{aligned}
&= \sum_\xi |\phi_\xi|^2 \int_T dx \\
&= \sum_\xi |\phi_\xi|^2 \\
&= \|\phi\|_0^2
\end{aligned}$$

Therefore, we have $C^0 \subset H_0$. For the nonzero case, we begin by a computation.

$$\begin{aligned}
\left(\frac{d\phi}{dx^j}\right)_\xi &= \int_T \frac{d\phi}{dx^j} e^{-i\langle \phi, x\rangle} dx \\
&= -\int_T \phi \frac{d(e^{-\langle \xi, x\rangle})}{dx^j} dx \\
&= -\int_T \phi e^{i\langle \xi, x\rangle} \cdot -i \cdot \xi^j \\
&= i\xi^j \int_T \phi e^{i\langle \xi, x\rangle} dx \\
&= i\xi^j \phi_\xi
\end{aligned}$$

Therefore, if we define $D_j := \frac{1}{i}\frac{d}{dx^j}$ in order to cancel out the i term, then we have for a multi-index α,

$$(D^\alpha \phi)_\xi = \xi^\alpha \phi_\xi.$$

Therefore, as

$$\|D^\alpha \phi\|_0^2 = \sum_\xi |\xi^\alpha \phi_\xi|^2 < \infty$$

by the $s = 0$ case, it remains to show this newly defined norm is equivalent to the standard Sobolev s-norm which will show complete the proof that $C^s \subset H_s$. Let us investigate why

$$\sum_{|\alpha| \le s} |\xi^\alpha|^2 \le (1 + \|\xi\|^2)^s$$

and this will give a clue on how to prove the second inequality. The cases $s = 1, 2$ will be enough to hint at the general case. For $s = 1$, we actually just get an equality

$$1 + \sum_{j=1}^n |\xi^j|^2 \le 1 + \sum_{j=1}^n |\xi^j|^2.$$

where the 1 on the LHS comes from $|\alpha| = 0$, the empty multi-index case. For $s = 2$, and this is the real clue of the general case,

$$\begin{aligned}
1 + \sum_{j=1}^n |\xi^j|^2 + \sum_{j,k=1^n} |\xi^j \xi^k|^2 &\le \left(1 + \sum_{j=1}^n |\xi^j|^2\right)^2 \\
&= \binom{2}{0} 1 + \binom{2}{1}\left(\sum_{j=1}^n |\xi^j|^2\right) + \binom{2}{2}\left(\sum_{j=1}^n |\xi^j|^2\right)^2
\end{aligned}$$

and now simply compare each term on the LHS to the corresponding term on the RHS. Notice that the only difference is the binomial coefficient, and clearly this choice is dependent only on s. Therefore, to get the reverse inequality, just maximize over these coefficients.

Let us now prove the Sobolev lemma.

Theorem 4.2. *$H_{s\lfloor \frac{n}{2}\rfloor + 1} \subset C^s$. Namely, every formal Fourier series $u \in H_{\lfloor \frac{n}{2}\rfloor + 1}$ is the Fourier series of some function $\phi \in C^r(T)$*

Proof. Again induction. For $s = 0$, consider

$$u = \sum_{\xi} u_\xi e^{i\langle \xi, x\rangle}$$

with

$$\|u\|^2_{\lfloor \frac{n}{2} \rfloor + 1} = \sum_{\xi} (1 + \|\xi\|^2)^{\lfloor \frac{n}{2} \rfloor + 1} |u_\xi|^2 < \infty.$$

Consider partial sums

$$S_R := \sum_{\|\xi\| < R} u_\xi e^{i\langle \xi, x\rangle}$$

and for $R \leq R'$ we have

$$\begin{aligned}
|S_R - S_{R'}| &\leq \sum_{\|\xi\| \geq R} |u_\xi| \\
&= \sum_{\xi \geq R} \frac{\left[(1 + \|\xi\|^2)^{\lfloor \frac{n}{2} \rfloor + 1} |u_{\|}\xi\||^2\right]^{1/2}}{\left(1 + \|\xi\|^2)^{\lfloor \frac{n}{2} \rfloor + 1}\right)^{1/2}} \\
&\leq \|u\|_{\lfloor \frac{n}{2} \rfloor + 1} \left(\sum_{\|\xi\| \geq R} \frac{1}{(1 + \|\xi\|^2)^{\lfloor \frac{n}{2} \rfloor + 1}} \right)
\end{aligned}$$

TO FINISH!!! $\square$

We expand the Laplacian in terms of Fourier coefficients. Note that we have concocted up the definition of D such that $D^2 := \sum_i D_i^2 = \Delta_d$. We compute

$$\begin{aligned}
\Delta_d \phi &= \sum_i D_i^2 \phi \\
&= -\sum_i \frac{d^2 \phi}{d(x^i)^2} \\
&= -\sum_i \sum_{\xi} (\xi^i)^2 \phi_\xi e^{i\langle \xi, x\rangle} \\
&= -\sum_{\xi} \sum_i (\xi^i)^2 \phi_\xi e^{i\langle \xi, x\rangle} \\
&= -\sum_{\xi} \|\xi\|^2 \phi_\xi e^{i\langle \xi, x\rangle} \\
&= -\sum_{\xi \neq 0} \|\xi\|^2 \phi_\xi e^{i\langle \xi, x\rangle}
\end{aligned}$$

To motivate the idea of a weak solution, let us think of what it means to solve the Laplacian. A smooth function ϕ is a solution to Laplace equation

$$\Delta_d \phi = \psi$$

iff for every smooth[8] function η, integrating against η yields the same result

$$\int_T \eta \cdot \Delta_d \phi = \int_T \eta \cdot \psi.$$

Of course the reverse direction is the more interesting one, and of course a sufficient check (by linearity!) is to check that $f = 0$ iff

$$\int \eta \cdot f = 0$$

for all smooth functions η. Notice how similar this is to an inner product! In the same way taking an inner product of a vector v in the direction of e_i "tests" the behavior of v in that direction by giving the component, η similarly "tests" f. To make this precise, we would like to have η, f be contained in the same function space.

Anyways, onto the proof of the fact above. We need to make some regularity assumption on f, and in fact the weakest one will work, namely continuity. Continuity alone is enough to guarantee that point-wise (say) positivity translates to local positivity. Namely, if f is positive at a point x, then it is in some neighborhood U_x. But then we can conjure up a bump function supported in U_x. So it must be 0 everywhere.

This proof shows us that for a function space to deserve the name "test function" it's necessary for it to contain bump functions and we would really like it to be a vector space. **I wonder if C_c^∞ is the "smallest" in this sense.**

If we reinterpret integration as a pairing, namely

$$\int_T \eta \cdot \phi = (\eta, \psi),$$

then it makes sense to ask when Δ_d a self-adjoint operator, and it turns out a sufficient condition is asking for smoothness of ϕ. Thus, we are motivated to make the definition of a weak solution. We say $\phi \in H_0(T) = L^2(T)$ is a **weak solution** to the inhomogenous Laplace equation if

$$\int_T \Delta_D \eta \cdot \phi = \int_T \eta \cdot \psi.$$

If we can prove regularity of ϕ, then a weak solution becomes a real solution.

[8] In fact it is true, and really we *should* do this, that we can relax η to be a smooth function with compact support, i.e. a test function. This is really the defining property of a test function.

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