Notes on Schauder Estimates

Paul Tee

The following are the *standard assumptions.* On bounded domains $\Omega \subset \mathbb{R}^n$, will consider 2nd order (uniformly) elliptic operators

$$Lu := \sum_{i,j} a^{ij} u_{ij} + \sum_i b^i u_i + cu = f,$$

where a^{ij}, b^i, c are functions with regularity to be specified later. We assume symmetry on the leading order coefficient terms $a^{ij} = a^{ji}$, and for all uniform ellipticity says $\xi \in \mathbb{R}^n$, for some positive constant $\lambda > 0$,

$$\sum_{i,j} a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2.$$

The regularity of this approach will require our functions u, f to lie in appropriate Hölder spaces. In this note, we will introduce a zoo of seminorms and norms on the way to proving Schauder estimates, each of which will slightly differently exploit the fact that Ω is bounded.

To start, we introduce the standard Hölder spaces and their respective norms and seminorms. This approach will be pointwise in nature, compared to the integral-based approach of Sobolev spaces last semester. Let $\alpha \in (0,1)$. With respect to Ω, we have the pointwise (and global) Hölder seminorms which detect pointwise (and uniform) α-Hölder continuity. We may also define local α-Hölder continuity by asking for every compact $K \subset \Omega$, that $[u]_{\alpha;K}$ is finite. In such a case, we say u is locally uniformly continuous.

1. **(Pointwise)** For $x_o \in \Omega$,
$$[u]_{0,\alpha;x_o} := \sup_{x \in \Omega} \frac{|u(x) - u(x_o)|}{|x - x_o|^\alpha}.$$
2. **(Local)** For all compact $K \subset \Omega$,
$$\sup_{x \neq y \in K} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$$
3. **(Global)**
$$[u]_{0,\alpha;\Omega} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

As usual, global implies local implies pointwise. The above gives rise to $C^{k,\alpha}(\bar{\Omega})$(and $C^{k,\alpha}(\Omega)$) consisting of functions whose k-th order derivatives are uniformly (and locally uniformly) continuous. In the global case, we get a seminorm as defined above. We can formally extend this to $\alpha = 0, 1$ which gives continuity and Lipschitz continuity respectively.

Hölder regularity gives a quantitative measurement of continuity. In fact, greater the α, the more regularity we have.[1] One can see this in the following illustrative example:

Example 0.0.1. *The function $f(x) = |x|^\beta$ is α-Hölder continuous at $x = 0$ iff $\alpha \leq \beta$.*

Proof. Let Ω be a domain containing 0. We compute

$$\sup_{x \neq 0 \in \Omega} \frac{|f(x) - f(0)|}{|x - 0|^\alpha} = \sup_{x \neq 0 \in \Omega} |x|^{\beta - \alpha} = \begin{cases} \text{finite} & \alpha \leq \beta \\ \infty & \alpha > \beta. \end{cases}$$

■

Recall the $C^k(\Omega)$ seminorm is defined by measuring how large the kth derivative of u on Ω is,

$$[u]_{k,0;\Omega} := |D^k u|_{0;\Omega} := \sup_{x \in \Omega, |\beta| = k} |D^\beta u|,$$

where β is a multi-index of length k. Similarly, we have the $C^k(\Omega)$ norm by adding the largest of all the derivatives to order k,

$$|u|_{k,0;\Omega} := \sum_{j=1}^{k} |D^j u|_{0,\Omega}.$$

These definitions admit easy extensions to include α-Hölder regularity, namely we have the *standard (semi)norm* of $C^{k,\alpha}(\bar{\Omega})$.

1. $[u]_{k,\alpha;\Omega} := [D^k u]_{0,\alpha;\Omega} := \sup_{|\beta| = k} [D^\beta u]$.
2. $|u|_{k,\alpha;\Omega} := |u|_{C^k(\bar{\Omega})} + [u]_{k,\alpha;\Omega}$.

One may check that $C^{k,\alpha}(\bar{\Omega})$ for $0 \leq \alpha < 1$ are Banach spaces with the standard norms. One should think of seminorms as taking the "highest degree" portion of the norm.

A short word on notation on our (semi)norms. From here on out, we drop the α from the notation when $\alpha = 0$, but never drop how many derivatives we take.

[1] We briefly mention that for $k = 0$, we have a Rellich-type result for Hölder spaces. Namely for $0 < \alpha < \beta \leq 1$, we have the compact inclusion $C^{0,\beta}(\Omega) \subset C^{0,\alpha}(\Omega)$.

1 The Classical Proof

We unabashedly follow Chapters $2, 4, 6$ of [2].

1.1 Reduction to the Laplacian

Roughly speaking, the Schauder interior estimates tell us that we can bound the $C^{2,\alpha}$ norm of u in terms of the C^0 norm of u and the $C^{0,\alpha}$ norm of f. Explicitly, we have the following theorem.

Theorem 1.1.1. ***(Interior Estimates)***[2] *Given the standard assumptions, if*

$$|a^{ij}|^{(0)}_{0,\alpha;\Omega}, |b^i|^{(1)}_{0,\alpha;\Omega}, |c|^{(2)}_{0,\alpha;\Omega} \leq \Lambda,$$

then the following estimate holds

$$|u|^*_{2,\alpha;\Omega} \leq C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}).$$

We now introduce the relevant norms so the above statement is legible. The following Hölder (semi)norms are weighted by their distances to the boundary. Namely for $x \neq y \in \Omega$, consider

$$d_x := d(x, \partial\Omega) \quad d_{x,y} := \min\{d_x, d_y\}.$$

We use the above two quantities to define the following (semi)norms. For $\alpha \in [0,1]$ and $u \in C^{k,\alpha}(\Omega)$, we first define the *0-homogeneous (semi)norm* of $u \in C^{k,\alpha}(\Omega)$.

1. **0-homogeneous seminorm.** For $\alpha = 0$,

$$[u]^*_{k;\Omega} := \sup_{x\in\Omega, |\beta|=k} d^k_x |D^\beta u(x)|.$$

   For $\alpha > 0$,

$$[u]^*_{k,\alpha;\Omega} := \sup_{x\neq y\in\Omega, |\beta|=k} d^{k+\alpha}_{x,y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

2. **0-homogeneous norm.** For $\alpha = 0$,

$$|u|^*_{k;\Omega} := \sum_{j=0}^{k} [u]^*_{j,\Omega}.$$

   For $\alpha > 0$,

$$|u|^*_{k,\alpha;\Omega} := |u|^*_{k,\Omega} + [u]^*_{k,\alpha;\Omega}.$$

[2] Compare Theorem 1.1.1 to the regularity statement of ([1], pg 329, "Interior H^2-regularity").

Next, we introduce the σ-*homogeneous (semi)norm* of $u \in C^{k,\alpha}(\Omega)$ for $\sigma \in \mathbb{R}$, generalizing the above. The 0-homogeneous (semi)norms can obviously be recovered by setting $\sigma = 0$.

1. **σ-homogeneous seminorm.** For $\alpha = 0$,
$$[u]^{(\sigma)}_{k;\Omega} := \sup_{x\in\Omega, |\beta|=k} d_x^{k+\sigma} |D^\beta u(x)|.$$
For $\alpha > 0$,
$$[u]^{(\sigma)}_{k,\alpha;\Omega} := \sup_{x\neq y\in\Omega, |\beta|=k} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

2. **σ-homogeneous norm.** For $\alpha = 0$,
$$|u|^{(\sigma)}_{k;\Omega} := \sum_{j=0}^{k} [u]^{(\sigma)}_{j,\Omega}.$$
For $\alpha > 0$,
$$|u|^{(\sigma)}_{k,\alpha;\Omega} := |u|^{(\sigma)}_{k,\Omega} + [u]^{(\sigma)}_{k,\alpha;\Omega}.$$

It's worth pointing out that in the $k = 0$ case, the σ-homogeneous (semi)norms agree with the standard ones. We show that the σ-inhomogenous norms enjoy a submultiplicative property.[3]

Lemma 1.1.2. *For $\sigma + \tau \geq 0$, we have*
$$|fg|^{(\sigma+\tau)}_{0,\alpha;\Omega} \leq |f|^{(\sigma)}_{0,\alpha;\Omega} \cdot |g|^{(\tau)}_{0,\alpha;\Omega}.$$

Proof. This is a Baby Rudin proof. Recall
$$|fg|^{(\sigma+\tau)}_{0,\alpha;\Omega} := \underbrace{\sup_{x\neq y\in\Omega} \left(d_{x,y}^{+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x-y|^\alpha} \right)}_{(*)} + \underbrace{\sup_{x\in\Omega} \left(d_x^{\sigma+\tau} |fg(x)| \right)}_{(**)}.$$

We estimate the first term by
$$\begin{aligned}
(*) &= \sup_{x\neq y\in\Omega} \left(d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x-y|^\alpha} \right) \\
&\leq \sup_{x\neq y\in\Omega} \left(d_{x,y}^{\sigma+\alpha+\tau} \cdot d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)|}{|x-y|^\alpha} \right) \\
&\leq \sup_{x\neq y\in\Omega} \left(d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)|}{|x-y|^\alpha} \right) + \sup_{x\neq y\in\Omega} \left(d_{x,y}^{\tau+\alpha+\sigma} \frac{|g(y)| \cdot |f(x) - f(y)|}{|x-y|^\alpha} \right) \\
&\leq |f|^{(\sigma)}_{0;\Omega} \cdot [g]^{(\tau)}_{0,\alpha;\Omega} + |g|^{(\tau)}_{0;\Omega} \cdot [f]^{(\sigma)}_{0,\alpha;\Omega}.
\end{aligned}$$

[3] (6.11) of [2].

We estimate the second term by

$$\begin{aligned}(**) &= \sup_{x\in\Omega}(d_x^{\sigma+\tau}|fg(x)|) \\ &\le \sup_{x\in\Omega}(d_x^{\sigma}|f(x)|)\cdot\sup_{x\in\Omega}(d_x^{\tau}|g(x)|) \\ &= |f|^{(\sigma)}_{0;\Omega}\cdot|g|^{(\tau)}_{0;\Omega}\end{aligned}$$

Combining the two estimates,

$$\begin{aligned}(*)+(**) &= |f|^{(\sigma)}_{0;\Omega}\cdot[g]^{(\tau)}_{0,\alpha;\Omega}+|g|^{(\tau)}_{0;\Omega}\cdot[f]^{(\sigma)}_{0,\alpha;\Omega}+|f|^{(\sigma)}_{0;\Omega}\cdot|g|^{(\tau)}_{0;\Omega} \\ &\le |f|^{(\sigma)}_{0,\alpha;\Omega}\cdot|g|^{(\tau)}_{0,\alpha;\Omega}\end{aligned}$$

■

Throughout the course of the proof, we will frequently employ the interpolation inequalities.[4] Unfortunately, we must omit the proof.

Lemma 1.1.3. *Let $u\in C^{2,\alpha}(\Omega)$, then for any $\varepsilon>0$, there exists a constant $C=C(\varepsilon)$ such that*

$$[u]^*_{j,\beta;\Omega}\le|u|^*_{j,\beta;\Omega}\le C|u|_{0;\Omega}+\varepsilon[u]^*_{2,\alpha;\Omega},$$

where $j=0,1,2$ and $0\le\alpha,\beta\le1$ and $j+\beta<2+\alpha$.

We first reduce showing Schauder estimates from the general case to the case of constant coefficients with no lower order terms.[5]

Proof. (of 1.1.1.) By using a proper, compact exhaustion, it suffices to show the equality holds for every compact subset of Ω. By the interpolation inequalities 1.1.3, it suffices to show the result where the LHS is $[u]^*_{2,\alpha;\Omega}$. Thus, the relevant quantity to consider is

$$d^{2+\alpha}_{x_o,y_o}\frac{|D^2u(x_o)-D^2u(y_o)|}{|x_o-y_o|^\alpha},$$

for arbitrary $x_o,y_o\in\Omega$. The idea here is to "freeze" one variable to reduce to the case of constant coefficients with no lower order terms. Suppose $d_{x_o}=d_{x_o,y_o}$, and consider

$$\sum_{i,j}a^{ij}(x_o)u_{ij}(x)=F(x),$$

where $F:=\sum_{i,j}a^{ij}(x_o)u_{ij}-Lu+f$. Assuming local Schauder estimates on the ball

$$|u|^*_{2,\alpha;B}\le C(|u|_{0;B}+|F|^{(2)}_{0,\alpha;B}),$$

[4] (6.8) and (6.9) of [2].
[5] Theorem 6.2 of [2].

where $B := B_{x_o}(\frac{d}{2})$ where $d := \mu d_{x_o}$ for some $\mu \leq \frac{1}{2}$ we will choose later.

We now perform a near-and-far estimate on the frozen case, and we estimate the "junk" term F in the sequel. For the near estimate, if $y_o \in B$, we have the estimate

$$\left(\frac{d}{2}\right)^{2+\alpha} \frac{|D^2u(x_o) - D^2u(y_o)|}{|x_o - y_o|^\alpha} \leq C(|u|_{0;B} + |F|^{(2)}_{0,\alpha;B}).$$

Since $\left(\frac{d}{2}\right)^{2+\alpha} = \left(\frac{d_{x_o}\mu}{2}\right)^{2+\alpha}$, we enlarge our domain to obtain the estimate

$$d_{x_o}^{2+\alpha} \frac{|D^2u(x_o) - D^2u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|^{(2)}_{0,\alpha;B}\right).$$

For the far estimate, if $y_o \notin B \iff 2|x_o - y_o| \geq d$, and so $(2|x_o - y_o|)^\alpha \geq d^\alpha$. Thus,

$$\begin{aligned}
d_{x_o}^{2+\alpha} \frac{|D^2u(x_o) - D^2u(y_o)|}{|x_o - y_o|^\alpha} &= \left(\frac{d}{\mu}\right)^{2+\alpha} \frac{|D^2u(x_o) - D^2u(y_o)|}{|x_o - y_o|^\alpha} \\
&\leq \frac{d^2 \cdot 2^\alpha}{\mu^{2+\alpha}} |D^2u(x_o) - D^2u(y_o)| \\
&= \frac{(d_{x_o}\mu)^2 \cdot 2^\alpha}{\mu^2 \cdot \mu^\alpha} |D^2u(x_o) - D^2u(y_o)| \\
&\leq \left(\frac{2}{\mu}\right)^\alpha \left(d_{x_o}^2 |D^2u(x_o)| + d_{y_o}^2 |D^2u(y_o)|\right) \\
&\leq \frac{2^{\alpha+1}}{\mu^\alpha} [u]^*_{2;\Omega} \\
&\leq \frac{4}{\mu^\alpha} [u]^*_{2;\Omega}.
\end{aligned}$$

Combining the two estimates, we have

$$d_{x_o}^{2+\alpha} \frac{|D^2u(x_o) - D^2u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|^{(2)}_{0,\alpha;B}\right) + \frac{4}{\mu^\alpha} [u]^*_{2;\Omega}$$

The next step is to estimate the "junk" term $|F|^{(2)}_{0,\alpha;B}$ in terms of $|u|_{0,\Omega}$ and $[u]^*_{2,\alpha;\Omega}$. This is a mildly annoying process, but straightforward conceptually. Specifically, we will estimate the terms on the RHS of

$$|F|^{(2)}_{0,\alpha;B} \leq \sum_{i,j} |(a^{ij}(x_o) - a^{ij}(x))u_{ij}|^{(2)}_{0,\alpha;B} + \sum_i |b^i u_i|^{(2)}_{0,\alpha;B} + |cu|^{(2)}_{0,\alpha;B} + |f|^{(2)}_{0,\alpha;B}.$$

We first provide a generic estimate local-to-global estimate for arbitrary $g \in C^{0,\alpha}(\Omega)$. For $x \in B$, define $d_x^{\partial B} := \operatorname{dist}(x, \partial B)$ and $d_{x,y}^{\partial B} := \min\{d_x^{\partial B}, d_y^{\partial B}\}$. Note that upon localizing to

B, we need to consider the distance to ∂B not to $\partial\Omega$.[6]

$$\begin{aligned}
|g|^{(2)}_{0,\alpha;B} &\le \sup_{x\in B}(d^{\partial B}_x)^2 \cdot \sup_{x\in B}|g(x)| + \sup_{x\neq y\in B}(d^{\partial B}_{x,y})^{2+\alpha}\cdot \sup_{x\neq y\in B}\frac{|g(x)-g(y)|}{|x-y|^\alpha}\\
&= d^2|g|_{0;B} + d^{2+\alpha}[g]_{\alpha;B}\\
&\le |g|^{(2)}_{0;\Omega} + [g]^{(2)}_{\alpha;\Omega}\\
&\le \left(\frac{\mu}{1-\mu}\right)^2 |g|^{(2)}_{0;\Omega} + \left(\frac{\mu}{1-\mu}\right)^{2+\alpha}[g]^{(2)}_{\alpha;\Omega}\\
&\le 4\mu^2|g|^{(2)}_{0;\Omega} + 8\mu^{2+\alpha}[g]^{(2)}_{0,\alpha;\Omega}
\end{aligned}$$

The third line follows since for any $x \in B$ which comes from the computation

$$d = \mu d_{x_o} \le (1-\mu)d_{x_o} \le d_x.$$

Replacing d with d_x is nondecreasing on B, so the result follows. The fifth line follows from $\mu \le \frac{1}{2} \implies \left(\frac{1}{1-\mu}\right)^2 \le 4$ and $\left(\frac{1}{1-\mu}\right)^{2+\alpha} \le 8$.

We now apply estimates in detail to the principal term; the lower order terms are treated similarly. By Lemma 1.1.2, we compute

$$\begin{aligned}
|(a^{ij}(x_o)-a^{ij})u_{ij}|^{(2)}_{0,\alpha;B} &\le |a^{ij}(x_o)-a^{ij}|^{(0)}_{0.\alpha;B}\cdot |u_{ij}|^{(2)}_{0,\alpha;B}\\
&\le \underbrace{|a^{ij}(x_o)-a^{ij}|^{(0)}_{0.\alpha;B}}_{(*)}\cdot(4\mu^2[u]^*_{2;\Omega} + 8\mu^{2+\alpha}[u]^*_{2,\alpha;\Omega}).
\end{aligned}$$

We also estimate the local term $(*)$ by

$$\begin{aligned}
(*) &= \sup_{x\in B}\left(|a^{ij}(x_o)-a^{ij}(x)|\right) + \sup_{x\neq y\in B}\left((d^{\partial B}_{x,y})^\alpha\frac{|a^{ij}(x_o)-a^{ij}(x)-(a^{ij}(x_o)-a^{ij}(y))|}{|x-y|^\alpha}\right)\\
&\le \sup_{x\neq x_o\in B}\left(\frac{|a^{ij}(x_o)-a^{ij}(x)|}{|x-x_o|^\alpha}\cdot|x-x_o|^\alpha\right) + d^\alpha[a^{ij}]_{0,\alpha;B}\\
&\le 2d^\alpha[a^{ij}]_{0,\alpha;B}\\
&\le 2(2\mu)^\alpha[a^{ij}]^*_{0,\alpha;\Omega}\\
&\le 4\Lambda\mu^\alpha,
\end{aligned}$$

where the second to last inequality follows since $d \le d_x$ for all $x \in B$, and $\mu \le \frac{1}{2}$. Altogether, we find

$$\begin{aligned}
\sum_{i,j}|a^{ij}(x_o)-a^{ij}|^{(2)}_{0,\alpha;B} &\le 4n^2\Lambda\mu^\alpha(4\mu^2[u]^*_{2;\Omega} + 8\mu^{2+\alpha}[u]^*_{2,\alpha;\Omega})\\
&\le 32n^2\Lambda\mu^{2+\alpha}([u]^*_{2;\Omega} + \mu^\alpha[u]^*_{2,\alpha;\Omega})\\
&\le 32n^2\Lambda\mu^{2+\alpha}(C(\mu)|\mu|_{0;\Omega} + 2\mu^\alpha[u]^*_{2,\alpha;\Omega}),
\end{aligned}$$

[6]Caution, [2] uses d_x for our $d^{\partial B}_x$, which may be a cause of confusion.

where the last step follows from the interpolation inequalities 1.1.3. Similarly, we have the estimates

$$\sum_i |b^i u_i|^{(2)}_{0,\alpha;B} \leq 8n\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]^*_{2,\alpha;\Omega}),$$

$$|cu|^{(2)}_{0,\alpha;B} \leq 8\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]^*_{2,\alpha;\Omega}),$$

$$|f|^{(2)}_{0,\alpha;\Omega} \leq 8\mu^2|f|^{(2)}_{0,\alpha;\Omega}.$$

Altogether, we estimate F by

$$|F|^{(2)}_{0,\alpha;B} \leq C\mu^{2+2\alpha}[u]^*_{2,\alpha;\Omega} + C(\mu)(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}).$$

Combining the two steps and using the interpolation inequalities 1.1.3, we get

$$[u]^*_{2,\alpha;\Omega} \leq C\mu^\alpha[u]^*_{2,\alpha;\Omega} + C(\mu)(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}).$$

Fixing μ so that $C\mu^\alpha \leq \frac{1}{2}$ yields the desired result. ■

We quickly sketch a proof of Schauder estimates for the case of constant coefficients with no lower order terms.[7] Consider

$$L_0 u := \sum_{i,j} A^{ij} u_{ij},$$

where $A := (A^{ij}) \in \mathrm{GL}_n(\mathbb{R})$ is a constant matrix. In this case, we diagonalize A by an orthogonal matrix[8] to a diagonal one, turning $L_0 u(x) = f(x)$ to $\Delta\tilde{u}(y) = \tilde{f}(y)$. Since our transformations are all linear, the norms are equivalent, and so it suffices to prove Schauder estimates for the Laplacian.

1.2 The Laplacian

We first introduce the *fundamental solution* and some basic ideas surrounding it. The upshot is that we obtain some estimates for future use, and the following *Green's representation formula*,[9] to prove the following theorem.

Theorem 1.2.1. *Let $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$ such that $\Delta u = f$. Then,*

$$u = h + w,$$

where h is some harmonic function, and w is the Newtonian potential of f.

[7] Lemma 6.1 of [2].
[8] This follows by spectral theorem, since A is a symmetric, real-valued, nonsingular matrix.
[9] (2.16) of [2].

Fix $x \in \Omega \subset \mathbb{R}^n$ for $n \geq 2$.[10] Let ω_n be the volume of the n-ball. Define the fundamental solution on $\Omega - x$ as the following radial function about x:

$$\Gamma(x-y) := \begin{cases} \frac{1}{n(2-n)\omega_n}|x-y|^{2-n} & n > 2 \\ \frac{1}{2\pi}\log|x-y| & n = 2. \end{cases}$$

We first obtain some basic estimates on Γ. The following derivatives will be taken with respect to the variable x. For convenience, define

$$S(x) := \sum_{k=1}^{n}(x_k - y_k)^2,$$

so that $|x-y| = S^{\frac{1}{2}}$, and $S_i = 2(x_i - y_i)$. We calculate for $n > 2$,

$$\begin{aligned}\Gamma_i(x-y) &= \frac{1}{n(2-n)\omega_n}\left(\frac{2-n}{2}S^{\frac{2-n}{2}-1}\cdot 2(x_i-y_i)\right) \\ &= \frac{1}{n\omega_n}|x-y|^{-n}(x_i-y_i).\end{aligned}$$

We calculate for $n = 2$,

$$\begin{aligned}\Gamma_i(x-y) &= \frac{1}{2\pi}\frac{\frac{1}{2}S^{-\frac{1}{2}}\cdot S_i}{S^{\frac{1}{2}}} \\ &= \frac{1}{2\pi}S^{-1}(x_i-y_i) \\ &= \frac{1}{2\pi}|x-y|^{-2}(x_i-y_i).\end{aligned}$$

Continuing, we calculate the second derivatives for $n \geq 2$,

$$\begin{aligned}\Gamma_{ij}(x-y) &= \frac{1}{n\omega_n}\left(\delta_{ij}S^{-\frac{n}{2}} + (x_i-y_i)\left(\frac{-n}{2}S^{-\frac{n}{2}-1}\cdot 2(x_j-y_j)\right)\right) \\ &= \frac{S^{-\frac{n}{2}-1}}{n\omega_n}\left(\delta_{ij}S^1 - n(x_i-y_i)(x_j-y_j)\right) \\ &= \frac{|x-y|^{-n-2}}{n\omega_n}\left(\delta_{ij}|x-y|^2 - n(x_i-y_i)(x_j-y_j)\right).\end{aligned}$$

Γ is indeed harmonic, since

$$\Delta\Gamma = \sum_i \Gamma_{ii} = \frac{|x-y|^{-n-2}}{n\omega_n}\sum_{i=1}^{n}(\delta_{ii}|x-y|^2 - n(x_i-y_i)^2) = 0.$$

For near future use, we record estimates on the fundamental solutions.

[10] Careful! [2] uses "y" to denote the singularity in Chapter 2, but switches to "x" to denote the singularity in Chapter 4.

Proposition 1.2.2. *Let $\rho = |x-y|$ and for any multi-index β, we estimate*

$$|D^\beta \Gamma(\rho)| \le C\rho^{2-n-|\beta|},$$

where $C = C(n, |\beta|)$. In fact, we calculate directly

$$\Gamma'(\rho) = \frac{1}{n\omega_n}\rho^{1-n}.$$

In Cartesian coordinates, we make the constant explicit to get

$$|\Gamma_i(x-y)| \le \frac{1}{n\omega_n}|x-y|^{1-n},$$

$$|\Gamma_{ij}(x-y)| \le \frac{1}{\omega_n}|x-y|^{-n}.$$

Next, we recall *Green's identities.* First, recall for $g \in C^\infty(\Omega)$ and $X \in \mathfrak{X}(\Omega)$,

$$\operatorname{div}(gX) = \nabla g \cdot X + g \operatorname{div} X.$$

In the case $X = \nabla u$ for $u \in C^2(\Omega)$, we get

$$\operatorname{div}(g\nabla u) = \nabla g \cdot \nabla u + g\,\Delta u.$$

Integrating and using *divergence theorem* yields *Green's first identity*

$$\int_\Omega \nabla g \cdot \nabla u + \int_\Omega g\Delta u = \int_{\partial\Omega} (g\nabla u)\cdot \nu = \int_{\partial\Omega} g\frac{\partial u}{\partial \nu}.$$

Switching the roles of g, u and subtracting yields *Green's second identity*

$$\int_\Omega g\Delta u - u\Delta g = \int_{\partial\Omega} g\frac{\partial u}{\partial\nu} - u\frac{\partial g}{\partial \nu}.$$

We are in a position to give a proof of Theorem 1.2.1. In general, we call

$$w(x) := \Gamma * f(x) = \int_\Omega \Gamma(x-y) f(y)dy$$

the Newtonian potential of f.

Proof. (of 1.2.1.) We would like to use Green's second identity for $g = \Gamma$. But due to the singularity at x, we must use it instead for $\Omega - B_x(\rho)$, for small ρ. We compute

$$\int_{\Omega - B_x(\rho)} \Gamma\Delta u = \int_{\partial\Omega} \Gamma\frac{\partial u}{\partial \nu} - u\frac{\partial\Gamma}{\partial\nu} + \underbrace{\int_{\partial B_x(\rho)} \Gamma\frac{\partial u}{\partial\nu} - u\frac{\partial \Gamma}{\partial\nu}}_{(*)}.$$

We show $(*) \to u(x)$ as $\rho \to 0$. Keeping in mind that Γ is radial, we proceed with estimating the first term of $(*)$,

$$\begin{aligned}\int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} &= \Gamma(\rho) \cdot \int_{\partial B_x(\rho)} \frac{\partial u}{\partial \nu} \\ &\leq C(n)\rho^{2-n} \cdot n\omega_n \rho^{n-1} \sup_{\partial B_x(\rho)} |Du|.\end{aligned}$$

Since $u \in C^2(\Omega)$, the exponents work out so that the last quantity vanishes as $\rho \to 0$.[11] For the second term, we calculate by 1.2.2,[12]

$$\begin{aligned}-\int_{\partial B_x(\rho)} u \frac{\partial \Gamma}{\partial \nu} &= \Gamma'(\rho) \int_{\partial B_x(\rho)} u \\ &= \frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_x(\rho)} u.\end{aligned}$$

Since u is continuous, the last quantity goes to $u(x)$ as $\rho \to 0$. Altogether, sending $\rho \to 0$ yields Green's representation formula[13]

$$u(x) = \underbrace{\int_{\partial\Omega} u(y)\frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu}(y)\, ds}_{h(x)} + \underbrace{\int_{\Omega} \Gamma(x-y)\Delta u(x)\, dx}_{w(x)}.$$

We argue h is harmonic. Since $\partial\Omega$ is compact, we can commute ∂_i past the integral. By IBP we can put both ∂_i derivatives on Γ, which is harmonic; by Clairaut's theorem, we can put ∂_i past any ∂_ν terms. IBP leaves no boundary term since $\partial(\partial\Omega) = \partial^2\Omega = 0$. ■

We give explicit formulas for the first two derivatives of the Newtonian potential.[14]

Proposition 1.2.3. *Suppose f is bounded, integrable in Ω with Newtonian potential w. Then $w \in C^1(\mathbb{R}^n)$ and its derivative given by the formula*

$$w_i(x) = \int_\Omega \Gamma_i(x-y)f(y)dy.$$

Proof. Set $v(x) := \int_\Omega \Gamma_i(x-y)f(y)dy$. By the asymptotic estimates of 1.2.2 and the fact that f is bounded, we find that v is well-defined. We consider the sequence $w^\varepsilon := (\Gamma\eta^\varepsilon) * f$, where $\eta^\varepsilon = \eta\left(\frac{|x-y|}{\varepsilon}\right)$. Here, $\eta \in C^1(\mathbb{R})$ is a function such that

$$\eta(t) = \begin{cases} 1 & t \geq 2 \\ 0 & t \leq 1, \end{cases} \quad \text{and} \quad 0 \leq \eta' \leq 2.$$

[11] At least for $n > 2$. But the $n = 2$ case follows similarly.

[12] Note since ν is outward pointing on $\Omega - B$, then ν into B. However, ρ is defined so that it is positive as you go away from x. Therefore, the sign changes.

[13] Observe for compactly supported u, the second term vanishes, while for harmonic u, the first term vanishes.

[14] Lemmas 4.1 and 4.2 of [2].

Therefore, $w^\varepsilon(x)$ has the effect of doing nothing far from the singularity of Γ (at distances $|x-y| \geq 2\varepsilon$), and killing off the function near the singularity of Γ (at distances $|x-y| \leq \varepsilon$). Therefore, it is clear that $w^\varepsilon(x)$ converges pointwise to $w(x)$. Let $K \subset \Omega$ be compact, and for any $x \in K$ we compute for $n > 2$,[15]

$$\begin{aligned}
|v(x) - w_i^\varepsilon(x)| &= \left| \int_\Omega \frac{\partial}{\partial x^i} \Gamma(x-y) f(y) dy - \frac{\partial}{\partial x^i} \int_\Omega \Gamma(x-y) \eta^\varepsilon(x,y) f(y) dy \right| \\
&= \left| \int_{B_e(2\varepsilon)} \left(\frac{\partial \Gamma(x-y)}{\partial x^i} - \frac{\partial \Gamma(x-y) \cdot \eta^\varepsilon}{\partial x^i} \right) f(y) dy \right| \\
&\leq \sup_{B_x(2\varepsilon)} |f| \int_{B_e(2\varepsilon)} |D\Gamma| + \left(|D\Gamma| + \frac{2}{\varepsilon^n} \cdot \varepsilon^{n-1} |\Gamma| \right) dy \\
&\leq \frac{\sup |f|}{n\omega_n} \int_{B_x(2\varepsilon)} 2|x-y|^{1-n} + \frac{2}{\varepsilon(2-n)} |x-y|^{2-n} dy \\
&= \frac{\sup |f|}{n\omega_n} \left(\frac{2n\omega_n (2\varepsilon)^{n-1}}{(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{1-n} (\rho^{n-1} d\rho) + \frac{2n\omega_n (2\varepsilon)^{n-1}}{\varepsilon(2-n)(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{2-n} (\rho^{n-1} d\rho) \right) \\
&= \sup |f| \left(2\varepsilon + \frac{2}{\varepsilon(n-2)} \cdot \frac{(2\varepsilon)^2}{2} \right) \\
&= \sup |f| \frac{2n\varepsilon}{n-2}.
\end{aligned}$$

The third to last equality is from co-area formula, along with change of variables. Since the upper bound is independent of x, the uniformity follows. For $n = 2$, we leave the proof to the reader. ∎

Proposition 1.2.4. *Suppose f is bounded and locally Hölder continuous (with exponent $\alpha \leq 1$) in Ω and let w be the Newtonian potential of f. Then w is $C^2(\Omega)$, $\Delta w = f$ in Ω and for any $x \in \Omega$,*

$$w_{ij}(x) = \int_{\Omega_0} \Gamma_{ij}(x-y)(f(y) - f(x)) dy - f(x) \int_{\partial \Omega_0} \Gamma_i(x-y) \nu_j(y) dz_y.$$

Here Ω_0 is domain containing Ω where the divergence theorem holds, and f is extended ot vanish outside Ω.

Proof. Similar to above. Come back to this if there's time. Proof techniques combine those in Theorems 1.2.3 and 1.2.5. ∎

We provide the fundamental estimate for the Schauder estimate for the Laplacian.[16]

Lemma 1.2.5. *Let $B_1 := B_R(x_o)$ and $B_2 := B_{2R}(x_o)$ be concentric balls in $\mathbb{R}^n$. If $f \in C^{0,\alpha}(\bar{B}_2)$ with w the Newtonian potential of f in B_2, then $w \in C^{2,\alpha}(\bar{B}_1)$. Furthermore,*

$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

[15] Check this computation...

[16] Lemma 4.4 of [2].

Proof. As a preliminary, we remark that $|B_2| = \omega_n(2R)^n$ and $|\partial B_2| = n\omega_n(2R)^{n-1}$. For $x \in B_1$, we obtain from 1.2.4 and 1.2.2, the first estimate

$$\begin{aligned}
|w_{ij}(x)| &= |f(x)| \int_{\partial B_2} |\Gamma_i(x-y)\nu_j(y)| ds_y + \int_{B_2} |\Gamma_{ij}(x-y)(f(y)-f(x))| dy \\
&\leq \frac{|f(x)|}{n\omega_n} \int_{\partial B_2} \underbrace{|x-y|^{1-n}}_{(*)} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \int_{B_2} |x-y|^{\alpha-n} dy \\
&\leq \frac{|f(x)|}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \underbrace{\int_{B_{3R}(x)} |x-y|^{\alpha-n} dy}_{(**)} \\
&\leq 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha [f]_{x;\alpha}, \\
&\leq C_1(|f(x)| + R^\alpha [f]_{\alpha;x}),
\end{aligned}$$

where $C_1 = C_1(n, \alpha)$. Here, $(*)$ follows since $R \leq |x-y|$, and so $|x-y|^{1-n} \leq R^{1-n}$ since $n \geq 2$. We calculate $(**)$ as

$$\begin{aligned}
(**) &= \frac{n\omega_n(3R)^{n-1}}{(3R)^{n-1}} \int_0^{3R} \rho^{\alpha-n}(\rho^{n-1} d\rho) \\
&= (n\omega_n) \frac{\rho^\alpha}{\alpha} \Big|_0^{3R} \\
&= \frac{n\omega_n}{\alpha}(3R)^\alpha.
\end{aligned}$$

Notice the idea here to shift the center of the ball from x_o to x; in doing so, we must enlarge the radius to $3R$ so that we capture B_2. The first estimate gives

$$|D^2 w|_{0;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

For our second estimate, for $x, \bar{x} \in B_1$, $\xi = \frac{1}{2}(x + \bar{x})$ and $\delta = |x - \bar{x}|$,

$$
\begin{aligned}
w_{ij}(\bar{x}) - w_{ij}(x) &= -f(\bar{x}) \int_{\partial B_2} \Gamma_i(\bar{x} - y)\nu_j(y) ds_y + f(x) \int_{\partial B_2} \Gamma_i(x - y)\nu_j(y) ds_y \\
&\quad + \int_{B_2} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy - \int_{B_2} \Gamma_{ij}(x - y)(f(y) - f(x)) dy \\
&= f(x) \underbrace{\int_{\partial B_2} \Gamma_i(x - y) - \Gamma_i(\bar{x} - y)\nu_j(y) ds_y}_{I_1} + (f(x) - f(\bar{x})) \underbrace{\int_{\partial B_2} \Gamma_i(\bar{x} - y)\nu_j(y) ds_y}_{I_2} \\
&\quad + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(x - y)(f(x) - f(y)) dy}_{I_3} + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy}_{I_4} \\
&\quad + (f(x) - f(\bar{x})) \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) dy}_{I_5} \\
&\quad + \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) - \Gamma_{ij}(\bar{x} - y)(f(\bar{x}) - f(y)) dy}_{I_6},
\end{aligned}
$$

where the second equality follows from (detailed) inspection. We estimate $I_1, ..., I_6$.[17]

1. We use mean value inequality for some $\hat{x}$ between $x, \bar{x}$, then follow with the same trick as the first term in the first estimate to compute

$$
\begin{aligned}
|I_1| &\leq |x - \bar{x}| \int_{\partial B_2} |D\Gamma_i(\hat{x} - y)| ds_y \\
&\leq |x - \bar{x}| \int_{\partial B_2} |\hat{x} - y|^{-n} ds_y \\
&\leq \frac{|x - \bar{x}|}{\omega_n} \int_{\partial B_2} R^{-n} ds_y \\
&= \frac{|x - \bar{x}| n 2^{n-1}}{R} \\
&\leq n 2^{n-\alpha} \left(\frac{\delta}{R}\right)^\alpha,
\end{aligned}
$$

where the last estimate follows by simple computation from $\delta = |x - \bar{x}| < 2R$.[18]

2. The estimate follows the ideas when estimating the first term of $w_{ij}(x)$. We record our estimates for ease of replication in I_5. Note that the final answer is completely

[17] My labelling of the six integrals follows that in [2].

[18] There is an extra factor of n in [2], which I don't see where it comes from. Similarly, this extra factor appears when estimating $|I_6|$.

independent of the ball.

$$\begin{aligned}|I_2| &\le \frac{1}{n\omega_n}\int_{\partial B_2} |x-y|^{1-n} ds_y \\ &\le \frac{1}{n\omega_n}\int_{\partial B_2} R^{1-n} ds_y = 2^{n-1}.\end{aligned}$$

3. $|I_3| \le \frac{n}{\alpha}\left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;x}$. The estimate follows the ideas when estimating the second term of $w_{ij}(x)$ with $B_\delta(x)$ in the place of B_2 and $B_{\frac{3\delta}{2}}$ in the place of $B_{3R}(x)$.

4. $|I_4| \le \frac{n}{\alpha}\left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha;\bar{x}}$. Ditto, but swap x for $\bar{x}$.

5. Since the estimate in I_2 is independent of the ball, the desired estimate follows immediately.

$$\begin{aligned}|I_5| &\le \left|\int_{\partial B_2} \Gamma_i(x-y)\nu_j(y) ds_y\right| + \left|\int_{\partial B_\delta(\xi)} \Gamma_i(x-y)\nu_j(y) ds_y\right| \\ &\le 2(2^{n-1}) = 2^n\end{aligned}$$

6. Using ideas as before, we calculate for $C = C(n)$ and $\hat{x}$ some point between $\bar{x}$ and x,

$$\begin{aligned}|I_6| &\le |x-\bar{x}|\int_{B_2 - B_\delta(\xi)} |D\Gamma_{ij}(\hat{x}-y)||f(\bar{x})-f(y)|dy \\ &\le C\delta\int_{B_2-B_\delta(\xi)} \frac{|f(\bar{x})-f(y)|}{|\hat{x}-y|^{n+1}} dy \\ &\le C\delta[f]_{\alpha,x}\left(\frac{3}{2}\right)^\alpha \int_{B_\delta(\xi)} |\xi-y|^{\alpha-n-1}dy \\ &\le \frac{C}{1-\alpha}\delta^\alpha[f]_{\alpha,x}\left(\frac{3}{2}\right)^\alpha\end{aligned}$$

Altogether, the second estimate gives

$$R^\alpha[D^2w]_{\alpha;B_1} \le C(|f|_{0;B_2} + R^\alpha[f]_{\alpha;B_2}).$$

■

Lemma 1.2.5 proves a Schauder estimate for concentric balls. We introduce our final norm. Let $\eta =$ diameter of Ω, then

$$|u|'_{k,\alpha;\Omega} := \sum_{j=0}^{k} \sup_{x\in\Omega} \eta^j |D^j u(x)| + \sup_{x\neq y\in\Omega} \eta^{k+\alpha}\frac{|D^k u(x) - D^k u(y)|}{|x-y|^\alpha}.$$

Again, observe in the $k=0$ case, this norm agree with the standard one. We now state the fundamental estimate to show Schauder estimates for the Laplacian.[19]

[19] Theorem 4.6 of [2].

Lemma 1.2.6. *Let $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$ satisfy $\Delta u = f$. Then $u \in C^{2,\alpha}(\Omega)$ and for any two concentric balls $B_1 := B_R(x_o)$, $B_2 := B_{2R}(x_o)$ compactly contained in Ω,*

$$|u|'_{2,\alpha;B_1} \le C(|u|_{0;B_2} + |f|'_{0,\alpha;B_2}).$$

Proof. By Theorem 1.2.1, we may write

$$u = v + w,$$

for harmonic h and w is the Newtonian potential of f. The estimates 1.2.3 and 1.2.5 together give the result. ■

Finally, we have arrived at the promised land.[20]

Theorem 1.2.7. *For $u \in C^2(\Omega)$ and $f \in C^{0,\alpha}(\Omega)$, with $\Delta u = f$,*

$$|u|^*_{2,\alpha;\Omega} \le C(|u|_{0,\Omega} + |f|^{(2)}_{0,\alpha;\Omega}).$$

Proof. We first obtain an estimate on the $|u|^*_{2,\Omega}$ by the RHS. Since $|u|_{0;\Omega}$ appears on the RHS, we only need to worry about bounding $[u]^*_{j;\Omega}$ for $j = 1, 2$v. We compute for $3R = d_x$ and $B_1 := B_R(x)$ and $B_2 := B_{2R}(x)$,

$$\begin{aligned} d_x|Du(x)| + d_x^2|D^2u(x)| &\le (3R)|Du|_{0;B_1} + (3R)^2|D^2u|_{0;B_1} \\ &\le C(|u|_{0;B_2} + R^2|f|'_{0,\alpha;B_2}) \\ &\le C(|u|_{0;\Omega} + R^2|f|_{0,\alpha;B_2}) \\ &\le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}), \end{aligned}$$

where the second inequality follows from Lemma 1.2.6. Therefore, we have

$$|u|^*_{2;\Omega} \le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}). \tag{$*$}$$

Next, we obtain an estimate on $[u]^*_{2,\alpha;\Omega}$ by the RHS. We compute for $x \neq y \in \Omega$, with $d_x = d_{x,y}$.

$$\begin{aligned} d_x^{2+\alpha}\frac{|D^2u(x) - D^2u(y)|}{|x-y|^\alpha} &\le (3R)^{2+\alpha}[D^2u]_{0,\alpha;B_1} + 3^{\alpha+1}(3R)^2 \sup_{x\in\Omega}|D^2u(x)| \\ &\le C(|u|_{0;B_2} + R^2|f|'_{0;\alpha;B_2}) + 6[u]^*_{2;\Omega} \\ &\le C(|u|_{0;\Omega} + |f|^{(2)}_{0,\alpha;\Omega}) \end{aligned}$$

where the first inequality follows from a near-and-far estimate on B_1, as in the proof of 1.1.1. The second inequality follows from Lemma 1.2.6. The last inequality follows from ($*$). ■

[20]Theorem 4.8 of [2].

References

[1] L. Evans. *Partial differential equations*. 2nd ed. Providence, R.I.: American Mathematical Society, 2010.

[2] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer Berlin Heidelberg, 2001.