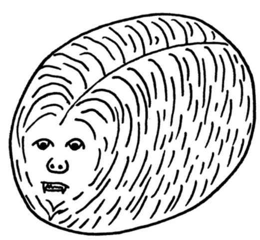
Notes on Characteristic Classes

2022-2023

Standard reference we follow is Milnor-Stasheff [16]. Hatcher also has a book on vector bundles and K-theory [14]. For lecture notes, see Debray [9] and [8], Guillou [12], and Quick [17].

1 Chapter 1

2 Chapter 2



Here's an analogy. Consider a rank k vector bundle $E \to B$.

1. Base Spaces : Domains of functions.
2. Sections : Functions to $\mathbb{R}^k$.
3. Total Space : Domain $\times$ Codomain where the graphs of functions live.

This analogy becomes on-the-nose if we consider the trivial bundle. In general, this analogy is locally true by local trivializations. Globally, we may have some nontrivial topology on the total space. Therefore we have the slogan:

Sections are to functions where manifolds are to $\mathbb{R}^n$.

I think this argument also shows that all invariants of bundles must be global, which is why (co)homology comes in.

Let us discuss trivialities of various classical bundles.

Theorem 2.1. *The canonical line bundle $E(\gamma_n^1) \to \mathbb{P}^n$ is nontrivial.*

Proof. We argue that this is nontrivial by showing every section has a nontrivial zero. This is essentially the analog of hairy ball theorem. Let $s : \mathbb{P}^n \to E(\gamma_n^1)$ denote a section, and consider

$$\mathbb{S}^n \to \mathbb{P}^n \xrightarrow{s} E(\gamma_n^1),$$

$$x \mapsto [x] \overset{s}{\mapsto} \{([x], t_x \cdot x) \in E(\gamma_n^1)\}.$$

Thus the map $x \mapsto t_x$ defines a real-valued function from the sphere. Following the antipodal point under the composition, we compute

$$-x \mapsto [x] \overset{s}{\mapsto} ([x], t_{-x} \cdot -x).$$

Since $x, -x \in \mathbb{S}^n$ both map to the same value under the first map, they must agree under the second map also. Setting the second components equal, we have $t_x \cdot x = t_{-x} \cdot -x$, for all $x \in \mathbb{S}^n$, and so

$$t_{-x} = -t_x.$$

By extreme value theorem, t must take on some zero; therefore so must s. $\square$

As a remark, for $n = 1$ note that this is neither the tangent nor the normal bundle of $\mathbb{P}^1 \cong S^1$. This is the Mobius bundle.

In chapter 4, we'll see the Stiefel–Whitney class of the canonical line bundle is $w(\gamma_n^1) = 1 + a$ for nonzero $a \in H^1(\mathbb{P}^n, \mathbb{Z}/2)$ which gives an alternative proof of this fact, since trivial bundles ϵ have Stiefel–Whitney class $w(\epsilon) = 1$. However, the analogous question with the tangent bundle of a sphere or projective space is classical. We have the following theorem:

Theorem 2.2. *For $X^n \in \{\mathbb{S}^n, \mathbb{P}^n\}$, we have $TX^n \to X^n$ is trivial iff $n = 0, 1, 3, 7$.*

Some remarks:

1. Hairy Ball Theorem in [13], Theorem 2.28, tells us even-dimensional spheres have nontrivial tangent bundles. This is given by the following no-go theorem:

   Proposition 2.3. *$\mathbb{S}^n$ has a continuous vector field iff n is odd.*

   Proof. (essentially) **Problem 2-B**. □

2. On odd-dimensional spheres, Adams [3] classified the maximum number of linearly independent vector fields $\mathbb{S}^{n-1}$ can carry.

   Theorem 2.4. *The sphere $\mathbb{S}^{n-1}$ supports at most $2^c + 8d - 1$ linearly independent vector fields, where $n = (2a+1)2^b$, and $b = c + 4d$ for $0 \leq c \leq 3$.*

   The proof introduces and uses cohomology operations in K-theory.

3. A sufficient result for triviality of the tangent bundle is to classify which X^n are Lie groups. This follows by pushing forward a basis at $\mathfrak{g}$ by left multiplication. Note that the following results misses out X^7. However, in Chapter 4 we'll see that Stiefel–Whitney classes will catch this remaining case.

   Proposition 2.5. *$\mathbb{S}^n$ is a Lie group iff $n = 0, 1, 3$.*

   Proof. We have multiplication structures given by *structure de transport* under the following diffeomorphisms:

   (a) $\mathbb{S}^0 \cong \mathbb{Z}/2$,

   (b) $\mathbb{S}^1 \cong \mathrm{SO}(2)$,

   (c) $\mathbb{S}^3 \cong \mathrm{SU}(2) \cong \mathrm{Spin}(3)$.

   We show these are the only Lie groups combining ideas from [10] and [7], Chapter 12. Since $G = \mathbb{S}^n$ is connected for $n \geq 1$, we may assume G is a connected Lie group. The idea is the define a bi-invariant form ($\Longrightarrow$ closed) using the Lie bracket and the metric which will give a nonzero cohomology class in H^3. Recall the Koszul formula: for vector fields $X, Y, Z \in \mathfrak{X}(G)$ we have

$$\begin{aligned} 2\langle \nabla_Y X, Z\rangle = X\langle Y, Z\rangle - Z\langle X, Y\rangle + Y\langle Z, X\rangle \\ - \langle [X, Y], Z\rangle + \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle. \end{aligned} \tag{2}$$

If we choose a bi-invariant metric[1] and take left-invariant vector fields $X, Y, Z \in \mathfrak{g}$ left-invariant vector fields then this formula simplifies dramatically. By left-invariance of the vector fields and the metric, the first three term of Equation 2 vanishes. Namely, for every $g \in G$,

$$\langle Y_g, Z_g\rangle = \langle (L_g)_* Y_e, (L_g)_* Z_e\rangle = \langle Y_e, Z_e\rangle.$$

The last two terms of Equation 2 also vanish by the following argument:

$$\begin{aligned} 0 &= Z\langle X, Y\rangle \\ &= \langle \mathcal{L}_Z X, Y\rangle + \langle X, \mathcal{L}_Z Y\rangle \\ &= \langle [Z, X], Y\rangle + \langle X, [Z, Y]\rangle \\ &= \langle [Z, X], Y\rangle - \langle [Y, Z], X\rangle, \end{aligned}$$

and in the second equality I used the fact that the Lie derivative of a right-invariant metric is zero. Namely, if we let g denote the metric,

$$\mathcal{L}_Z g = \left.\frac{d}{dt}\right|_{t=0} R^*_{\exp(tZ)} g = \left.\frac{d}{dt}\right|_{t=0} g = 0,$$

where right-invariance of the metric is used on the second equality. Therefore, for left-invariant vector fields $X, Y \in \mathfrak{g}$, we have the simplified Koszul formula

$$2\nabla_X Y = [X, Y].$$

Note that this is a $(1, 2)$ tensor, so we use the bi-invariant metric to make it into the $(0, 3)$ tensor $T := \langle [X, Y], Z\rangle$. It is clear that T is anti-symmetric with respect to X, Y. We calculate anti-symmetry with respect to Y, Z and X, Z for left-invariant vector fields $X, Y, Z \in \mathfrak{g}$

$$\begin{aligned} \langle [X, Y], Z\rangle &= 2\langle \nabla_X Y, Z\rangle \\ &= \cancel{2X\langle Y, Z\rangle} - 2\langle Y, \nabla_X Z\rangle \\ &= -\langle [X, Z], Y\rangle, \end{aligned}$$

[1]Existence on compact groups follows from the standard averaging trick.

$$
\begin{aligned}
\langle [X,Y],Z\rangle &= -\langle [Y,X],Z\rangle \\
&= \cancel{-2Y\langle X,Z\rangle} + 2\langle X,\nabla_Y Z\rangle \\
&= -\langle [Z,Y],X\rangle.
\end{aligned}
$$

Therefore, T defines an anti-symmetric 3-form at T_eG and we extend this to G by left multiplication. Left-invariance of T comes by definition; right-invariance comes from the right-invariance of the metric, and

$$(R_g)_*[X,Y] = [(R_g)_*X,(R_g)_*Y]$$

since right multiplication is a diffeomorphism. T is closed since it is bi-invariant, so it descends down to a class in $H^3_{dR}(G)$. Now assume G is non-abelian, thus so is $\mathfrak{g}$, i.e. for some $X,Y \in \mathfrak{g}$ we have $[X,Y] \neq 0$. In this case,

$$T(X,Y,[X,Y]) = \|[X,Y]\|^2 \neq 0,$$

so T is a non-zero form. In fact, we argue $[T]$ must be a nonzero class. Since G is nonabelian, $n \geq 2$. But then $H^1_{dR}(G) = 0$, and so $H^2_{dR}(G) = 0$. Therefore, if $T = dS$ for some 2-form S, then $T = d^2R = 0$ for some 1-form R. This forces $G \cong \mathbb{S}^3$ by cohomological considerations.

If G is abelian, then we use the classification of abelian Lie groups to get $G \cong \mathbb{R}^i \times (\mathbb{S}^1)^j$. Since G is a sphere, $G \cong \mathbb{S}^1$. □

Before turning to the corresponding question for projective spaces, let us remark on some relaxations.

(a) $\mathbb{S}^n$ is a topological group iff $n = 0,1,3$. This is one statement of Hilbert's 5th problem, and follows by Gleason in [11].

(b) $\mathbb{S}^n$ is a H-space iff $n = 0,1,3,7$. This is proved by Adams in [2], then drastically simplified by Adams-Atiyah in [4]. We now turn to the corresponding question for $\mathbb{P}^n$.

Proposition 2.6. *$\mathbb{P}^n$ is a Lie group iff $n = 0,1,3$.*

Proof. Again, we transport the multiplicative structure using diffeomorphisms:

(a) $\mathbb{P}^0 \cong e$,

(b) $\mathbb{P}^1 \cong \mathrm{SO}(2)$,

(c) $\mathbb{P}^3 \cong \mathrm{SO}(3)$.

Conversely, assume $G = \mathbb{P}^n$ is a Lie group. In particular, it is a topological group. Consider the map composite map f

$$\mathbb{S}^n \times \mathbb{S}^n \xrightarrow{p \times p} G \times G \xrightarrow{m} G,$$

where p denotes the canonical projection. Since $f_*(\pi_1(\mathbb{S}^n \times \mathbb{S}^n)) \subset p_*(\pi_1(\mathbb{S}^n))$, we get a unique lift $\tilde{f}$ which commutes the following diagram

$$\begin{tikzcd} & \mathbb{S}^n \arrow[d, "p"] \\ \mathbb{S}^n \times \mathbb{S}^n \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & G \end{tikzcd}$$

Since $\tilde{f}$ gives $\mathbb{S}^n$ a topological group structure, we must have $n = 0, 1, 3$. $\square$

4. Characteristic classes will catch out the remaining $n = 7$ case quite easily.[2] What isn't easy is to show no X^n is parallelizable for $n > 7$. This is done by Kervaire in [15] and by Milnor in [6], both relying on Bott's famous periodicity theorem. One may also prove this via K-theory as by Atiyah-Hirzebruch in [5].

[2]This is where the discussion on division algebras enters.

3 Chapter 3

Where will Estonia go with category theory? (artwork Daniel Clark)

Warning 3.1. *Milnor-Stasheff bundle maps are always fiberwise isomorphisms!*

For example, we have no bundles maps between total spaces E, F if rank of E is not equal to rank of F. Also, the zero map is never considered a bundle map.

Given the diagram

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

we define the **pullback bundle** as following bundle over M

$$f^*E := \{(m, e) \in M \times E : f(m) = \pi(e)\}.$$

This comes with natural projection maps into M and E which commute the following square.

$$\begin{tikzcd}
f^*E \arrow[r, dashed] \arrow[d, dashed] & E \arrow[d, "\pi"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

We remark that the rank of E is the rank of f^*E. This concept generalizes many natural constructions. Here are some examples.

1. Pullback over a point is trivial. This follows since there is only one map to a point, namely the constant map.

$$\begin{tikzcd}
M \times V \arrow[r, dashed] \arrow[d, dashed] & V \arrow[d, "\pi"] \\
M \arrow[r, "f"] & \bullet
\end{tikzcd}$$

2. More generally, pullbacks along constant maps are trivial. Suppose $f(m) \equiv n$, then the pullback bundle would be isomorphic to $M \times \pi^{-}(n)$.

3. Pullback along the identity is identity.

$$\begin{tikzcd}
E \arrow[r, dashed, "\mathbb{1}"] \arrow[d, dashed, "\pi"] & E \arrow[d, "\pi"] \\
M \arrow[r, "\mathbb{1}"] & M
\end{tikzcd}$$

4. Pullback along inclusion maps are restrictions of the base.[3] For the inclusion map $i : M \subset N$ with bundle $E \to N$, we have $i^*E := \{(m, e) : m = \pi(e)\}$. In particular, if E is trivial, so is $E|_M$.

$$\begin{tikzcd}
E|_M \arrow[r, dashed] \arrow[d, dashed, "\pi|_M"'] & E \arrow[d, "\pi"] \\
M \arrow[r, "i"] & N
\end{tikzcd}$$

[3] NOT of the bundle!

5. A special case of the previous is when we embed $M \subset \mathbb{R}^N$ and look at the pullback of the $T\mathbb{R}^N$. In this case, we may split the bundle as $\mathbb{R}^n \cong f^*(T\mathbb{R}^N) \cong TM \oplus \nu_f$, where ν_f is the normal bundle of the embedding.

6. More generally, the same splitting holds when f is an immersion into a Riemannian manifold N.

In fact pullbacks are so general because of the following proposition – *any* bundle over M can be realized as a pullback bundle, once you supply a map $f : M \to N$.[4] Precisely,

Proposition 3.2. *Given the following diagram*

$$\begin{tikzcd} D \arrow[r, "\tilde{f}"] \arrow[d, "\pi_M"] & E \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

the bundle D is canonically isomorphic to the pullback bundle f^*E.

Proof. On the level of elements, the commmutative diagram gives the equation

$$f(\pi_M(d)) = \pi_N(\tilde{f}(d)),$$

for each $d \in D$. This builds the canonical bundle isomorphism

$$D \to f^*E$$

$$d \mapsto (\pi_M(d), \tilde{f}(d)).$$

□

Warning 3.3. *Careful! **Product bundles** and **Whitney bundles** are not the same; they're defined over different bases.*

Onto investigating categorical properties of pullbacks. Among other things, we'll show that they commute with everything on mother nature's green earth. It shouldn't be a surprise that a high-brow categorical proof exists for

[4]See the naturality axiom in Chapter 4.

each of the following statements,[5] but we'll take the peasant's way out and use sets.

Proposition 3.4. *Pullbacks are functorial.*

Proof. We checked above that the identity pulls back to the identity. To show pullback composes, we need to check for any bundle $E \to O$, as well as maps $f : M \to N$ and $g : N \to O$,

$$(g \circ f)^* E = f^*(g^* E).$$

The high-brow way would be to use the uniqueness from the universal property of the pullback. The low-brow way is left as an exercise. $\square$

In the following sequence of propositions, assume $i \in \{1, 2\}$.

Proposition 3.5. *Pullbacks commute with products. Given bundles $\pi_i : E_i \to B_i$, and maps $f_i : M_i \to N_i$, we have*

$$(f_1 \times f_2)^*(E_1 \times E_2) \cong f_1^* E_1 \times f_2^* E_2.$$

Proof. The proof is immediate once you expand out the sets.

$$f_1^* E_1 \times f_2^* E_2 = \{((m_1, e_1), (m_2, e_2)) : f_1(m_1) = \pi_1(e_1) \,\&\, f_2(m_2) = \pi_2(e_2)\},$$

$$(f_1 \times f_2)^*(E_1 \times E_2) = \{((m_1, m_2), (e_1, e_2)) : (f_1 \times f_2)(m_1, m_2) = (\pi_1 \times \pi_2)(e_1, e_2)\},$$

so just take the map (of sets) which swaps E_1 with M_2. $\square$

Proposition 3.6. *Pullbacks commute with direct sums. Given bundles $\pi_i : E_i \to N$ and map $f : M \to N$, we have*

$$f^*(E_1 \oplus E_2) = f^* E_1 \oplus f^* E_2.$$

[5] Apparently, the correct setting where pullbacks commute with everything in *Cartesian closed categories.*

Proof. We compute

$$\begin{aligned}
f^*(E_1 \oplus E_2) &= f^*(\Delta^*(E_1 \times E_2)) \\
&= (\Delta \circ f)^*(E_1 \times E_2) \\
&= ((f \times f) \circ \Delta)^*(E_1 \times E_2) \\
&= \Delta^*((f \times f)^*(E_1 \times E_2)) \\
&= (\Delta^*(f^*E_1)) \times (\Delta^*(f^*E_2)) \\
&= f^*E_1 \oplus f^*E_2,
\end{aligned}$$

where the third equality follows from the commutativity of the following diagram.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\Delta"] & N \arrow[d, "\Delta"] \\
M \times M \arrow[r, "f \times f"] & N \times N
\end{tikzcd}$$

□

Proposition 3.7. *Pullbacks commute with tensor products. Given bundles $\pi_i : E_i \to M$, and map $f : M \to N$, we have*

$$f^*(E_1 \otimes E_2) \cong f^*E_1 \otimes f^*E_2.$$

Proof. TODO □

Finally, we remark vector bundles display a certain rigidity phenomenon. In fact, one can use the following proposition to show homotopic spaces have isomorphic classes of vector bundles.

Proposition 3.8. *Given $f_0, f_1 : M \to N$ homotopic maps, and a bundle $E \to N$, then*

$$f_0^*E \cong f_1^*E.$$

Proof. [14], Theorem 1.6. □

Two reasons for performing these abstract dances. One is in anticipation for calculation simplifications down the line. The second is to prepare for the eventual discussion on K-theory.

For the question of when is a subbundle a direct summand, the answer is *always, but* it depends on your metric. Even on the level of vector spaces, a vector subspace subspace $U \subset V$ isn't canonically a direct summand of V until a metric is chosen. We will discuss the converse question shortly.

4 Chapter 4

Fix a bundle $E \to M$ for general discussion. Stiefel–Whitney (cohomology) classes $w_i(E)$ always take *values* in the base, namely $w_k(E) \in H^k(M, \mathbb{Z}/2)$. We denote the *total* Stiefel–Whitney class by the polynomial

$$w(E) := \sum_{i=0}^{n} w_i(E).$$

Trivial bundles will always have trivial Stiefel–Whitney classes because the cohomology of a point is trivial. Combining with the product axiom, we see that Stiefel–Whitney classes fail to detect taking direct sums with the trivial bundle. Two questions naturally arise from this observation.

1. Is there a splitting $E \cong F \oplus \epsilon^k$, where ϵ^k is a trivial bundle? Observe this is equivalent to asking for k linearly independent sections.[6,7]
2. When does E sit as the subbundle of a trivial bundle?[8]

Note that the second question implicitly tells us the following: *Subbundles of trivial bundles may not be trivial.* Indeed, we've already seen this phenomenon when observing discussing embeddings of manifolds into Euclidean space. The issue is the standard problem of the inability of transition from local-to-global, and this is illustrated nicely by the hairy ball theorem.

From the point of view of Stiefel–Whitney classes, answering the first question reduces the calculation of Stiefel–Whitney classes from a high rank bundle to a low rank bundle. Answering second question gives a constraint on what Stiefel–Whitney classes your bundle can be.

[6]See [1] for a sketch of large rank bundles splitting.
[7]See Theorem 3.2 in [17] for a proof.
[8]Recall that computationally, metric + subbundle $\implies$ direct sum. Converse?

Let's see both instances of the above in action. Using the standard embedding of $\mathbb{S}^2 \subset \mathbb{R}^3$, one sees that Stiefel–Whitney classes fail to distinguish the tangent bundle from the trivial one. This follows since $\epsilon^3 \cong T\mathbb{S}^2 \oplus \nu$, and $\nu \cong \epsilon$. Therefore, $w(\mathbb{S}^2) = 1$.

Let's compute the Stiefel–Whitney classes of the other bundles we've discusses so far, namely the canonical bundle and the tangent bundle of $\mathbb{P}^n$. The canonical bundle can be calculated quite easily. Recall we have a natural inclusion map $i : \mathbb{P}^1 \to \mathbb{P}^n$ given by the natural cell structure on $\mathbb{P}^n$.[9] In fact, $i^*\gamma_n^1 = \gamma_1^1$, so that $w(\gamma_n^1) = 1 + a$.

Calculating $w(\mathbb{P}^n)$ is trickier. The first step is to show

$$T\mathbb{P}^n \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$$

where $\gamma_n^1 \oplus \gamma^\perp = \epsilon^{n+1}$.[10] The basic idea of the identification is identifying functions with their graphs. Namely we have the following identification

$$\{f : X \to Y\} \longleftrightarrow \{(x, y) \in X \times Y : \text{ distinct x}\}.$$

Let $x \in \mathbb{S}^n$ and $v \in (T_x\mathbb{S}^n)^\perp$. Since the antipodal map is orthogonal, the natural map $\mathbb{S}^n \to \mathbb{P}^n$ also identifies velocities by the equivalence relation

$$(x, v) \sim (-x, -v),$$

so we may write $T\mathbb{P}^n = \{(\pm x, \pm v)\}$. Since each point in $\mathbb{P}^n$ is tautologically contained in the line it defines, we may realize $x \in L_{[x]}$ as a unit vector. We identify $(\pm x, \pm v)$ with the function $x \mapsto v$ and then extend this linearly to a map $L_{[x]} \to L_{[x]}^\perp$. Note that this is equivalent to the linear extension of $-x \mapsto -v$, so this process is well-defined on $\mathbb{P}^n$. Globalize this to get the desired result.

The second step is to show $T\mathbb{P}^n \oplus \epsilon \cong (\gamma_n^1)^{\oplus n+1}$ and so,

$$w(\mathbb{P}^n) = (1 + a)^{n+1}.$$

[9] See 0.4 in [13] for details.

[10] Recall γ_n^1 was *defined* as a subbundle of the trivial bundle $\mathbb{P}^1 \times \mathbb{R}^{n+1}$.

We claim $\epsilon \cong \mathrm{Hom}(\gamma_n^1, \gamma_n^1)$. By rank considerations, $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is a line bundle, and the identity map forms a global section. Therefore,

$$\begin{aligned} T\mathbb{P}^n \oplus \epsilon &\cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) \\ &\cong \mathrm{Hom}(\gamma_n^1, \epsilon^{\oplus n+1}) \\ &\cong (\mathrm{Hom}(\gamma_n^1, \epsilon))^{\oplus n+1} \\ &\cong (\gamma_n^1)^{\oplus n+1} \end{aligned}$$

As a corollary, we conclude the only projective spaces with trivial tangent bundles are those where all its binomial coefficeints vanish mod 2. These are exactly $\mathbb{P}^{n-1}$ such that $n = 2^k$ for some k. We now show for $n = 0, 1, 3, 7$, the tangent bundle of $\mathbb{P}^n$ is trivial. The result follows from considering $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ as real division algebras. Let $X^n \in \{\mathbb{S}^n, \mathbb{P}^n\}$.

Proposition 4.1. *A nondegenerate pairing $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ implies the tangent bundle of X^{n-1} is trivial.*

Proof. For $X^{n-1} = \mathbb{S}^{n-1}$, the result follows TODO

Nondegeneracy implies for every nonzero $b_i \in \mathbb{R}^n$, the map $p_{b_i} := p(-, b_i)$ is injective map in $\mathrm{End}(\mathbb{R}^n)$, so it is an isomorphism. Consider a basis $b_1, ..., b_n \in \mathbb{R}^n$, and let $v_i \in \mathrm{Iso}(\mathbb{R}^n)$ be defined by the equation $v_i \circ p_{b_1} = p_{b_i}$.

We claim v_i induces $n-1$ linearly independent sections $\bar{v}_i$ of $T\mathbb{P}^{n-1} \cong \mathrm{Hom}(\gamma_{n-1}^1, \gamma^\perp)$. For each line $L \subset \mathbb{R}^n$, the section $\bar{v}_i$ is constructed as the composite linear map

$$L \xrightarrow{i} \mathbb{R}^n \xrightarrow{v_i} \mathbb{R}^n \xrightarrow{\pi} L^\perp,$$

where i, π respectively denote the natural inclusion and projection map. Since $v_1 = \mathbb{1}$, the induced section $\bar{v}_1$ is the zero map. For $x \in \mathbb{R}^n$ such that $L = L_{[x]}$ and $i \neq 1$, we argue $\bar{v}_i(x)$ forms a linearly independent set in $L^\perp$, which concludes the argument. Setting $w_i := (v_i \circ i)(x)$, we compute

$$0 = \sum_{i=2}^{n} a_i \pi(w_i) = \pi\left(\sum_{i=2}^{n} a_i w_i\right).$$

Since the kernel of π is exactly $L = \mathrm{Span}(w_1)$,

$$\sum_{i=2}^{n} a_i w_i = -a_1 w_1$$

for some $a_1 \in \mathbb{R}$. Therefore, all $a_i = 0$ since w_i are linearly independent. $\square$

We've introduced Stiefel–Whitney classes, now let's introduce an even coarser invariant - Stiefel–Whitney numbers. We will justify the existence of these numbers later; we start by computing Stiefel–Whitney numbers for $\mathbb{P}^n$. Recall

$$w(\mathbb{P}^n) = (1+a)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} a^i,$$

so that $w_i(\mathbb{P}^n) = \binom{n+1}{i} a^i$. So for example,

1. $w_n[\mathbb{P}^n] = w_1^n[\mathbb{P}^n] = (n+1)^n$. So for even n, these Stiefel–Whitney number are nonzero.

2. The Stiefel–Whitney numbers of odd-dimensional projective spaces all vanish. If $n = 2k-1$, then

$$w(\mathbb{P}^n) = (1+a)^{2k} = (1+a^2)^k = \sum_{i=0}^{2k-1} \binom{2k-1}{i} a^{2i}.$$

   It follows that $w_j(\mathbb{P}^n) = 0$ for odd j. But since the total Stiefel–Whitney monomial is always of odd-degree, the Stiefel–Whitney number must contain some odd w_j.

The last computation may lead one to doubt the usefulness of such a set of invariants. However, the justification is as follows:

Theorem 4.2. *All Stiefel–Whitney numbers of M^n vanish iff M^n is the boundary of some manifold N^{n+1}.*

The forward direction (hard) is due to Thom, while the reverse direction (easy) is due to Pontrjagin. We show the reverse direction. As a corollary, we see that the set of Stiefel–Whitney numbers forms a complete cobordidsm invariant.

Proof. ($\Longleftarrow$) Every Stiefel–Whitney class on M comes from a class on N. The collar neighborhood theorem implies that $TN|_M \cong TM \oplus \epsilon$, and the result follows. As a result, in the LES of pairs,

$$\ldots \to H^n(N) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(N, M) \to \ldots$$

any Stiefel–Whitney monomial $w := \prod_i w_i^{r_i} \in H^n(M)$ pulls back to a class in $H^n(N)$. Recall we have the pairing on N between the fundamental class and top cohomology classes as

$$\langle \partial \mu_N, w \rangle = \langle \mu_N, \delta w \rangle.^{11}$$

The RHS vanishes by previous considerations, so $w = 0$ since the pairing is nondegenerate. □

TODO : exercises!

5 Chapter 5

Upshot: Forget about bundles, talk about maps into Gr_n instead.

A word on notation. We understand the ambient space is $\mathbb{R}^{n+k}$ or $\mathbb{R}^\infty$, so we suppress it. The universal bundles portion holds only for the ∞ case, while everything else holds for both.

Let V_n be the set of n-frames in $\mathbb{R}^{n+k}$ which is naturally topologized as as a of $(\mathbb{R}^{n+k})^{\times n}$. There is a natural surjective map

$$V_n \xrightarrow{\mathfrak{S}pan} \mathrm{Gr}_n$$

which takes a n-frame to the n-dimensional subspace it spans. We topologize Gr_n by requiring $\mathfrak{S}pan$ to be a quotient map. One can alternatively topologize Gr_n as a quotient from the set of orthonormal frames V_n^0 to Gr_n. The

[11] Now *that's* what I call generalized Stokes' theorem! This follows from the following formula (from Wikipedia don't @ me):

$$\partial(\sigma \frown \psi) = (-1)^{q+1}(\sigma \frown \delta\psi - \partial\sigma \frown \psi),$$

where $\psi \in C^q(M; \mathbb{Z}/2)$. Taking $\mathbb{Z}/2$ coefficients and $\sigma = \mu_N$, the result follows.

upshot of this method is that it is on immediately sees that the Grassmannian is compact. Gram-Schmidt process defines a retraction between V_n to V_n^0, so both topologizations agree.

One can show Gr_n is Hausdorff by constructing a continuous function (least squares) which separates out distinct n-planes.

Now claim the dimension of Gr_n is nk. Fix an n-plane X, and take the neighborhood

$$\{Y \in \mathrm{Gr}_n : \pi : Y \xrightarrow{\cong} X\},$$

where π is the restriction of the orthogonal projection $\mathbb{R}^{n+k} \to X$. We show this set is canonical identified with $\mathrm{Hom}(X, X^\perp)$. Once we show continuity of this identification, the claim follows by considering dimensions. A n-plane $Y \in \mathrm{Gr}_n$ is the graph of a linear map $T_Y : X \to X^\perp$ iff it satisfies the generalized vertical line test[12] for all $y \in Y$. By linearity, we just need to check the generalized vertical line test at $y = 0$. But since $T_Y(0) = 0$, this is satisfied exactly when $Y \cap X^\perp = \{0\}$. Since $\ker \pi = X^\perp$, the conclusion follows by rank-nullity.

We must show this identification is a continuous one. The idea is to fix some basis $x_i \in X$, and this induces a basis y_i by taking the isomorphism $\pi^- : X \to Y$. This new basis satisfies the equation

$$y_i = x_i + T_Y(x_i),$$

where T_Y corresponds to the linear map with graph Y. The continuity of this process shows that the identification is a homeomorphism.

Thus, Gr_n is a compact manifold of dimension nk. One can also show the map

$$\mathrm{Gr}_n \to \mathrm{Gr}_k$$
$$X \to X^\perp$$

is a homeomorphism. We construct the **universal vector bundle** γ^n as the following vector bundle over Gr_n

$$\{(X, x) \in \mathrm{Gr}_n \times \mathbb{R}^{n+k} : X \in \mathrm{Gr}_n, x \in X\}.$$

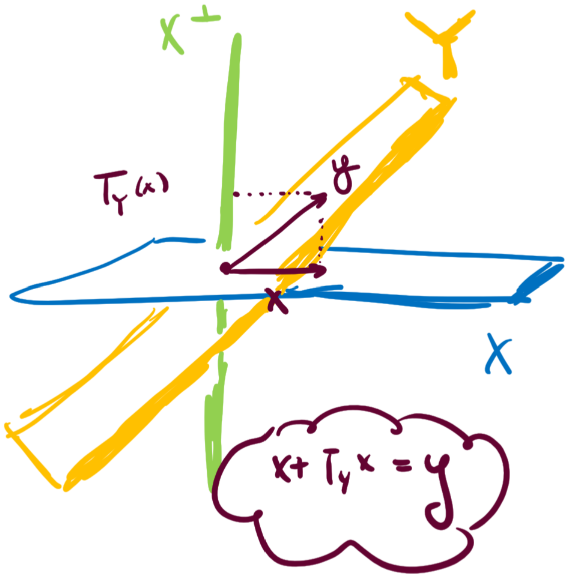
[12] $Y \subset X \times X^\perp$ satisfies the generalized vertical line test iff for all $(x, x'), (x, x'') \in Y$ we have $x' = x''$.

Proposition 5.1. *The universal bundle is locally trivial.*

Proof. For an n-plane $X \in \mathrm{Gr}_n$, consider the same neighborhood as before, namely

$$\{Y \in \mathrm{Gr}_n : \pi : Y \xrightarrow{\cong} X\}.$$

All the ingredients of the trivialization are already present, so we just present a picture.



$\square$

The next two propositions show the universality of γ^n.

Proposition 5.2. *For $\pi : E \to M$ a rank k bundle, we have a bundle map*

$$E \to \gamma^k(\mathbb{R}^L)$$

for large enough L.

Proof. We prove in the case M is compact. One can easily modify for the paracompact case. We show that E admits a linear, injective map on each fiber into $\mathbb{R}^L$. Injectivity implies we don't collapse any k-plane.

Choose a finite cover of the base U_i where we may trivialize the bundle over each U_i. From here, the proof uses two ingredients. For each i,

1. Consider the map $h_i : \pi^{-1}(U_i) \to \mathbb{R}^k$ by trivialization composed with the projection.
2. Consider $U_i \supset V_i \supset W_i$, where W_i form a cover of B, and construct a continuous cutoff function $\lambda_i : B \to \mathbb{R}$ with respect to these inclusions.

We combine these two ingredients together to form the map $h_i^\lambda : E \to \mathbb{R}^k$, where

$$h_i^\lambda := \begin{cases} \lambda_i(\pi(e)) \cdot h_i(e) & \pi(e) \in U_i \\ 0 & \pi(e) \notin V_i \end{cases}$$

and $L = rk$ and the map $e \mapsto (h_1^\lambda(e), ..., h_r^\lambda(e))$ is the injective, linear map we seek. This follows since each e projects on some W_i since W_i form a covering. □

Proposition 5.3. *Any two bundle maps $E \to \gamma^n$ are homotopic.*

Proof. Let $\tilde{f}, \tilde{g} : E \to \gamma^n$ be two bundle maps. Suppose for all nonzero e,

$$\tilde{f}(e) \neq -\lambda \tilde{g}(e),$$

for any $\lambda > 0$. In this case, the linear homotopy

$$\tilde{h} = \tilde{h}_t(e) := (1-t)\tilde{f}(e) + t\tilde{g}(e)$$

avoid 0 and is a desired homotopy. To show $\tilde{h}$ is continuous, it suffices to show the induced homotopy on base spaces $h : M \to \mathrm{Gr}_n$ is continuous. This is done in a local trivialization.

The general case is done by perturbing the above argument. □

As a corollary of the above two arguments:

Theorem 5.4. *Any rank n bundle $E \to M$ determines a unique homotopy class of maps $[f] \in [M, Gr_n]$.*

As per the discussion above, any cohomology class $c \in H^k(\mathrm{Gr}_n, \Lambda)$ determines a unique cohomology class $c(E) := f^*c \in H^k(M, \Lambda)$ for any coefficient ring

Λ. The assignment $E \mapsto c(E)$ is natural in E, meaning pullbacks commute with c. To see this, take bundle maps

$$\begin{tikzcd} & \gamma^n \\ E \arrow[ur, "f"] \arrow[r, "\phi"'] & F \arrow[u, "g"'] \end{tikzcd}$$

and recalling $E = \phi^* F$, we see the induced map on cohomology satisfies

$$c(E) = f^* c = \phi^* g^* c = \phi^* c(F).$$

Conversely, consider any natural assignment $E \mapsto c(E)$. Since we may always realize $E = f^* \gamma^n$ for some bundle map f, naturality implies

$$c(E) = f^* c(\gamma^n).$$

Therefore, this is the most general construction, and so the naming is justified.

TODO (exercises, especially 5E)

6 Chapter 6

The cell structure on the Grassmannian is based off the concept of a **flag**, namely a sequence of subspaces V_i where $V_i \subset V_j$ for $i \leq j$. Specifically, we will be considering subflags of the canonical flag of $\mathbb{R}^m$, namely

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset ... \subset \mathbb{R}^m,$$

where $\mathbb{R}^i := \{(\xi^1, ..., \xi^i, 0, ..., 0) \in \mathbb{R}^n : \xi^i \in \mathbb{R}\}$. We will also consider subflags of the canonical closed half-flag, namely

$$\overline{\mathbb{H}}^1 \subset \overline{\mathbb{H}}^2 \subset ... \subset \overline{\mathbb{H}}^m,$$

where $\overline{\mathbb{H}}^i := \{(\xi^1, ..., \xi^i, 0, ..., 0) \in \mathbb{R}^n : \xi^i \in \mathbb{R}^{\geq 0}\}$. We also consider (open) half-spaces $\mathbb{H}^i \subset \overline{\mathbb{H}}^i$, but these don't form a flag.

Subflags will be determined by a **Schubert symbol**, that is a tuple

$$\sigma := (\sigma_1, ..., \sigma_n),$$

such that $n \leq m$ and $1 \leq \sigma_1 < ... < \sigma_n$. σ determines the subflags

$$\mathbb{R}^{\sigma_1} \subset ... \subset \mathbb{R}^{\sigma_n},$$

$$\overline{\mathbb{H}}^{\sigma_1} \subset ... \subset \overline{\mathbb{H}}^{\sigma_n}.$$

In fact, σ determines an unique orthonormal basis $x_1, ..., x_n$, where $x_i \in \mathbb{H}^{\sigma_i}$.

We assign to each subspace n-dimensional subspace X, the Schubert cell σ in the following fashion. First, consider the intersection of X with the canonical flag,

$$X \cap \mathbb{R}^1 \subset X \cap \mathbb{R}^2 \subset ... \subset X \cap \mathbb{R}^m.$$

For each i, observe there exists a linear sequence

$$0 \to X \cap \mathbb{R}^{i-1} \to X \cap \mathbb{R}^i \xrightarrow{\pi_i} \mathbb{R},$$

where π_i is project onto the i-th factor. Note

$$\pi_i \equiv 0 \iff \text{exact at } X \cap \mathbb{R}^i,$$
$$\pi_i \text{ surjective} \iff \text{not exact at } X \cap \mathbb{R}^i.$$

By rank-nullity,

$$\dim(X \cap \mathbb{R}^i) - \dim(X \cap \mathbb{R}^{i-1}) = \begin{cases} 1 & \pi_i \equiv 0, \\ 0 & \pi_i \text{ surjective.} \end{cases}$$

Fix a Schubert cell σ. Let $e(\sigma)$ be the set of all n-planes $X \in \mathrm{Gr}_n$ corresponding to σ. Let $f(\sigma) := V_n^0(\sigma) \cap (\mathbb{H}^{\sigma_1} \cap ... \cap \mathbb{H}^{\sigma_n})$ (respectively $\bar{f}(\sigma)$) denote the set of orthonormal n-frames x_i such that $x_i \in \mathbb{H}^{\sigma_i}$ (respectively $\overline{\mathbb{H}}^{\sigma_i}$).

It is clear that $e(\sigma) \xrightarrow{\mathfrak{S}pan} f(\sigma)$ is a homeomorphism, with the inverse being take a basis as discussed above.[13] The technical lemma of this section is a computation of the dimension of the closed cell $\bar{f}(\sigma)$.

Lemma 6.1. $\dim(\bar{f}(\sigma)) = \sum_{i=1}^{n} \sigma_i - i$.

Proof. The proof is by induction. The base case $n = 1$ is when X is a line, where some $x_1 \in X \subset \mathbb{H}^{\sigma_1} - \mathbb{H}^{\sigma_1 - 1}$. In this case,

$$\bar{f}(\sigma) = \{x_1 \in \overline{\mathbb{H}}^{\sigma_1} : \|x_1\| = 1\}.$$

For example,

1. $\sigma_1 = 1$, then $\bar{f}(\sigma)$ is the right point.
2. $\sigma_1 = 2$, then $\bar{f}(\sigma)$ is the upper semi-circle.
3. $\sigma_1 = 3$, then $\bar{f}(\sigma)$ is the upper half-sphere.

Continuing, we see that for $\sigma_1 = n$, then $\bar{f}(\sigma)$ is the hemisphere of dimension $\sigma_1 - 1$ contained in $\overline{\mathbb{H}}^{\sigma_n}$, proving the base case.

Before continuing with the induction step, we setup some preliminaries. For unit vectors $u, v \in \mathbb{R}^m$ such that $u \neq -v$, consider the linear transformation $T(u, v) \in \mathrm{End}(\mathbb{R}^m)$ characterized by the two properties:

1. $T(u, v) : u \mapsto v$.
2. $T(u, v)$ fixes $u^\perp \cap v^\perp$.

This data is m-dimensional, so the prescription extends to a linear operator on $\mathbb{R}^m$. Furthermore, one easily sees its inverse is given by $T(v, u)$ and fixes the unit sphere, so $T \in \mathrm{O}(m)$.

We now proceed with the induction case. Let $b_i \in \mathbb{H}^{\sigma_i}$ be the standard σ_i-th basis vector. Consider the set D consisting of unit vectors $u \in \overline{\mathbb{H}}^{\sigma_{n+1}}$ such

[13] Warning, this is wildly false if we replace the codomain by $\bar{f}(\sigma)$. Example?

that $u \cdot b_i = 0$ for all $1 \leq i \leq n$. Clearly, D is a hemisphere of dimension $\sigma_{n+1} - 1 - n = \sigma_{n+1} - (n+1)$. We show the following map is a homeomorphism

$$\bar{f}(\sigma_1, ..., \sigma_n) \times D \to \bar{f}(\sigma_1, ..., \sigma_{n+1})$$

$$(x_1, ..., x_n, u) \mapsto (x_1, ..., x_n, Tu),$$

where $T = T(b_n, x_n) \circ ... \circ T(b_1, x_1)$, which will conclude the induction step. As T is orthogonal, we compute

$$Tu \cdot Tu = u \cdot u = 1,$$

$$Tu \cdot x_i = Tu \cdot Tb_i = u \cdot b_i = 0,$$
$$Tu \in \overline{\mathbb{H}}^{\sigma_{n+1}},$$

where the last line follows from TODO , so that $(x_1, ..., x_n, Tu) \in \bar{f}(\sigma)$. We can explicitly construct an inverse, sending x_{n+1} to

$$T^{-1} x_{n+1} = T(x_1, b_1) \circ ... \circ T(x_n, b_n) x_{n+1},$$

and so the result is shown. $\square$

For $\sigma = (\sigma_1, ..., \sigma_n)$ with the underlying space being $\mathbb{R}^m$, note that there are $\binom{m}{n}$ ways of producing distinct strictly increasing sequences. By the technical lemma, each produces a distinct cell $e(\sigma)$. The collection of all cells forms the structure of a CW complex to $\mathrm{Gr}_n(\mathbb{R}^m)$. Taking $m \to \infty$ give the CW structure to $\mathrm{Gr}_n(\mathbb{R}^\infty)$.

Now consider $m = \infty$. The number of r-cells corresponds to the number of partitions of r into at most n integers. This follows since each σ with $\dim(e(\sigma)) = r$ corresponds to the partition of r as

$$r = \sigma_1 - 1, ..., \sigma_n - n,$$

but some of these numbers might be 0 so we take them out.

7 Chapter 7

The main result of this chapter is identifying the $\mathbb{Z}/2$ cohomology of the infinite Grassmanian as an algebra, namely

Theorem 7.1.

$$H^*(Gr_n, \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1(\gamma^n), ..., w_n(\gamma^n)].$$

Note $\deg w_i(\gamma^n) = i$. We start off with a lemma which says the cohomology classes $w_i(\gamma^n)$ are *algebraically independent* over $\mathbb{Z}/2$.

Lemma 7.2. $\{w_1(\gamma^n), ..., w_n(\gamma^n)\}$ *satisfies no polynomial relation.*

Proof. Suppose $p(w_1(\gamma^n), ..., w_n(\gamma^n)) = 0$ for some polynomial $p \in \mathbb{Z}/2[X]$. For any rank n bundle $E \to M$, consider its classifying map $f : M \to \mathrm{Gr}_n$. Then,

$$\begin{aligned} p(w_1(E), ..., w_n(E)) &= p(f^* w_1(\gamma^n), ..., f^* w_n(\gamma^n)) \\ &= f^* p(w_1(\gamma^n), ..., w_n(\gamma^n)), \end{aligned}$$

where the last equality follows since f^* commutes with $+$ and $\smile$ since it's a ring homomorphism.[14] Consider the n-fold product of the canonical bundle $(\gamma^1)^{\times n} \to (\mathbb{P}^\infty)^{\times n}$. Recall $H^*(\mathbb{P}^\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[a]$ where $a \in H^1(\mathbb{P}^\infty, \mathbb{Z}/2)$, so by Kunneth,

$$\begin{aligned} H^*(\bigtimes_n \mathbb{P}^\infty, \mathbb{Z}/2) &\cong \bigotimes_n H^*(\mathbb{P}^\infty, \mathbb{Z}/2) \\ &= (\mathbb{Z}/2[a])^{\otimes n} \\ &= \mathbb{Z}/2[a_1, ..., a_n], \end{aligned}$$

where $a_i = \pi_i^* a$ and $\pi_i : \bigtimes_n \mathbb{P}^\infty \to \mathbb{P}^\infty$ is the projection onto the i-th factor. Note the $\deg a_i = 1$. For Stiefel–Whitney classes,

$$\begin{aligned} w(\bigtimes_n \gamma^1) &\cong (1+a) \times ... \times (1+a) \\ &= (1+a_1) \smile ... \smile (1+a_n), \end{aligned}$$

so

1. $w_1(\bigtimes_n \gamma^1) = a_1 + ... + a_n$,
2. $w_2(\bigtimes_n \gamma^1) = a_1 \smile a_2 + ... + a_{n-1} \smile a_n$,

   $\vdots$

[14] See [13], Proposition 3.10.

3. $w_n(\bigtimes_n \gamma^1) = a_1 \smile ... \smile a_n$.

Thus, $w_i(\bigtimes_n \gamma^1)$ is the i-th elementary symmetric polynomial. By the fundamental theorem of elementary symmetric polynomials,

$$p(w_1(\gamma^1), ..., w_n(\gamma^1)) = 0 \implies p = 0.$$

$\square$

Thus, we have proven the existence of a free $\mathbb{Z}/2$ subalgebra of $H^*(\mathrm{Gr}_n, \mathbb{Z}/2)$ generated by $w_i(\gamma^n)$. We move on to show the main theorem by showing this subalgebra is the full algebra.

Proof. (of 7.1) We will count to show

$$\text{rank } H^r(\mathrm{Gr}_n, \mathbb{Z}/2) = \text{rank } \mathcal{H}_r(\mathbb{Z}/2[w_1(\gamma^n), ..., w_n(\gamma^n)]),$$

where $\mathcal{H}_r$ denotes the homogeneous polynomials of degree r. Since the base ring is a field, the result follows.

Recall that we argued the number of r-cells of Gr_n is the number of ways you can partition r into at most n integers. Therefore,

$$\begin{aligned} \#r\text{-cells} &= \text{rank } C^r(\mathrm{Gr}_n, \mathbb{Z}/2) \\ &\geq \text{rank } Z^r(\mathrm{Gr}_n, \mathbb{Z}/2) \\ &\geq \text{rank } H^r(\mathrm{Gr}_n, \mathbb{Z}/2). \end{aligned}$$

However, consider the set of all possible monomials in $H^r(\mathrm{Gr}_n, \mathbb{Z}/2)$

$$w_1(\gamma^n)^{r_1} \smile ... \smile w_n(\gamma^n)^{r_n},$$

where $\sum_{i=1}^n i \cdot r_i = r$. We claim this set also naturally in bijection with the number of ways to partition r into at most n integers. This is given by

$$r_n, r_n + r_{n-1}, ..., r_n + ... + r_n,$$

where as before, we delete any 0's which might occur. Since monomials are linearly independent and generate $\mathcal{H}_r$, the result follows. $\square$

Now to prove uniqueness. Honestly pretty standard argument. TODO

8 Chapter 8

For a rank n bundle $E \to M$ with generic fiber F, consider $E_0 \subset E$ and $F_0 \subset F$ to be nonzero elements.

$$H^i(F, F_0; \mathbb{Z}/2) = \begin{cases} 0 & i \neq n \\ \mathbb{Z}/2 & i = n \end{cases}$$

$$H^i(E, E_0; \mathbb{Z}/2) = \begin{cases} 0 & i < n \\ H^{i-n}(M, \mathbb{Z}/2) & i \geq n \end{cases}$$

Milnor-Stasheff gives some intuition with cell complexes I can't make sense out of. I'll give my own intuition.

Recall the rough idea cohomology of a pair (X, A) is to quotient out by A. The cohomology of the F's, since we only look in the fiber direction, is just the cohomology of the pair $(\mathbb{R}^n, \mathbb{R}^{n-1})$ which is the cohomology of the n-sphere. The cohomology of the E's intuitively measures the cohomology of the zero section of E which is a copy of the base space. The bundle being rank n means that we add n extra directions, which explains why we need to subtract off n dimensions when taking cohomology.

We will rigorously prove the following statement in chapter 10. The coefficeints are $\mathbb{Z}/2$.

Theorem 8.1. *$H^i(E, E_0) = 0$ for $i < n$, and $H^n(E, E_0)$ contains a unique class u such that the restriction $u|_{(F,F_0)}$*[15] *is the unique nonzero class for every F. Furthermore,*

$$H^n(E) \to H^{n+k}(E, E_0)$$
$$x \mapsto x \smile u$$

is an isomorphism for every k.

We call u the *fundamental cohomology class*. Clearly, $\pi^* : H^k(B) \to H^k(E)$ induces an isomorphism since the zero section defines a retract. We define the Thom isomorphism ϕ as the composite map

$$H^n(B) \to H^n(E) \to H^{n+k}(E, E_0).$$

We introduce *squaring operations* $\mathrm{Sq}^i : H^n(X, A) \to H^{n+i}(X, A)$ axiomatically.

[15]Namely $u|_{(F,F_0)}$ is pullback along the inclusion map $\iota : (F, F_0) \to (E, E_0)$

1. Additive.
2. Naturality. For $f : (X, A) \to (Y, B)$, the induced map on cohomology $f^* : H^k(Y, B) \to H^k(X, A)$ commutes with squaring.
3. For $i = 0$ and $i \geq n$, Sq^i is compute by the formulae

$$\mathrm{Sq}^i := \begin{cases} \mathbb{1} & i = 0 \\ x \smile x & i = n \\ 0 & i > n \end{cases}$$

   The case $i = n$ justifies the name.
4. Henri Cartan's formula:

$$\mathrm{Sq}^k(a \smile b) = \sum_{i+j=k} \mathrm{Sq}(a) \smile \mathrm{Sq}(b).$$

We define $w_i(E) := \phi^{-1}(\mathrm{Sq}^i(\phi(1)))$ where $\phi : H^0(M) \to H^n(E, E_0)$ the Thom isomorphism. If we expand out the definition of ϕ, we see that $w_i(E) \in H^i(M)$ is unique class which satisfies

$$\pi^*(w_i(E)) \smile u = \mathrm{Sq}^i(u).$$

Here is a diagram corresponding to $w_i(E)$.

$$\begin{tikzcd} H^0(M) \arrow[d, "\phi"] & H^i(M) \arrow[d, "\phi"] \\ H^n(E, E_0) \arrow[r, "\mathrm{Sq}^i"'] & H^{n+i}(E, E_0) \end{tikzcd}$$

We verify $w_i(E)$ satisfies the axioms in Chapter 4.

1. For $i = 0$, then $\mathrm{Sq}^i = \mathbb{1}$, so the composed map is the identity. For $i > n$, $\mathrm{Sq}^i = 0$, so the composed map is 0.

2. Axiomatically, Sq^i are natural. It suffices then to show the Thom isomorphism is also natural, meaning the following diagram commute,

$$\begin{tikzcd} H^0(M') \arrow[r, "f^*"] \arrow[d, "\phi'"] & H^0(M) \arrow[d, "\phi"] \\ H^n(E', E_0') \arrow[r, "\tilde{f}^*"] & H^n(E, E_0) \end{tikzcd}$$

Here, $\tilde{f}$ is the bundle map $E \to E'$ which induces a map of pairs $(E, E_0) \to (E', E_0')$ since $\tilde{f}$ is an isomorphism. Expanding out the Thom isomorphisms, we want to see the following equalities on the level of elements,

$$\tilde{f}^*(\pi^*(1) \smile u') = \pi^*(f^*(1)) \smile u.$$

We first note that $\tilde{f}^*(u') = u$ as bundle maps are fiberwise isomorphisms.[16] Therefore, it suffices to show

$$\pi \circ \tilde{f} = f \circ \pi.$$

Indeed, this is the case since f is a map covered by the bundle map $\tilde{f}$.

[16] I think this part of the argument is correct.

9 Appendix A: Singular (Co)homology

Standard n-simplex: convex set $\Delta^n \subset \mathbb{R}^{n+1}$ consisting of all $(n+1)$–tuples $(t_0, \ldots, t_n)$ with

$$t_i \geq 0, \quad t_0 + t_1 + \ldots + t_n = 1.$$

Any continuous map from Δ^n to a topological space X is called a *singular n–simplex* in X. The i-th *face* of a singular n-simplex $\sigma : \Delta^n \to X$ is the singular $(n-1)$–simplex

$$\sigma \circ \phi_i : \Delta^{n-1} \to X$$

where the embedding $\phi_i : \Delta^{n-1} \to \Delta^n$ is defined by

$$\phi_i(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n).$$

For each $n \geq 0$ the *singular chain group* $C_n(X; R)$ with coefficients in any commutative ring R is the free R–module having one generator $[\sigma]$ for each singular n–simplex $\sigma \in X$. For $n < 0$, the group $C_n(X; R)$ is defined to be zero. The *boundary homomorphism*

$$\partial : C_n(X; R) \to C_{n-1}(X; R)$$

is defined by

$$\partial[\sigma] = [\sigma \circ \phi_0] - [\sigma \circ \phi_1] + \ldots + (-1)^n[\sigma \circ \phi_n].$$

We have the identity $\partial^2 = 0$. Define the n-th *singular homology group* $H_n(X; R)$ to be the quotient module $Z_n(X; R)/B_n(X; R)$ ("homology is cycles modulo boundaries"), where $Z_n(X; R)$ is the kernel of ∂_n and $B_n(X; R)$ is the image of ∂_{n+1}.

By dualizing appropriately, we pass from homology to cohomology. That is, we define the n-th *cochain group* $C^n(X; R)$ to be the dual R-module $\mathrm{Hom}_R(C_n(X; R), R)$ (that is, consisting of all R–linear maps from $C_n(X; R)$ to R.)

The value of a cochain $c \in C^n(X; R)$ on a chain $\gamma \in C_n(X; R)$ will be denoted by $\langle c, \gamma \rangle \in R$. The *coboundary* of a cochain $c \in C^n(X; R)$ is defined to be the cochain $\delta c \in C^{n+1}(X;R)$ whose value on each $(n+1)$-chain α is determined by the formula

$$\langle \delta c, \alpha \rangle + (-1)^n \langle c, \partial \alpha \rangle = 0.$$

This yields the *singular cohomology groups*, obtained by modding cocycles out by coboundaries:

$$H^n(X;R) = Z^n(X;R)/B^n(X;R) = (\ker \delta^n)/\,(im\delta^{n-1}).$$

TODO : cohomology of $\mathbb{P}^n$

TODO : universal coefficients (esp.for $\mathbb{Z}/2$ coefficients)

Recall a manifold is orientable if $H_n(M;\mathbb{Z}) = \mathbb{Z}$. An orientation is a choice of a generator. However, any closed manifold has a unique $\mathbb{Z}/2$ orientation. Uniqueness implies that for orientable manifolds,

$$1 = -1 \implies 2 = 0,$$

which gives an indication of why we take coefficients in $\mathbb{Z}/2$.

10 Appendix B: Bernoulli Numbers

The **Bernoulli numbers** $B_1, B_2, \cdots$ appear as coefficients of the following power series whose radius of convergence is π:

$$\frac{x}{\tanh x} = \sum_{n\geq 0} (-1)^n \frac{B_n}{(2n)!}(2x)^{2n} =$$

$$\tanh x = \frac{2}{\tanh 2x} - \frac{1}{\tanh x} \implies \tanh x = \sum_{n\geq 1} (-1)^n (2^{2n})(2^{2n}-1)\frac{B_n}{(2n)!}(x)^{2n-1}$$

Using the identity

$$\tanh iy = i \tanh y$$

and the preceding expansion, we obtain:

Lemma 10.1. *For each n, $\frac{2^{2n}(2^{2n}-1)B_n}{2n}$ is a positive integer.*

Proof. The Taylor expansion above indicates that the following holds:

$$\frac{d^{2n-1}}{dy^{2n-1}} = \frac{2^{2n}(2^{2n}-1)B_n}{2n}$$

$\square$

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