

GMT Notes

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My two primary sources are [Mag12] and [Sim14].

1 Lebesgue equals Hausdorff (3.9.23)

We follow (Thm 2.9, [Sim14]). Recall the definitions of Lebesgue and Hausdorff measures.

$$\mathcal{L}^n(A) = |A| := \inf \left\{ \sum_{j=1}^{\infty} |R_j| : R_j \text{ is a open rectangle, } A \subset \bigcup_{j=1}^{\infty} R_j \right\},$$

$$\mathcal{H}_{\delta}^n(A) := \inf \left\{ \sum_{j=1}^{\infty} \omega_n \left(\frac{\text{diam} C_j}{2} \right)^n : \text{diam}(C_j) < \delta, A \subset \bigcup_{j=1}^{\infty} C_j \right\},$$

and $\mathcal{H}^n(A) := \lim_{\delta \searrow 0} \mathcal{H}_{\delta}^n(A)$.

Theorem 1.1. *For every $\delta > 0$ and $A \subset \mathbb{R}^n$,*

$$\mathcal{H}^n(A) = \mathcal{H}_{\delta}^n(A) = |A|.$$

Proof. Taking $\delta \searrow 0$, it suffices to show $\mathcal{H}_{\delta}^n = \mathcal{L}^n$. Since taking closure does not affect diameter, we may assume C_k are closed. Note also $|E| = 0 \implies \mathcal{H}_{\delta}^n(E) = 0$ since we may inscribe a ball B_j with diameter $< \delta$ in each rectangle R_j .

($\mathcal{H}_{\delta}^n \leq \mathcal{L}^n$) The idea here is to “uniformly eat up” all the measure A with finitely many pairwise disjoint balls, then iterate this algorithm *ad infinitum*.¹ Fix an arbitrary cover $\{R_j\}$ of A by open rectangles.

Step $k = 1$. For each j , consider a disjoint family of cubes $\{I_k\}$ ² with $\text{diam}(I_k) < \delta$ such that the interiors of I_k are pairwise disjoint and $\bigcup I_k = R_j$. This is possible since R_j is an open set (Thm 1.4, [SS05]). For each k , inscribe a *closed* ball B_k into I_k such that $\text{diam}(B_k) > \frac{1}{2}s_k$ where s_k is the side length of I_k [Figure 1]. This is obviously not sharp, but it doesn't matter. What does matter is that the most measure we've left off out is

$$|I_k - B_k| \leq s_k^n - \omega_n \left(\frac{s_k}{4} \right)^n = \left(1 - \frac{\omega_n}{4^n} \right) s_k^n.$$

¹i.e. Step 1 leaves $\frac{1}{2}$ the measure, step 2 leaves $\frac{1}{4}$ the measure, etc (these are not the correct constants, but the idea is right).

² I_k depends on j also, which we suppress from the notation. Note that I_k^j may intersect $I_k^{j'}$ for $j \neq j'$ - the pairwise disjoint condition is with respect to a fixed rectangle.

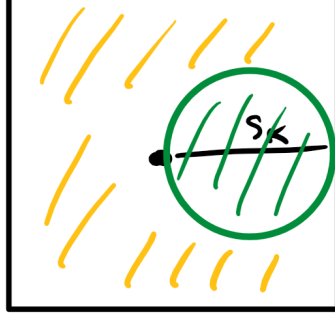


Figure 1: B_k must be bigger than the green ball.

Note the constant in front of s_k is uniform in k and strictly smaller than 1. Therefore,

$$|R_j - \cup_k B_k| = |\cup_k I_k - B_k| \leq (1 - \frac{\omega_n}{4^n}) \sum_k s^k = (1 - \frac{\omega_n}{4^n}) |R_j|.$$

Thus, for each j , we may choose finitely many ball B_j^1, \dots, B_j^N such that the same inequality holds.


Step $k \geq 2$. For $k = 2$, we repeat the above argument but for $R_j - \cup_{k=1}^N B_j^k$ instead of R_j , which is *again an open set*. Repeat this construction for $k > 2$.

In total, for each j we get a countable disjoint collection of balls $\{B_j^k\}$ of R_j with radius $< \delta$ such that $|R_j - \cup_k B_j^k| = 0$, so $\mathcal{H}_\delta^n(R_j - \cup_k B_j^k) = 0$. Therefore,

$$\mathcal{H}_\delta^n(R_j) = \mathcal{H}_\delta^n(\cup_k B_j^k) \leq \sum_{k=1}^{\infty} \omega_n \left(\frac{\text{diam} B_j^k}{2} \right)^n = \sum_{k=1}^{\infty} |B_j^k| = |R_j|.$$

Thus

$$\mathcal{H}_\delta^n(A) \leq \sum_{j=1}^{\infty} \mathcal{H}_\delta^n(R_j) \leq \sum_{j=1}^{\infty} |R_j|,$$

and since R_j was an arbitrary cover, the result is shown. 

($\mathcal{H}_\delta^n \geq \mathcal{L}^n$) The idea here is to use *Steiner symmetrization* to prove the isodiametric inequality.

Lemma 1.2. *For any $A \subset \mathbb{R}^n$,*

$$|A| \leq \omega_n \left(\frac{\text{diam} A}{2} \right)^n.$$

Assuming the lemma, take C_j an arbitrary collection of sets which cover A , with $\text{diam} C_j < \delta$. Then,

$$|A| \leq |\cup_j C_j| \leq \sum_j |C_j| \leq \sum_j \omega_n \left(\frac{\text{diam} C_j}{2} \right)^n.$$

(of Lemma 1.2). It suffices to prove A compact since taking closure does not increase diameter. We sketch a symmetrization process as follows. For the hyperplane $H_i := \{x^i = 0\}$ for $i = 1, \dots, n$, let $\xi_i \in H_i$ parameterize A in the sense that the fibers (under orthogonal projection to H_i) of A are at most 1-dimensional. We record the Lebesgue measure (length) of the fibers, the symmetrically distribute the length across ξ^i . This symmetrizes A across the hyperplane H_i , is diameter non-increasing, and the Lebesgue measure of A is constant. Furthermore, this preserves symmetrizations across other hyperplanes. Thus, symmetrizing A across each hyperplane H_i produces a subset of the ball of radius $\frac{\text{diam} A}{2}$, proving the lemma. ■

Both inequalities are proven, and so the result follows. ■

2 Riesz Representation Theorem II (4.6.23)

We state and prove RRT.

Theorem 2.1. *If L is a bounded linear functional on $C_c(\mathbb{R}^n, \mathbb{R}^m)$, then its variation $|L|$ is a (scaler-valued) Radon measure on \mathbb{R}^n and there exists a $|L|$ -measurable function $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $g = 1$ $|L|$ -a.e., and for all $\phi \in C_c(\mathbb{R}^n, \mathbb{R}^m)$*

$$\langle L, \phi \rangle = \int_{\mathbb{R}^n} (\phi \cdot g) d|L|.$$

Moreover,

$$|L|(A) = \sup \left\{ \int_{\mathbb{R}^n} \phi \cdot g d|L| : \phi \in C_c(A, \mathbb{R}^m), |\phi| \leq 1 \right\}.$$

Proof. Recall the definition of the total variation (measure) of L .

$$|L| :=$$

Zack had previously shown that $|L|$ is Radon, and that a version of RRT holds for L^1 . ■

3 Compactness and Regularization (4.27.23)

Maggi proves this by intertwining the functional analysis with the geometric measure theory. I think it's more clear to separate them out. ☹ ————— 🎉🎉🎉🎉

Let $(V, \|\cdot\|)$ be a normed linear space, and $B \subset V$ the unit ball defined by $\|\cdot\|$. Recall the following basic fact about the strong topology

Proposition 3.1. *A normed linear space V is finite dimensional iff the unit ball is compact.*


The proof is easy but irrelevant. What matters is the moral of the story - if we're out looking for compact sets in infinite dimensional space, the strong topology is not the correct one to look in. Even the most basic candidate for a compact set is not compact, so there are way too many open sets. We need a coarser topology.

Look not in V , but in its dual V^* together with its unit ball B^* defined by the operator norm. We see

$$B^* := \{\mu : V \rightarrow \mathbb{R} : |\mu(\phi)| \leq \|\phi\|\} = \{L : V \rightarrow \mathbb{R} : |\mu(\phi)| \leq 1 \text{ for } \|\phi\| \leq 1\}.$$

In other words, $\mu : B \rightarrow [-1, 1]$ and so $\mu \in [-1, 1]^B$; in other words, B^* canonically sits inside $[-1, 1]^B$, so it is compact in the subspace topology by Tychonoff's. However, the subspace topology agrees does *not* agree with the topology induced by the operator norm. The subspace topology oversees only that the evaluation pairing between V, V^* is continuous, and so it agrees with the weak star topology. That is, a sequence $\{\mu_i\}$ in V^* converges weak-* to μ iff for every $\phi \in V$,

$$\mu_i \xrightarrow{*} \mu \iff \langle \phi, \mu_i \rangle \rightarrow \langle \phi, \mu \rangle,$$

where $\langle -, - \rangle$ denotes the evaluation pairing. This is known as *Banach-Alaoglu* and will serve as the backbone functional analysis tool for the compactness result for the space of Radon measures (which by RRT is dual to $V = C_c$ with a mildly strange topology). Truthfully, it's important that we have the sequential version of Banach-Alaoglu - this is true if V is *separable* (counterexamples exist otherwise) which is true in our case. 

Theorem 3.2. *Given a sequence $\{\mu_i\}$ of Radon measures which are locally uniformly bounded, i.e. for $K \subset \mathbb{R}^n$ compact*

$$\sup_i \mu_i(K) < \infty,$$

then there exists a subsequence, upon reindexing, $\{\mu_k\}$ which weak- converge to a Radon measure. That is, by RRT,*

$$\int_K \phi d\mu_i \rightarrow \int_K \phi d\mu,$$

for each $\phi \in C_c(\mathbb{R}^n)$.

Proof. The idea is basically sequential Banach-Alaoglu and RRT. We first give a construction of $\{\mu_k\}$ via a diagonalization procedure. Consider an exhaustion of \mathbb{R}^n by balls $\{B_j\}$ centered at 0. Define functionals

$$F_{i,j}(\phi) := \int_{B_j} \phi d\mu_i,$$

for $\phi \in C_c(B_j)$. Linearity of $F_{i,j}$ is clear, and boundedness follows from the local uniformity assumption as

$$|F_{i,j}(\phi)| \leq \sup \phi \cdot \mu_i(B_j) \leq C \sup \phi,$$

where $C = C(j)$. By sequential Banach-Alaoglu, there is a functional $F_j : C_c(B_j) \rightarrow \mathbb{R}$ and a subsequence $\{F_{i',j}\}$ such that $F_{i',j} \xrightarrow{*} F_j$. By extracting subsequences subsequently, then selecting the diagonal subsequence, we extract the desired $\{\mu_k\}$ upon relabeling. We claim that $F = \lim_k \mu_k$. By construction, this limit is well-defined. Furthermore, by RRT we have

$$F(\phi) := \int_{\mathbb{R}^n} \phi d\mu,$$

for some Radon measure μ . For $\phi \in C_c(\mathbb{R}^n)$, take $\text{spt} \phi \subset B_j$. Linearity follows since $F(\phi) = F_j(\phi)$, and F_j is linear. Boundedness follows from

$$|F(\phi)| \leq \sup \phi \cdot \mu(B_j) \leq C \sup \phi.$$

There are two obvious requirements that need to hold for the above formula to be true. First, if $E := \{x \in \mathbb{R}^n : Jf(x) = 0\}$, then $\mathcal{H}^n(f(E)) = 0$ (injectivity is dropped for this statement). This is the content of Proposition 8.7. The second is that under the above assumptions, $f(E)$ had better be \mathcal{H}^n measurable.

Proposition 4.2. *If E is Lebesgue measurable in \mathbb{R}^n and $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is an injective Lipschitz function, then $f(E)$ is \mathcal{H}^n measurable in \mathbb{R}^m .*

Proof. Assume E is bounded, and exhaust E by a sequence of compact sets $\{K_i\}$ wrt \mathcal{L}^n , so $|E - \cup K_i| = 0$. Since f is Lipschitz, it's continuous and so $f(K_i)$ is compact and $\cup f(K_i)$ is Borel. Then,

$$\mathcal{H}^n(f(E) - \cup f(K_i)) = \mathcal{H}^n(f(E - \cup K_i)) \leq \text{Lip} f^n \cdot \mathcal{H}^n(E - \cup K_i) = \text{Lip} f^n \cdot |E - \cup K_i| = 0.$$

Note that injectivity is used in the first equality. ■

Here, we used that Lipschitz functions play nicely with the Hausdorff measure in the sense that $\mathcal{H}^s(f(E)) \leq \text{Lip} f^s \mathcal{H}^s(E)$, and is equal to the Lebesgue measure. For future reference, recall also that Hausdorff measure plays nicely with scaling in that $\mathcal{H}^s(rE) = r^s \mathcal{H}^s(E)$.

The next statement is to prove that area formula holds on arbitrary (not necessarily injective) linear functions $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Proposition 4.3. *If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear, then for every $E \subset \mathbb{R}^n$,*

$$\mathcal{H}^n(T(E)) = JT \cdot |E|.$$

For this, we must first recall some linear algebra. Set $\text{Lin}(n, m) := \{T : \mathbb{R}^n \rightarrow \mathbb{R}^m : T \text{ linear}\}$ and $\text{Isom}(n, m) := \{P \in \text{Lin}(n, m) : \langle Px, Py \rangle = \langle x, y \rangle\}$. We think of $\text{Isom}(n, m)$ as the set of linear isometric embeddings - clearly all must be injective.

Lemma 4.4. *For $P \in \text{Isom}(\mathbb{R}^n, \mathbb{R}^m)$, we have $P^*P = id_n$.*

Proof. Recall $P^* := (\mathbb{R}^m)^* \rightarrow (\mathbb{R}^n)^*$ is precomposition with P . The composition P^*P is understood as the following isomorphism:

$$\mathbb{R}^n \xrightarrow{P} \mathbb{R}^m \rightarrow (\mathbb{R}^m)^* \xrightarrow{P^*} (\mathbb{R}^n)^* \rightarrow \mathbb{R}^n$$

$$x \mapsto Px \mapsto \langle Px, - \rangle \mapsto \langle Px, P- \rangle \mapsto x$$

where the last map is justified since the covector $y \mapsto \langle Px, Py \rangle = \langle x, y \rangle$ is metrically equivalent to the vector $y \in \mathbb{R}^n$. ■

Since $P \in \text{Isom}(n, m)$ is an isometry, both P and its adjoint P^* (orthogonal projections) have Lipschitz constant 1. Recall also the spectral theorem - every symmetric real-valued matrix diagonalizes with real eigenvalues (and orthogonal eigenspaces) and the polar decomposition - every linear map $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be realized as

$$T = PS$$

with $S \in \text{Lin}(n, n)$ symmetric and $P \in \text{Isom}(n, m)$ (both S, T are given explicitly in terms of the diagonalization of T). It turns out linear isometric embeddings do not change the fine structure of the geometry, and in a sense we'll make precise, all the change in the geometry happens with S .

(of 4.3.) Set $\kappa := D_{\mathcal{L}^n} \nu = \frac{\mathcal{H}^n(T(B))}{|B|}$ for $\nu(E) := \mathcal{H}^n(T(E))$ to be the density. The last equality is justified by scaling since for all $r > 0$,

$$\begin{aligned}\mathcal{H}^n(T(rB)) &= r^n \mathcal{H}^n(T(B)) \\ |rB| &= r^n |B|.\end{aligned}$$

It suffices to show the following two statements:


1. $\mathcal{H}^n(T(E)) = \kappa |E|$,
2. $JT = \kappa$.

First assume $\kappa = 0$ (T is non-injective). Then, $\mathcal{H}^n(T(B)) = 0 \implies \mathcal{H}^n(T(rB)) = 0$ for every $r > 0$ by linearity of T and so in the limit, $\mathcal{H}^n(\mathbb{R}^n) = 0$. By monotonicity of the measure, $\mathcal{H}^n(T(E)) = 0$ for every E .

Next, assume $\kappa > 0$ (T is injective). Since $E \mapsto \mathcal{H}^n(T(E))$ is Radon, by the decomposition theorem for measures, it suffices to show the equality is true on balls $B_r(x)$ for $r > 0$ and $x \in \mathbb{R}^n$. This follows by translation invariance and scaling, as

$$\begin{aligned}\mathcal{H}^n(T(B_r(x))) &= \mathcal{H}^n(T(x + rB)) \\ &= r^n \mathcal{H}^n(T(B)) \\ &= r^n \kappa |B| \\ &= \kappa |B_r(x)|,\end{aligned}$$

which shows $\mathcal{H}^n(T(-)) \ll \mathcal{L}^n$ and identifies κ as the density.

Remark 4.5. For linear functions, non-injectivity is very easy to deal with as both sides evaluates to 0. We don't need to consider multiplicity for this case. 

To show $JT = \kappa$, we first show for $T = PS$ and $S(Q) = E$ for a cube Q , we have $\frac{\mathcal{H}^n(P(E))}{|E|} = 1$, since

$$|E| = |P^*P(E)| \leq \text{Lip}(P^*)^n |P(E)| = |P(E)| = \mathcal{H}^n(P(E)) \leq \text{Lip}(P)^n |E| = |E|.$$

Therefore,

$$\kappa = \frac{\mathcal{H}^n(PS(Q))}{|Q|} = \frac{\mathcal{H}^n(P(E))}{|E|} \frac{|S(Q)|}{|Q|} = \frac{|S(Q)|}{|Q|}.$$

Note we can switch to other sets when defining density (Thm 3.22, [Fol13]). By homogeneity, we take Q to be the unit cube, so we must prove $JT = |S(Q)|$. Consider now the spectral decomposition of $Sv_i = \lambda_i v_i$. Using the fact that $\nabla T = T$ as T is linear, pullback is contravariant, and

$$\langle S^*Sv_i, v_j \rangle = \langle Sv_i, Sv_j \rangle = \begin{cases} \lambda_i^2 & i = j \\ 0 & i \neq j, \end{cases}$$

we compute

$$JT = \sqrt{\det T^*T} = \sqrt{\det(S^*P^*PS)} = \sqrt{\det(S^*S)} = \sqrt{\prod_i |\lambda_i|^2} = |\det(S)| = |S(Q)|.$$

■

The next proof shows that the area formula does not see the *singular set* $\mathcal{S} := \{Jf = 0\}$. For notation, balls without mention to the dimension will be assumed to be dimension n , and without mention to the center will be assumed to be centered at the origin.

Proposition 4.6. *If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is Lipschitz, then on the singular set \mathcal{S} ,*

$$\mathcal{H}^n(f(\mathcal{S})) = 0.$$

Proof. We will deduce $\mathcal{H}_\infty^n(f(\mathcal{S} \cap B_R)) = 0$ for arbitrary $R > 0$ (recall $\mathcal{H}^n \ll \mathcal{H}_\infty^n$). For $x \in \mathcal{S}$ and $1 > \epsilon > 0$,³ by definition $\nabla f(x)$ is the linear map which satisfies the inequality

$$|f(x+v) - f(x) - \nabla f(x)v| < \epsilon|v|,$$

for $|v| < r(\epsilon, x)$. We can reinterpret the above (think of $f(x) = 0$.) in terms of small balls⁴: for $r = |v| < r(\epsilon, x)$, we have

$$f(B_r(x)) \subset f(x) + B_{\epsilon r}(\nabla f(x)(B_r)).$$

It would serve us well, therefore, to find a bound on the (translation-invariant) size of $B_{\epsilon r}(\nabla f(x)(B_r))$

Lemma 4.7. *For $0 < \epsilon < 1$ and $C = C(n, \text{Lip} f)$,*

$$\mathcal{H}_\infty^n(B_{\epsilon r}(\nabla f(x)(B_r))) \leq Cr^n \epsilon.$$

The proof will use that \mathcal{S} is singular by using a dimension drop argument, which produces the ϵ . Assuming this lemma, we get for $x \in \mathcal{S}$,

$$\mathcal{H}_\infty^n(f(B_r(x))) \leq Cr^n \epsilon. \tag{2}$$

We take the family \mathcal{F} of balls with centers in $\mathcal{S} \cap B_R$,

$$\mathcal{F} := \{B_r(x) \subset \mathbb{R}^n : x \in \mathcal{S} \cap B_R, 0 < r < r(\epsilon, x)\}.$$

³As far as I can tell, the restriction to $1 > \epsilon$ is for polish.

⁴Compare this to the definition of continuity in terms of balls. f is continuous at x if for $r < r(\epsilon, x)$,

$$f(B_r(x)) \subset f(x) + B_\epsilon.$$

So if you can take a derivative, the information you gain is having a modulus of control over the error tolerance ϵr in terms on your parameter r .

We cutoff \mathcal{S} by B_R to have bounded centers; we may therefore apply Besicovitch covering theorem to extract subfamilies $\mathcal{F}_1, \dots, \mathcal{F}_{\xi(n)}$ such that each \mathcal{F}_i is countable, pairwise disjoint and $\mathcal{S} \cap B_R \subset \cup_{\xi(n)} \mathcal{F}_i$. By inequality 2,

$$\begin{aligned}
\mathcal{H}_\infty^n(f(\mathcal{S} \cap B_R)) &\leq \sum_{i=1}^{\xi(n)} \sum_{B_r(x) \in \mathcal{F}_i} \mathcal{H}_\infty^n(f(B_r(x))) \\
&\leq C\omega_n \epsilon \sum_{i=1}^{\xi(n)} \sum_{B_r(x) \in \mathcal{F}_i} r^n \\
&\leq \frac{C\epsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \sum_{B_r(x) \in \mathcal{F}_i} |B_r(x)| \\
&= \frac{C\epsilon}{\omega_n} \sum_{i=1}^{\xi(n)} \left| \bigcup_{B_r(x) \in \mathcal{F}_i} B_r(x) \right| \\
&\leq \frac{C\epsilon \xi(n)}{\omega_n} |B_1(\mathcal{S} \cap B_R)|.
\end{aligned}$$

Therefore, it suffices to prove the lemma.

(of Lemma 4.7). We first identify that $\nabla f(x)(B_r) \subset B_{r\text{Lip}f}^m \cap \nabla f(x)(\mathbb{R}^n)$, where the right hand side is a disk $D_{r\text{Lip}f}$ of dimension $k < n$ since $x \in \mathcal{S}$. This follows from homogeneity and $\|\nabla f\| \leq \text{Lip}f$, since we may characterize $\text{Lip}f$ as

$$\text{Lip}f = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Therefore, we must obtain a bound of the form

$$\mathcal{H}_\infty^n(B_\epsilon(D_s)) \leq C(n, s)\epsilon, \quad (3)$$

since taking a neighborhood of a disk is linear,

$$\begin{aligned}
\mathcal{H}_\infty^n(B_{\epsilon r}(\nabla f(x)(B_r))) &\leq \mathcal{H}_\infty^n(B_{\epsilon r}(B_{r\text{Lip}f}^m \cap \nabla f(x)(\mathbb{R}^n))) \\
&\leq r^n \mathcal{H}_\infty^n(B_\epsilon(B_{\text{Lip}f}^m \cap \nabla f(x)(\mathbb{R}^n))) \\
&\leq r^n C(n, \text{Lip}f)\epsilon.
\end{aligned}$$

To obtain inequality 3, we produce a covering of $B_\epsilon(D_s) \subset \mathbb{R}^k \times \mathbb{R}^{m-k}$ by translates of

$$B_{\epsilon s}^k \times B_\epsilon^{m-k} \subset \mathbb{R}^k \times \mathbb{R}^{m-k}.$$

Note that we need $C\epsilon^{-k}$ number of translates to cover $B_\epsilon(D_s) \approx B_s^k \times B_\epsilon^{m-k}$ for small ϵ , since area of B_s^k grows like s^k .⁵ By Pythagorean theorem,

$$\text{diam}(B_{\epsilon s}^k \times B_\epsilon^{m-k})^2 = \text{diam}(B_{\epsilon s}^k)^2 + \text{diam}(B_\epsilon^{m-k})^2 = 4\epsilon^2(1 + s^2).$$

⁵e.g. For 2-dimensional disks, need (up to a dimensional constant) one-hundred small disks $D_{1/10}$ to cover one big disk D_1 .

Since $k < n$, we have

$$\begin{aligned}
\mathcal{H}_\infty^n(B_\epsilon(D_s)) &\leq \omega_n \sum_{\# \text{ translates}} \left(\frac{\text{diam}(B_{\epsilon s}^k \times B_\epsilon^{m-k})}{2} \right)^n \\
&= \omega_n C \epsilon^{-k} (2\epsilon^2(1+s^2))^{\frac{n}{2}} \\
&= C(n, s) \epsilon^{n-k} \\
&\leq C\epsilon.
\end{aligned}$$

■

By the lemma, $\mathcal{H}^n(\mathcal{S}) = 0$.

■

5 Approximate Tangent Spaces (9.20.23)

We follow [Mag12] Chapters 10.1-10.2.

Set $k \leq n$. We say a set $M \subset \mathbb{R}^n$ is **k -rectifiable** if there are countably many Lipschitz maps $f_i : \mathbb{R}^k \rightarrow \mathbb{R}^n$ such that

$$\mathcal{H}^k(M \setminus \cup_{i=1}^\infty f_i(\mathbb{R}^k)) = 0.$$

The point is that a k -rectifiable set is a measure-theoretic version of a k -dimensional manifold. We say M is **locally k -rectifiable** if for every compact K , $\mathcal{H}^k(K \cap M) < \infty$. The point here is that $\mathcal{H}^k \llcorner M$ is Radon (not just Borel) iff M is locally k -rectifiable. By Kirezbraun theorem and regularity properties of Hausdorff measure, M is k -rectifiable iff there exists Borel M_0 which is \mathcal{H}^k -null such that

$$M = M_0 \cup \bigcup_{i=1}^\infty f_i(E_i),$$

where E_i are bounded, Borel sets, and f_i is Lipschitz. This decomposition is highly non-unique, and we will exploit this in the following lemma by asking for a decomposition with nice regularity properties. Namely, for each $i \in \mathbb{N}$, the pair (f_i, E_i) will form a regular Lipschitz image. We say (f, E) form a **regular Lipschitz image** if

1. f is injective (so $k \leq n$) and differentiable on E , and $Jf > 0$ (i.e. $Jf \neq 0$ anywhere) on E ($\implies f$ is an injective immersion on E).
2. All points in E form a point of density 1 for E .
3. All point in E are Lebesgue points of ∇f (\implies all points are Lebesgue points of Jf).
4. There exists a lower bound for $\text{Lip}(f)$ on E .

Morally speaking, this is saying you are a Lipschitz injective immersion with good in C^1 control. Such a decomposition of a rectifiable set M always exists (Theorem 10.1) by a number of previous theorems. Here are the quoted ones from [Mag12].

1. **Theorem 2.10:** Borel sets admit inner/outer approximations, if the measure is a locally finite Borel measure.
2. **Theorem 8.7:** Singular values of a Lipschitz function are \mathcal{H}^k -null.
3. **Theorem 8.8:** The regular points of a Lipschitz function admit a countable, almost flat partition, on the function is injective on each leaf.
4. **Rademacher's:** Lipschitz functions are differentiable almost everywhere.
5. **Theorem 5.16:** Almost every point of a $L^1_{\text{loc}}(\mu)$ function satisfies MVP, if μ is Radon.

The point of the above four axioms is that they are sufficient conditions to make the following lemma true. This lemma classifies the approximate tangent space for the image of a regular Lipschitz function.

Lemma 5.1. *Suppose $M^k = f(E)$ with (f, E) a regular Lipschitz image. Then, then for any $x \in M$, the approximate tangent space is given by*

$$T_x M = (\nabla f|_z)(\mathbb{R}^k),$$

where $x = f(z)$.

Proof. Recall that the approximate tangent plane to M at x is the k -plane $T_x M \subset \mathbb{R}^n$ such that

$$\lim_{r \downarrow 0} \frac{1}{r^k} \int_M \phi \left(\frac{y - x}{r} \right) d\mathcal{H}^k(y) = \int_{T_x M} \phi(y) d\mathcal{H}^k(y),$$

for all $\phi \in C_c(\mathbb{R}^n)$. This is the measure-theoretic version of a tangent space. By pulling back along f , and using change of variables,

$$\begin{aligned} \frac{1}{r^k} \int_M \phi \left(\frac{y - x}{r} \right) d\mathcal{H}^k(y) &= \frac{1}{r^k} \int_E \phi \left(\frac{f(\tilde{w}) - f(z)}{r} \right) Jf(\tilde{w}) d\tilde{w} \\ &= \underbrace{\int_{\mathbb{R}^k} \mathbb{1}_E(z + rw) \cdot \phi \left(\frac{f(z + rw) - f(z)}{r} \right) Jf(z + rw) dw}_{:= u_r(w)} \end{aligned}$$

Since $z \in E$ is a point of density 1 for E ,

$$\lim_{r \downarrow 0} \int_{\mathbb{R}^k} \mathbb{1}_E(z + rw) dw = \mathbb{1}_E(z) = 1.$$

Since $z \in E$ is Lebesgue point for Jf ,

$$\lim_{r \downarrow 0} \int_{\mathbb{R}^k} Jf(z + rw) dw = J(z).$$

Take the limit as $r \downarrow 0$ on both sides. For the moment, take on faith that DCT can be justified, and switch $\lim_{r \downarrow 0}$ with \int . Since ϕ is continuous and f is differentiable at $z \in E$,

$$\phi \left(\frac{f(z + rw) - f(z)}{r} \right) \xrightarrow{r \downarrow 0} \phi(\nabla f|_z w).^6$$

⁶ *Type check:* note that $\nabla f|_z$ is a linear map from $\mathbb{R}^k \rightarrow \mathbb{R}^n$ (the Jacobian), and w is a vector in \mathbb{R}^k .

Altogether,⁷

$$\begin{aligned}
\lim_{r \downarrow 0} \int_{\mathbb{R}^k} u_r(w) \, dw &= \int_{\mathbb{R}^k} \phi(\nabla f|_z w) Jf(z) \, dw \\
&= \int_{\mathbb{R}^k} \phi(\nabla f|_z w) J(\nabla f|_z)(w) \, dw \\
&= \int_{\nabla f|_z(\mathbb{R}^k)} \phi \, d\mathcal{H}^k,
\end{aligned}$$

where the last step follows from area formula of injective, Lipschitz maps. Now to justify DCT. Observe the L^∞ (not pointwise, as values of Jf can take ∞ on a null set) upper bound

$$\|u_r\|_\infty \leq \left(\sup_{\mathbb{R}^n} \phi \right) \cdot \text{Lip}(f)^k,$$

and notice the RHS is a constant function. It suffices then to show that for all $r > 0$, $\text{spt} u_r \subset B_R$, for some R which is independent of r . Since $\text{spt} u_r := \overline{\{u_r > 0\}}$, and a set is bounded iff its closure is, it suffices to show $\{u_r > 0\} \subset B_R$. Take $w \in \{u_r > 0\}$, so that

$$0 < \mathbb{1}_E(z + rw) \cdot \phi \left(\frac{f(z + rw) - f(z)}{r} \right) Jf(z + rw);$$

in particular, $z + rw \in E$. Thus, last condition of a regular Lipschitz image immediately gives for some $\lambda = \lambda(E) > 0$,

$$|f(z + rw) - f(z)| \geq \lambda r |w|. \quad (4)$$

On the other hand, since

$$0 < \mathbb{1}_E(z + rw) \cdot \phi \left(\frac{f(z + rw) - f(z)}{r} \right) Jf(z + rw),$$

it follows that

$$\frac{f(z + rw) - f(z)}{r} \in \text{spt} \phi.$$

Since $\text{spt} \phi \subset B_\Lambda$ for some $\Lambda = \Lambda(\phi)$,

$$\left| \frac{f(z + rw) - f(z)}{r} \right| \leq \Lambda. \quad (5)$$

Together, inequalities 4 and 5 together give the conclusion upon setting $R := \frac{\Lambda}{\lambda}$, as

$$\lambda r |w| \leq r \Lambda \implies w \in B_{\frac{\Lambda}{\lambda}}.$$

■

⁷The gradient of the linear map $\nabla f|_z$ is $\nabla f|_z$. Thus

$$J(\nabla f|_z)(w) = \sqrt{\det((\nabla(\nabla f|_z)|_w)^* \cdot (\nabla(\nabla f|_z)|_w))} = \sqrt{\det(\nabla f|_z^* \cdot \nabla f|_z)} = Jf(z).$$

Theorem 5.2. Suppose $M \subset \mathbb{R}^n$ is a locally k -rectifiable set. Then for \mathcal{H}^k -a.e. point in M , there exists a unique k -dimensional plane $T_x M$ such that:

1. The blow-up (defined below) of the measure $\mathcal{H}^k \llcorner M$ weak-star converges to $\mathcal{H}^k \llcorner T_x M$. That is, as $r \downarrow 0$,

$$\mathcal{H}^k \llcorner \left(\frac{M - x}{r} \right) \xrightarrow{*} \mathcal{H}^k \llcorner T_x M.$$

2. For \mathcal{H}^k -a.e. $x \in M$,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^k(M \cap B_r(x))}{\omega_k r^k} = 1.$$

More precisely, this happens whenever x admits an approximate tangent space.

Proof. In the proof, we will need the notion of a blow-up. For fixed $x \in M, r > 0$, we set

$$\eta_{x,r}(y) = \eta(y) := \frac{y - x}{r}.$$

1. This would immediately follow from Lemma 5.1 upon setting $T_x M = \nabla f|_z(\mathbb{R}^k)$, if not for the \mathcal{H}^k -null set M_0 . This will be taken care of by the *upper density theorem* which tells us for each i , as M_i is locally k -rectifiable, for a.e. $x \notin M_i$,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^k(M_i \cap B_r(x))}{\omega_k r^k} = 0.$$

Note that since the RHS is 0, in fact the numerator and denominator need not decay at the same rate. So more generally, we have for $r, R > 0$,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^k(M_i \cap B_{rR}(x))}{\omega_k r^k} = 0.$$

We use this as follows. Let $\phi \in C_c(\mathbb{R}^n)$ such that $\text{spt} \phi \subset B_R(0)$. Then, for $x \in M_i$,

$$\begin{aligned} \left| \frac{1}{r^k} \int_{M \setminus M_i} \phi \circ \eta_{x,r} d\mathcal{H}^k \right| &= \left| \int_{\eta_{x,r}(M \setminus M_i)} \phi d\mathcal{H}^k \right| \\ &\leq \mathcal{H}^k(B_R(0) \cap \eta_{x,r}(M \setminus M_i)) \cdot \sup_{\mathbb{R}^n} |\phi| \\ &= \omega_k \sup_{\mathbb{R}^n} |\phi| \cdot \frac{\mathcal{H}^k(B_{rR}(0) \cap (M \setminus M_i) - x)}{\omega_k r^k} \\ &= \omega_k \sup_{\mathbb{R}^n} |\phi| \cdot \frac{\mathcal{H}^k(B_{rR}(x) \cap (M \setminus M_i))}{\omega_k r^k} \\ &\xrightarrow{r \downarrow 0} 0. \end{aligned}$$

2. This follows from algebraic set-theoretic interactions of the blow-up with Hausdorff measure. Choose a sequence $\{\phi_i\} \subset C_c(\mathbb{R}^n)$ such that $\phi_i \xrightarrow{i \rightarrow \infty} \mathbb{1}_{B_1^n(0)}$. On one hand,

$$\lim_{r \downarrow 0} \int_M \phi_i \circ \eta(y) d\mathcal{H}^k(y) = \lim_{r \downarrow 0} \int_{\eta M} \phi_i(y) d\mathcal{H}^k(y) \xrightarrow{i \rightarrow \infty} \lim_{r \downarrow 0} \mathcal{H}^k(\eta M \cap B_1^n(0)).$$

On the other hand,

$$\int_{T_x M} \phi_i(y) d\mathcal{H}^k(y) \xrightarrow{i \rightarrow \infty} \mathcal{H}^k(T_x M \cap B_1(0)) = \mathcal{H}^k(B_1^k(0)) = \omega_k.$$

As the two expressions are equal, we calculate

$$\begin{aligned} 1 &= \lim_{r \downarrow 0} \frac{\mathcal{H}^k(\eta M \cap B_1^n(0))}{\omega_k} \\ &= \lim_{r \downarrow 0} \frac{r^k \mathcal{H}^k(\eta M \cap B_1^n(0))}{\omega_k r^k} \\ &= \lim_{r \downarrow 0} \frac{\mathcal{H}^k((M - x) \cap B_r^n(0))}{\omega_k r^k} \\ &= \lim_{r \downarrow 0} \frac{\mathcal{H}^k(M \cap B_r^n(x))}{\omega_k r^k}. \end{aligned}$$

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