

# Notes on Schauder Estimates

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The following are the *standard assumptions*. On bounded domains  $\Omega \subset \mathbb{R}^n$ , will consider 2nd order (uniformly) elliptic operators

$$Lu := \sum_{i,j} a^{ij} u_{ij} + \sum_i b^i u_i + cu = f,$$

where  $a^{ij}, b^i, c$  are functions with regularity to be specified later. We assume symmetry on the leading order coefficient terms  $a^{ij} = a^{ji}$ , and for all uniform ellipticity says  $\xi \in \mathbb{R}^n$ , for some positive constant  $\lambda > 0$ ,

$$a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2.$$

The regularity of this approach will require our functions  $u, f$  to lie in appropriate Hölder spaces. In this note, we will introduce a zoo of seminorms and norms on the way to proving Schauder estimates, each of which will slightly differently exploit the fact that  $\Omega$  is bounded.



To start, we introduce the standard Hölder spaces and their respective norms and seminorms. This approach will be pointwise in nature, compared to the integral-based approach of Sobolev spaces last semester. Let  $\alpha \in (0, 1)$ . With respect to  $\Omega$ , we have the pointwise (and global) Hölder seminorms which detect pointwise (and uniform)  $\alpha$ -Hölder continuity. We may also define local  $\alpha$ -Hölder continuity by asking for every compact  $K \subset \Omega$ , that  $[u]_{\alpha;K}$  is finite. In such a case, we say  $u$  is locally uniformly continuous.

1. **(Pointwise)** For  $x_o \in \Omega$ ,

$$[u]_{\alpha, x_o} := \sup_{x \in \Omega} \frac{|u(x) - u(x_o)|}{|x - x_o|^\alpha}.$$

2. **(Local)** For all compact  $K \subset \Omega$ ,

$$\sup_{x \neq y \in K} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty$$

3. **(Global)**

$$[u]_{\alpha; \Omega} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

As usual, global implies local implies pointwise. The above gives rise to  $C^{k,\alpha}(\bar{\Omega})$  (and  $C^{k,\alpha}(\Omega)$ ) consisting of functions whose  $k$ -th order derivatives are uniformly (and locally uniformly) continuous. In the global case, we get a seminorm as defined above. We can formally extend this to  $\alpha = 0, 1$  which gives continuity and Lipschitz continuity respectively.

Hölder regularity gives a quantitative measurement of continuity. In fact, greater the  $\alpha$ , the more regularity we have.<sup>1</sup> One can see this in the following illustrative example:

**Example 0.0.1.** *The function  $f(x) = |x|^\beta$  is  $\alpha$ -Hölder continuous at  $x = 0$  iff  $\alpha \leq \beta$ .*

*Proof.* Let  $\Omega$  be a domain containing 0. We compute

$$\sup_{x \neq 0 \in \Omega} \frac{|f(x) - f(0)|}{|x - 0|^\alpha} = \sup_{x \neq 0 \in \Omega} |x|^{\beta-\alpha} = \begin{cases} 0 & \alpha \leq \beta \\ \infty & \alpha > \beta. \end{cases}$$

□

Recall the  $C^k(\Omega)$  seminorm is defined by measuring how large the  $k$ th derivative of  $u$  on  $\Omega$  is,

$$[u]_{k,0;\Omega} := |D^k u|_{0;\Omega} := \sup_{x \in \Omega, |\beta|=k} |D^\beta u|,$$

where  $\beta$  is a multi-index of length  $k$ . Similarly, we have the  $C^k(\Omega)$  norm by adding the largest of all the derivatives to order  $k$ ,

$$\|u\|_{k,0;\Omega} := \sum_{j=1}^k |D^j u|_{0,\Omega}.$$

These definitions admit easy extensions to include  $\alpha$ -Hölder regularity, namely we have the *standard (semi)norm* of  $C^{k,\alpha}(\bar{\Omega})$ .

1.  $[u]_{k,\alpha;\Omega} := [D^k u]_{\alpha;\Omega} := \sup_{|\beta|=k} [D^\beta u].$
2.  $\|u\|_{k,\alpha;\Omega} := \|u\|_{C^k(\bar{\Omega})} + [u]_{k,\alpha;\Omega}.$

One may check that  $C^{k,\alpha}(\bar{\Omega})$  for  $0 \leq \alpha < 1$  are Banach spaces with the standard norms. For  $\alpha = 0$ , one can think of seminorm as taking the “top degree” portion of the norm. In the  $\alpha > 0$  case, one can think of the seminorm as taking the “Hölder portion of the norm.”<sup>2</sup>

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<sup>1</sup>We briefly mention that for  $k = 0$ , we have a Rellich-type result for Hölder spaces. Namely for  $0 < \alpha < \beta \leq 1$ , we have the compact inclusion

$$C^{0,\beta}(\Omega) \subset C^{0,\alpha}(\Omega).$$

<sup>2</sup>As a mental schematic, the Hölder exponents  $\alpha \in [0, 1)$  allow us to expand our regularity index from  $\mathbb{Z}^{\geq 0}$  to the half ray  $[0, \infty)$ . It's unfortunate for this schematic that Lipschitz functions have a difficult relationship with  $C^1$  functions. Recall that  $C^1$  implies locally Lipschitz, while Lipschitz implies differentiable a.e.

# 1 The Classical Proof

We unabashedly follow Chapters 2, 4, 6 of [2].

## 1.1 Reduction to the Laplacian

Roughly speaking, the Schauder interior estimates tell us that we can bound the  $C^{2,\alpha}$  norm of  $u$  in terms of the  $C^0$  norm of  $u$  and the  $C^{0,\alpha}$  norm of  $f$ . Explicitly, we have the following theorem.

**Theorem 1.1.1. (*Interior Estimates*)**<sup>3</sup> *Given the standard assumptions, if*

$$|a^{ij}|_{0,\alpha;\Omega}^{(0)}, |b^i|_{0,\alpha;\Omega}^{(1)}, |c|_{0,\alpha;\Omega}^{(2)} \leq \Lambda,$$

*then the following estimate holds*

$$|u|_{2,\alpha;\Omega}^* \leq C(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

We now introduce the relevant norms so the above statement is legible. The following Hölder (semi)norms are weighted by their distances to the boundary. Namely for  $x \neq y \in \Omega$ , consider

$$d_x := d(x, \partial\Omega) \quad d_{x,y} := \min\{d_x, d_y\}.$$

We use the above two quantities to define the following (semi)norms. For  $\alpha \in [0, 1]$  and  $u \in C^{k,\alpha}(\Omega)$ , we first define the  $\theta$ -homogeneous (semi)norm of  $u \in C^{k,\alpha}(\Omega)$ .

1. **0-homogeneous seminorm.** For  $\alpha = 0$ ,

$$[u]_{k;\Omega}^* := \sup_{x \in \Omega, |\beta|=k} d_x^k |D^\beta u(x)|.$$

For  $\alpha > 0$ ,

$$[u]_{k,\alpha;\Omega}^* := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

2. **0-homogeneous norm.** For  $\alpha = 0$ ,

$$|u|_{k;\Omega}^* := \sum_{j=0}^k [u]_{j,\Omega}^*.$$

For  $\alpha > 0$ ,

$$|u|_{k,\alpha;\Omega}^* := |u|_{k;\Omega}^* + [u]_{k,\alpha;\Omega}^*.$$

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<sup>3</sup>Compare Theorem 1.1.1 to the regularity statement of ([1], pg 329, “Interior  $H^2$ -regularity”).

Next, we introduce the  $\sigma$ -homogeneous (semi)norm of  $u \in C^{k,\alpha}(\Omega)$  for  $\sigma \in \mathbb{R}$ , generalizing the above. The 0-homogeneous (semi)norms can obviously be recovered by setting  $\sigma = 0$ .

1.  **$\sigma$ -homogeneous seminorm.** For  $\alpha = 0$ ,

$$[u]_{k;\Omega}^{(\sigma)} := \sup_{x \in \Omega, |\beta|=k} d_x^{k+\sigma} |D^\beta u(x)|.$$

For  $\alpha > 0$ ,

$$[u]_{k,\alpha;\Omega}^{(\sigma)} := \sup_{x \neq y \in \Omega, |\beta|=k} d_{x,y}^{k+\alpha+\sigma} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\alpha}.$$

2.  **$\sigma$ -homogeneous norm.** For  $\alpha = 0$ ,

$$|u|_{k;\Omega}^{(\sigma)} := \sum_{j=0}^k [u]_{j,\Omega}^{(\sigma)}.$$

For  $\alpha > 0$ ,

$$|u|_{k,\alpha;\Omega}^{(\sigma)} := |u|_{k,\Omega}^{(\sigma)} + [u]_{k,\alpha;\Omega}^{(\sigma)}.$$

We show that the  $\sigma$ -inhomogeneous norms enjoy a submultiplicative property.<sup>4</sup>

**Lemma 1.1.2.** *For  $\sigma + \tau \geq 0$ , we have*

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} \leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}.$$

*Proof.* This is a Baby Rudin proof. Recall

$$|fg|_{0,\alpha;\Omega}^{(\sigma+\tau)} := \underbrace{\sup_{x \neq y \in \Omega} \left( d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x - y|^\alpha} \right)}_{(*)} + \underbrace{\sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|)}_{(**)}.$$

We estimate the first term by

$$\begin{aligned} (*) &= \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\sigma+\tau+\alpha} \frac{|fg(x) - fg(y)|}{|x - y|^\alpha} \right) \\ &\leq \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\sigma+\alpha+\tau} \cdot d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)|}{|x - y|^\alpha} \right) \\ &\leq \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\tau+\alpha+\sigma} \frac{|f(x)| \cdot |g(x) - g(y)|}{|x - y|^\alpha} \right) + \sup_{x \neq y \in \Omega} \left( d_{x,y}^{\tau+\alpha+\sigma} \frac{|g(y)| \cdot |f(x) - f(y)|}{|x - y|^\alpha} \right) \\ &\leq |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot |f|_{0,\alpha;\Omega}^{(\sigma)}. \end{aligned}$$

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<sup>4</sup>(6.11) of [2].

We estimate the second term by

$$\begin{aligned}
(**) &= \sup_{x \in \Omega} (d_x^{\sigma+\tau} |fg(x)|) \\
&\leq \sup_{x \in \Omega} (d_x^\sigma |f(x)|) \cdot \sup_{x \in \Omega} (d_x^\tau |g(x)|) \\
&= |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)}
\end{aligned}$$

Combining the two estimates,

$$\begin{aligned}
(*) + (**) &= |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)} + |g|_{0;\Omega}^{(\tau)} \cdot |f|_{0,\alpha;\Omega}^{(\sigma)} + |f|_{0;\Omega}^{(\sigma)} \cdot |g|_{0;\Omega}^{(\tau)} \\
&\leq |f|_{0,\alpha;\Omega}^{(\sigma)} \cdot |g|_{0,\alpha;\Omega}^{(\tau)}
\end{aligned}$$

□

Throughout the course of the proof, we will frequently employ the interpolation inequalities.<sup>5</sup> Unfortunately, we must omit the proof.

**Lemma 1.1.3.** *Let  $u \in C^{2,\alpha}(\Omega)$ , then for any  $\varepsilon > 0$ , there exists a constant  $C = C(\varepsilon)$  such that*

$$[u]_{j,\beta;\Omega}^* \leq |u|_{j,\beta;\Omega}^* \leq C|u|_{0;\Omega} + \varepsilon[u]_{2,\alpha;\Omega}^*,$$

where  $j = 0, 1, 2$  and  $0 \leq \alpha, \beta \leq 1$  and  $j + \beta < 2 + \alpha$ .



We first reduce showing Schauder estimates from the general case to the case of constant coefficients with no lower order terms.<sup>6</sup>

*Proof.* (of 1.1.1.) By using a proper, compact exhaustion, it suffices to show the equality holds for every compact subset of  $\Omega$ . By the interpolation inequalities 1.1.3, it suffices to show the result where the LHS is  $[u]_{2,\alpha;\Omega}^*$ . Thus, the relevant quantity to consider is

$$d_{x_o, y_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha},$$

for arbitrary  $x_o, y_o \in \Omega$ . The idea here is to “freeze” one variable to reduce to the case of constant coefficients with no lower order terms. Suppose  $d_{x_o} = d_{x_o, y_o}$ , and consider

$$\sum_{i,j} a^{ij}(x_o) u_{ij} = F(x),$$

where  $F := \sum_{i,j} a^{ij}(x_o) u_{ij} - Lu + f$ . Assuming local Schauder estimates on the ball

$$[u]_{2,\alpha;B}^* \leq C(|u|_{0;B} + |f|_{0,\alpha;B}^{(2)}),$$

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<sup>5</sup>(6.8) and (6.9) of [2].

<sup>6</sup>Theorem 6.2 of [2].

where  $B := B_{x_o}(\frac{d}{2})$  where  $d := \mu d_{x_o}$  for some  $\mu \leq \frac{1}{2}$  we will choose later. Thus, if  $y_o \in B$ , we have the estimate

$$\left(\frac{d}{2}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq [u]_{2,\alpha;B}^* \leq C(|u|_{0;B} + |F|_{0,\alpha;B}^{(2)}).$$

Since  $(\frac{d}{2})^{2+\alpha} = \left(\frac{d_{x_o}\mu}{2}\right)^{2+\alpha}$ , we enlarge our domain to yield the estimate

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right).$$

For  $y_o \notin B \iff 2|x_o - y_o| \geq d$ , and so  $(2|x_o - y_o|)^\alpha \geq d^\alpha$ . Thus,

$$\begin{aligned} d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} &= \left(\frac{d}{\mu}\right)^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \\ &\leq \frac{d^2 \cdot 2^\alpha}{\mu^{2+\alpha}} |D^2 u(x_o) - D^2 u(y_o)| \\ &= \frac{(d_{x_o}\mu)^2 \cdot 2^\alpha}{\mu^2 \cdot \mu^\alpha} |D^2 u(x_o) - D^2 u(y_o)| \\ &\leq \left(\frac{2}{\mu}\right)^\alpha (d_{x_o}^2 |D^2 u(x_o)| + d_{y_o}^2 |D^2 u(y_o)|) \\ &\leq \frac{2^{\alpha+1}}{\mu^\alpha} [u]_{2;\Omega}^* \\ &\leq \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*. \end{aligned}$$

Combining the two estimates, we have

$$d_{x_o}^{2+\alpha} \frac{|D^2 u(x_o) - D^2 u(y_o)|}{|x_o - y_o|^\alpha} \leq \frac{C}{\mu^{2+\alpha}} \left(|u|_{0;\Omega} + |F|_{0,\alpha;B}^{(2)}\right) + \frac{4}{\mu^\alpha} [u]_{2;\Omega}^*$$



The next step is to estimate  $|F|_{0,\alpha;B}^{(2)}$  in terms of  $|u|_{0;\Omega}$  and  $[u]_{2,\alpha;\Omega}^*$ . Specifically, we will estimate the terms on the RHS of

$$|F|_{0,\alpha;B}^{(2)} \leq \sum_{i,j} |(a^{ij}(x_o) - a^{ij}(x))u_{ij}|_{0,\alpha;B}^{(2)} + \sum_i |b^i u_i|_{0,\alpha;B}^{(2)} + |cu|_{0,\alpha;B}^{(2)} + |f|_{0,\alpha;B}^{(2)}.$$

We first prove a generic estimate for arbitrary  $g \in C^{0,\alpha}(\Omega)$  which relates the local 2-homogeneous norm on  $B$  to the global one. For  $x \in B$ , define  $d_x^{\partial B} := \text{dist}(x, \partial B)$  and  $d_{x,y}^{\partial B} := \min\{d_x^{\partial B}, d_y^{\partial B}\}$ . Note that upon localizing to  $B$ , we need to consider the distance to

$\partial B$  not to  $\partial\Omega$ .

$$\begin{aligned}
|g|_{0,\alpha;B}^{(2)} &\leq \sup_{x \in B} (d_x^{\partial B})^2 \cdot \sup_{x \in B} |g(x)| + \sup_{x \neq y \in B} (d_{x,y}^{\partial B})^{2+\alpha} \cdot \sup_{x \neq y \in B} \frac{|g(x) - g(y)|}{|x - y|^\alpha} \\
&= d^2 |g|_{0;B} + d^{2+\alpha} [g]_{\alpha;B} \\
&\leq [g]_{0;\Omega}^{(2)} + [g]_{\alpha;\Omega}^{(2)} \\
&\leq \left( \frac{\mu}{1-\mu} \right)^2 [g]_{0;\Omega}^{(2)} + \left( \frac{\mu}{1-\mu} \right)^{2+\alpha} [g]_{\alpha;\Omega}^{(2)} \\
&\leq 4\mu^2 [g]_{0;\Omega}^{(2)} + 8\mu^{2+\alpha} [g]_{\alpha;\Omega}^{(2)}
\end{aligned}$$

The third line follows since for any  $x \in B$  which comes from the computation

$$d = \mu d_{x_o} \leq (1 - \mu) d_{x_o} \leq d_x.$$

The fifth line follows from  $\mu \leq \frac{1}{2} \implies \left( \frac{1}{1-\mu} \right)^2 \leq 4$  and  $\left( \frac{1}{1-\mu} \right)^{2+\alpha} \leq 8$ . Since all our norms are defined as a supremum, and replacing  $d$  with  $d_x$  is nondecreasing on  $B$ , the result follows. We now apply estimates in detail to the principal term; the lower order terms are treated similarly. By Lemma 1.1.2, we compute

$$\begin{aligned}
|(a^{ij}(x_o) - a^{ij})u_{ij}|_{0,\alpha;B}^{(2)} &\leq |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)} \cdot |u_{ij}|_{0,\alpha;B}^{(2)} \\
&\leq \underbrace{|a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(0)}}_{(*)} \cdot (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*).
\end{aligned}$$

We also estimate the local term  $(*)$  by

$$\begin{aligned}
(*) &= \sup_{x \in B} (|a^{ij}(x_o) - a^{ij}(x)|) + \sup_{x \neq y \in B} \left( (d_{x,y}^{\partial B})^\alpha \frac{|a^{ij}(x_o) - a^{ij}(x) - (a^{ij}(x_o) - a^{ij}(y))|}{|x - y|^\alpha} \right) \\
&\leq \sup_{x \neq x_o \in B} \left( \frac{|a^{ij}(x_o) - a^{ij}(x)|}{|x - x_o|^\alpha} \cdot |x - x_o|^\alpha \right) + d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2d^\alpha [a^{ij}]_{0,\alpha;B} \\
&\leq 2(2\mu)^\alpha [a^{ij}]_{0,\alpha;\Omega}^* \\
&\leq 4\Lambda\mu^\alpha,
\end{aligned}$$

where the second to last inequality follows since  $d \leq d_x$  for all  $x \in B$ , and  $\mu \leq \frac{1}{2}$ . Altogether, we find

$$\begin{aligned}
\sum_{i,j} |a^{ij}(x_o) - a^{ij}|_{0,\alpha;B}^{(2)} &\leq 4n^2 \Lambda \mu^\alpha (4\mu^2 [u]_{2;\Omega}^* + 8\mu^{2+\alpha} [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} ([u]_{2;\Omega}^* + \mu^\alpha [u]_{2,\alpha;\Omega}^*) \\
&\leq 32n^2 \Lambda \mu^{2+\alpha} (C(\mu) |u|_{0;\Omega} + 2\mu^\alpha [u]_{2,\alpha;\Omega}^*),
\end{aligned}$$

where the last step follows from the interpolation inequalities 1.1.3. Similarly, we have the estimates

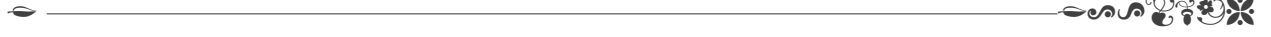
$$\sum_i |b^i u_i|_{0,\alpha;B}^{(2)} \leq 8n \Lambda \mu^2 (C(\mu) |u|_{0;\Omega} + \mu^{2\alpha} [u]_{2,\alpha;\Omega}^*),$$

$$|cu|_{0,\alpha;B}^{(2)} \leq 8\Lambda\mu^2(C(\mu)|u|_{0;\Omega} + \mu^{2\alpha}[u]_{2,\alpha;\Omega}^*),$$

$$|f|_{0,\alpha;\Omega}^{(2)} \leq 8\mu^2|f|_{0,\alpha;\Omega}^{(2)}.$$

Altogether, we estimate  $F$  by

$$|F|_{0,\alpha;B}^{(2)} \leq C\mu^{2+2\alpha}[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$



Combining the two steps and using the interpolation inequalities 1.1.3, we get

$$[u]_{2,\alpha;\Omega}^* \leq C\mu^\alpha[u]_{2,\alpha;\Omega}^* + C(\mu)(|u|_{0;\Omega} + |f|_{0,\alpha;\Omega}^{(2)}).$$

Fixing  $\mu$  so that  $C\mu^\alpha \leq \frac{1}{2}$  yields the desired result.  $\square$

We quickly sketch a proof of Schauder estimates for the case of constant coefficients with no lower order terms.<sup>7</sup> Consider

$$L_0 u := \sum_{i,j} A^{ij} u_{ij},$$

where  $A := (A^{ij}) \in \text{GL}_n(\mathbb{R})$  is a constant matrix. In this case, we diagonalize  $A$  by an orthogonal matrix<sup>8</sup> to a diagonal one, turning  $L_0 u(x) = f(x)$  to  $\Delta \tilde{u}(y) = \tilde{f}(y)$ . Since our transformations are all linear, the norms are equivalent, and so it suffices to prove Schauder estimates for the Laplacian.

## 1.2 The Laplacian

We first introduce the *fundamental solution* and some basic ideas surrounding it. The upshot is that we obtain some estimates for future use, and the following *Green's representation formula*,<sup>9</sup> to prove the following theorem.

**Theorem 1.2.1.** *Let  $u \in C^2(\Omega)$  and  $f \in C^{0,\alpha}(\Omega)$  such that  $\Delta u = f$ . Then,*

$$u = h + w,$$

*where  $h$  is some harmonic function, and  $w$  is the Newtonian potential of  $f$ .*

Fix  $x \in \Omega \subset \mathbb{R}^n$  for  $n \geq 2$ .<sup>10</sup> Let  $\omega_n$  be the volume of the  $n$ -ball. Define the fundamental solution on  $\Omega - x$  as the following radial function about  $x$ :

$$\Gamma(x - y) := \begin{cases} \frac{1}{n(2-n)\omega_n} |x - y|^{2-n} & n > 2 \\ \frac{1}{2\pi} \log |x - y| & n = 2. \end{cases}$$

<sup>7</sup>Lemma 6.1. of [2].

<sup>8</sup>This follows by spectral theorem, since  $A$  is a symmetric, real-valued, nonsingular matrix.

<sup>9</sup>(2.16) of [2].

<sup>10</sup>Careful! [2] uses “y” to denote the singularity in Chapter 2, but switches to “x” to denote the singularity in Chapter 4.



We first obtain some basic estimates on  $\Gamma$ . The following derivatives will be taken with respect to the variable  $x$ . For convenience, define

$$S(x) := \sum_{k=1}^n (x_k - y_k)^2,$$

so that  $|x - y| = S^{\frac{1}{2}}$ , and  $S_i = 2(x_i - y_i)$ . We calculate for  $n > 2$ ,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{n(2-n)\omega_n} \left( \frac{2-n}{2} S^{\frac{2-n}{2}-1} \cdot 2(x_i - y_i) \right) \\ &= \frac{1}{n\omega_n} |x - y|^{-n} (x_i - y_i). \end{aligned}$$

We calculate for  $n = 2$ ,

$$\begin{aligned} \Gamma_i(x - y) &= \frac{1}{2\pi} \frac{\frac{1}{2} S^{-\frac{1}{2}} \cdot S_i}{S^{\frac{1}{2}}} \\ &= \frac{1}{2\pi} S^{-1} (x_i - y_i) \\ &= \frac{1}{2\pi} |x - y|^{-2} (x_i - y_i). \end{aligned}$$

Continuing, we calculate the second derivatives for  $n \geq 2$ ,

$$\begin{aligned} \Gamma_{ij}(x - y) &= \frac{1}{n\omega_n} \left( \delta_{ij} S^{-\frac{n}{2}} + (x_i - y_i) \left( \frac{-n}{2} S^{-\frac{n}{2}-1} \cdot 2(x_j - y_j) \right) \right) \\ &= \frac{S^{-\frac{n}{2}-1}}{n\omega_n} (\delta_{ij} S^1 - n(x_i - y_i)(x_j - y_j)) \\ &= \frac{|x - y|^{-n-2}}{n\omega_n} (\delta_{ij} |x - y|^2 - n(x_i - y_i)(x_j - y_j)). \end{aligned}$$

$\Gamma$  is indeed harmonic, since

$$\Delta \Gamma = \sum_i \Gamma_{ii} = \frac{|x - y|^{-n-2}}{n\omega_n} \sum_{i=1}^n (\delta_{ii} |x - y|^2 - n(x_i - y_i)^2) = 0.$$

For near future use, we record estimates on the fundamental solutions.

**Proposition 1.2.2.** *Let  $\rho = |x - y|$  and for any multi-index  $\beta$ , we estimate*

$$|D^\beta \Gamma(\rho)| \leq C \rho^{2-n-|\beta|},$$

where  $C = C(n, |\beta|)$ . In fact, we calculate directly

$$\Gamma'(\rho) = \frac{1}{n\omega_n} \rho^{1-n}.$$

In Cartesian coordinates, we make the constant explicit to get

$$|\Gamma_i(x - y)| \leq \frac{1}{n\omega_n} |x - y|^{1-n},$$

$$|\Gamma_{ij}(x - y)| \leq \frac{1}{\omega_n} |x - y|^{-n}.$$

Next, we recall *Green's identities*. First, recall for  $f \in C^\infty(\Omega)$  and  $X \in \mathfrak{X}(\Omega)$ ,

$$\operatorname{div}(fX) = \nabla f \cdot X + f \operatorname{div} X.$$

In the case  $X = \nabla u$  for  $u \in C^2(\Omega)$ , we get

$$\operatorname{div}(f \nabla u) = \nabla f \cdot \nabla u + f \Delta u.$$

Integrating and using *divergence theorem* yields *Green's first identity*

$$\int_{\Omega} \nabla f \cdot \nabla u + \int_{\Omega} f \Delta u = \int_{\partial\Omega} (f \nabla u) \cdot \nu = \int_{\partial\Omega} f \frac{\partial u}{\partial \nu}.$$

Switching the roles of  $f, u$  and subtracting yields *Green's second identity*

$$\int_{\Omega} f \Delta u - u \Delta f = \int_{\partial\Omega} f \frac{\partial u}{\partial \nu} - u \frac{\partial f}{\partial \nu}.$$

We would like to use Green's second identity for  $f = \Gamma$ , but because of the singularity at  $x$ , we use it instead for  $\Omega - B_x(\rho)$  for small  $\rho$ . We compute

$$\int_{\Omega - B_x(\rho)} \Gamma \Delta u = \int_{\partial\Omega} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu} + \underbrace{\int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} - u \frac{\partial \Gamma}{\partial \nu}}_{(*)}.$$

We show  $(*) \rightarrow u(x)$  as  $\rho \rightarrow 0$ . Keeping in mind that  $\Gamma$  is radial, we proceed with estimating the first term of  $(*)$ ,

$$\begin{aligned} \int_{\partial B_x(\rho)} \Gamma \frac{\partial u}{\partial \nu} &= \Gamma(\rho) \cdot \int_{\partial B_x(\rho)} \frac{\partial u}{\partial \nu} \\ &\leq C(n) \rho^{2-n} \cdot n \omega_n \rho^{n-1} \sup_{\partial B_x(\rho)} |Du|. \end{aligned}$$

Since  $u \in C^2(\Omega)$ , the exponents work out so that the last quantity vanishes as  $\rho \rightarrow 0$ .<sup>11</sup> For the second term, we calculate by 1.2.2,<sup>12</sup>

$$\begin{aligned} - \int_{\partial B_x(\rho)} u \frac{\partial \Gamma}{\partial \nu} &= \Gamma'(\rho) \int_{\partial B_x(\rho)} u \\ &= \frac{1}{n \omega_n \rho^{n-1}} \int_{\partial B_x(\rho)} u. \end{aligned}$$

Since  $u$  is continuous, the last quantity goes to  $u(x)$  as  $\rho \rightarrow 0$ . Altogether, sending  $\rho \rightarrow 0$  yields Green's representation formula

$$u(x) = \underbrace{\int_{\partial\Omega} u(y) \frac{\partial \Gamma}{\partial \nu}(x-y) - \Gamma(x-y) \frac{\partial u}{\partial \nu}(y) ds}_{h(x)} + \underbrace{\int_{\Omega} \Gamma(x-y) \Delta u(x) dx}_{w(x)}.$$

<sup>11</sup>At least for  $n > 2$ . But the  $n = 2$  case follows similarly.

<sup>12</sup>Note since  $\nu$  is outward pointing on  $\Omega - B$ , then  $\nu$  into  $B$ . However,  $\rho$  is defined so that it is positive as you go away from  $y$ . Therefore, the sign changes.

Observe for compactly supported  $u$ , the second term vanishes, while for harmonic  $u$ , the first term vanishes. In general, we call

$$w(x) := \Gamma * f(x) = \int_{\Omega} \Gamma(x-y)f(y)dy$$

the Newtonian potential of  $f$ .



We now give formulas for the first two derivatives of the Newtonian potential.

**Proposition 1.2.3.** *Suppose  $f$  is bounded, integrable in  $\Omega$  with Newtonian potential  $w$ . Then  $w \in C^1(\mathbb{R}^n)$  and its derivative given by the formula*

$$w_i(x) = \int_{\Omega} \Gamma_i(x-y)f(y)dy.$$

*Proof.* Set  $v(x) := \int_{\Omega} \Gamma_i(x-y)f(y)dy$ . By the asymptotic estimates of 1.2.2 and the fact that  $f$  is bounded, we find that  $v$  is well-defined. We consider the sequence  $w^\varepsilon := (\Gamma\eta^\varepsilon) * f$ , where  $\eta^\varepsilon = \eta\left(\frac{|x-y|}{\varepsilon}\right)$ . Here,  $\eta \in C^1(\mathbb{R})$  is a function such that

$$\eta(t) = \begin{cases} 1 & t \geq 2 \\ 0 & t \leq 1, \end{cases} \text{ and } 0 \leq \eta' \leq 2.$$

Therefore,  $w^\varepsilon(x)$  has the effect of doing nothing far from the singularity of  $\Gamma$  (at distances  $|x-y| \geq 2\varepsilon$ ), and killing off the function near the singularity of  $\Gamma$  (at distances  $|x-y| \leq \varepsilon$ ). Therefore, it is clear that  $w^\varepsilon(x)$  converges pointwise to  $w(x)$ . Let  $K \subset \Omega$  be compact, and for any  $x \in K$  we compute for  $n > 2$ ,<sup>13</sup>

$$\begin{aligned} |v(x) - w_i^\varepsilon(x)| &= \left| \int_{\Omega} \frac{\partial}{\partial x^i} \Gamma(x-y)f(y)dy - \frac{\partial}{\partial x^i} \int_{\Omega} \Gamma(x-y)\eta^\varepsilon(x,y)f(y)dy \right| \\ &= \left| \int_{B_\varepsilon(2\varepsilon)} \left( \frac{\partial \Gamma(x-y)}{\partial x^i} - \frac{\partial \Gamma(x-y) \cdot \eta^\varepsilon}{\partial x^i} \right) f(y)dy \right| \\ &\leq \sup_{B_x(2\varepsilon)} |f| \int_{B_\varepsilon(2\varepsilon)} |D\Gamma| + \left( |D\Gamma| + \frac{2}{\varepsilon^n} \cdot \varepsilon^{n-1} |\Gamma| \right) dy \\ &\leq \frac{\sup |f|}{n\omega_n} \int_{B_x(2\varepsilon)} 2|x-y|^{1-n} + \frac{2}{\varepsilon(2-n)} |x-y|^{2-n} dy \\ &= \frac{\sup |f|}{n\omega_n} \left( \frac{2n\omega_n(2\varepsilon)^{n-1}}{(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{1-n}(\rho^{n-1}d\rho) + \frac{2n\omega_n(2\varepsilon)^{n-1}}{\varepsilon(2-n)(2\varepsilon)^{n-1}} \int_0^{2\varepsilon} \rho^{2-n}(\rho^{n-1}d\rho) \right) \\ &= \sup |f| \left( 2\varepsilon + \frac{2}{\varepsilon(n-2)} \cdot \frac{(2\varepsilon)^2}{2} \right) \\ &= \sup |f| \frac{2n\varepsilon}{n-2}. \end{aligned}$$

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<sup>13</sup>Check this computation...

The third to last equality is from co-area formula, along with change of variables. Since the upper bound is independent of  $x$ , the uniformity follows. For  $n = 2$ , we leave the proof to the reader.  $\square$

**Proposition 1.2.4.** *Suppose  $f$  is bounded and locally Hölder continuous (with exponent  $\alpha \leq 1$ ) in  $\Omega$  and let  $w$  be the Newtonian potential of  $f$ . Then  $w$  is  $C^2(\Omega)$ ,  $\Delta w = f$  in  $\Omega$  and for any  $x \in \Omega$ ,*

$$w_{ij}(x) = \int_{\Omega_0} \Gamma_{ij}(x-y)(f(y) - f(x))dy - f(x) \int_{\partial\Omega_0} \Gamma_i(x-y)\nu_j(y)dz_y.$$

Here  $\Omega_0$  is domain containing  $\Omega$  where the divergence theorem holds, and  $f$  is extended to vanish outside  $\Omega$ .

*Proof.* Similar to above. Come back to this if there's time. Proof techniques combine those in Theorems 1.2.3 and 1.2.5.  $\square$

We provide the fundamental estimate for the Schauder estimate for the Laplacian.

**Lemma 1.2.5.** *Let  $B_1 := B_R(x_o)$  and  $B_2 := B_{2R}(x_o)$  be concentric balls in  $\mathbb{R}^n$ . If  $f \in C^{0,\alpha}(\bar{B}_2)$  with  $w$  the Newtonian potential of  $f$  in  $B_2$ , then  $w \in C^{2,\alpha}(\bar{B}_1)$ . Furthermore,*

$$|D^2 w|_{0;B_1} + R^\alpha [D^2 w]_{\alpha;B_1} \leq C(|f|_{0;B_2} + R^\alpha [f]_{\alpha;B_2}).$$

*Proof.* As a preliminary, we remark that  $|B_2| = \omega_n(2R)^n$  and  $|\partial B_2| = n\omega_n(2R)^{n-1}$ . For  $x \in B_1$ , we obtain from 1.2.4 and 1.2.2, the first estimate

$$\begin{aligned} |w_{ij}(x)| &= |f(x)| \int_{\partial B_2} |\Gamma_i(x-y)\nu_j(y)|ds_y + \int_{B_2} |\Gamma_{ij}(x-y)(f(y) - f(x))|dy \\ &\leq \frac{|f(x)|}{n\omega_n} \int_{\partial B_2} \underbrace{|x-y|^{1-n}}_{(*)} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \int_{B_2} |x-y|^{\alpha-n} dy \\ &\leq \frac{|f(x)|}{n\omega_n} R^{1-n} \int_{\partial B_2} ds_y + \frac{[f]_{x;\alpha}}{\omega_n} \underbrace{\int_{B_{3R}(x)} |x-y|^{\alpha-n} dy}_{(**)} \\ &\leq 2^{n-1}|f(x)| + \frac{n}{\alpha}(3R)^\alpha [f]_{x;\alpha}, \\ &\leq C_1(|f(x)| + R^\alpha [f]_{\alpha;x}), \end{aligned}$$

where  $C_1 = C_1(n, \alpha)$ . Here,  $(*)$  follows since  $R \leq |x-y|$ , and so  $|x-y|^{1-n} \leq R^{1-n}$  since  $n \geq 2$ . We calculate  $(**)$  as

$$\begin{aligned} (**) &= \frac{n\omega_n(3R)^{n-1}}{(3R)^{n-1}} \int_0^{3R} \rho^{\alpha-n}(\rho^{n-1}d\rho) \\ &= (n\omega_n) \frac{\rho^\alpha}{\alpha} \Big|_0^{3R} \\ &= \frac{n\omega_n}{\alpha} (3R)^\alpha. \end{aligned}$$

Notice the idea here to shift the center of the ball from  $x_o$  to  $x$ ; in doing so, we must enlarge the radius to  $3R$  so that we capture  $B_2$ . For our second estimate, for  $x, \bar{x} \in B_1$ ,  $\xi = \frac{1}{2}(x + \bar{x})$  and  $\delta = |x - \bar{x}|$ ,

$$\begin{aligned}
w_{ij}(\bar{x}) - w_{ij}(x) &= -f(\bar{x}) \int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y + f(x) \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \\
&\quad + \int_{B_2} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy - \int_{B_2} \Gamma_{ij}(x - y)(f(y) - f(x)) dy \\
&= f(x) \underbrace{\int_{\partial B_2} \Gamma_i(x - y) - \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_1} + (f(x) - f(\bar{x})) \underbrace{\int_{\partial B_2} \Gamma_i(\bar{x} - y) \nu_j(y) ds_y}_{I_2} \\
&\quad + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(x - y)(f(x) - f(y)) dy}_{I_3} + \underbrace{\int_{B_\delta(\xi)} \Gamma_{ij}(\bar{x} - y)(f(y) - f(\bar{x})) dy}_{I_4} \\
&\quad + (f(x) - f(\bar{x})) \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) dy}_{I_5} \\
&\quad + \underbrace{\int_{B_2 - B_\delta(\xi)} \Gamma_{ij}(x - y) - \Gamma_{ij}(\bar{x} - y)(f(\bar{x}) - f(y)) dy}_{I_6},
\end{aligned}$$

where the second equality follows from (detailed) inspection. We estimate  $I_1, \dots, I_6$ .<sup>14</sup>

1. We use mean value inequality for some  $\hat{x}$  between  $x, \bar{x}$ , then follow with the same trick as the first term in the first estimate to compute

$$\begin{aligned}
|I_1| &\leq |x - \bar{x}| \int_{\partial B_2} |DD_i \Gamma(\hat{x} - y)| ds_y \\
&\leq |x - \bar{x}| \int_{\partial B_2} |\hat{x} - y|^{-n} ds_y \\
&\leq \frac{|x - \bar{x}|}{\omega_n} \int_{\partial B_2} R^{-n} ds_y \\
&= \frac{|x - \bar{x}| n 2^{n-1}}{R} \\
&\leq n 2^{n-\alpha} \left( \frac{\delta}{R} \right)^\alpha,
\end{aligned}$$

where the last estimate follows by simple computation from  $\delta = |x - \bar{x}| < 2R$ .

2. The estimate follows the ideas when estimating the first term of  $w_{ij}(x)$ . We record our estimates for ease of replication in  $I_5$ . Note that the final answer is completely

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<sup>14</sup>My labelling of the six integrals follows that in [2].

independent of the ball.

$$\begin{aligned} |I_2| &\leq \frac{1}{n\omega_n} \int_{\partial B_2} |x - y|^{1-n} ds_y \\ &\leq \frac{1}{n\omega_n} \int_{\partial B_2} R^{1-n} ds_y = 2^{n-1}. \end{aligned}$$

3.  $|I_3| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha; \bar{x}}$ . The estimate follows the ideas when estimating the second term of  $w_{ij}(x)$  with  $B_\delta(x)$  in the place of  $B_2$  and  $B_{\frac{3\delta}{2}}$  in the place of  $B_{3R}(x)$ .
4.  $|I_4| \leq \frac{n}{\alpha} \left(\frac{3\delta}{2}\right)^\alpha [f]_{\alpha; \bar{x}}$ . Ditto, but swap  $x$  for  $\bar{x}$ .
5. Since the estimate in  $I_2$  is independent of the ball, the desired estimate follows immediately.

$$\begin{aligned} |I_5| &\leq \left| \int_{\partial B_2} \Gamma_i(x - y) \nu_j(y) ds_y \right| + \left| \int_{\partial B_\delta(\xi)} \Gamma_i(x - y) \nu_j(y) ds_y \right| \\ &\leq 2(2^{n-1}) = 2^n \end{aligned}$$

6.

□

## References

- [1] L. Evans. *Partial differential equations*. 2nd ed. Providence, R.I.: American Mathematical Society, 2010.
- [2] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. Springer Berlin Heidelberg, 2001.