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Geometric Controls of a Quadrotor UAV with Decoupled Yaw Control

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Abstract— **This paper presents a geometric control system for a quadrotor unmanned aerial vehicle with decoupled attitude controls. In particular, the attitude control system on the special orthogonal group is decomposed into the reduced attitude controls for the total thrust direction evolving on the two-dimensional unit sphere, and for the remaining one-dimensional rotations about the thrust vector corresponding to the yawing motion. Consequently, the yaw dynamics are controlled separately from the roll and pitch dynamics that are critical for the stability of the translational dynamics of the quadrotor. As such, the proposed controller exhibits improved position tracking capabilities especially for large-angle yawing motions. These are constructed directly on the two-sphere and the one-sphere to avoid complexities and singularities associated with local coordinates. Furthermore, the control systems are augmented with integral terms to deal with fixed disturbances. The efficacy of the proposed method is illustrated by numerical simulation.**

I. INTRODUCTION

Geometric control systems for a quadrotor unmanned aerial vehicle are developed in [1], [2], and they have been utilized in various flight experiments, such as agile maneuvers [3], laser-guided landing [4], and aerial transportation [5]. They are designed based on the assumption that the attitude dynamics of a quadrotor is full-actuated, i.e., there are three degrees of freedom in the control moment. This is critical in the quadrotor dynamics, as the direction of the resultant force is fixed to the body-fixed frame. Therefore, to change the direction of the total thrust, the attitude of a quadrotor should be rotated accordingly.

In the geometric control systems proposed in [1], [2], the desired attitude is constructed such that the direction of the total thrust is identical to the thrust required to follow a given position tracking command. Then, the control moment is designed to follow the desired attitude accordingly. As constructed directly on the configuration manifold, these geometric control systems avoid singularities associated with local coordinates, and exhibit excellent tracking performances especially for complex aerial maneuvers.

However, the actual mechanism behind generating control moments is not carefully considered in the control system design, and consequently, they may excite the yawing dynamics unnecessarily excessively. In any quadrotor, the pitching moment and the rolling moment are generated by the discrepancy between the thrusts of two rotors along the direction normal to the rotational axis. In contrast, the yawing moment is produced by the weak reactive torque from each rotor. Therefore, to generate a moderate yawing moment without causing any roll or pitch moment, it is required that two rotors are accelerated, and the other two rotors decelerated rapidly exactly in the same rate. Consequently, any imbalance in the motor or the propeller may deteriorate the roll and pitch dynamics that are critical for the overall stability of the translational dynamics.

To address this issue, this paper presents an attitude control system that is decomposed into the roll/pitch dynamics and the yaw dynamics. Instead of defining a desired attitude on the attitude configuration space of the special orthogonal group, the desired direction of the total thrust is defined in the two-sphere, which is the quotient space where the attitudes with varying yawing are identified, and a roll/pitch controller is designed accordingly. Next, the rotations about the thrust direction are controlled by a yaw controller defined for one-dimensional rotations.

The desirable, distinctive feature is that the error in the yawing motion does not cause an additional error in the roll/pitch dynamics. In the proposed controller, the roll/pitch dynamics are controlled only to follow the given position trajectory, and it is decoupled from the yawing error. As such, it exhibits improved position tracking performances even for large-angle yawing motion. Moreover, they are designed globally on the two-sphere, and the unit-circle, respectively, to avoid singularities, and integral controls terms are added to deal with fixed disturbances. The efficacy of the proposed approach is illustrated by numerical examples.

In short, the main contribution of this paper is presenting a geometric control system for a quadrotor that explicitly considers the physical mechanism behind moment generation in controller design. Instead of assuming that the attitude dynamics of a quadrotor is fully actuated and the control moment can be generated arbitrarily, the challenges in producing the yawing moment is noted in the controller design. The proposed control system improves reliability and consistency in practice, while utilizing the desirable intrinsic, global analysis from the geometric formulation.

II. PROBLEM FORMULATION

A. Quadrotor Dynamics

We consider the quadrotor system model defined in [1]. Define the inertial frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, and the body-fixed frame $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$. The inertial frame is chosen such that its third axis $\vec{e}_3$ points downward along the gravity. The body-fixed frame is selected such that the origin of the body-fixed frame is located at the center of gravity; the third axis $\vec{b}_3$

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is opposite to the rotor thrust; the first and the second axes point toward the corresponding rotor.

The configuration space is the special Euclidean group that is the semidirect product of $\mathbb{R}^3$ for the position, and the special orthogonal group for the attitude:

$$\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times3} \,|\, R^T R = I_{3\times3},\ \det[R] = 1\}.$$

Let $x \in \mathbb{R}^3$ be the location of the mass center resolved in the inertial frame, and let $R \in \mathsf{SO}(3)$ be the attitude corresponding to the linear transformation of the representation of a vector from the body-fixed frame to the inertial frame. The kinematics equations are given by

$$\dot{x} = v, \tag{1}$$

$$\dot{R} = R\hat{\Omega}, \tag{2}$$

where the *hat* map, $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ is chosen such that $\hat{x}y = x \times y$ and $\hat{x}^T = -\hat{x}$ for any $x, y \in \mathbb{R}^3$.

Let the mass of the quadrotor be $m \in \mathbb{R}$, and the positive-definite inertia matrix be $J \in \mathbb{R}^{3\times3}$. The equations of motion are given by

$$m\dot{v} = mge_3 - fRe_3 + \Delta_x, \tag{3}$$

$$J\dot{\Omega} + \Omega \times J\Omega = M + \Delta_R, \tag{4}$$

where $f \in \mathbb{R}$ and $M \in \mathbb{R}^3$ are the total thrust and the resultant moment generated by four rotors. The fixed uncertainties in the translational dynamics and the rotational dynamics are denoted by Δ_x and Δ_R, respectively. It is assumed that M and Δ_R are resolved in the body-fixed frame.

The relation between the thrust from each rotor and the resultant force/moment is as follows. Let $f_i \in \mathbb{R}$ be the thrust from the i-th rotor for $i \in \{1, 2, 3, 4\}$, which is positive when the thrust is opposite to $\vec{b}_3$. For $d > 0$, the rotors are located at $d\vec{b}_1, d\vec{b}_2, -d\vec{b}_1$, and $-d\vec{b}_2$, respectively for the ascending order of the motor index (the displacements along $\vec{b}_3$ do not matter). Assume the i-th rotor generates a reactive torque $\tau_i \in \mathbb{R}$ with the magnitude proportional to the thrust, i.e., $|\tau_i| = c_{\tau f}|f_i|$ for $c_{\tau f} > 0$. Also, assume the first and the third rotors rotate clockwise for a positive thrust, and the other rotors rotate counterclockwise. We have

$$\begin{bmatrix} f \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -d & 0 & d \\ d & 0 & -d & 0 \\ -c_{\tau f} & c_{\tau f} & -c_{\tau f} & c_{\tau f} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}. \tag{5}$$

B. Tracking Control Problem

Suppose that the desired position trajectory is given by $x_d(t) \in \mathbb{R}^3$, and also the desired direction of the first body fixed axis is prescribed as $b_{1_d}(t) \in \mathsf{S}^2 = \{q \in \mathbb{R}^3 \,|\, \|q\| = 1\}$.

We make the following assumptions.

Assumption 1: The mass distribution of the quadrotor is symmetric such that the inertia matrix is given by $J = \mathrm{diag}[J_1, J_1, J_3]$ for $J_1, J_3 > 0$.

Assumption 2: The uncertainty in the position dynamics is bounded by

$$\|\Delta_x\| \leq \delta_x. \tag{6}$$

for a known constant $\delta_x > 0$.

Assumption 3: There is a constant $B_1 > 0$ such that

$$\|mge_3 - m\ddot{x}_d\| \leq B_1. \tag{7}$$

The first assumption is satisfied in general, as the quadrotor arms are almost identical. This is required to separate the yaw dynamics. Specifically, let $\Omega = (\Omega_1, \Omega_2, \Omega_3) \in \mathbb{R}^3$, $M = (M_1, M_2, M_3) \in \mathbb{R}^3$, and $\Delta_R = (\Delta_1, \Delta_2, \Delta_3) \in \mathbb{R}^3$. It is straightforward to show (4) is rewritten as

$$J_1\dot{\Omega}_1 = -(J_3 - J_1)\Omega_2\Omega_3 + M_1 + \Delta_1, \tag{8}$$

$$J_1\dot{\Omega}_2 = (J_3 - J_1)\Omega_3\Omega_1 + M_2 + \Delta_2, \tag{9}$$

$$J_3\dot{\Omega}_3 = M_3 + \Delta_3. \tag{10}$$

The second assumption is to limit the effects of the attitude controls on the translational dynamics in the stability analysis. The last assumption states that the acceleration of the desired trajectory is bounded.

Under these assumptions, we wish to design the thrust of each rotor (f_1, f_2, f_3, f_4) such that $x(t) = x_d(t)$ becomes exponentially stable.

III. Control System Design

A. Position Control System

As shown in (3), the resultant control force is given by $-fRe_3 \in \mathbb{R}^3$, where the magnitude of the thrust f can be arbitrarily adjusted through (5), but the direction of the thrust is $-Re_3$ in the inertial frame, which is opposite to the third body-fixed axis, $\vec{b}_3$. Therefore, to control the translational dynamics, the attitude should be controlled accordingly.

In this section, we relax this restriction, and assume that the resultant control force can be adjusted arbitrarily. More specifically, we suppose the term $-fRe_3$ is replaced by a fictitious control input $A \in \mathbb{R}^3$, and design the desired thrust for A.

Let the error variables be

$$e_x = x - x_d, \tag{11}$$

$$e_v = v - \dot{x}_d. \tag{12}$$

Also, the integral term is chosen as

$$e_i(t) = \int_0^t e_v(\tau) + c_1 e_x(\tau) d\tau, \tag{13}$$

where $c_1 > 0$. For controller gains $k_x, k_v, k_i > 0$, we choose

$$A = -k_x e_x - k_v e_v - k_i \mathrm{sat}_\sigma(e_i) - mge_3 + m\ddot{x}_d, \tag{14}$$

where a saturation function $\mathrm{sat}_\sigma : \mathbb{R} \to [-\sigma, \sigma]$ is introduced as

$$\mathrm{sat}_\sigma(y) = \begin{cases} \sigma & \text{if } y > \sigma \\ y & \text{if } -\sigma \leq y \leq \sigma \\ -\sigma & \text{if } y < -\sigma \end{cases},$$

for $\sigma > 0$. If the input is a vector $y \in \mathbb{R}^n$, then the above saturation function is applied element-wise.

Proposition 1: Consider the translational dynamics given by (3). Suppose the term $-fRe_3$ in (3) is replaced by A defined in (14). If,

$$k_i\sigma > \delta_x, \tag{15}$$

$$c_1 < \min\left\{\sqrt{\frac{k_x}{m}}, \frac{4k_xk_v}{k_v^2+4mk_x}\right\}, \tag{16}$$

then the equilibrium of the tracking errors for the translational dynamics, namely $(e_x, e_v, e_i) = (0, 0, \frac{\Delta_x}{k_i})$ is stable in the sense of Lyapunov, and e_x, e_v asymptotically converge to zero. Also,

$$\|A\| \leq k_x\|e_x\| + k_v\|e_v\| + \sqrt{3}k_i\sigma + B_1. \tag{17}$$

Proof: Substituting (14) into (3),

$$m\dot{e}_v = -k_xe_x - k_ve_v - k_i\mathrm{sat}_\sigma(e_i) + \Delta_x. \tag{18}$$

Let a Lyapunov function be

$$\begin{aligned}\mathcal{V}_x = &\frac{1}{2}m\|e_v\|^2 + c_1me_v\cdot e_x + \frac{1}{2}k_x\|e_x\|^2\\ &+\int_{\frac{\Delta_x}{k_i}}^{e_i}(k_i\mathrm{sat}_\sigma(\mu) - \Delta_x)\cdot d\mu,\end{aligned} \tag{19}$$

which is positive-definite about the equilibrium $(e_x, e_v, e_i) = (0, 0, \frac{\Delta_x}{k_i})$ from (15) and (16).

Substituting (18) and rearranging, we have

$$\dot{\mathcal{V}}_x = -(k_v - c_1m)\|e_v\|^2 - c_1k_ve_x\cdot e_v - c_1k_x\|e_x\|^2, \tag{20}$$

which is negative-semidefinite for c_1 satisfying (16). Consequently, the equilibrium is stable in the sense of Lyapunov, and $(e_x, e_v) \to (0, 0)$ as $t \to \infty$ due to LaSalle-Yoshizawa theorem [6]. ■

B. Roll/Pitch Control System

The preceding proposition states that when $-fRe_3 \equiv A$, the control objectives are achieved. However, that is impossible unless the attitude of the quadrotor can be changed instantaneously. Here we construct a control system such that $-fRe_3 \to A$ as $t \to \infty$ instead, and analyze the stability of the resulting coupled system.

In the prior development [1], [2], a desired attitude R_d is formulated first, and an attitude tracking control system is designed on $\mathsf{SO}(3)$. This is based on the assumption that all elements of the control moment, namely (M_1, M_2, M_3) can be adjusted arbitrarily according to (5). While that is mathematically correct, as the 4×4 matrix in (5) is nonsingular as long as $d, c_{\tau f} \neq 0$, the constant $c_{\tau f}$ representing reactive torque is typically at the level of $10^{-2}\,\mathrm{m}$. Consequently, a moderate amount of M_3 may yield a excessive discrepancy between f_1/f_3 and f_2/f_4. This can amplify the errors in motor calibration, and cause undesired vibration in practice.

To address this issue, we propose to decompose the attitude controller into two parts, namely the roll/pitch controller and the yaw controller. As shown in the preceding subsection, the complete attitude does not have to be controlled for position tracking. Instead, the direction of the third body-fixed axis, namely $b_3 = Re_3 \in \mathsf{S}^2$ should be controlled. More precisely, the desired direction of b_3, namely $b_{3_d}(t) \in \mathsf{S}^2$ is given by

$$b_{3_d} = -\frac{A}{\|A\|}. \tag{21}$$

The *reduced* attitude control, to ensure $b_3 \to b_{3_d}$ as $t \to \infty$, is referred to as the roll/pitch control, as adjusting b_3 causes rolling and pitching maneuvers in the conventional flight dynamics.

Mathematically, this corresponds to the design of the attitude controller in the quotient space: define an equivalent relation such that $R_1 \sim R_2$ when $R_1e_3 = R_2e_3$ for $R_1, R_2 \in \mathsf{SO}(3)$; the resulting quotient space is the two-sphere $\mathsf{SO}(3)/\!\sim\, = \mathsf{S}^2$.

In other words, we will design a roll/pitch controller with the first two elements of M, namely (M_1, M_2) to control the third body-fixed axis b_3. The remaining component M_3 will be used for a yaw controller presented in the next subsection.

a) Dynamics of b_3: We first construct the equations of motion for b_3 on S^2. Let $b_i = Re_i \in \mathsf{S}^2$ for $i \in \{1, 2, 3\}$, i.e., b_i is the representation of $\vec{b}_i$ in the inertial frame.

We have

$$\begin{aligned}\dot{b}_3 &= \dot{R}e_3 = R\hat{\Omega}e_3 = \widehat{R\Omega}b_3 = \sum_{i=1}^{3}\Omega_ib_i \times b_3\\ &= \omega_{12} \times b_3,\end{aligned} \tag{22}$$

where $\omega_{12} \in \mathbb{R}^3$ is

$$\omega_{12} = \Omega_1b_1 + \Omega_2b_2, \tag{23}$$

satisfying $\omega_{12}\cdot b_3 = \dot{\omega}_{12}\cdot b_3 = 0$.

Following the similar development as (22) and rearranging,

$$\begin{aligned}J_1\dot{\omega}_{12} = &(-J_3\Omega_2\Omega_3 + M_1 + \Delta_1)b_1\\ &+ (J_3\Omega_3\Omega_1 + M_2 + \Delta_2)b_2.\end{aligned}$$

We apply feedback linearization to define a fictitious control input $\tau \in \mathbb{R}^3$ as

$$\tau = (-J_3\Omega_2\Omega_3 + M_1)b_1 + (J_3\Omega_3\Omega_1 + M_2)b_2, \tag{24}$$

satisfying $\tau\cdot b_3 = 0$. Substituting this, the equation of motion for ω_{12} is written as

$$J_1\dot{\omega}_{12} = \tau + \Delta_1b_1 + \Delta_2b_2. \tag{25}$$

In short, the roll/pitch dynamics for b_3 evolving on S^2 are described by (22) and (25), where the control input τ is defined in terms of (M_1, M_2).

b) Control System for b_3: We invoke a slight variation of a PID controller on S^2 [7]. From (21), we can formulate the desired angular velocity $\omega_{12_d} \in \mathbb{R}^3$ such that

$$\dot{b}_{3_d} = \omega_{12_d} \times b_{3_d}. \tag{26}$$

Define error variables $e_b, e_\omega \in \mathbb{R}^3$ and $e_{I_1}, e_{I_2} \in \mathbb{R}$ as

$$e_b = b_{3_d} \times b_3, \tag{27}$$

$$e_\omega = \omega_{12} + \hat{b}_3^2\omega_{12_d}. \tag{28}$$

Also, the integral terms are chosen as, for $c_2 > 0$,

$$e_{I_1} = \int_0^t (e_\omega + c_2 e_b) \cdot b_1 \, dt, \tag{29}$$

$$e_{I_2} = \int_0^t (e_\omega + c_2 e_b) \cdot b_2 \, dt. \tag{30}$$

The control moment for the roll/pitch dynamics is chosen as

$$\begin{aligned} \tau = &-k_b e_b - k_\omega e_\omega - k_I e_{I_1} b_1 - k_I e_{I_2} b_2 \\ &- J_1 (b_3 \cdot \omega_{12_d}) \dot{b}_3 - J_1 \hat{b}_3^2 \dot{\omega}_{12_d}, \end{aligned} \tag{31}$$

for controller gains $k_b, k_\omega, k_I > 0$.

Proposition 2: Consider the quadrotor dynamics given by (3) and (4). The total thrust is chosen as

$$f = -A \cdot b_3, \tag{32}$$

and the moments (M_1, M_2) are defined by combining (24) and (31).

For $0 < \alpha < 1$, $0 < \beta, \sigma, k_x, k_v$, choose positive control parameters $k_i, k_b, k_\omega, k_I, c_1, c_2$ such that

$$k_i \sigma > \delta_x, \tag{33}$$

$$c_1 < \min\left\{ \sqrt{\frac{k_x}{m}}, \frac{4(1-\alpha)^2 k_x k_v}{(1+\alpha)^2 k_v^2 + 4(1-\alpha) m k_x} \right\}, \tag{34}$$

$$c_2 < \min\left\{ \sqrt{\frac{k_b}{J_1}}, \frac{4 k_b k_\omega}{(k_\omega + J_1 B_2)^2 + 4 k_b J_1} \right\}, \tag{35}$$

$$\lambda_{\min}(W_2) > \frac{\|W_{12}\|^2}{4 \lambda_{\min}(W_1)}, \tag{36}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of a symmetric matrix, and the matrices $W_1, W_{12}, W_2 \in \mathbb{R}^{2\times 2}$ are defined as

$$W_1 = \begin{bmatrix} (1-\alpha) c_1 k_x & -\frac{1}{2}(1+\alpha) c_1 k_v \\ -\frac{1}{2}(1+\alpha) c_1 k_v & (1-\alpha) k_v - c_1 m \end{bmatrix}, \tag{37}$$

$$W_{12} = \begin{bmatrix} c_1(\sqrt{3} k_i \sigma + B_1) & 0 \\ \sqrt{3} k_i \sigma + B_1 + \beta k_x & 0 \end{bmatrix}, \tag{38}$$

$$W_2 = \begin{bmatrix} c_2 k_b & -\frac{1}{2} c_2 (k_\omega + J_1 B_2) \\ -\frac{1}{2} c_2 (k_\omega + J_1 B_2) & k_\omega - c_2 J_1 \end{bmatrix}. \tag{39}$$

Then the equilibrium of the tracking errors for the coupled dynamics, namely $(e_x, e_v, e_i, e_b, e_w, e_{I_1}, e_{I_2}) = (0, 0, \frac{\Delta_x}{k_i}, 0, 0, \frac{\Delta_1}{k_I}, \frac{\Delta_2}{k_I})$ is stable in the sense of Lyapunov, and the tracking errors e_x, e_v, e_b, e_w asymptotically converge to zero.

Proof: Define the following domain about the equilibrium.

$$\begin{aligned} D = \{&(x, v, R, \Omega, e_i, e_{I_1}, e_{I_2}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathsf{SO(3)} \times \\ &\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^2 \mid \|e_b\| < \alpha, \|e_x\| < \beta, \|\omega_{12_d}\| < B_2\}, \end{aligned} \tag{40}$$

for $0 < \alpha < 1$ and $0 < \beta, B_2$. The subsequent Lyapunov stability analysis is performed within D.

Translation Dynamics: In contrast to Proposition 1, the error dynamics for e_v should account the difference between $-fRe_3$ and A. Therefore, the error dynamics (18) is revised into

$$m\dot{e}_v = -k_x e_x - k_v e_v - k_i \mathrm{sat}_\sigma(e_i) + \Delta_x + X, \tag{41}$$

where the additional term $X \in \mathbb{R}^3$ represents the discrepancy between the actual thrust vector $-f b_3$ and the ideal one A:

$$X = -f b_3 - A.$$

Substituting (32) and $A = -\|A\| b_{3_d}$,

$$X = -(I_{3\times 3} - b_3 b_3^T) A = \|A\| b_3 \times e_b.$$

Therefore, from (17),

$$\|X\| \leq (k_x \|e_x\| + k_v \|e_v\| + \sqrt{3} k_i \sigma + B_1) \|e_b\|. \tag{42}$$

Consider the Lyapunov function defined in (19), which is positive definite from (34). The time-derivative of the Lyapunov function presented in (20) is augmented with X as

$$\begin{aligned} \dot{\mathcal{V}}_x = &-(k_v - c_1 m)\|e_v\|^2 - c_1 k_v e_x \cdot e_v - c_1 k_x \|e_x\|^2 \\ &+ (e_v + c_1 e_x) X. \end{aligned}$$

From (42), and the domain D defined in (40), we can show

$$\dot{\mathcal{V}}_x \leq -z_1^T W_1 z_1 + z_1^T W_{12} z_2, \tag{43}$$

for any state in D, where $z_1 = (\|e_x\|, \|e_v\|)$, $z_2 = (\|e_b\|, \|e_\omega\|) \in \mathbb{R}^2$.

Roll/Pitch Dynamics: Next, for the roll/pitch dynamics, consider the following Lyapunov function.

$$\begin{aligned} \mathcal{V}_{b_3} = &\frac{1}{2} e_\omega \cdot J_1 e_\omega + c_2 e_b \cdot J_1 e_\omega + k_b (1 - b_3 \cdot b_{3_d}) \\ &+ \sum_{j=1}^{2} \frac{1}{2k_I} (k_I e_{I_j} - \Delta_j)^2, \end{aligned}$$

which is positive-definite for c_2 satisfying (35).

We have

$$\dot{e}_\omega = \dot{\omega}_{12} + \hat{b}_3^2 \dot{\omega}_{12_d} + \dot{b}_3 (b_3 \cdot \omega_{12_d}) + b_3 (\dot{b}_3 \cdot \omega_{12_d}).$$

Therefore, substituting (31) into (25),

$$\begin{aligned} J_1 \dot{e}_\omega = &-k_b e_b - k_\omega e_\omega - \sum_{j=1}^{2} (k_I e_{I_j} - \Delta_j) b_i \\ &+ J_1 b_3 (\dot{b}_3 \cdot \omega_{12_d}). \end{aligned}$$

From the results of [7, Proposition 2.1],

$$\begin{gathered} e_b \cdot b_3 = e_\omega \cdot b_3 = 0, \\ \frac{1}{2}\|e_b\|^2 \leq (1 - b_3 \cdot b_{3_d}), \\ \frac{d}{dt}(1 - b_3 \cdot b_{3_d}) = e_b \cdot e_\omega, \\ \dot{e}_b \cdot e_\omega \leq B_2 \|e_b\| \|e_\omega\| + \|e_\omega\|^2. \end{gathered}$$

Using these, the time-derivative of $\mathcal{V}_{b_3}$ is bounded by

$$\dot{\mathcal{V}}_{b_3} \leq -z_2 W_2 z_2. \tag{44}$$

c) Stability of Couple Dynamics: Finally, let the Lyapunov function for the coupled dynamics be $\mathcal{V} = \mathcal{V}_x + \mathcal{V}_{b_3}$. From (43) and (44),

$$\dot{\mathcal{V}} = -z^T \begin{bmatrix} \lambda_{\min}(W_1) & -\frac{1}{2}\|W_{12}\| \\ -\frac{1}{2}\|W_{12}\| & \lambda_{\min}(W_2) \end{bmatrix} z,$$

for $z = (\|z_1\|, \|z_2\|) \in \mathbb{R}^2$, which is negative definite from (36). Therefore, the given equilibrium is stable in the sense of Lyapunov, and $z \to 0$ as $t \to \infty$. ■

C. Yaw Control System

This proposition states that the translational dynamics of a quadrotor can be controlled by three control inputs, namely (f, M_1, M_2). In this section, we develop a yaw control system utilizing the last control moment M_3.

As shown at the last row of (5), which is copied below, the yawing moment is generated by the sum of the reactive torques that are assumed to be proportional to the thrust.

$$M_3 = c_{\tau f}((f_2 + f_4) - (f_1 + f_3)).$$

However, the parameter $c_{\tau f}$ is quite small, and the accurate estimation of it's values is difficult. For example, it is determined as $c_{\tau f} = 0.0135\,\mathrm{m}$ for the particular motor/propeller combination studied in the authors' lab. Consequently, a moderate level of M_3 may cause large and possibly inaccurate rotor thrusts that can adversely affect the roll/pitch dynamics.

In contrast to the prior developments in [1], [2], here the yawing control M_3 is designed independently from the preceding roll/pitch control. Therefore, we can adjust the controller gains for the yaw dynamics to have slower timescale without sacrificing the agility of the roll/pitch dynamics that is critical for the stability of the translational dynamics. In other words, the proposed scheme has two distinct attitude control systems that can be adjusted properly depending on the underlying dynamic characteristics and the role in the overall stability.

As formulated in Section II-B, we assume that the desired direction of the first body-fixed axis is given by $b_{1_d}(t)$. Since there is a constraint of $b_1 \cdot b_3 = 0$, and b_3 should be adjusted for the position tracking command, we cannot guarantee that b_1 asymptotically follows b_{1_d} in general. Instead, we project b_{1_d} to the plane normal to b_3 to construct

$$b_{1_c}(t) = -(\hat{b}_3(t))^2 b_{1_d}(t) = (I_{3\times 3} - b_3(t) b_3(t)^T) b_{1_d}(t). \tag{45}$$

Therefore, $b_{1_c} \cdot b_3 \equiv 0$. The objective of the yaw control is to design M_3 such that $b_1 \to b_{1_c}$ as $t \to \infty$. While b_1 is a two-dimensional unit vector in S^2, there is one-dimensional degree of freedom as both b_1 and b_{1_c} are constrained to be normal to b_3. Not surprisingly, this is equivalent to a control design on S^1 or the unit-circle.

Similar with (22), we have

$$\dot{b}_1 = (\Omega_2 b_2 + \Omega_3 b_3) \times b_1 = -\Omega_2 b_3 + \Omega_3 b_2. \tag{46}$$

And, the kinematics equation for b_{1_c} can be written as

$$\begin{aligned} \dot{b}_{1_c} &= \omega_c \times b_{1_c} \\ &= (\omega_{c_1} b_1 + \omega_{c_2} b_2 + \omega_{c_3} b_3) \times b_{1_c}, \end{aligned} \tag{47}$$

where the angular velocity of b_{1_c} is given by $\omega_c \in \mathbb{R}^3$ when resolved in the inertial frame, and the coordinates of ω_c are $(\omega_{c_1}, \omega_{c_2}, \omega_{c_3})$ when represented with respect to the body-fixed frame.

Let the error function for the yawing control be

$$\Psi_y(b_1, b_{1_c}) = \frac{1}{2}\|b_1 - b_{1_c}\|^2 = 1 - b_1 \cdot b_{1_c}.$$

From (46) and (47), its time-derivative

$$\begin{aligned} \dot{\Psi}_y &= -(-\Omega_2 b_3 + \Omega_3 b_2) \cdot b_{1_c} \\ &\quad - b_1 \cdot \{(\omega_{c_1} b_1 + \omega_{c_2} b_2 + \omega_{c_3} b_3) \times b_{1_c}\} \\ &= (\Omega_3 - \omega_{c_3}) \cdot (-b_2 \cdot b_{1_c}). \end{aligned} \tag{48}$$

Motivated by this, the error variables for the yaw control, namely $e_y, e_{\omega_y} \in \mathbb{R}$ are defined as

$$e_y = -b_2 \cdot b_{1_c}, \tag{49}$$

$$e_{\omega_y} = \Omega_3 - \omega_{c_3}. \tag{50}$$

Also, for $c_3 > 0$, the integral term is chosen as

$$e_{I_y} = \int_0^t e_{\omega_y} + c_3 e_y dt. \tag{51}$$

For positive controller gains $k_y, k_{\omega_y}, k_{I_y}$, the control moment is chosen as

$$M_3 = -k_y e_y - k_{\omega_y} e_{\omega_y} - k_{I_y} e_{I_y} + J_3 \dot{\omega}_{c_3}. \tag{52}$$

Proposition 3: Consider the yawing dynamics defined by (10). For positive controller gains $k_y, k_{\omega_y}, k_{I_y}$, choose $c_3 > 0$ such that

$$c_3 < \min\left\{\sqrt{\frac{k_y}{J_3}}, \frac{f k_y k_{\omega_y}}{k_{\omega_y}^2 + 4 k_y J_3}\right\}. \tag{53}$$

Then, the equilibrium $(e_y, e_{\omega_y}, e_{I_y}) = (0, 0, \frac{\Delta_3}{k_{I_y}})$ is stable in the sense of Lyapunov, and $e_y, e_{\omega_y} \to 0$ as $t \to 0$.

Proof: Let the Lyapunov function be

$$\begin{aligned} \mathcal{V}_y &= \frac{1}{2} J_3 e_{\omega_y}^2 + c_3 J_3 e_{\omega_y} e_y + k_y (1 - b_1 \cdot b_{1_c}) \\ &\quad + \frac{1}{2 k_{I_y}} (k_{I_y} e_{I_y} - \Delta_3)^2, \end{aligned}$$

which is positive-definite for c_3 satisfying (53), as $\frac{1}{2}\|e_y\|^2 \leq (1 - b_1 \cdot b_{1_c})$.

From (10) and (52),

$$J_3 \dot{e}_{\omega_y} = -k_y e_y - k_{\omega_y} e_{\omega_y} - k_{I_y} e_{I_y} + \Delta_3,$$

and similar with (48),

$$\begin{aligned} \dot{e}_y &= -(\Omega_1 b_3 - \Omega_3 b_1) \cdot b_{1_c} \\ &\quad - b_2 \cdot \{(\omega_{c_1} b_1 + \omega_{c_2} b_2 + \omega_{c_3} b_3) \times b_{1_c}\} \\ &= (\Omega_3 - \omega_{c_3})(b_1 \cdot b_{1_c}) = e_{\omega_y}(b_1 \cdot b_{1_c}) \end{aligned}$$

which implies $\dot{e}_y e_{\omega_y} \leq e_{\omega_y}^2$ as $|b_1 \cdot b_{1_c}| \leq 1$.

Using these, we can show

$$\dot{\mathcal{V}}_y \leq -z_y^T \begin{bmatrix} c_3 k_y & -\frac{1}{2} c_3 k_{\omega_y} \\ -\frac{1}{2} c_3 k_{\omega_y} & k_{\omega_y} - c_3 J_3 \end{bmatrix} z_y,$$

where $z_y = (|e_y|, |e_{\omega_y}|) \in \mathbb{R}^2$. The matrix in the above expression is positive-definite for c_3 satisfying (53). Therefore, the equilibrium is stable in the sense of Lyapunov, and $z_y \to 0$ as $t \to \infty$. ■

IV. NUMERICAL EXAMPLE

The parameters of a quadrotor are given by

$$J = [2, 2, 4] \times 10^{-3}\,\mathrm{kgm}^2,\ m = 2\,\mathrm{kg},\ d = 0.169\,\mathrm{m},$$
$$c_{\tau f} = 0.0135\,\mathrm{m}.$$

The controller gains are selected as

$$k_x = 4,\ k_v = 2.8,\ k_i = 2,\ c_1 = 1,\ \sigma = 10,$$
$$k_b = 16,\ k_\omega = 5.6,\ k_I = 10,\ c_2 = 1,$$
$$k_y = 1,\ k_{\omega_y} = 1.4,\ k_{I_y} = 10.$$

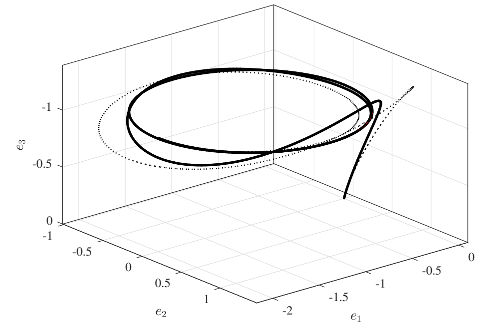
Disturbances of $\Delta_x = [0.5, 0.8, -1]\,\mathrm{N}$ and $\Delta_R = [0.2, 1, -0.1]\,\mathrm{Nm}$ are considered.

The desired trajectories for the position and the yawing direction are given by

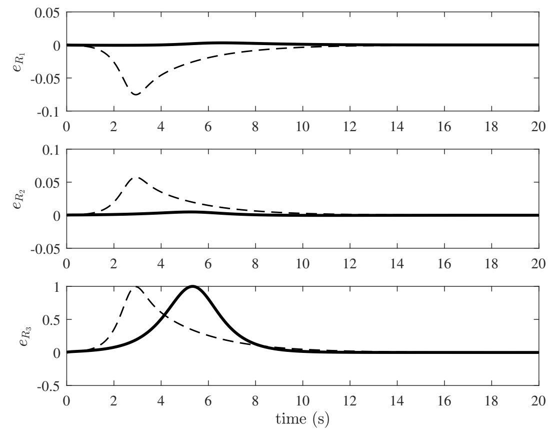
$$x_d(t) = \left[\cos\left(\frac{\pi}{4}t\right) - 1,\ \sin\left(\frac{\pi}{4}t\right),\ -1\right]\ \mathrm{m},$$
$$b_{1_d}(t) = \left[\cos\left(\frac{\pi}{5}t\right),\ \sin\left(\frac{\pi}{5}t\right),\ 0\right].$$

The initial conditions are $x(0) = [0, 0, 0]\,\mathrm{m}$, $R = \exp(\pi\hat{e}_3)$ with zero velocities.

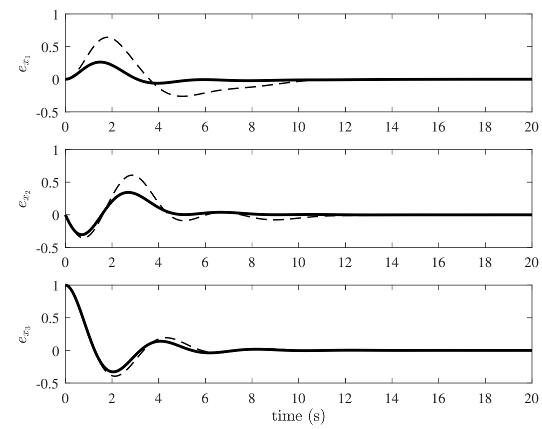
The corresponding numerical simulation results are illustrated at Figure 1. The results of the proposed method are denoted by thick, solid lines, and for comparison, the results of the controller proposed in [1] are represented by thin, dashed lines. In particular, Figure 1(b) shows the attitude tracking error computed by $e_R = \frac{1}{2}(R_d^T R - R^T R_d)^\vee \in \mathbb{R}^3$. While the proposed controller yields a delayed error in the yawing direction, maximized at around $t = 5\,\mathrm{sec}$, the errors in the roll and pitch that are critical for the accuracy of the resultant control force remain close to zero. Consequently, it exhibits superior position tracking performance. In contrast, for the controller presented in [1], the error in the yawing direction is coupled to the roll and pitch dynamics near $t = 3\,\mathrm{sec}$, and they result in a greater error in the position tracking.



(a) Position trajectory (m)



(b) Attitude error $e_R = \frac{1}{2}(R_d^T R - R^T R_d)^\vee \in \mathbb{R}^3$



(c) Position error $e_x = x - x_d$ (m)

Fig. 1. Simulation results (solid,thick:proposed; dashed,thin: [1])

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