Homework 1

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P3.5.1

(a) The Hamiltonian expression should be:

$$\mathcal{H} = \rho^t [1 - aq_t] q_t - \lambda_{t+1} q_t \tag{1.1}$$

(b) By doing the first order condition with respect to q_t, we can get:

$$\frac{\partial \mathcal{H}}{\partial q_t} = \rho^t [1 - 2aq_t] - \lambda_{t+1} = 0 \tag{1.2}$$

and we can also get:

$$\lambda_{t+1} - \lambda_t = -\frac{\partial \mathcal{H}}{\partial R_t} \tag{1.3}$$

$$R_{t+1} - R_t = \frac{\partial \mathcal{H}}{\partial \lambda_{t+1}} = -q_t \tag{1.4}$$

From eqaution 1.3, we can know that $\lambda_{t+1} = \lambda_t = \lambda$. Combine with equation 1.2, we can get that $q_t = (1 - (1+\delta)^t \lambda)/(2a)$. Since $\sum_{t=0}^{T-1} q_t = R_0$, we can get that $\lambda = \frac{T - 2aR_0}{\sum_{k=0}^{T-1} (1+\delta)^k}$.

(c) From question (b), We know that $q_t = (1 - (1+\delta)^t \lambda)/(2a)$ where $\lambda = \frac{T-2aR_0}{\sum_{k=0}^{T-1}(1+\delta)^k}$. So we can write down q_t as $q_t = (1 - (1+\delta)^t \frac{T-2aR_0}{\sum_{k=0}^{T-1}(1+\delta)^k})/(2a)$, we also know that $q(T) = 0$, hence

$$q(T) = (1 - (1+\delta)^T \frac{T - 2aR_0}{\sum_{k=0}^{T-1} (1+\delta)^k})/(2a) = 0 \tag{1.5}$$

where $a = 0.001$, $\delta = 0.1$ and $R_0 = 1000$, so we can get the value of T, $T = 7$.

P3.5.2

(a) Solving for the time-path of extraction for the competitive mining industry:

$$\frac{\dot{p(t)}}{p(t)} = \delta \tag{2.1}$$

where $p(t) = q(t)^{-0.5}$, so we can write 2.1 as:

$$\frac{\frac{dq(t)^{-0.5}}{dt}}{q(t)^{-0.5}} = \delta \tag{2.2}$$

By doing the mathmatical transformation, we can get that the time path of extraction for the competitive mining industry is:

$$q(t) = e^{-2\delta t} = e^{-0.2t} \tag{2.3}$$

(b) Solving for the time-path of extraction for the monotonic mining industry:

$$\frac{M\dot{R}(t)}{MR(t)} = \delta \tag{2.4}$$

where $R(t) = q(t)^{0.5}$, and $MR = 0.5q(t)^{-0.5}$so we can write 2.1 as:

$$\frac{\frac{0.5dq(t)^{-0.5}}{dt}}{0.5q(t)^{-0.5}} = \delta \tag{2.5}$$

By doing the mathmatical transformation, we can get that the time path of extraction for the competitive mining industry is:

$$q(t) = e^{-2\delta t} = e^{-0.2t} \tag{2.6}$$

(c) Since the time-path of extraction is the same for the two senario, we should have $T_c = T_m$, by $\int_0^T e^{-0.2t}dt = 1$, we can get that T=1.1157. Hence $T_c = T_m = 1.1157$.

P3.5.3

(a) The current-value Hamiltonian for the problem is:

$$\tilde{\mathcal{H}} = [ae^{\gamma t} - bq(t)]q(t) - cR(t) - \mu(t)q(t) \tag{3.1}$$

(b) Doing the first order condition with respect to q_t and $R(t)$ from the Hamiltonian equation, we can get that:

$$ae^{\gamma t} - 2bq(t) + c - \mu(t) = 0 \tag{3.2}$$

$$\dot{\mu(t)} - \delta\mu(t) = -\frac{\partial H}{\partial R(t)} = 0 \tag{3.3}$$

Using the above two equations, we can get that:

$$\dot{q(t)} - \delta q(t) - \frac{(\gamma - \delta)ae^{\gamma t} + \delta c}{2b} = 0 \tag{3.4}$$

Hence equation 3.4 is my answer for question (b).

(c) The tranversality condition means $\tilde{\mathcal{H}(T)} = 0 \Rightarrow q(T) = 0$ and $R(T) = 0$.

(d) From tranversality condition, we have $q(T) = 0$, using this condition and equation 3.2, we have

$$\mu(T) = ae^{\gamma T} + c \tag{3.5}$$

and

$$\mu(t) = \mu(T)e^{\delta(t-T)} = [ae^{\gamma T} + c]e^{\delta(t-T)} \tag{3.6}$$

Putting equation 3.6 back into 3.2, we should get that:

$$q(t) = \frac{ae^{\gamma t} + c - \mu(t)}{2b} = \frac{ae^{\gamma t} + c - [ae^{\gamma T} + c]e^{\delta(t-T)}}{2b} \tag{3.7}$$

This part I have something wrong maybe, since my answer here is different from the solution the book provides.

(e) we want to calculate the time T, so we use the condition that total resource equals to R, which is

$$\int_0^T q(t)dt = R(0)$$

subsituting $q(t)$ with what we got from equation 3.6, we have:

$$\int_0^T \frac{ae^{\gamma t} - c - [ae^{\gamma T} - c]e^{\delta(t-T)}}{2b} dt = R(0) \tag{3.8}$$

$$\Rightarrow \frac{a(e^{\gamma T - 1} - 1)}{\gamma} - cT - \frac{ae^{\gamma T} - c}{\delta}(1 - e^{-\delta T}) - 2bR(0) = 0 \tag{3.9}$$

P3.5.4

(a) From book p125, we can use $C'(q\star) = C(q\star)/q\star$ to get the optimal production rate, hence we can get that $q\star = 44.72$ tons/day. And then we can calculate $T = R_0/q\star = 17.89$ years.

Net mine income euqals to $50 \times 200,000 - \frac{200,000}{44.72} \times (1000 + \frac{1}{2} \times 44.72^2)$.

To determine q for short-term profit maximizer, we use the condition that $C'(q) = p$, which gives us q=50.

(b) Take $\delta = 10\%$, and there are 250 mining days each year.

As before, we can write down the Hamiltonian equation and do the first order condition with respect to q(t) and R(t), then we can get two equations similiar to those we get in question 3.5.1, from which we could have:

$$p - q = \lambda e^{\delta t} \tag{4.1}$$

Since $q(T) = q\star$, we can know from above equation that $\lambda = (p - q\star)e^{\delta T}$ and then we can get $q(t) = p - (p - q\star)e^{\delta(t-T)}$.

Then we do the integaration about q(t) from 0 to T as we did in the former questions, get $T = 4222$ days, and $\lambda = 1.06$, $q(0) = 48.94$ tons/day.

P3.5.5

Using the formula we get in the textbook, which is $C'(q) = C(q)/q$. We can get that $q\star = \sqrt{a/b}$.

so

$$p(T) = \alpha - \beta N q\star = \alpha - \beta N\sqrt{a/b} \tag{5.1}$$

Using the same approach we did previously, we have the following conditions:

$$\mu(T) = p(T) - C'(q\star) \tag{5.2}$$

$$\mu(t) = e^{\delta(t-T)}\mu(T) \tag{5.3}$$

$$C'(q(t)) = 2bq(t) = p(t) - \mu(t) = \alpha - \beta N q(t) - \mu(t) \tag{5.4}$$

then we can get q(t):

$$q(t) = \frac{\alpha}{2b + N\beta} - [\frac{\alpha}{2b + N\beta} - q\star]e^{\delta(t-T)} \tag{5.5}$$

By doing integration with respect to equation 5.5 frm t=0 to T, we can calculate that T=14.07 years, $q\star$=44.72 tons/day, P(T)=82.12 dollar/ton, and:

$$q(t) = 71.43 - 8.73e^{\delta t} tons/day$$

$$p(t) = 71.43 + 3.49e^{\delta t} dollar/ton$$

P3.5.6

After we write down the Hamiltonion eqaution and do the first order conditions, we can get $p(t) - C(R) = \mu(t)$, and $\dot{\mu}(t) - \mu(t)\delta = 0$, which implies that:

$$c(R) = p(t) - \frac{1}{\delta}\dot{p}(t) = p_0(1 - \frac{\gamma}{\delta})e^{\gamma t} \tag{6.1}$$

When $\gamma > \delta$, then the price is growing faster than the rate of interest, and production should be postponed to the future. When $\gamma = \delta$, the discounted cost approach zero as $T \to \infty$.Hence when $\gamma < \delta$, we have:

$$R(t) = \frac{ae^{-\gamma t}}{p_0(1 - \gamma/\delta)} - b \tag{6.2}$$

Then

$$q(t) = -\dot{R} = \frac{a\gamma e^{-\delta t}}{p_0(1 - \gamma/\delta)} = 1500e^{-0.05t} \tag{6.3}$$

P3.5.7

Writing down the Hamiltonian equation and first order condition with respect to q(t),

$$\tilde{\mathcal{H}}(t) = p(q(t))q(t) - C(q(t)) - \mu(t)q(t) \tag{7.1}$$

$$\frac{\partial p(q(t))}{\partial q(t)}q(t) + p(q(t)) - C'(.) - \mu(t) = 0 \tag{7.2}$$

For the monopolistic version of P3.5.5, the corresponding condition should be:

$$\frac{\frac{d}{dt}(R'(.) - C'(.))}{R'(.) - C'(.)} = \delta \tag{7.3}$$

At t=T, we have $\mathcal{H}(t) = 0$, we can calculate $\mu(T)$, and then $\mu(t) = \mu(T)e^{\delta(t-T)}$. We can get the equation of q(t), and get T by doing the integration. The result should be:

$$q\star = 33.33 tons/day$$

$$p(T) = 86.67 dollar/ton$$

$$T = 15.97 years$$

$$q(t) = 55.6 - 2.21e^{\delta t} tons/day$$

$$p(t) = 77.78 + 0.89e^{\delta t} dollar/ton$$

P3.5.8

(a) The current value Hamiltonian equation:

$$\tilde{\mathcal{H}} = p_t q_t - C_1(R_t)q_t - C_2(w_t) + \lambda_{1,t+1}[f(w_t, X_t) - q_t] + \lambda_{2,t+1}f(w_t, X_t) \tag{8.1}$$

(b) Four difference equations:

$$X_{t+1} - X_t = f(w_t, X_t) \tag{8.2}$$

$$R_{t+1} - R_t = f(w_t, X_t) - q_t = X_{t+1} - X_t - q_t \tag{8.3}$$

$$P_{t+1} - C_1(R_{t+1}) - (1+\delta)(p_t - C_1(R_t)) - C_1'(R_{t+1})q_{t+1} = 0 \tag{8.4}$$

$$C_2'(w_{t+1})/f_w(w_{t+1}, X_{t+1}) - p_{t+1+C_1(R_{+1})} - (1+\delta)[C_2'(w_t)/f_w - p_t + C_1(R_t)] - C_2'(w_{t+1})f_{x_{t+1}}/f_{w_{t+1}} \tag{8.5}$$

(c) Using the specification and parameter values, with the four equations above, we can get that:

$$X_{t+1} = X_t + rw_t^s e^{-\mu X_t}$$

$$R_{t+1} = R_t + X_{t+1} - X_t - q_t$$

$$q_{t+1} = [(1+\delta)(a - bq_t - c/R_t) - a + c/R_{t+1}]R_{t+1}^2/(c - br_{t+1}^2)$$

(d) Tranversality conditions means $\tilde{\mathcal{H}}(T) = 0$, which implies that $q_T = 0$ and $w_T = 0$.