1. A $Volterra operator V$ on $L^2(0,1)$ is an operator of integration such as the one shown below
$$(Vx)(t) = \int_0^t x(s)ds, 0 \leq t \leq 1$$
(c) An operator is said to have $rank 1$ if its range is one dimensional. Show that $V + V^*$ has rank 1.

Proof: First we will compute the adoint of V. In particular, we need to find the operator $V^*$ such that $(Vx, y) = (x, V^*y)$ for all $x, y \in L^2(0,1)$. Hence,
$$(Vx, y) = \int_0^1 \int 0^t x(s)\overline{y(t)}dsdt$$
Integrating by parts, we observe that:
$$\begin{aligned}
(Vx, y) &= \int_0^1 \int_0^t x(s)\overline{y(t)}dsdt \\
&= (\int_0^t x(s)ds|_0^1)(\int_0^t \overline{y(s)}ds|_0^1) - \int_0^1 x(t) \int_0^t \overline{y(s)}dsdt \\
&= (\int_0^1 x(s)ds)(\int_0^1 \overline{y(s)}ds) - \int_0^1 x(t) \int_0^t \overline{y(s)}dsdt \\
&= \int_0^1 x(t) \int_0^1 \overline{y(s)}dsdt - \int_0^1 x(t) \int_0^t \overline{y(s)}dsdt \\
&= \int_0^1 x(t) \int_t^1 \overline{y(s)}dsdt \\
&= \int_0^1 x(t) \overline{\int_t^1 y(s)dsdt} = (x, V^*y)
\end{aligned} \tag{1}$$
where
$$(V^*y)(t) = \int_t^1 y(s)ds$$
Therefore,
$$((V + V^*)x)(t) = \int_0^t x(s)ds + \int_t^1 x(s)ds = \int_0^1 x(s)ds$$
Therefore $((V + V^*)x)(t)$ is a constant. Thus the range of $V + V^*$ consists of the constant functions, a one-dimensional set. Therefore $V + V^*$ has rank 1.

2. Let T be a bounded linear operator of rank 1 on a hilbert space H, and let $\psi$ be a non-zerp vector in the range of T.
(a) Show that there exists $\varphi \in H$ such that
$$\forall x \in H, Tx = \langle x, \varphi \rangle \varphi$$

and that

$$||T|| = ||\varphi||||\psi||$$

Proof: Suppose T is a rank-one bounded linear operator on a Hilbert space H, and let $\psi$ be any nonzero vector in the range of T. Then for and $x \in H$, there exists some $\gamma \in C$ such that

$$Tx = \gamma\psi$$

Define the linear functional $f : H \to C$ with $Tx = \gamma\psi$. Since From the Riesz Representation Theorem, there is some $\psi \in H$ such that

$$f(x) = \langle x, \psi \rangle$$

for all $x \in H$.
Hence we can compute $||T||$ in terms of $\psi$ and $f$:

$$||T|| = \sup_{||x||<1} ||Tx|| = \sup_{||x||<1} ||f(x)\psi|| = \sup_{||x||<1} |f(x)|||\psi|| = ||\psi||||f||$$

From Riezs Representation Theorem, we can also know that $||f|| = ||\varphi||$, hence

$$||T|| = ||\varphi||||\psi||$$

3. An operator K on $L^2(0,1)$ is defined by

$$(Kx)(t) = \int_0^1 e^{t-s}x(s)ds, 0 < t < 1$$

Use the result from Q2(a) to show that

$$||K|| = \sinh 1$$

Proof: To see that K is a rank-one operator, note that for all $t \in (0,1)$,

$$(kx)(t) = \int_o^1 e^{t-s}x(s)ds = e^t \int_0^1 e^{-s}x(s)ds = e^t(x, e^{-s})$$

From Q2(a), we have $\varphi(t) = e^t$ and $\psi(s) = e^{-t}$, hence $||K|| = ||e^t||||e^{-s}||$. We can simplify this formula:

$$||e^t||||e^{-s}|| = (\int_0^1 e^{2t}dt)^{\frac{1}{2}}(\int_0^1 e^{-2t}dt)^{\frac{1}{2}} = \frac{1}{2}(e^2-1)^{\frac{1}{2}}(1-e^{-2})^{\frac{1}{2}} = \frac{1}{2}(e-e^{-1}) = \sinh 1$$

Therefore, $||K|| = \sinh 1$

4. Let

$$g_n(x) = \frac{e^{inx}}{\sqrt{2\pi(1+n^2)}}, n \in \mathbb{Z}, x \in [-\pi, \pi]$$

Show that $(g_n)_\infty^\infty$ is an orthonormal sequence in $W[-\pi,\pi]$. Let

$$W_0 = \{f \in W[-\pi,\pi] | f(\pi)\} = f(-\pi)$$

Show that sinh is orthogonal to every function in $W_0$. Deduce that $(g_n)_\infty^\infty$ is not a total orthonormal sequence in $W[-\pi,\pi]$.
Proof: $(f,g)_W$ - inner product on $W[-\pi,\pi]$. $(f,g)$ - usual $L^2(-\pi,\pi)$ inner product.

$$e_n = \frac{e^{inx}}{\sqrt{2\pi}}$$

$$g_n(x) = \frac{e^{inx}}{\sqrt{2\pi(1+n^2))}} = \frac{e^{inx}}{\sqrt{1+n^2}}$$

$W[-\pi,\pi]$.

$$(g_n^{'}, g_m^{'}) = (2\pi(1+n^2))^{-1}(ine^{inx}, ime^{-imx})$$

$$\Rightarrow (g_n, g_n)_W = \frac{1}{1+n^2} + \frac{n^2}{1+n^2} = 1$$

Hence, $(g_n)$ forms an orthonormal sequence in the space.
Since $\sinh x = \frac{1}{2}(e^x - e^{-x})$, suppose $g(x) = \sinh x$

$$(f^{'}, g^{'}) = \int_{-\pi}^{\pi} f^{'}(\frac{1}{2}(e^x + e^{-x}))dx = -(f,g)$$

Hence,

$$(f,g)_W = (f,g) + (f^{'} + g^{'}) = (f,g) - (f,g) = 0$$

Since all the basis function $g_n$ satisfy $g_n(\pi) = g_n(-\pi)$, we see that sinh is a nontrivial function that is orthogonal to all $g_n$, therefore $g(n)$ is not a total orthonormal sequence in $W[-\pi,\pi]$