ECSE-500-01-1 Assignment 9 4.11.2005

1. For n=1,2,3,... , and for x real, put

$$f_n(x) = \frac{x}{1+nx^2}$$

Show that $\{f_n\}$ converges uniformly (on $\mathbb{R}$) to a function f. What is this limit function f ?

Proof: For fixed $x \in \mathbb{R}, f_n(x) \to 0$ as n $\to \infty$. To show uniform convergence, qwe need to show $\sup_{x\in\mathbb{R}} |f_n(x)| \to 0$ as $n \to \infty$. We see $\lim_{|x|\to\infty} |f_n(x)| = 0$. As f_n is differentiable everywhere, then:

$$f_n^{'}(x) = \frac{1-nx^2}{(1+nx^2)^2}$$

We can find that $f_n^{'}(x)$ is 0 at $x = \pm\frac{1}{\sqrt{n}}$. We substitute $x = \pm\frac{1}{\sqrt{n}}$ into $f_n(x)$, then:

$$f_n(\frac{1}{\sqrt{n}}) = \frac{\frac{1}{\sqrt{n}}}{1+n(\frac{1}{\sqrt{n}})^2} = \frac{1}{2\sqrt{n}}$$

$$f_n(\frac{-1}{\sqrt{n}}) = -\frac{1}{2\sqrt{n}}$$

Therefore, as $n \to \infty$, $\sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{2\sqrt{n}} \to 0$

Hence $\{f_n\}$ convenges uniformly to a function f on $\mathbb{R}$. The limit of function f is $\frac{1}{2\sqrt{n}}$.

2. Let $a < b$ in $\mathbb{R}$. Show that the space $W[a,b]$ of continuously differentiable functions on $[a,b]$ with values in $\mathbb{R}$, is an inner product space with respect to pointwise addition $\big((f+g)(t) = f(t)+g(t)\big)$ and scalar multiplication $\big((\alpha f)(t) = \alpha f(t)\big)$, and inner product

$$\langle f,g\rangle = \int_a^b f(t)g(t) + f'(x)g'(x)dt$$

Proof: For any $f,g \in W[a,b]$ and $\alpha \in \mathbb{R}$, $f+g$ and αf are both continuously differentiable, so W[a,b] is closed under addition and scalar multiplication.
If $f,g \in W[a,b]$, then f and g are continuously differentiable, and hence fg and $f^{'}g^{'}$ are both bounded on the compact interval $[a,b]$. Hence,

$$\langle f,g\rangle_W = \int_a^b f(t)g(t) + f'(x)g'(x)dt$$

is a finite quantity. By breaking the integr and into real and imaginary parts, we can have that:

$$\int_a^b f(t)\overline{g(t)} = \overline{\int_a^b \overline{f(t)}g(t)dt}$$

It is also clear from properties of the integral that $(\alpha f, g)_W = \alpha(f,g)_W$ and $(f+g,h)_W = (f,h)_W + (g,h)_W$ for any $f,g,h \in W[a,b]$.
Then we need to prove that $(f,f)_W > 0$ for $f \neq 0$. Since $f \in W[a,b]$ is continuous on $[a,b]$, if f differs from 0 at some $x_* \in [a,b]$, there must be some finite interval over which $f(x)\overline{f(x)} = |f(x)|^2 > 0$. Therefore,

$$\int_a^b f(x)\overline{f(x)}dt > 0$$

Therefore W[a,b] is a vector space and $(\cdot,\cdot)_W$ satifies all the properties of an inner product.

3. Prove that in $W[0,1]$

$$\langle f, \cosh \rangle = f(1)\sinh 1$$

and deduce that

$$\{f \in W[0,1] \mid f(1) = 0\}$$

is a closed subspace of $W[0,1]$.

Proof: Given $W[0,1] : \mathbb{R} \to \mathbb{C}$ defined as the space of continuously differentiable functions with inner product $\langle f,g \rangle = \int_0^1 f(t)g(t) + f'(x)g'(x)dt$

$$\begin{aligned}\langle f, \cosh \rangle &= \int_0^1 f(t)\cosh(t) + f^{'}(t)\cosh^{'}(t)dt \\ &= [F(t)\cosh(t)]_0^1 - \int_0^1 F(t)\cosh^{'} dt + [f(t)\cosh^{'}]_0^1 - \int_0^1 f(t)\cosh^{''}(t)dt \\ &= [f(t)\cosh^{'}(t)]_0^1 \\ &= f(1)\sinh(1)\end{aligned} \tag{1}$$

To show that M is closed, as we have proved that $\langle f, \cosh \rangle = f(1)\sinh 1$, hence

$$f(1) = 0$$

$$\Rightarrow \langle f, \cosh \rangle_W = 0$$

Therefore M is a closed subspace of $W[0,1]$

4. Let V be an inner product space and let $x, y \in V$. Prove Pythagoras' theorem:

$$\langle x,y \rangle = 0 \;\Rightarrow\; \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof: As $\langle x, y\rangle = 0$, we can know that x and y are orthogonal.Hence

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x, x\rangle + \langle v, v\rangle \\ &= \|x\|^2 + \|y\|^2 \end{aligned} \tag{2}$$

5. ℓ^1 denotes the complex vector space of all absolutely summable sequences of complex numbers. That is all sequences $\{x_n\} \subset \mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |x_n| < \infty$$

with componentwise addition and scalar multiplication. Show that

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$$

defines a norm on ℓ^1.
Proof: ℓ^1 denotes the space of all absolutely summable sequences $\{x_n\} \subset \mathbb{C}$, that is:

$$\ell^1 = \left\{x = (x_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |x_k| < \infty\right\}$$

This set is a vector space. In particular, it is closed under the operations of addition of sequences and multiplication of a sequence by a scalar. Further, a straightforward calculation shows that the function d defined by

$$d(x, y) = ||x-y||_1 = \sum_{n=1}^{\infty} |x_n - y_n|, x, y \in \ell^1$$

is a metric on ℓ^1, so ℓ^1 is both a vector space and a metric space.
Therefore, there exists a sequence of real or complex scalars

$$x = (x_n)_{n\in\mathbb{N}} = (x_1, x_2, ...)$$

with the define of the ℓ^1-norm of x to be

$$||x||_1 = ||(x_n)_{n\in\mathbb{N}}||_1 = \sum_{n=1}^{\infty} |x_n|$$

6. Show that the subspace of polynomial functions is not closed in $C[0,1]$ (continuous real functions on $[0,1]$) with respect to either the supremum norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$$

or the inner product norm

$$\|f\|_2 = \langle f, f\rangle^{1/2} = \left(\int_0^1 f(x)^2 dx\right)^{1/2}$$

(*hint*: find a limit point that is not a polynomial)

Proof: from

$$\begin{aligned} \|f\|_\infty &= \sup_{x\in[0,1]} |f(x)| \\ \|f\|_2 &= \langle f, f\rangle^{1/2} = \left(\int_0^1 f(x)^2 dx\right)^{1/2} \end{aligned} \tag{3}$$

Hence,

$$\|f\|_2 \leq \|f\|_\infty . \left(\int_0^1 dx\right)^{1/2} = \|f\|_\infty \tag{4}$$

Suppose a non-polynomial function $f = e^x$.
Denote the partial sum of its Taylor series expansion as Tn. $(T_n)_{n=1}^\infty$ is a a sequence of polynomials, therefore,

$$\|Pn - f\|_\infty \to 0$$

$$\Rightarrow \|P_n - f\|_2 \leq \|P_n - f\|_\infty \to 0$$

Therefore the subspace of polynomial functions is not closed in C[0,1], neither the inner product norm.

7. c_0 denotes the subspace of ℓ^∞ comprising all sequences $\{x_n\}$ which converge to zero as $n \to \infty$. Prove that c_0 is closed in ℓ^∞ (with respect to $\|\cdot\|_\infty$).

Proof: We need to prove that the limit of sequence in c_0 that converges in l^∞ is actually in c_0. Considering a sequence of elements $((x_n^m)_n)_m$ of c_0, denote $(x_n)_n$ as the limit of that sequence in l^∞.
Let $\varepsilon > 0$, $\exists M \in \mathbb{N}$ s.t. $\forall m \geq M$,

$$||(x_n^m)_n - (x_n)_n||_\infty < \frac{\varepsilon}{2}$$

As we know that sequences $\{x_n\}$ converge to zero as $n \to \infty$, then $\lim_{n\to\infty} x_n^m = 0$. Hence,

$$|x_n^m| < \frac{\varepsilon}{2}$$

$$|x_n| \leq |x_n - x_n^m| + |x_n^m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, $\lim_{n\to\infty} x_n = 0$, c_0 is closed in l^∞

8. Prove that ℓ^∞ with the supremum norm $\|\cdot\|_\infty$ is complete.

   Proof: l^∞ is said complete if every Cauchy sequence of points in l^∞ has a limit that is also in l^∞ or alternatively, if every Cauchy sequence in l^∞ converges in l^∞. Hence we have to show that a Cauchy sequence in l^∞ converges.
   Consider a Cauchy sequence (x_n) in l^∞. Each x_n is itself a sequence, and denote it by

   $$x_n(0), x_n(1), ...$$

   and $x_n(k)$ means the kth term in the nth element of the sequence.
   We nned to find an element x in l^∞ such that x_n converges to x. Let $\varepsilon > 0$, as x_n is a Cauchy sequence, $\exists N \in \mathbb{N}, N > 0$ such that

   $$||x_n - x_m||_\infty < \varepsilon$$

   for all $n, m > N$
   Hence,

   $$|x_n^k - x_m^k| < \varepsilon$$

   for all $n, m > N$ Therefore, every Cauchy sequence in l^∞ converges. l^∞ with the supremum norm $\|\cdot\|_\infty$ is complete.