1. Show that if $s_n \leq t_n$ for all $n \geq N$, where N is some fixed integer, then

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

Proof: Since $s_n \leq t_n$ for all $n \geq N$, when $n \to \infty$, n must $\geq$ the fixed integer N. Then $s_n \leq t_n$. $\lim_{n\to\infty}$inf s_n is the lowest limit of sequence s_n, hence:

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$

Similarly, we know that $\lim_{n\to\infty}$sup s_n is the greatest limit of sequence s_n, the upper limit s^* and t^* belong to s_n, and $s_n \leq t_n$ for all $n \geq N$, therefore:

$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty}$$

2. Find the upper and lower limits of the sequence $\{s_n\}$ defined by:

$$s_1 = 0, \quad s_{2m} = \frac{s_{2m-1}}{2}, \quad s_{2m+1} = \frac{1}{2} + s_{2m}$$

Proof: We show that

$$(s_{2n-1}, s_{2n}) = (\frac{2^{n-1}-1}{2^n-1}, \frac{2^{n-1}-1}{2^n}), n \geq 1.$$

In fact, by the definition,

$$(s_1, s_2) = (0, \frac{0}{2}) = (0, 0).$$

Suppose the formula holds for $n = k$. Then

$$\begin{aligned}(s_{2(k+1)-1}, s_{2(k+1)}) &= (\frac{1}{2} + s_{2k}, \frac{s_{2k+1}}{2}) = (\frac{1}{2} + \frac{2^{k-1}-1}{2^k}, \frac{\frac{1}{2}+s_{2k}}{2}) \\ &= (\frac{2^{k+1}-2}{2^{k+1}}, \frac{1}{4}, \frac{1}{2}\frac{2^{k-1}-1}{2^k}) \\ &= (\frac{2^{(k+1)-1}-2}{2^{(k+1)-1}}, \frac{2^{(k+1)-1}}{2^{k+1}})\end{aligned} \tag{1}$$

The proved expression for s_n gives:

$$\lim_{n\to\infty} s_{2n-1} = 1, \lim_{n\to\infty} s_{2n-1} = \frac{1}{2}.$$

Hence, by the definitions of upper and lower limits, we have

$$\limsup_{n\to\infty} s_n = \overline{\lim_{n\to\infty}} s_n = 1, \liminf_{n\to\infty} = \underline{\lim_{n\to\infty}} s_n = \frac{1}{2}$$

3. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequences $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

   Proof: Since p_n is a Cauchy sequence and p_{n_i} converges to a point $p \in X$, given $\varepsilon > 0$ there exists N_1, N_2 such that

   $$d(p_n, p_m) < \frac{\varepsilon}{2}, n, m \geq N_1$$

   $$d(p_{n_i}, p) < \frac{\varepsilon}{2}, i \geq N_2$$

   Assume that $n_i \leq n_{i+1}$, hence

   $$d(p_n, p) \leq d(p_n, p_{n_i}) + d(p_{n_i}, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

   if $n, n_i \geq N = max(N-1, n_{N_2})$, since $\varepsilon > 0$ is arbitrary, $p_n \to p$

4. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

   Hint: For any m, n

   $$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

   Proof: For any $m, n, d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$, it follows that $|d(p_n, q_n) - d(p_m, q_m)$ is small if m and n are large.
   For any $\varepsilon > 0$, there exits N such that $d(p_n, p_m) < \varepsilon$ and $d(q_m, q_n) < \varepsilon$ whenever $m, n \geq N$. Note that

   $$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

   It allows that

   $$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n) < 2\varepsilon$$

   Thus $d(p_n, q_n)$ is a Cauchy sequence in X. Hence $d(p_n, q_n)$ converges.

5. If f is a continuous mapping of a metric space X into a metric space Y, prove that

   $$f(\overline{E}) \subset \overline{f(E)}$$

   for every set $E \subset X$. ($\overline{E}$ denotes the closure of E).
   Proof: $\overline{f(E)}$ is a closed subset of Y and f is continuous, so $f^{-1}(\overline{f(E)})$ is a closed subset of X. Clearly $E \subset f^{-1}(\overline{f(E)})$, and the latter closed, so $\overline{E} \subset f^{-1}(\overline{f(E)})$, or $f(\overline{E}) \subset \overline{f(E)}$.

6. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

   Proof: To check that $f(E)$ is dense in $f(X)$, it suffices to check that for every open subset $U \subset Y$ such that $U \cap f(X) \neq \emptyset$, $f(E) \cap (U \cap f(X)) \neq \emptyset$.
   But $f^{-1}(U \cap f(X)) = f^{-1}(U)$ is then nonempty and is open in X because f is continuous. E is dense in X, so $E \cap f^{-1}(U) \neq \emptyset$.
   Therefore if $x \in E \cap f^{-1}(U)$ then $f(x) \in f(E) \cap U = f(E) \cap (U \cap f(X))$, so the latter set is nonempty.
   Let $S = x \in X | f(x) = g(x)$. Firstly, S is closed by checking its complement $S^c = x \in X | f(x) \neq g(x)$ is an open subset of X. If $x \in X$ is such that $f(x) \neq g(x), i.e. x \in S^c$, then let $r = d_Y(f(x), g(x)) > 0$.
   Then $B_{r/2}(f(x)) \cap B_{r/2}(g(x)) = \emptyset$ in Y, because if $z \in B_{r/2}(f(x))$ then $r = d(f(x), g(x)) \leq d(f(x), z) + d(z, g(x))$, or $d(g(x), z) \geq d(f(x), g(x)) - d(f(x), z) = r - d(f(x), z) > r - \frac{r}{2} = \frac{r}{2}$, hence $z \notin B_{r/2}(g(x))$.
   Since f, g are both continuous, $V = f^{-1}(B_{r/2}(g(x)))$ is an open subset of X containing x which has the property that if $y \in V$ then $f(y) \in B_{r/2}(g(x))$ and $g(y) \in B_{r/2}(g(x))$ so that $f(y) \neq g(y)$ since $B_{r/2}(f(x)) \cap B_{r/2}(g(x)) = \emptyset$.
   Therefore $x \in V \subset S^c$, so S^c is open.
   Therefore S is closed.¡¡ Now that $E \subset S$, and S is a closed subset of X, so $X = \overline{E} \subset \overline{S} = S \subset X$.
   Therefore $S = X$, and $f(x) = g(x)$ for all $x \in X$.

7. Let f be a real, uniformly continuous function on a bounded set E in R^1. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

   Proof: Since E is a bounded set in $\mathbb{R}$, there is some $R > 0$ such that $E \subset [-R, R]$.
   Set $\varepsilon = 1$. Since f is uniformly continuous, there is some $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon = 1$. Let $n \in \mathbb{N}$ be the smallest integer such that $\frac{1}{n} < \delta$, and let

$$[-R, -R + \frac{1}{n}], [-R + \frac{1}{n}, -R + \frac{2}{n}], ..., [R - \frac{2}{n}, R - \frac{1}{n}], [R - \frac{1}{n}, R]$$

   be a collection S of $2nR$ invervals that exactly line up and cover $[-R, R]$.
   By omitting some if necessary, let $I_1, I_2, ..., I_N$ be those intervals from the collection that overlap with E, i.e. such that $I_k \cap E \Rightarrow |x_k - x| < \frac{1}{n} < \delta \Rightarrow |f(x)| < 1 + |f(x_k)|$
   Letting $M = max_{1 \leq k \leq N}(1 + f(x_k))$, for any $x \in E$ we have $|f(x)| < M$, so f is a bounded function on E.
   The second part asks us to show the conclusion is false if boundedness of E is omitted from the hypothesis. Let $E = \mathbb{R}$, which is unbounded, and let $f(x)$, which is uniformly continuous because for any $\varepsilon > 0$, let $\delta = \varepsilon$ and we have

$|x-y| < \delta \Rightarrow |f(x) - f(y)| = |x-y| < \delta = \varepsilon$
However, $f(x) = x$ is not bounded on $E = \mathbb{R}$

8. If f is defined on E, the *graph* of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane. Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

   Proof: $(\Rightarrow)$ Let $G = (x, f(x)) : x \in E$. Since f is a continuous mapping of a compact set E into $f(E)$, from Theorem 4.14 in Lecture 6: A mapping f of a set E into R^k is said to be bounded if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.
   Hence f is also compact. Suppose the product of finitely many compact sets is compact. Thus $G = E \times f(E)$ is also compact.
   $(\Leftarrow)$ Define
   $$g(x) = (x, f(x))$$
   from E to G for $x \in E$. Suppose that $g(x)$ is continuous on E.Consider $h(x, f(x)) = x$ from G to E. Hence h is injective, continuous on a compact set G.
   Therefore its inverse function $g(x)$ is injective and continuous on a compact set E. Since $g(x)$ is continuous on E, the component of $g(x)$ is continuous on a compact set E. That means $f(x)$ is continuous on a compact set E.

9. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* or *kernel* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.
   Proof: Since $Z(f) = f^{-1}(0)$, and 0 is a closed set in $\mathbb{R}$, by the Corollary of Theorem 4.8 from lecture 6: A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.
   Hence $Z(f)$ is closed if f is continuous.
   Let p be a limit point of $Z(f)$. Then there exists a sequence p_n in $Z(f)$ such that $d(p_n, o) \to 0$. Since f is continuous at p, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, p) < \delta$ implies $|f(x) - f(p)| < \varepsilon$.
   By $d(p_n, p) \to 0$, we know that there exists N such that $n \geq N$ implies $d(p_n, p) < \delta$. Hence, if $n \geq N$, then $|f(p_n) - f(p)| < \varepsilon$. Since $f(p_n) = 0$, we know that $|f(p)| < \varepsilon$ for any $\varepsilon > 0$. This implies $f(p) = 0$, or $p \in Z(f)$. Therefore $Z(f)$ is closed since it contains all its limit points.