Counting:

1. *A monkey types a 280-character message on Twitter, using only the 26 upper-case English letters. A research assistant is trying to determine if there is any hidden meaning in the message, and tries partitioning the message into 3 segments (the first x letters, the next y letters, and the last 280-x-y letters). How many ways could the message be partitioned, assuming each segment has at least one letter?*

Example: ABCDE | FGHIJKLM | NOPQ
We need two cuts to separate the message into three segments.

280 letters => 279 places to cut

Thus, we will need to choose 2 cuts from the 279 possible cuts.

$$\binom{279}{2} = \frac{279!}{2!\,277!} = 38781$$

There are 38781 ways

2. *How many different strings are there of length 10, made by using 10 of the letters in MISSISSIPPI?*

There are 11 letters in MISSISSIPPI. We need take out one letter.

Take out a M, we have four I's , four S's, and two P's:

$$\frac{10!}{4!\,4!\,2!}\ ways$$

Take out an I, we have four S's, three I's, two P's and one M

$$\frac{10!}{4!\,3!\,2!\,1!}\ ways$$

Take out an S, we have four I's, three S's, two P's and one M

$$\frac{10!}{4!\,3!\,2!\,1!}\ ways$$

Take out an P, we have four S's, four I's, one P's and one M

$$\frac{10!}{4!\,4!\,1!\,1!}\ ways$$

In total, we have:

$$\begin{aligned}&\frac{10!}{4!\,4!\,2!} + \frac{10!}{4!\,3!\,2!\,1!} + \frac{10!}{4!\,3!\,2!\,1!} + \frac{10!}{4!\,4!\,1!\,1!}\\ &= 3150 + 12600 + 12600 + 6300\\ &= 34650\ ways\end{aligned}$$

3. *How many ways can n+1 distinguishable objects be placed into n indistinguishable boxes, so that no box is empty?*

No box is empty => n – 1 boxes have one object and one box have two objects

This is basically to choose which two objects will be in the same box.

Thus, there are

$$\binom{n}{2} = \frac{n!}{(n-2)!\,2!}\ ways$$

4. There are 10 people in line, and each person is either a Ninja, a Pirate, or an Avenger (no single person is two of these things). There is no more than 1 Ninja, and there is at least 1 Pirate. How many possible lines could there be?

At least 1 Pirate => We need to assign the rest 9 people to either Ninja, Pirate, or Avenger

No more than 1 ninja => either 0 Ninja or 1 Ninja
0 Ninja:
Assign all 10 people to either Pirate or Avengers
At least 1 Pirate = ways of 0 Ninja * $\sum_{k=1}^{10} ways\ of\ k\ Pirates\ and\ 10-k\ Avengers$

$$= 1 * [\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \binom{10}{5} + \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}]$$

$$= 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1$$

$$= 1023$$

1 Ninja:

$$\binom{10}{1}\ ways\ to\ choose\ this\ Ninja$$

Assign the rest 9 people to either Pirate or Avenger
At least 1 Pirate = ways of 1 Ninja * $\sum_{k=1}^{9} ways\ of\ k\ Pirates\ and\ 9-k\ Avengers$

$$= \binom{10}{1} * [\binom{9}{1} + \binom{9}{2} + \binom{9}{3} + \binom{9}{4} + \binom{9}{5} + \binom{9}{6} + \binom{9}{7} + \binom{9}{8} + \binom{9}{9}]$$

$$= 10 * (9 + 36 + 84 + 126 + 126 + 84 + 36 + 9 + 1)$$

$$= 10 * 511$$

$$= 5110$$

In total:

$$1023 + 5110 = 6133\ ways$$

5. There is a Binary Search Tree with 12 nodes. Each node has a distinct value between 1 and 12. The root has value 9, and its left child has value 5. How many possible Binary Search Trees could this be?

Root is 9 => its left subtree has 8 nodes; its right subtree has 3 nodes
Root's left child is 5 => 5's left subtree has 4 nodes; 5's right subtree has 3 nodes

# of BSTs for n nodes:
1 node: 1 BST
2 nodes: 2 BSTs
    (root + left child) or (root + right child) => 2 BSTs
3 nodes: 5 BSTs
    root + 2-node left subtree = 2 BSTs
    root + 2-node right subtree = 2 BSTs
    root + 1-node left subtree + 1-node right subtree = 1 BSTs
    in total = 2 + 2 + 1 = 5 BSTs
4 nodes:
    root + 3-node left subtree = 5 BSTs
    root + 3-node right subtree = 5 BSTs
    root + 2-node left subtree + 1-node right subtree = 2 * 1 = 2 BSTs
    root + 1-node left subtree + 2-node right subtree = 1 * 2 = 2 BSTs
    in total: 5 + 5 + 2 + 2 = 14 BSTs

For this 12 nodes BST with 9 as root and its left child as 5,
Total BSTs
= # of 4-node BSTs * # of 3-node BSTs * # of 3-node BST
= 14 * 5 * 5
= 350
There could be 350 BSTs

Probability:

1. A family has 5 children, where each child is equally likely to be a boy or a girl. E is the event where there are at least 3 boys. F is the event where all children are the same gender. Are these events independent?

Total ways:
(5 girls) + (1 boy + 4 girls) + (2 boys + 3 girls) + (3 boys + 2 girls) + (4 boys + 1 boys) + (5 boys)
= $1 + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + 1$
= 1 + 5 + 10 + 10 + 5 + 1
= 32

P(E) = ((3 boys + 2 girls) + (4 boys + 1 boys) + (5 boys))/ total ways
= $\frac{10+5+1}{32} = \frac{1}{2}$
P(F) = ((5 girls) + (5 boys)) / total ways = = $\frac{1+1}{32} = \frac{1}{16}$
P(E∩F) = (5 boys) / totay ways = $\frac{1}{32}$
P(E) * P(F) = $\frac{1}{2} * \frac{1}{16} = \frac{1}{32} =$ P(E∩F)

Thus, E and F are independent

2. A hash table has 10 indices, and 7 pieces of data are placed into it. Each piece of data is assigned an index, independently of each other, and uniformly at random. What is the probability that no two pieces of data are assigned the same index?

Suppose we put the 7 pieces of data in to the hash table one by one. When placing each piece, we don't want any collisions.

P(no same index) = $\frac{10}{10} * \frac{9}{10} * \frac{8}{10} * \frac{7}{10} * \frac{6}{10} * \frac{5}{10} * \frac{4}{10} = 0.06048$

3. There are n jelly beans in a box of Bertie Bott's jelly beans. Exactly one is your desired flavor (hot chocolate). You draw one bean from the box at a time until you find your desired jelly bean. What is the expected number of beans you draw?

Let X = number of beans drew until hot chocolate

$$E(X) = \sum_{s \in S} p(s) \cdot X(s) = \sum_{k=1}^{n} P(k \text{ beans drew}) * k$$
$$= \frac{1}{n} * 1 + \frac{n-1}{n} * \frac{1}{n-1} * 2 + \frac{n-1}{n} * \frac{n-2}{n-1} * \frac{1}{n-2} * 3 + \cdots +$$
$$\left(\frac{n-1}{n} * \frac{n-2}{n-1} * \frac{1}{n-2} \ldots \ldots * \frac{1}{n-(n-1)}\right) * n$$
$$= \frac{1}{n} * 1 + \frac{1}{n} * 2 + \cdots + \frac{1}{n} * n$$
$$= \frac{1}{n} * (1 + 2 + \cdots + n)$$
$$= \frac{1}{n} * \frac{n * (n+1)}{2}$$
$$= \frac{n+1}{2}$$

4. A *triangle* in an undirected graph is a set of 3 nodes that all have edges to each other. Let G be a graph with n nodes. Each of the (n choose 2) possible edges exist with probability p. What is the expected number of triangles in G?

Within G, let's pick three random nodes A, B, C
Given this, the probability of forming a triangle is
P(A, B have an edge) * P(A, C have an edge) * P(B,C have an edge)
= $p * p * p = p^3$

Let X = number of triangles in G

$$E(X) = \sum_{s \in S} p(s) \cdot X(s)$$
$$= \sum_{n\ choose\ 3} P(triangle\ within\ three\ nodes) * 1$$
$$= \binom{n}{3} * p^3$$
$$= \frac{n!}{(n-3)! * 3!} * p^3$$

5. Two hundred kilometers above the coast of Brazil lies the center of the South Atlantic Anomaly (SAA) which is known to be a very hazardous zone of high energetic particles. This region exposes orbiting satellites to much higher-than-usual levels of radiation and can potentially destroy electronics onboard a satellite. Flying through the SAA, an ordinary satellite has a survival rate of 80%. However, if the satellite has radiation-hardened (rad-hard) components, the survival rate is 97%. Four satellites (A, B, C, D) flew through the SAA and one of them didn't survive. Satellite D was the only one without rad-hard components. What is the probability that satellite D was the one that didn't survive?

E = D didn't survive, F = one of the satellite didn't survive
P(E) = 1 – 80% = 20% = 0.2

P(E∩F) = P(D didn't survive and others survive) = (1 – 80%) * 97% * 97% * 97% = 0.1825346

P(F) = P(A didn't survive and others survive) + P(B didn't survive and others survive) + P(C didn't survive and others survive) + P(D didn't survive and others survive)
= (1 – 97%) * 97% * 97% * 80% + (1 – 97%) * 97% * 97% * 80% + (1 – 97%) * 97% * 97% * 80% + (1 – 80%) * 97% * 97% * 97%
= 0.2502794

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$
$$= \frac{0.1825346}{0.2502794}$$
$$\approx 0.729$$