Notations

- $Y \in \mathcal{Y}$: outcome variable
- $X = (X_1, X_2)$ two subsets of covariate varibles
  - Study how much different model depend on X_1 to predict Y
- $Z = (Y, X_1, X_2) \in \mathcal{Z}$
- data set $\mathbf{Z} = \left[\mathbf{y}^{n\times 1}, \mathbf{X}^{n\times p}\right]$
- Z is iid
- Model Class $\mathcal{F} \subset \{f | f : \mathcal{X} \rightarrow \mathcal{Y}\}$
  - $f(x)$ or $f(x_1, x_2)$ as a prediction model
- Loss function L
  - Squared Error Loss $L(f, z) = (y - f(x))^2$
- Algorithm : $\mathcal{A} : \mathcal{Z}^n \rightarrow \mathcal{F} : f = A(\mathbf{Z})$

Alogorithm Reliance （Existing Metic）

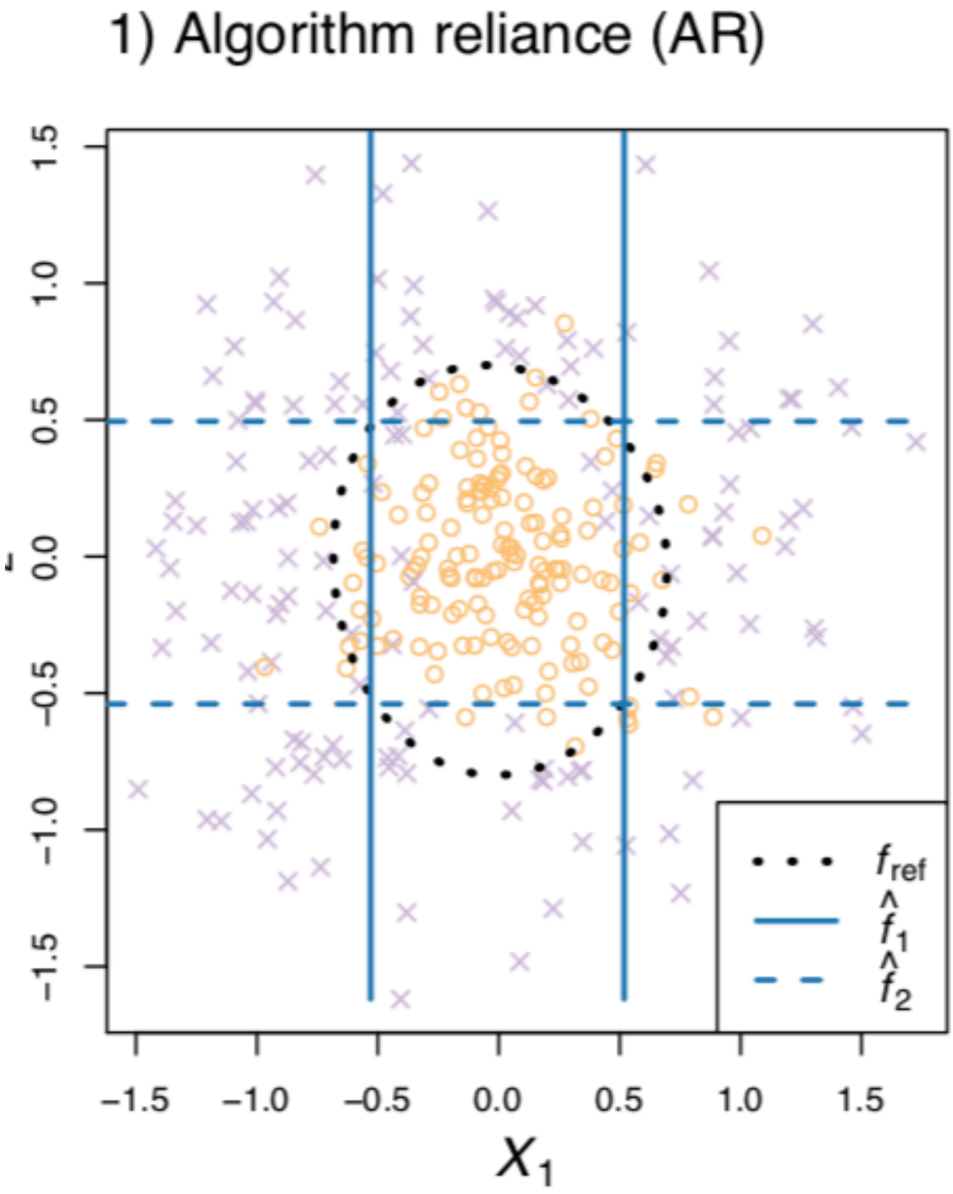
Algorithm （model-fitting algorithm） reliance (AR) on X_1:

- run a model-fitting algorithm A twice,
  - first on all of the data,
  - and then again after removing X_1 from the dataset .
- The losses for the two resulting models are then compared to determine the importance, or “necessity,” of X_1

AR property

- This measure is a function of ~~two prediction models rather than one,~~
  - it does not measure how much either individual model relies on X1.

- AR can result in seemingly incoherent behavior. Specifically, if several prediction models fit the data well, then removing a variable may not substantially change the prediction loss even if **the best prediction model relies heavily on that variable.**
  - can identify the strongest performing prediction model with no reliance on a given variable, but they generally do not bound how much any well-performing model may rely on that variable.



MR： Model Reliance

How much a model, a model-fitting algorithm, or a model class, relies on a subset of covariates X_1.

$X = [X_1 : (height), X_2 : (age)]$, $x^a = (1.6, 21)$, $x^b = (1.8, 22)$

- 'Switched Loss'： be the loss of model f on $z^{(b)}$, if $x_1^{(b)}$ was first replaced with $x_1^{(a)}$:
  - Like anchors? fixed some feature and permutate

$$h_f(z^a, z^b) := L(f, (y^b, x_1^{(a)}, x_2^{(b)}))$$
$$e_{switch}(f) := \mathbb{E}h_f(Z^{(a)}, Z^{(b)})$$
$$\hat{e}_{switch}(f) := \frac{1}{(n-1)(n)} \sum_{i=1}^{n} \sum_{j \neq i} h_f(\mathbf{Z}_{[i,:]}, \mathbf{Z}_{[j,:]})$$
$$= \frac{1}{(n-1)(n)} \sum_{i=1}^{n} \sum_{j \neq i} L(f, (\mathbf{y}_{[i]}, \mathbf{X}_{1[i,:]}, \mathbf{X}_{2[j,:]}))$$

- 'Original Loss ' i.e, the standard loss

$$e_{orig}(f) := \mathbb{E}h_f(Z^{(a)}, Z^{(a)})$$
$$\hat{e}_{orig}(f) := \frac{1}{n} \sum_{i=1}^{n} L(f, \mathbf{Z}_{[i,:]}) = \hat{\mathbb{E}}L(f, Z)$$

- Model Reliance : the ratio of switched loss and original loos

$$MR(f) =: \frac{e_{switch}(f)}{e_{orig}(f)}$$
$$\hat{M}R(f) =: \frac{\hat{e}_{switch}(f)}{\hat{e}_{orig}(f)}$$

Model Reliance and Causal Effects

Causal Inferene Literature

$$\text{Rename: } (Y, X_1, X_2) \Rightarrow (Y, T, C)$$
$$T : X_1 \text{ represents a binary treatment indicator } \{0,1\}$$
$$C : X_2 \text{ a set of baseline covariates}$$
$$f^*(t, c) = \mathbb{E}(Y|C = c, T = t)$$

- Y_1, Y_0: potential outcomes under treatment and control respectively
  - $Y = Y_0(1-T) + Y_1 T.$
    - i.e, if $T = 0$, $Y = Y_0$, if $T = 1$, $Y = Y_1$
- Treatment Effect: $Y_1 - Y_0$
- Average treatment effect : $\mathbb{E}[Y_1 - Y_0]$
- Conditional average treatment effect :

$\mathcal{TE}(c) : \mathbb{E}[Y_1 - Y_0|C = c] = \mathbb{E}[Y_1|C = c, T = 1] - \mathbb{E}[Y_0|C = c, T = 0] = f^*(t = 1, c) - f^*(t = 0, c)$

- $P(T = 1|C = c) \in (0, 1)$

Relation between $\mathcal{TE}(c)$ and the model reliance $e_{orig}(f^*) - e_{switch}(f^*)$

suppose the loss function is $L(f, (y, t, c)) := (y - f(t, c))^2$

Relation

$$e_{orig}(f^*) - e_{switch}(f^*) = \text{Var}(T) \sum_{t \in \{0,1\}} \mathbb{E}_{C|T=t} \mathcal{TE}(C)^2$$

Model reliance for linear models

For any prediction model f and squared error loss , and $f_\beta(x) = X_1^T \beta_1 + X_2^T \beta_2$

then,

$$e_{switch}(f) - e_{orig}(f) = 2Cov(Y, X_1)\beta_1 - 2\beta_2^T Cov(X_2, X_1)\beta_1$$

- Model reliance for linear models can be interpreted in terms of the **population covariances and the model coefficients**.

Model Class Reliance

$$\text{subset of good models: } \mathcal{R}(\epsilon, f^*, \mathcal{F}) := \{f \in \mathcal{F} : e_{orig}(f) \leq e_{orig}(f^*) + \epsilon\}$$

$$MCR_+(\epsilon, f^*, \mathcal{F}) := \max_{f \in \mathcal{R}(\epsilon, f^*, \mathcal{F}):} MR(f)$$
$$MCR_-(\epsilon, f^*, \mathcal{F}) := \min_{f \in \mathcal{R}(\epsilon, f^*, \mathcal{F}):} MR(f)$$

- if MCR_+ is low, no well performing model exists that places high importance on X_1
- if MCR_- is large, then every well performing model must rely substantially on X_1
- In practice, we need to use f_{ref} to approximate the optimal f^*

Finite-sample bounds for model class reliance

Assumptions :

- Bounded individual loss

$$f \in \mathcal{F} : L(f_{ref}, (y, x_1, x_2) \leq B_{ind}$$

- Bounded relative loss)

$$|L(f, (y, x_1, x_2) - L(f_{ref}, (y, x_1, x_2)| \leq B_{ref}$$

- Bounded average loss)

$$\mathbb{P}(0 \leq b_{avg} \leq \hat{e}_{orig}(f) \leq B_{avg}) = 1$$

Suppose $f_{+,\epsilon} = \in \arg\max_{\mathcal{R}(\epsilon)} MR(f)$ and $f_{-,\epsilon} = \in \arg\min_{\mathcal{R}(\epsilon)} MR(f)$ satisfy the above assumptions, then

$$\mathbb{P}(MCR_+(\epsilon) > \widehat{MCR}_+(\epsilon_1) + \mathcal{Q}_1) \leq \delta$$
$$\mathbb{P}(MCR_-(\epsilon) < \widehat{MCR}_-(\epsilon_1) - \mathcal{Q}_1) \leq \delta$$
$$\epsilon_1 := \epsilon + 2B_{ref}\sqrt{\frac{\log(3\delta^{-1})}{2n}}$$
$$\mathcal{Q}_1 := \frac{B_{avg}}{b_{avg}} - \frac{B_{avg} - B_{ind}\sqrt{\frac{\log(3\delta^{-1})}{n}}}{b_{avg} + B_{ind}\sqrt{\frac{\log(3\delta^{-1})}{2n}}}$$

r-margin-expection-cover : a function set $\mathcal{G}_r$: if for any $f \in \mathcal{F}$, and distribution D, there exists a $g \in \mathcal{G}_r$ such taht

$$\mathbb{E}_{Z \sim D}|L(f, Z) - L(g, Z)| \leq r$$
$$\mathcal{N}(\mathcal{F}, r) : \text{covering number: the size of smallest } \mathcal{G}_R$$

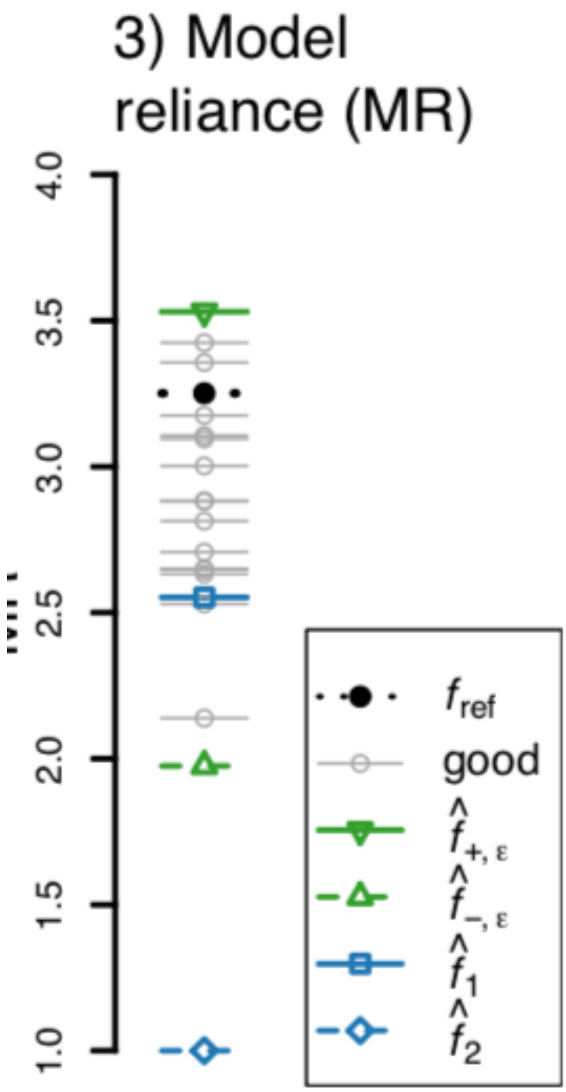
$$\mathbb{P}(MCR_+(\epsilon) < \widehat{MCR}_+(\epsilon_3) + \mathcal{Q}_3) \leq \delta$$
$$\mathbb{P}(MCR_-(\epsilon) > \widehat{MCR}_-(\epsilon_3) - \mathcal{Q}_3) \leq \delta$$
$$\epsilon_3 = \epsilon - 2B_{ref}\sqrt{\frac{\log(4\delta^{-1}\mathcal{N}(\mathcal{F}, r\sqrt{2}))}{n}} - 2r$$
$$\mathcal{Q}_3 := \frac{B_{avg}}{b_{avg}} - \frac{B_{avg} - B_{ind}\sqrt{\log(8\delta^{-1}\mathcal{N}(\mathcal{F}, r\sqrt{2})/n} - 2r\sqrt{2}}{b_{avg} + B_{ind}\sqrt{\log(8\delta^{-1}\mathcal{N}(\mathcal{F}, r)/2n} - 2r}$$

Illustrations

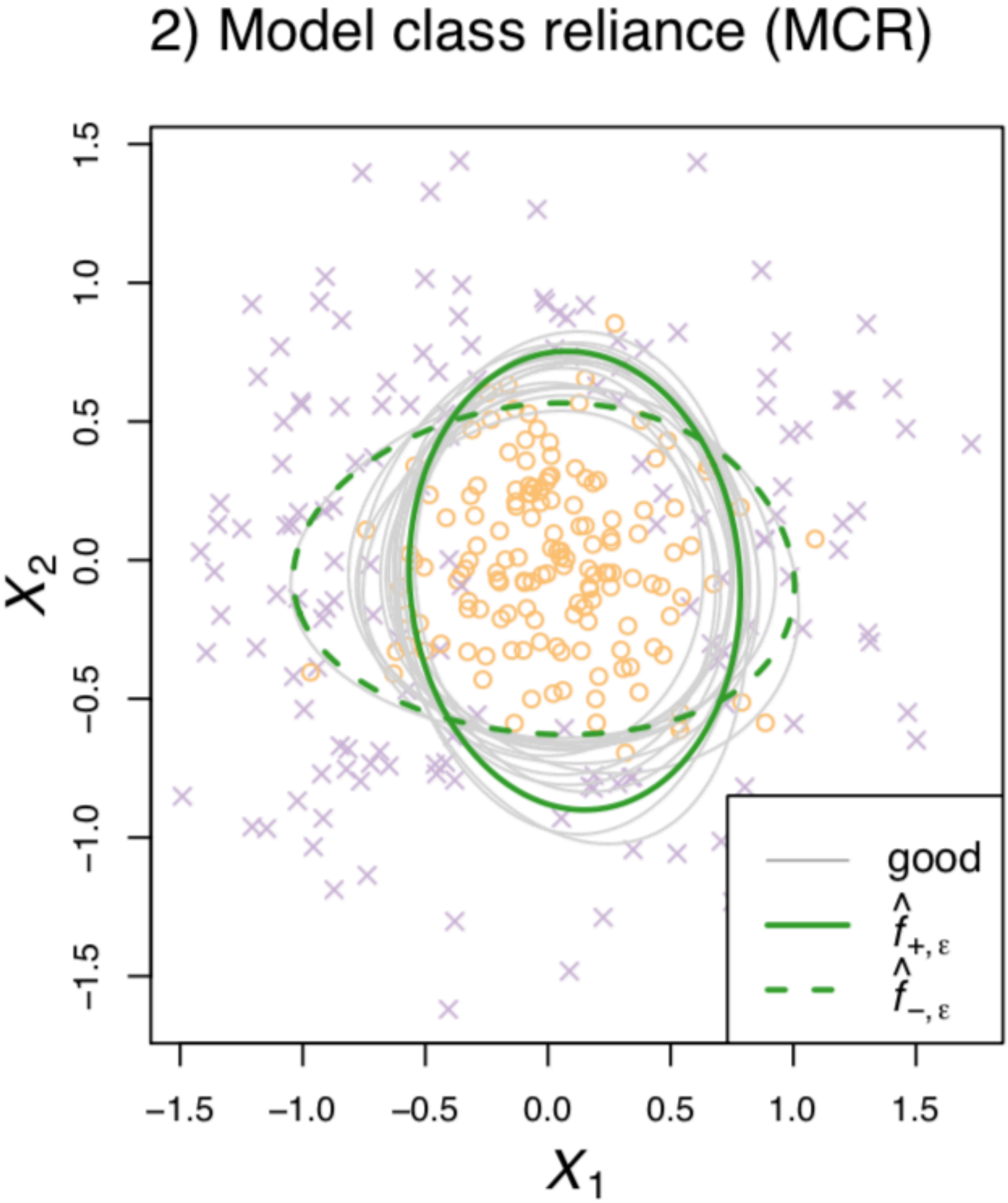
- $X = (X_1, X_2) \in \mathbb{R}^2$ and $Y \in \{-1, 1\}$
- $\mathbb{E}[X_1|Y=-1] = \mathbb{E}[X_2|Y=-1] = 0$, $\mathbb{V}(X_1|Y=-1) = \mathbb{V}(X_2|Y=-1) = 1/9$
- For $Y = 1$, X are drawn as $Y = -1$, but additional $(\cos(U, \sin(U))$ are added
- $\mathcal{F}$: 3-degree polynomial classifers

$$f_\theta(x_1, x_2) = \theta_1 + \theta_2 x_1 \ldots \theta_4 x_1^2 + \theta_5 x_2^2 + \theta_6 x_1 x_2 \ldots \theta_{10} x_1 x_2^2$$

MR



The classifier $\hat{f}_2$ fit without using X_1 (blue dashed line) has a model reliance=1



- $\mathbf{v}$: vector
  - $\mathbf{v}_{[j]}$: the jth element of $\mathbf{v}$
  - $\mathbf{v}_{[-j]}$: all elements except the jth
- $\mathbf{A}$: matrix
  - $\mathbf{A}^T$: TRANSPOSE

$e_{switch}(f)$: expected loss in a pair of observations in which **the treatment has been switched**?.

$$e_{orig}(f) = \mathbb{E}(L(f, y, T, C)) = p(t=1) \int\int L(f, y, t=1, c) g(y, c|t=1) d_y d_c + p(t=0) \int\int L(f, y, t=0, c) g(y, c|t=0) d_y d_c$$

$$e_{switch}(f) = p(t=1) \int\int L(f, y, t=0, c) g(y, c|t=1) d_y d_c +$$
$$p(t=0) \int\int L(f, y, t=1, c) g(y, c|t=0) d_y d_c$$

$$e_{switch}(f^*) - e_{orig}(f^*) = p(t=1) \int\int [(y - \mathbb{E}[y|0, c])^2 - (y - \mathbb{E}[y|1, c])^2] g(y, c|t=1) d_y d_c$$
$$+ (1 - p(t=1)) \int\int [(y - \mathbb{E}[y|1, c])^2 - (y - \mathbb{E}[y|0, c])^2] g(y, c|t=0) d_y d_c$$

$$\text{subset of good models: } \mathcal{R}(\epsilon, f^*, \mathcal{F}) := \{f \in \mathcal{F} : e_{orig}(f) \leq e_{orig}(f^*) + \epsilon\}$$

$$\mathcal{TE}(C) : \mathbb{E}[Y_1 - Y_0 | C = c] = \mathbb{E}[Y_1 | C = c, T = 1] - \mathbb{E}[Y_0 | C = c, T = 0] = f^*(T = 1, C) - f^*(T = 0, C)$$

$$\sum_{t \in \{0,1\}} \mathbb{E}_{C|T=t} \mathcal{TE}(C)^2 = \int\int (\mathbb{E}[y|1, c] - \mathbb{E}[y|0, c])^2 g(y, c|1) d_y d_c + \int\int (\mathbb{E}[y|1, c] - \mathbb{E}[y|0, c])^2 g(y, c|0) d_y d_c$$

$$\mathrm{Var}(T) = \mathbb{E}[(T - p(t=1))^2] = p(t=1)(1 - p(t=1))$$