CS5800 Algorithms Spring 2014 Solution to Assignment #6

1. (30 points) A train is given to two children as a gift. A train consists of a sequence of cars connected in a single line. There is an even number of cars, and the two children wish to divide them up. They do this by taking one car at a time from one end of the train. The older child goes first. Each car has a value to each child. and they want to maximize the value of the cars that they select. Note that the same car could be valued differently by the two children. Develop an algorithm for finding the best strategy for each child. The time complexity of your algorithm must be at most quadratic in the number of cars. If there happens to be a situation where the two choices for a child have the same value, then the child will make the choice which is best for the other child. It is not necessary to prove that your algorithm is correct, but you must prove that it is quadratic.

This can be solved using DP. Let n be the number of cars in the initial train. Let v(i,k) be the value of car i to child k.

1. The subproblem is to maximize the (total) value for the subsequence of cars from car i to car j, inclusive. Call this SP(i,j).
2. The labels are the pairs (i,j) such that 1≤i≤j≤n.
3. The formula is SP(i,j) = max(SP(i+1,j)+v(i,k), SP(i,j-1)+v(j,k)) where k is the child making the choice.
4. The directed graph has edges from (i,j) to (i+1,j) and (i,j-1) when j>i.
5. The sink nodes in the dependency graph are (i,i) and SP(i,i) = value of car i for the younger child.
6. The solution to the problem is SP(1,n).

The strategy is obtained by examining which of the two choices were made in each formula. It can happen that there are several optimal strategies.

There are n * (n+1) / 2 subproblems. Each can be computed in constant time, so the time complexity is $\Theta(n^2)$. The space complexity is not the same since one arrange the iterative solution so that only a diagonal is saved. So the space complexity is $\Theta(n)$. However, it is okay to say that the space complexity is $O(n^2)$, i.e., at most quadratic.

One must explicitly state that one is solving the problem with DP. It cannot be solved using the Greedy Algorithm. Another important point is that one is maximizing the total value of the cars chosen by both children. This is not a zero-sum game, so mini-max solutions will necessarily be wrong. In fact, if the two children value the cars equally, then it would not matter which ones they chose, as the value of the selected cars would always be the same.

2. (25 points) In this problem we look at two translations of Aristotle's Metaphysics. The first one (by W.D. Ross) begins like this:

"All men by nature desire to know. An indication of this is the delight we take in our senses; for even apart from their usefulness they are loved for themselves; and above all others the sense of sight. For not only with a view to action, but even when we are not going to do anything, we prefer seeing (one might say) to everything else. The reason is that this, most of all the senses, makes us know and brings to light many differences between things."

Another (by Hugh Tredennick) begins like this:

"All men naturally desire knowledge. An indication of this is our esteem for the senses; for apart from their use we esteem them for their own sake, and most of all the sense of sight. Not only with a view to action, but even when no action is contemplated, we prefer sight, generally speaking, to all the other senses. The reason of this is that of all the senses sight best helps us to know things, and reveals many distinctions."

To compare them, we align them word-by-word so as to maximize the alignment score. A matching word scores 10 points. Two words that have the same meaning but differ only in a part of speech (e.g., singular versus plural) are also considered a match but score only 5 points. A gap (insertion or deletion) scores -5 points for a noun, pronoun or verb, and scores 0 points for other parts of speach. Perform alignment separately for each major clause. Show the alignments and the scores for the first two major clauses (i.e., up to the word "senses").

The first sentence has 7 words in the first quotation and 5 in the second, so it requires a 6x8 table.

	_	All	men	by	nature	desire	to	know
_	0	-5	-10	-10	-15	-20	-20	-25
All	-5	10	5	5	0	-5	-5	-10
men	-10	5	20	20	15	10	10	5
naturally	-10	5	20	20	25	20	20	15
desire	-15	0	15	15	20	35	35	30
knowledge	-20	-5	10	10	15	30	30	40

The alignment is:

All	men	by	nature	desire	to	know
All	men	_	naturally	desire	_	knowledge
10	10	0	5	10	0	5

The score is 40. The words "nature" and "naturally" match with 5 points as do "know" and "knowledge". The word "all" is a pronoun.

The second has 12 words in the first translation and 10 words in the second, so it requires an 11x13 table. Note that it is not necessary to have a detailed definition of part of speech, since the only two cases are these two, and they are rather obvious.

	_	An	indication	of	this	is	the	delight	we	take	in	our	senses
_	0	0	-5	-5	-10	-15	-15	-20	-25	-30	-30	-35	-40
An	0	10	5	5	0	-5	-5	-10	-15	-20	-20	-25	-30
indication	-5	5	20	20	15	10	10	5	0	-5	-5	-10	-15

of	-5	5	20	30	25	20	20	15	10	5	5	0	-5
this	-10	0	15	25	40	35	35	30	25	20	20	15	10
is	-15	-5	10	20	35	50	50	45	40	35	35	30	25
our	-20	-10	5	15	30	45	45	40	35	30	30	45	40
esteem	-25	-15	0	10	25	40	40	35	30	25	25	40	35
for	-25	-15	0	10	25	40	40	35	30	25	25	40	35
the	-25	-15	0	10	25	40	50	45	40	35	35	40	35
senses	-30	-20	-5	5	20	35	45	40	35	30	30	35	50

An	indication	of	this	is	the	delight	we	take	in	our	_	_	_	senses
An	indication	of	this	is	_	_	_	_	_	our	esteem	for	the	senses
10	10	10	10	10	0	-5	-5	-5	0	10	-5	0	0	10

The score is 50.

3. (25 points) Compute the all-pairs shortest path lengths for the following graph:

Edge	**Length**	**Edge**	**Length**	**Edge**	**Length**	**Edge**	**Length**
(1,3)	-3	(1,4)	1	(2,4)	7	(2,5)	8
(3,4)	8	(3,5)	9	(4,2)	2	(4,3)	10
(4,5)	9	(5,3)	-1				

Show your answer as a sequence of 5x5 matrices. You do not have to show the shortest paths.

	1	**2**	**3**	**4**	**5**
1	∞	∞	-3	1	∞
2	∞	∞	∞	7	8
3	∞	∞	∞	8	9
4	∞	2	10	∞	9
5	∞	∞	-1	∞	∞

	1	**2**	**3**	**4**	**5**
1	∞	∞	-3	1	∞
2	∞	∞	∞	7	8

3	∞	∞	∞	8	9
4	∞	2	10	∞	9
5	∞	∞	-1	∞	∞

	1	**2**	**3**	**4**	**5**
1	∞	∞	-3	1	∞
2	∞	∞	∞	7	8
3	∞	∞	∞	8	9
4	∞	2	10	9	9
5	∞	∞	-1	∞	∞

	1	**2**	**3**	**4**	**5**
1	∞	∞	-3	1	6
2	∞	∞	∞	7	8
3	∞	∞	∞	8	9
4	∞	2	10	9	9
5	∞	∞	-1	7	8

	1	**2**	**3**	**4**	**5**
1	∞	3	-3	1	6
2	∞	9	17	7	8
3	∞	10	18	8	9
4	∞	2	10	9	9
5	∞	9	-1	7	8

	1	**2**	**3**	**4**	**5**
1	∞	3	-3	1	6
2	∞	9	7	7	8
3	∞	10	8	8	9
4	∞	2	8	9	9

5	∞	9	-1	7	8

4. (20 points) Suppose that one has a collection of coins with denominations $d_1, d_2, ..., d_n$. The denominations are distinct positive integers, and one has an unlimited supply of coins of each denomination. You wish to determine whether it is possible to select a set of coins so as to make a total value v. Moreover, if it is possible, then you want to know the minimum number of coins that are needed to make the value v. Express this problem as a DP problem, and prove that your DP algorithm is correct.

1. The subproblem is Ch(w) = "The minimum number of coins needed to make the total value v if it is possible or ∞ if not".
2. The labels are the values w where $0 \leq w \leq v$.
3. The computation of Ch(w) in terms of other subproblems is $Ch(w) = \min(1+Ch(w-d_i) | 1 \leq i \leq n \text{ and } d_i \leq w)$.
4. There is an edge from w to $w-d_i$ for each i such that $1 \leq i \leq n$ and $d_i \leq w$.
5. A node labeled with w in the directed graph is a sink node if there is no d_i such that $d_i \leq w$. In other words, if the minimum of the denominations d_i is greater than w. For the sink nodes. Ch(w) = 0 if w = 0 and Ch(w) = ∞ for w >0.
6. The main problem is Ch(v).

The proof is by induction:

1. Initialization. For the sink node w = 0, Ch(w) is clearly equal to 0 since a set of no coins will produce this value. For the other sink nodes, it is not possible to produce the required value since there is no denomination small enough.
2. Maintenance. Suppose that w is not a sink node and that Ch(u) is the correct number of coins for all u<w. If Ch(w)=∞, then one cannot produce w using the set of denominations. It follows that $Ch(w-d_i)=\infty$ for every i such that $d_i \leq w$. If Ch(w)<∞, then $Ch(w) \leq \min(1+Ch(w-d_i) | 1 \leq i \leq n \text{ and } d_i \leq w)$ because one can add a coin of denomination d_i to a collection of coins that produces $w-d_i$ to get w. On the other hand, if $Ch(w) < \min(1+Ch(w-d_i) | 1 \leq i \leq n \text{ and } d_i \leq w)$, then removing one of the coins from a collection producing w will use fewer coins than is possible for every value $w-d_i$. (Note that this is where we use the fact that w is not a sink node.)
3. Termination. The value of Ch(v) is the answer to the main problem.

Finally, the directed graph of dependencies is a DAG, because for every edge the label decreases. Thus no cycle is possible. The proof now follows by induction on the DAG.