On The Phase Space Representations of Finite-State Quantum System

Abstract

The phase space representation of quantum mechanics transforms QM into a quasi-classical probability form. If the time evolution generated by the Hamiltonian operator is contained in the symmetry group of the phase space, classical trajectories are induced by the symplectic structure of the phase space. Three type of phase space, Weyl-Stratonovich phase space, a SU(F) phase space and the phase space of Meyer-Miller model are talked about in this document. The only difference between the last two phase space is a global phase, which leads to a huge difference on the complexity of equation of motion.

1 Introduction

When a phase space representation of quantum mechanics is employed to perform dynamical problems, we usually want to transform the operator-state form into a classical probability form, i.e., to get the expectation value of some operator $\hat{A}$ at time t or the value of corrlation function $\langle\hat{A}(0)\hat{B}(t)\rangle$ by an integral over phase space (as an differential manifold, noted M):

$$\langle\hat{A}(t)\rangle = \int \mathrm{d}\mu(z)\, f_{\hat{\rho}}(z)\tilde{f}_{\hat{A}(t)}(z) = \int \mathrm{d}\mu(z)\, \tilde{f}_{\hat{\rho}}(z)f_{\hat{A}(t)}(z), \tag{1}$$

$$\langle\hat{A}(0)\hat{B}(t)\rangle = \int \mathrm{d}\mu(z)\, f_{\hat{\rho}\hat{A}}(z)\tilde{f}_{\hat{B}(t)}(z) = \int \mathrm{d}\mu(z)\, \tilde{f}_{\hat{\rho}\hat{A}}(z)f_{\hat{B}(t)}(z). \tag{2}$$

Here $\hat{A}(t) \equiv \mathrm{e}^{\mathrm{i}\hat{H}t}\hat{A}\mathrm{e}^{-\mathrm{i}\hat{H}t}$ is the Heisenberg operator at time t of operator $\hat{A}$, and $f_{\hat{A}(t)}(z)$ is its mapped function in phase space. A point in phase space is noted z, it has components of the same number as the dimension of the phase space. The mapped function $f_{\hat{A}}(z)$ in phase space of an operator $\hat{A}$ is defined

$$f_{\hat{A}}(z) \equiv \mathrm{tr}(\hat{K}(z)\hat{A}), \tag{3}$$

where $\hat{K}(z)$ is a smooth Hermitian operator function defined on the phase space satisfing $\mathrm{tr}\hat{K}(z) = 1$. $\tilde{f}_{\hat{A}}(z)$ is the dual mapped function defined as

$$\tilde{f}_{\hat{A}}(z) = \mathrm{tr}(\hat{K}^{-1}(z)\hat{A}), \tag{4}$$

where $\hat{K}^{-1}(z)$ is the inverse mapping kernel as a smooth Hermitian operator function on the phase space, and satisfies $\mathrm{tr}\hat{K}^{-1}(z) = 1$. Since the time evolution performs as continous unitary transformations of operators, to perform it on the phase space, an action of some Lie group G is defined on the phase space, and there is a covariance property of mapping kernel [15, 3]:

$$\hat{D}(g)\hat{K}(z)\hat{D}^{\dagger}(g) = \hat{K}(g \cdot z), g \in G, z \in M. \tag{5}$$

$\hat{D}(g)$ is the unitary representation of group element g in the same space of operators considered, for example, $\hat{K}(z)$. The integral measure $\mathrm{d}\mu(z)$ is an invariant volumn element under the group action $g\cdot$.

On even-dimensional phase space, there exists a natural symplectic structure on it (see e.g.[5, 2], Arbgaryan *et.al.* noted the relationship between the symplectic structure and the unitary transform in their 2021 paper [1]), and from this symplectic structure we cen derive a classical trajectory-form evolution z_t from z (notice that in classical mechanics, the evolution in phace space is defined by the symplectic structure of the phace space). This classical trajectory-form evolution is exact when the quantum evolution $\mathrm{e}^{-\mathrm{i}\hat{H}t}$ generated by the system's Hamiltonian $\hat{H}$ is a unitary representation of some element g in the Lie group G chosen in the last paragraph, i.e., the Hamiltonian is an generator of the Lie group G. In general cases, we employ this classical trajectory-form evolution as a approximation of the exact evolution of operator:

$$f_{\hat{A}(t)}(z) \approx f_{\hat{A}}(z_t). \tag{6}$$

Now the expectation value and correlation function becomes:

$$\langle \hat{A}(t) \rangle = \int \mathrm{d}\mu(z)\, \tilde{f}_{\hat{\rho}}(z) f_{\hat{A}}(z_t), \tag{7}$$

$$\langle \hat{A}(0)\hat{B}(t) \rangle = \int \mathrm{d}\mu(z)\, \tilde{f}_{\hat{\rho}\hat{A}}(z) f_{\hat{B}}(z_t). \tag{8}$$

Employing Monte-Carlo method to do the integral over phase space, the dynamical calculation is performed as follows:

1. initial condition: sampling phase space points satisfying some distribution, for example, uniformly distribution on the phase space or the distribution defined by $\tilde{f}_{\hat{\rho}}(z)$.

2. evolution: calculation z_t for each initial state z by classical trajectory-form equation of motion.

3. expectation: perform Monte-Carlo integral of Eqs.(7) and (8), values of physical obsevable $\hat{A}$ is $f_{\hat{A}}(z_t)$.

For odd-dimensional phase space, there exists a structure similar to the symplectic structure of even-dimensional phase space. In classical mechanics, there is an odd-dimensional phase space as the oringinal phase space added with a variable expressing time. We will see a similarity of mapping phase space of quantum mechanics in the discussion following.

For finite-state quantum system, we will talk about Weyl-Stratonovich mapping phase space for spin systems (defined with the Lie group SU(2)) [15], the mapping phase space defined with SU(F) (F is the number of states in the quantum system) [10, 18], and the phase space in Meyer-Miller mapping model [12, 13]. This document is constructed as follows: in section 2, we will present the difinition of these mapping phase space representations, containing the mapping kernels, the phace space with its volumn element, and the inverse kernels; in section 3, we will give a general discussion on the symplectic structure on phase space, together with the similar structure on odd-dimensional phase space; in section 4, we will talk about the explicit form of classical equation of motion for these mapping phase space and the connections between them.

2 Difinitions of Mapping Phase Space Representations

2.1 Weyl-Stratonovich Phase Space

For spin systems, the Hamiltonians are linear to angular momentum operators $\hat{J}_x, \hat{J}_y$ and $\hat{J}_z$ most of the time, so the phase space related to the Lie group SU(2) is useful. This mapping phase space representation is intoduced by Stratonovich for 2-state (spin half) system at first, and is extended to general spin cases. The mapping kernel for spin half system is [15]

$$\hat{K}(\theta, \phi) = -3^{(1+s)/2} \left|\theta, \phi\right\rangle\!\left\langle\theta, \phi\right| + \frac{\hat{1}}{2}[1 + 3^{(1+s)/2}]. \tag{9}$$

Here the notation $|\theta, \phi\rangle$ refers to a spin coherent state, which has the form in nomally used basis $\{|\uparrow\rangle, |\downarrow\rangle\}$

$$|\theta, \phi\rangle = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi/2}\cos(\theta/2) \\ -\mathrm{e}^{-\mathrm{i}\phi/2}\sin(\theta/2) \end{pmatrix}. \tag{10}$$

The value of s can be any real number, however, replacing $(1+F)^{(1+s)/2}$ with $-(1+F)^{(1+s)/2}$ is also allowed. It plays a similar role with that γ in the mapping kernel of Meyer-Miller model.

The mapping kernel of spin system with a spin $j > 1/2$ (there are $F = (2j+1)$ states) has a complex form [19, 4, 11]:

$$\hat{K}(\theta, \phi) = \sqrt{\frac{4\pi}{2j+1}} \sum_{l=0}^{2j} \sum_{m=-l}^{l} (C^{jj}_{jj,l0})^{-s} Y^*_{lm}(\theta, \phi) \hat{T}^j_{lm}. \tag{11}$$

Here $Y_{lm}(\theta, \phi)$ is spherical harmonics, $\hat{T}^j_{lm}$ are the irreducible tensor operators of spin j [6]:

$$\hat{T}^j_{lm} = \sqrt{\frac{2l+1}{2j+1}} \sum_{m_1, m_2=-j}^{j} C^{jm_2}_{jm_1,lm} \left|j, m_2\right\rangle\!\left\langle j, m_1\right|. \tag{12}$$

The notation $C^{jm}_{j_1m_1,j_2m_2}$ refers to Clebsch-Gordan coefficients for the addition (coupling) of two angular momentums, is the overlap between the product state $|j_1m_1\rangle\otimes|j_2m_2\rangle$ and the coupling state $|j_1j_2,jm\rangle$ with angular momentum number j and z-component m. The partucular values are available on handbooks or websites.

The phase space of Weyl-Stratonovich representation is sphere $\mathbb{S}^2$, i.e., the range of the variables are $\theta\in[0,\pi]$ and $\phi\in[0,2\pi)$. The invariant integral measure (volumn element) of the phase space is

$$\mathrm{d}\mu(\theta,\phi)=\frac{2j+1}{4\pi}\sin\theta\mathrm{d}\theta\mathrm{d}\phi. \tag{13}$$

Since the group SU(2) is a matrix group, each element g in SU(2) can be written as (in Eular angle parameterization)

$$g_{(\theta,\phi,\psi)}=\mathrm{e}^{\mathrm{i}\phi\sigma_z/2}\mathrm{e}^{\mathrm{i}\theta\sigma_y/2}\mathrm{e}^{\mathrm{i}\psi\sigma_z/2}. \tag{14}$$

σ_z and σ_y are Pauli matrices

$$\sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix};\quad\sigma_y=\begin{pmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{pmatrix}. \tag{15}$$

The action of element g on phase space can be taken by the natural action (matrix multiplication) on the vactor space $\mathbb{C}^2$, i.e., the action of g on point (θ,ϕ) returns (θ',ϕ') such that for a real number ψ' it holds that

$$\mathrm{e}^{\mathrm{i}\psi'}\begin{pmatrix}\mathrm{e}^{\mathrm{i}\phi'/2}\cos(\theta'/2)\\-\mathrm{e}^{-\mathrm{i}\phi'/2}\sin(\theta'/2)\end{pmatrix}=g\cdot\begin{pmatrix}\mathrm{e}^{\mathrm{i}\phi/2}\cos(\theta/2)\\-\mathrm{e}^{-\mathrm{i}\phi/2}\sin(\theta/2)\end{pmatrix}. \tag{16}$$

Another understanding of the group action on the phase space is considering each point of the phase space as a coset of the subgroup

$$H\cong U(1)\equiv\{h=\mathrm{e}^{\mathrm{i}\psi\sigma_z/2}\in SU(2)\mid\psi\in[0,2\pi)\} \tag{17}$$

of SU(2). Each coset can be expressed as

$$(\theta,\phi)=g_{(\theta,\phi,0)}H=\{\mathrm{e}^{\mathrm{i}\phi\sigma_z}\mathrm{e}^{\mathrm{i}\theta\sigma_y}\mathrm{e}^{\mathrm{i}\psi\sigma_z}\in SU(2)\mid\psi\in[0,2\pi)\}. \tag{18}$$

The action of group element a on phase space point (θ,ϕ) is induced by the group multiplication on cosets

$$(\theta',\phi')=g_{(\theta',\phi',0)}H=a\cdot g_{(\theta,\phi,0)}H. \tag{19}$$

In this perspective, the phase space is isomophic to a quotient: $\mathbb{S}^2\cong SU(2)/U(1)$.

For any spin j, the inversed kernel of the mapping kernel with the value s is the mapping kernel with the value $-s$.

2.2 Mapping Phase Space Associated to the Group SU(F)

The mapping phase space representation related to the Lie group SU(F) is similarly defined as SU(2) [10, 18]. The $(2F-2)$-dimensional phase space $SU(F)/U(F-1)$ is parameterized by $(2F-2)$ angles $(\theta_1, \theta_2, \ldots, \theta_{F-1}, \phi_1, \phi_2, \ldots, \phi_{F-1})$, the value range of each angle θ_i is $[0, \pi/2]$, and of each angle ϕ_i is $[0, 2\pi)$. The group action is similarly defined by the group multiplication over cosets of $U(F-1)$ as in Weyl-Stratonovich mapping phase space. For simplicity, a point on phase space is noted z. The invariant integral measure (volumn element) $\mathrm{d}\mu(z)$ is

$$\begin{aligned} \mathrm{d}\mu(z) &= \frac{F!}{2\pi^{F-1}} \prod_{1\le i\le F-1} K_i \,\mathrm{d}\theta_i \mathrm{d}\phi_i, \\ K_i &= \begin{cases} \sin(2\theta_1), & i=1, \\ (\cos\theta_i)^{2i-1}\sin\theta_i, & i \neq F-1, \\ \cos\theta_{F-1}(\sin\theta_{F-1})^{2F-3}, & i = F-1. \end{cases} \end{aligned} \tag{20}$$

Although this mapping phase space representation can be also defined for F'-state system $(F' > F)$, we only talk about the situation of F-state system. For this situation, the mapping kernel has the form

$$\hat{K}(z) = (1+F)^{(1+s)/2} \left|\psi_z\right\rangle\!\left\langle\psi_z\right| + \frac{\hat{1}}{F}\left(1 - (1+F)^{(1+s)/2}\right). \tag{21}$$

$|\psi_z\rangle$ is a state related to phase space point z, it can be also considered as a parameterization of an arbitrary normalized state in F-state system up to a global phase [16, 17]:

$$|\psi_z\rangle = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\phi_1+\phi_2+\cdots+\phi_{F-1})}\cos(\theta_1)\cos(\theta_2)\cdots\cos(\theta_{F-2})\sin(\theta_{F-1}) \\ -\mathrm{e}^{\mathrm{i}(-\phi_1+\phi_2+\cdots+\phi_{F-1})}\sin(\theta_1)\cos(\theta_2)\cdots\cos(\theta_{F-2})\sin(\theta_{F-1}) \\ -\mathrm{e}^{\mathrm{i}(\phi_3+\phi_4+\cdots+\phi_{F-1})}\sin(\theta_2)\cos(\theta_3)\cdots\cos(\theta_{F-2})\sin(\theta_{F-1}) \\ \vdots \\ -\mathrm{e}^{\mathrm{i}(\phi_{F-3}+\phi_{F-2}+\phi_{F-1})}\sin(\theta_{F-4})\cos(\theta_{F-3})\cos(\theta_{F-2})\sin(\theta_{F-1}) \\ -\mathrm{e}^{\mathrm{i}(\phi_{F-2}+\phi_{F-1})}\sin(\theta_{F-3})\cos(\theta_{F-2})\sin(\theta_{F-1}) \\ -\mathrm{e}^{\mathrm{i}(\phi_{F-1})}\sin(\theta_{F-2})\sin(\theta_{F-1}) \\ \cos(\theta_{F-1}) \end{pmatrix}. \tag{22}$$

For the situation $F = 3$, it is

$$|\psi_z\rangle = \begin{pmatrix} \mathrm{e}^{\mathrm{i}(\phi_1+\phi_2)}\sin\theta_2\cos\theta_1 \\ -\mathrm{e}^{\mathrm{i}(-\phi_1+\phi_2)}\sin\theta_2\sin\theta_1 \\ \cos\theta_2 \end{pmatrix}. \tag{23}$$

Similar to the situation of SU(2), s can be any real number and $(1+F)^{(1+s)/2}$ can be replaced by $-(1+F)^{(1+s)/2}$. The inverse kernel for the kernel with parameter s is the kernel with parameter $-s$.

2.3 Meyer-Miller Model

Although globar phase of the state $|\psi_z\rangle$ does not contribute in the mapping kernel in the last subsection, we can still include it in the phase space, and as we can see in section 4, this will provide some convience in dynamics. Now the phase space is a $(2F-1)$-dimensional sphere [7]

$$\Big\{ z = (x^{(1)}, p^{(1)}, \dots, x^{(n)}, p^{(n)}, \dots, x^{(F)}, p^{(F)}), \\ \sum_{n=1}^{F} \left(\left(x^{(n)}\right)^2 + \left(p^{(n)}\right)^2 \right) = 2(1+F\gamma) \Big\}. \tag{24}$$

related to the mapping kernel [12, 13]

$$\hat{K}(z) = \sum_{m,n=1}^{F} \left[\frac{(x^{(m)} - \mathrm{i}p^{(m)})(x^{(n)} + \mathrm{i}p^{(n)})}{2} - \gamma\delta_{mn} \right] |n\rangle\langle m| = |\tilde{\psi}_z\rangle\langle\tilde{\psi}_z| - \gamma\hat{1}. \tag{25}$$

The non-normalized state $|\tilde{\psi}_z\rangle$ is (where $|n\rangle$ is the n-th basis state of the F-state system)

$$|\tilde{\psi}_z\rangle = \sum_{n=1}^{F} \frac{x^{(n)} + \mathrm{i}p^{(n)}}{\sqrt{2}} |n\rangle . \tag{26}$$

Although the phase space is $(2F-1)$-dimensional, $2F$ components of coodination are used to describe it. The invariant integral measure (volumn element) is proportional to the natural measure of the sphere

$$\mathrm{d}\mu(z) = \frac{F}{\Omega(\gamma)} \prod_{i=1}^{F} \mathrm{d}x^{(i)} \mathrm{d}p^{(i)} \delta \left(\sum_{n=1}^{F} \frac{(x^{(n)})^2 + (p^{(n)})^2}{2} - (1+F\gamma) \right), \tag{27}$$

$\Omega(\gamma)$ is the area of the sphere

$$\Omega(\gamma) = \int \prod_{i=1}^{F} \mathrm{d}x^{(i)} \mathrm{d}p^{(i)} \left(\frac{(x^{(n)})^2 + (p^{(n)})^2}{2} - (1+F\gamma) \right). \tag{28}$$

The value of gamma can be in range $[-1/F, +\infty)$. If γ is chosen smaller than $-1/F$, the constraint of sphere should be change into

$$\sum_{n=1}^{F} \left(\left(x^{(n)}\right)^2 + \left(p^{(n)}\right)^2 \right) = -2(1+F\gamma), \tag{29}$$

and the signs in the equation of motion should be changed (talked in section 4).

The related Lie group of this phase space is U(F) instead of SU(F), since global phase is added. The phase space is isomophic to U(F)/U(F-1). The group action on phase space is defined by the matrix multiplycation of group element g as a F-dimensional unitary matrix with the state $|\tilde{\psi}_z\rangle$ as a vector in $\mathbb{C}^F$.

The inverse kernel has the form

$$\hat{K}^{-1}(z) = \sum_{m,n=1}^{F} \left[\frac{1+F}{2(1+F\gamma)^2} (x^{(m)} - \mathrm{i}p^{(m)})(x^{(n)} + \mathrm{i}p^{(n)}) - \frac{1-\gamma}{1+F\gamma} \delta_{mn} \right] |n\rangle\langle m| \,. \tag{30}$$

Since the only difference between the SU(F) phase space and the phase space of Meyer-Miller model is a global phase added to the state $|\psi_z\rangle$ in eq.(22), the map between these two phase space is obvious: a point $(\theta_1, \phi_1, \dots, \theta_{F-1}, \phi_{F-1})$ of the SU(F) phase space is related to a circle in the phase space of Meyer-Miller, which satisfies that when $n \neq F$ (the explicit expression of $|\psi_z\rangle$ is in eq.(22))

$$\begin{aligned} x^{(n)} &= \sqrt{2} \times (1+F)^{(1+s)/4} \mathrm{Re}\, \langle n|\psi_z\rangle \,, \\ p^{(n)} &= \sqrt{2} \times (1+F)^{(1+s)/4} \mathrm{Im}\, \langle n|\psi_z\rangle \,; \end{aligned} \tag{31}$$

when $n = F$ the constrain of the circle is

$$(x^{(F)})^2 + (p^{(F)})^2 = 2(1+F)^{(1+s)/2} \cos^2\theta_{F-1}. \tag{32}$$

The relationship between the parameter s in SU(F) phase space and the parameter γ in the pahse space of Meyer-Miller is

$$2(1+F\gamma) = (1+F)^{(1+s)/2}. \tag{33}$$

In terms of the mapped function $f_{\hat{A}}$ of an operator $\hat{A}$, it has the same value on the circle just defined and the related point $(\theta_1, \phi_1, \dots, \theta_{F-1}, \phi_{F-1})$ in SU(F) phase space.

3 Classical Trajectories on the Phase Space

In this section we will do some general discussion for the classical trajectories in a phase space representation of quantum mechanics. The Lie group related to the covariance property of mapping kernels is noted G, and the phase space is (isomorphic to) the quotient set G/H, where H is a Lie subgroup of G.

3.1 Symplectic Structure on Even-Dimensional Phase Space

Recall the covariance property of mapping kernel $\hat{K}(z)$

$$\hat{D}(g)\hat{K}(z)\hat{D}^\dagger(g) = \hat{K}(g \cdot z). \tag{34}$$

Since the phase space is the quotient set G/H, for any phase space point z' in the neighborhood of phase space point z, there exists a group element $g \in G$ such that $g \cdot z = z'$. Taking the difference between z and z' infinitesimal, i.e., $z' = z + \mathrm{d}z$, g is infinitesimal and $\hat{D}(g)$ is a infinitesimal unitary operator. Choosing a set of generators $\{\hat{X}_1, \dots, \hat{X}_N, \hat{Y}_1, \dots, \hat{Y}_M\}$ of G, in which $\{\hat{Y}_l\}$ are generators of the isotropic subgroup H_z, $N = \dim G - \dim H$ is the dimension of phase space, and $M = \dim H_z = \dim H$ is the dimension of group H_z, the transformation from $\hat{K}(z)$ to $\hat{K}(z+\mathrm{d}z)$ can be expanded as

$$\hat{K}(z+\mathrm{d}z) = \mathrm{e}^{-\mathrm{i}\hat{X}_k \mathrm{d}z^k}\hat{K}(z)\mathrm{e}^{\mathrm{i}\hat{X}_k \mathrm{d}z^k} = \hat{K}(z) - \mathrm{i}\mathrm{d}z^k[\hat{X}_k, \hat{K}(z)]. \tag{35}$$

Since H_z is different for different phase space point, the choice of basis is different at different point. The coordinate z has N components, and Einstein's sum rule is employed here, which implies an automatic summation between a pair of superscription and subscription. Quantities with superscriptions are different from those with subscriptions, each of them follow different law of coordinate transforms, and a quantity with superscription can not be transformed into one with subscription, vise versa.

In eq.(35) generators $\hat{Y}_k$ do not contribute since for any group element $h \in H_z$ the action on phase space point $h \cdot z = z$. The covariance property related to this is

$$\hat{D}(h)\hat{K}(z)\hat{D}^\dagger(h) = \hat{K}(z). \tag{36}$$

Since eq.(36) holds for any group element $h \in H_z$, we can derive that

$$[\hat{K}(z), \hat{Y}_l] = 0, \quad \forall\, l = 1, 2, \ldots, M. \tag{37}$$

It is worth noticing that the statement here is general enough for a well defined phase space. Since the mapping kernel $\hat{K}(z)$ is a smooth Hermitian operator function on phase space which satisfies $\mathrm{tr}\hat{K}(z) = 1$, the mapping kernel $\hat{K}(z+\mathrm{d}z)$ is always connected by $\hat{K}(z)$ in the form of eq.(35), for any infinitesimal unitary operator is exponential of some infinitesimal Hermitian operator. With some Hermitian operators $\hat{Y}_l$ that commute with $\hat{K}(z)$ at z, $\{\hat{X}_k, \hat{Y}_l\}$ form a set of generators of some Lie group, and $\{\hat{Y}_l\}$ is the generators of the isotropy subgroup H_z at z.

After discussion the local structure of mapping kernels, we will locally construct the classical evolution. Although the discussion is always focused on local structure, it is appropriate to any choice of basis, and all difference of coodinate choice or the local structure of different phase space point can be considered coming from the choice of basis. So the result of evolution equation is general.

Considere the quantum mechanical time evolution of the mapped function of operator $\hat{A}(t) = \mathrm{e}^{\mathrm{i}\hat{H}t}\hat{A}\mathrm{e}^{-\mathrm{i}\hat{H}t}$ in Heisenberg picture

$$f_{\hat{A}(t)}(z) = \mathrm{tr}(\hat{K}(z)\mathrm{e}^{\mathrm{i}\hat{H}t}\hat{A}\mathrm{e}^{-\mathrm{i}\hat{H}t}) = \mathrm{tr}(\mathrm{e}^{-\mathrm{i}\hat{H}t}\hat{K}(z)\mathrm{e}^{\mathrm{i}\hat{H}t}\hat{A}), \tag{38}$$

if the Hamiltonian operator $\hat{H}$ of the system has the form

$$\hat{H} = h^k \hat{X}_k + \tilde{h}^l \hat{Y}_l, \tag{39}$$

$e^{-i\hat{H}t}$ is a representation matrix of some element in the group G. According to the covariance property, there exists another point z_t in the phase space such that

$$\hat{K}(z_t) = e^{-i\hat{H}t}\hat{K}(z)e^{i\hat{H}t}, \tag{40}$$

and this is a trajectory-form evolution. The differential form of the trajectory can be derived by comparing differential form of eq.(41)

$$\hat{K}(z+\dot{z}\mathrm{d}t) = \hat{K}(z) - \mathrm{i}\mathrm{d}t[\hat{H}, \hat{K}(z)] = \hat{K}(z) - \mathrm{i}h^k\mathrm{d}t[\hat{X}_k, \hat{K}(z)] \tag{41}$$

with eq.(35). The result is

$$\dot{z}^k = h^k. \tag{42}$$

In classical mechanics, the Hamiltonian equation of motion has the form that the time derivative of coordinate is related to some gradients of the Hamiltonian function. In mapping phase space representation of quantum mechanics, if the Hamiltonion function is the mapped function $f_{\hat{H}}(z) = \mathrm{tr}(\hat{K}(z)\hat{H})$ of the Hamiltonian operator $\hat{H} = h^k\hat{X}_k + \tilde{h}^l\hat{Y}_l$ of the system, the gradients of it can be derived by considering a infinitesimal difference of coodinate z:

$$\begin{aligned} f_{\hat{H}}(z+\mathrm{d}z) - f_{\hat{H}}(z) & \\ &= -\mathrm{i}\,\mathrm{d}z^k\mathrm{tr}\left(\hat{H}[\hat{X}_k, \hat{K}(z)]\right) = -\mathrm{i}\,\mathrm{d}z^k\mathrm{tr}\left([\hat{H}, \hat{X}_k]\hat{K}(z)\right) \\ &= -\mathrm{i}\,\mathrm{d}z^k h^j\mathrm{tr}\left(\hat{K}(z)[\hat{X}_j, \hat{X}_k]\right) \equiv \sigma_{kj}h^j\mathrm{d}z^k, \end{aligned} \tag{43}$$

this gives

$$\partial_k f_{\hat{H}}(z) \equiv \frac{\partial}{\partial z^k} f_{\hat{H}}(z) = \sigma_{kj}h^j. \tag{44}$$

Here $\sigma_{jk} \equiv \mathrm{i}\,\mathrm{tr}\left(\hat{K}(z)[\hat{X}_j, \hat{X}_k]\right)$ is the (components of) symplectic structure on the phase space [5]. Here we state without provement that σ_{jk} is a skew-symmetric, invertable matrix in the situation that the phase space is even-dimensional. The inverse matrix of σ_{jk} is σ^{jk}. Those who are interested in more rigorous discussion are advised to Appendix.(A) or refs[5, 2].

From eq.(44) h^k can be expressed by gradients of the Hamiltonian function $f_{\hat{H}}(z)$. The differential form of the trajectory now becomes the form of classical Hamiltonian equation of motion:

$$\dot{z}^k = \sigma^{kl}\partial_l f_{\hat{H}}(z), \tag{45}$$

this is why we call the trajectory "classical".

The trajectory talked about here is suited for the form in eq.(7) and eq.(8). However, if it is wanted to use the form $\tilde{f}_{\hat{A}}(z_t)$ instead of $f_{\hat{A}}(z_t)$, the trajectories are exactly the same as in eq.(45). The expectation value of correlation function can be expressed by the form

$$\mathrm{tr}(\hat{A}(0)\hat{B}(t)) \equiv \mathrm{tr}(\hat{A}\mathrm{e}^{\mathrm{i}\hat{H}t}\hat{B}\mathrm{e}^{-\mathrm{i}\hat{H}t}) = \int \mathrm{d}z\, f_{\hat{A}}(z)\tilde{f}_{\hat{B}(t)}(z). \tag{46}$$

It can be reformulated by the properties of trace as

$$\mathrm{tr}(\hat{A}(0)\hat{B}(t)) = \mathrm{tr}(\mathrm{e}^{-\mathrm{i}\hat{H}t}\hat{A}\mathrm{e}^{\mathrm{i}\hat{H}t}\hat{B}) = \int \mathrm{d}z\, f_{\hat{A}(-t)}(z)\tilde{f}_{\hat{B}}(z). \tag{47}$$

Since the trajectory form of $f_{\hat{A}(t)}(z)$ is known, eq.(47) can be expressed by trajectory form as

$$\mathrm{tr}(\hat{A}(0)\hat{B}(t)) = \int \mathrm{d}z\, f_{\hat{A}}(z_{-t})\tilde{\hat{B}}(z) = \int \mathrm{d}z\, f_{\hat{A}}(z)f_{\hat{B}}(z_t). \tag{48}$$

The last equivalence comes from the inverse map of $z \to z_t$ is $z \to z_{-t}$, and this map does not change the invariant volumn element.

3.2 Extension in Odd-Dimensional Phase Space

In classical mechanics, an extended phase space can be generated by including time t in the phase space for closed system [2]. In quantum mechanics, a globar phase plays the similar role as time t. Here is a discussion for an extension of the even-dimensional phase space defined at the start of this section from a Lie group G and a Lie subgroup H with a global phase φ. Phase space point of the extended phase space is noted $\tilde{z}$, while of the oringinal phase space is noted z. The discussion is restriced on the situation that the mapping kernel is a linear combination of a projection operator on a pure state $|\psi_{\tilde{z}}\rangle\langle\psi_{\tilde{z}}|$ and the identity operator $\hat{1}$. The pure state $|\psi_{\tilde{z}}\rangle$ here is related with an original pure state $|\psi_z\rangle$ in the original phase space representation by $|\psi_{\tilde{z}}\rangle = \mathrm{e}^{-\mathrm{i}\varphi}\,|\psi_z\rangle$.

For this type of mapping kernel, a global phase φ of the pure state does not contribute to the mapped function of any operator. This allows us to assert any smooth evolution to φ. Specially, this evolution can be choised the same as that defined by Schrodinger equation. In detail, at a phase space point z, the generators of G is chosed as $\{\hat{X}_0, \hat{X}_1, \ldots, \hat{X}_N, \hat{Y}_1, \ldots, \hat{Y}_M\}$, these generators satisfy that only $\langle\psi_z|\,\hat{X}_0\,|\psi_z\rangle$ is not zero, and $\hat{X}_0$ itself is the identity operator. $\{\hat{X}_0, \hat{Y}_1, \ldots, \hat{Y}_M\}$ are generaters of the oringinal isotropy subgroup H_z at point z, and $N = \dim G - \dim H_z$ is the dimension of the original even-dimensional phase space, where the extra dimension is the global phase φ , related to the generater $\hat{X}_0$. A infinitesimal unitary transform on the extended phase space now becomes $\hat{U}(\mathrm{d}\varphi, \mathrm{d}z) = \mathrm{e}^{-\mathrm{i}\hat{X}_0\mathrm{d}\varphi - \mathrm{i}\hat{X}_k\mathrm{d}z^k}$. Taking $\langle\psi_{\tilde{z}}|$ on the both side of Schrodinger equation

$$\hat{U}(\mathrm{d}\varphi, \mathrm{d}z)\,|\psi_{\tilde{z}}\rangle = \hat{H}\mathrm{d}t\,|\psi_{\tilde{z}}\rangle\,, \tag{49}$$

we get

$$\dot{\varphi} = \langle\psi_{\tilde{z}}|\,\hat{H}\,|\psi_{\tilde{z}}\rangle\,. \tag{50}$$

The evolution of φ can not be expressed by a derivative on the phase space, i.e., the limit

$$\lim_{z'\to z}\frac{f_{\hat{H}}(z')-f_{\hat{H}}(z)}{z'-z}, \tag{51}$$

$z'-z$ here is the subtraction of coordinates. There exists no Hamiltonian-form equation of motion in the case that the number of variables is odd. This fact comes from there exists no invertable skew-symmetric odd-dimensional matrix, from some properties of determinent:

$$\det(\sigma)=\det(\sigma^{\mathrm{T}})=\det(-\sigma)=(-1)^{\dim}\det(\sigma), \tag{52}$$

if dim is odd, the determinent of σ must be zero, i.e., there is a zero eigenvalue in $sigma$. This zero eigenvalue is correceponded with φ.

The method to generate a Hamiltonian equation of motion is employing an additional variable to conjugate with φ. This variable is noted λ, and the mapping kernel becomes (μ is a constant)

$$\hat{K}(\lambda,\tilde{z})=\lambda\,|\psi_{\tilde{z}}\rangle\langle\psi_{\tilde{z}}|+\mu\hat{1}. \tag{53}$$

It is easy to varify that there exists a Hamiltonian equation of motion describing the exact trajectories on this phase space:

$$\begin{pmatrix}\dot{\lambda}\\ \dot{\varphi}\\ \dot{z}^k\end{pmatrix}=\begin{pmatrix}0 & 1 & \\ -1 & 0 & \\ & & \sigma^{kl}\end{pmatrix}\begin{pmatrix}\partial_\lambda\\ \partial_\varphi\\ \partial_l\end{pmatrix}f_{\hat{H}}(\lambda,\tilde{z}). \tag{54}$$

σ^{kl} here the inverse matrix of σ_{jk} which is defined $\sigma_{kl}=\mathrm{i}\,\mathrm{tr}(\hat{K}(\lambda,\tilde{z})[\hat{X}_k,\hat{X}_l])$. Since the mapped Hamiltonian function $f_{\hat{H}}(\lambda,\tilde{z})$ is independent of φ, λ is in fact a constant, so we can define the phase space just in the phase space $\tilde{z}$ and take $\mu=(1-\lambda)/F$ such that $\mathrm{tr}\hat{K}=1$.

4 Explicit Formulars for Trajectories in Mapping Phase Space Representations

4.1 Weyl-Stratonovich Phase Space

In Weyl-Stratonovich phase space, the mapping kernel can be written as

$$\hat{K}(\theta,\phi)=\hat{U}(\theta,\phi)\hat{K}(0,0)\hat{U}^{\dagger}(\theta,\phi), \tag{55}$$

where the rotation operator is

$$\hat{U}=\mathrm{e}^{\mathrm{i}\phi\hat{J}_z}\mathrm{e}^{\mathrm{i}\theta\hat{J}_y}. \tag{56}$$

To get the generators at point z in this global coordination, consider a coordination transform

$$\hat{K}(z+\Delta z)=\hat{U}(z+\Delta z)\hat{K}(0)\hat{U}^{\dagger}(z+\Delta z)=\hat{U}'(\Delta z)\hat{K}(z)\hat{U}'^{\dagger}(\Delta z). \tag{57}$$

Taking Δz as a infinitesimal, $\hat{U}'(\Delta z)$ has the form $\mathrm{e}^{-\mathrm{i}\hat{X}_k \Delta z^k}$. Choosing Δz such that it has the same components as z, the generator $\hat{X}_k$ at z can be derived from $\hat{U}'(\Delta z)$:

$$-\mathrm{i}\hat{X}_k = \partial_k \hat{U}'(\Delta z)\Big|_{\Delta z\to 0} = \partial_k \left[\hat{U}(z+\Delta z)\right]_{\Delta z \to 0} \hat{U}^\dagger(z). \tag{58}$$

After some calculations, we get

$$\begin{cases} -\mathrm{i}\hat{X}_\theta = \mathrm{e}^{\mathrm{i}\phi \hat{J}_z} \mathrm{i}\hat{J}_y \mathrm{e}^{-\mathrm{i}\phi \hat{J}_z}, \\ -\mathrm{i}\hat{X}_\phi = \mathrm{i}\hat{J}_z. \end{cases} \tag{59}$$

The symplectic structure is

$$\begin{aligned} \sigma_{\theta\phi} = -\sigma_{\phi\theta} &= \mathrm{i}\,\mathrm{tr}\left(\hat{K}(z)[\hat{X}_\theta, \hat{X}_\phi]\right) \\ &= \mathrm{i}\,\mathrm{tr}\left(\hat{U}(\theta,\phi)\hat{K}(0)\hat{U}^\dagger(\theta,\phi)\mathrm{e}^{\mathrm{i}\phi\hat{J}_z}[\hat{J}_y,\hat{J}_z]\mathrm{e}^{-\mathrm{i}\phi\hat{J}_z}\right) \\ &= -\mathrm{tr}\left(\hat{K}(0)\mathrm{e}^{-\mathrm{i}\theta\hat{J}_y}\hat{J}_x\mathrm{e}^{\mathrm{i}\theta\hat{J}_y}\right) = -j\left(\frac{j+1}{j}\right)^{(1+s)/2}\sin\theta. \end{aligned} \tag{60}$$

For the situation $j = 2$ (2-state system), the result is

$$\sigma_{\theta\phi} = -\frac{3^{\frac{1+s}{2}}}{2}\sin\theta. \tag{61}$$

The equation of motion of trajectories is [8, 9]

$$\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = 3^{-(1+s)/2} \times \frac{2}{\sin\theta} \begin{pmatrix} \partial_\phi \\ -\partial_\phi \end{pmatrix} f_{\hat{H}}(\theta,\phi). \tag{62}$$

4.2 Phase Space Associated to SU(F) for F-State System

The mapping kernel has the form that

$$\hat{K}(z) = (1+F)^{(1+s)/2}\left|\psi_z\middle\rangle\!\middle\langle\psi_z\right| + \frac{\hat{1}}{F}\left(1-(1+F)^{(1+s)/2}\right). \tag{63}$$

For this form of mapping kernel, the symplectic structure can be calculated by ($\lambda = (1+F)^{(1+s)/2}$ here) [14]

$$\sigma_{jk} = \mathrm{i}\,\mathrm{tr}\left(\hat{K}(z)[\hat{X}_j,\hat{X}_k]\right) = \lambda\left\langle\psi_z\right|[\hat{X}_j,\hat{X}_k]\left|\psi_z\right\rangle = -2\lambda\,\mathrm{Im}\left\langle\partial_j\psi_z\middle|\partial_k\psi_z\right\rangle, \tag{64}$$

where

$$\left|\partial_j\psi_z\right\rangle = \partial_j\left(\mathrm{e}^{-\mathrm{i}y^k\hat{X}_k}\left|\psi_z\right\rangle\right)\Big|_{y=0} = -\mathrm{i}\hat{X}_j\left|\psi_z\right\rangle. \tag{65}$$

The provement can be performed by direct calculations. Taking the explicit form of the state $|\psi_z\rangle$ in to this formula, the result is

$$\sigma^{ij} = (1+F)^{-\frac{1+s}{2}}\begin{pmatrix} 0 & -\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{A} & 0 \end{pmatrix}, \tag{66}$$

the raw index i of matrix $\boldsymbol{A}$ relates to ϕ_i, and the column index j relates to θ_j,

$$\boldsymbol{A}^{ij} = \begin{cases} \frac{1}{2}\csc(2\theta_1)\csc^2\theta_{F-1}\prod_{k=2}^{F-2}\sec^2\theta_k, & i=j=1; \\ -\frac{1}{2}\cot(2\theta_1)\csc^2\theta_{F-1}\prod_{k=2}^{F-2}\sec^2\theta_k, & (i-1)=j=1; \\ \csc(2\theta_i)\csc^2\theta_{F-1}\prod_{k=i+1}^{F-2}\sec^2\theta_k, & 2\le i=j\le F-2; \\ -\frac{1}{2}\cot(\theta_i)\csc^2\theta_{F-1}\prod_{k=i+1}^{F-2}\sec^2\theta_k, & 2\le(i-1)=j\le F-2; \\ -\csc(2\theta_{F-1}), & i=j=(F-1). \end{cases} \tag{67}$$

There are $(2F-3)$ non-zero elements in the $(F-1)\times(F-1)$ matrix $\boldsymbol{A}$. A simpliest example is the situation that $F=3$:

$$\sigma^{jk} = \begin{pmatrix} & & -\csc^2\theta_2\csc(2\theta_1)/2 & \csc^2\theta_2\cot(2\theta_1)/2 \\ & & 0 & \csc(2\theta_2) \\ \csc^2\theta_2\csc(2\theta_1)/2 & 0 & & \\ -\csc^2\theta_2\cot(2\theta_1)/2 & -\csc(2\theta_2) & & \end{pmatrix}. \tag{68}$$

The Hamiltonian equation of motion is

$$\begin{aligned} \dot{\theta}_1 &= (1+F)^{-\frac{1+s}{2}} \times \frac{\cos(2\theta_1)\partial_{\phi_2} - \partial_{\phi_1}}{2\sin^2\theta_2\sin(2\theta_1)} f_{\hat{H}}(z), \\ \dot{\theta}_2 &= (1+F)^{-\frac{1+s}{2}} \times \frac{\partial_{\theta_2}}{\sin(2\theta_2)} f_{\hat{H}}(z), \\ \dot{\phi}_1 &= (1+F)^{-\frac{1+s}{2}} \times \frac{\partial_{\theta_1}}{2\sin^2\theta_2\sin(2\theta_1)} f_{\hat{H}}(z), \\ \dot{\phi}_2 &= (1+F)^{-\frac{1+s}{2}} \times \left(\frac{-\cos(2\theta_1)\partial_{\theta_1}}{2\sin^2\theta_2\sin(2\theta_1)} + \frac{-\partial_{\theta_2}}{\sin(2\theta_2)}\right) f_{\hat{H}}(z). \end{aligned} \tag{69}$$

4.3 Meyer-Miller Model

The phase space in Meyer-Miller model is odd-dimensional. Comparing the state $|\tilde{\psi}_z\rangle$ in the difinition eq.(26) and the state in eq.(22), the difference of the phase space of Meyer-Miller model and the phase space from SU(F) group is just a global phase. These facts implies that the phase space in Meyer-Miller model is just the extension of the phase space from SU(F) group with a coordinate transform.

To get the equation of motion of trajectories in this phase space, one can simply do direct calculation. However, here we will still use the method in subsection 3.2 to perform the correcepondence between these two phase space representations more explicitly.

Recall the evolution equation eq.(54)

$$\begin{pmatrix} \dot{\lambda} \\ \dot{\varphi} \\ \dot{z}^k \end{pmatrix} = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & \sigma^{kl} \end{pmatrix} \begin{pmatrix} \partial_\lambda \\ \partial_\varphi \\ \partial_l \end{pmatrix} f_{\hat{H}}(\lambda, \tilde{z}).$$

Noting the extended coordinate $(\lambda, \varphi, \tilde{z})$ as ζ, and the transformed coordinate ξ, the transform of time derivatives and gradients induced by the coordinate transform $(\zeta \to \xi)$ are

$$\dot{\xi}^k = \frac{\partial \xi^k}{\partial \zeta^l} \dot{\zeta}^l \tag{70}$$

and

$$\tilde{\partial}_k \equiv \frac{\partial}{\partial \xi^k} = \frac{\partial \zeta^l}{\partial \xi^k} \frac{\partial}{\partial \zeta^l} \equiv \frac{\partial \zeta^l}{\partial \xi^k} \partial_l. \tag{71}$$

Noting the symplectic matrix in eq.(54) ω^{kl}, eq.(54) becomes

$$\dot{\xi}^k = \frac{\partial \zeta^k}{\partial \xi^i} \frac{\partial \zeta^l}{\partial \xi^j} \omega^{ij} \tilde{\partial}_l f_{\hat{H}}(\xi) = \tilde{\omega}^{kl} \tilde{\partial}_l f_{\hat{H}}(\xi). \tag{72}$$

The explicit form of ξ is $\xi = (x^{(1)}, p^{(1)}, \ldots, x^{(F)}, p^{(F)})$. The coordinate transform from ζ to ξ is defined such that the state $|\tilde{\psi}_z\rangle$ in eq.(26) equals to the state $\sqrt{\lambda} e^{-i\varphi} |\psi_z\rangle$, where $|\psi_z\rangle$ is defined in eq.(22). The result of the matrix ω^{kl} is

$$\omega^{kl} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \tag{73}$$

which gives the pleasing equation of motion

$$\begin{aligned} \dot{x}^{(n)} &= +\frac{\partial}{\partial p^{(n)}} f_{\hat{H}}(x, p), \\ \dot{p}^{(n)} &= -\frac{\partial}{\partial x^{(n)}} f_{\hat{H}}(x, p). \end{aligned} \tag{74}$$

The form of equation of motion is much simplier that that of the phase space associated to SU(F), although the only difference between these two phase space is only a global phase.

5 Conclusion

In this document we discussed the difinitions and the trajectory-form dynamics of three type of phase space representation of quantum mechanics. According to the covariance property of mapping kernel, the mapping phase space is defined based on some Lie group devided by one of its Lie subgroup. The group action on phase space is defined the action on the quotient set. Since the evolution generated by Hamiltonian operator is unitary, when the Hamiltonian of the system is a generator of the Lie group there exist trajectory, and this trajectory can be described by a classical Hamiltonian equation of motion, which is associated with the symplectic structure of the phase space. The Weyl-Stratonovich phase space is defind on the SU(2) group, devided by U(1). A phase space can be defined by SU(F) group, added with a global phase variable, the phase space becomes the phase space in Meyer-Miller model. The addition of global phase variable largely simplifies the equation of motion.

A Properties of the Symplectic Structure

A tangent vector v at point z of differential manifold M is a linear map from smooth function on the manifold to $\mathbb{R}$. With a choice of coordinate z^k at the neighborhood of z, the action of v on a function $f(z)$ is

$$v(f) = v^k \frac{\partial}{\partial z^k} f(z). \tag{75}$$

$\{\partial_k \equiv \partial/\partial z^k\}$ forms a set of basis of the linear space $T_z M$ of tangent vectors v at point z. The set of tangent spaces of all point on the manifold is noted TM (fiber bundle).

A differential 1-form ω^1 is a smooth function on the manifold M, the value of which at each point z is a linear map from $T_z M$ to $\mathbb{R}$. Correceponded to the basis $\{\partial_k\}$ of $T_z M$, the basis of the linear space of 1-form $T_z^* M$ is $\mathrm{d}z^k$, which satisfies $\mathrm{d}z^j(\partial_k) = \delta_k^j$, δ_k^j is Kronecker delta.

A differential 2-form ω^2 is a skew-symmetric bilinear map from $TM \otimes TM$ to $\mathbb{R}$, i.e., for $u, v, w \in T_z M, a, b \in \mathbb{R}$,

1. $\omega^2(au + bv, w) = a\,\omega^2(u, w) + b\,\omega^2(v, w)$;
2. $\omega^2(v, w) = -\omega^2(w, v)$.

Correceponded to the basis $\{\partial_k\}$ of $T_z M$, the basis of the linear space of ω^2 is $\{\mathrm{d}z^j \wedge \mathrm{d}z^k\}$, each of which satisfies $\mathrm{d}z^j \wedge \mathrm{d}z^k(\partial_l, \partial_m) = \delta_l^j \delta_m^k - \delta_m^j \delta_l^k$.

Similarly, we can define differential k-form ω^k on the manifold. The basis of the linear space of k-form at point z is all the k-forms $\mathrm{d}z^{i_1} \wedge \mathrm{d}z^{i_2} \wedge \cdots \wedge \mathrm{d}z^{i_k}$ defined as

$$(\mathrm{d}z^{i_1} \wedge \mathrm{d}z^{i_2} \wedge \cdots \wedge \mathrm{d}z^{i_k})(v^{(1)}, \ldots, v^{(k)}) \equiv \begin{vmatrix} \mathrm{d}z^{i_1}(v^{(1)}) & \cdots & \mathrm{d}z^{i_k}(v^{(1)}) \\ \vdots & & \vdots \\ \mathrm{d}z^{i_1}(v^{(k)}) & \cdots & \mathrm{d}z^{i_k}(v^{(k)}) \end{vmatrix}. \tag{76}$$

It is easy to varify that $\mathrm{d}z^{i_1} \wedge \mathrm{d}z^{i_2} \wedge \cdots \wedge \mathrm{d}z^{i_k} = \varepsilon(P)\mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \cdots \wedge \mathrm{d}z^k$, in which P is a permutation operation from $(1, 2, \ldots, k)$ to $(i_1, i_2, \ldots, i_k)$, and $\varepsilon(P)$ is 1 if the permutation is even and is -1 if odd.

The exterior product operation $\wedge$ for a k-form ω^k and a l-form ω^l is defined (in terms of basis) by

$$(\omega^k \wedge \omega^l)(v^{(1)}, \dots, v^{(k+l)}) \equiv \sum_{i_1<\cdots<i_k, j_1<\cdots<j_l} \varepsilon(P)\omega^k(v^{(i_1)}, \dots, v^{(i_k)})\omega^l(v^{(j_1)}, \dots, v^{(j_l)}). \tag{77}$$

Here P is the permutation operation from $(1, 2, \dots, k+l)$ to $(i_1, \dots, i_k, j_i, \dots, j_l)$. The result of the exterior product of ω^k and ω^l is a $k+l$-form. It can be varified that the exterior product satisfies

1. $\omega^k \wedge \omega^l = (-1)^{kl}\omega^l \wedge \omega^k$,
2. $(\lambda_1\omega^k_{(1)} + \lambda_2\omega^k_{(2)}) \wedge \omega^l = \lambda_1\omega^k_{(1)} \wedge \omega^l + \lambda_2\omega^k_{(2)} \wedge \omega^l$,
3. $(\omega^k \wedge \omega^l)\omega^m = \omega^k \wedge (\omega^l \wedge \omega^m)$.

Another operation for differential forms exterior differentiation d is defined as $\mathrm{d}\big(f(z)\,\mathrm{d}z^{i_1} \wedge \mathrm{d}z^{i_2} \wedge \cdots \wedge \mathrm{d}z^{i_k}\big) = \sum_j \partial_j f(z)\,\mathrm{d}z^j \wedge \mathrm{d}z^{i_1} \wedge \mathrm{d}z^{i_2} \wedge \cdots \wedge \mathrm{d}z^{i_k}$. This operation is linear.

A classical mechanics in Hamiltonian form can be defined by an even dimensional manifold M, a symplectic structure and a hamitonion function f_H on the manifold. A symplectic structure σ is a closed non-degenerate differential 2-form on the manifold, i.e.,

1. for any tangent vector v at point z, there exists another tangent vector w at z, such that $\sigma(v, w) \neq 0$ (non-degenerate).
2. $\mathrm{d}\sigma = 0$ (closed).

From the symplectic structure σ, a map I from a 1-from $\omega \in T^*_z M$ to a tangent vector $w \in T_z M$ is defined

$$w = I\omega \quad \text{s.t. } \sigma(v, w) = \omega(v),\ \forall v \in T_z M. \tag{78}$$

In terms of coordinate $\{z^k\}$ and the related basis $\{\partial_k\}$ for tangent vectors and $\{\mathrm{d}z^k\}$ for 1-forms, the map I can be expressed in a matrix form:

$$\begin{aligned}\sum_{j<k}(\sigma_{jk}\mathrm{d}z^j \wedge \mathrm{d}z^k)(w^l\partial_l, v^m\partial_m) = \sigma_{kj}w^j v^k = \omega_l\,\mathrm{d}z^l(v^k\partial_k) = \omega_k v^k \\ \Rightarrow w^j = \sigma^{jk}\omega_k.\end{aligned} \tag{79}$$

Here σ^{jk} is the inverse matrix of the matrix σ_{jk}.

The evolution in classical mechanics is a map g_H^t (as a single-parameter Lie group) in terms of the Hamiltonian function f_H that transforms each point z on the manifold into another point $g_H^t(z)$. The velosity $\dot{z}$ is a tangent vector such that for any function $f_A(z)$

$$\dot{z}(f_A)(z) = \frac{\mathrm{d}}{\mathrm{d}t} f_A\left(g_H^t(z)\right)\Big|_{t=0}. \tag{80}$$

The Hamiltonian equation of motion is related to the map I ($\mathrm{d}f_H$ is a 1-form)

$$\dot{z} = I\mathrm{d}f_H. \tag{81}$$

The form of the Hamiltonian equation of motion in the coordinate z^k is

$$\dot{z}^k = \sigma^{kl}\,\partial_l f_H(z). \tag{82}$$

In the main text, we assert without provement that the 2-form $\sigma = \sigma_{jk}\mathrm{d}z^j\mathrm{d}z^k$ is a symplectic structure on the even-dimensional phase space manifold. Now we will give a brief provement for that this 2-form is non-degenerate and closed.

The non-degeneracy can be proved by contradiction. If there exists a tangent vector $w = w^k\partial_k$ such that $\sigma(v, w) = 0$ holds for all tangent vectors v, recall the difinition of σ, this condition equals to that $\mathrm{tr}\left(\hat{K}(z)[\hat{X}_j, \hat{X}_k w^k]\right) = 0$ for all generators $\hat{X}_j$. From the definition of the generators, $\hat{X}_k$ are independent generators of the group G other than the isotropic subgroup H_z. If $\mathrm{tr}\left(\hat{K}(z)[\hat{X}_j, \hat{X}_k w^k]\right) = 0$ for all generators $\hat{X}_j$, $\hat{X}_k w^k$must be an generator of the isotropic subgroup H_z, this is contradict to the difinition of $\hat{X}_k$.

The closedness can be proved by the consideration that any group of unitary operations is a subgroup of the group containg all unitary operations of the F-state space. Noting the symplectic structure of the phase space related to the maximum group σ_0. Since $\mathrm{d}\sigma_0$ is a 3-form, it is zero if and only if it is zero on any three tangent vectors u, v, w. Noticing that σ_0 is invariant under any unitary transform, we can employ a unitary transform to reverse these three vectors, from the invariance of σ_0 we get

$$\mathrm{d}\sigma_0(u, v, w) = \mathrm{d}\sigma_0(-u, -v, -w) = -\mathrm{d}\sigma_0(u, v, w). \tag{83}$$

This gives $\mathrm{d}\sigma_0 = 0$, i.e., σ_0 is closed. For a general phase space, the symplectic structure σ is connected with σ by the pull-back φ^* of the map φ from the the smaller space to the full phase space: $\sigma = \varphi^*\sigma_0$. Since φ^* commutes with exterior difference d, σ is closed form that σ_0 is closed.

Correceponded to a map φ from manifold M to manifold N, the pull-back φ^* is a map from differential forms of manifold N T^*N to T^*M. With the expression of basis, $y = \varphi(x)$ with coordinates y^α for M and x^k for N, the pull-back φ^* mappes $\mathrm{d}y^\alpha$ into a 1-form $(\partial y^\alpha/\partial x^k)\mathrm{d}x^k$ on M. The provement of commutation between φ^* and d can be found in any differential geometry textbook.

Another property of classical mechanics is that the evolution $g_H^t : M \to M$ generated by a Hamiltonian function f_H conserves the symplectic structure σ, i.e., ${g_H^t}^*\sigma = \sigma$. Noticing that $\sigma\wedge\sigma\wedge\cdots\wedge\sigma$ is a volumn element of the phase space, and it is the invariant volumn element up to a constant. From this method, we gan get the invariant volumn element of the phase space representation of quantum mechanics.

References

[1] Vahagn Abgaryan, Arsen Khvedelidze, and Astghik Torosyan. Kenfack - zyczkowski indicator of nonclassicality for two non-equivalent representations of wigner function of qutrit. *Physics Letters A*, 412, 2021.

[2] V. I. Arnold. *Symplectic manifolds*, pages 201–232. Springer New York, New York, NY, 1989.

[3] C. Brif and A. Mann. A general theory of phase-space quasiprobability distributions. *Journal of Physics a-Mathematical and General*, 31(1):L9–L17, 1998.

[4] C. Brif and A. Mann. Phase-space formulation of quantum mechanics and quantum-state reconstruction for physical systems with lie-group symmetries. *Physical Review A*, 59(2):971–987, 1999.

[5] Pavel Bóna. *Classical Mechanical Projections of QM*, pages 35–48. Springer International Publishing, Cham, 2020.

[6] U. Fano. Geometrical characterization of nuclear states and the theory of angular correlations. *Phys. Rev.*, 90:577–579, May 1953.

[7] Xin He, Zhihao Gong, Baihua Wu, and Jian Liu. Negative zero-point-energy parameter in the meyer–miller mapping model for nonadiabatic dynamics. *The Journal of Physical Chemistry Letters*, 12(10):2496–2501, 2021. PMID: 33667108.

[8] A. B. Klimov and S. M. Chumakov. On the su(2) wigner function dynamics. *Revista Mexicana De Fisica*, 48(4):317–324, 2002.

[9] A. B. Klimov and P. Espinoza. Moyal-like form of the star product for generalized su(2) stratonovich-weyl symbols. *Journal of Physics a-Mathematical and General*, 35(40):8435–8447, 2002.

[10] Andrei B. Klimov and Hubert de Guise. General approach to su(n) quasi-distribution functions. *Journal of Physics a-Mathematical and Theoretical*, 43(40), 2010.

[11] Bálint Koczor, Robert Zeier, and Steffen J. Glaser. Continuous phase-space representations for finite-dimensional quantum states and their tomography. *Phys. Rev. A*, 101:022318, Feb 2020.

[12] H. D. Meyer and W. H. Miller. Classical-models for electronic degrees of freedom: Derivation via spin analogy and application to f*+h2→f+h2. *The Journal of Chemical Physics*, 71(5):2156–2169, 1979.

[13] Hans-Dieter Meyer and William H. Miller. A classical analog for electronic degrees of freedom in nonadiabatic collision processes. *The Journal of Chemical Physics*, 70(7):3214–3223, 1979.

[14] J. P. Provost and G. Vallee. Riemannian structure on manifolds of quantum states. *Communications in Mathematical Physics*, 76(3):289–301, 1980.

[15] R. L. Stratonovich. On distributions in represenation space. *Zh. Eksp. Teor. Fiz.*, 31:1012, 1956.

[16] T. Tilma and E. C. G. Sudarshan. Generalized euler angle parametrization for su(n). *Journal of Physics a-Mathematical and General*, 35(48):10467–10501, 2002.

[17] T. Tilma and E. C. G. Sudarshan. Generalized euler angle parameterization for u(n) with applications to su(n) coset volume measures. *Journal of Geometry and Physics*, 52(3):263–283, 2004.

[18] Todd Tilma and Kae Nemoto. Su(n)-symmetric quasi-probability distribution functions. *Journal of Physics a-Mathematical and Theoretical*, 45(1), 2012.

[19] Joseph C Várilly and JoséM Gracia-Bondía. The moyal representation for spin. *Annals of Physics*, 190(1):107–148, 1989.