F100-PW-220 Non-Ideal Nozzle Model

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1 Model

1.1 Geometry and Parameterization

An axisymmetric converging-diverging geometry is assumed. The shape of the interior wall of the nozzle can be parameterized using a linear converging-diverging function, a cubic spline, or a 3rd-degree basis spline (B-spline). The 3rd-degree B-spline is recommended as it has several advantages over the other parameterizations. By specifying the knots vector and the coefficients matrix which contains the control points of the B-spline, the following more standard nozzle geometric properties can be controlled:

- nozzle length
- throat location
- inlet diameter
- diameter, $D(x)$
- various area ratios

In addition, the shape of the exterior wall of the nozzle is specified using a piecewise-linear thickness function $t(x)$. The radius of the outer wall of the nozzle $r_o(x)$ can be calculated as:

$$r_o(x) = r_i(x) + t(x) \tag{1}$$

Note that $r_i(x)$ is given by the B-spline parameterization.

1.2 Model Inputs

The only required model inputs are:

- $T_t(0)$, stagnation temperature at inlet, used in $T_t(x)$ calculations
- $P_t(0)$, stagnation pressure at inlet, used in $P_t(x)$ calculations
- T_∞, freestream temperature, used in thermal analysis
- k_w, thermal conductivity of wall material
- α, thermal expansion coefficient of wall material
- E, elastic modulus of wall material
- ν, Poisson's ratio of wall material

- h_∞, generalized heat transfer coefficient from outside nozzle wall to freestream
- properties of fluid in nozzle (may be temperature dependent)
- nozzle geometry

In addition, various tolerances for iterative solvers and ODE solvers must be set.

1.3 Determination of Nozzle State

A quick analysis of the corresponding ideal nozzle (*i.e.* adiabatic and frictionless) is made to determine whether there is a shock present in the nozzle. The current MATLAB implementation does not calculate heat transfer and friction effects with the presence of a shock in the nozzle. However, more likely than not, the nozzle will be designed/controlled to not have a shock under normal operating conditions. In these cases, as well as for subsonic flow, the method presented in the remainder of this report is valid.

The ideal nozzle state is dependent on the pressure ratio P_{t7}/P_∞[1]. With nozzle geometry given, the critical subsonic and supersonic Mach numbers ($M^*_{\text{sub/super}}$) can be determined from the area-Mach relation found in equation 6, where choking occurs at the nozzle throat. Then, the corresponding critical pressure ratio for each critical Mach number can be determined:

$$\left(\frac{P_t}{P}\right)^*_{\text{sub/super}} = \left(1 + \frac{\gamma - 1}{2}\left(M^*_{\text{sub/super}}\right)^2\right)^{\frac{\gamma}{\gamma-1}} \tag{2}$$

1. Subsonic flow throughout nozzle

$$1 < \frac{P_{t7}}{P_\infty} < \left(\frac{P_t}{P}\right)^*_{\text{sub}}$$

2. Shock in nozzle

   If P_{t7}/P_∞ increases further, a shock appears in the nozzle downstream of the throat and moves towards the nozzle exit. Thus, a shock in the nozzle will occur when:

$$\left(\frac{P_t}{P}\right)^*_{\text{sub}} < \frac{P_{t7}}{P_\infty} < \left(\frac{P_t}{P_\infty}\right)_{\text{normal shock at exit}}$$

3. Overexpanded flow

$$\left(\frac{P_t}{P_\infty}\right)_{\text{normal shock at exit}} < \frac{P_{t7}}{P_\infty} < \left(\frac{P_t}{P}\right)^*_{\text{super}}$$

4. Fully expanded flow

$$\frac{P_{t7}}{P_\infty} = \left(\frac{P_t}{P}\right)^*_{\text{sub}}$$

5. Underexpanded flow

$$\frac{P_{t7}}{P_\infty} > \left(\frac{P_t}{P}\right)^*_{\text{sub}}$$

[1] Cantwell, B. J., "AA 210 Course Notes"

Once the state of the nozzle is determined (and if it does not have a shock), then the governing equation of motion below can be integrated numerically.

1.4 Governing Equation of Motion

Manipulating the quasi-1D area-averaged equations of motion with the absence of mass addition leads to the following differential form[2]:

$$\left(\frac{1-M^2}{2(1+\frac{\gamma-1}{2})M^2}\right)\frac{dM^2}{M^2} = \frac{-dA}{A} + \frac{\gamma M^2}{2}\left(\frac{4C_f dx}{D}\right) + \left(\frac{1+\gamma M^2}{2}\right)\frac{dT_t}{T_t} \tag{3}$$

where all variables above, except for γ are functions of x. Note that if $A(x)$, $C_f(x)$, $D(x)$, and $T_t(x)$ are specified for the nozzle, then equation (3) can integrated numerically.

1.4.1 Solution of Governing Equation

`ode45` is used to integrate equation (3) from the nozzle throat to the nozzle exit. Equation (3) is modified to enable `ode45` to solve for M^2 from the nozzle throat to the nozzle inlet:

$$-\left(\frac{1-M^2}{2(1+\frac{\gamma-1}{2})M^2}\right)\frac{dM^2}{M^2} = \frac{-dA}{A} + \frac{\gamma M^2}{2}\left(\frac{4C_f dx}{D}\right) + \left(\frac{1+\gamma M^2}{2}\right)\frac{dT_t}{T_t} \tag{4}$$

The initial condition for each integration is $M^2 = 1$. `ode45` uses a 4th-order accurate Runge-Kutta method. Tolerances as low as 0.01 were deemed of sufficient accuracy, and result in faster solution.

1.4.2 Calculation of Other Properties

Once $M(x)$ is known, $T(x)$ is easily calculated from the definition of stagnation temperature since $T_t(x)$ is specified. $P_t(x)$ is calculated using the following mass conservation formula between station x and the inlet, where $P_t(0)$ is specified:

$$\frac{P_t(x)A(x)f(M(x))}{\sqrt{T_t(x)}} = \frac{P_t(0)A(0)f(M(0))}{\sqrt{T_t(0)}} \tag{5}$$

where $f(M)$ is defined as:

$$f(M) = \frac{A^*}{A} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{M}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \tag{6}$$

$P(x)$ is then easily calculated from the definition of stagnation temperature, and ρ is calculated from ideal gas law. U, Re_x, and any other flow properties are also easily calculated.

1.5 Heat Transfer Calculations

Heat transfer phenomena is modeled in the transverse direction only using thermal resistances. The model estimates convection and conduction from the nozzle fluid to the nozzle wall (h_f), conduction through the nozzle wall (k_w), and sums convection, conduction, and radiation effects from the exterior nozzle wall to the environment using a generalize heat transfer coefficient h_∞. It is interesting to note, that so long as $h_\infty \gg 0$, it has little impact on the stagnation pressure and temperature ratios across the nozzle. However h_∞ has a large impact on the wall temperature distribution.

Recall the definition of the dimensionless Stanton number:

$$St = \frac{Q_w}{\rho U C_p (T_w - T_t)} \tag{7}$$

[2]Cantwell, B. J., "AA 210 Course Notes"

where T_t in this instance is the adiabatic wall recovery temperature, which is taken to be the nozzle flow stagnation temperature for a recovery factor of 1, and with $Pr = 1$.

Note also that the change in enthalpy along the length of the nozzle can be written as:

$$dh_t = \frac{Q_w \pi D dx}{\rho U A} = 4C_p St(T_w - T_t)\frac{dx}{D} \tag{8}$$

1.5.1 Determination of h_f

Using Newton's Law of Cooling, which is closely linked to the definition of a convective thermal resistance, the heat flux Q_w through the nozzle wall can also be written:

$$Q_w = h_f(T_w - T_t) \tag{9}$$

Solving equation 8 for Q_w and comparing with equation gives the convective heat transfer coefficient h_f.

$$h_f = \rho U C_p St \tag{10}$$

Lastly, the Chilton-Colburn analogy, a modified form of the Reynolds analogy, can be used to estimate St in terms of the friction coefficient C_f:

$$St Pr^{\frac{2}{3}} = \frac{1}{2}C_f \tag{11}$$

Thus, the convective heat transfer coefficient h_f can be written as:

$$h_f = \frac{1}{2}\rho U C_p Pr^{-\frac{2}{3}} C_f \tag{12}$$

1.5.2 Determination of Heat Flux Q_w and Stagnation Temperature T_t

Using the method of thermal resistances, the following equation can be written:

$$Q_w A = \frac{\Delta T}{R_{total}} = \frac{T_\infty - T_t}{\frac{1}{h_f A} + \frac{t}{k_w A} + \frac{1}{h_\infty A}} \tag{13}$$

Assuming the area A is approximately the same corresponding to each thermal resistance, the simplification is obtained:

$$Q_w = \frac{T_\infty - T_t}{\frac{1}{h_f} + \frac{t}{k_w} + \frac{1}{h_\infty}} = \frac{T_\infty - T_t}{R'_{tot}} \tag{14}$$

Using the equation for change in stagnation enthalpy (equation 8) and noting that $dh_t = C_p dT_t$, the following equation for change in stagnation temperature can be written:

$$\frac{dT_t}{dx} = \left(\frac{T_\infty - T_t}{R'_{tot}}\right)\frac{4}{C_p \rho U D} \tag{15}$$

Integrating equation 15 yields the following result for stagnation temperature along the nozzle:

$$T_t = T_\infty(1 - e^\alpha) + T_t(0)e^\alpha \tag{16}$$

where

$$\alpha = -\int_0^x \frac{4}{C_p \rho U D R'_{tot}} dx'$$

The integration for α can be carried out numerically, and thus T_t can be determined. Once T_t is known, $\frac{dT_t}{dx}$ can be determined from equation 15. Finally, Q_w may be determined from equation 14.

1.5.3 Determination of Wall Temperatures

Calculating the wall temperature is now trivial given the heat flux and thermal resistances. Using the method of thermal resistances, the interior nozzle wall temperature can be calculated as:

$$T_w = T_t + \frac{Q_w}{h_f} \tag{17}$$

The exterior wall temperature can be calculated as:

$$T_{ext} = T_\infty - \frac{Q_w}{h_\infty} \tag{18}$$

1.6 C_f Calculation

Sommer and Short's modified T' method is used to determine C_f[3]. The method in short is:

1. Evaluate

$$\frac{T'}{T_f} = 1 + 0.035 M_f^2 + 0.45 \left(\frac{T_w}{T_f} - 1 \right) \tag{19}$$

2. Evaluate

$$\frac{Re'}{Re_f} = \frac{1}{\left(\frac{T'}{T_f}\right)\left(\frac{\mu'}{\mu_f}\right)} \tag{20}$$

where Sutherland's law is used to calculate the dynamic viscosity ratio:

$$\frac{\mu'}{\mu_f} = \left(\frac{T'}{T_f}\right)^{1.5} \left(\frac{T_f + S}{T' + S}\right) = \left(\frac{T'}{T_f}\right)^{1.5} \left(\frac{1 + S/T_f}{T'/T_f + S/T_f}\right) \tag{21}$$

where $S = 216°\mathrm{R} = 110.4$ K

3. Evaluate the incompressible skin friction coefficient, C_{fi}

$$C_{fi} = \frac{0.074}{Re^{0.2}} \tag{22}$$

4. Calculate C_f

$$\frac{C_f}{C_{fi}} = \frac{T_f}{T'} \left(\frac{Re_f}{Re'}\right)^{0.2} \tag{23}$$

1.7 Thermofluid Coupling

Since, the calculation of the Mach number M depends on the prescribed stagnation temperature and friction coefficient profile along the length of the nozzle, several iterations must be made to converge the calculation. Convergence is achieved when the change in static temperature at the outlet of the nozzle is less than a specified value; for speed this was chosen to be 5%. Usually 7 iterations or less is sufficient.

[3]Sommer, S. C. and Short, B. J., "Free-flight measurements of turbulent-boundary-layer skin friction in the presence of severe aerodynamic heating at Mach numbers from 2.8 to 7.0," 1995.

1.8 Structural Model

A simple structural model is used where the maximum principle stress $\sigma_1(x)$ present in the nozzle wall is determined as a sum of the hoop stress due to the internal nozzle pressure and the thermal stress due to the temperature mismatch across the nozzle wall. Thus, a maximum principal stress $\sigma_1(x)$ is determined at each station x:

$$\sigma_1(x) = \sigma_{hoop}(x) + \sigma_{thermal}(x) \tag{24}$$

The hoop stress can be written as:

$$\sigma_{hoop}(x) = \frac{P(x)D(x)}{2t(x)} \tag{25}$$

The thermal stress can be adapted from that of thermal stress in a cylinder:

$$\sigma_{thermal}(x) = \frac{E\alpha\left(T_{inside}(x) - T_{outside}(x)\right)}{2(1-\nu)\left({}^{1}/_{log({}^{r_o(x)}/_{r_i(x)})}\right)\left(1 - {}^{2r_i(x)^2}/_{(r_o(x)^2 - r_i(x)^2)} log({}^{r_o(x)}/_{r_i(x)})\right)} \tag{26}$$

where E is the elastic modulus, α is the thermal expansion coefficient, $T_{inside}(x)$ is the interior temperature of the wall, $T_{outside}(x)$ is the exterior temperature of the wall, ν is Poisson's ratio, $r_o(x)$ is the radius of the outside of the wall, and $r_i(x)$ is the radius of the inside of the wall.

1.9 Cycles to Failure Model

A very rough cycles to failure model is implemented for the nozzle based on experimental data for a Nextel 312/Silicon Oxycarbide ceramic matrix composite (CMC) material[4]. The data for the 8X filtration test material is used. The calculations based on this material's data are extremely approximate as characterizing the properties of CMCs in high-temperature, high-stress environments is still an active research area[5].

According to the Gonczy and Sikonia paper, the CMC under examination was found to have degradation of material properties due to three separate effects: fatigue, creep, and oxidation effects which in turn affect the fatigue and creep properties of the material. It is unclear what is meant by oxidation effects since the material is already oxidized.

The purpose of the cycles to failure model is to estimate N_f, the number of cycles the material can endure before failing due to a cyclic stress $\Delta\sigma$ and a cyclic temperature ΔT. For the purposes of this model, the stress and thermal cycles were further simplified by assuming the max stress and max temperature occurred as a pulse, *i.e.* the cyclic stresses and temperatures when plotted look like a square wave. In addition, the thermal and stress cycles were assumed to be synchronized, *i.e.* the maximum stress occurs at the same time as the maximum temperature, which in general is not unreasonable for an aircraft engine nozzle.

A number of cycles to failure is estimated for each separate effect and then all are combined into a total cycles to failure N_f using a Miner's rule:

$$\frac{1}{N_f} = \frac{1}{N_f^{fatigue}} + \frac{1}{N_f^{oxidation}} + \frac{1}{N_f^{creep}} \tag{27}$$

Note that the Miner's rule has no real physical basis (to my knowledge), and has only been used for metals.

The cycles to failure due to fatigue $N_f^{fatigue}$ can be estimated using an S-N curve (stress vs. cycles to failure). An S-N curve is estimated from two data points in Gonczy and Sikonia's paper, by drawing a straight line in the $log\left(N_f^{fatigue}\right)$ vs. $log\left(\Delta\sigma\right)$ space. A maximum number of cycles was set to be 120,000.

The cycles to failure due to creep N_f^{creep} is based on the equation for strain rate, which when simplified yields:

[4]Gonczy, S. T. and Sikonia, J. G., "Nextel 312/Silicon Oxycarbide Ceramic Composites", Handbook of Ceramic Composites, Kluwer Academic Publishers 2005.

[5]Dave Marshall, a CMC expert from Teledyne, affirmed that the approach taken below is a reasonable first guess at estimating the material properties of a CMC, however its accuracy is another matter. *Phone discussion on April 12, 2016*

$$\dot{\epsilon} = A\sigma^n e^{Q/RT} \tag{28}$$

where $\dot{\epsilon}$ is the strain rate, A is a constant, σ is the applied stress, n is a constant, Q is the activation energy for creep, R is the universal gas constant, and T is the temperature. By taking the log of the strain rate equation, a slope and intercept can be estimated using data from Gonczy and Sikonia's paper. Then, by assuming a typical value for the creep activation energy, the creep rate can be calculated as a function of temperature and stress. Finally, the creep cycles to failure can be calculated as:

$$N_f^{creep} = \frac{\epsilon_{fail}}{\dot{\epsilon}\Delta t} \tag{29}$$

where ϵ_{fail} is the failure strain and Δt is the amount of time corresponding to one cycle. Failure is assumed to occur at a strain of 0.3% since experimental data shows failure occurring between 0.1% and 0.5% strain.

Finally, the cycles to failure due to oxidation $N_f^{oxidation}$ is estimated by interpolating and extrapolating flexural strength from a plot of flexural strength vs. oxidation time for several temperatures as given in Gonczy and Sikonia's paper. A rule of thumb for polymers is used to translate flexural strength into yield strength, namely $\sigma_{yield} = \sigma_{flexural}/1.5$. Thus, for any given max temperature and stress equal to the material's yield stress, a number of cycles to failure due to oxidation effects can be estimated.

In summary, the assumed failure criteria states: when the maximum stress exceeds the yield strength at the given maximum temperature, failure occurs. It cannot be emphasized enough that the above calculations are extremely approximate, and only serve to given "ballpark" values of N_f.