Complex Numbers

1. Definition:
   A complex number, z, is expressed in the form

$$z = a + jb$$

   where a and b are real numbers, j (sometimes we use i) is the imaginary unit that satisfies the relation

$$j^2 = -1$$

   - a is called the real part of z, $\mathrm{Re}(z) = a$,
   - b is called the imaginary part of z, $\mathrm{Im}(z) = b$.

2. Properties:

   (a) Two complex numbers, $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$ are equal if and only if $a_1 = a_2$ and $b_1 = b_2$.

   (b) Arithmetic operations on complex numbers. Consider two complex numbers, $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$,

   - Addition/Subtraction:

$$z_1 \pm z_2 = (a_1 + jb_1) \pm (a_2 + jb_2) = (a_1 \pm a_2) + j(b_1 \pm b_2)$$

   - Multiplication:

$$\begin{aligned} z_1 z_2 &= (a_1 + jb_1)(a_2 + jb_2) \\ &= a_1a_2 + ja_1b_2 + ja_2b_1 - b_1b_2 \\ &= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1) \end{aligned}$$

   - Division:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + jb_1}{a_2 + jb_2} \\ &= \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + j\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2} \end{aligned}$$

<u>Exercise</u>: Simplify $\dfrac{(-1+5j)^2(3-4j)}{1+3j} + \dfrac{10+7j}{5j}$.

<u>Ans</u>: $10 + 38.2j$

- Calculation with complex numbers are reduced to calculation with real numbers.
- Addition and multiplication are commutative and associative, i.e.

$$z_1 + z_2 = z_2 + z_1$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

- Distributive law also holds

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

(c) Argand Diagram:
Complex numbers can be represented as points in the $x - y$ plane (the complex plane) as shown in Figure 1. The x-axis is called the real axis (Re) and the y-axis the imaginary axis (Im).

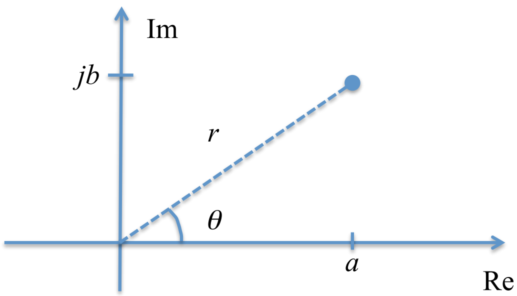

Figure 1: Argand diagram

<u>Exercise</u>: Given the $\arg(a + jb) = \theta$, where $a > 0, b > 0$, find in terms of θ and π, the value of (a) $\arg(a - jb)$, (b)$\arg(-a + jb)$ and (c) $\arg(-a - jb)$.

From Figure 1, we can also represent the complex number in polar notation where r is the radius (magnitude) and angle (phase or argument) of the complex number in the form: $r\angle\theta$. The two representations are related by

$$\boxed{\begin{aligned} r &= \sqrt{a^2 + b^2} \\ \theta &= \tan^{-1}\left(\frac{b}{a}\right) \\ a &= r\cos\theta \\ b &= r\sin\theta \end{aligned}}$$

3. Euler's formula
   Euler's formula relates complex exponentials and trigonometric functions:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- If we multiply Euler's formula by a constant, $r > 0$, we get the two forms of complex numbers:

$$z = re^{j\theta} = r\cos\theta + jr\sin\theta$$

Similar to Phasors in our AC circuit analysis

- Examples:

  Convert the following complex numbers from rectangular to polar form:
  (a) $1 + j$
  (b) $-1 - j$
  (c) $-5 + j12$

  Convert the following complex numbers from polar to rectangular form:
  (a) $5e^{j\pi/4}$
  (b) $e^{-3\pi/2}$
  (c) $10e^{j2.618}$

Also learn how to perform the conversions using a calculator that supports complex number operations!

- More examples on arithmetic operations:

  Addition and subtraction are easier in rectangular form.
  Multiplication and division are easier in polar form.

$$r_1e^{j\theta_1}r_2e^{j\theta_2} = (r_1r_2)e^{j(\theta_1+\theta_2)}$$

  Note that the magnitude of a product is the product of the magnitudes. Argument of the product of complex numbers is the sum of the arguments.

  Similarly, for division, we have:

$$\frac{r_1e^{j\theta_1}}{r_2e^{j\theta_2}} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}$$