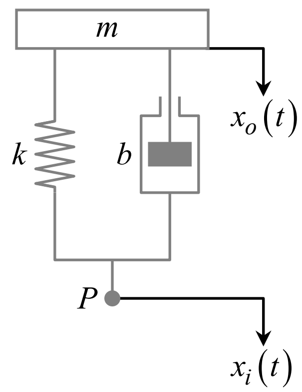
CG2023 TUTORIAL 4 (SOLUTIONS)

Solution to Q.1



Transfer function of suspension system:

$$\tilde{H}(s) = \frac{\tilde{X}_o(s)}{\tilde{X}_i(s)} = \frac{bs+k}{ms^2+bs+k}$$

Substututing $m = 1\ kg,\ k = 1\ Nm^{-1},\ b = \sqrt{2}\ Nm^{-1}s$, we get

$$\tilde{H}(s) = \frac{s\sqrt{2}+1}{s^2+s\sqrt{2}+1}$$

Frequency response of suspension system:

$$\tilde{H}(j\omega) = \frac{1+j\omega\sqrt{2}}{1-\omega^2+j\omega\sqrt{2}}$$

$$\rightarrow \begin{cases} \text{Magnitude Response: } \left|\tilde{H}(j\omega)\right| = \left(\dfrac{1+2\omega^2}{\left(1-\omega^2\right)^2+2\omega^2}\right)^{1/2} = \left(\dfrac{1+2\omega^2}{1+\omega^4}\right)^{1/2} \\ \text{Phase Response: } \angle\tilde{H}(j\omega) = \angle\left(1+j\omega\sqrt{2}\right) - \angle\left(1-\omega^2+j\omega\sqrt{2}\right) \\ \qquad = \tan^{-1}\left(\dfrac{\omega\sqrt{2}}{1}\right) - \tan^{-1}\left(\dfrac{\omega\sqrt{2}}{1-\omega^2}\right) \end{cases}$$

Fourier series expansion of input: $x_i(t) = \frac{4}{\pi}\left[\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\cdots\right]$

- **Steady-state** response of system due to sinusoidal input $\sin(t)$ is given by

$$\left|\tilde{H}(j1)\right| \cdot \sin\left(t + \angle\tilde{H}(j1)\right) = 1.2247\sin(t - 0.6155)$$

- **Steady-state** response of system due to sinusoidal input $\sin(3t)$ is given by

$$\left|\tilde{H}(j3)\right| \cdot \sin\left(3t + \angle\tilde{H}(j3)\right) = 0.4814\sin(3t - 1.3147)$$

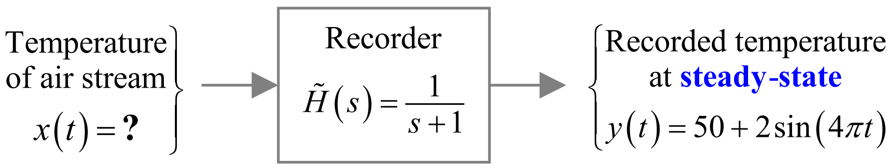
- **Steady-state** response of system due to sinusoidal input $\sin(5t)$ is given by

$$\left|\tilde{H}(j5)\right| \cdot \sin\left(5t + \angle\tilde{H}(j5)\right) = 0.2854\sin(5t - 1.4248)$$

Since system is linear, the output of the system can be obtained by superposition. Hence, at **steady state**:

$$\begin{aligned} x_o(t) &= \frac{4}{\pi}\left[1.2247\sin(t-0.6155) + \frac{1}{3}\times 0.4814\sin(3t-1.3147) + \frac{1}{5}\times 0.2854\sin(5t-1.4248) + \cdots\right] \\ &= \frac{4}{\pi}\left[1.2247\sin(t-0.6155) + 0.1065\sin(3t-1.3147) + 0.05708\sin(5t-1.4248) + \cdots\cdots\cdots\right] \end{aligned}$$

Solution to Q.2



$$\tilde{H}(j\omega) = \frac{1}{j\omega + 1} \;\cdots\cdots\; \begin{cases} \text{Magnitude response:} & \left|\tilde{H}(j\omega)\right| = \left(\omega^2 + 1\right)^{-1/2} \\ \text{Phase response:} & \angle\tilde{H}(j\omega) = -\tan^{-1}(\omega) \end{cases}$$

At STEADY-STATE:

The system output is $y(t) = 50 + 2\sin(4\pi t)$.

At $\omega = 0$: $\begin{cases} \left|\tilde{H}(j0)\right| = (0+1)^{-1/2} = 1 \\ \angle\tilde{H}(j0) = -\tan^{-1}(0) = 0 \end{cases}$

At $\omega = 4\pi$: $\begin{cases} \left|\tilde{H}(j4\pi)\right| = \left(16\pi^2 + 1\right)^{-1/2} = 0.0793 \\ \angle\tilde{H}(j4\pi) = -\tan^{-1}(4\pi) = -1.4914 \end{cases}$

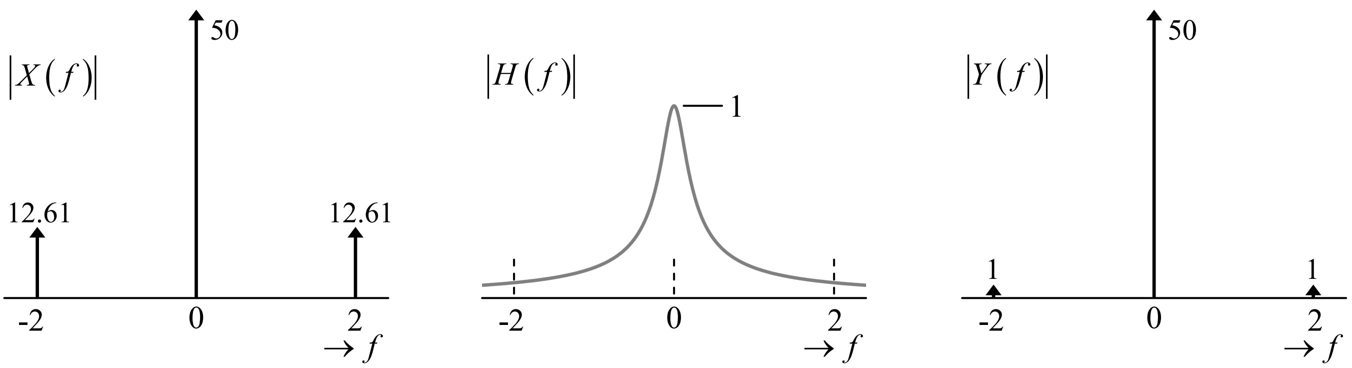
The system input is then given by

$$\begin{aligned} x(t) &= \frac{50}{\left|\tilde{H}(j0)\right|} + \frac{2}{\left|\tilde{H}(j4\pi)\right|}\sin\left(4\pi t - \angle\tilde{H}(j4\pi)\right) = 50 + \frac{2}{0.0793}\sin\left(4\pi t - (-1.4914)\right). \\ &= 50 + 25.22\sin(4\pi t + 1.4914) \end{aligned}$$

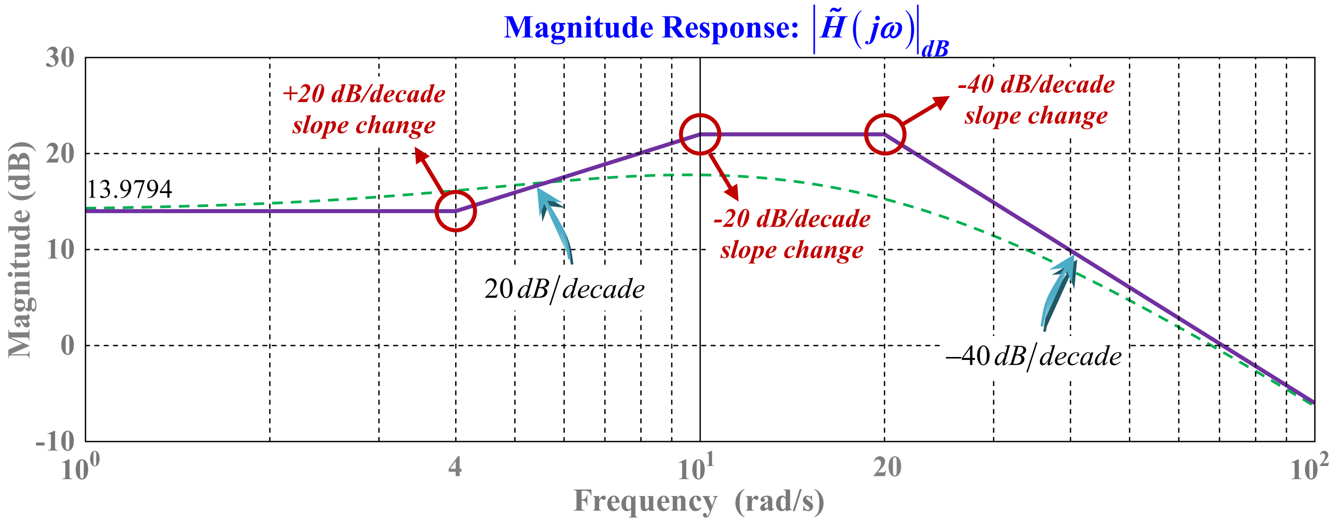
which shows that:

Maximum air temperature: $\mathbf{50 + 25.22 = 75.22°C}$
Minimum air temperature: $\mathbf{50 - 25.22 = 24.78°C}$

We note that the recorded maximum and minimum temperatures are **52°C** and **48°C**. Clearly, the recorder does not have sufficient bandwidth.



Solution to Q.3



Transfer function: $\tilde{H}(s) = \dfrac{K(s+\alpha)}{(s+\beta)(s+\gamma)(s+\lambda)} = \dfrac{K_{dc}\left(\frac{s}{\alpha}+1\right)}{\left(\frac{s}{\beta}+1\right)\left(\frac{s}{\gamma}+1\right)\left(\frac{s}{\lambda}+1\right)}$; $\quad K = \dfrac{\beta\gamma\lambda}{\alpha}K_{dc}$

(a)

- At $\omega = 4\ rad/s$, there is a **slope-change** of $20\ \text{dB/decade}$. This indicates the presence of the zero factor $\left(\dfrac{s}{4}+1\right)$.

- At $\omega = 10\ rad/s$, there is a **slope-change** of $-20\ \text{dB/decade}$. This indicates the presence of the pole factor $\left(\dfrac{s}{10}+1\right)^{-1}$.

- At $\omega = 20\ rad/s$, there is a **slope-change** of $-40\ \text{dB/decade}$. This indicates the presence of the double pole factor $\left(\dfrac{s}{20}+1\right)^{-2}$.

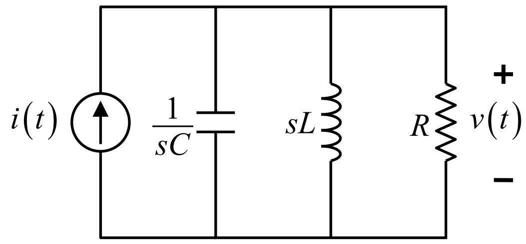
- $\text{DC (or Static) gain}:\ \left[20\log_{10}K_{dc} = \left|G(j0)\right|_{dB} = 13.9794\text{ dB}\right]$ or $\left[K_{dc} = 10^{13.9794/20} = 5\right]$.

Hence, the transfer function is

$$\tilde{H}(s) = \frac{5\left(\frac{s}{4}+1\right)}{\left(\frac{s}{10}+1\right)\left(\frac{s}{20}+1\right)^2} = \frac{5000(s+4)}{(s+10)(s+20)^2}$$

$\therefore\ \boldsymbol{K = 5000,\ \ \alpha = 4,\ \ \beta = 10,\ \ \gamma = \lambda = 20}$

Solution to Q.4



s-Domain Circuit $\left(R = 40\Omega \quad C = 10^{-4}F \quad L = 1H\right)$

(a) Transfer function:

$$\tilde{H}(s) = \frac{\tilde{V}(s)}{\tilde{I}(s)} = \frac{\frac{1}{sC}sLR}{\frac{1}{sC}sL + \frac{1}{sC}R + sLR} = sL\frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

$$= \frac{10000s}{s^2 + 250s + 10000}$$

$$= \frac{10000s}{(s+50)(s+200)}$$

(b) Given: $i(t) = 5\cos(100t)$.

Frequency response:

$$\tilde{H}(j\omega) = \frac{j10000\omega}{\left(10000 - \omega^2\right) + j250\omega} \quad \rightarrow \quad \begin{cases} \left|\tilde{H}(j\omega)\right| = \dfrac{10000\omega}{\sqrt{\left(10000-\omega^2\right)^2 + (250\omega)^2}} \\ \angle\tilde{H}(j\omega) = 90° - \tan^{-1}\left(\dfrac{250\omega}{10000-\omega^2}\right) \end{cases}$$

$$\left|\tilde{H}(j100)\right| = 40$$
$$\angle\tilde{H}(j100) = 0°$$

$$v(t) = 5\left|\tilde{H}(j100)\right|\cos\left(100t + \angle\tilde{H}(j100)\right) = 200\cos(100t)$$

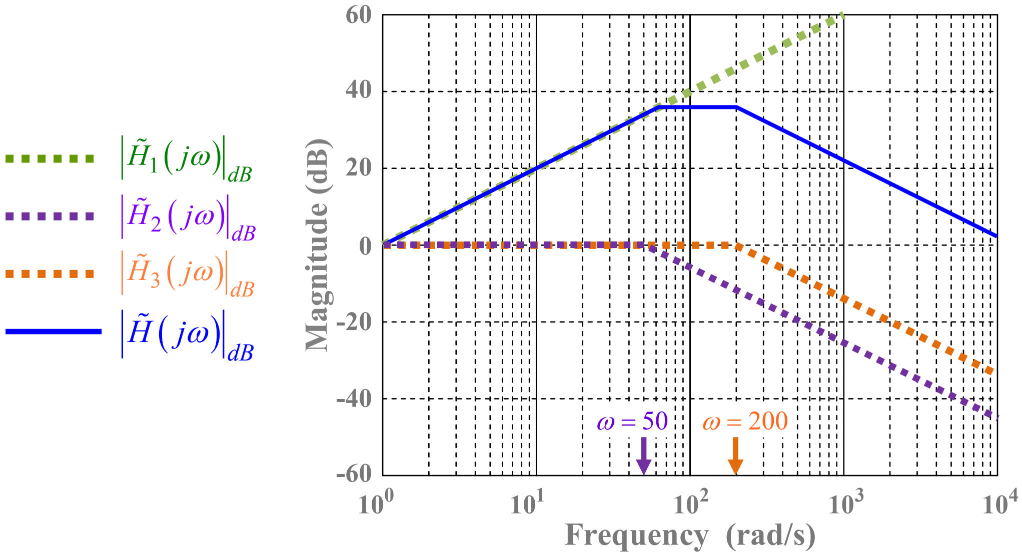
(c) $\tilde{H}(s)$ is an overdamped system comprising two distinct pole factors and a unity gain differentiator:

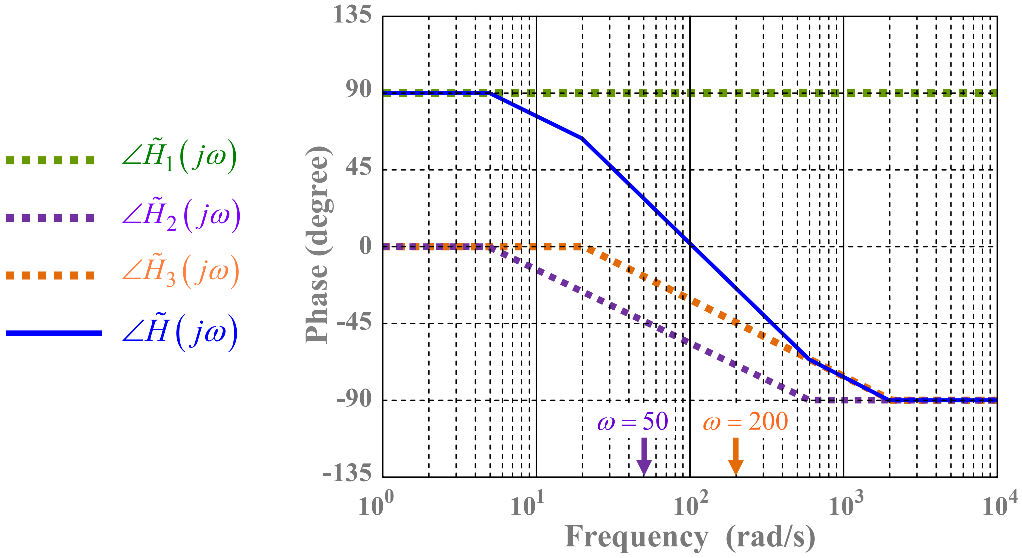
$$\tilde{H}(s) = s\cdot\frac{10000}{(s+50)(s+200)} = s\cdot\frac{1}{\left(\frac{s}{50}+1\right)\left(\frac{s}{200}+1\right)}$$
$$= \tilde{H}_1(s)\tilde{H}_2(s)\tilde{H}_3(s)$$

where

$$\tilde{H}_1(s) = s \qquad \tilde{H}_2(s) = \frac{1}{\frac{s}{50}+1} \qquad \tilde{H}_3(s) = \frac{1}{\frac{s}{200}+1}$$

$$\tilde{H}_1(j\omega) = j\omega \qquad \tilde{H}_2(j\omega) = \frac{1}{j\frac{\omega}{50}+1} \qquad \tilde{H}_3(j\omega) = \frac{1}{j\frac{\omega}{200}+1}$$





Solution to S.1

$$\tilde{H}(j\omega) = \left.\frac{2}{0.2s+1}\right|_{s=j\omega} = \frac{2}{1+j0.2\omega} \quad \begin{cases} \left|\tilde{H}(j\omega)\right| = \dfrac{2}{\sqrt{1+0.04\omega^2}} \\ \angle\tilde{H}(j\omega) = -\tan^{-1}\left(\dfrac{0.2\omega}{1}\right) \end{cases}$$

The system response to $\sin(3t)$ is

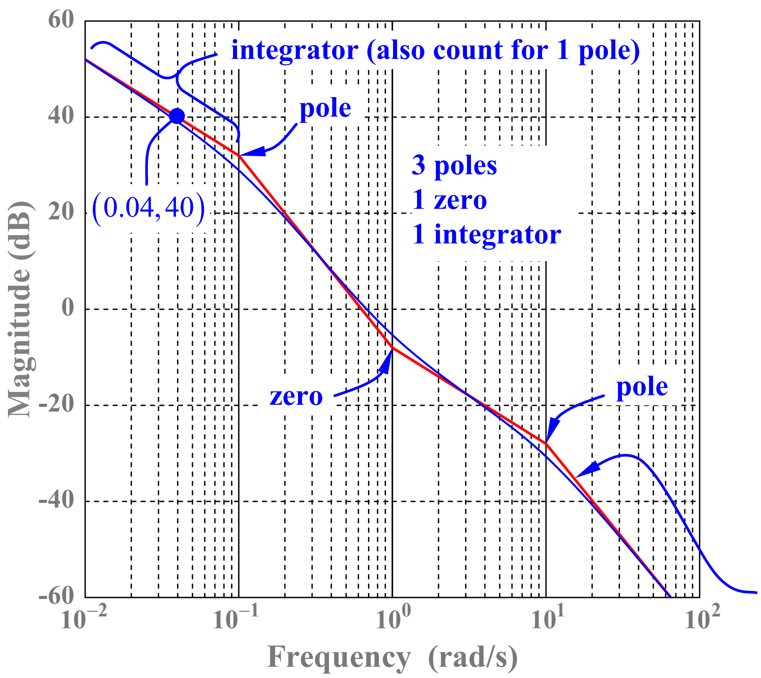
$$\left|\tilde{H}(j3)\right|\sin\left(3t+\angle\tilde{H}(j3)\right) = 1.715\sin(3t-0.54)$$

Solution to S.2

With $\left[\left|\tilde{H}(j5)\right| = 0.75\right]$ and $\left[\angle\tilde{H}(j5) = -68°\right]$, the system output $4.5\sin(5t-38°)$ is due to an input signal given by

$$\frac{4.5}{\left|\tilde{H}(j5)\right|}\sin\left(5t-38°-\angle\tilde{H}(j5)\right) = \frac{4.5}{0.75}\sin(5t-38°+68°)$$
$$= 6\sin(5t+30°)$$

Solution to S.3



Low frequency response:

$$\left|K/j\omega\right|_{dB} = 20\log_{10}(K) - 20\log_{10}(\omega) \quad \cdots\cdots \ (\spadesuit)$$

Substitute the coordinates of any point on the low frequency asymptote into $(\spadesuit)$ to solve for K.

For example, choose the point (0.04,40):

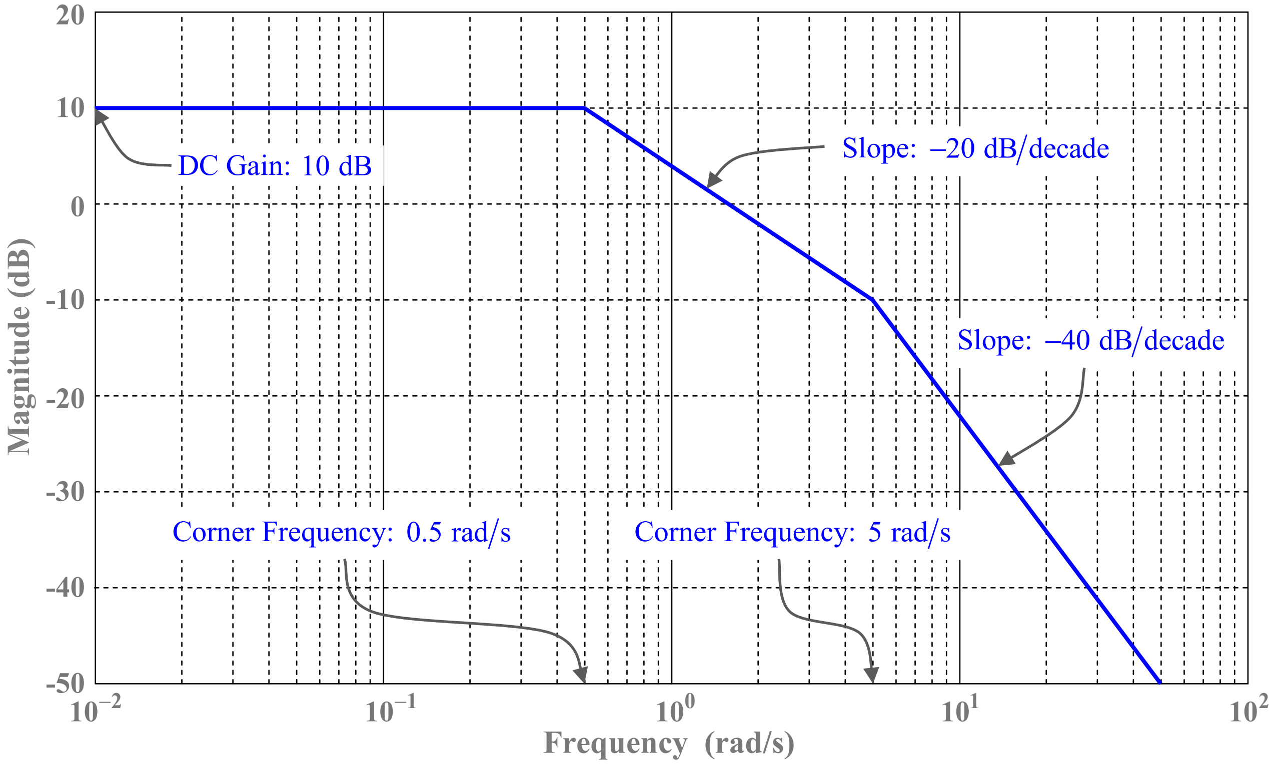
$$40 = 20\log_{10}(K) - 20\log_{10}(0.04)$$
$$\rightarrow \log_{10}\left(\frac{K}{0.04}\right) = 2 \quad \text{or} \quad \boldsymbol{K=4}$$

Solution to S.4

$$\tilde{H}(j\omega) = \frac{50}{5 - 2\omega^2 + j11\omega}$$

$$\tilde{H}(s) = \tilde{H}(j\omega)\Big|_{\omega=\frac{s}{j}} = \frac{50}{5 + 2s^2 + 11s} = \frac{50}{2(s+5)(s+0.5)} = \frac{10}{\left(\frac{s}{5}+1\right)\left(\frac{s}{0.5}+1\right)} \qquad \cdots\cdots\cdots\cdots\cdots\cdots \quad (\clubsuit)$$

In $(\clubsuit)$, we observe that the system has a dc gain of 10, and two pole factors with corner frequencies $0.5\ rad/s$ and $5\ rad/s$. Th straight-line Bode magnitude plot is shown below.



Solution to S.5

System transfer function: $\tilde{H}(s) = \dfrac{K(s+a)}{(s+b)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$

Existence of resonance peak indicated that 2nd-order factor is underdamped $\left(i.e.\ 0 \le \zeta < 1/\sqrt{2}\right)$.

By inspection: $\boldsymbol{a = 30}$ $\boldsymbol{b = 9}$ $\boldsymbol{\omega_n = 2}$.

$$K_{dc} = \tilde{H}(j\omega)\Big|_{\omega=0} = \tilde{H}(s)\Big|_{s=0} = \frac{aK}{b\omega_n^2} = \frac{5}{6}K$$

With $20\log_{10}(K_{dc}) = 20\,\text{dB}$ where $K_{dc} = \frac{5}{6}K$, we get $\boldsymbol{K = 12}$

With $\underbrace{\omega_r = \omega_n\sqrt{1-2\zeta^2}}_{\text{Resonant Frequency}}$ where $\left\{\begin{matrix}\omega_n = 2 \\ \omega_r = 1.87\end{matrix}\right\}$, we get $\boldsymbol{\zeta = 0.25}$