National University of Singapore

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EE2023 Signals & Systems Revision Notes

1 Laplace Transforms

Laplace Transforms (LT) are an integral part of systems and control. It is used widely in solving ordinary differential equations (ODE) by transforming them into algebraic equations involving the complex variable s. Via the LT, the time domain ODEs are converted into algebraic equations in frequency domain where s is the frequency variable. This also gives the fundamental link between time and frequency domains of signals and systems. Via the frequency domain, many properties of linear time invariant systems can be described and characterised without the need to solve the original ODEs. This leads to a generalization of system behaviour for this class of systems.

In mathematics, the Laplace Transform is an integral transform defined by :

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

where $F(s)$ and $f(t)$ are Laplace Transform pair. The lower limit of 0^- is required in cases where there are non-zero initial conditions. Example on how the Laplace integral is used :

$$\begin{aligned}
\mathcal{L}\{\sin\omega t\} &= \int_{0^-}^{\infty} \sin\omega t e^{-st}dt \\
&= \int_{0^-}^{\infty} \frac{1}{2j}\{e^{j\omega t} - e^{-j\omega t}\}e^{-st}dt \\
&= \int_{0^-}^{\infty} \frac{1}{2j}\{e^{-(s-j\omega)t} - e^{-(s+j\omega t)}\}dt \\
&= \frac{1}{2j}\left[\frac{1}{-s+j\omega}e^{-(s-j\omega)t} + \frac{1}{s+j\omega}e^{-(s+j\omega)t}\right]\Bigg|_{0^-}^{\infty} \\
&= \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

△ Can you find $\mathcal{L}\{\cos\omega t\}$?

Laplace transforms are linear operators which has the following properties. Note that $F(s)$ and $G(s)$ are used to denote the Laplace Transforms of $f(t)$ and $g(t)$.

- Linearity : $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$, α and β are constants.

- Transform of derivatives : $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$ where $f(0^-)$ denotes the initial condition of $f(t)$. In general,

$$\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^k(0^-)$$

An example :

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - sf(0^-) - \left.\frac{df}{dt}\right|_{t=0^-}$$

- Transform of an Integral : $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$

- Derivatives of Transforms : $\mathcal{L}\{-tf(t)\} = \frac{dF(s)}{ds}$. In general,

$$(-1)^n \mathcal{L}\{t^n f(t)\} = \frac{d^n F(s)}{ds^n}$$

Example : Since $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$, then according to the formula :

$$\mathcal{L}\{t \sin \omega t\} = -\frac{d}{ds}\{\frac{\omega}{s^2+\omega^2}\} = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

- Shift in Time Domain : $\mathcal{L}\{f(t-t_0)\} = e^{-st_0}F(s)$. An example of a signal shifted in time is given in Figure 1.

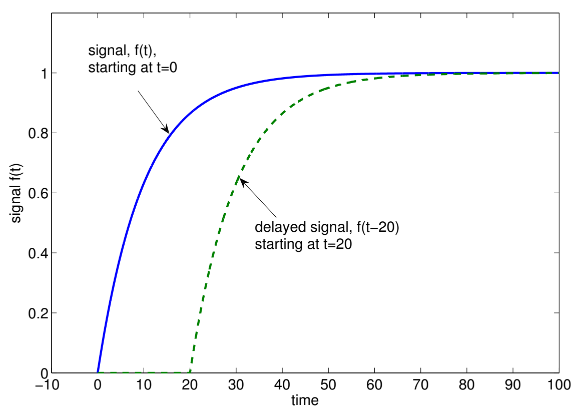

Fig. 1: Signal delayed by 20 time units

Notice how a shifted time function is written. Its Laplace transform has a simple relationship with the original LT of $f(t)$. This can be shown as follows :

$$\begin{aligned}
\mathcal{L}\{f(t-t_0)\} &= \int_0^\infty f(t-t_0)e^{-st}dt \\
&= \int_0^\infty f(t-t_0)e^{-s(t-t_0)}e^{-st_0}dt \\
&= e^{-st_0}\int_0^\infty f(t-t_0)e^{-s(t-t_0)}dt \\
&= e^{-st_0}F(s) \quad \text{after making the substitution } \lambda = t-t_0
\end{aligned}$$

where $F(s) = \mathcal{L}\{f(t)\}$.

- Shift in the Frequency Domain : $\mathcal{L}\{e^{-\alpha t}f(t)\} = F(s+\alpha)$.

  You should be able to prove this following the steps in the delayed signal case.

- Final Value Theorem : For a time domain function which has a finite steady state value, the final value theorem is given as :

  $$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

  Proof :

  $$\begin{aligned}
  \mathcal{L}\{\frac{df(t)}{dt}\} &= \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = sF(s) - f(0) \\
  \lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}e^{-st}dt &= \lim_{s\to 0}[sF(s) - f(0)] \\
  \lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}dt &= \lim_{s\to 0}[sF(s) - f(0)] \\
  f(\infty) - f(0) &= \lim_{s\to 0}[sF(s) - f(0)] \\
  \lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s)
  \end{aligned}$$

  This is a convenient formula to use when you want to determine the steady state value of a system's output response. You can obtain this steady state value from $F(s)$ instead of from $f(t)$. Applicable only when the $f(t)$ has a constant steady state value. Not applicable when $f(t)$ is always changing with time. Examples where the final value theorem fails is when $f(t) = \sin\omega t$, $f(t) = t$, etc.

Fortunately, we do not need to evaluate the LT integral each time we want to find the Laplace Transform of a function $f(t)$, at least for the kind of $f(t)$ that are generally encountered in systems and control. We rely on tables to determine them. Some examples are given in Table 1.

Table 1: Some examples of Laplace Transform pairs

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
$e^{-at}t$	$\frac{1}{(s+a)^2}$

2 Inverse Laplace Transform and Partial Factorization

Inverse Laplace Transform is used to recover the time domain signal from its s-domain equivalent. The inverse Laplace Transform is defined as :

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2j\pi} \int_{-\infty}^{\infty} F(s)e^{st} ds$$

It can be seen that this is a complex integral. In most cases, we do not need to find the inverse Laplace Transform using this integral. We should be able to use some well known Laplace Transform pairs such as those in Table 1 to help us find the inverse.

Example 1 : Find the inverse Laplace Transform of $\dfrac{1}{s^2+2s+4}$.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+4}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4-1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+3}\right\} \\
&= \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s+1)^2+(\sqrt{3})^2}\right\} \\
&= \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s_1^2+(\sqrt{3})^2}\right\} \text{ where } s_1 = s+1
\end{aligned}$$

Since

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2} \text{ and } \mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a),$$

it follows that

$$\mathcal{L}\left\{\frac{1}{s^2+2s+4}\right\} = \frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t$$

Laplace Transforms which are encountered in linear systems theory are of the form :

$$F(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in the s-variable. Examples are $F(s) = 1/s$, $F(s) = \frac{s+1}{(s+1)^2+3}$, $F(s) = \frac{1}{s+2}$, etc. For simple numerator and denominator polynomials of order 2 or lower, the inverse Laplace Transform of $F(s)$ can be obtained by inspection from Table 1. However, when $N(s)$ and $D(s)$ are polynomials of higher order, the inverse may not be so easily obtainable from the table.

The approach to obtain the inverse LT of a general $F(s)$ is to first partial factorize $F(s)$ so that the partial factors consists of only polynomials of orders at most 2. The inverse LT of $F(s)$ is then the inverse LT of each partial factor which can be easily deduced from Table 1. This approach works because of the linearity property of the LT.

There are 3 different forms of partial factors :

- Functions involving only <u>distinct</u> linear factors in the $D(s)$:

$$F(s) = \frac{N(s)}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_n)} = \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2} + \ldots + \frac{A_n}{s+\alpha_n}$$

where $\alpha_i \neq \alpha_j,\ i \neq j$.

For these distinct linear factors, the inverse LT is given by :

$$f(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} + \ldots + A_n e^{-\alpha_n t}.$$

- Functions involving repeated linear factors in the $D(s)$:

$$F(s) = \frac{N(s)}{(s+\alpha)^n} = \frac{A_1}{s+\alpha} + \frac{A_2}{(s+\alpha)^2} + \ldots + \frac{A_n}{(s+\alpha)^n}$$

For these repeated linear factors, the inverse LT is given by :

$$f(t) = \left(A_1 + A_2 t + \frac{1}{2} A_3 t^2 + \ldots + \frac{1}{n!} A_n t^{n-1} \right) e^{-\alpha t}.$$

- Functions involving quadratic factors :

$$F(s) = \frac{N(s)}{(s^2 + 2\beta_1 s + \gamma_1^2)(s^2 + 2\beta_2 s + \gamma_2^2)} = \frac{A_1 s + B_1}{(s^2 + 2\beta_1 s + \gamma_1^2)} + \frac{A_2 s + B_2}{(s^2 + 2\beta_2 s + \gamma_2^2)}$$

For such linear factors, the general form of the inverse LT is given by :

$$f(t) = R_1 e^{-\beta_1 t} \sin\left(\sqrt{\gamma_1^2 - \beta_1^2}\, t + \phi_1 \right) + R_2 e^{-\beta_2 t} \sin\left(\sqrt{\gamma_2^2 - \beta_2^2}\, t + \phi_2 \right).$$

The R_i are functions of β_i and γ_i. In this quadratic form, it should be presumed that each quadratic factor admits complex roots.

Each of these different forms of partial factors require a different approach to determine A_i. These methods are briefly given as follows :

- Functions with distinct linear factors :

$$F(s) = \frac{N(s)}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_n)} = \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2} + \ldots + \frac{A_n}{s+\alpha_n}$$

To find A_k, use the following formula :

$$A_k = \left. \frac{\cancel{(s+\alpha_k)} N(s)}{(s+\alpha_1)(s+\alpha_2)\cancel{(s+\alpha_k)}\ldots(s+\alpha_n)} \right|_{s=-\alpha_k}$$

Example : Find $f(t)$ from its LT $F(s) = \frac{2}{(s+1)(s+2)}$.

$$F(s) = \frac{2}{(s+1)(s+2)} \quad = \quad \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Using the above method :

$$\begin{aligned}
A_1 &= \left.\frac{\cancel{(s+1)}2}{\cancel{(s+1)}(s+2)}\right|_{s=-1} = 2 \\
A_2 &= \left.\frac{\cancel{(s+2)}2}{(s+1)\cancel{(s+2)}}\right|_{s=-2} = -2 \\
F(s) &= \frac{2}{s+1} - \frac{2}{s+2} \\
f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} = 2e^{-t} - 2e^{-2t}
\end{aligned}$$

- Functions with repeated factors

$$F(s) = \frac{N(s)}{(s+\alpha)^n} = \frac{A_1}{s+\alpha} + \frac{A_2}{(s+\alpha)^2} + \ldots + \frac{A_n}{(s+\alpha)^n}$$

To find A_k, use the following formula :

$$A_k = \frac{1}{(n-k)!}\left[\frac{d^{n-k}}{ds^{n-k}}\left((s+\alpha)^n F(s)\right)\right]\Bigg|_{s=-\alpha}$$

where $\frac{d^{n-k}}{ds^{n-k}}(.)$ denotes differentiating with respect to s, $(n-k)$ times.

Example : Find $f(t)$ from its LT $F(s) = \frac{2}{(s+1)(s+2)^2}$.

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad = \quad \frac{A_1}{s+1} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)}$$

Using the above method :

$$\begin{aligned}
A_1 &= \left.\frac{\cancel{(s+1)}2}{\cancel{(s+1)}(s+2)^2}\right|_{s=-1} = 2 \\
A_2 &= \left.\frac{\cancel{(s+2)^2}2}{(s+1)\cancel{(s+2)^2}}\right|_{s=-2} = -2 \\
A_3 &= \frac{1}{(1)!}\left[\frac{d}{ds}\left((s+2)^2 F(s)\right)\right]\Bigg|_{s=-2} = -2 \\
F(s) &= \frac{2}{s+1} - \frac{2}{(s+2)^2} - \frac{2}{s+1} \\
f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} = 2e^{-t} - 2te^{-2t} - 2e^{-2t}
\end{aligned}$$

- Functions with quadratic factors

$$F(s) = \frac{N(s)}{(s^2+2\beta_1 s+\gamma_1^2)(s^2+2\beta_2 s+\gamma_2^2)} = \frac{A_1 s + B_1}{(s^2+2\beta_1 s+\gamma_1^2)} + \frac{A_2 s + B_2}{(s^2+2\beta_2 s+\gamma_2^2)}$$

Each of the quadratic factors can be further factorized as :

$$\begin{aligned}\frac{As+B}{(s^2+2\beta s+\gamma^2)} &= \frac{As+B}{(s+\beta+j\sqrt{\gamma^2-\beta^2})(s+\beta-j\sqrt{\gamma^2-\beta^2})} \\ &= \frac{P}{(s+\beta+j\sqrt{\gamma^2-\beta^2})}+\frac{Q}{(s+\beta-j\sqrt{\gamma^2-\beta^2})}\end{aligned}$$

P and Q can be obtained in the same way as those with distinct linear factors.

△ Find the inverse Laplace Transform of $\dfrac{10}{s(s+2)(s+3)^2}$ and $\dfrac{1}{s^2+s+1}$.

3 Using Laplace Transform to solve Ordinary Differential Equation

Consider the first order RC circuit given in Figure 2. Assume a general input voltage $v(t)$

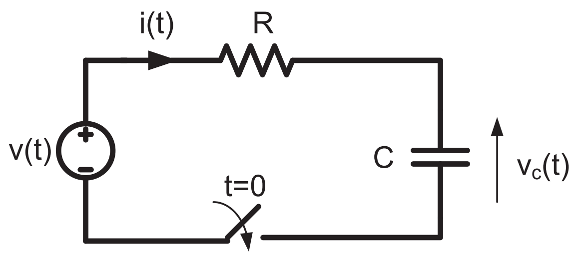

Fig. 2: First order RC circuit.

and an output voltage $v_c(t)$ across the capacitor. Deriving the model in time domain using differential equation and by applying circuit laws :

$$\begin{aligned}v(t) &= i(t)R+v_c(t) \\ i(t) &= C\frac{dv_c(t)}{dt} \\ v(t) &= RC\frac{dv_c(t)}{dt}+v_c(t) \qquad (1)\end{aligned}$$

Equation (1) may be solved using mathematical techniques and it can also be solved using Laplace Transform. Using the LT approach, since the input voltage is a general input $v(t)$, the LT of $v(t)$ is written generally as $V(s)$ while the LT of the output voltage $v_c(t)$ is written as $V_c(s)$. Taking the LT of (1),

$$
\begin{aligned}
V(s) &= RCsV_c(s) - RCv_c(0) + V_c(s) \\
&= RCsV_c(s) - RCv_c(0) + V_c(s) \\
V_c(s) &= \frac{V(s)}{RCs+1} + \frac{RCv_c(0)}{RCs+1} \qquad (2)
\end{aligned}
$$

If the input voltage $v(t)$ is a constant DC source with value $v(t) = V$, then $V(s) = \frac{V}{s}$ and (2) becomes

$$
V_c(s) = \frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}
$$

Then $v_c(t) = \mathcal{L}^{-1}\{V_c(s)\}$ can be found as follows :

$$
\begin{aligned}
v_c(t) &= \mathcal{L}^{-1}\left\{\frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{V}{s} - \frac{VRC}{RCs+1}\right\} + \mathcal{L}^{-1}\left\{\frac{v_c(0)}{s+\frac{1}{RC}}\right\} \\
&= V - Ve^{-\frac{t}{RC}} + v_c(0)e^{-\frac{t}{RC}} \\
&= V + [v_c(0) - V]\, e^{-\frac{t}{RC}} \qquad (3)
\end{aligned}
$$

From (3), it is clear that $\lim_{t\to\infty} v_c(t) = V$ which is the final value which the capacitor will be charged up to when a DC voltage of V is applied. This same result can be obtained by applying the Final Value Theorem to $V_c(s)$ as follows :

$$
\lim_{t\to\infty} v_c(t) = \lim_{s\to 0} s\left\{\frac{V}{s(RCs+1)} + \frac{RCv_c(0)}{RCs+1}\right\} = V.
$$

The output voltage in (3) can also be rewritten as :

$$
v_c(t) = v_c(0)e^{-\frac{t}{RC}} + V\left(1 - e^{-\frac{t}{RC}}\right)
$$

The different components in $v_c(t)$ can be seen in Figure 3.

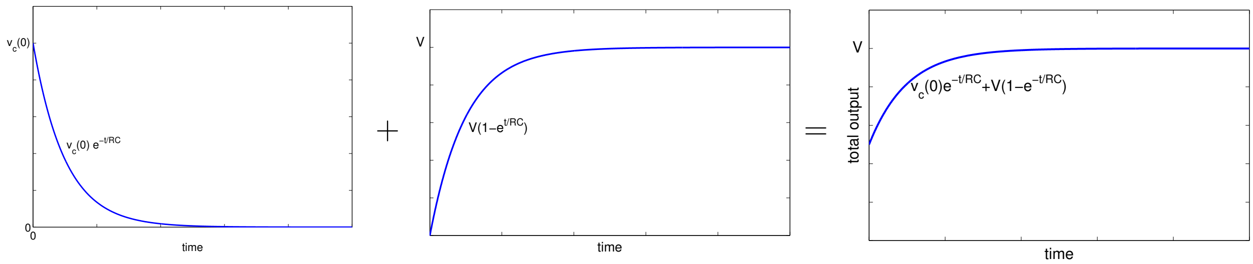

Fig. 3: Plotting individual components of $v_c(t)$.

Notice how the initial condition $v_c(0)$ decays over time. The plot in the middle is the typical charging characteristic of a capacitor with zero initial condition. Finally, the sum total of the first two plots gives the actual charging trajectory of the capacitor, including the non-zero initial condition.