National University of Singapore

Department of Electrical & Computer Engineering

EE2023 Signals & Systems Notes 1

1 Signals

In this module, we are only interested in continuous time signals, as opposed to discrete time signals. We begin with their mathematical description and their representations. In general, signals are represented mathematically as functions of one or more independent variable. For example, a speech signal can be represented by variations in acoustic pressure as a function of time, and a computer image can be represented by brightness as a function of two spacial variables as well as of time. We will focus only on signals involving time in this module.

Signals can be classified as periodic or aperiodic. A periodic continuous time signal $x(t)$ has the property that

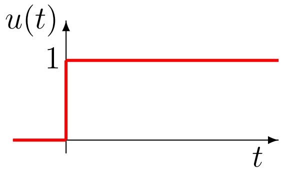
$$x(t) = x(t + T)$$

where T is the period of the signal. We say that $x(t)$ is periodic with period T. Typical examples of periodic signals include a sinusoidal function such as $x(t) = A\sin\omega t$ where $\omega = 2\pi/T$ is the frequency in rad/s. Others include a square wave signal which can be expanded in terms of sine and cosine functions in a Fourier series. Some of these ought to have been covered in your year 1 Math modules.

Aperiodic signals are signals which do not obey the periodic relationship. There are many examples of such aperiodic signals. Some of the fundamental ones include the following :

1. Unit Step Function : This is generally represented by

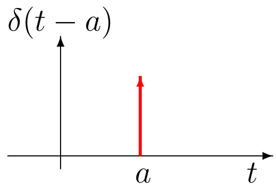
$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



Step functions can also take arbitrary magnitudes.

2. Unit Impulse Function or the Dirac Delta function : This is defined as follows :

$$\begin{aligned}\delta(t-a) &= \lim_{k\to 0} f_k(t-a) \\ \text{where } f_k(t-a) &= \begin{cases} \frac{1}{k} & a < t < a+k \\ 0 & elsewhere \end{cases}\end{aligned}$$



The impulse function satisfies the property that

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t) \text{ and } \frac{du(t)}{dt} = \delta(t)$$

where $u(t)$ is the unit step function defined previously.

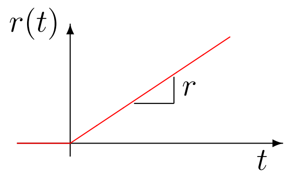
This impulse function also has an integral equivalent as follows :

$$\delta(t) = \int_{-\infty}^{\infty} e^{\pm j2\pi ft} df \quad \text{or} \quad \delta(f) = \int_{-\infty}^{\infty} e^{\pm j2\pi ft} dt.$$

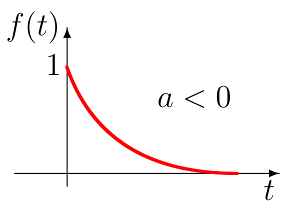
Impulse functions are useful in representing instantaneous non-zero signals such as an impact of a ball on the ground, crash of a car against a wall, a fall, etc.

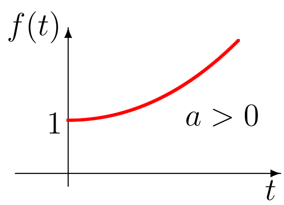
3. Ramp function : This is an increasing function with time, t. A ramp function with gradient r is represented by :

$$r(t) = \begin{cases} 0 & t < 0 \\ rt & t \geq 0 \end{cases}$$



4. Exponential function : $f(t) = e^{at}$. If $a > 0$, then $f(t)$ is an exponentially growing signal and hence it is unbounded. If $a < 0$, then $f(t)$ is an exponentially decreasing signal which will decay to zero over time.





5. Complex exponential function : $f(t) = Ae^{zt}$ where z is a complex number. This is an extension of the exponential function above where $z = a$ is a real number. In this case,

z is a complex number and hence we have the following :

$$\begin{aligned} f(t) &= Ae^{zt} = Ae^{(\sigma + j\omega)t} \\ &= Ae^{\sigma t}e^{j\omega t} \\ &= Ae^{\sigma t}(\cos\omega t + j\sin\omega t) \\ \text{Real}\left\{Ae^{zt}\right\} &= Ae^{\sigma t}\cos\omega t \\ \text{Imag}\left\{Ae^{zt}\right\} &= Ae^{\sigma t}\sin\omega t \end{aligned}$$

If $\sigma < 0$, then each signal ($Ae^{\sigma t}\cos\omega t$ or $Ae^{\sigma t}\sin\omega t$) turns out to be a decaying sinusoidal signal where the decay envelope is given by $e^{\sigma t}$. If $\sigma > 0$, then it is an exponentially growing sinusoid. Illustrations of these signals are shown in Figure 1. These signals are often encountered in control systems, as you will discover later in the course.

2 Frequency Domain Representation of Signals

Continuous time domain signals can be decomposed into its frequency components using either the Fourier Series or the Fourier Transform. The Fourier Series is used for decomposing periodic signals while the Fourier Transform is used for decomposing aperiodic or non-periodic signals. Later in the module, you will also find that the Fourier Transform is also applicable to periodic signals.

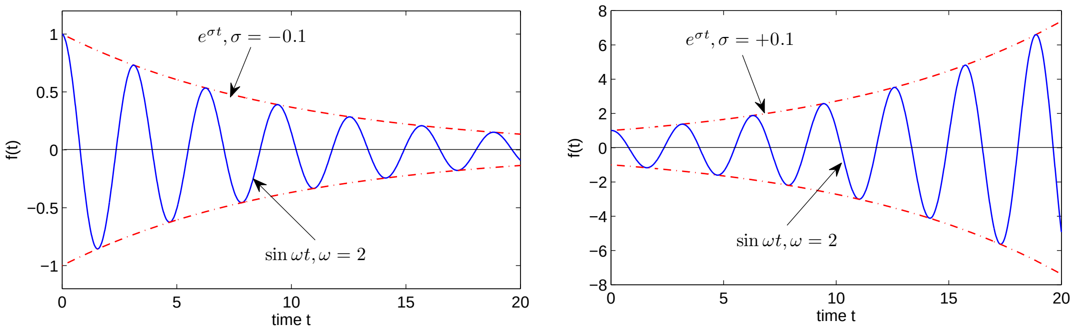


Fig. 1: Complex exponential functions

Periodic Signals : A periodic signal $x(t)$ with *fundamental* period T_0 can be represented by its Fourier Series (FS) expansion given in the table below. There are 3 possible forms of the FS.

Periodic signal, $x(t)$	
Fundamental frequency	$f_0 = \dfrac{1}{T_0}$ Hz or $\omega_0 = \dfrac{2\pi}{T_0}$ rad/s
$x(t)$ satisfies	$x(t) = x(t + T_0)$
$x(t)$ even implies	$x(t) = x(-t)$
$x(t)$ odd implies	$x(t) = -x(-t)$
Fourier Series of $x(t)$ has three forms :	
Trigonometric form :	$x(t) = a_0 + 2\Sigma_{k=0}^{\infty} a_k \cos 2\pi k f_0 t + 2\Sigma_{k=0}^{\infty} b_k \sin 2\pi k f_0 t$
	$a_k = \dfrac{1}{T_0}\displaystyle\int_{-T_0/2}^{T_0/2} x(t)\cos 2\pi k f_0 t dt$
	$b_k = \dfrac{1}{T_0}\displaystyle\int_{-T_0/2}^{T_0/2} x(t)\sin 2\pi k f_0 t dt$
Alternatively,	$x(t) = a_0 + 2\Sigma_{k=0}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k)$
	$A_k = \sqrt{a_k^2 + b_k^2}, \quad \phi_k = -\tan^{-1} b_k/a_k$
Exponential form :	$x(t) = \Sigma_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$
	$c_k = \dfrac{1}{T_0}\displaystyle\int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi k f_0 t} dt$
	$c_0 = a_0, \quad c_k = a_k - jb_k$
a_k, b_k, A_k, c_k	these are called the Fourier coefficients of $x(t)$
kf_0	this is the k^{th} harmonic of $x(t)$

The spectrum of a signal is the representation of that signal in the frequency domain. For a periodic signal, the spectrum is obtained via its FS representation. There are two types of spectra : single-sided and double-sided. The trigonometric form leads to a one-sided spectrum in which only components in the positive frequencies are plotted. The exponential form leads to a two-sided spectrum because in this representation, negative frequencies are considered. For the single-sided spectrum, the plot of $2|A_k|$ versus frequency gives the amplitude spectrum while the plot of ϕ_m versus frequency gives the phase spectrum. In the double-sided spectrum, the plot of $|c_k|$ and $\angle c_k$ versus frequency gives the amplitude and phase spectrum, respectively.

Note that the spectrum of a periodic signal is discrete in frequency ie only frequencies $kf_0, k = 0, 1, 2, \ldots$ exist where f_0 is the fundamental frequency. Frequencies kf_0 are referred to as the k^{th} harmonic. For example, the lowest A-note on the piano which has 88 keys has a frequency of 27.5 Hz (A0 note). Subsequently all other higher A-notes are doubled in frequency and hence are harmonics of the A0 note.

If a periodic signal is real, then $c_{-k} = c_k^*$ where * denotes conjugate. Hence this implies that a real signal has a two-sided spectrum which is symmetric in its amplitude but anti-symmetric (or asymmetric) in its phase. This is because $|c_{-k}| = |c_k^*| = |c_k|$ while $\angle c_{-k} = \angle c_k^* = -\angle c_k$.

If a periodic signals is even, then $b_k = 0$ for all k and likewise, if it is odd, $a_k = 0$. This is because cosine functions are even while sine functions are odd. Hence an even periodic signal is a sum of cosine functions while an odd periodic signal is a sum of sine functions. This further implies the following about their spectra.

- For even functions, $b_k = 0$ and hence $c_k = a_k - jb_k = a_k$. Thus c_k consists of only real numbers ie not complex and thus the spectrum of an even signal will have a zero phase spectrum. It is also symmetric.

- For odd functions, $a_k = 0$, $c_k = a_k - jb_k = -jb_k$. Thus c_k consists of purely imaginary numbers. Thus the spectrum of an odd signal will have a phase spectrum that consists of $\pm 0.5\pi$. The amplitude spectrum is symmetric but the phase spectrum is anti-symmetric.

Periodic signals always have discrete frequency components and hence has a discrete spectra (both amplitude and phase). However, not all signals with discrete spectra are periodic in time domain.

If a periodic signal is made up of a number of other periodic signals (eg sinusoids) added together, how does one determine its fundamental frequency? For example, if

$$x(t) = \sin 2\pi t + \sin 5\pi t.$$

What is the fundamental frequency of $x(t)$? First assume that $x(t)$ is periodic with fundamental period T_0. Hence $x(t) = x(t + T_0)$. Therefore

$$\sin 2\pi t + \sin 5\pi t = \sin 2\pi(t + T_0) + \sin 5\pi(t + T_0) = \sin(2\pi t + 2\pi T_0) + \sin(5\pi t + 5\pi T_0)$$

In order to satisfy the above equation, we require :

$$2\pi T_0 = 2n\pi, \quad \text{and} \quad 5\pi T_0 = 2k\pi$$

where n and k are integers. This implies that

$$T_0 = n, \quad \text{or } T_0 = \frac{2}{5}k$$

In order to find the smallest T_0, we need to find the smallest n and k integer values which satisfy $5n = 2k$ and this turns out to be $n = 2$ and $k = 5$. This leads to the smallest $T_0 = 2$ or a fundamental frequency of π rad/s.

This fundamental frequency can also be obtained from our method of finding the Highest Common Factor (HCF) between the frequencies of the components in $x(t)$. Since the components are 2π and 5π rad/s, the highest common factor ie $HCF\{2\pi, 5\pi\} = \pi$ and hence frequency is π rad/s.

If a fundamental frequency cannot be found, then the signal is not periodic. Example of such a signal is $x(t) = \sin 2\pi t + \sin 5t$ where the HCF of 2π and 5 does not exist because π is an irrational number.

Example 1. *Show that the signal $x(t) = 3\cos(20t + \pi/4) + \sin 10t$ is periodic. Find its fundamental frequency.*

$$\begin{aligned} x(t + T_0) &= 3\cos(20(t + T_0) + \pi/4) + \sin(10(t + T_0)) \\ &= 3\cos(20t + \pi/4 + 20T_0) + \sin(10t + 10T_0) \qquad (1) \end{aligned}$$

In order to satisfy (1), we require :

$$20T_0 = 2n\pi, \quad \textit{and} \quad 10T_0 = 2k\pi$$

where n *and* k *are integers. This implies that*

$$T_0 = \frac{2n\pi}{20} = \frac{2k\pi}{10}, \quad \textit{or}\, n = 2k$$

In order to find the smallest T_0*, we need to find the smallest* n *and* k *integer values which satisfy* $n = 2k$ *and this turns out to be* $n = 2$ *and* $k = 1$*. This leads to* $T_0 = \pi/5$ *or a fundamental frequency of 10 rad/s.*

This fundamental frequency can also be obtained from our method of finding the Highest Common Factor (HCF) between the frequencies of the components in $x(t)$*. Since the components are 20 and 10 rad/s, the highest common factor ie* $HCF\{20, 10\} = 10$ *rad/s. Hence the fundamental frequency of* $x(t)$ *is 10 rad/s.*

Example 2. *Determine whether each of the following signal is periodic :*

(i) $x_1(t) = \cos(8\pi t) + \sin(20\pi t)$

(ii) $x_2(t) = e^{j2\pi^2 t} + \cos(2\pi^3 t)$

(iii) $x_3(t) = \sin(3t)\sin(5t)$

(i) *Check whether the HCF of* 8π *and* 20π *exists. Factors of* 8π *and* 20π *are* $\{\pi, 2\pi, 4\pi, 8\pi\}$ *and* $\{\pi, 2\pi, 4\pi, 5\pi, 10\pi, 20\pi\}$*, respectively. Hence, the highest common factor is* 4π *and thus the fundamental frequency is* 4π *rad/s. Thus* $x_1(t)$ *is periodic with fundamental frequency of* 4π *rad/s or 2 Hz.*

(ii) *Proceeding the same way as above, the HCF of* $2\pi^2$ *and* $2\pi^3$ *does not exists because* π *is an irrational number. Hence* $x_2(t)$ *is not periodic.*

(iii) *When two periodic signals multiply together, the method of find HCF does not work immediately for determining the fundamental frequency. In this case,* $x_3(t)$ *has to be manipulated to see if we can find components of periodic signals which add together. If*

you can find an alternate expression for the signal consisting of periodic signals which add together, then the HCF method can be applied to this alternate form.

$$\begin{aligned} x_3(t) &= 0.5\left(\cos(5-3)t - \cos(5+3)t\right) \\ &= 0.5\left(\cos(2t) - \cos(8t)\right) \end{aligned}$$

making use of the trigonometric identity $\cos(A+B)t = \cos At \cos Bt - \sin At \sin Bt$. *Since* $x_3(t)$ *can now be written as :*

$$x_3(t) = 0.5\left(\cos(2t) - \cos(8t)\right),$$

the HCF method can be used to find the fundamental frequency. In this case, the fundamental frequency will be 2 rad/s since the $HCF\{2,8\}$ *is 2. The fundamental period is therefore* π *or approximately 3.141. See Figure 2.*

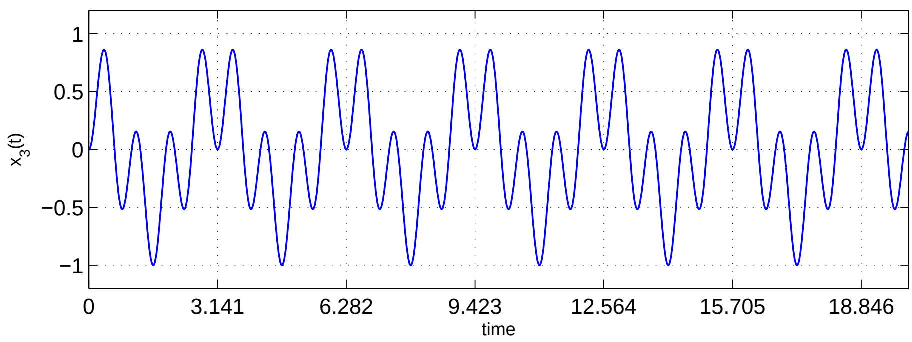


Fig. 2: Plot of $x_3(t)$

Example 3. *Sketch the amplitude and phase spectra of* $x(t) = 2\sin(8\pi t + \pi/6)$.

$$\begin{aligned} x(t) &= 2\sin(8\pi t + \pi/6) \\ &= 2\frac{1}{2j}\left[e^{j(8\pi t+\pi/6)} - e^{-j(8\pi t+\pi/6)}\right] \\ &= e^{-j\pi/2}\left[e^{j\pi/6}e^{j8\pi t} - e^{-j\pi/6}e^{-j8\pi t}\right] \\ &= e^{-j\pi/3}e^{j8\pi t} + e^{-j\pi}e^{-j2\pi/3}e^{-j8\pi t} \\ &= e^{-j\pi/3}e^{j8\pi t} + e^{+j\pi/3}e^{-j8\pi t} \end{aligned}$$

The frequency components of $x(t)$ *are* 8π *and* -8π *rad/s or 4 and -4 Hz respectively. The Fourier coefficients corresponding to these frequencies are* $e^{-j\pi/3}$ *and* $e^{+j\pi/3}$ *and hence the amplitude and phase spectra of* $x(t)$ *can be plotted as in Figure 3.*

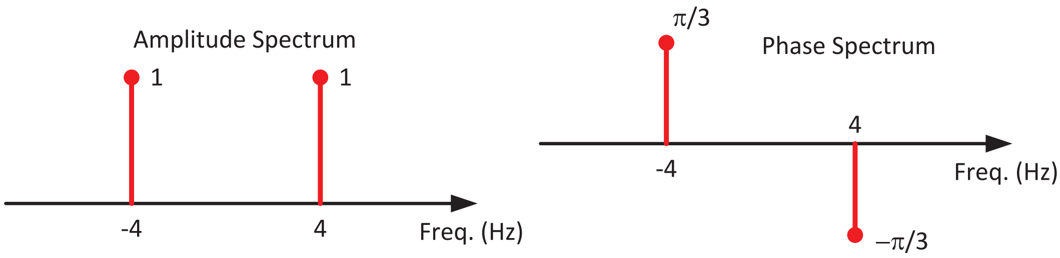


Fig. 3: Amplitude and Phase Spectra

Example 4. *The discrete frequency spectrum of $x(t)$ is given in Figure 4.*

(i) Write the mathematical expression for $x(t)$.

(ii) Let $x(t) = \Sigma_{k=-\infty}^{\infty} X_k e^{j2\pi k f_0 t}$ denote the Fourier Series expansion of $x(t)$, where f_0 is the fundamental frequency. Evaluate its Fourier coefficients, X_k.

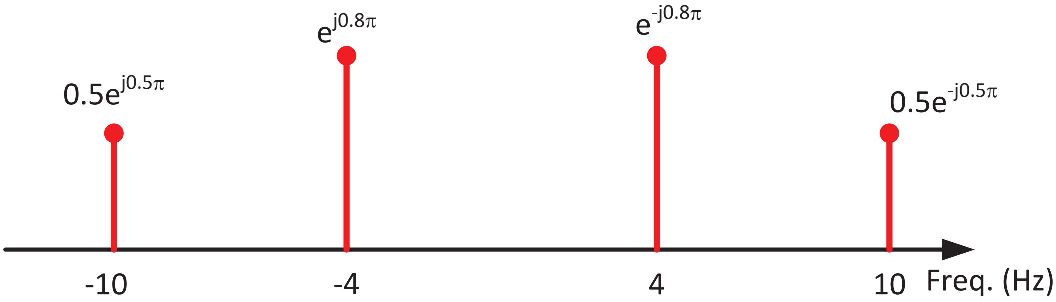


Fig. 4: Discrete Frequency Spectrum of $x(t)$.

(i) From the spectrum given in Figure 4, $x(t)$ can be written as :

$$\begin{aligned} x(t) &= 0.5e^{j0.5\pi}e^{-j20\pi t} + e^{j0.8\pi}e^{-j8\pi t} + e^{-j0.8\pi}e^{j8\pi t} + 0.5e^{-j0.5\pi}e^{j20\pi t} \qquad (2) \\ &= \cos(20\pi t - 0.5\pi) + \cos(8\pi t - 0.8\pi) \end{aligned}$$

(ii) $x(t)$ contains 2 frequency components : 10 and 4 Hz. The HCF of 4 and 10 is 2 and hence the fundamental frequency of $x(t)$ is 2 Hz or 4π rad/s. Comparing (2) with $x(t) = \Sigma_{k=-\infty}^{\infty} X_k e^{j2\pi k f_0 t}$, you get the following :

$$X_{-5} = 0.5e^{j0.5\pi},\ X_{-2} = e^{j0.8\pi},\ X_2 = e^{-j0.8\pi},\ X_5 = 0.5e^{-j0.5\pi}, \textit{all other } X_k = 0$$

Aperiodic signals : For an aperiodic or non-periodic signal, $x(t)$, the Fourier transform applies and is defined as:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

The Fourier transform, $X(f)$, is a continuous function of frequency and hence an aperiodic signal has a continuous spectrum ie non-discrete.

The properties of the Fourier transform can be summarized in the following table :

Scaling	$x(\beta t) \Leftrightarrow \frac{1}{\lvert\beta\rvert} X\left(\frac{f}{\beta}\right)$
Duality	$X(t) \Leftrightarrow x(-f)$
Time shift	$x(t-t_0) \Leftrightarrow X(f)e^{-j2\pi f t_0}$
Frequency shift	$X(f-f_0) \Leftrightarrow x(t)e^{j2\pi f_0 t}$
Differentiation in time	$\frac{dx(t)}{dt} \Leftrightarrow j2\pi f X(f)$
Integration in time	$\int_{-\infty}^{t} x(t)dt \Leftrightarrow \frac{1}{j2\pi f} X(f) + 0.5X(0)\delta(f)$
Multiplication in time	$x_1(t)x_2(t) \Leftrightarrow \int_{-\infty}^{\infty} X_1(f-\zeta)X_2(\zeta)d\zeta = X_1(f) * X_2(f)$
Convolution in time	$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau \Leftrightarrow X_1(f)X_2(f)$

The operator * denotes convolution which is defined as :

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau$$

In the following, we show the proof for the following two points :

(1) <u>If</u> the signal in time domain, $x(t)$ is real, then its spectrum in frequency domain is symmetric in amplitude and anti-symmetric in phase.

<u>Proof</u> : Writing the continuous frequency spectrum as :

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Then $X^*(f) = \int_{-\infty}^{\infty} x^*(t)e^{j2\pi ft}dt$ - after conjugating all the terms in the integral.

Also, $X(-f) = \int_{-\infty}^{\infty} x(t)e^{j2\pi ft}dt$.

It follows that $X^*(f) = X(-f)$ since $x(t)$ is real ie $x^*(t) = x(t)$. Therefore the amplitude spectrum satisfies $|X(f)| = |X^*(f)| = |X(-f)|$, which implies symmetry in amplitude.

For the phase spectrum, $\angle X(f) = -\angle X^*(f) = -\angle X(-f)$ and this shows the anti-symmetry since $\angle X(f) = -\angle X(-f)$.

Note that the above proof requires the knowledge in complex number theory where if $z = x + jy$ where x and y are positive real numbers, then $z^* = x - jy$. Hence $|z| = |z^*|$ and $\angle z = -\angle z^*$.

(2) <u>If</u> the spectrum, $X(f)$ is real, <u>and</u> $x(t)$ is also real, then $x(t)$ turns out to be symmetric.

<u>Proof</u> : Writing the inverse Fourier transform for $x(t)$,

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Then $x(-t) = \int_{-\infty}^{\infty} X(f)e^{-j2\pi ft}df$.

Also $x^*(t) = \int_{-\infty}^{\infty} X^*(f)e^{-j2\pi ft}df$.

It follows that $x(-t) = x^*(t)$ since $X(f)$ is real. As $x(t)$ is also assumed to be real, it follows that $x(-t) = x^*(t) = x(t)$ and this proves symmetry for $x(t)$.

Common Signals and their Transforms

The following are some of the properties related to the Dirac delta function, $\delta(t)$.

Sifting	$\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)$ Proof : Recall that when you sample a function at $t=\lambda$, the sampling process is written as $x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda)$ Hence $\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)\int_{-\infty}^{\infty} \delta(t-\lambda)dt = x(\lambda)$. There are other ways of proving this.
White Spectrum	$\mathcal{F}\{\delta(t)\} = 1$
Replication	$x(t) * \delta(t-\xi) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\xi-\tau)d\tau$ $= \int_{-\infty}^{\infty} x(\tau)\delta(\tau+\xi-t)d\tau$ $= \int_{-\infty}^{\infty} x(\tau)\delta(\tau-(t-\xi))d\tau$ because $\delta(t)=\delta(-t)$ $= x(t-\xi)$ using the sifting property $X(f) * \delta(f-f_0) = X(f-f_0)$ - same as above but in f domain

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As a consequence of the properties of the Dirac delta function, there are two types of periodicities that can be derived :

(a) Periodicity in time domain If a periodic signal, $x_p(t)$ (with period T_p) is formed from a generating function $g(t)$, then we can write $x_p(t)$ as follows :

$$x_p(t) = \Sigma_{n=-\infty}^{n=\infty} g(t-nT_p)$$

This can be further rewritten in another form using the replication property of the Dirac

Table 1: Table of Common Transform Pairs

DC or constant signal	$x(t) = K \Leftrightarrow K\delta(f)$ Proof : $\mathcal{F}\{K\delta(t)\} = K$ Using duality, $\mathcal{F}\{K\} = K\delta(f)$
Exponential signal	$x(t) = Ke^{j2\pi f_0 t} \Leftrightarrow K\delta(f - f_0)$ Proof : Comes from frequency shifting
Cosine signal	$x(t) = A\cos 2\pi f_0 t \Leftrightarrow X(f) = \dfrac{A}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$
Sine signal	$x(t) = A\sin 2\pi f_0 t \Leftrightarrow X(f) = \dfrac{A}{2j}\left[\delta(f - f_0) + \delta(f + f_0)\right]$
Rectangular signal	$x(t) = A\mathrm{rect}\left(\dfrac{t}{T}\right) \Leftrightarrow AT\mathrm{sinc}(fT)$ $\mathrm{sinc}(fT) = \dfrac{\sin(\pi fT)}{\pi fT}$
Arbitrary periodic signal	$x_p(t)$ with period, T_p, Fundamental frequency, $f_p = \frac{1}{T_p}$ $X(f) = \Sigma_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$ $x_p(t) = \Sigma_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t}$ c_k are the Fourier coefficients of $x_p(t)$.
Comb function	$\xi(t) = \Sigma_{k=-\infty}^{\infty} \delta(t - kT_p)$ $\Xi(f) = \dfrac{1}{T_p}\Sigma_{k=-\infty}^{\infty} \delta(f - kf_p)$

delta function :

$$x_p(t) = g(t) * \Sigma_{k=-\infty}^{\infty} \delta(t - kT_p)$$

When written in the above form, the Fourier Transform of $x_p(t)$ becomes easier to find because (1) convolution in time domain leads to multiplication in frequency domain, and (2) the Fourier Transform of the comb function (in time domain) is another comb function (in frequency domain), as given in Table 1. Hence the Fourier Transform of $x_p(t)$ becomes :

$$X_p(f) = G(f)\frac{1}{T_p}\Sigma_{k=-\infty}^{\infty} \delta(f - kf_p)$$

The above should be viewed as a sampling process in frequency domain because $G(f)$ multiplies with the comb function. Hence

$$X_p(f) = \frac{1}{T_p}\Sigma_{k=-\infty}^{\infty} G(kf_p)\delta(f - kf_p) = \Sigma_{k=-\infty}^{\infty} \frac{G(kf_p)}{T_p}\delta(f - kf_p).$$

The relationship between the Fourier Transform of the periodic function ($x_p(t)$) and the Fourier Transform of its generating function $g(t)$ can be summarised as follows :

$$\begin{aligned} g(t) &\leftrightarrow G(f) \\ x_p(t) &\leftrightarrow X_p(f) = \Sigma_{k=-\infty}^{\infty} \frac{G(kf_p)}{T_p}\delta(f - kf_p) \end{aligned} \tag{3}$$

Equation (3) tells us that $X_p(f)$ has discrete frequency components which are the same as $G(f)$ at discrete frequencies $f = kf_p$. Accordingly, we can form the periodic signal $x_p(t)$ using the Fourier series representation as follows :

$$x_p(t) = \Sigma_{k=-\infty}^{\infty} \frac{G(kf_p)}{T_p} e^{j2\pi k f_p t}.$$

The conclusion here is that if you sample an aperiodic signal ($g(t)$) in <u>frequency domain</u>, the result is that the <u>time domain</u> of the resulting signal (after synthesis, ($x_p(t)$) is periodic.

Example 5. *Consider a generating function given by* $g(t) = rect\left(\frac{t}{3}\right)$. *Hence* $G(f) = 3sinc(3f)$. $G(f)$ *can be used to generate a square wave periodic function* $x_p(t)$ *with period* $T_p = 5$. *This can be done as follows :*

$$x_p(t) = \Sigma_{k=-\infty}^{\infty} \frac{3sinc(3kf_p)}{T_p} e^{j2\pi k f_p t}.$$

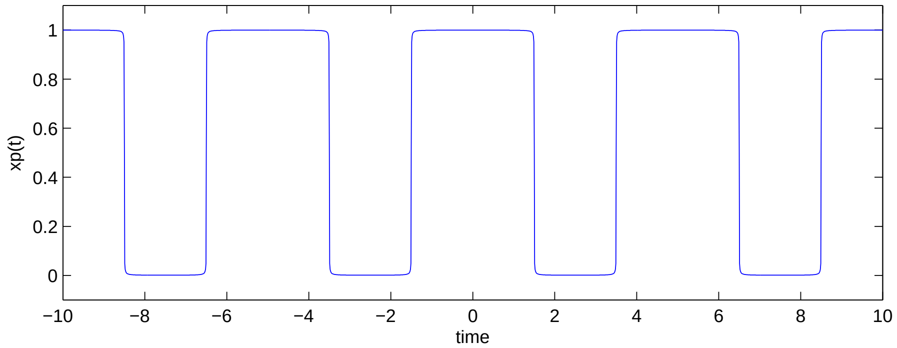

Fig. 5: Periodic function $x_(t)$ generated from $g(t) = \text{rect}\left(\frac{t}{3}\right)$.

The resulting $x_p(t)$ is shown in Figure 5. The Matlab code for calculating $x_p(t)$ from $g(t)$ is as follows :

```
clear;
n=2;
T=3; %T=width of the rect function : rect(t/T)
tp=5; %tp=period of the periodic function
fp=1/tp;
j=sqrt(-1);
k=-500:500;
vk=T*fp*sinc(T*fp*k); %forming the fourier coefficients by sampling the sinc
t=-n*tp:0.01:n*tp;
for i=1:size(t,2);
yk=exp(j*2*pi*fp*k*t(1,i));
xp(i)=vk*yk';
end;
plot(t,xp)
axis([-n*tp n*tp -0.1 1.1])
xlabel('time')
ylabel('xp(t)')
```

(b) Periodicity in frequency domain Suppose you have an aperiodic signal $x(t)$ which is sampled at every T_s to form $x_s(t)$. Then we write

$$x_s(t) = x(t)\Sigma_{k=-\infty}^{\infty}\delta(t-kT_s) = \Sigma_{k=-\infty}^{\infty}x(kT_s)\delta(t-kT_s).$$

In frequency domain,

$$\begin{aligned} X_s(f) &= X(f) * \frac{1}{T_s}\Sigma_{k=-\infty}^{\infty}\delta(f-kf_s) \\ &= \frac{1}{T_s}\Sigma_{k=-\infty}^{\infty}X(f-kf_s) \quad \text{a result of replication} \end{aligned} \tag{4}$$

Equation (4) tells us that $X_s(f)$ is replicated at every kf_s and thus is periodic in frequency with period f_s.

The conclusion here is that if you sample an aperiodic signal in time domain at frequency f_s, the resulting signal in frequency domain is periodic with period f_s.

Example 6. *Consider sampling $x(t) = rect(t/2)$ at frequency of $f_s = 10$ Hz. According to (4), we should expect to see the $X(f) = 2sinc(2f)$ replicated at every kf_s. Figure 6 shows the spectrum, $X_s(f)$.*

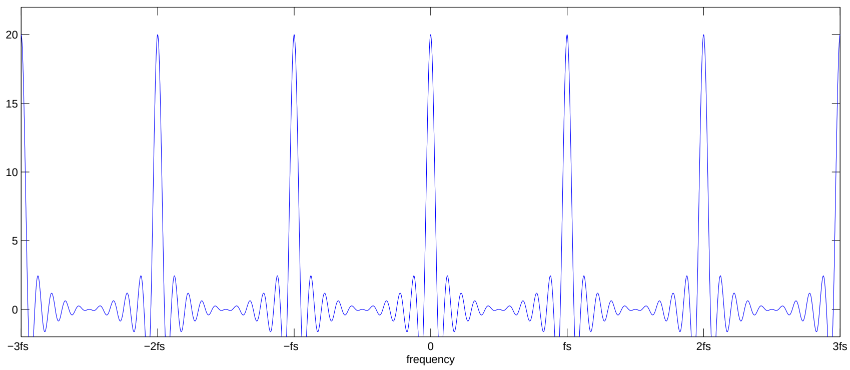

Fig. 6: Spectrum of sampled $x_(t)$, showing periodicity in frequency.

The Matlab code to generate the above plot is given below.

```
clear;
n=500;
n0=3; %number of fs to replicate
T=2; %T=width of the rect function :rect(t/T)
ts=.1; %ts=sampling period
fs=1/ts;
j=sqrt(-1);
fk=-n*fs:fs:n*fs; %to sum over n*fs where n should tend towards infinity
f=-n0*fs:0.01:n0*fs;
xp1=[];
for k=1:size(f,2);
xp1(k)=fs*T*sinc((f(1,k)-fk(1,1))*T);
for i=2:size(fk,2);
xp1(k)=xp1(k)+fs*T*sinc((f(1,k)-fk(1,i))*T);
end;
end;
plot(f,xp1)
axis([-n0*fs n0*fs -2 1.1*fs*T])
xlabel('frequency')
ylabel('Xs(f)')
```

Energy and Power Spectral Densities

The energy of a signal, $x(t)$, is defined as :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

A signal is an *energy* signal if $E < \infty$.

The power of a signal $x(t)$, is defined as :

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt.$$

A signal is a *power* signal if $P < \infty$.

A signal cannot be both an energy and power signal at the same time. It can only be either one or the other. However, it can be neither ie not an energy and neither is it a power signal. For example, a rectangular pulse is an energy signal with its power, $P = 0$, while a sinusoidal signal is a power signal ie $P < \infty$ and its $E = \infty$. All periodic signals are power signals.

In frequency domain, energy and power can be computed as follows :

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} E_x(f) df \\ P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(f)|^2 df = \int_{-\infty}^{\infty} P_x(f) df \end{aligned}$$

The quantities $E_x(f) = |X(f)|^2$ and $P_x(f) = \lim_{T\to\infty} \frac{1}{2T}|X(f)|^2$ are known as *energy* and *power* spectral densities respectively.

By virtue of Parseval's theorem,

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \\ P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(f)|^2 df \end{aligned}$$

For a periodic signal, $E = \infty$ but $P = \Sigma_{k=-\infty}^{\infty} |c_k|^2$. In this case, the power spectral density, $P_x(f) = |c_k|^2$ defines the power contained in each frequency component, $f = kf_0$.

For a non-periodic energy signal, $E = \int_{k=-\infty}^{\infty} |X_f|^2 df < \infty$ but $P = 0$. In this case, the energy spectral density is $E_x(f) = |X(f)|^2$ defines the energy contained in each frequency component, f.

Sampling and the Spectrum of Sampled Signals

When a signal is multiplied by a comb function, the process results in the sampling of the signal. Assume that the sampling period is T_s seconds and the sampling frequency is thus $f_s = 1/T_s$ Hz. In other words,

$$x(t)\Sigma_{-\infty}^{\infty}\delta(t - nT_s) = x(nT_s), n = 0, 1, 2, \ldots = x_s(t)$$

where $x_s(t)$ denotes the sampled signal of $x(t)$. In the frequency domain, $X_s(f)$, which is the Fourier transform of $x_s(t)$), is derived as follows :

$$
\begin{aligned}
x_s(t) &= x(t)\sum_{-\infty}^{\infty}\delta(t-nT_s) \\
X_s(f) &= X(f)*\Xi(f) \\
&= \int_{-\infty}^{\infty} X(\gamma)\Xi(f-\gamma)d\gamma \quad \text{where } \Xi(f)=f_s\sum_{k=-\infty}^{\infty}\delta(f-kf_s) \\
&= \int_{-\infty}^{\infty} X(\gamma)f_s\sum_{k=-\infty}^{\infty}\delta(f-kf_s-\gamma)d\gamma \\
&= f_s\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(\gamma)\delta(f-kf_s-\gamma)d\gamma \\
&= f_s\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(\gamma)\delta(\gamma-(f-kf_s))d\gamma \\
&= f_s\sum_{k=-\infty}^{\infty} X(f-kf_s)
\end{aligned}
$$

Hence the spectrum of $X_s(f)$ is replicated at every integer multiple of f_s. This implies that even though $X(f)$ which is the spectrum of the original signal ($x(t)$) may be bandlimited, the spectrum of the sampled signal ($x_s(t)$) has infinite frequency components. In order to recover the original signal, $x(t)$ from the sampled signal, $x_s(t)$, appropriate filtering has to be applied to $x_s(t)$.

This result also leads to the necessary requirement that a continuous time low pass signal which has maximum frequency components up to f_m Hz has to be sampled at a minimum sampling frequency of $f_s = 2f_m$ in order for the original signal to be reconstructed completely or accurately. f_s is also known as the Nyquist sampling frequency. If $f_s < 2f_m$, then $x_s(t)$ will have overlapping spectra and reconstruction or low pass filtering will not be able to recover the signal completely ie distortion occurs. This phenomenon is also known as aliasing. If the original signal is not bandlimited, then it is better to pre-filter the signal using anti-aliasing filters before sampling at the Nyquist sampling frequency.

For bandlimited signal, the sampling frequency need not satisfy the Nyquist sampling frequency. However, the recovery of the bandlimited signal requires bandpass filters.