National University of Singapore

Department of Electrical & Computer Engineering

EE2023 Signals & Systems Revision Notes

Signals, circuits and systems are all integral parts of electrical and computer engineering. The previous chapters of this preparatory course covers mostly the mathematical concepts which are most applicable to signals. In this chapter, we will focus on circuit elements such as resistors (R), capacitors (C) and inductors (L). In particular, we will revisit their models ie how do they behave in time and frequency domains. We will then go on to revise some basic circuit analysis to solve circuit problems. Some examples of circuit analysis will be provided.

1 Circuit Elements and their Models

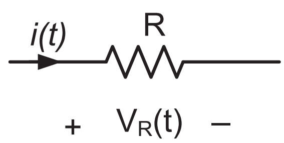
We start with the models of R, C and L in terms of time domain and frequency domains. Then we look at their series and parallel combinations to derive equivalent impedances. Finally we revisit the current and voltage division laws that are often used to analyze circuits.

1.1 Resistors, Capacitors and Inductors

Resistors

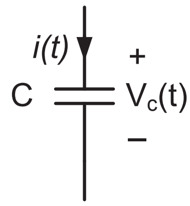
$$\text{Time Domain} : i(t) = \frac{V_R(t)}{R}$$

$$\text{Frequency Domain} : I(s) = \frac{V_R(s)}{R}$$



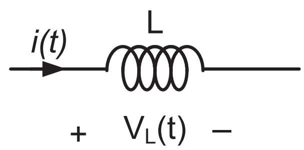
<u>Capacitors</u>

$$\begin{aligned}\text{Time Domain} : i(t) &= C\frac{dV_c(t)}{dt} \\ V_c(t) &= \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau \\ \text{Frequency Domain} : V_c(s) &= \frac{I(s)}{sC} \\ Z_c(s) &= \frac{V_c(s)}{I(s)} = \frac{1}{sC}\end{aligned}$$



<u>Inductors</u>

$$\begin{aligned}\text{Time Domain} : V_L(t) &= L\frac{di(t)}{dt} \\ i(t) &= \frac{1}{L}\int_{-\infty}^{t} V_L(\tau)d\tau \\ \text{Frequency Domain} : V_L(s) &= sLI(s) \\ Z_L(s) &= \frac{V_L(s)}{I(s)} = sL\end{aligned}$$



1.2 Parallel and series combinations of R, C and L

Series Combination :

$$\begin{aligned}\text{Resistance} : R_{total} &= R_1 + R_2 + \ldots + R_n \\ \text{Impedances} : Z_{c,total}(s) &= \frac{1}{sC_1} + \frac{1}{sC_2} + \ldots + \frac{1}{sC_n} \\ Z_{L,total}(s) &= sL_1 + sL_2 + \ldots + sL_n\end{aligned}$$

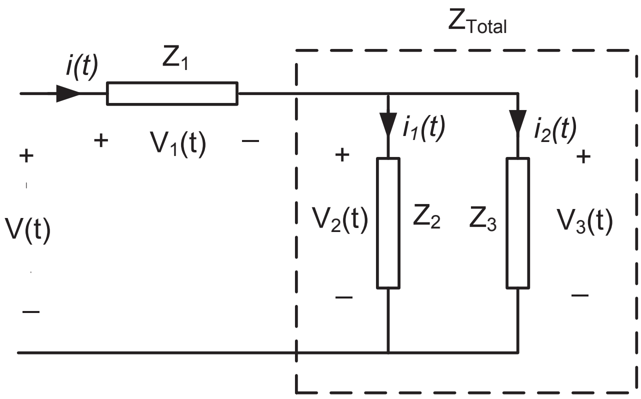
Parallel Combination :

$$\begin{aligned}\text{Resistance} : \frac{1}{R_{total}} &= \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n} \\ \text{Impedances} : Z_c(s) &= \frac{1}{sC_1 + sC_2 + \ldots + sC_n} \\ \frac{1}{Z_L(s)} &= \frac{1}{sL_1} + \frac{1}{sL_2} + \ldots + \frac{1}{sL_n}\end{aligned}$$

$Z_c(s)$ and $Z_L(s)$ are complex impedances ie they are complex numbers which behave like resistances except that their impedance values change with frequency when $s = j\omega$ where

ω is the frequency of operation. For example, in the case of a capacitor with capacitance C Farad, its impedance value is $Z_c(s = j\omega) = \frac{1}{j\omega C}$ Ω and therefore, this impedance value, $Z_c(j\omega)$ will decrease as ω increases. At $\omega = 0$ or at DC, the capacitor operates like an open circuit since $Z_c(j0) = \infty$. A similar behaviour can be said of an inductor with inductance L Henry except that the impedance of an inductor is proportional to ω. Hence an inductor has zero impedance at DC. This is in contrast to resistances, R, which have real values which are not affected very much by frequencies of operation.

1.3 Current and Voltage Division Laws



$$\frac{1}{Z_{Total}} = \frac{1}{Z_2} + \frac{1}{Z_3}$$

Fig. 1: Current and Voltage Division

Applying Kirchoff's Current and Voltage Laws in Figure 1, and working in the frequency domain, we have :

$$\begin{aligned} I(s) &= I_1(s) + I_2(s) \\ I_1(s) &= \frac{Z_3(s)}{Z_2(s) + Z_3(s)} I(s) \\ I_2(s) &= \frac{Z_2(s)}{Z_2(s) + Z_3(s)} I(s) \end{aligned}$$

$$\begin{aligned} V_2(s) &= V_3(s) \\ V(s) &= V_1(s) + V_2(s) \\ V_1(s) &= \frac{Z_1(s)}{Z_1(s) + Z_{Total}(s)} V(s) \\ V_2(s) &= \frac{Z_{Total}(s)}{Z_1(s) + Z_{Total}(s)} V(s) \end{aligned}$$

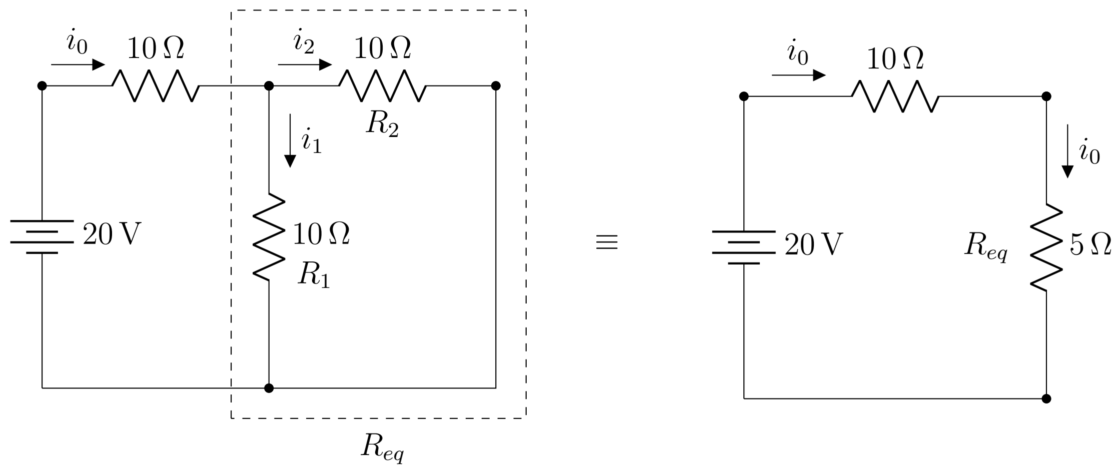
These principles are useful in analysing circuits containing resistors, inductors and capacitors. The above equations, however, are only valid in the frequency domain. In time domain, differential equations are needed to model the currents and voltages in such circuits.

2 Circuit Analysis

In this section, we will make use of all the concepts in Section1 to solve circuit problems. "Solving circuit problems" typically means that we calculate either current and/or voltage quantities in different branches of the circuit or across different components.

2.1 Circuits Involving only Resistors

We start with circuits which consist of only resistors as shown below.



The circuit on the left can be replaced by the circuit on the right if we compute the equivalent resistance (R_{eq}) of the two branches that are in the dashed box. According to the formula for parallel combination rule from Section 1.2,

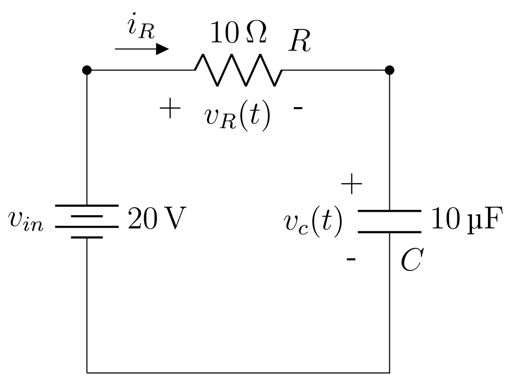
$$\begin{aligned} \frac{1}{R_{eq}} &= \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{5} \\ R_{eq} &= 5\ \Omega \end{aligned}$$

So it is now easy to see how current i_0 can be calculated. $i_0 = 20/15 = 4/3$ A.

In order to calculate i_1 and i_2, we use the current division rule given in Section 1.3.

$$
\begin{aligned}
i_1 &= \frac{R_2}{R_1+R_2} i_0 = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}\ \mathrm{A} \\
i_2 &= \frac{R_1}{R_1+R_2} i_0 = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}\ \mathrm{A}
\end{aligned}
$$

Next, consider an $R - C$ circuit as shown below. Find the differential equation that relates the input DC voltage v_{in} to the voltage across the capacitor v_c.



Apply Kirchoff's voltage law around the circuit :

$$
\begin{aligned}
v_{in} &= v_R + v_c \\
&= i_R R + v_c \\
i_R &= C \frac{dv_c(t)}{dt} \\
\therefore v_{in} &= RC \frac{dv_c(t)}{dt} + v_c \qquad (1)
\end{aligned}
$$

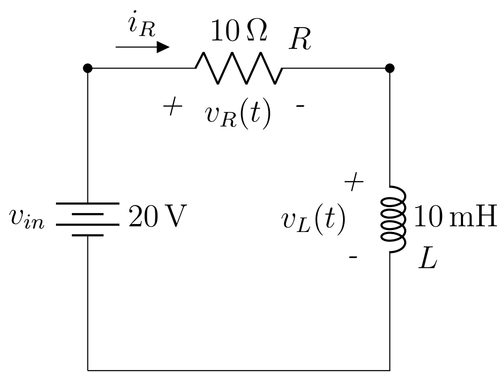
Equation (1) is a typical differential equation which describes the behaviour of a simple $R - C$ circuit. It is a first order differential equation. Inserting the values of the battery source, resistance and capacitance, we get the following differential equation which can be solved using mathematical tools.

$$
\begin{aligned}
20 &= 10 \times 10 \times 10^{-6} \frac{dv_c(t)}{dt} + v_c \\
&= 10^{-4} \frac{dv_c(t)}{dt} + v_c \qquad (2)
\end{aligned}
$$

Solving (2) will give us the voltage across the capacitor as time progresses. We expect that as time goes to infinity, the capacitor will charge to a voltage equal to the battery source. At steady state, $v_c = 20$ V. How do we derive this value from (2)?

Since we expect the capacitor to charge to a steady state value as time goes to infinity, we can set $\dfrac{dv_c(t)}{dt} = 0$ in (2) since at steady state, there should be no more change to $v_c(t)$. If we do this, we arrive at $v_c = 20$ V as time goes to infinity.

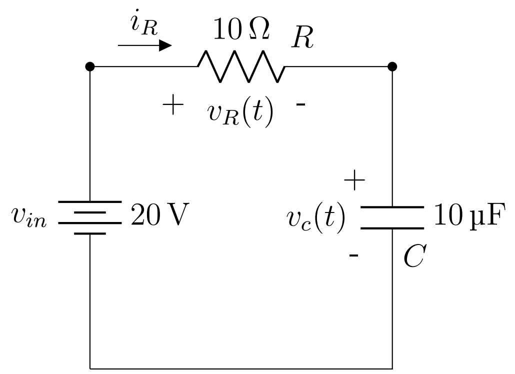
Exercise 1. *Replace the capacitor in the $R - C$ circuit with an inductor. Rewrite the differential equation which describes the relationship between v_{in} and v_L. What happens to v_L as time goes to infinity? What about i_R?*



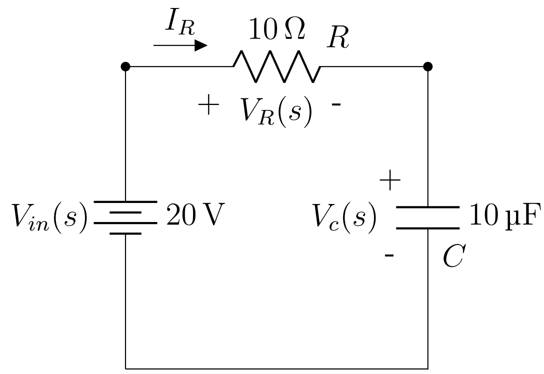
The examples above show the time domain approach to solving circuit problems. This is clear from the time-domain differential equation that is set up in terms of t. We can also use frequency domain to solve circuit problems. Let's revisit the $R - C$ circuit, reproduced below in Figure 2a. Figure 2b is the equivalent circuit in frequency or s-domain. Notice that all the quantities have been replaced by "big caps" to denote that those quantities are in frequency domain. For example, instead of $v_c(t)$ which is in time domain, we denote the equivalent in s-domain by $V_c(s)$.

Recall that the impedance of a capacitor is given by $Z_c = \dfrac{1}{sC}$ where $s = j\omega$ and ω is the frequency of the signal that is going through the C. Hence by applying simple voltage division rule, we can straight away write the voltage across the capacitor in s-domain as follows :

$$\begin{aligned} V_c(s) &= \frac{Z_c(s)}{Z_c(s) + R} \times V_{in}(s) \\ \therefore V_c(j\omega) &= \frac{1}{1 + j\omega RC}\, V_{in}(s) \end{aligned}$$



(a) $R - C$: Time Domain



(b) $R - C$: Frequency Domain

We can then use Laplace Transform to recover $v_c(t)$ from $V_c(s)$.

Let's analyze a RLC circuit that is more complicated.

Example 1. *Consider the circuit in Figure 3 below. Derive the differential equation which relates the current source i_i to the output voltage $v_0(t)$.*

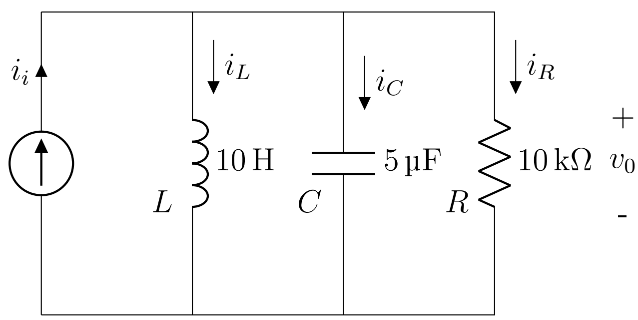


Fig. 3: Parallel RLC circuit

Writing all the current and voltage relationships for L, R and C, we get the following :

$$\begin{aligned} v_0(t) &= L\frac{di_L(t)}{dt} && \text{voltage} - \text{current relationship across L} \\ i_R &= \frac{v_0(t)}{R} && \text{Ohm's Law} \\ i_C(t) &= C\frac{dv_o(t)}{dt} && \text{voltage} - \text{current relationship across C} \end{aligned}$$

Applying Kirchoff's current law : $i_i(t) = i_L(t) + i_C(t) + i_R(t)$, we get the following :

$$i_i(t) = \underbrace{\frac{1}{L}\int_0^t v_0(\tau)d\tau}_{i_L(t)} + \underbrace{C\frac{dv_0(t)}{dt}}_{i_C(t)} + \underbrace{\frac{v_0(t)}{R}}_{i_R(t)} \tag{3}$$

Equation (3) contains a mix of integrals and derivatives which are not convenient to work with. Hence consider differentiating with respect to t on both sides of the equation.

$$\frac{d_i(t)}{dt} = \frac{1}{L}v_0(t) + C\frac{dv_0^2(t)}{dt^2} + \frac{1}{R}\frac{dv_0(t)}{dt} \tag{4}$$

Equation (4) is a second order differential equation which relates the input current $i_i(t)$ to the output voltage $v_0(t)$.

This example is an illustration of how you can write differential equations to model simple circuits involving R, L and C.

We can also deal with everything using complex impedances. Redrawing Figure 3 and combining the impedances of all the R, L and C,

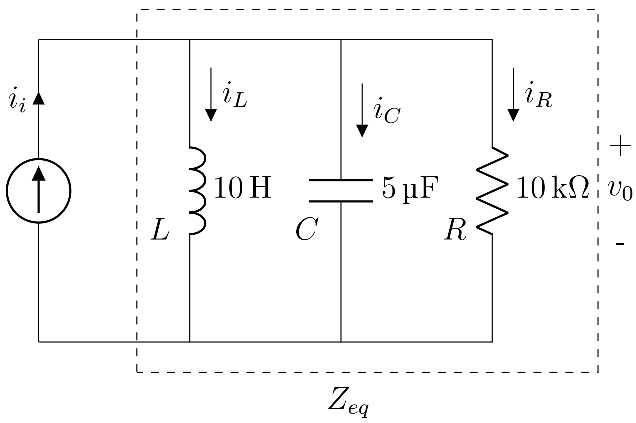


Fig. 4: Parallel RLC circuit

By using the rules to determine Z_{eq}, we have

$$\begin{aligned} \frac{1}{Z_{eq}} &= \frac{1}{R} + \frac{1}{sL} + sC \\ Z_{eq} &= \frac{sLR}{s^2RLC + sL + R} \end{aligned}$$

Hence, $V_0(s)$ can be computed as follows :

$$\begin{aligned} V_0(s) &= I_i(s)Z_{eq} \\ &= \frac{sLR}{s^2RLC + sL + R}I_i(s) \end{aligned} \tag{5}$$

Equation (5) gives the complex relationship between output voltage $V_0(s)$ and input current $I_i(s)$, which is similar to Ohm's Law which applies to a voltage-current relationship in a resistor. Notice that this is so much easier to derive than using differential equations.