3. Continuous-Frequency Spectrum (Fourier Transform)

3.1 Fourier Transform

- In the preceding chapter, we have shown that the ***discrete-frequency spectrum*** of a ***periodic signal***, $x_p(t)$, is given by its complex exponential Fourier series coefficients, c_k .

- Unlike periodic signals, the spectrum of an aperiodic signals has a continuous-frequency domain and do not exhibit discrete spectral lines.

- The ***continuous-frequency spectrum*** of an ***aperiodic signal***, $x(t)$, is given by its Fourier transform, $X(f)$, which is a generalization of the complex exponential Fourier series in the limit $T_p \to \infty$, $k \to \infty$ and $k/T_p \to f$.

- Stating without proof, the relationship between $x(t)$ and $X(f)$ is given by:

$$\left[\begin{array}{ll} \left.\begin{array}{c} \textit{forward} \\ \textbf{FOURIER TRANSFORM} \\ t\textit{-domain to } f\textit{-domain} \end{array}\right\} & X(f) = \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft)\, dt \\ \left.\begin{array}{c} \textit{inverse} \\ \textbf{FOURIER TRANSFORM} \\ f\textit{-domain to } t\textit{-domain} \end{array}\right\} & x(t) = \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft)\, df \end{array}\right] \tag{3.1}$$

- ***Dirichlet's Conditions***

For the Fourier transform of $x(t)$ to exist, the following conditions must be satisfied:

(1) $x(t)$ *is single-valued within any finite time interval.*

(2) $x(t)$ *have at most a finite number of maxima and minima in any finite time interval.*

(3) $x(t)$ *have at most a finite number of discontinuities in any finite time interval.*

(4) $x(t)$ *is absolutely integrable, i.e.* $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$*, or an energy signal.*

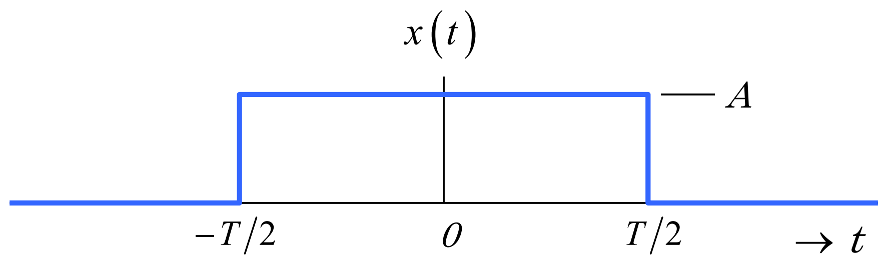
Conditions (1) to (3) are usually satisfied by signals of interest.

Condition (4) appears to exclude power signals from the application of the Fourier transform. However, as we shall see later, the *Fourier transform can also be applied to power signals with the help of the Dirac* δ *function*.

Example 3-1(a):

Spectrum of a rectangular pulse, $x(t)$.

$$x(t) = A \cdot \mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} A; & |t| < T/2 \\ 0; & |t| > T/2 \end{cases}$$



The Fourier transform of $x(t)$ is given by

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt$$

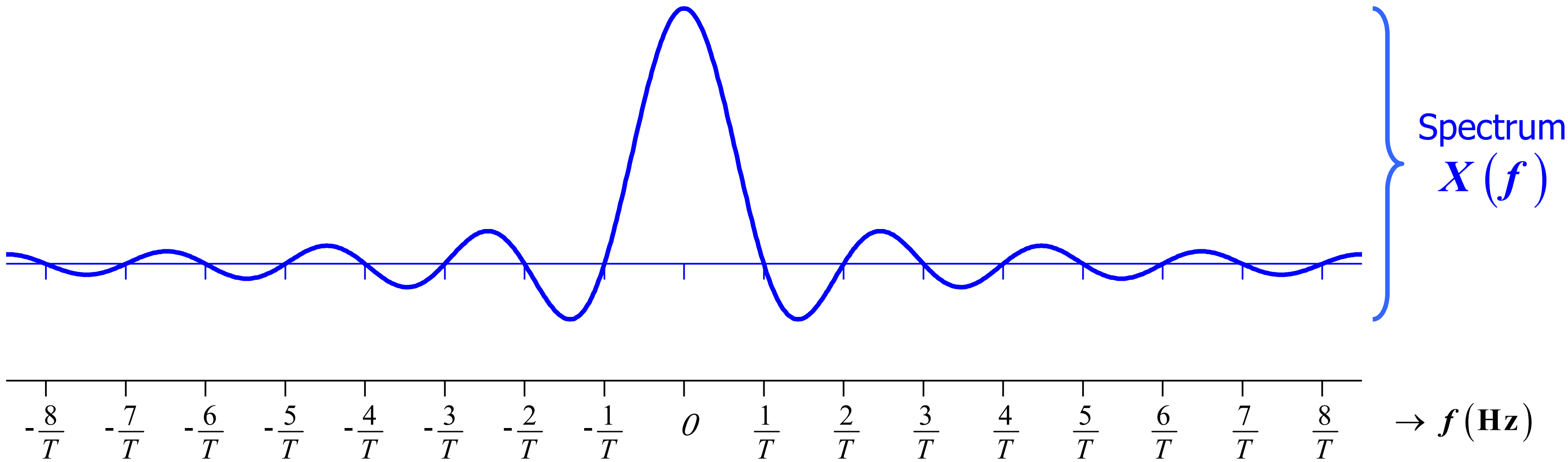
$$= \int_{-T/2}^{T/2} A\exp(-j2\pi ft)\,dt$$

$$= \left(\begin{array}{ll} AT\dfrac{\sin(\pi fT)}{\pi fT}; & f \neq 0 \\ AT; & f = 0 \end{array}\right)$$

$$= AT\,\mathrm{sinc}(fT)$$

$$\left[\boldsymbol{A \cdot \mathrm{rect}\left(\frac{t}{T}\right) \Longleftrightarrow AT\,\mathrm{sinc}(fT)} \right]$$

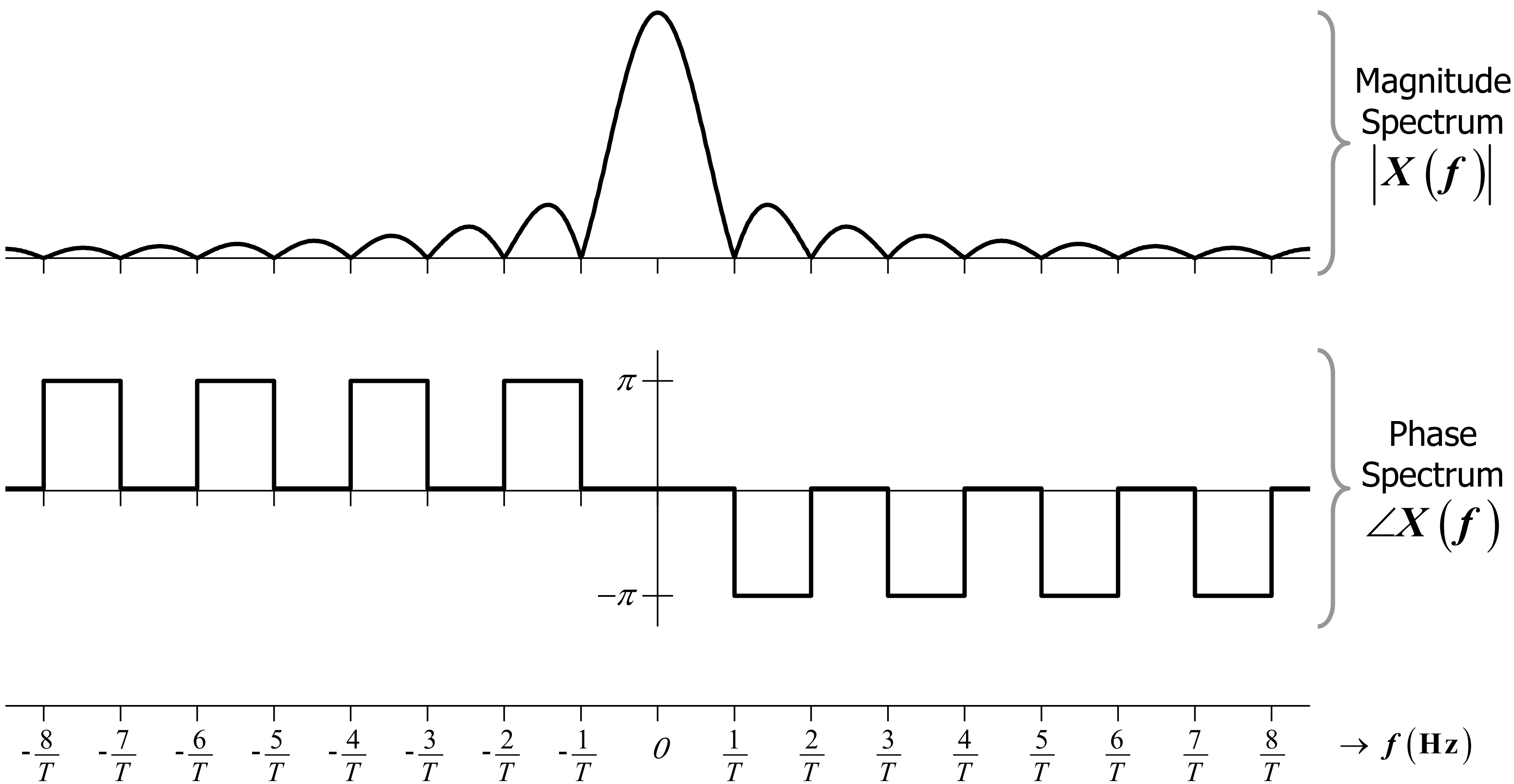
Since $X(f)$ *is real, the spectrum can be depicted by a* ***single*** *spectral plot:*



We may also write $X(f)$ *in terms of its magnitude and phase:*

$$X(f) = AT\,\mathrm{sinc}(fT) = |X(f)|\exp\left(j\angle X(f)\right) \quad \textit{where} \quad \begin{cases} |X(f)| = AT\,|\mathrm{sinc}(fT)| \\ \angle X(f) = \begin{cases} 0; & X(f) \geq 0 \\ \pm\pi; & X(f) < 0 \end{cases} \end{cases}$$

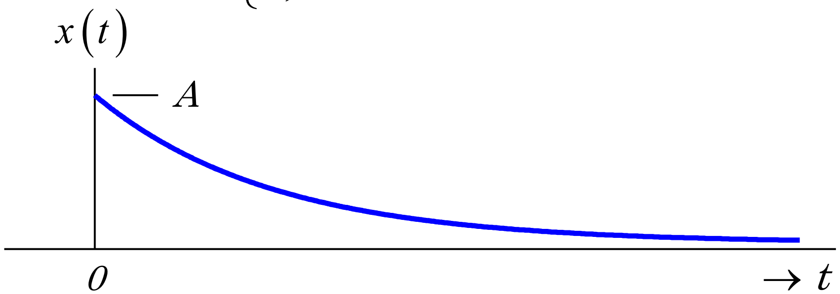
where the corresponding plots are shown below:



Example 3-1(b):

Spectrum of an exponentially decaying pulse, $x(t)$.

$$x(t) = A\exp(-\alpha t)u(t) = \begin{cases} A\exp(-\alpha t); & t \geq 0 \\ 0; & t < 0 \end{cases} \quad \cdots\cdots \quad \text{ASSUME } \alpha > 0$$



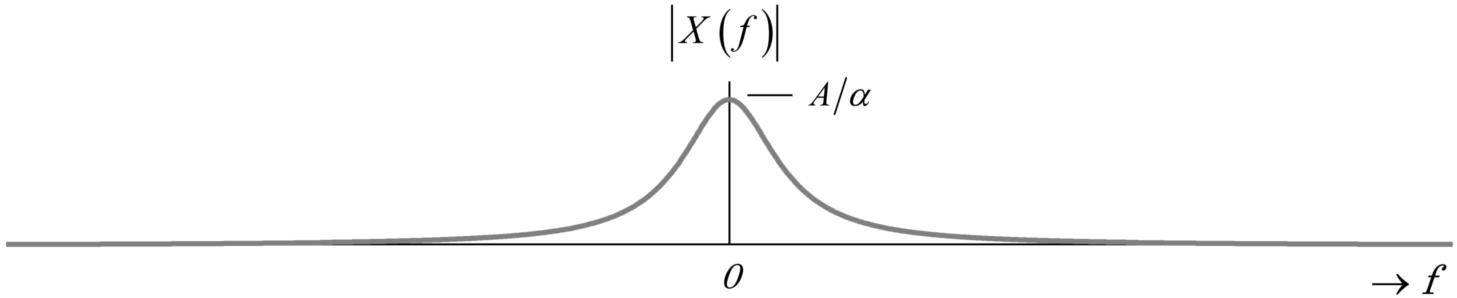
The Fourier transform of $x(t)$ is given by

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt = \int_{0}^{\infty} A\exp(-\alpha t)\exp(-j2\pi ft)\,dt$$

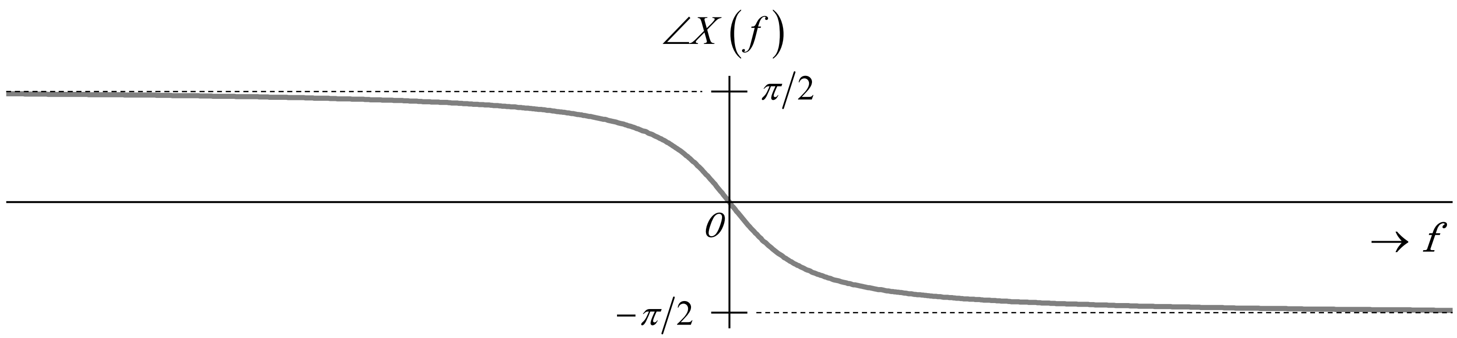
$$= \int_{0}^{\infty} A\exp\big(-(\alpha + j2\pi f)t\big)\,dt = A\left[\frac{\exp\big(-(\alpha + j2\pi f)t\big)}{-(\alpha + j2\pi f)}\right]_0^{\infty}$$

$$= \frac{A}{\alpha + j2\pi f}$$

Magnitude Spectrum: $\left|X(f)\right| = \sqrt{X(f)\,X^*(f)} = \dfrac{A}{\sqrt{\alpha^2 + 4\pi^2 f^2}}$



Phase Spectrum: $\angle X(f) = \tan^{-1}\left(\dfrac{\mathrm{Im}\left[X(f)\right]}{\mathrm{Re}\left[X(f)\right]}\right) = -\tan^{-1}\left(\dfrac{2\pi f}{\alpha}\right)$



Since $X(f)$ is complex, the magnitude and phase spectra ***cannot*** *be combined into a single spectral plot.*

3.2 Properties of Fourier Transform

Let $\begin{cases} X(f) = \Im\{x(t)\} & \textit{denote the Fourier transform of } x(t) \\ x(t) \longleftrightarrow X(f) & \textit{denote a Fourier transform pair} \end{cases}$

A. *Linearity*

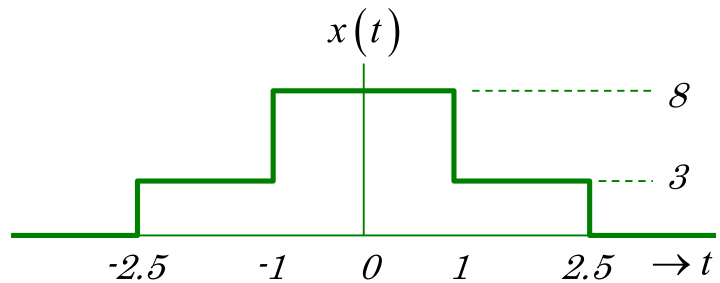
$$\alpha x_1(t) + \beta x_2(t) \longleftrightarrow \alpha X_1(f) + \beta X_2(f) \tag{3.2}$$

Example 3-2(A)

Given $\Im\{\mathrm{rect}(t/T)\} = T\,\mathrm{sinc}(Tf)$

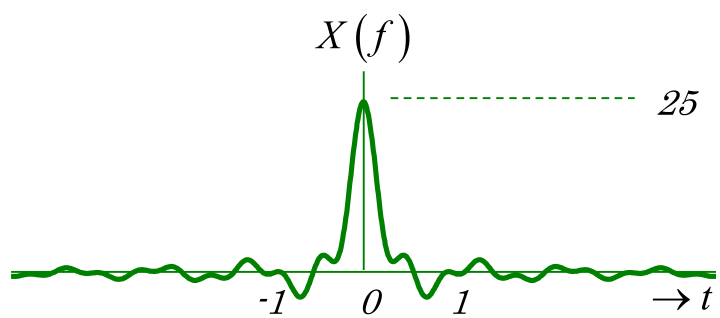
Find the Fourier transform of

$$x(t) = 3\cdot\mathrm{rect}(t/5) + 5\cdot\mathrm{rect}(t/2).$$



Applying the LINEARITY property, we have:

$$\begin{aligned} X(f) &= 3\cdot\Im\{\mathrm{rect}(t/5)\} + 5\cdot\Im\{\mathrm{rect}(t/2)\} \\ &= 15\cdot\mathrm{sinc}(5f) + 10\cdot\mathrm{sinc}(2f) \end{aligned}$$

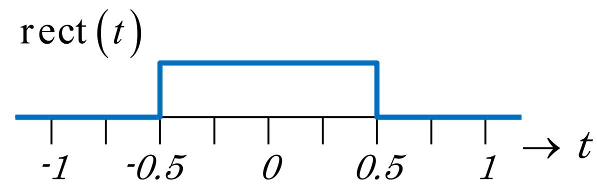


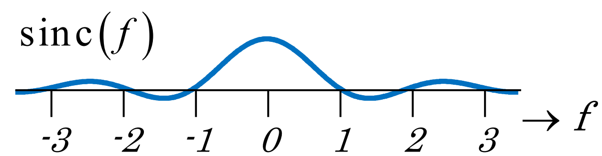
B. *Time Scaling*

$$x(\beta t) \;\square\; \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right) \qquad (3.3)$$

Example 3-2(B):

Given $\Im\{\mathrm{rect}(t)\} = \mathrm{sinc}(f)$.

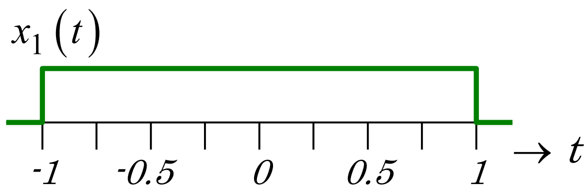


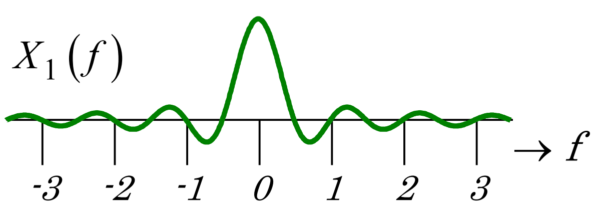


Find the Fourier transform of $x_1(t) = \mathrm{rect}(0.5t)$ *and* $x_2(t) = \mathrm{rect}(2t)$.

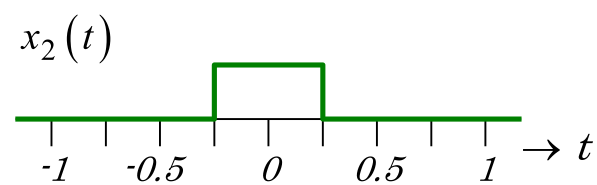
Applying the TIME SCALING property we have:

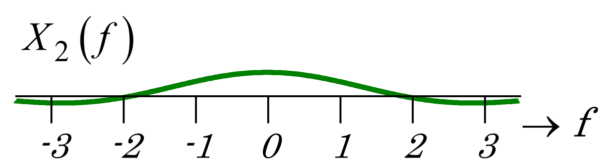
$$X_1(f) = \Im\{\mathrm{rect}(0.5t)\} = 2 \cdot \mathrm{sinc}(2f)$$





$$X_2(f) = \Im\{\mathrm{rect}(2t)\} = \frac{1}{2} \cdot \mathrm{sinc}\left(\frac{f}{2}\right)$$





Observation: *In general,* ***time-spread*** *is inversely proportional to* ***frequency-spread****.*

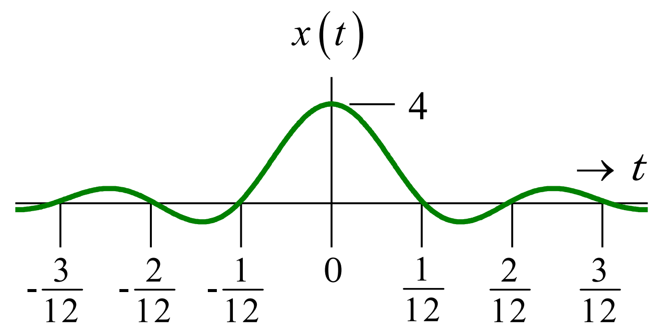
C. *Duality*

$$X(t) \Leftrightarrow x(-f) \quad \text{or} \quad X(-t) \Leftrightarrow x(f) \tag{3.4}$$

Example 3-2(C):

Given $\Im\left\{A \ \mathrm{rect}\left(\frac{t}{T}\right)\right\} = AT \ \mathrm{sinc}(Tf)$.

Find the Fourier transform of $x(t) = 4 \ \mathrm{sinc}(12t)$.



Let $T = 12$ *and* $A = \frac{1}{3}$. *Then*

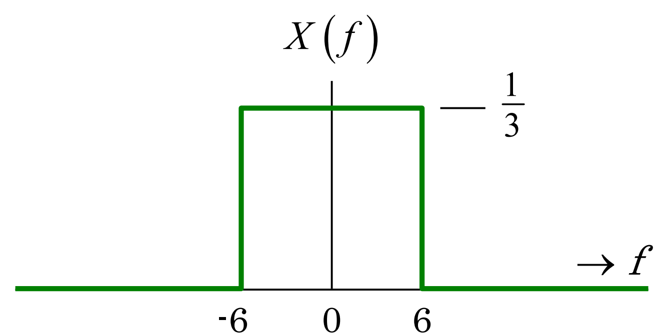
$$\Im\left\{\frac{1}{3} \ \mathrm{rect}\left(\frac{t}{12}\right)\right\} = 4 \ \mathrm{sinc}(12f).$$

Applying the DUALITY property, we have:

$$X(f) = \Im\{4 \ \mathrm{sinc}(12t)\} = \frac{1}{3} \ \mathrm{rect}\left(-\frac{f}{12}\right)$$

$$\cdots \ \textit{since} \ \mathrm{rect}(\cdot) \ \textit{is symmetric} \quad .$$

$$= \frac{1}{3} \ \mathrm{rect}\left(\frac{f}{12}\right)$$



D. *Time Shifting*

$$x(t-t_0) \Leftrightarrow X(f)\exp(-j2\pi f t_0) \qquad (3.5)$$

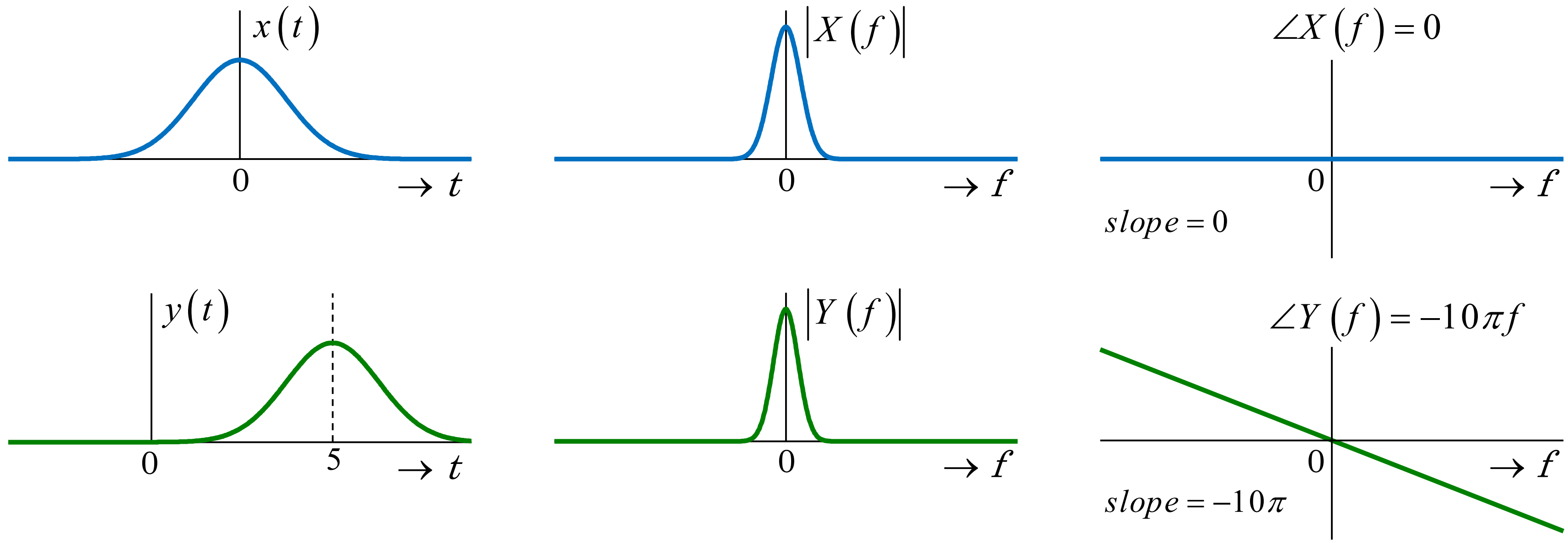
Example 3-2(D):

Given $x(t)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)$ *and* $X(f)=\exp\left(-2\pi^2 f^2\right)$.

Find the Fourier transform of $y(t)=x(t-5)$.

Applying the TIME-SHIFTING property:

$$Y(f)=X(f)\exp(-j2\pi f5)=\exp\left(-2\pi^2 f^2\right)\exp(-j10\pi f)$$



E. *Frequency Shifting (Modulation)*

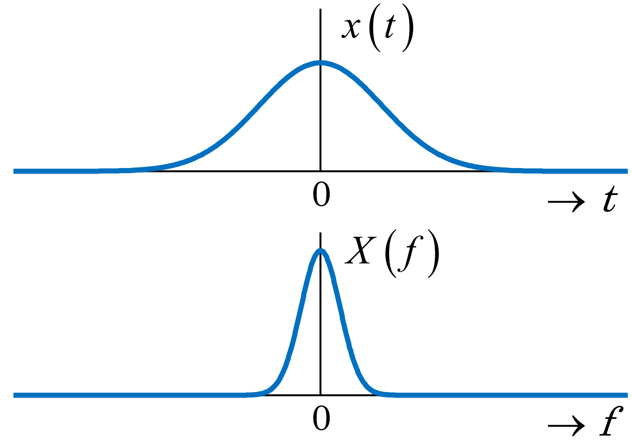
$$x(t)\exp(j2\pi f_0 t) \leftrightarrow X(f-f_0) \tag{3.6}$$

Example 3-2(E):

Given

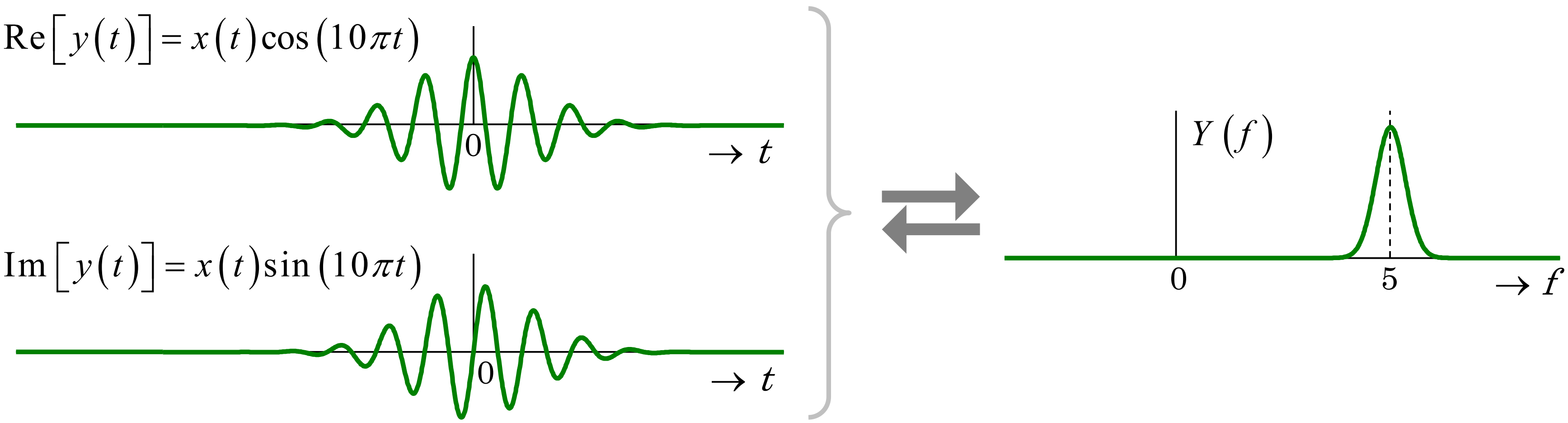
$x(t)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)$ *and* $X(f)=\exp\left(-2\pi^2 f^2\right)$.

Find the Fourier transform of $y(t)=x(t)\exp(j2\pi 5t)$.



Applying the FREQUENCY-SHIFTING property:

$$Y(f)=X(f-5)=\exp\left[-2\pi^2(f-5)^2\right]$$



F. *Differentiation in the Time Domain*

$$\frac{d}{dt}x(t) \Leftrightarrow j2\pi f \cdot X(f) \qquad (3.7)$$

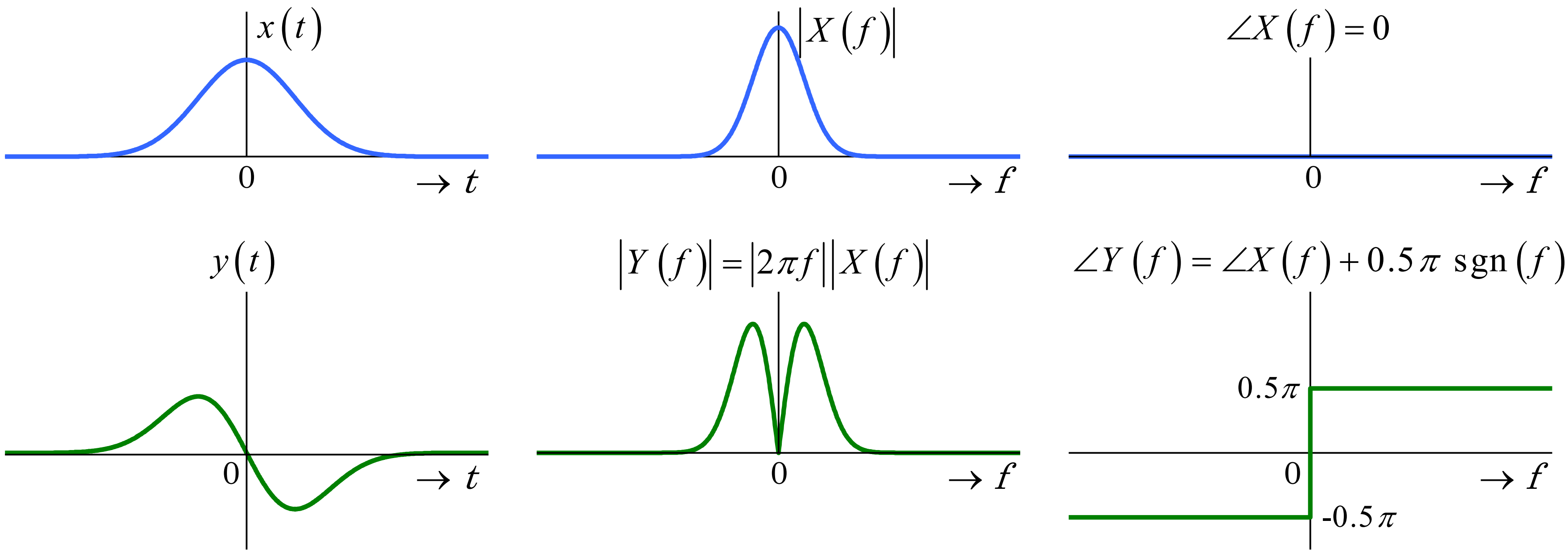
Example 3-2(F):

Given $x(t) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)$ *and* $X(f) = \exp\left(-2\pi^2 f^2\right)$.

Find the Fourier transform of $y(t) = dx(t)/dt$.

Applying the DIFFERENTIATION in t-DOMAIN property:

$$Y(f) = j2\pi f \cdot X(f) = j2\pi f \exp\left(-2\pi^2 f^2\right)$$



G. *Integration in the Time Domain*

$$\int_{-\infty}^{t} x(\tau)\, d\tau \quad \Leftrightarrow \quad \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\delta(f). \tag{3.8}$$

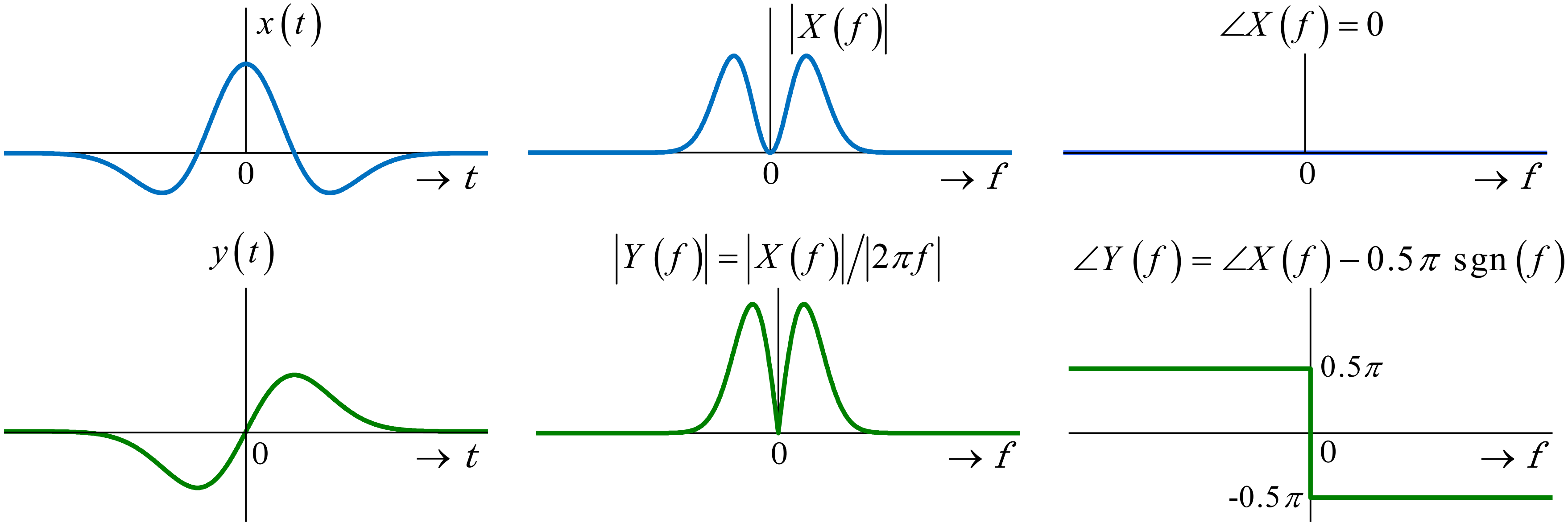
Example 3-2(G):

Given $x(t) = (2\pi)^{-0.5}\left(1 - t^2\right)\exp\left(-t^2/2\right)$ *and* $X(f) = (2\pi f)^2 \exp\left(-2\pi^2 f^2\right)$.

Find the Fourier transform of $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ *.*

Applying the INTEGRATION in t-DOMAIN property:

$$Y(f) = \frac{1}{j2\pi f} \cdot X(f) + \frac{1}{2} \underbrace{X(0)}_{=0} \delta(f) = -j2\pi f \exp\left(-2\pi^2 f^2\right)$$



H. *Convolution in the Time Domain* (*or Multiplication in the Frequency-Domain*)

$$\underbrace{\int_{-\infty}^{\infty} x_1(\zeta)\, x_2(t-\zeta)\, d\zeta}_{x_1(t) * x_2(t)} \rightleftharpoons X_1(f)\, X_2(f) \tag{3.9}$$

Remarks:

- $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$ is defined as the *convolution* of $x_1(t)$ and $x_2(t)$.
- Convolution is **Commutative** : $x_1(t) * x_2(t) = x_2(t) * x_1(t)$.
- Convolution is **Associative** : $\left[x_1(t) * x_2(t)\right] * x_3(t) = x_1(t) * \left[x_2(t) * x_3(t)\right]$
- Convolution is **Distributive** : $x_1(t) * \left[x_2(t) + x_3(t)\right] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$
- This Fourier transform property plays a central role in the analysis and design of continuous-time LTI systems.

Example 3-2(H):

Given $\begin{cases} x_1(t) = \exp(-t)\, u(t) \quad \textit{and} \quad X_1(f) = \dfrac{1}{1 + j2\pi f} \\ x_2(t) = \exp(t)\, u(-t) \quad \textit{and} \quad X_2(f) = \dfrac{1}{1 - j2\pi f} \end{cases}.$

Find the Fourier transform of $y(t) = x_1(t) * x_2(t)$.

Applying the CONVOLUTION in t-DOMAIN property:

$$Y(f) = X_1(f)X_2(f) = \left(\frac{1}{1+j2\pi f}\right)\left(\frac{1}{1-j2\pi f}\right) = \frac{1}{1+4\pi^2 f^2}$$

Verifying the result:

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \exp(-\tau)u(\tau)\exp(t-\tau)u(\tau-t)\,d\tau$$

$$= \begin{cases} \int_t^{\infty} \exp(t-2\tau)\,d\tau = 0.5\exp(-t); & t \geq 0 \\ \int_0^{\infty} \exp(t-2\tau)\,d\tau = 0.5\exp(\ t\ ); & t < 0 \end{cases} = 0.5\exp(-|t|)$$

$$Y(f) = \Im\{0.5\exp(-|t|)\} = 0.5\left[\int_{-\infty}^{\infty} \exp(-|t|)\exp(-j2\pi f t)\,dt\right]$$

$$= 0.5\left[\int_0^{\infty} \exp(-t)\exp(-j2\pi f t)\,dt + \int_{-\infty}^{0} \exp(t)\exp(-j2\pi f t)\,dt\right]$$

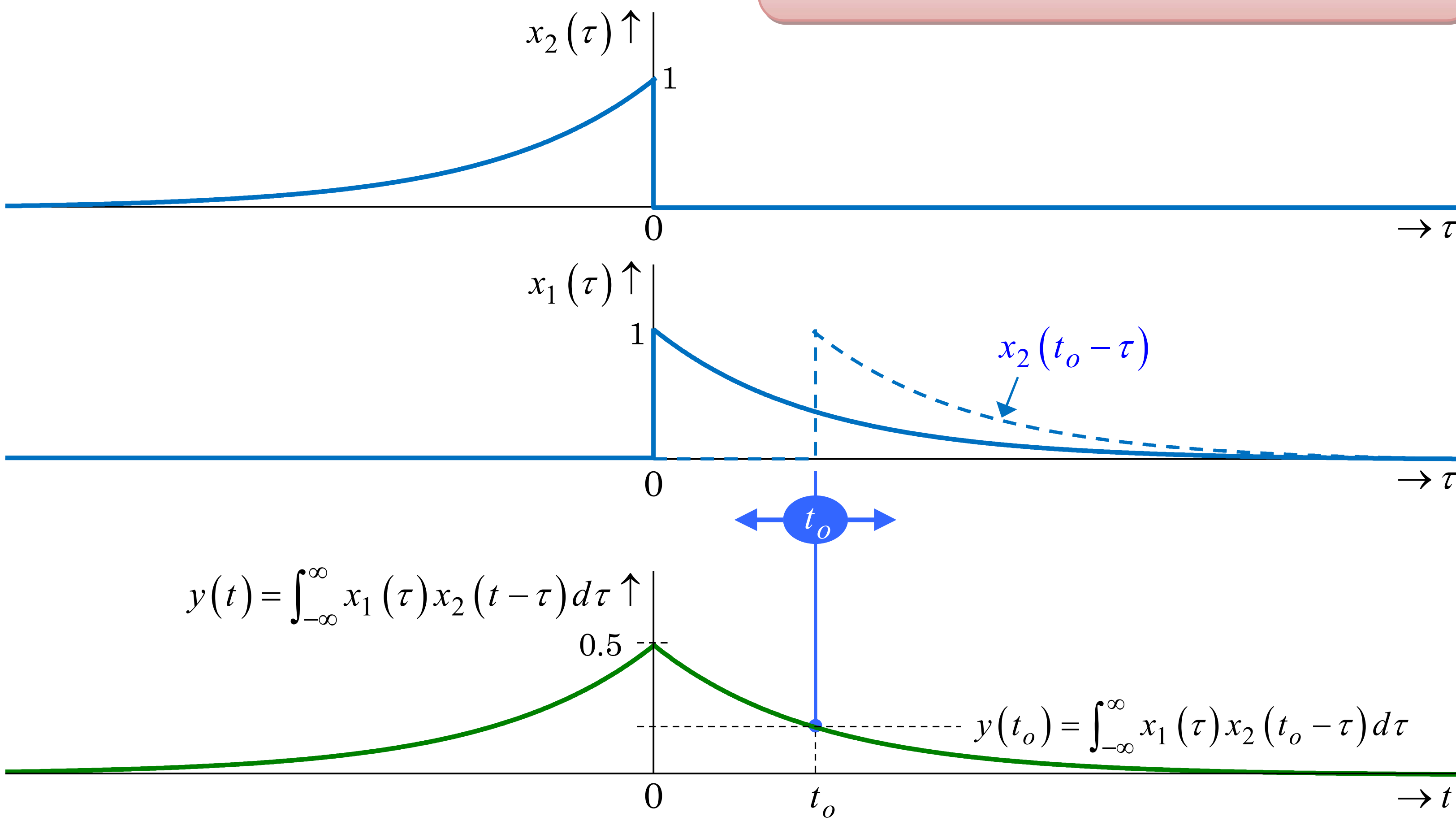
$$= 0.5\left[\int_0^{\infty} \exp(-(1+j2\pi f)t)\,dt + \int_{-\infty}^{0} \exp((1-j2\pi f)t)\,dt\right]$$

$$= 0.5\left[\left.\frac{\exp(-(1+j2\pi f)t)}{-(1+j2\pi f)}\right|_0^{\infty} + \left.\frac{\exp((1-j2\pi f)t)}{(1-j2\pi f)}\right|_{-\infty}^{0}\right]$$

$$= 0.5\left[\frac{1}{1+j2\pi f} + \frac{1}{1-j2\pi f}\right] = \frac{1}{1+4\pi^2 f^2}$$

Graphical evaluation of $y(t)=\int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau$:

www.jhu.edu/~signals/index.html



I. *Multiplication in the Time Domain* (*or Convolution in the Frequency-Domain*)

$$x_1(t)x_2(t) \Leftrightarrow \int_{-\infty}^{\infty} X_1(\zeta)X_2(f-\zeta)d\zeta \qquad (3.10)$$

Example 3-2(I):

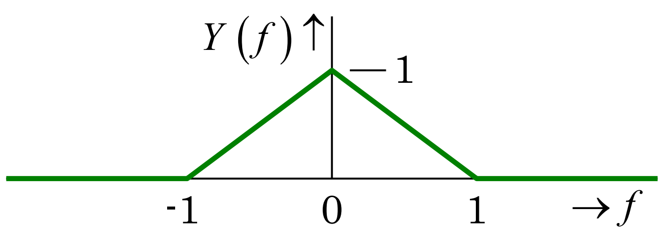
Given $x(t)=\operatorname{sinc}(t)$ *and* $X(f)=\operatorname{rect}(f)$.

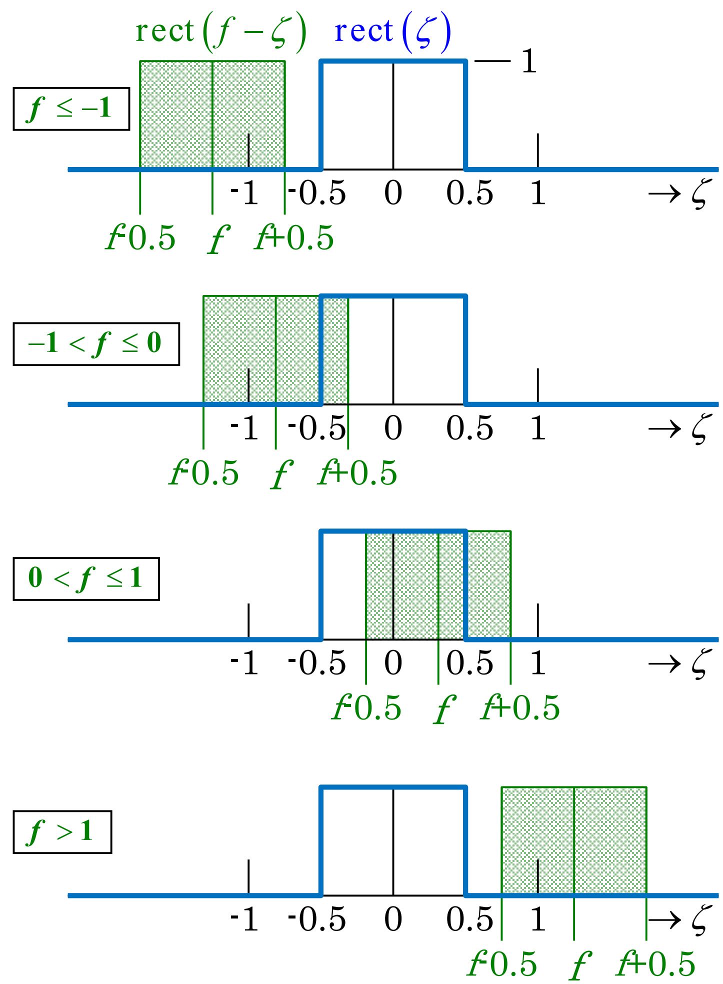
Find the Fourier transform of $y(t)=x^2(t)$.

Applying the MULTIPLICATION in t-DOMAIN property:

$$Y(f)=X(f)*X(f)=\int_{-\infty}^{\infty}\operatorname{rect}(\zeta)\operatorname{rect}(f-\zeta)d\zeta$$

$$=\begin{cases} 0; & f\le -1 \\ \int_{-0.5}^{f+0.5} d\zeta = 1+f; & -1<f\le 0 \\ \int_{f-0.5}^{0.5} d\zeta = 1-f; & 0<f\le 1 \\ 0; & f>1 \end{cases} = \mathbf{tri}(f)$$





3.3 Spectral Properties of a REAL Signal

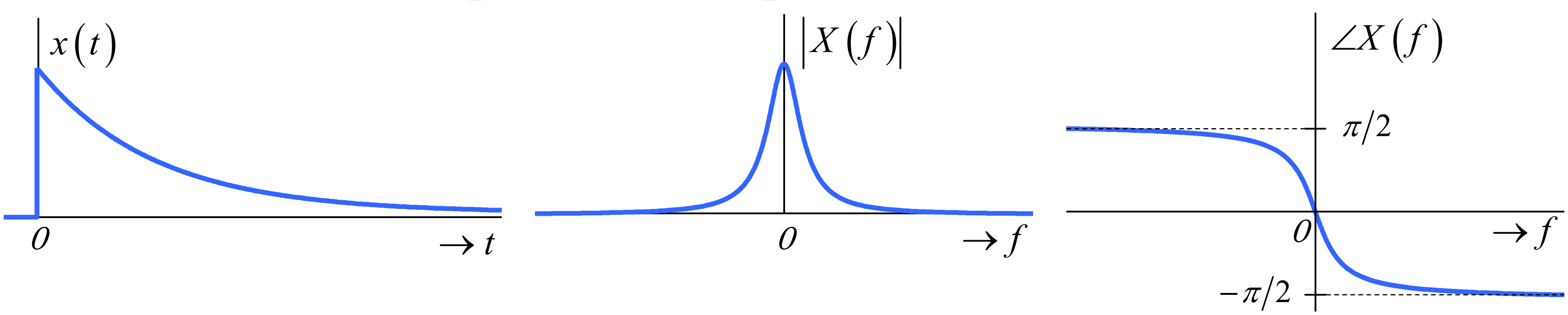
- $x(t)$ ***is REAL:*** $\left[x^*(t) = x(t)\right]$

$$\left.\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \\ X(-f) &= \int_{-\infty}^{\infty} x(t)\exp(j2\pi ft)\,dt \\ X^*(f) &= \int_{-\infty}^{\infty} \underbrace{x^*(t)}_{x(t)}\exp(j2\pi ft)\,dt \end{aligned}\right\} \rightarrow \left\{\begin{array}{c} \underbrace{X^*(f) = X(-f)}_{X(f)\ \textbf{is Conjugate Symmetric}} \\ \Downarrow \\ \underbrace{|X(f)| = |X(-f)|}_{\textbf{EVEN Symmetry}} \ \textit{and} \ \underbrace{\angle X(f) = -\angle X(-f)}_{\textbf{ODD Symmetry}} \end{array}\right. \qquad (3.11)$$

Proof : $\left.\begin{aligned} X(f) &= |X(f)|\exp(j\angle X(f)) \\ X(-f) &= |X(-f)|\exp(j\angle X(-f)) \\ X^*(f) &= |X(f)|\exp(-j\angle X(f)) \end{aligned}\right\}$ *with* $X^*(f) = X(-f)$ *we have* $\left\{\begin{aligned} |X(f)| &= |X(-f)| \\ \angle X(f) &= -\angle X(-f) \end{aligned}\right.$

Example 3-3:

$\left[x(t) = \exp(-4t)u(t)\right]$ □ $\left[X(f) = \dfrac{1}{4 + j2\pi f}\right]$

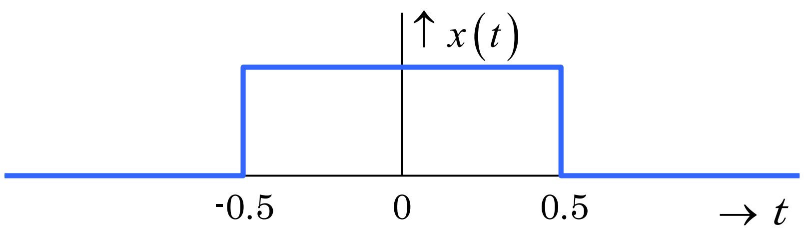


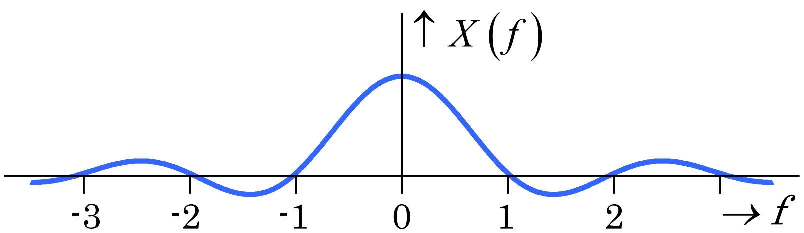
- ***$x(t)$ is REAL and EVEN:*** $\left[x^*(t) = x(t) \ \ \textbf{\textit{and}} \ \ x(t) = x(-t) \right]$

$$\left.\begin{array}{l} \left(\underbrace{x^*(t) = x(t) \ \ \Rightarrow \ \ X^*(f) = X(-f)}_{x(t)\text{ is REAL, see (3.11)}} \right. \\ \left(\begin{array}{l} x(-t) \rightleftharpoons X(-f) \ \cdots \text{ Scaling Property} \\ \underbrace{x(t) = x(-t)}_{x(t)\text{ is EVEN}} \ \ \Rightarrow \ \ X(f) = X(-f) \end{array} \right. \end{array} \right\} \Rightarrow \underbrace{\overbrace{X^*(f) = X(f)}^{\text{Real}} \text{ and } \overbrace{X(f) = X(-f)}^{\text{Even}}}_{X(f)\textit{ is REAL and EVEN}} \qquad (3.12)$$

Example 3-4:

$$\left[x(t) = \mathrm{rect}(t) \right] \rightleftharpoons \left[X(f) = \mathrm{sinc}(f) \right]$$





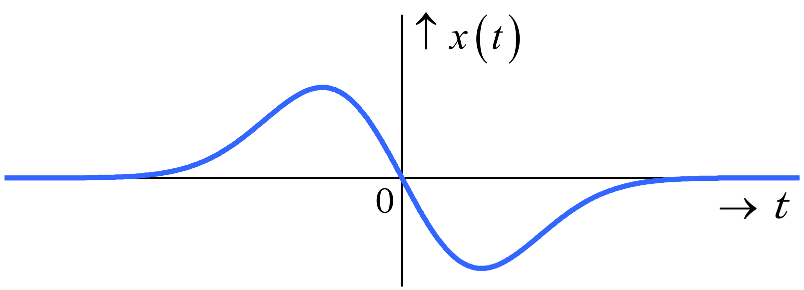
- $x(t)$ ***is REAL and ODD:*** $\left[x^*(t)=x(t) \text{ and } x(-t)=-x(t)\right]$

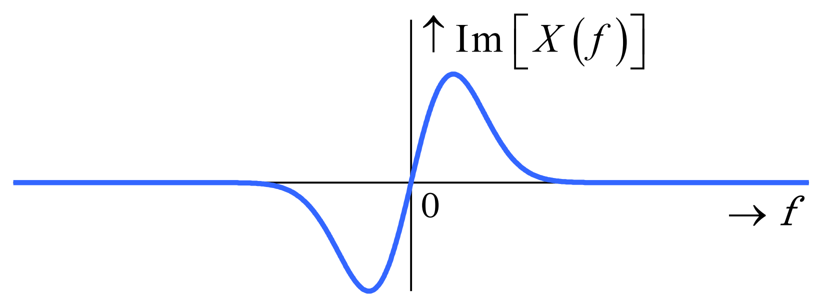
$$\left.\begin{aligned}&\left(\underbrace{x^*(t)=x(t) \quad \Rightarrow \quad X^*(f)=X(-f)}_{x(t)\text{ is REAL, see (3.11)}}\right.\\&\left(\begin{aligned}&x(-t)\Leftrightarrow X(-f) \cdots \text{Scaling Property}\\&\underbrace{x(t)=-x(-t)}_{x(t)\text{ is ODD}} \quad \Rightarrow \quad X(f)=-X(-f)\end{aligned}\right.\end{aligned}\right\} \Rightarrow \underbrace{\overbrace{X^*(f)=-X(f)}^{\text{Imaginary}} \text{ and } \overbrace{X(f)=-X(-f)}^{\text{Odd}}}_{X(f)\textit{ is IMAGINARY and ODD}} \qquad (3.13)$$

Example 3-5:

$$\left[x(t)=-(2\pi)^{-0.5}\, t\exp\left(-t^2/2\right)\right] \Leftrightarrow \left[X(f)=j2\pi f\exp\left(-2\pi^2 f^2\right)\right]$$

see Example 2-4(F)





The above results are also applicable to the Fourier series coefficients of periodic signals, as summarized below:

- $x_p(t)$ ***is REAL:***

$$\left(\underbrace{X_k^* = X_{-k}}_{X_k \textbf{ is Conjugate Symmetric}} \quad \underbrace{|X_k| = |X_{-k}|}_{\textbf{EVEN Symmetry}} \quad \underbrace{\angle X_k = -\angle X_{-k}}_{\textbf{ODD Symmetry}} \right) \tag{3.14}$$

- $x_p(t)$ ***is REAL and EVEN:***

$$\left(\underbrace{\overbrace{X_k^* = X_k}^{\text{Real}} \quad \text{and} \quad \overbrace{X_k = X_{-k}}^{\text{Even}}}_{X_k \textbf{ is REAL and EVEN}} \right) \tag{3.15}$$

- $x_p(t)$ ***is REAL and ODD:***

$$\left(\underbrace{\overbrace{X_k^* = -X_k}^{\text{Imaginary}} \quad \text{and} \quad \overbrace{X_k = -X_{-k}}^{\text{Odd}}}_{X_k \textbf{ is IMAGINARY and ODD}} \right) \tag{3.16}$$

REMARKS: Either the *positive frequency* portion or the *negative frequency* portion of a spectrum would suffice to specify a real signal completely in the frequency domain because one can be derived from the other through the conjugate symmetry property of the spectrum. **Spectrum analyzers** usually display only the positive frequency portion of a spectrum.

3.4 The Dirac-δ and Spectrum of Periodic Signals

3.4.1 The Continuous-time Unit Impulse (Dirac-δ function)

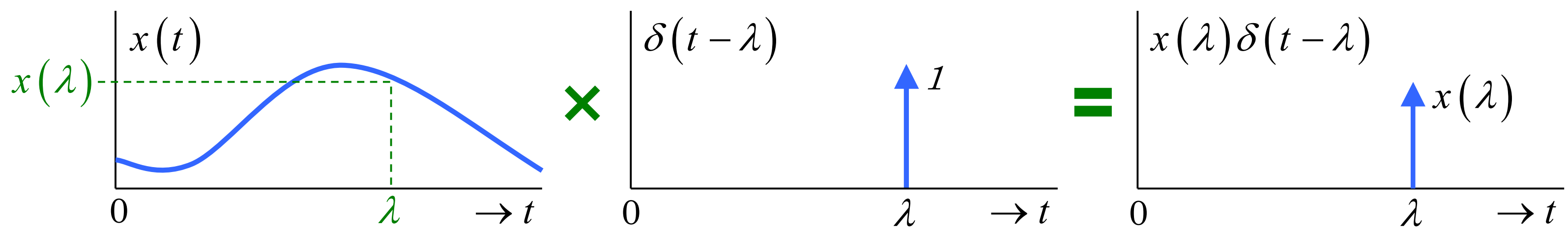
The ***continuous-time unit impulse***, also known as the ***Dirac-δ function***, is defined as (see Chapter 1, Section 1.2)

$$\delta(t)=\begin{cases}\infty; & t=0\\ 0; & t\neq 0\end{cases} \quad \text{and} \quad \int_{-\varepsilon}^{\varepsilon}\delta(t)\,dt=1;\quad \forall\varepsilon>0 \qquad (3.17)$$

- **Properties of $\delta(t)$**

1. ***Symmetry:*** $\delta(t)=\delta(-t)$ (3.18)

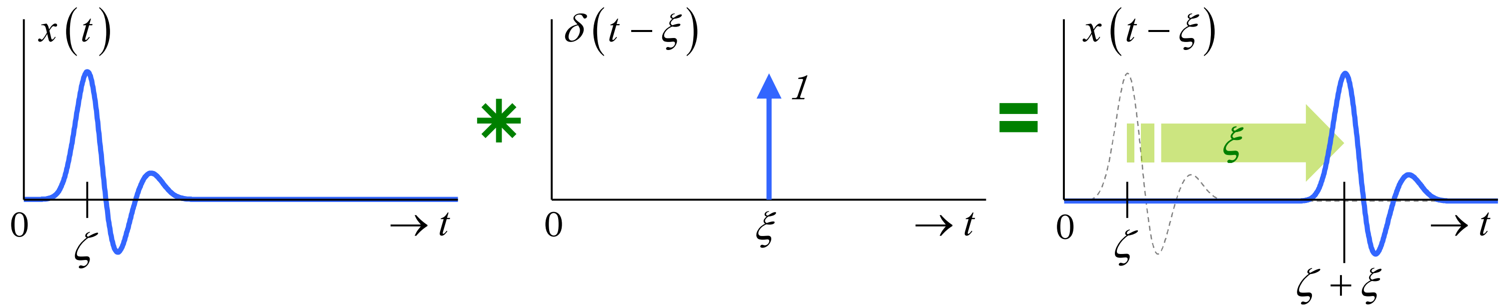
2. ***Sampling:*** $x(t)\delta(t-\lambda)=x(\lambda)\delta(t-\lambda)$ (3.19)



3. ***Sifting:*** $\underbrace{\int_{-\infty}^{\infty}x(t)\delta(t-\lambda)\,dt=x(\lambda)\int_{-\infty}^{\infty}\delta(t-\lambda)\,dt}_{\textit{from the sampling property of } \delta(\cdot)}=x(\lambda)$ (3.20)

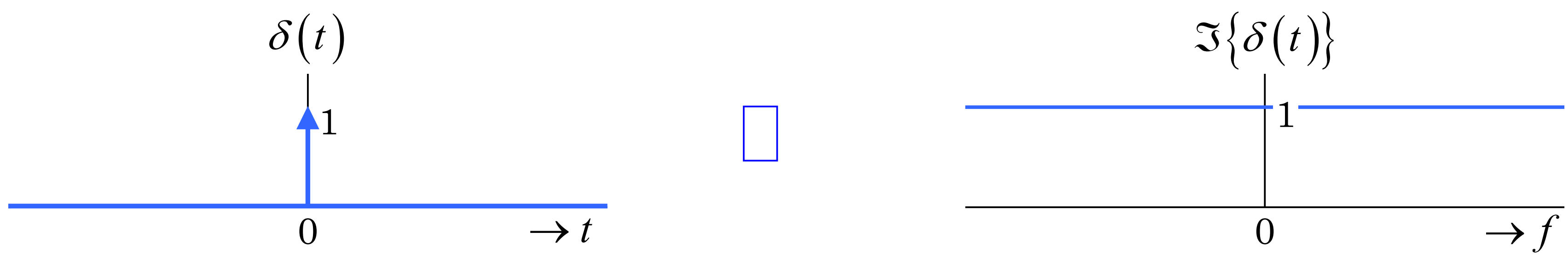
4. ***Replication:***

$$\begin{cases} x(t)*\delta(t-\xi) = \overbrace{\int_{-\infty}^{\infty} x(\zeta)\,\delta(t-\zeta-\xi)\,d\zeta = \underbrace{\int_{-\infty}^{\infty} x(\zeta)\,\delta\big(\zeta-(t-\xi)\big)\,d\zeta = x(t-\xi)}_{\text{apply } \textbf{sifting} \text{ property}}}^{\text{apply } \textbf{symmetry} \text{ property}} \\ \textbf{Note:}\ x(t)*\delta(t) = x(t) \end{cases} \tag{3.21}$$



5. ***White Spectrum:***

$$\begin{cases} \Im\{\delta(t)\} = \underbrace{\int_{-\infty}^{\infty} \delta(t)\exp(-j2\pi ft)\,dt = 1}_{\text{apply } \textbf{sifting} \text{ property}} \end{cases} \tag{3.22}$$



Example 3-6 (Spectra of Unit Step and Signum):

Unit Step function:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Let $\Delta(f) = \Im\{\delta(t)\} = 1$ *. Applying the integration property (3.8), we get*

$$\Im\{u(t)\} = \frac{1}{j2\pi f}\underbrace{\Delta(f)}_{=1} + \frac{1}{2}\underbrace{\Delta(0)}_{=1}\delta(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

Signum function:

$$\operatorname{sgn}(t) = \begin{cases} 1, & t \ge 0 \\ -1, & t < 0 \end{cases}$$

Rewriting $\operatorname{sgn}(t) = 2u(t) - 1$ *and applying the linearity property (3.2), we get*

$$\Im\{\operatorname{sgn}(t)\} = 2 \cdot \Im\{u(t)\} - \Im\{1\} = 2 \cdot \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] - \delta(f) = \frac{1}{j\pi f}.$$

3.4.2 Spectrum of Periodic Signals

Recap:

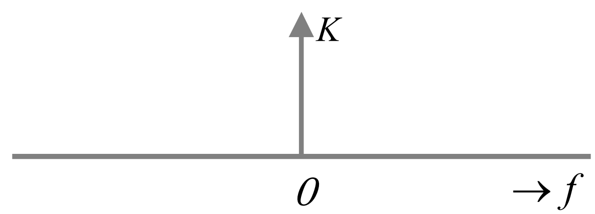
- Fourier series analysis of a periodic signal leads to a discrete-frequency spectrum.
- Dirichlet's 4th condition: A signal must be absolutely integrable for its Fourier transform to exist. *(This will exclude the application of Fourier transform to periodic signals, which have infinite total energy.)*

With the help of the unit impulse function, Fourier transform can be applied to a periodic signal to obtain its continuous-frequency spectrum (thus violating Dirichlet's 4th condition).

- ***DC:*** $\left\{x(t)=K\right\}$

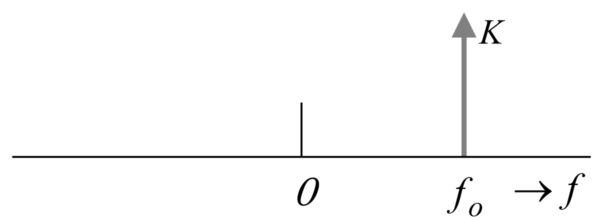
  (A dc signal may be viewed as a periodic signal of arbitrary period)

$$\left(\begin{array}{c}\Im\left\{K\delta(t)\right\}=K \xrightarrow{\text{Duality Property}} \Im\left\{K\right\}=K\delta(f) \\ \left[x(t)=K\right] \Leftrightarrow \left[X(f)=K\delta(f)\right]\end{array}\right)$$



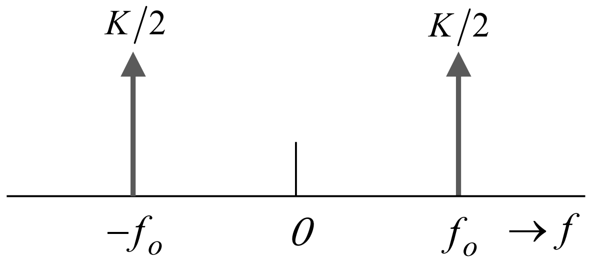
- ***Complex Exponential:*** $\left\{x(t)=K\exp\left(j2\pi f_o t\right)\right\}$

$$\left(\underbrace{\left[x(t)=K\exp\left(j2\pi f_o t\right)\right] \Leftrightarrow \left[X(f)=K\delta(f-f_o)\right]}_{\text{Frequency-shifting (or Modulation) Property}}\right)$$



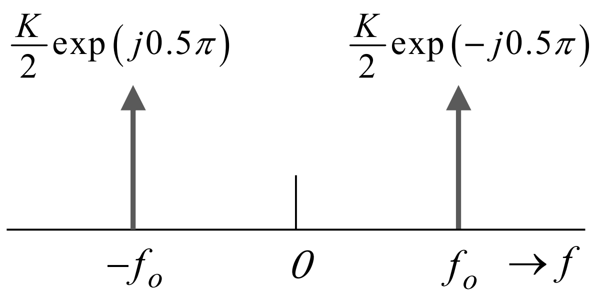
- ***Cosine:*** $\left\{x(t) = K\cos(2\pi f_o t)\right\}$

$$\left(\begin{array}{c} x(t) = \frac{K}{2}\exp(j2\pi f_o t) + \frac{K}{2}\exp(-j2\pi f_o t) \\ \left[x(t) = K\cos(2\pi f_o t)\right] \square \quad \left[X(f) = \frac{K}{2}\delta(f - f_o) + \frac{K}{2}\delta(f + f_o)\right] \end{array}\right)$$



- ***Sine:*** $\left\{x(t) = K\sin(2\pi f_o t)\right\}$

$$\left(\begin{array}{c} x(t) = \frac{K}{j2}\exp(j2\pi f_o t) - \frac{K}{j2}\exp(-j2\pi f_o t) \\ \left[x(t) = \sin(2\pi f_o t)\right] \square \quad \left[\begin{array}{l} X(f) = \frac{K}{j2}\delta(f - f_o) - \frac{K}{j2}\delta(f + f_o) \\ = \left(\begin{array}{l} \frac{K}{2}\exp\left(-j\frac{\pi}{2}\right)\delta(f - f_o) \\ + \frac{K}{2}\exp\left(j\frac{\pi}{2}\right)\delta(f + f_o) \end{array}\right) \end{array}\right] \end{array}\right)$$



- ***Arbitrary periodic signals:*** $\left\{x_p(t):\ Period = T_p\right\}$

Fourier series: $$\left[\underbrace{c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\exp\left(-j2\pi\frac{k}{T_p}t\right)dt}_{\text{ANALYSIS}} \qquad \underbrace{x_p(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(j2\pi\frac{k}{T_p}t\right)}_{\text{SYNTHESIS}}\right]$$

Applying Fourier transform to $x_p(t)$:

$$X_p(f) = \Im\left\{x_p(t)\right\} = \underbrace{\Im\left\{\sum_{k=-\infty}^{\infty} c_k \exp\left(j2\pi\frac{k}{T_p}t\right)\right\} = \sum_{k=-\infty}^{\infty} c_k \Im\left\{\exp\left(j2\pi\frac{k}{T_p}t\right)\right\}}_{\text{Linearity property of Fourier transform}} \tag{3.23}$$

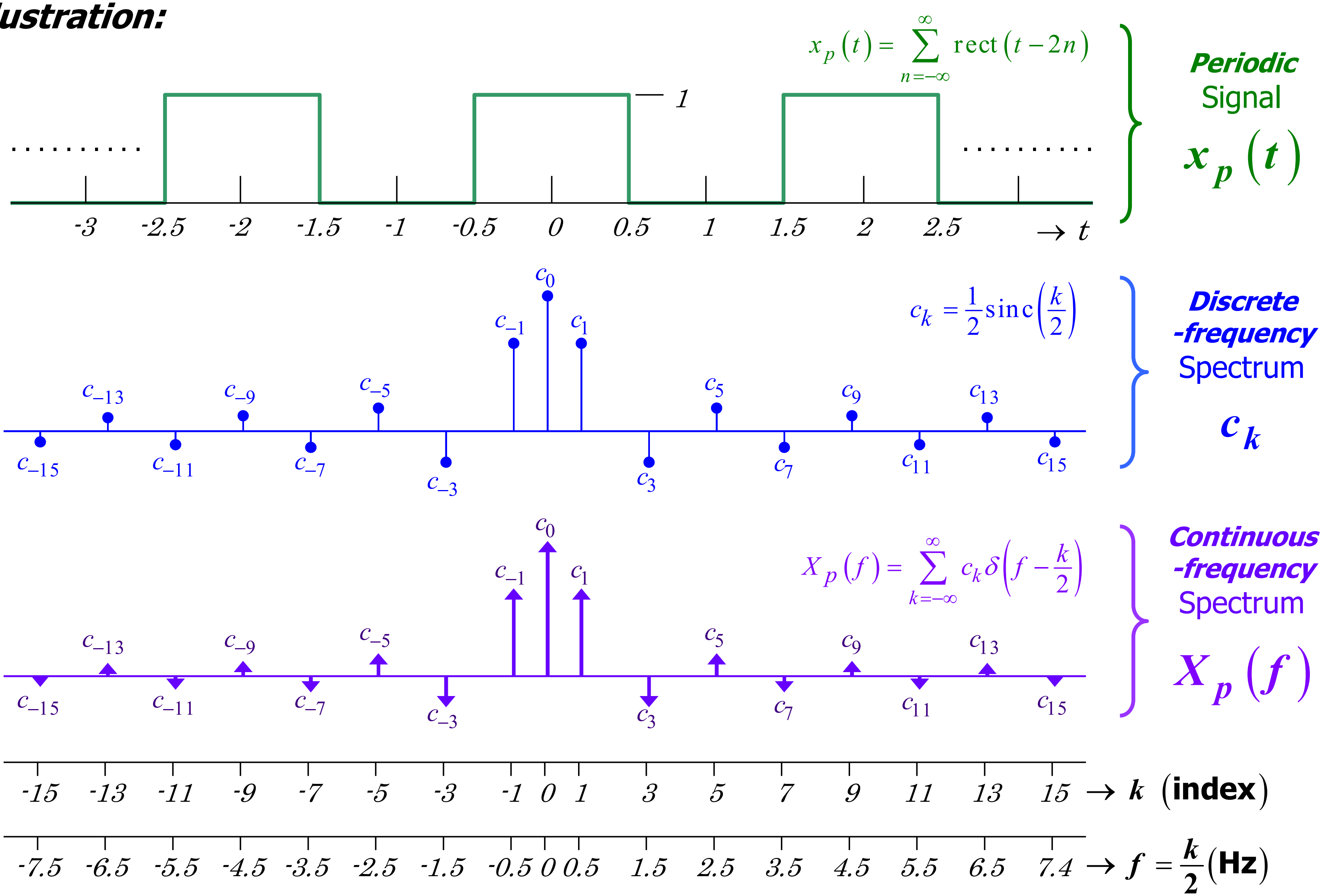
Substituting $\Im\left\{\exp\left(j2\pi\frac{k}{T_p}t\right)\right\} = \delta\left(f-\frac{k}{T_p}\right)$ into (3.23) yields:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f-\frac{k}{T_p}\right) \tag{3.24}$$

Inference:

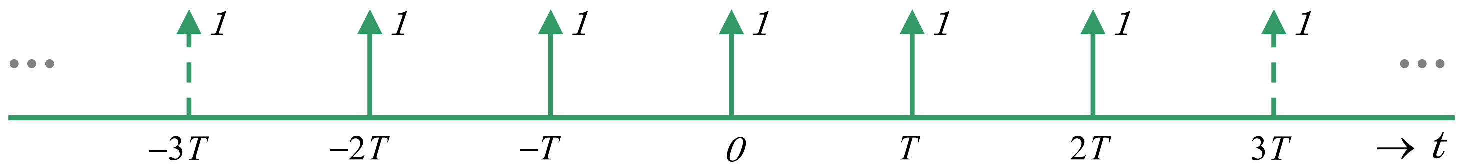
The Fourier transform, $X_p(f)$, of a periodic signal, $x_p(t)$, can be obtained by first computing the Fourier series coefficients, c_k, of $x_p(t)$ and then substituting them into (3.24).

Illustration:



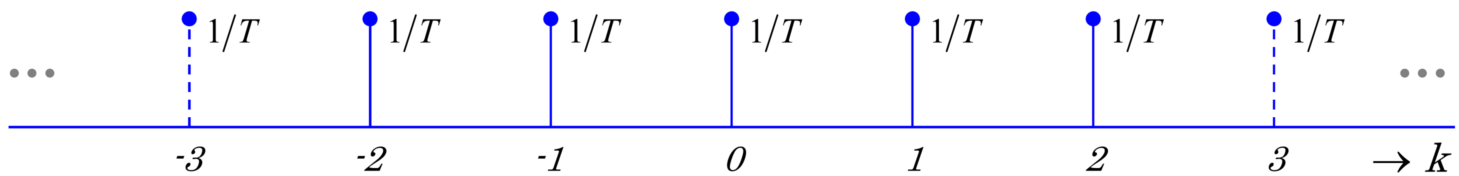
Example 3-7:

Spectrum of a COMB FUNCTION, $\xi_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$



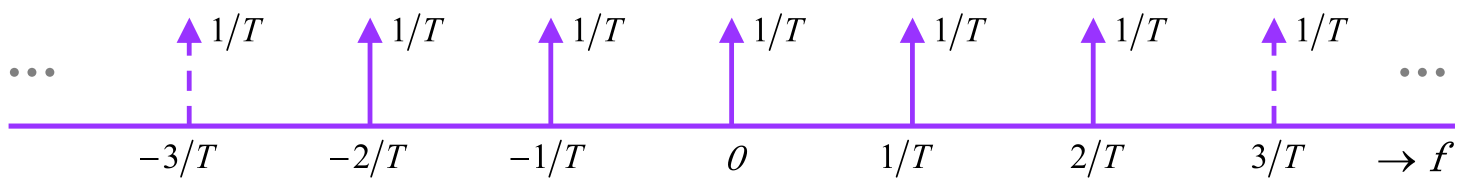
Discrete-frequency spectrum, $\Xi_{T,k}$ $\left[\textbf{Fourier series coefficients of } \xi_T(t)\right]$*:*

$$\Xi_{T,k} = \frac{1}{T}\int_{-T/2}^{T/2} \xi_T(t)\exp\left(-j2\pi\left(\frac{k}{T}\right)t\right)dt = \frac{1}{T}\int_{-\infty}^{\infty} \delta(t)\exp\left(-j2\pi\left(\frac{k}{T}\right)t\right)dt = \frac{1}{T}$$

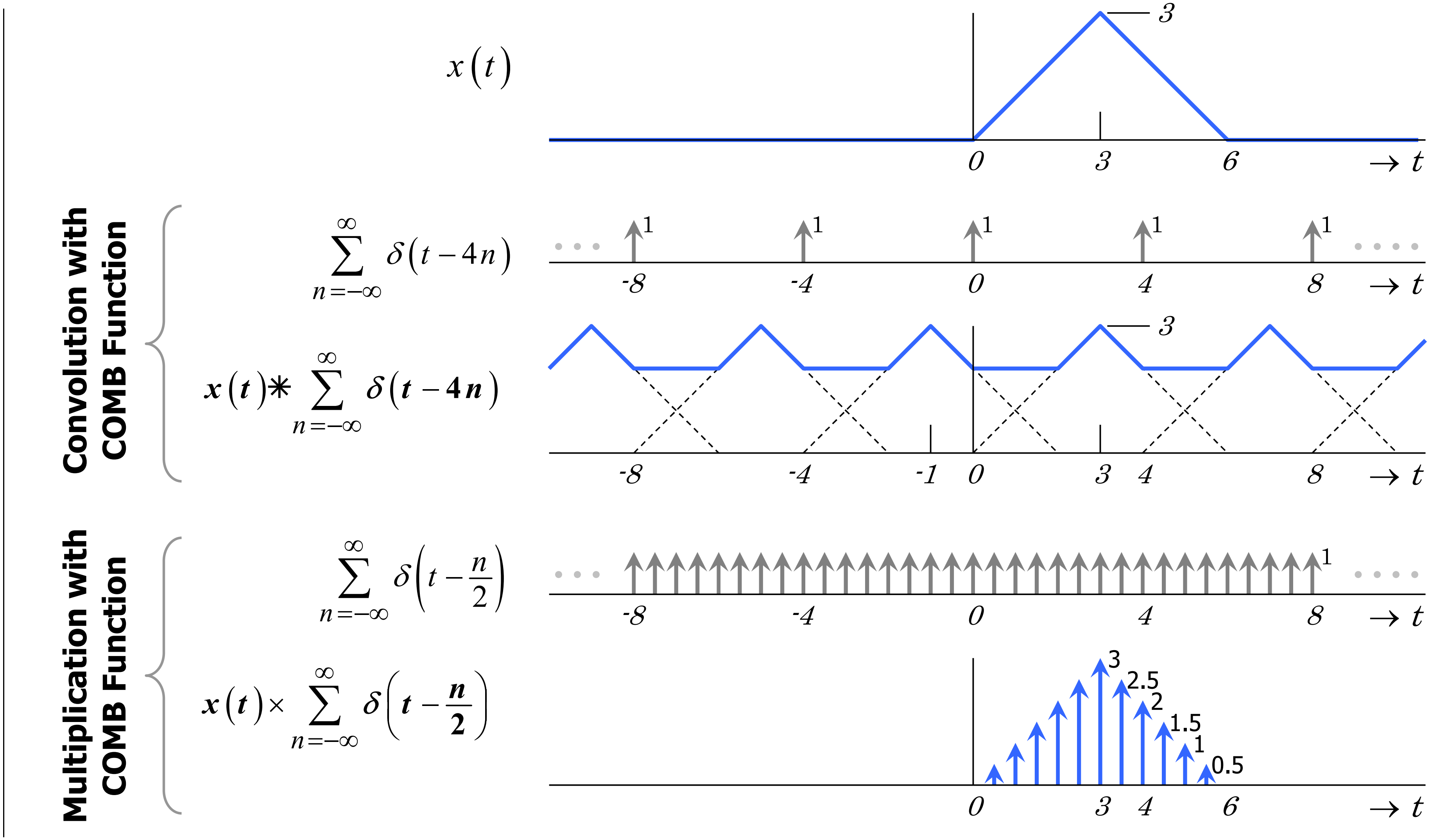


Continuous-frequency spectrum, $\Xi_T(f)$ $\left[\textbf{Fourier transform of } \xi_T(t)\right]$*:*

$$\Xi_T(f) = \sum_{k=-\infty}^{\infty} \Xi_{T,k}\,\delta\left(f - \frac{k}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right)$$



Example 3-8:


$x(t)$
3
0
3
6
$\to t$
Convolution with COMB Function
$\sum_{n=-\infty}^{\infty} \delta(t-4n)$
1
-8
-4
0
4
8
$\to t$
$x(t) * \sum_{n=-\infty}^{\infty} \delta(t-4n)$
3
-8
-4
-1
0
3
4
8
$\to t$
Multiplication with COMB Function
$\sum_{n=-\infty}^{\infty} \delta\left(t-\frac{n}{2}\right)$
1
-8
-4
0
4
8
$\to t$
$x(t) \times \sum_{n=-\infty}^{\infty} \delta\left(t-\frac{n}{2}\right)$
3
2.5
2
1.5
1
0.5
0
3
4
6
$\to t$