4. ESD, PSD and Bandwidth

4.1 Energy Spectral Density (ESD) - - - *a.k.a. Energy Spectrum*

Energy spectral density describes how the energy of a signal is distributed across its frequency components. In the time-domain, the energy of a signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \quad (Joules). \tag{4.1}$$

The Rayleigh's energy theorem provides us an alternate method for computing the energy of a signal in the frequency-domian, namely

$$E = \overbrace{\int_{-\infty}^{\infty} |x(t)|^2 \, dt}^{\textit{time-domain}} = \overbrace{\int_{-\infty}^{\infty} |\boldsymbol{X}(\boldsymbol{f})|^{\mathbf{2}} \, df}^{\textit{frequency-domain}} \quad \cdots\cdots \quad \textbf{Rayleigh Energy Theorem} \tag{4.2}$$

where $X(f) = \Im\{x(t)\}$ is the spectrum of the signal.

Since the integral on the right-hand side of (4.2) is the total energy of $x(t)$, the integrand $|X(f)|^2$ can be interpreted as the energy density of the signal at frequency f. In light of this, the energy spectral density of a signal $x(t)$ is defined as

$$E_x(f) = |X(f)|^2 \quad (Joules/Hz) \quad \cdots\cdots \quad \textbf{Energy Spectral Density}. \tag{4.3}$$

Properties of $E_x(f)$:

(a) $E_x(f)$ is a real function of f.

(b) $E_x(f) \ge 0 \quad \forall f$.

(c) $E_x(f)$ is an even function of f if $x(t)$ is real.

Reasons: - (a) & (b) are due to the fact that $|magnitude|$ is always real and non-negative.

- (c) is due to the fact that $|X(f)|$ is an even function of f because $x(t)$ is real.

Example 4-1:

Consider the energy signal $\mathbf{x(t) = 2\exp(-4t)u(t)}$.

Find the spectrum, $X(f)$, and energy spectral density, $E_x(f)$, of $x(t)$. Calculate the total energy, E, of $x(t)$ using the time-domain and frequency-domain definitions of E.

Spectrum:
$$\begin{cases} X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt = \int_{0}^{\infty} 2\exp(-4t)\exp(-j2\pi ft)\,dt \\ \quad = 2\left.\dfrac{\exp\left[-(j2\pi f+4)t\right]}{-(j2\pi f+4)}\right|_0^{\infty} = \dfrac{1}{2+j\pi f} \end{cases}.$$

ESD: $$E_x(f) = |X(f)|^2 = \left|\frac{1}{2+j\pi f}\right|^2 = \frac{1}{4+\pi^2 f^2}.$$

Energy calculated using the time-domain definition:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_0^{\infty} 4\exp(-8t)\,dt = \left.\frac{4\exp(-8t)}{-8}\right|_0^{\infty} = \frac{1}{2} \quad \cdots\cdots\cdots\cdots\cdots \quad (\clubsuit)$$

Energy calculated using the frequency-domain definition:

$$\left.\begin{aligned} E &= \int_{-\infty}^{\infty} E_x(f)\,df = \int_{-\infty}^{\infty} \frac{1}{4+\pi^2 f^2}\,df = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1}{1+(0.5\pi f)^2}\,df \\ &\cdots\cdots \quad let\ \tan(\theta) = 0.5\pi f, \ \therefore \sec^2(\theta)\,d\theta = 0.5\pi\,df \\ &= \frac{1}{4}\int_{-0.5\pi}^{0.5\pi} \frac{1}{1+\tan^2(\theta)} \frac{\sec^2(\theta)}{0.5\pi}\,d\theta = \frac{1}{4}\int_{-0.5\pi}^{0.5\pi} \frac{1}{0.5\pi}\,d\theta = \frac{1}{2} \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (\blacklozenge)$$

Clearly, the results in $(\clubsuit)$ and $(\blacklozenge)$ are consistent with the Rayleigh Energy theorem. The complexity of calculating E in the time- or frequency-domain is dependent on the nature of the signal and its spectrum. For the above signal, we observe that the calculation of E in the time-domain is simpler.

4.2 Power Spectral Density (PSD) - - - *a.k.a. Power Spectrum*

The definition of energy spectral density in Section 4.1 is applied to energy signals, of which the Fourier transforms exist.

For continuous-time signals that have infinite total energy, ~~namely~~ for example power signals, it makes more sense to define a power spectral density, which describes how the power of a signal is distributed across its frequency components.

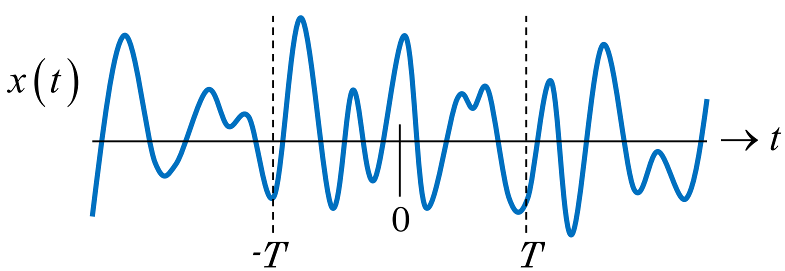
In the time-domain, power is defined as the time-average of the squared magnitude of the signal:

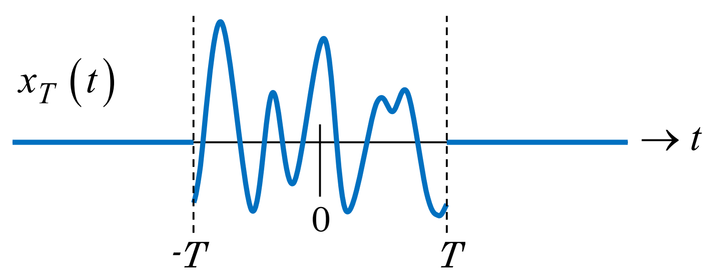
$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left| x(t) \right|^2 dt . \tag{4.4}$$

In analyzing the frequency content of a power signal, it is advantageous to begin with a truncated version of $x(t)$:

$$x_T(t) = x(t)\,\mathrm{rect}\left(\frac{t}{2T}\right) \tag{4.5}$$

noting that $\lim_{T \to \infty} x_T(t) = x(t)$.





Now, $x_T(t)$ is an energy signal which has a Fourier transform

$$X_T(f) = \int_{-\infty}^{\infty} x_T(t)\exp(-j2\pi ft)\,dt\,. \tag{4.6}$$

Applying the Rayleigh energy theorem given in (4.2), the energy of $x_T(t)$ can be expressed as

$$\int_{-\infty}^{\infty} |x_T(t)|^2\,dt = \int_{-\infty}^{\infty} |\boldsymbol{X}_T(\boldsymbol{f})|^{\boldsymbol{2}}\,df\,. \tag{4.7}$$

But $\int_{-\infty}^{\infty} |x_T(t)|^2\,dt = \int_{-T}^{T} |x(t)|^2\,dt$. Hence, we may rewrite (4.7) as

$$\int_{-T}^{T} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |\boldsymbol{X}_T(\boldsymbol{f})|^{\boldsymbol{2}}\,df \tag{4.8}$$

Dividing (4.8) by $2T$ and then taking the limit $T \to \infty$ leads to the Parseval power theorem:

$$\underbrace{P = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt}_{\text{Average Power [see (4.4)]}} = \int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{2T}|\boldsymbol{X}_T(\boldsymbol{f})|^{\boldsymbol{2}}\,df \quad \cdots\cdots \quad \textbf{Parseval Power Theorem}. \tag{4.9}$$

Since the integral on the right-hand side of (4.9) is the average power of $x(t)$, the integrand $\lim_{T\to\infty}\frac{1}{2T}\left|\boldsymbol{X}_T(\boldsymbol{f})\right|^{2}$ can be interpreted as the power density of the signal at frequency f. In light of this, the power spectral density of a signal $x(t)$ is defined as

$$P_x(f)=\lim_{T\to\infty}\frac{1}{2T}\left|\boldsymbol{X}_T(\boldsymbol{f})\right|^{2}\quad (Watts/Hz) \cdots\cdots \textbf{ Power Spectral Density}. \tag{4.10}$$

Properties of $P_x(f)$:

(a) $P_x(f)$ is a real function of f.

(b) $P_x(f)\geq 0 \quad \forall f$.

(c) $P_x(f)$ is an even function of f if $x(t)$ is real.

Reasons:
- (a) & (b) are due to the fact that $|magnitude|$ is always real and non-negative.
- (c) is due to the fact that $\left|X_T(f)\right|$ is an even function of f because $x_T(t)$ is real due to $x(t)$ being real.

Remarks:

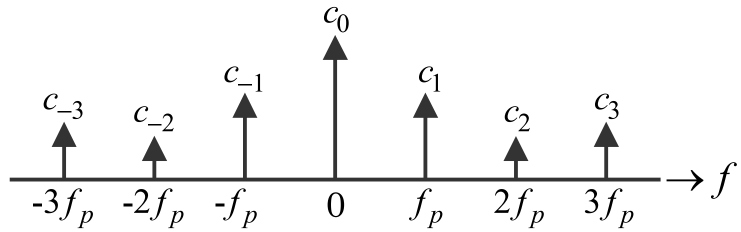
The derivation of power spectral density for a deterministic power signal involves solving (4.10), which may be difficult. However, if the signal is periodic, the solution becomes straightforward, as we shall see in Section 4.2.1.

4.2.1 PSD of Periodic Signals

Let f_p, T_p and c_k denote the fundamental frequency, period and Fourier series coefficient of a periodic signal $x_p(t)$, where c_k is in general complex even if $x_p(t)$ is real.

The continuous-frequency **spectrum** of $x_p(t)$ has the form

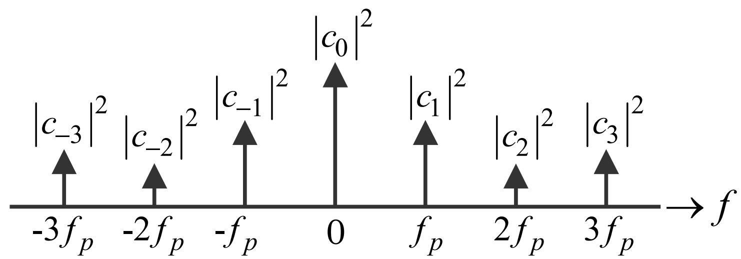
$$\underbrace{X(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)}_{see\ Chapter\ 3,\ equation\ (3.24)} \tag{4.11}$$



Claim:

Power Spectral Density of $x_p(t)$:

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta\left(f - \frac{k}{T_p}\right) \tag{4.12}$$



Average Power of $x_p(t)$:

$$P = \int_{-\infty}^{\infty} P_x(f)\,df = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{4.13}$$

Proof - (4.12) and (4.13) [optional]:

$$\boldsymbol{P} = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} \left|x_p(t)\right|^2 dt \qquad \left(\begin{array}{l} Since\ x_p(t)\ is\ periodic,\ its\ power\ may\ be\ obtained\ by \\ averaging\ over\ 1\ period. \end{array}\right)$$

$$= \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} \Im^{-1}\left\{\sum_{k=-\infty}^{\infty} c_k\delta\left(f - k/T_p\right)\right\}\left[\Im^{-1}\left\{\sum_{l=-\infty}^{\infty} c_l\delta\left(f - l/T_p\right)\right\}\right]^* dt$$

$$= \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p}\left[\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_k\delta\left(f - k/T_p\right)\cdot e^{j2\pi ft}df\right]\left[\int_{-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_l^*\delta\left(\tilde{f} - l/T_p\right)\cdot e^{-j2\pi\tilde{f}t}d\tilde{f}\right]dt$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^*\delta\left(f - k/T_p\right)\left[\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p}\left\{\int_{-\infty}^{\infty}\delta\left(\tilde{f} - l/T_p\right)e^{j2\pi\left(f-\tilde{f}\right)t}d\tilde{f}\right\}dt\right]\right\}df$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^*\delta\left(f - k/T_p\right)\left[\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} e^{j2\pi\left(f-l/T_p\right)t}dt\right]\right\}df$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^*\delta\left(f - k/T_p\right)\mathrm{sinc}\left(fT_p - l\right)\right\}df$$

$$= \int_{-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} c_k c_l^*\delta\left(f - k/T_p\right)\mathrm{sinc}\left(k - l\right)\right\}df$$

$$= \int_{-\infty}^{\infty}\underbrace{\left\{\sum_{k=-\infty}^{\infty}\left|c_k\right|^2\delta\left(f - k/T_p\right)\right\}}_{\mathbf{PSD:}\,\boldsymbol{P_x}(f)}df = \sum_{k=-\infty}^{\infty}\left|c_k\right|^2\int_{-\infty}^{\infty}\delta\left(f - k/T_p\right)df = \underbrace{\sum_{k=-\infty}^{\infty}\left|c_k\right|^2}_{\mathbf{Power:}\,\boldsymbol{P}}$$

Example 4-2:

Consider the periodic signal $\boldsymbol{x(t) = 2 + 4\exp(j8\pi t) + 6\cos(16\pi t)}$.

Find the spectrum, $X(f)$***, and power spectral density,*** $P_x(f)$***, of*** $x(t)$***. Calculate the average power,*** P ***, of*** $x(t)$***.***

$$x(t) = 2 + 4\exp(j8\pi t) + 6\cos(16\pi t) = 3\exp\left(j2\pi(-8)t\right) + 2 + 4\exp\left(j2\pi(4)t\right) + 3\exp\left(j2\pi(8)t\right)$$

$$\cdots\ Fundamental\ frequency:\ \mathrm{HCF}\{4,8\} = 4\ Hz$$

Comparing the above with the Fourier series expansion $x(t) = \sum_k c_k \exp\left(j2\pi(4k)t\right)$*, we have*

$$c_k = \begin{cases} 2; & k = 0 \\ 4; & k = 1 \\ 3; & k = \pm 2 \\ 0; & otherwise \end{cases} \qquad \cdots\cdots\ by\ inspection$$

Spectrum: $X(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - 4k) = 3\delta(f+8) + 2\delta(f) + 4\delta(f-4) + 3\delta(f-8)$.

PSD: $P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - 4k) = 9\delta(f+8) + 4\delta(f) + 16\delta(f-4) + 9\delta(f-8)$.

Power: $P = \int_{-\infty}^{\infty} P_x(f)\,df = 9 + 4 + 16 + 9 = 38\ \cdots \left(which\ is\ essentially\ \sum_k |c_k|^2\right)$.

4.3 Bandwidth

The bandwidth of a signal $x(t)$ is a measure of the width of the range of frequencies occupied by its spectrum $|X(f)|$.

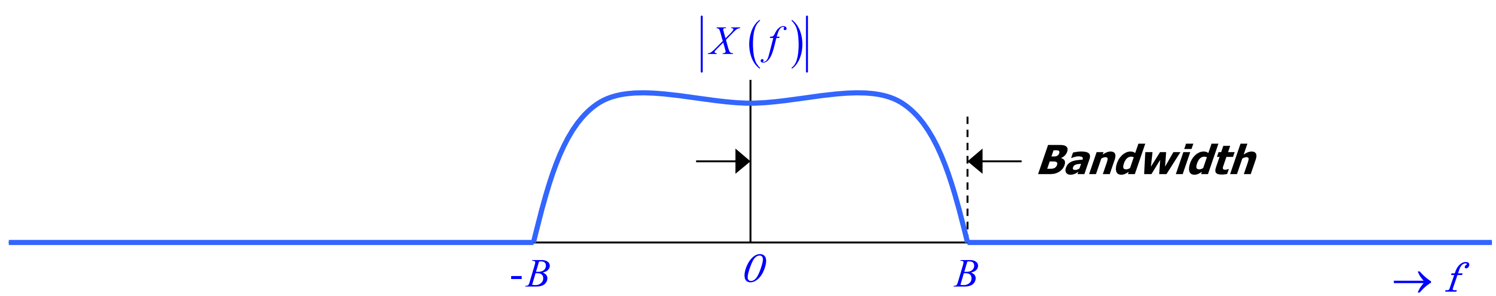
4.3.1 Bandlimited Signals

LOWPASS SIGNAL

A signal $x(t)$ is said to be a **bandlimited lowpass signal** if all of its frequency components are zero above a certain finite frequency, i.e.

$$\left|\boldsymbol{X}(\boldsymbol{f})\right| = \boldsymbol{0}; \quad |\boldsymbol{f}| > \boldsymbol{B} \tag{4.14}$$

where $\boldsymbol{B}$ is called the **bandwidth** of the signal.



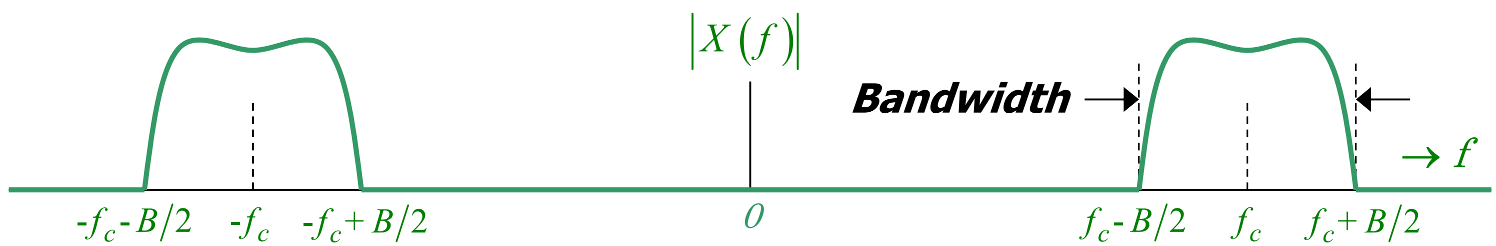
Note: *When* $x(t)$ *is real,* $|X(f)|$ *is symmetric about* $f=0$.

BANDPASS SIGNAL

A signal $x(t)$ is said to be a **bandlimited bandpass signal** if all of its frequency components are zero outside a certain finite frequency range, i.e.

$$|X(f)| = 0; \quad |f| < f_c - B/2 \quad or \quad |f| > f_c + B/2 \tag{4.15}$$

where f_c and B are, respectively, called the **center frequency** and **bandwidth** of the signal.



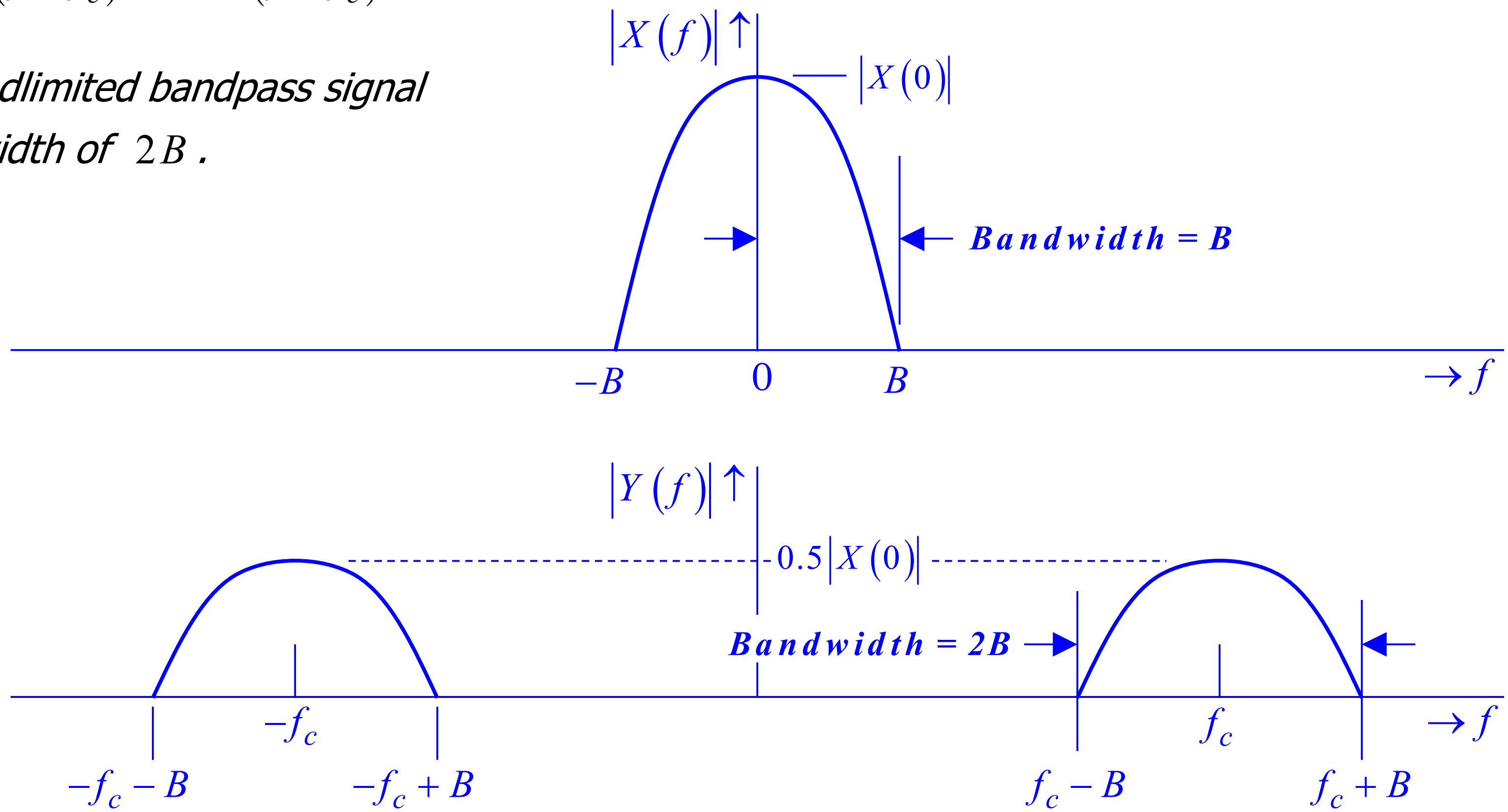
Note: *When $x(t)$ is real, $|X(f)|$ is symmetric about $f = 0$. To convenient discussion, we have further assume that $|X(f)|u(f)$ is symmetric about $f = f_c$, and $|X(f)|u(-f)$ is symmetric about $f = -f_c$. This is usually the case in many practical situations.*

Example 4-3:

Let B be the bandwidth of a bandlimited lowpass signal $x(t)$. Express the bandwidth of $y(t) = x(t)\cos(2\pi f_c t)$ in terms of B, assuming that $f_c >> 2B$.

$$Y(f) = X(f) * 0.5\left[\delta(f - f_c) + \delta(f + f_c)\right]$$
$$= 0.5X(f - f_c) + 0.5X(f + f_c)$$

$y(t)$ is a bandlimited bandpass signal with a bandwidth of $2B$.



4.3.2 Signals with Unrestricted Band

In general, practical signals are seldom strictly bandlimited but have infinite frequency extent. Such signals are said to have unrestricted band.

The concept of infinite bandwidth presents difficulties in signal processing. For instant, consider the propagation of a signal with unrestricted band through a system with bandwidth B_s:

- If B_s is finite, the signal spectrum will be truncated and this may lead to an unacceptable level of signal distortion. In order to avoid signal distortion, B_s must be greater or equal to the signal bandwidth.

- If B_s approaches infinity to accommodate the signal spectrum, the system noise will approach infinity. In this case, the system noise will completely mask out the signal. (Here, we are referring to white noise inherently generated by the system, where the noise power is proportional to B_s.)

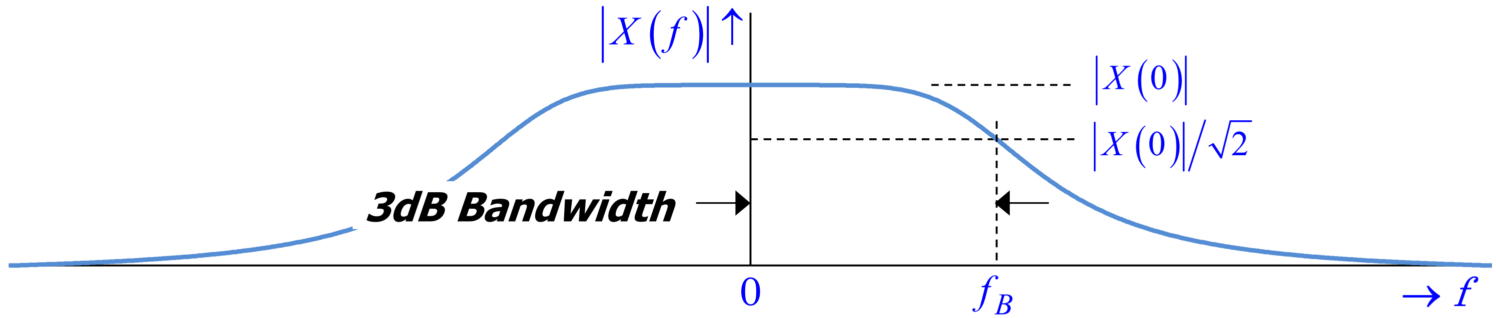
In signal processing, it is often useful to define a bandwidth measure to include only the "important part" of the signal spectrum. This bandwidth measure can then be used to estimate the bandwidth of the signal. It is important to understand that there is no bandwidth measure that is universally applicable to all signals. The choice of bandwidth measure is dependent on what we consider as the "important part" of the signal spectrum.

In this section, we will introduce the notions of $\mathbf{3}\boldsymbol{dB}$ bandwidth and *1*st***-null*** bandwidth.

- **$3dB$ BANDWIDTH**

LOWPASS SIGNAL $x(t)$:

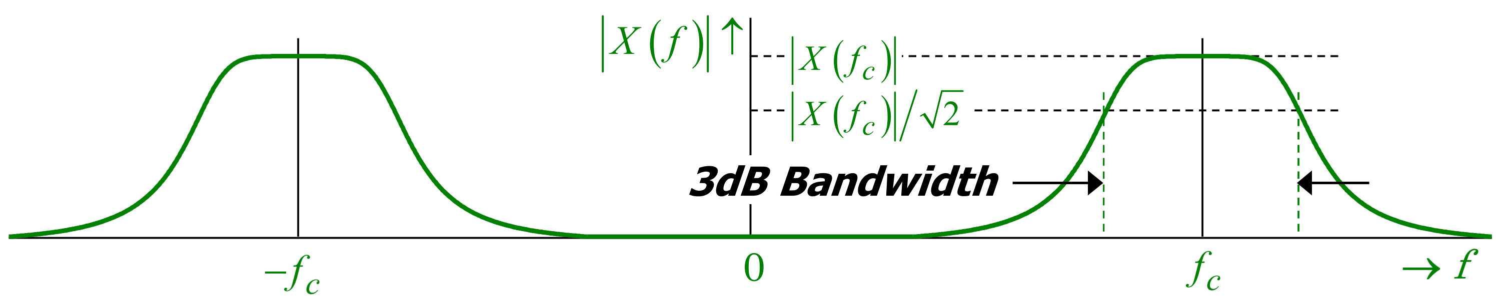
The 3dB bandwidth of a lowpass signal $x(t)$ is defined as the frequency at which $|X(f)| = |X(0)|/\sqrt{2}$ first occurs when f is increased from 0:



f_B is called the $3dB$ frequency because $20\log_{10}\left(\left|X(f_B)/X(0)\right|\right) = 20\log_{10}\left(1/\sqrt{2}\right) \approx -3.01dB$.

BANDPASS SIGNAL $x(t)$:

Likewise, the 3dB bandwidth of a bandpass signal $x(t)$ with center frequency f_c is defined as illustrated below:



Example 4-4:

Consider the lowpass Gaussian pulse $\mathbf{x}(t)=\exp\left(-t^2/2\right)$ which has an energy spectral density given by

$$\boldsymbol{E_x}(\boldsymbol{f})=\mathbf{2}\pi\exp\left(-\mathbf{4}\pi^{\mathbf{2}}\boldsymbol{f}^{\mathbf{2}}\right).$$

Find the 3-dB bandwidth of $\mathbf{x}(t)$.

Let f_B be the 3-dB bandwidth of $x(t)$. By definition,

$$\frac{|X(f_B)|}{|X(0)|}=\frac{1}{\sqrt{2}} \quad \textit{or} \quad \frac{|X(f_B)|^2}{|X(0)|^2}=\frac{1}{2}$$

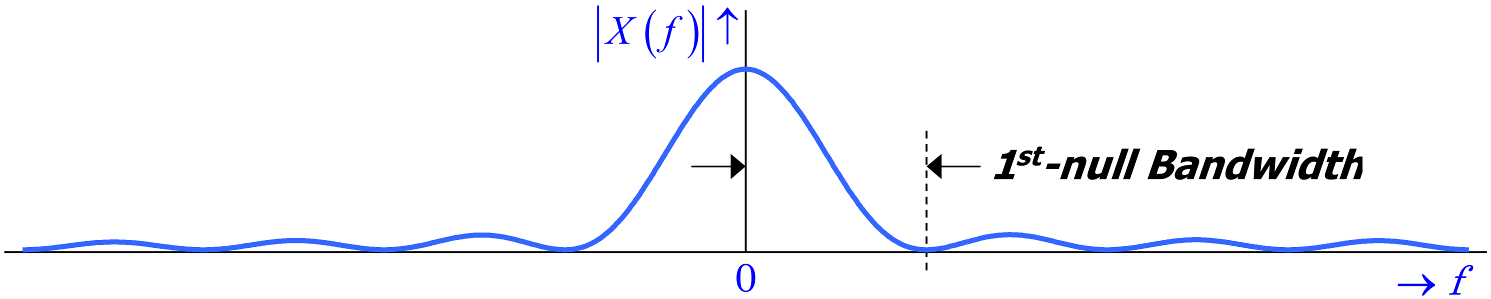
Since $E_x(f)=|X(f)|^2$, it follows that

$$\left.\frac{E_x(f_B)}{E_x(0)}=\frac{2\pi\exp\left(-4\pi^2 f_B^2\right)}{2\pi}=\frac{1}{2}\right\} \rightarrow f_B=\frac{\sqrt{\ln(2)}}{2\pi}$$

- **1^{st}-null BANDWIDTH**

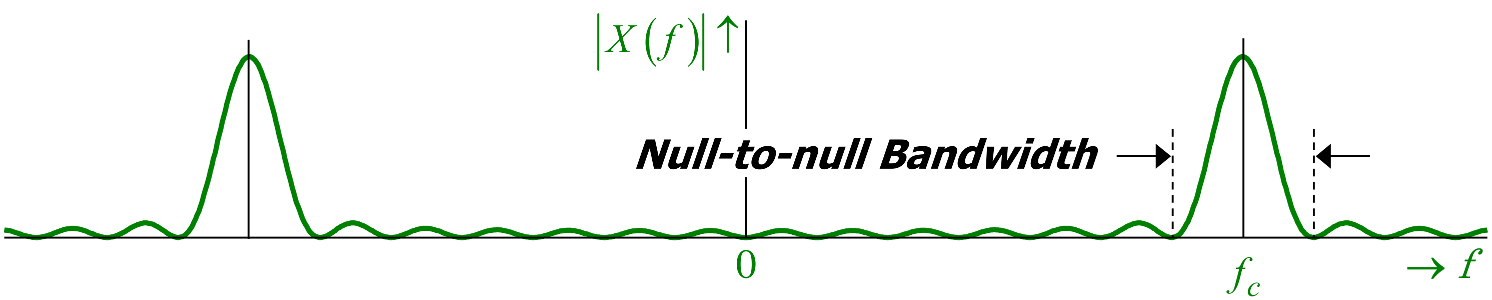
LOWPASS SIGNAL $x(t)$:

The 1^{st}-null bandwidth of a lowpass signal $x(t)$ is defined as the frequency at which $|X(f)| = 0$ first occurs when f is increased from 0:



BANDPASS SIGNAL $x(t)$:

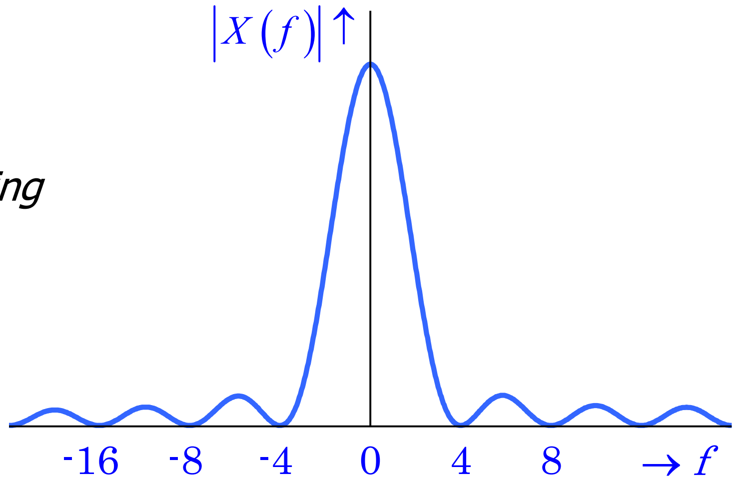
Likewise, the 1^{st}-null (a.k.a. null-to-null) bandwidth of a bandpass signal $x(t)$ with center frequency f_c is defined as illustrated below:



Example 4-5:

What is the 1st-null bandwidth of $x(t) = 5 \cdot \mathrm{tri}(4t-8)$?

Note that $x(t) = 5 \cdot \mathrm{tri}(4t-8) = 5 \cdot \mathrm{tri}\left(\dfrac{t-2}{0.25}\right)$



Applying $\Im\left\{\mathrm{tri}\left(\dfrac{t}{T}\right)\right\} = T\,\mathrm{sinc}^2(Tf)$ *and the time-shifting property of Fourier transform, we obtain*

$$X(f) = \frac{5}{4} \cdot \mathrm{sinc}^2\left(\frac{f}{4}\right) e^{-j4\pi f}$$

which yields a magnitude spectrum of

$$|X(f)| = \frac{5}{4} \cdot \mathrm{sinc}^2\left(\frac{f}{4}\right).$$

The nulls of $|X(f)|$ *occur at* $f = \pm 4, \pm 8, \pm 12, \cdots\cdots\cdots$ *Hz.*

Since the 1st-null of $|X(f)|$ *occurs at 4 Hz, the 1st-null bandwidth of* $x(t)$ *is* $4\ Hz$.