National University of Singapore

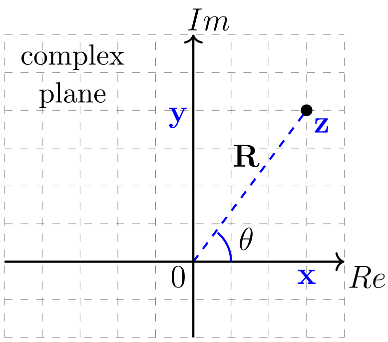
Department of Electrical & Computer Engineering

EE2023 Signals & Systems Revision Notes

1 Complex Numbers

Complex numbers play a very important role in the analysis of signals and electrical circuits in electrical engineering and elsewhere. Hence it is important that you are familiar with the correct manipulation of complex numbers and complex functions.

Complex numbers, z, are numbers that take the following form $z = x + jy$ where $j = \sqrt{-1}$. x and y denote the real and imaginary parts of z, respectively and they are both real numbers. z can be plotted on a complex plane as follows :



$\Longrightarrow$ Cartesian and polar coordinates of z :

$$z = x + jy = Re^{j\theta} = R\cos\theta + jR\sin\theta$$

$$R = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

It is clear that the point z is some distance away from the origin of the complex plane. This distance is given by $R = \sqrt{x^2 + y^2}$, and the line between the origin and z (blue dashed line) makes an angle, θ, with the real axis. This angle is given by $\theta = \tan^{-1}(y/x)$.

z can now be written in polar coordinates : $z = Re^{j\theta}$. The relationships between x, y, R and θ are as follows:

- $z = x + jy = Re^{j\theta} = R\cos\theta + jR\sin\theta$. Hence $x = R\cos\theta$, $y = R\sin\theta$.
- Magnitude of z, denoted as $|z| = R = \sqrt{x^2 + y^2} > 0$
- Argument or phase of z, denoted as $\angle z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$. θ may be in radians or degrees. However, in the polar coordinates, θ should be in radians.

1.1 Multiplication and Division of Complex Numbers

Since complex numbers can be expressed in polar coordinates, there are some convenient ways to deal with the multiplication and division of 2 complex numbers. This is illustrated as follows :

Suppose you have two complex numbers : $z_1 = x_1 + jy_1 = R_1e^{j\theta_1}, z_2 = x_2 + jy_2 = R_2e^{j\theta_2}$. What are the values of $z_3 = z_1 \times z_2$ and $z_4 = z_1/z_2$? What are their magnitudes and phases?

In Cartesian coordinates,

$$\begin{aligned} z_3 &= z_1 \times z_2 \\ &= (x_1 + jy_1) \times (x_2 + jy_2) \\ &= (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) \\ |z_3| &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\ \angle z_3 &= \tan^{-1}\left(\frac{x_1y_2 + x_2y_1}{x_1x_2 - y_1y_2}\right) \end{aligned}$$

In polar coordinates,

$$\begin{aligned} z_3 &= z_1 \times z_2 \\ &= R_1e^{j\theta_1} \times R_2e^{j\theta_2} \\ &= R_1R_2e^{j(\theta_1+\theta_2)} \\ |z_3| &= R_1R_2 & (1) \\ \angle z_3 &= \theta_1 + \theta_2 & (2) \end{aligned}$$

From (1) and (2), what can you conclude about the ease of multiplying two complex numbers - should you use the cartesian or polar coordinates when you are multiplying 2 complex numbers? This example shows that when 2 complex numbers multiply together, their magnitudes also multiply together but their phases add. This gives an extremely convenient way of dealing with the multiplication of complex numbers which should always be done in polar coordinates and not Cartesian coordinates.

Likewise, if you divide two complex numbers, you should always do so using polar coordinates as the following shows :

In Cartesian coordinates,

$$\begin{aligned} z_4 &= z_1/z_2 \\ &= \frac{(x_1 + jy_1)}{(x_2 + jy_2)} \\ &= \text{? complicated expression} \\ |z_4| &= \sqrt{\text{complicated expression}} \\ \angle z_4 &= \tan^{-1}(\text{complicated expression}) \end{aligned}$$

In polar coordinates,

$$\begin{aligned} z_3 &= z_1/z_2 \\ &= \frac{R_1e^{j\theta_1}}{R_2e^{j\theta_2}} \\ &= \frac{R_1}{R_2}e^{j(\theta_1-\theta_2)} \\ |z_4| &= \frac{R_1}{R_2} & (3) \\ \angle z_4 &= \theta_1 - \theta_2 & (4) \end{aligned}$$

From (3) and (4), the magnitude $|z_4| = R_1/R_2 = |z_1|/|z_2|$ and phase $\angle z_4 = \theta_1 - \theta_2 = \angle z_1 - \angle z_2$, once again shows the convenience of dealing with division of complex numbers using polar coordinates.

In summary, it is important to note that :

1. If $z = z_1 \times z_2 \times z_3 ... \times z_n$, then
   - z has a magnitude $|z| = R_1 R_2 R_3 .... R_n$ where $|z_i| = R_i$.
   - $\angle z = \theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n$ where $\angle z_i = \theta_i$.
2. If $z = z_1/z_2$, then
   - z has a magnitude $|z| = R_1/R_2$ where $|z_i| = R_i$.
   - $\angle z = \theta_1 - \theta_2$ where $\angle z_i = \theta_i$
3. When you have a complex number $z = C/z_1$, where C is a positive real constant, then
   - $|z| = C/R_1$ where $R_1 = |z_1|$
   - $\angle z = \angle(C) - \angle z_1 = 0 - \theta_1 = -\theta_1$ because the phase of a positive C is zero.
   - If $C < 0$, then $\angle z = \angle(C) - \angle z_1 = \pm\pi - \theta_1$ because the phase of a negative C is either $+\pi$ or $-\pi$.

Exercise 1. *Find the magnitudes and phases of the following complex numbers :*

(a) $z = \frac{1+j1}{1+2j}$

(b) $z = (1 + j1) \times (1 + 2j)$

(c) $z = \frac{1-j1}{1+2j}$

(d) $z = \frac{1+j1}{1-2j}$

(e) $z = (-1 + j1) \times (1 + 2j)$

(f) $z = (1 + j1) \times (-1 + 2j)$

In complex numbers, there is also the idea of complex conjugate which is denoted by z^*. If we have a complex number $z = x + jy$ where x and y are real numbers, then its conjugate is $z^* = x - jy$. Notice that the imaginary part is negated. If the complex number is $z = x - jy$, then $z^* = x + jy$ is the conjugate of z. What then is the relationship between z and z^* in terms of the magnitude and phases of these two complex numbers? We demonstrate this through an example.

Example 1. *Suppose $z = 1 + 2j$. What is z^*? What are the magnitudes and phases of z and z^*?*

Solution

$$\begin{aligned} z &= 1 + 2j \\ &= \sqrt{1^2 + 2^2}\, e^{j\tan^{-1}(2)} \\ &= \sqrt{5}\, e^{j1.1} \end{aligned} \qquad \begin{aligned} z^* &= 1 - 2j \\ &= \sqrt{1^2 + 2^2}\, e^{j\tan^{-1}(-2)} \\ &= \sqrt{5}\, e^{-j1.1} \end{aligned}$$

What can you conclude about the magnitudes and phases of z and z^*? It is obvious that the conjugates of any complex number will have the same magnitudes as the original complex number, but its phase will be negative of one another. So in general you have the following :

$$\begin{aligned} z &= x + jy \\ |z| &= \sqrt{x^2 + y^2} = |z^*| \\ \angle z &= \tan^{-1}\frac{y}{x} \\ &= -\angle z^* \end{aligned} \qquad \begin{aligned} z^* &= x - jy \\ |z^*| &= \sqrt{x^2 + y^2} = |z| \\ \angle z^* &= -\tan^{-1}\frac{y}{x} \\ &= -\angle z \end{aligned}$$

Hence complex numbers and their conjugates have the same magnitudes but their phases are negative of one another.

1.2 Beware of the Phases of Complex Numbers

In this section, for convenience, x and y are assumed positive, unless otherwise stated. We will first take a look at some special complex numbers in Table 1 and these are also illustrated on the complex plane in Figure 1.

Table 1: Examples of Complex Numbers and their Polar Coordinates

Complex No.	Cartesian Coord	Polar Coord	Magnitude	Phase
$z_1 = j$	$z_1 = 0 + j$	$z = e^{j0.5\pi}$	$\|z_1\| = 1$	$\angle z_1 = 0.5\pi = 90^0$
$z_2 = -j$	$z_2 = 0 - j$	$z_2 = e^{-j0.5\pi}$	$\|z_2\| = 1$	$\angle z_2 = -0.5\pi = -90^0$
$z_3 = 1/j = -j$	$z_3 = 0 - j$	$z_3 = e^{-j0.5\pi}$	$\|z_3\| = 1$	$\angle z_3 = -0.5\pi = -90^0$
$z_4 = -1.5/j = 1.5j$	$z_4 = 0 + 1.5j$	$z_4 = 1.5e^{j0.5\pi}$	$\|z_4\| = 1.5$	$\angle z_4 = 0.5\pi = 90^0$
$z_5 = j \times j = -1$	$z_5 = -1 + 0j$	$z_5 = e^{\pm j\pi}$	$\|z_5\| = 1$	$\angle z_5 = \pm\pi = \pm 180^0$
$z_6 = 1.5$	$z_6 = 1.5 + j0$	$z_6 = 1.5e^{j0}$	$\|z_6\| = 1.5$	$\angle z_6 = 0$
$z_7 = 1 + j$	$z_7 = 1 + 1j$	$z_7 = \sqrt{2}e^{j0.25\pi}$	$\|z_7\| = \sqrt{2}$	$\angle z_7 = 0.25\pi = 45^0$
$z_8 = -1 - j$	$z_8 = -1 - j$	$z_8 = \sqrt{2}e^{-j0.75\pi}$	$\|z_8\| = \sqrt{2}$	$\angle z_8 = -0.75\pi = -135^0$

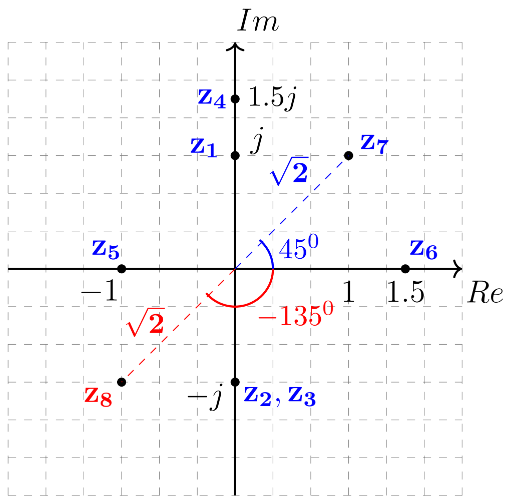


Fig. 1: Points z_1 to z_8 on the complex plane

Following this, we show some examples of how complex numbers can be converted between Cartesian (in terms of x and y) and polar coordinates (in terms of R and θ). This skill is required when you deal with frequency response plots and other calculations involving complex numbers in which familiarity in the conversions is crucial. The key problem lies in the computation of the phase values.

Example 2. *Suppose you are given a complex number $z = (1+2j)^2$. What is the magnitude and phase of z?*

<u>*Solution*</u> *: First convert $(1+2j)$ into polar form :*

$$
\begin{aligned}
|(1+2j)| &= \sqrt{1^2+2^2} = \sqrt{5} \\
\angle(1+2j) &= \tan^{-1}(2) = 1.11\,\text{rad} = \left(1.11 \times \frac{180}{\pi}\right)^0 = 63.45^0 \quad (\pi \text{ rad} = 180^0) \\
1+2j &= \sqrt{5}e^{j1.11} \\
z &= (1+2j)^2 = \left(\sqrt{5}e^{j1.11}\right)^2 = (\sqrt{5}\times\sqrt{5})e^{j(2\times 1.11)} = 5e^{j2.22} \\
|z| &= 5, \quad \angle z = 2.22 \text{ radians} = 126.9^0
\end{aligned}
$$

z is shown on the complex plane in Figure 2.

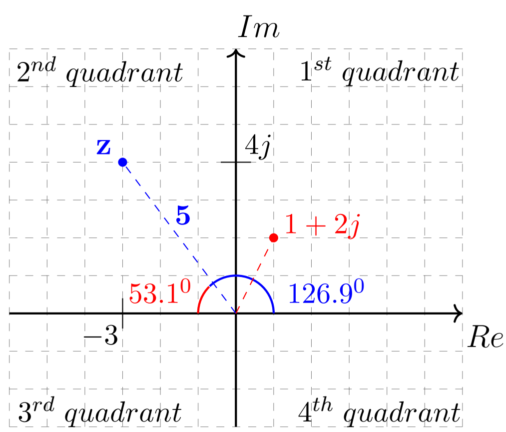


Fig. 2: Point z on the complex plane

The idea here is not to multiply $(1+2j)\times(1+2j)$ because doing so may lead to careless mistakes in the multiplication and eventually also issues with calculation of phase. The computation below illustrates this problem.

$$
\begin{aligned}
z &= (1+2j)^2 = (1+2j)\times(1+2j) = 1-4+2j+2j = -3+4j \\
|z| &= \sqrt{(-3)^2+4^2} = 5,\ \angle z = \tan^{-1}(-4/3)
\end{aligned}
$$

From the calculator, $\tan^{-1}(-4/3) = -0.927$ rad which is equivalent to -53.1^0 (4^{th} quadrant). Clearly this phase is not correct because $z = -3+4j$ (where the real part is negative and imaginary part is positive) is in the 2nd quadrant of the complex plane and hence its phase should be 126.9^0.

Calculators will always give arctan values between $-0.5\pi < \theta < 0.5\pi$ *which means that the phase angles given by calculators are only in the first or fourth quadrant. However, in this case,* $z = -3 + 4j$ *is in the second quadrant. Therefore, the value obtained from the calculator will not be the correct phase and has to be adjusted by making this adjustment : actual phase* $= 180^0 - 53.1^0 = 126.9^0$.

Some tips for ensuring the correct calculation of phase of a complex number.

- $z = (x + jy)^2$: $|z| = |x + jy|^2 = x^2 + y^2$ and $\angle z = 2 \times \angle(x + jy) = 2\tan^{-1}\frac{y}{x}$.
- $z = \frac{1}{x+jy}$: $|z| = \frac{1}{|x+jy|} = \frac{1}{\sqrt{x^2+y^2}}$ and $\angle z = \angle 1 - \angle(x + jy) = 0 - \tan^{-1}\frac{y}{x} = -\tan^{-1}\frac{y}{x}$.
- $z = -x + jy$: $|z| = \sqrt{x^2 + y^2}$ and $\angle z = \tan^{-1}\frac{y}{-x} = -\tan^{-1}\frac{y}{x}$.

  In this case, $z = -x + jy$ is in the second quadrant assuming x and y are positive. Therefore, the value obtained from the calculator will not be the correct phase and has to be adjusted as follows :

$$\angle z = \pi - \tan^{-1}\frac{y}{x} \text{ or } -(\pi + \tan^{-1}\frac{y}{x}).$$

- $z = x - jy$: $|z| = \sqrt{x^2 + y^2}$ and

$$\angle z = \tan^{-1}\frac{-y}{x} = -\tan^{-1}\frac{y}{x} \tag{5}$$

  The phase given by the calculator will be correct since this complex number lies in the fourth quadrant.

- $z = \frac{1}{x-jy}$: $|z| = \frac{1}{\sqrt{x^2+y^2}}$ and $\angle z = -\left\{\tan^{-1}\frac{-y}{x}\right\}$. It is not easy to see what the phase should be without further manipulation as follows :

$$\begin{aligned} z &= \frac{1}{x - jy} \\ &= \frac{1}{x - jy} \times \frac{x + jy}{x + jy} \\ &= \frac{x + jy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + j\frac{y}{x^2 + y^2} \end{aligned} \tag{6}$$

  Hence, although $z = \frac{1}{x-jy}$, z is actually a complex number in the first quadrant, deducing from (6) where both the real and imaginary parts are positive. Thus

$$\angle z = -\left\{\tan^{-1}\frac{-y}{x}\right\} = \tan^{-1}\frac{y}{x}.$$

Another way to determine the phase is to consider :

$$\begin{aligned} z &= \frac{1}{x - jy} \\ &= \frac{1}{Re^{-j\theta}} = \frac{1}{R}e^{j\theta} \end{aligned}$$

where $-\theta$ is the angle in the fourth quadrant as in (5) above. Thus $\angle z = \theta$ which is in the first quadrant : same conclusion earlier.

Exercise 2.

(a) Convert $z = -3 - 4j$ *into polar coordinates*

(b) Convert $z = 3 + 4j$ *into polar coordinates*

(c) Convert $z = (-3 - 4j)^3$ *into polar coordinates*

(d) Convert $z = (-3 - 4j)^{1/3}$ *into polar coordinates*

(e) Convert $z = \dfrac{1}{-3 - 4j}$ *into polar coordinates*

(f) Convert $z = 5e^{-j2}$ *into Cartesian coordinates*

(g) Convert $z = \left(5e^{-j2}\right)^{0.5}$ *into Cartesian coordinates*

(h) Find all the roots of $z = (-128j)^{1/5}$. *There should be 5 roots.*

By now, you should appreciate the difficulties involved in determining the polar coordinates of complex numbers, particularly with respect to the phase values. The key here is to identify on which quadrant the complex number lies and then adjust your phase computations accordingly. Some calculators are able to give the correct phase if the complex number is presented in cartesian coordinates.

In the next section, we will explore complex exponential functions. As you recall, functions are not the same as numbers. Functions give a series of numbers generated when a parameter(s) in the function changes. For example, in a straight line equation ($y = mx + c$), you have the independent parameter, x. Changing x generates values of y which can be plotted against the x-axis. Hence we say that y is a function of x because the values of y depend on x. In the following section, we will look at functions of time t, which gives complex values. Specifically, we are interested in complex exponential functions of time which you will encounter in EE2023.

2 Complex Exponential Functions

In EE2023, we often deal with functions in time that are of the form $Ae^{\alpha t}$ where A and α are constants and α may be a complex number. For example, you may have $y_1(t) = 2e^{3t}$ or $y_2(t) = 2e^{j3t}$. In this example, do you know the difference between $y_1(t)$ and $y_2(t)$? What do they look like if you plot their values against the t-axis?

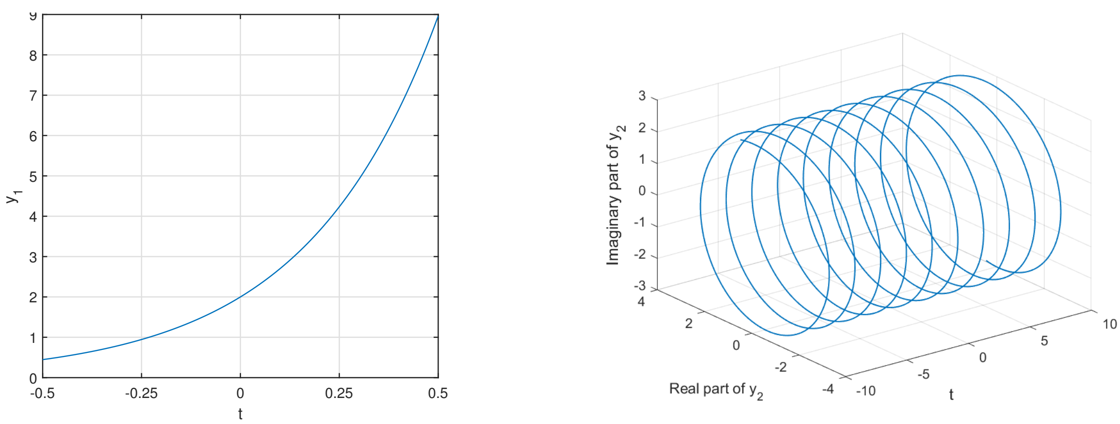


Fig. 3: Plots of $y_1(t)$ and $y_2(t)$

Notice how different the two plots in Figure 3 look. One is a 2D plot while the other is a 3D plot. Why is this so?

The values of $y_1(t)$ and $y_2(t)$ are evaluated in Table 2. It is clear that all values of $y_1(t)$ are positive real while $y_2(t)$ are mostly complex. Hence it is easy to plot $y_1(t)$ on a 2D plot (see left figure on Figure 3) since $\angle y_1(t) = 0$ when $y_1(t) > 0$. Any real positive number sits on the positive real axis on the complex plane and hence their phases are always zero.

Table 2 shows that $y_2(t)$ is complex for most values of t. Hence they have magnitudes and phases. Thus $y_2(t)$ cannot be visualized on a normal 2D plot. You need a 3D plot because, aside from t, $y_2(t)$ has real part and imaginary parts. In fact the 3D plot is a 'spiral' of radius

Table 2: Values of $y_1(t)$ and $y_2(t)$

t	0	1	2	3	...	t	Remarks
$y_1(t)$	2	$2e^3$	$2e^6$	$2e^9$	...	$2e^{3t}$	real, positive values
$\|y_1(t)\|$	2	$2e^3$	$2e^6$	$2e^9$	...	$2e^{3t}$	growing exponentially, always > 0
$\angle y_1(t)\|$	0	0	0	0	...	0	zero phase for all t
$y_2(t)$	2	$2e^{3j}$	$2e^{6j}$	$2e^{9j}$	...	$2e^{3jt}$	mostly complex values
$\|y_2(t)\|$	2	2	2	2	...	2	constant $\|y_2(t)\| = 2$ for all t
$\angle y_2(t)$	0	3	6	9	...	$3t$	radians - increases linearly with t

2 which corresponds to the magnitude of $y_2(t)$ when you view along the t-axis - see right figure in Figure 3. If $y_2(t)$ is projected on the complex plane, you will only see a circle of radius 2. You can thus imagine that a complex exponential signal $y_2(t)$ is a rotating phasor when viewed on the complex plane. See Figure 4.

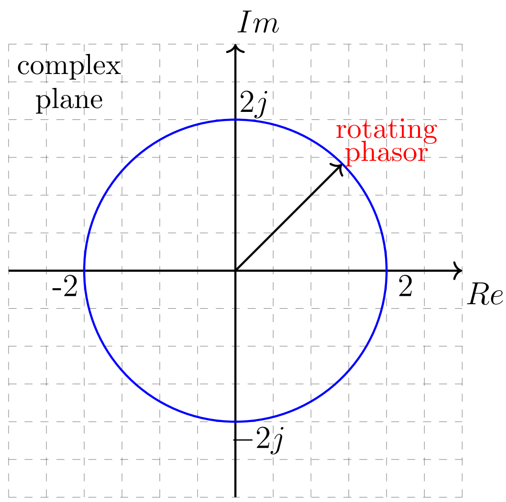


Fig. 4: $y_2(t)$ projected on the complex plane, seen as a rotating phasor

The rotating phasor depicts the phase of the complex number as t changes. The rotation is due to the increasing phase ie $\angle y_2(t) = 3t$. As t increases, $\angle y_2(t)$ increases but since $|y_2(t)| = 2$, the complex numbers that are generated by $y_2(t)$ as t changes, are points on the circle in Figure 4 with phase $3t$ rad. This also means that the phase increases by 3 radians per second ie for every second, the phase changes by 3 radians in the anti-clockwise direction since phase angles are positive as you move in an anti-clockwise direction. "3 rad/sec" is also a measure of angular frequency. We therefore conclude that the complex exponential signal $y_2(t)$ is a complex sinusoid with frequency 3 rad/sec.

In general, when you see a complex exponential of the form $s(t) = Ae^{j\omega t}$, $A > 0$, you can safely conclude that $|s(t)| = A$ while the frequency of $s(t)$ is ω rad/sec, generating phases, $\angle s(t) = \omega t$ rad, as t changes. You can visualize $s(t)$ as a rotating phasor, similar to that in Figure 4.

It is worth noting that complex exponential signals do not occur in practice. However, you will encounter such hypothetical signals often in EE2023 because they can be suitably combined to form real sinusoidal signals. For example :

$$A\cos(\omega t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \tag{7}$$

$$R\sin(\omega t) = \frac{R}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \tag{8}$$

You have now seen the difference between $2e^{3t}$ and $2e^{j3t}$. What about a signal $y_3(t) = e^{(-1+2j)t}$? What are the magnitudes and phases of the complex numbers generated by $y_3(t)$ as t changes?

Example 3. *Suppose $y_3(t) = e^{(-1+2j)t}$. What are the magnitudes and phases of $y_3(t)$?*

Table 3: Table of values of $y_3(t)$

t	0	1	2	3	$\ldots$	t
$y_3(t)$	$e^{(-1+2j)0}$ $=1$	$e^{(-1+2j)1}$ $= e^{-1}e^{2j}$	$e^{(-1+2j)2}$ $= e^{-2}e^{4j}$	$e^{(-1+2j)3}$ $= e^{-3}e^{6j}$	$\ldots$ $\ldots$	$e^{(-1+2j)t}$ $= e^{-t}e^{j2t}$
$\lvert y_3(t)\rvert$	1	e^{-1}	e^{-2}	e^{-3}	$\ldots$	e^{-t}
$\angle y_3(t)$	0	2	4	6	$\ldots$	$2t$

$y_3(t)$ are complex numbers for all t except at $t = 0$. Hence $y_3(t)$ cannot be plotted on a 2D plot but it can be plotted on a 3D plot as shown in left figure in Figure 5. The 3D plot looks strange because $|y_3(t)| \to 0$ exponentially as $t \to \infty$. It starts at $y_3(0) = 1$. When projected on the complex plane, the phasor diagram looks like the right figure in Figure 5. It shows the plot spiralling into the origin as $t \to \infty$.

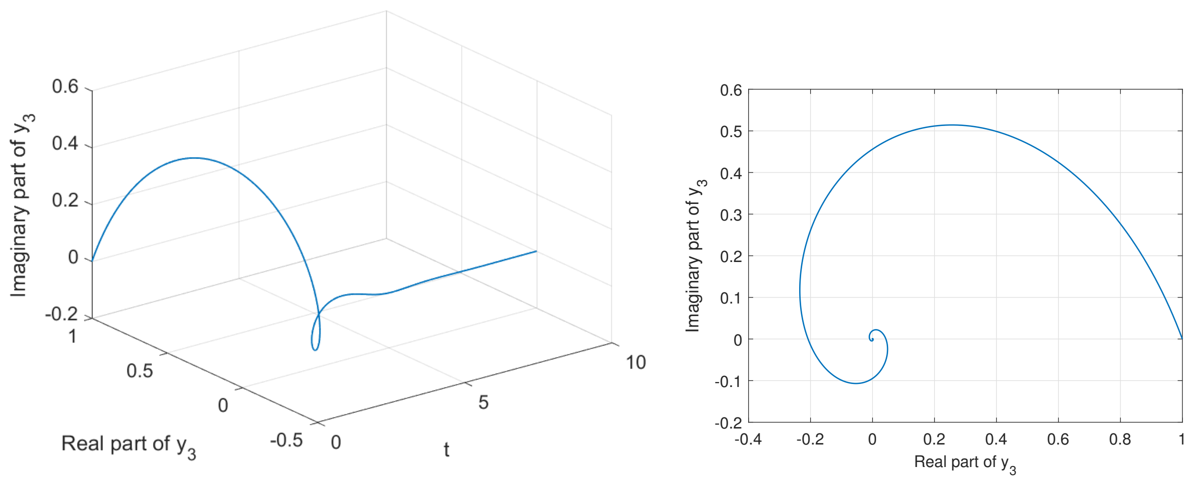


Fig. 5: Plots of $y_3(t)$

Plots such as those in Figure 5 do not allow us to visualize clearly how $|y_3(t)|$ *and* $\angle y_3(t)$ *are behaving as time progresses. But if we plot* $|y_3(t)|$ *and* $\angle y_3(t)$ *(versus time) on two separate plots as shown in Figure 6, we have a better idea of how the magnitude of the signal is changing over time. You can now see that* $|y_3(t)|$ *decreases exponentially ie* $|y_3(t)| = e^{-t}$ *while the phase increases linearly ie* $\angle y_3(t) = 2t$ *rad.*

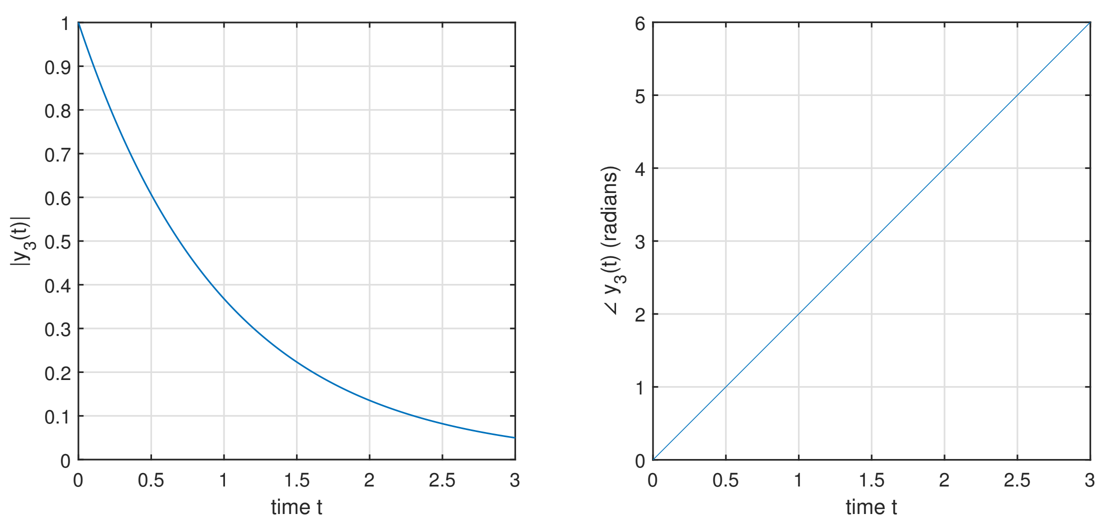


Fig. 6: Plots of $|y_3(t)|$ and $\angle y_3(t)$

In summary, we have the following observations about complex numbers and functions :

1. Complex numbers are of the form $z = x + jy$ where $j = \sqrt{-1}$. x and y are real numbers.

2. Complex numbers can be written in polar form : $z = x + jy = Re^{j\theta}$ where $R = \sqrt{x^2 + y^2}$ is the magnitude of z, while $\theta = \tan^{-1}(y/x)$ is the argument or phase of z in radians.

3. Multiplication and division of complex numbers should always be done using their polar forms. Addition and subtractions are better done in cartesian coordinates.

4. Just as we have real numbers and real functions, we also have complex numbers and complex functions. The most commonly encountered functions in EE2023 is the complex exponential function of the form $s(t) = Ae^{zt}$ where $A > 0$ is real while z may be complex. An example of such a signal is $y_3(t)$ in Example 3.

5. Complex functions are difficult to visualize because they can only be plotted on 3D plots. However, we can also plot their magnitudes and phases on separate plots, against t. See Figure 6. In EE2023, you will often encounter the need to plot magnitudes and phases of complex quantities on separate plots. For example, the spectrum of a signal is often a complex function of frequency, and the only way to visualize this is via magnitude and phase plots.

6. Complex exponential functions can be used to construct real sinusoidal functions. See equations (7) and (8) earlier. While complex exponential functions do not occur in the real world, real sinusoidal functions do. Hence we end up making use of complex functions to help us deal with real sinusoids in an easier way in signal processing.

Exercise 3.

(a) Write down the magnitude and phase of $z(t) = 2e^{-(1-0.5j)t}$, in terms of t. Sketch these quantities.

(b) Consider the function $z(t) = \mathrm{rect}\left(\frac{t}{8}\right) e^{(-1+2j)t}$. Sketch $|z(t)|$ and $\angle z(t)$.

(c) Plot the graph of $z(t) = 5e^{-2t}\sin(2\pi t)$ for $t > 0$.

(d) Plot the graph of $z(t) = e^{jt} + e^{-jt}$. What is the frequency of $z(t)$? Is $z(t)$ real or complex?