CG2023 ASSIGNMENT 4 (ESD and PSD) by Sahoo Sanjib Kumar

SOLUTIONS

1. For the rectangle signal given below in time domain, determine its

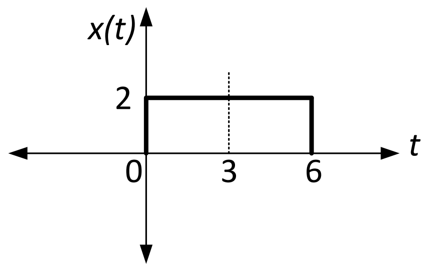

Figure 4.1

a) Energy spectral density (ESD), $E_x(f)$.

$x(t) = 2\, rect\left(\frac{t-3}{6}\right)$ i.e. it is a rectangle function which is time-shifted by 3 and is scaled by factor 6.

$X(f) = 2 \times 6 \times sinc(6f) \times e^{-j2\pi f \times 3} = 12 \sin c(6f) \times e^{-j6\pi f}$

$E_x(f) = |X(f)|^2 = 144 sinc^2(6f)$

b) Total energy, E

$E = \int_{t=-\infty}^{\infty} |x(t)|^2 dt = \int_{t=0}^{6} 4dt = 4 \times 6 = 24$ J

c) 1^{st}-null bandwidth.

$$|X(f)| = 0 \to sinc(6f) = 0$$
$$6f = 1 \to f = \frac{1}{6}\ Hz$$

The first null bandwidth for the signal is $\frac{1}{6}$ Hz.

2. For the triangular spectrum of a signal $X(f)$ given below, determine its

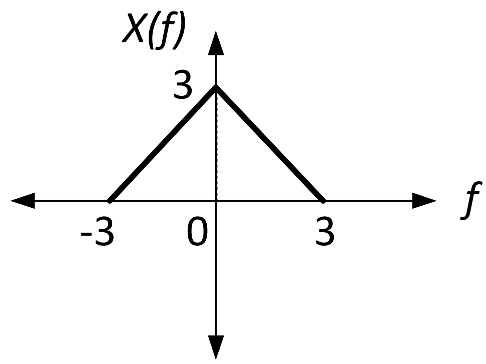

Figure 4.2

a) Energy spectral density (ESD), $E_x(f)$

$$X(f) = 3\,tri\left(\frac{f}{3}\right) = \begin{cases} 3\left(1-\frac{f}{3}\right) & for\ 0 \le f < 3 \\ 3\left(1+\frac{f}{3}\right) & for -3 < f < 0 \end{cases}$$

$$E_x(f) = |X(f)|^2 = \begin{cases} 9\left(1-\frac{f}{3}\right)^2 & for\ 0 \le f < 3 \\ 9\left(1+\frac{f}{3}\right)^2 & for -3 < f < 0 \end{cases}$$

b) Total energy, E

As the ESD is symmetric about $f = 0$, total energy can be obtained as:

$$E = 2 \times \int_0^3 |X(f)|^2 df =$$

$$2 \times \int_0^3 9\left(1-\frac{f}{3}\right)^2 df = 18 \times \left[\left(1-\frac{f}{3}\right)^3 \times \frac{1}{3} \times (-3)\right]_0^3 = 18\,J$$

c) $3dB$ bandwidth.

It is a low-pass signal.

At 3-dB bandwidth, $|X(f_B)| = \frac{|X(0)|}{\sqrt{2}} \rightarrow\ 3\left(1-\frac{f_B}{3}\right) = \frac{3}{\sqrt{2}}$

$1-\frac{f_B}{3} = \frac{1}{\sqrt{2}} \rightarrow f_B = 0.879\ Hz$

3. Find the total energy of the signals described below.

a) $X(f) = sinc(f)$

Total energy $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

Since it is not easy to integrate in frequency domain, we shall first determine the time-domain expression i.e. $x(t) = rect(-t) = rect(t)$.

Now, $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-0.5}^{0.5} 1\, dt = 1$ J

b) $y(t) = sinc(t)$

In this question, we need to find the frequency domain expression of the signal.

$$X(f) = rect(f)$$

Hence, total energy $E = \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-0.5}^{0.5} 1\, df = 1$ J

4. If a periodic signal $q(t) = \sum_{n=-\infty}^{\infty} x(t-12n)$ is obtained by repeating the rectangle signal $x(t)$ in Fig.1, determine its

a) Power spectral density (PSD), $P_q(f)$

Since $q(t)$ is a periodic signal with $T_p = 12$. We can find its Fourier series coefficients as:

$$c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} q(t)\, e^{-j2\pi\frac{k}{T_p}t}\, dt = \frac{1}{12}\int_0^6 2\, e^{-j2\pi\frac{k}{12}t}\, dt = \left[\frac{2}{12}\frac{e^{-j\frac{k\pi}{6}t}}{-j\frac{k\pi}{6}}\right]_0^6$$

$$= \frac{j}{k\pi}\left(e^{-jk\pi} - 1\right)$$

$$c_k = \begin{cases} 1\ for\ k=0 \\ \dfrac{-2j}{k\pi}\ for\ k\ odd \\ 0\ for\ k\ even \end{cases}$$

The power spectral density,

$$P_q(f) = \sum_k |c_k|^2 \delta\left(f - \frac{k}{12}\right) = \begin{cases} 1, k=0 \\ \dfrac{4}{k^2\pi^2}, k\ odd \\ 0, k\ even \end{cases}$$

b) Average power, P

Average power $P = \sum_k |c_k|^2 = \sum_k \frac{1}{\pi^2 k^2}$ which is difficult to calculate.

Instead, we shall find the energy in one period and divide by the time period.

$P = \frac{E_{one\ period}}{T_p} = \frac{1}{12}\int_0^6 2^2 dt = \frac{6\times 4}{12} = 2$ W

5. Given a signal $v(t) = 2 + (3+j)e^{j4\pi t} + 4e^{j8\pi t} + 5\,e^{j(10\pi t\ \frac{\pi}{4})}$, determine

a) Power spectral density, $P_v(f)$

The signal is a periodic signal, as we can find a fundamental frequency as:

$$HCF(2{,}4{,}5) = 1 \rightarrow f_p = 1$$

$$P_q(f) = \sum_k |c_k|^2 \delta(f - kf_p) =$$
$$= 2^2\delta(f) + (3^2 + 1^2)\delta(f - 2) + 4^2\delta(f - 4) + 5^2\delta(f - 5)$$
$$= 4\delta(f) + 10\delta(f - 2) + 16\delta(f - 4) + 25\delta(f - 5)$$

b) Average power, P

$$P = \sum_k |c_k|^2 = 4 + 10 + 16 + 25 = 55W$$

End