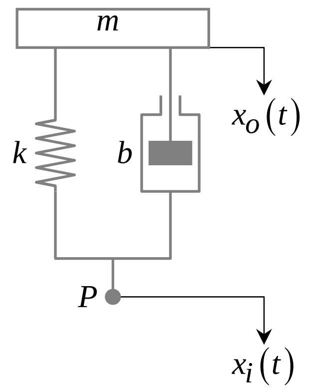
EE2023 TUTORIAL 8 (SOLUTIONS)

Solution to Q.1



Transfer function of suspension system:

$$H(s) = \frac{X_o(s)}{X_i(s)} = \frac{bs+k}{ms^2+bs+k}$$

Substututing $m = 1\ kg,\ k = 1\ Nm^{-1},\ b = \sqrt{2}\ Nm^{-1}s$, we get

$$H(s) = \frac{s\sqrt{2}+1}{s^2+s\sqrt{2}+1}$$

Frequency response of suspension system:

$$H(j\omega) = H(s)\big|_{s=j\omega} = \frac{1+j\omega\sqrt{2}}{1-\omega^2+j\omega\sqrt{2}}$$

$$\rightarrow \begin{cases} \text{Magnitude Response: } \left|H(j\omega)\right| = \left(\dfrac{1+2\omega^2}{1+\omega^4}\right)^{1/2} \\ \text{Phase Response: } \angle H(j\omega) = -\tan^{-1}\left(\dfrac{\omega^3\sqrt{2}}{1+\omega^2}\right) \end{cases}$$

Fourier series expansion of input: $x_i(t) = \dfrac{4}{\pi}\left[\sin(t) + \dfrac{1}{3}\sin(3t) + \dfrac{1}{5}\sin(5t) + \cdots\cdots\right]$

- **Steady-state** response of system due to sinusoidal input $\sin(t)$ is given by

$$\left|H(j1)\right|\cdot\sin\left(t+\angle H(j1)\right) = 1.2247\sin(t-0.6155)$$

- **Steady-state** response of system due to sinusoidal input $\sin(3t)$ is given by

$$\left|H(j3)\right|\cdot\sin\left(3t+\angle H(j3)\right) = 0.4814\sin(3t-1.3147)$$

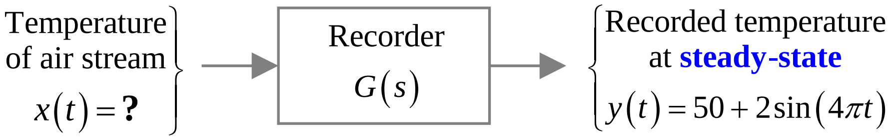
- **Steady-state** response of system due to sinusoidal input $\sin(5t)$ is given by

$$\left|H(j5)\right|\cdot\sin\left(5t+\angle H(j5)\right) = 0.2854\sin(5t-1.4248)$$

Since system is linear, the output of the system can be obtained by superposition. Hence, at **steady state**:

$$\begin{aligned} x_{o,ss}(t) &= \frac{4}{\pi}\left[1.2247\sin(t-0.6155) + \frac{1}{3}\times 0.4814\sin(3t-1.3147) + \frac{1}{5}\times 0.2854\sin(5t-1.4248) + \cdots\right] \\ &= \frac{4}{\pi}\left[1.2247\sin(t-0.6155) + 0.1065\sin(3t-1.3147) + 0.05708\sin(5t-1.4248) + \cdots\cdots\cdots\right] \end{aligned}$$

Solution to Q.2



$G(s) = \dfrac{1}{s+1}$ since it is given that $G(s)$ has $\left\{\begin{array}{l} \bullet \text{ DC gain } = 1 \\ \bullet \text{ First order dynamics} \\ \bullet \text{ Time constant} = 1 \text{ min} \end{array}\right\}$

Therefore $G(j\omega) = \dfrac{1}{j\omega + 1}$ ······ $\left\{\begin{array}{ll} \text{Magnitude response:} & |G(j\omega)| = \left(\omega^2 + 1\right)^{-1/2} \\ \text{Phase response:} & \angle G(j\omega) = -\tan^{-1}(\omega) \\ \text{Gain at } \omega = 0 \text{ rad/min:} & |G(0)| = 1 \cdots \textit{DC gain} \end{array}\right.$

At STEADY-STATE:

- The DC component of the output $y(t)$ is 50. Since the system has unity DC gain, the DC component of the input $x(t)$ must also be 50.

- The sinusoidal component of the output $y(t)$ is $2\sin(4\pi t)$. Since the magnitude and phase responses of the system at $\omega = 4\pi$ rad/min are

$$\begin{array}{l} |G(j4\pi)| = \left(16\pi^2 + 1\right)^{-1/2} = 0.0793 \\ \angle G(j4\pi) = -\tan^{-1}(4\pi) = -1.4914 \end{array},$$

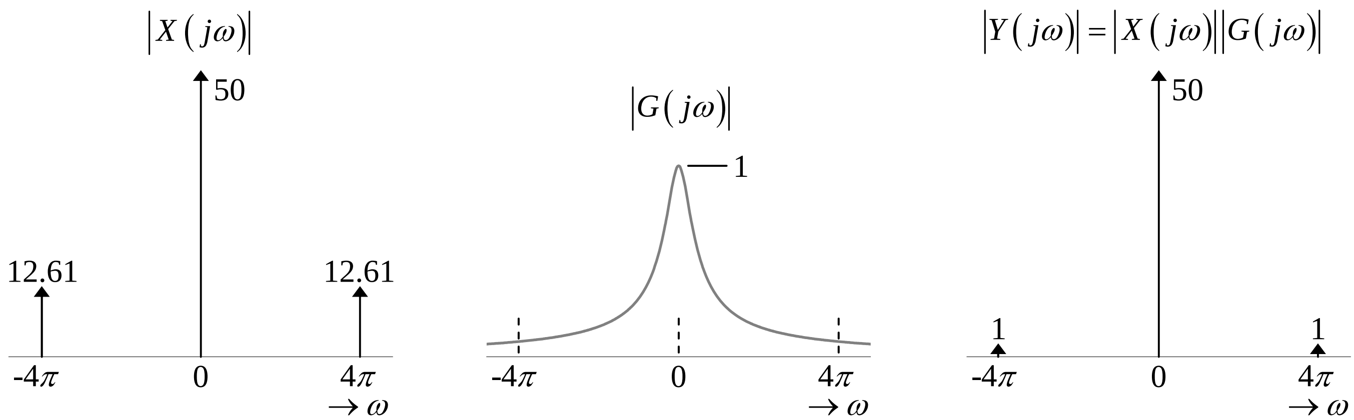
respectively, the corresponding input component must be a sinusoid given by

$$\frac{2}{|G(j4\pi)|}\sin\left(4\pi t - \angle G(j4\pi)\right) = \frac{2}{0.0793}\sin(4\pi t + 1.4914) = 25.22\sin(4\pi t + 1.49)$$

Combining these components, we have $x(t) = 50 + 25.22\sin(4\pi t + 1.49)$ which shows that:

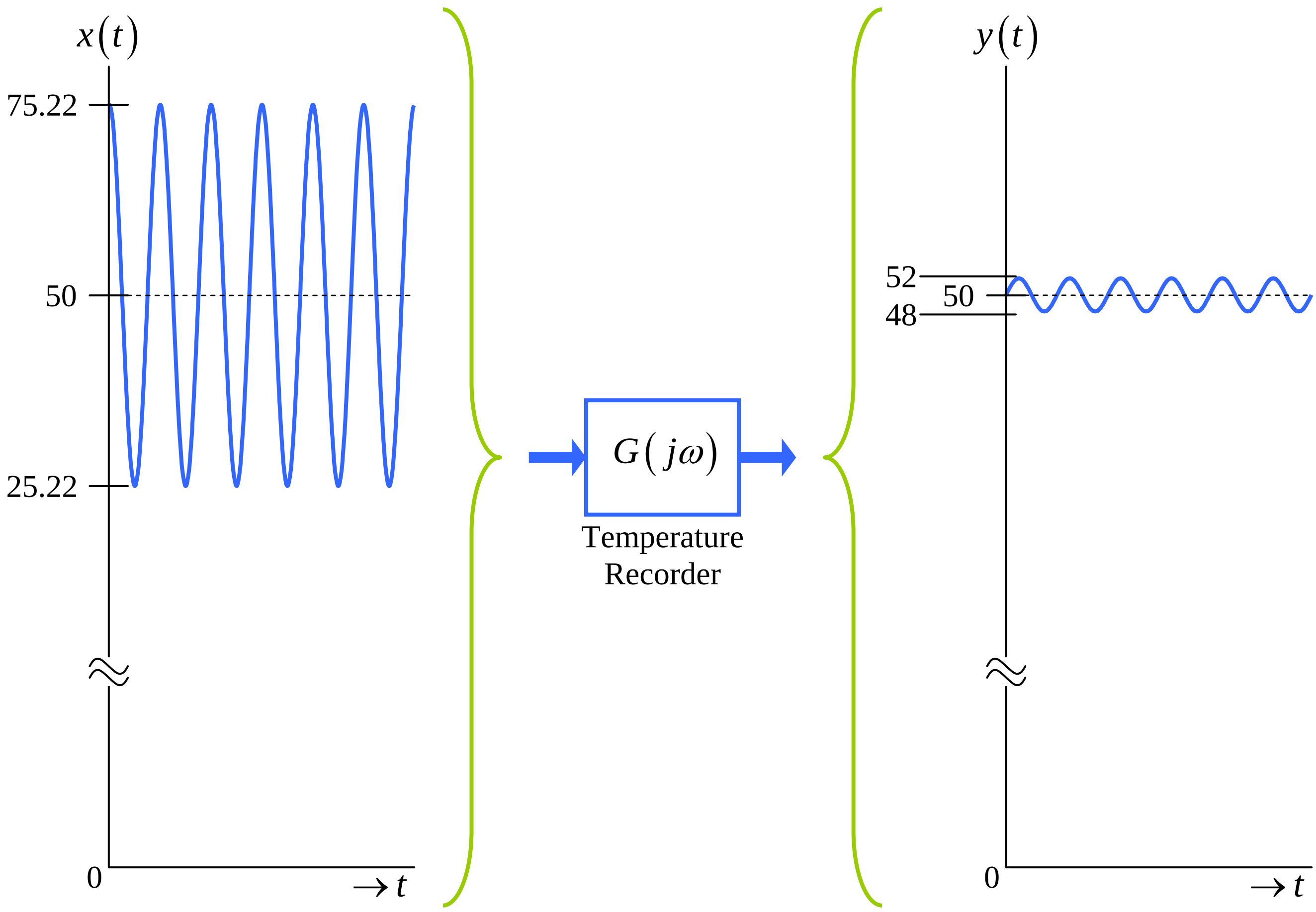
Estimated **maximum** air temperature: $\mathbf{50 + 25.22 = 75.22°C}$
Estimated **minimum** air temperature: $\mathbf{50 - 25.22 = 25.22°C}$

We note that the recorded maximum and minimum temperatures are $\mathbf{52°C}$ and $\mathbf{48°C}$. Clearly, the recorder does not have sufficient bandwidth.



In the time domain:

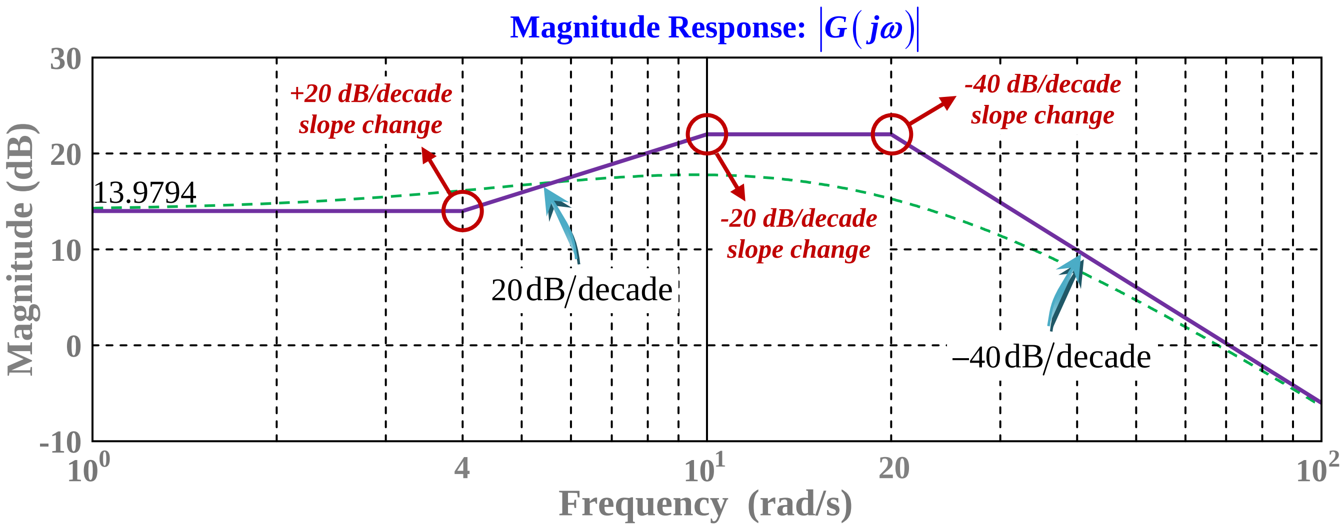
$$x(t) = 50 + 25.22\sin(4\pi t + 1.49) \qquad y(t) = 2\sin(4\pi t)$$



ACTUAL
air-stream steady state temperature

MEASURED
air-stream steady state temperature

Solution to Q.3



$$\textbf{Transfer function: } G(s)=\frac{K(s+\alpha)}{(s+\beta)(s+\gamma)(s+\lambda)}=\frac{K_{dc}\left(\frac{s}{\alpha}+1\right)}{\left(\frac{s}{\beta}+1\right)\left(\frac{s}{\gamma}+1\right)\left(\frac{s}{\lambda}+1\right)};\qquad K=\frac{\beta\gamma\lambda}{\alpha}K_{dc}$$

(a)

- At $\omega = 4\ rad/s$, there is a **slope-change** of 20 dB/decade. This indicates the presence of the zero factor $\left(\frac{s}{4}+1\right)$ in $G(s)$.

- At $\omega = 10\ rad/s$, there is a **slope-change** of -20 dB/decade. This indicates the presence of the pole factor $\left(\frac{s}{10}+1\right)^{-1}$ in $G(s)$.

- At $\omega = 20\ rad/s$, there is a **slope-change** of -40 dB/decade. This indicates the presence of the double pole factor $\left(\frac{s}{20}+1\right)^{-2}$.

- DC (or Static) gain: $\left[20\log_{10}K_{dc} = |G(j0)|_{dB} = 13.9794 \text{ dB}\right]$ or $\left[K_{dc}=10^{13.9794/20}=5\right]$.

Hence, the transfer function is

$$G(s)=\frac{5\left(\frac{s}{4}+1\right)}{\left(\frac{s}{10}+1\right)\left(\frac{s}{20}+1\right)^2}=\frac{5000(s+4)}{(s+10)(s+20)^2}$$

$\therefore\ \boldsymbol{K=5000,\ \alpha=4,\ \beta=10,\ \gamma=\lambda=20}$

(b)

CASE A: Magnitude response is invariant to transport delay (or dead-time).

Example: $\tilde{G}(s)=\dfrac{5000(s+4)}{(s+10)(s+20)^2}e^{-sL}$ and $G(s)=\dfrac{5000(s+4)}{(s+10)(s+20)^2}$ have the same magnitude response.

Proof:

$$\left|\tilde{G}(j\omega)\right|=\frac{5000|j\omega+4|}{|j\omega+10||j\omega+20|^2}\cdot\underbrace{\left|e^{-j\omega L}\right|}_{=1}=\frac{5000|j\omega+4|}{|j\omega+10||j\omega+20|^2}=\left|G(j\omega)\right|$$

REMARKS: For *unilateral* Laplace transform, this holds only for $L>0$.
For *bilateral* Laplace transform, L can take any real value.

CASE B: Magnitude response is invariant to the reflection of any zero about the ω axis on the *s*-plane

Example: $\tilde{G}(s)=\dfrac{5000(s-4)}{(s+10)(s+20)^2}$ and $G(s)=\dfrac{5000(s+4)}{(s+10)(s+20)^2}$ have the same magnitude response.

Proof:

$$\left|\tilde{G}(j\omega)\right|=\frac{5000|j\omega-4|}{|j\omega+10||j\omega+20|^2}=\frac{5000|j\omega+4|}{|j\omega+10||j\omega+20|^2}=\left|G(j\omega)\right|$$

REMARKS: At first sight, it appears that we may also preserve the magnitude response by reflecting any pole about the ω axis on the *s*-plane. However, this will cause the system to become unstable. For unstable systems, it is erroneous to set $s=j\omega$ in $G(s)$ to obtain the frequency response because $s=j\omega$ lies outside the region of convergence of $G(s)$. Stated another way, frequency response of an unstable system does not exist. It is thus quite meaningless to talk about magnitude response of an unstable system.

Suppose $G(s)$ is stable and has frequency response $G(j\omega)$. By reflecting one or more poles of $G(s)$ to the right-half of the s-plane to form $\tilde{G}(s)$, all we can say is that $\left|\tilde{G}(j\omega)\right|=\left|G(j\omega)\right|$. It is incorrect to call $\left|\tilde{G}(j\omega)\right|$ the magnitude response of $\tilde{G}(s)$ because $\tilde{G}(s)$ is unstable.