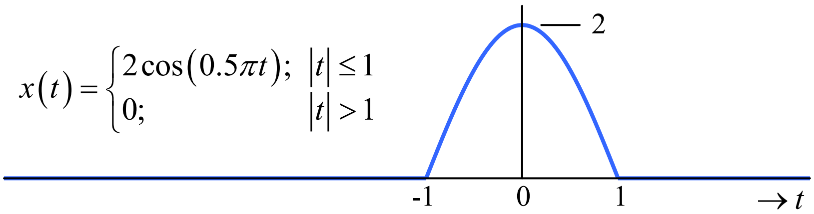
CG2023 TUTORIAL 3 (SOLUTIONS)

Solution to Q.1

$$x(t)=\begin{cases}2\cos(0.5\pi t); & |t|\le 1\\ 0; & |t|>1\end{cases}$$



(a)

Method 1: By applying direct Fourier transform:

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt\\
&= \int_{-\infty}^{-1} 0\exp(-j2\pi ft)\,dt + \int_{-1}^{1} 2\cos(0.5\pi t)\exp(-j2\pi ft)\,dt + \int_{1}^{\infty} 0\exp(-j2\pi ft)\,dt\\
&= 2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\cos(2\pi ft)}_{even\ function\ of\ t}dt - \cancel{j2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\sin(2\pi ft)}_{odd\ function\ of\ t}dt}\\
&= 4\int_{0}^{1}\cos(0.5\pi t)\cos(2\pi ft)\,dt\\
&\cdots\ applying\ \cos(A)\cos(B) = \frac{1}{2}\cos\left(\frac{A-B}{2}\right)+\frac{1}{2}\cos\left(\frac{A+B}{2}\right)\\
&= 2\int_{0}^{1}\cos((2\pi f-0.5\pi)t)+\cos((2\pi f+0.5\pi)t)\,dt\\
&= 2\left[\frac{\sin((2\pi f-0.5\pi)t)}{2\pi f-0.5\pi}+\frac{\sin((2\pi f+0.5\pi)t)}{2\pi f+0.5\pi}\right]_0^1\\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{2\pi f-0.5\pi}+\frac{\sin(2\pi f+0.5\pi)}{2\pi f+0.5\pi}\right)\\
&= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5}+\frac{\cos(2\pi f)}{2f+0.5}\right) = \frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}
\end{aligned}$$

Method 2: By applying Fourier transform properties:

The half-cosine pulse can be modeled as $x(t)=2\cos(0.5\pi t)\cdot\mathrm{rect}(0.5t)$

$$\Im\{2\cos(0.5\pi t)\}=\delta(f-0.25)+\delta(f+0.25)$$

$$\Im\{\mathrm{rect}(0.5t)\}=2\mathrm{sinc}(2f)$$

Applying the 'Multiplication in time-domain' property of the Fourier transform

$$\left[x(t)=\underbrace{2\cos(0.5\pi t)\cdot\mathrm{rect}(0.5t)}_{\text{Multiplication in time-domain}}\right] \rightleftarrows \left[X(f)=\underbrace{\Im\{2\cos(0.5\pi t)\}*\Im\{\mathrm{rect}(0.5t)\}}_{\text{Convolution in frequency-domain}}\right]$$

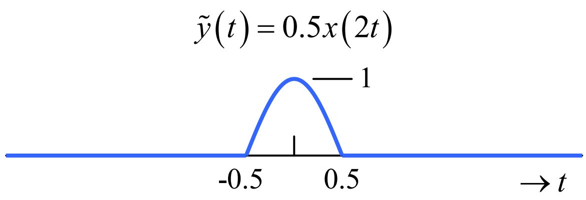
we get

$$
\begin{aligned}
X(f) &= \left[\delta(f-0.25)+\delta(f+0.25)\right] * 2\operatorname{sinc}(2f) \\
&= 2\operatorname{sinc}\left(2(f-0.25)\right)+2\operatorname{sinc}\left(2(f+0.25)\right) \\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{\pi(2f-0.5)}+\frac{\sin(2\pi f+0.5\pi)}{\pi(2f+0.5)}\right) \\
&= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5}+\frac{\cos(2\pi f)}{2f+0.5}\right) \\
&= \frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)} \quad \cdots\cdots \text{ Same result obtained by } \textbf{Method 1}
\end{aligned}
$$

(b) From Part (a): $X(f)=\dfrac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}$

Define an intermediate function $\tilde{y}(t)=0.5x(2t)$

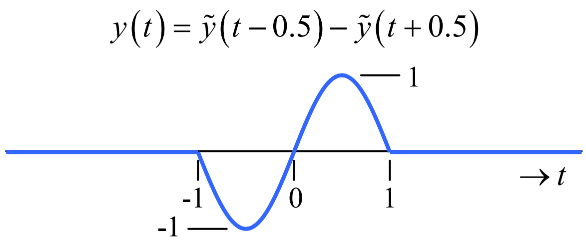
Applying the ***scaling property***:



$$
\begin{aligned}
\tilde{Y}(f) &= \Im\{0.5x(2t)\} \\
&= 0.5\left[\frac{1}{2}X\left(\frac{f}{2}\right)\right]=\frac{1}{4}X\left(\frac{f}{2}\right) \quad \cdots\cdots \quad (*)
\end{aligned}
$$

Now, $y(t)=\tilde{y}(t-0.5)-\tilde{y}(t+0.5)$

Applying the ***time-shifting property***:



$$
\begin{aligned}
Y(f) &= \tilde{Y}(f)\exp\left(-j2\pi f\left(\frac{1}{2}\right)\right) \\
&\quad -\tilde{Y}(f)\exp\left(j2\pi f\left(\frac{1}{2}\right)\right) \quad \cdots\cdots\cdots \quad (**)
\end{aligned}
$$

Substituting $(*)$ into $(**)$, we get

$$
\begin{aligned}
Y(f) &= \frac{1}{4}X\left(\frac{f}{2}\right)\exp(-j\pi f)-\frac{1}{4}X\left(\frac{f}{2}\right)\exp(j\pi f) \\
&= \frac{1}{4}X\left(\frac{f}{2}\right)\left\{\underbrace{\left[\cos(\pi f)-j\sin(\pi f)\right]}_{\exp(-j\pi f)}-\underbrace{\left[\cos(\pi f)+j\sin(\pi f)\right]}_{\exp(j\pi f)}\right\} \\
&= -j\frac{1}{2}X\left(\frac{f}{2}\right)\sin(\pi f) \\
&= \frac{1}{j2}\underbrace{\left[\frac{2\cos(\pi f)}{\pi\left(0.25-f^2\right)}\right]}_{X(f/2)}\sin(\pi f)=\frac{1}{j2}\left[\frac{\sin(2\pi f)}{\pi\left(0.25-f^2\right)}\right]
\end{aligned}
$$

OBSERVATION: $y(t)$ is real & odd and $Y(f)$ is pure imaginary & odd.

Solution to Q.2

(a) Fig.Q.2(a)(I) is a plot of $u(t-\gamma)$ against t:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \xrightarrow[t \text{ to } t-\gamma]{\textit{change}} \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \to \underbrace{\left[u(t-\gamma)=\begin{cases}1; & t\ge \gamma\\ 0; & t<\gamma\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } t \text{ while}\\ \text{treating } \gamma \text{ as a parameter}\\ \text{Fig.Q2(a)(I)}}}$$

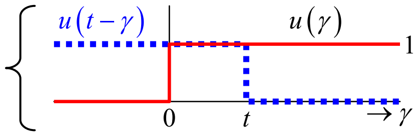
Fig.Q.2(a)(II) is a plot of $u(t-\gamma)$ against γ:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \xrightarrow[t \text{ to } t-\gamma]{\textit{change}} \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \to \underbrace{\left[u(t-\gamma)=\begin{cases}1; & \gamma\le t\\ 0; & \gamma>t\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } \gamma \text{ while}\\ \text{treating } t \text{ as a parameter}\\ \text{Fig.Q2(a)(II)}}}$$

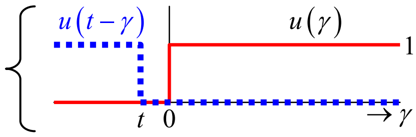
(b) $\cos(t)u(t) * u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)\,d\gamma$

Consider the term $u(\gamma)u(t-\gamma)$ in the integrand as a function of γ.

When $t\ge 0$, we have $u(\gamma)u(t-\gamma)=\begin{cases}1; & 0\le\gamma\le t\\ 0; & \textit{elsewhere}\end{cases}$



When $t<0$, we have $u(\gamma)u(t-\gamma)=0 \quad \forall\gamma$



$$\therefore \cos(t)u(t) * u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)\,d\gamma=\begin{cases}\int_0^t\cos(\gamma)\,d\gamma; & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\begin{cases}\sin(t); & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\sin(t)u(t)$$

Solution to Q.3

Spectrum of $x(t)$:

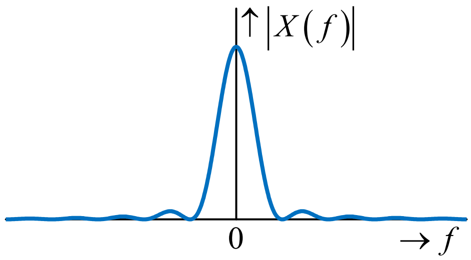
The given triangular pules may be expressed as $x(t) = \alpha \cdot \text{tri}\left(\frac{t}{\alpha}\right)$. Applying the Fourier transform pair $\text{tri}\left(\frac{t}{T}\right) \rightleftarrows T\text{sinc}^2(Tf)$, it is easy to see that

$$X(f) = \Im\left\{\alpha \cdot \text{tri}\left(\frac{t}{\alpha}\right)\right\} = \alpha^2 \text{sinc}^2(\alpha f).$$

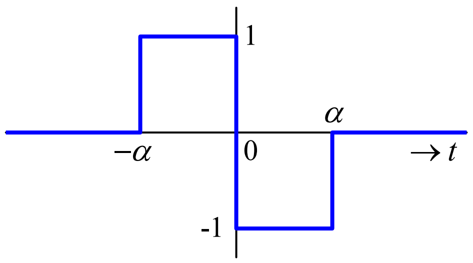
Hence,

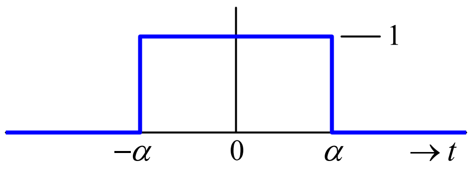
Magnitude spectrum: $|X(f)| = \alpha^2 \text{sinc}^2(\alpha f)$.

Phase spectrum: $\angle X(f) = 0$



ESD and Energy of $y(t) = \frac{dx(t)}{dt}$:





Applying the 'Differentiation in Time-Domain' property of the Fourier transform:

$$Y(f) = j2\pi f \cdot X(f) = j2\pi f \alpha^2 \text{sinc}^2(\alpha f)$$

Hence,

ESD:
$$E_y(f) = |Y(f)|^2 = Y(f)Y^*(f) = 4\pi^2 f^2 \alpha^4 \text{sinc}^4(\alpha f).$$

Total Energy:
$$E = \overbrace{\int_{-\infty}^{\infty} E_y(f)\,df = \int_{-\infty}^{\infty} |y(t)|^2\,dt}^{\text{Rayleigh energy theorem}} \underbrace{= 2\alpha}_{\text{By inspection of the plot of } |y(t)|^2}$$

Solution to Q.4

Spectrum: $X(f) = \exp(-\alpha|f|); \ \ \alpha > 0$

ESD of $\boldsymbol{x}(\boldsymbol{t})$: $E_x(f) = |X(f)|^2 = \exp(-2\alpha|f|)$

(a) ***Energy of $x(t)$ contained within a bandwidth of B:***

$$e(B) = \int_{-B}^{B} E_x(f)\,df = 2\int_{0}^{B} \exp(-2\alpha f)\,df = 2\left[\frac{\exp(-2\alpha f)}{-2\alpha}\right]_0^B = \frac{1}{\alpha}\left[1-\exp(-2\alpha B)\right] \quad \cdots\cdots \ (\clubsuit)$$

Total energy of $x(t)$ is equal to energy of $x(t)$ contained within a bandwidth of ∞.

Substituting $B = \infty$ into $(\clubsuit)$, we get

$$e(\infty) = \int_{-\infty}^{\infty} E_x(f)\,df = \frac{1}{\alpha} \quad \cdots\cdots \ \text{TOTAL ENERGY}$$

Let W denote the 99% energy containment bandwidth of $x(t)$. Then

Energy contained in bandwidth $W = 0.99 \times$ Total Energy

$$\rightarrow \ e(W) = 0.99 \times e(\infty)$$

$$\rightarrow \ \frac{1}{\alpha}\left[1-\exp(-2\alpha \cdot W)\right] = 0.99 \times \frac{1}{\alpha}\left[1-\exp(-2\alpha \cdot \infty)\right]$$

$$\rightarrow \ \exp(2\alpha W) = 100$$

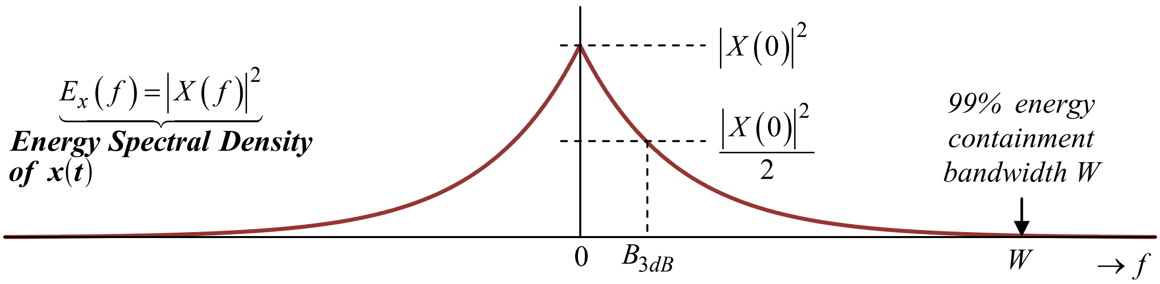
$$\rightarrow \ W = \frac{1}{\alpha}\ln(10) \ \text{Hz}$$

(b) ***Let B_{3dB} denote the 3dB bandwidth of $x(t)$. Then***

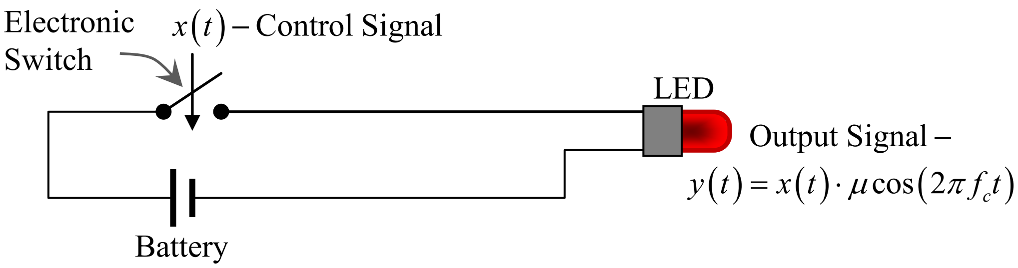
$$\frac{E_x(B_{3dB})}{E_x(0)} = \frac{1}{2} \quad \rightarrow \quad \frac{\exp(-2\alpha B_{3dB})}{\exp(0)} = \frac{1}{2} \quad \rightarrow \quad B_{3dB} = \frac{1}{2\alpha}\ln(2) \ \text{Hz}$$

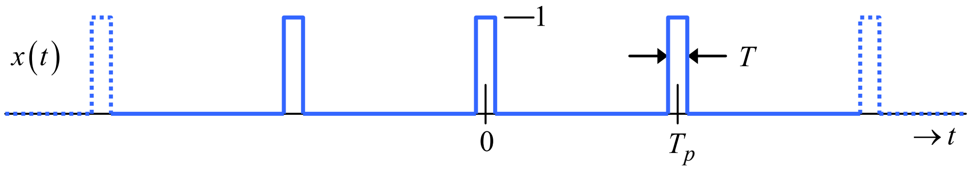
Percent energy contained within the 3dB bandwidth:

$$\frac{e(B_{3dB})}{e(\infty)} \times 100 = \frac{\cancel{\frac{1}{\alpha}}\left[1-\exp\left(-\cancel{2\alpha}\,\frac{\ln(2)}{\cancel{2\alpha}}\right)\right]}{\cancel{\frac{1}{\alpha}}} \times 100 = 50\%$$



Solution to Q.5





(a) Expressing $x(t)$ as $x(t)=\text{rect}\left(\frac{t}{T}\right)*\sum_n \delta(t-nT_p)$, the Fourier transform of $x(t)$ is obtained as

$$X(f)=T\text{sinc}(fT)\times\frac{1}{T_p}\sum_k \delta\left(f-\frac{k}{T_p}\right)=\sum_k \underbrace{\frac{T}{T_p}\text{sinc}\left(k\frac{T}{T_p}\right)}_{c_k}\delta\left(f-\frac{k}{T_p}\right)$$

PSD of $\boldsymbol{x(t)}$ $\quad : P_x(f)=\sum_{k=-\infty}^{\infty}\frac{T^2}{T_p^2}\text{sinc}^2\left(k\frac{T}{T_p}\right)\delta\left(f-\frac{k}{T_p}\right)$

(b) The average power of $x(t)$ can be computed using

$$P=\int_{-\infty}^{\infty}P_x(f)\,df=\sum_{k=-\infty}^{\infty}|c_k|^2=\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p}|x(t)|^2\,dt\,.$$

Based on the $x(t)$ given, and the expressions for c_k and $P_x(f)$ found in Part (a), it would be easier to compute P in the time-domain, i.e.

$$P=\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p}|x(t)|^2\,dt=\frac{1}{T_p}\int_{-0.5T}^{0.5T}dt=\frac{T}{T_p}$$

(c) Since $y(t)$ is periodic with period T_p, its average power, P, may be computed by averaging over one period:

$$\begin{aligned}P&=\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p}|y(t)|^2\,dt=\frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p}|x(t)|^2\mu^2\cos^2(2\pi f_c t)\,dt\\&=0.5\mu^2\frac{1}{T_p}\int_{-0.5T}^{0.5T}\left[1+\cos(4\pi f_c t)\right]dt=0.5\mu^2\frac{1}{T_p}\int_{-0.5T}^{0.5T}dt+0.5\mu^2\frac{1}{T_p}\int_{-0.5T}^{0.5T}\cos(4\pi f_c t)\,dt\\&=0.5\mu^2\frac{T}{T_p}\left[1+\frac{\sin(2\pi f_c T)}{2\pi f_c T}\right]=\underbrace{0.5\mu^2\frac{T}{T_p}\left[1+\text{sinc}(2f_cT)\right]=0.5\mu^2\frac{T}{T_p}}_{\textit{because } f_cT \textit{ is an integer}}\end{aligned}$$

Solution to S.1

(a) $x(t) = \cos(2\pi f_c t)u(t)$

$$\begin{cases} X(f) = \Im\{\cos(2\pi f_c t)\} * \Im\{u(t)\} = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] * \left[\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right] \\ \quad = \frac{1}{4}\left[\frac{1}{j\pi(f - f_c)} + \delta(f - f_c) + \frac{1}{j\pi(f + f_c)} + \delta(f + f_c)\right] \\ \quad = \frac{1}{4}\left[\delta(f - f_c) + \delta(f + f_c)\right] - \frac{jf}{2\pi\left(f^2 - f_c^2\right)} \end{cases}$$

(b) $x(t) = \sin(2\pi f_c t)u(t)$

$$X(f) = \frac{j}{4}\left[\delta(f + f_c) - \delta(f - f_c)\right] - \frac{f_c}{2\pi\left(f^2 - f_c^2\right)} \quad \cdots \quad (\text{Same approach as part } (a))$$

(c) $x(t) = \exp(-\alpha t)\cos(2\pi f_c t)u(t); \quad \alpha > 0$

$$\begin{cases} X(f) = \Im\{\exp(-\alpha t)\cos(2\pi f_c t)u(t)\} = \Im\{\exp(-\alpha t)u(t)\} * \Im\{\cos(2\pi f_c t)\} \\ \quad = \frac{1}{\alpha + j2\pi f} * \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] \\ \quad = \frac{1}{2}\left[\frac{1}{\alpha + j2\pi(f - f_c)} + \frac{1}{\alpha + j2\pi(f + f_c)}\right] \\ \quad = \frac{\alpha + j2\pi f}{\left[\alpha^2 - 4\pi^2\left(f^2 - f_c^2\right)\right] + j4\alpha\pi f} \end{cases}$$

(d) $x(t) = \exp(-\alpha t)\sin(2\pi f_c t)u(t); \quad \alpha > 0$

$$\begin{cases} X(f) = \Im\{\exp(-\alpha t)\sin(2\pi f_c t)u(t)\} = \Im\{\exp(-\alpha t)u(t)\} * \Im\{\sin(2\pi f_c t)\} \\ \quad = \frac{1}{\alpha + j2\pi f} * \frac{1}{j2}\left[\delta(f - f_c) - \delta(f + f_c)\right] \\ \quad = \frac{1}{j2}\left[\frac{1}{\alpha + j2\pi(f - f_c)} - \frac{1}{\alpha + j2\pi(f + f_c)}\right] \\ \quad = \frac{2\pi f_c}{\left[\alpha^2 - 4\pi^2\left(f^2 - f_c^2\right)\right] + j4\alpha\pi f} \end{cases}$$

Solution to S.2

(a) $x(t) = \text{rect}\left(\frac{t}{4}\right) + \text{tri}(t) \rightleftarrows X(f) = 4\text{sinc}(4f) + \text{sinc}^2(f)$

(b) $y(t) = 4\,\text{tri}\left(\frac{t}{2}\right) - \text{rect}\left(\frac{t}{2}\right) - \text{rect}(t) \rightleftarrows Y(f) = 8\,\text{sinc}^2(2f) - 2\,\text{sinc}(2f) - \text{sinc}(f)$

Solution to S.3

Given: $\Im\{x(t)\} = \text{rect}(\pi f)$ and $y(t) = \dfrac{dx(t)}{dt}$

Applying the **Differentiation** property of the FT: $Y(f) = \Im\left\{\dfrac{d}{dt}x(t)\right\} = j2\pi f \cdot \text{rect}(\pi f)$

Applying the **Rayleigh energy theorem**:

$$E = \int_{-\infty}^{\infty} |y(t)|^2\, dt = \int_{-\infty}^{\infty} |Y(f)|^2\, dt = \int_{-\infty}^{\infty} 4\pi^2 f^2 \text{rect}^2(\pi f)\, df$$
$$= \int_{-1/(2\pi)}^{1/(2\pi)} 4\pi^2 f^2 df = \left[\frac{4\pi^2 f^3}{3}\right]_{-1/(2\pi)}^{1/(2\pi)} = \frac{1}{3\pi}$$

Solution to S.4

Given: $\Im\left\{\dfrac{\pi}{\alpha}\exp(-2\pi\alpha|t|)\right\} = \dfrac{1}{\alpha^2 + f^2}$ and $x(t) = \dfrac{1}{\alpha^2 + t^2}$.

Applying the **duality** property of FT: $\dfrac{1}{\alpha^2 + t^2} \rightleftarrows \dfrac{\pi}{\alpha}\exp(-2\pi\alpha|f|)$

$\therefore\ X(f) = \dfrac{\pi}{\alpha}\exp(-2\pi\alpha|f|)$

Let B be the 99% energy containment bandwidth of $x(t) = \dfrac{1}{\alpha^2 + t^2}$. It follows that

$$0.99 = \frac{\int_0^B |X(f)|^2\, df}{\int_0^\infty |X(f)|^2\, df} = \frac{\int_0^B \exp(-4\pi\alpha f)\, df}{\int_0^\infty \exp(-4\pi\alpha f)\, df} = 1 - \exp(-4\pi\alpha B)$$

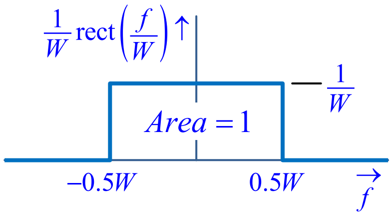
which yields

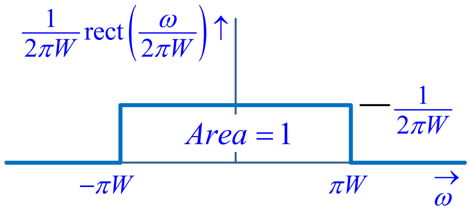
$$\exp(-4\pi\alpha B) = 0.01 \quad \text{or} \quad B = \frac{\ln(100)}{4\pi\alpha} = \frac{0.366}{\alpha}.$$

Solution to S.5

Method A:

$$\delta(f) = \lim_{W\to 0}\frac{1}{W}\operatorname{rect}\left(\frac{f}{W}\right) \xrightarrow[f=\frac{\omega}{2\pi}]{} \lim_{W\to 0}\frac{1}{W}\operatorname{rect}\left(\frac{\omega}{2\pi W}\right) = 2\pi \underbrace{\lim_{W\to 0}\frac{1}{2\pi W}\operatorname{rect}\left(\frac{\omega}{2\pi W}\right)}_{\delta(\omega)} = 2\pi\delta(\omega)$$





Method B:

$$\Im\{\delta(t)\} = 1 \;\rightarrow\; [\text{duality property}] \;\rightarrow\; \Im\{1\} = \delta(f)$$

$$\left[\begin{array}{c}\downarrow\\ \text{scaling}\\ \text{property}\\ \downarrow\end{array}\right]$$

$$\Im\{\delta(2\pi t)\} = \frac{1}{2\pi} \;\rightarrow\; [\text{duality property}] \;\rightarrow\; \begin{cases}\Im\left\{\frac{1}{2\pi}\right\} = \delta(2\pi f)\\ \frac{1}{2\pi}\Im\{1\} = \delta(\omega)\\ \Im\{1\} = 2\pi\delta(\omega)\end{cases}$$

$$\rightarrow\; \delta(f) = 2\pi\delta(\omega)$$

In GENERAL:

$$\delta(\beta x) = \frac{1}{|\beta|}\delta(x) \;\rightarrow\; \begin{cases}\delta(\omega) = \delta(2\pi f) = \frac{1}{2\pi}\delta(f)\\ \delta(f) = \delta\left(\omega/2\pi\right) = 2\pi\delta(\omega)\end{cases}$$

Solution to S.6

Forward FT:
$$\begin{cases}X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt \;\rightarrow\; \left[f = \frac{\omega}{2\pi}\right] \;\rightarrow\; X\left(\frac{\omega}{2\pi}\right) = \int_{-\infty}^{\infty} x(t)\exp\left(-j2\pi\frac{\omega}{2\pi}t\right)dt\\ \therefore\; \tilde{X}(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)\,dt\end{cases}$$

Inverse FT:
$$\begin{cases}x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)\,df \;\rightarrow\; \left[\begin{array}{l}f = \frac{\omega}{2\pi}\\ df = \frac{d\omega}{2\pi}\end{array}\right] \;\rightarrow\; x(t) = \int_{-\infty}^{\infty} X\left(\frac{\omega}{2\pi}\right)\exp\left(j2\pi\frac{\omega}{2\pi}t\right)\frac{d\omega}{2\pi}\\ \therefore\; x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{X}(\omega)\exp(j\omega t)\,d\omega\end{cases}$$
