CG2023 TUTORIAL 1 (SOLUTIONS)

Solution to Q.1

Write z in polar form:

$$z = x + jy = |z|\exp(j\angle z).$$

Since adding integer multiples of 2π to $\angle z$ does not affect the value of z, we may also express z as

$$z = |z|\exp(j(\angle z + 2k\pi)); \quad \forall k(\text{integer})$$

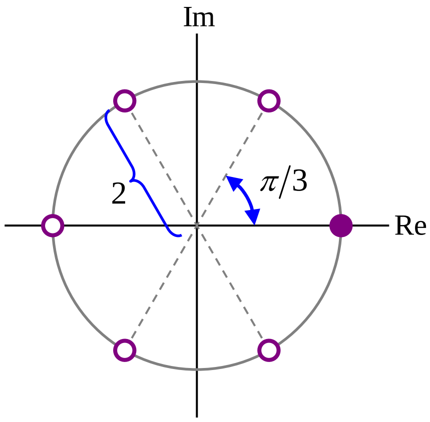
where k is an integer. This leads to

$$z^{1/N} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right); \quad \forall k(\text{integer}).$$

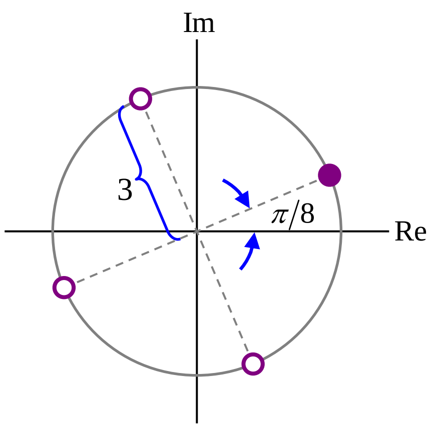
where the N distinct values of $z^{1/N}$ are

$$z^{1/N} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right); \quad k = 0, 1, \cdots, N-1.$$

$$64^{1/6}: \begin{cases} z = 64 \;\rightarrow\; \begin{cases} |z| = 64 \\ \angle z = 0 \end{cases} \\ 64^{1/6} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right)\Big|_{z=64,\, N=6} \\ \quad = 2\exp\left(j\left(\frac{k\pi}{3}\right)\right); \quad k = 0, 1, \cdots, 5 \\ \quad = \begin{cases} 2;\; 2\exp\left(j\left(\frac{\pi}{3}\right)\right);\; 2\exp\left(j\left(\frac{2\pi}{3}\right)\right); \\ -2;\; 2\exp\left(j\left(\frac{4\pi}{3}\right)\right);\; 2\exp\left(j\left(\frac{5\pi}{3}\right)\right) \end{cases} \end{cases}$$



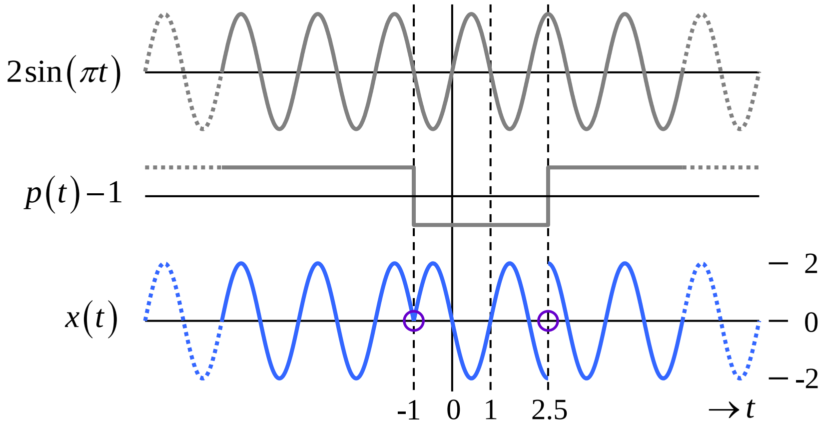
$$(j81)^{1/4}: \begin{cases} z = j81 \;\rightarrow\; \begin{cases} |z| = 81 \\ \angle z = \frac{\pi}{2} \end{cases} \\ (j81)^{1/4} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right)\Big|_{z=81,\, N=4} \\ \quad = 3\exp\left(j\left(\frac{\pi}{8} + \frac{k\pi}{2}\right)\right); \quad k = 0, 1, \cdots, 3 \\ \quad = \begin{cases} 3\exp\left(j\left(\frac{\pi}{8}\right)\right),\; 3\exp\left(j\left(\frac{5\pi}{8}\right)\right), \\ 3\exp\left(j\left(\frac{9\pi}{8}\right)\right),\; 3\exp\left(j\left(\frac{13\pi}{8}\right)\right) \end{cases} \end{cases}$$



Solution to Q.2

(a) $p(t) = 2 - 2\,\text{rect}\left(\dfrac{t-0.75}{3.5}\right)$

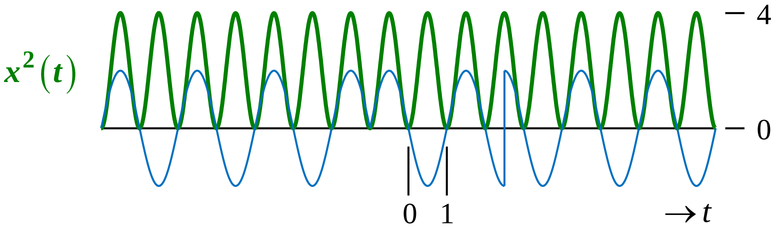
(b) By inspection, $x(t)$ is not periodic.



Notice the π rad $(\text{or } 180°)$ phase jumps in $x(t)$ occurring at the zero crossings of $p(t)-1$.

(c)

$$
\begin{aligned}
x^2(t) &= 4\sin^2(\pi t)\underbrace{\left(p(t)-1\right)^2}_{1} \\
&= 4\sin^2(\pi t) \\
&= 2\left(1-\cos(2\pi t)\right)
\end{aligned}
$$



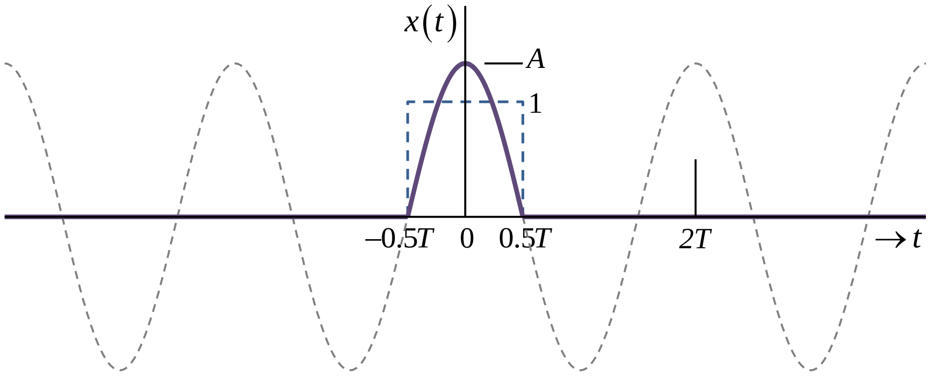
Note that $x^2(t)$ is periodic with a period of $T = 1$.

Average Power: $\left\{ P = \underbrace{\dfrac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt}_{\substack{x^2(t) \text{ is periodic. } \therefore \\ P \text{ can be obtained} \\ \text{by averaging over} \\ \text{one period.}}} = \int_{-0.5}^{0.5} 2\left(1-\cos(2\pi t)\right)dt = 2 \right.$

(d) Since the average power of $x(t)$ is finite, its total energy must be infinite. $x(t)$ is an aperiodic power signal.

Solution to Q.3

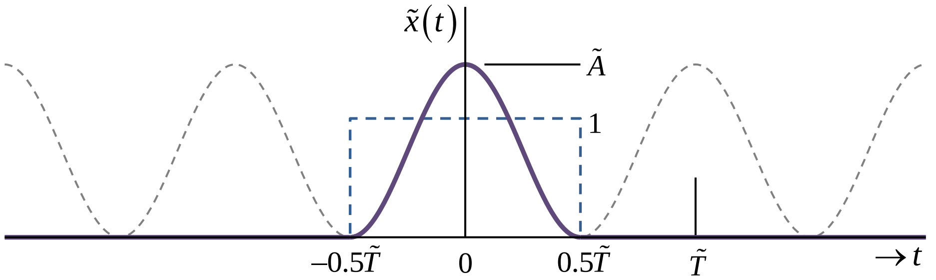
Half-cosine pulse: $x(t) = A\cos\left(\frac{\pi t}{T}\right)\text{rect}\left(\frac{t}{T}\right)$



$$x^2(t) = A^2\cos^2\left(\frac{\pi t}{T}\right)\text{rect}^2\left(\frac{t}{T}\right) = \frac{A^2}{2}\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]\text{rect}\left(\frac{t}{T}\right)$$

Energy : $E = \dfrac{A^2}{2}\displaystyle\int_{-0.5T}^{0.5T} 1 + \underbrace{\cos\left(\frac{2\pi t}{T}\right)}_{\int \text{over one period} = 0} dt = \frac{1}{2}A^2 T$

Raised-cosine pulse: $\tilde{x}(t) = \frac{\tilde{A}}{2}\left(1+\cos\left(\frac{2\pi t}{\tilde{T}}\right)\right)\text{rect}\left(\frac{t}{\tilde{T}}\right)$



$$\tilde{x}^2(t) = \frac{\tilde{A}^2}{4}\left[1+\cos\left(\frac{2\pi t}{\tilde{T}}\right)\right]^2\text{rect}^2\left(\frac{t}{\tilde{T}}\right)$$

$$= \frac{\tilde{A}^2}{4}\left[\frac{3}{2}+2\cos\left(\frac{2\pi t}{\tilde{T}}\right)+\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)\right]\text{rect}\left(\frac{t}{\tilde{T}}\right)$$

Energy : $\tilde{E} = \dfrac{\tilde{A}^2}{4}\displaystyle\int_{-0.5\tilde{T}}^{0.5\tilde{T}} \frac{3}{2} + \underbrace{2\cos\left(\frac{2\pi t}{\tilde{T}}\right)}_{\int \text{over one period} = 0} + \underbrace{\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)}_{\int \text{over two periods} = 0} dt = \frac{3}{8}\tilde{A}^2\tilde{T}$

Both $x(t)$ and $\tilde{x}(t)$ will have the same energy if $A^2T = \frac{3}{4}\tilde{A}^2\tilde{T}$.

Solution to Q.4

$$x(t) = \cos(3.2t) + \sin(1.6t) + \exp(j2.8t) \quad \cdots \quad \begin{cases} \cos(3.2t) & \text{has a frequency of } 3.2 \ \ rad/s \\ \sin(1.6t) & \text{has a frequency of } 1.6 \ \ rad/s \\ \exp(j2.8t) & \text{has a frequency of } 2.8 \ \ rad/s \end{cases}$$

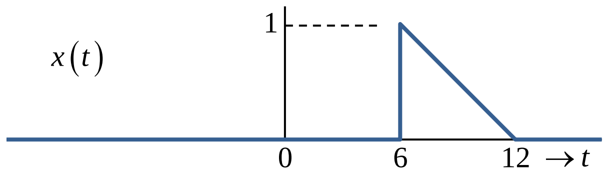
Highest common factor (HCF) of $\{3.2,\ 1.6,\ 2.8\}$ exists and is equal to 0.4. Thus, $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$. The period of $x(t)$ is $\dfrac{2\pi}{0.4} = 5\pi\ s$.

REMARKS: *Although $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$, it does not contain the fundamental frequency component itself.*

Solution to Q.5

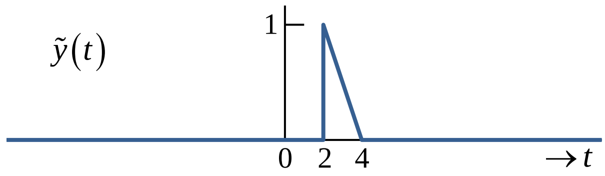
We observe that $y(t)$ is a time-scaled, -reversed and -shifted version of $x(t)$.

For problems of this nature, we should start with time-scaling first since it involves linear warping of the time axis. If we were to start with time-shifting and/or time-reversal, we may have to redo them after time-scaling. However, this sequence of operation need not be followed if we are sketching the signal from the mathematical expression.



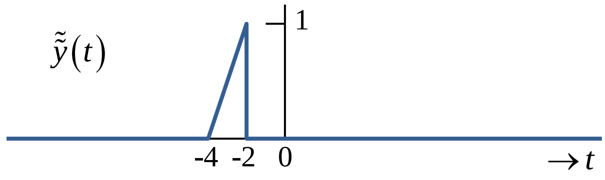
Time-scale $x(t)$ by a factor of 3:

$$\tilde{y}(t) = x(3t)$$



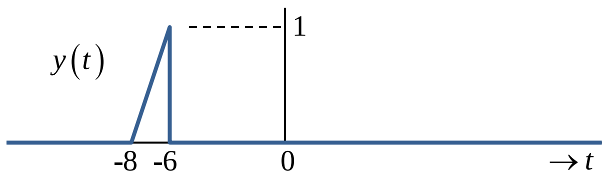
Time-reverse $\tilde{y}(t)$:

$$\tilde{\tilde{y}}(t) = \tilde{y}(-t) = x(-3t)$$



Time-shift (advance) $\tilde{\tilde{y}}(t)$ by 4 units:

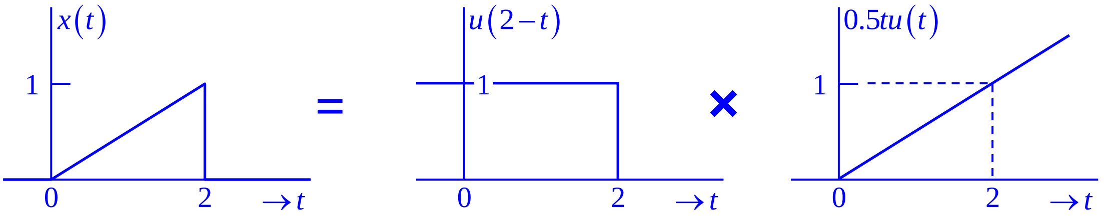
$$y(t) = \tilde{\tilde{y}}(t+4) = x(-3(t+4))$$



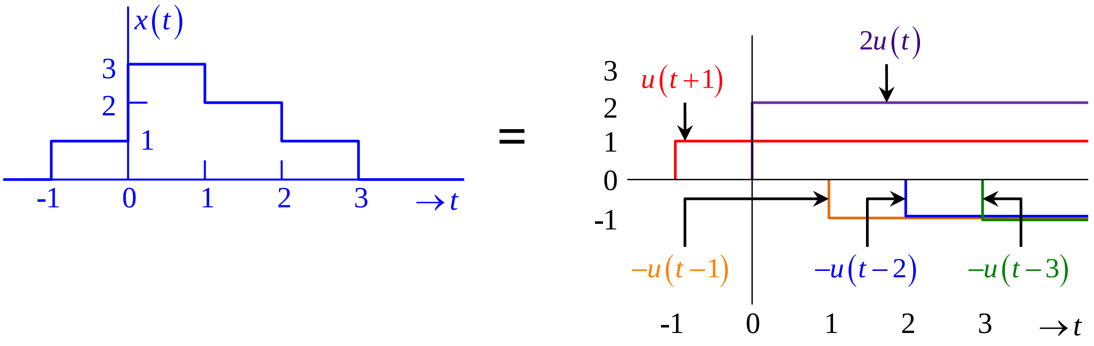
Therefore, $y(t) = x(-3t - 12)$

Solution to S.1

(a) $x(t) = u(2-t)\cdot 0.5tu(t) = u(2-t)\cdot \int_{-\infty}^{t} 0.5u(\tau)d\tau$



(b) $x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$



Solution to S.2

(a) When $t<0$: $\int_{-\infty}^{t}\cos(\tau)u(\tau)d\tau=0$

When $t\geq 0$: $\int_{-\infty}^{t}\cos(\tau)u(\tau)d\tau=\int_{0}^{t}\cos(\tau)d\tau=\sin(\tau)\big|_{0}^{t}=\sin(t)$

Combining the 2 cases: $\int_{-\infty}^{t}\cos(\tau)u(\tau)d\tau=\sin(t)u(t)$

(b) When $t<0$: $\int_{-\infty}^{t}\cos(\tau)\delta(\tau)d\tau=0$

When $t\geq 0$: $\int_{-\infty}^{t}\cos(\tau)\delta(\tau)d\tau=1$

Combining the 2 cases: $\int_{-\infty}^{t}\cos(\tau)\delta(\tau)d\tau=u(t)$

(c) $\int_{-\infty}^{\infty}\cos(t)u(t-1)\delta(t)dt=0$ because $u(t-1)\delta(t)=0\ \forall t$

(d) $\underbrace{\int_{0}^{2\pi}t\sin\left(\frac{t}{2}\right)\delta(\pi-t)dt=\pi\sin\left(\frac{\pi}{2}\right)}_{\text{sifting property of }\delta\text{-function}}=\pi$

Solution to S.3

(a) $x(t)=u(t)=\begin{cases}1; & t\geq 0\\ 0; & t<0\end{cases}$

$x_e(t)=0.5\left[u(t)+u(-t)\right]=\begin{cases}1; & t=0\\ 0.5; & t\neq 0\end{cases}$

$x_o(t)=0.5\left[u(t)-u(-t)\right]=\begin{cases}0; & t=0\\ 0.5; & t>0\\ -0.5; & t<0\end{cases}$

(b) $x(t)=\sin\left(\omega_c t+\frac{\pi}{4}\right)$

$$x_e(t)=0.5\left[\sin\left(\omega_c t+\frac{\pi}{4}\right)+\sin\left(-\omega_c t+\frac{\pi}{4}\right)\right]$$
$$=\sin\left(\frac{\pi}{4}\right)\cos(\omega_c t)=\frac{1}{\sqrt{2}}\cos(\omega_c t)$$

$$x_o(t)=0.5\left[\sin\left(\omega_c t+\frac{\pi}{4}\right)-\sin\left(-\omega_c t+\frac{\pi}{4}\right)\right]$$
$$=0.5\left[\sin\left(\omega_c t+\frac{\pi}{4}\right)+\sin\left(\omega_c t-\frac{\pi}{4}\right)\right]$$
$$=\sin(\omega_c t)\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\sin(\omega_c t)$$

where we make use of the trigonometric relationship $\sin(A)+\sin(B)=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{B-A}{2}\right)$.
