2. Discrete-Frequency Spectrum (Fourier Series)

2.1 What is a Spectrum in the Context of Signals?

- The frequency-domain representation of a signal is called the **spectrum** of the signal.
- Any energy or power signal has a corresponding spectrum. This includes familiar signals such as visible light (color), musical notes and radio/TV transmissions.
- From a spectrum, certain physical descriptions of the signal characteristics become much simpler. For example:

Why are infrared sensors blind to ultraviolet light?
Ans: *Infrared and ultraviolet lights have different frequencies.*

How do we describe Julie Andrews' coloratura soprano voice before it was damaged by a throat operation in 1997?
Ans: *It spanned an astonishing and thrilling four-octave.*

How do we explain why Internet and Cable TV signals can be concurrently brought to a subscriber home using the same cable without interfering with each other.
Ans: *They occupy nonoverlapping frequency bands.*

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- The mathematical model of a spectrum is, in general, a *complex* function of frequency. The graphical representation of a spectrum thus consists of two plots: the *magnitude spectrum* and the *phase spectrum*. In some cases, these may be combined into a single spectral plot.

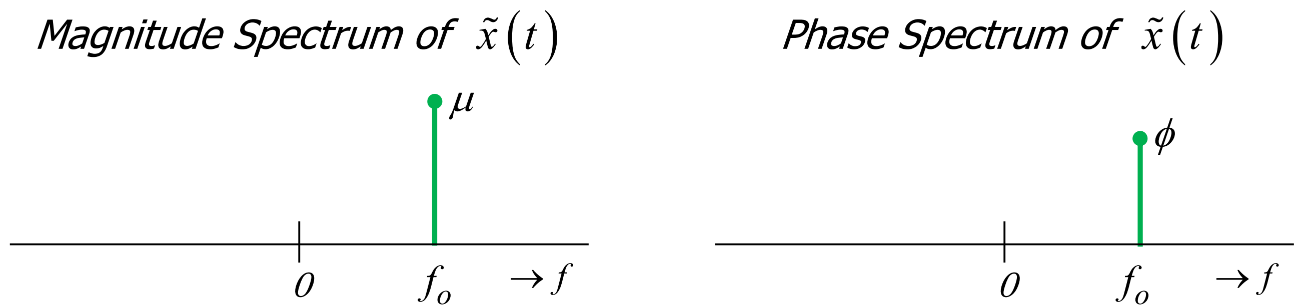
2.2 Spectrum of a Sinusoid

- ***Spectrum of a complex exponential signal***

Signal Model:

$$\tilde{x}(t) = \underbrace{\mu \exp\left[j\left(2\pi f_p t + \phi\right)\right]}_{\text{complex exponential}} = \underbrace{\mu \exp\left(j\phi\right)}_{\text{spectrum}} \exp\left(j2\pi f_p t\right) \quad \begin{cases} \text{Magnitude Spectrum:} & \mu \\ \text{Phase Spectrum:} & \phi \\ \text{Frequency:} & f_p \end{cases}$$

Spectral Plot:



Exercise: Plot the magnitude and phase spectrum of $\tilde{x}^*(t)$.

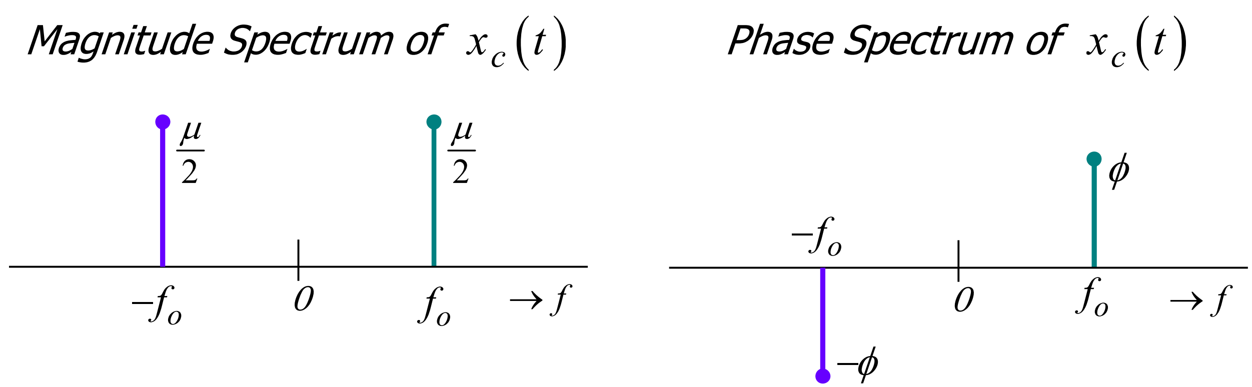
- ***Spectrum of a cosine signal***

Signal Model:

$$x_c(t) = \overbrace{\underbrace{\mu\cos\left(2\pi f_o t+\phi\right)}_{\text{cosine}} = \frac{1}{2}\underbrace{\mu\exp\left[j\left(2\pi f_o t+\phi\right)\right]}_{\tilde{x}(t)} + \frac{1}{2}\underbrace{\mu\exp\left[-j\left(2\pi f_o t+\phi\right)\right]}_{\tilde{x}^*(t)}}^{\text{applying Euler's formula}}$$

$$= \frac{\mu}{2}\exp\left(j\phi\right)\exp\left(j2\pi f_o t\right) + \frac{\mu}{2}\exp\left(j\left(-\phi\right)\right)\exp\left(j2\pi\left(-f_o\right)t\right)$$

Spectral Plot:



- ***Spectrum of a sine signal***

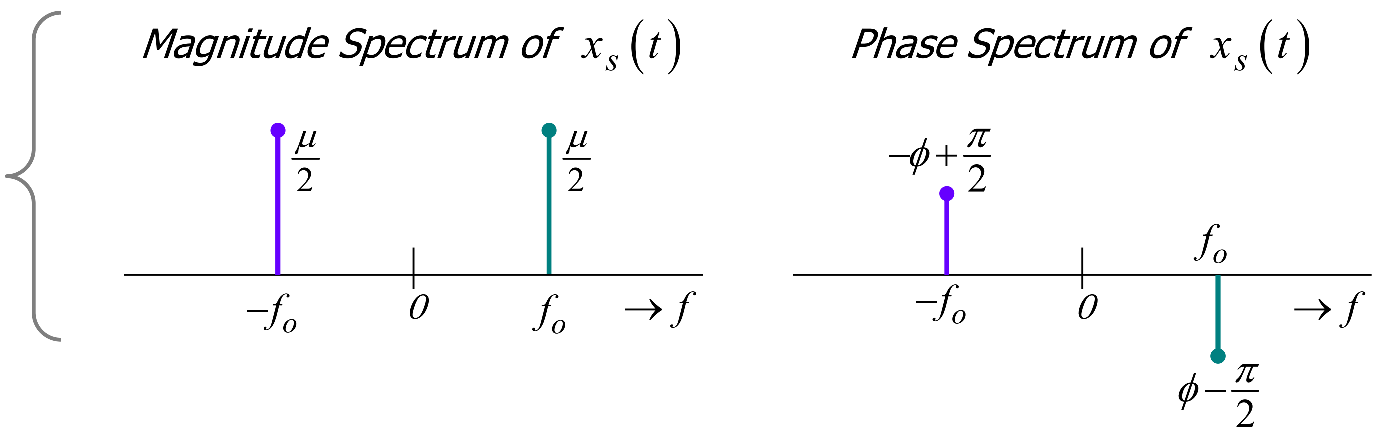
Signal Model:

$$x_s(t) = \overbrace{\underbrace{\mu \sin(2\pi f_o t + \phi)}_{\text{sine}} = \frac{1}{j2}\underbrace{\mu \exp\left[j(2\pi f_o t + \phi)\right]}_{\tilde{x}(t)} - \frac{1}{j2}\underbrace{\mu \exp\left[-j(2\pi f_o t + \phi)\right]}_{\tilde{x}^*(t)}}^{\text{applying Euler's formula}}$$

$$\cdots\cdots \quad with \quad j = \exp\left(j\frac{\pi}{2}\right)$$

$$= \frac{\mu}{2}\exp\left[j\left(\phi - \frac{\pi}{2}\right)\right]\exp(j2\pi f_o t) + \frac{\mu}{2}\exp\left[j\left(-\phi + \frac{\pi}{2}\right)\right]\exp\left(j2\pi(-f_o)t\right)$$

Spectral Plot:



Example 2-1:

Sketch the magnitude and phase spectra of $\mathbf{x(t) = 2\sin\left(8\pi t + \frac{\pi}{6}\right)}$***.***

Using Euler's formula

$$\exp(j\theta) = \cos(\theta) + j\sin(\theta) \;\rightarrow\; \begin{cases} \cos(\theta) = 0.5\left[\exp(j\theta) + \exp(-j\theta)\right] \\ \sin(\theta) = \dfrac{0.5}{j}\left[\exp(j\theta) - \exp(-j\theta)\right]' \end{cases}$$

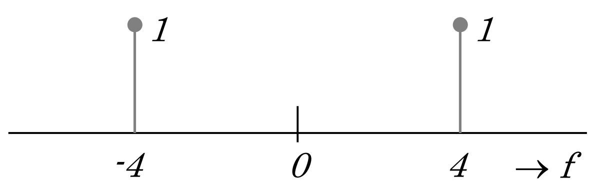
we express $x(t)$ *in terms of complex exponentials:*

$$x(t) = 2\sin\left(8\pi t + \frac{\pi}{6}\right) = 2\cdot\frac{1}{j2}\left\{\exp\left[j\left(2\pi(4)t + \frac{\pi}{6}\right)\right] - \exp\left[-j\left(2\pi(4)t + \frac{\pi}{6}\right)\right]\right\}$$

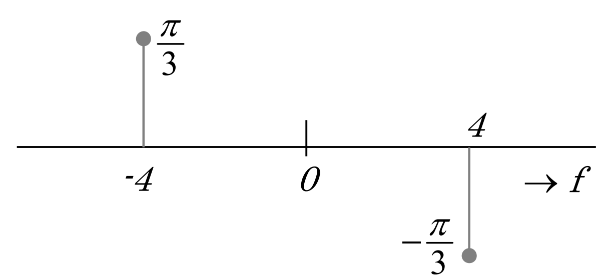
$$= \exp\left(-j\frac{\pi}{2}\right)\exp\left(j\frac{\pi}{6}\right)\exp\left(j2\pi(4)t\right) + \exp\left(j\frac{\pi}{2}\right)\exp\left(-j\frac{\pi}{6}\right)\exp\left(j2\pi(-4)t\right)$$

$$= \exp\left(-j\frac{\pi}{3}\right)\exp\left(j2\pi(4)t\right) + \exp\left(j\frac{\pi}{3}\right)\exp\left(j2\pi(-4)t\right)$$

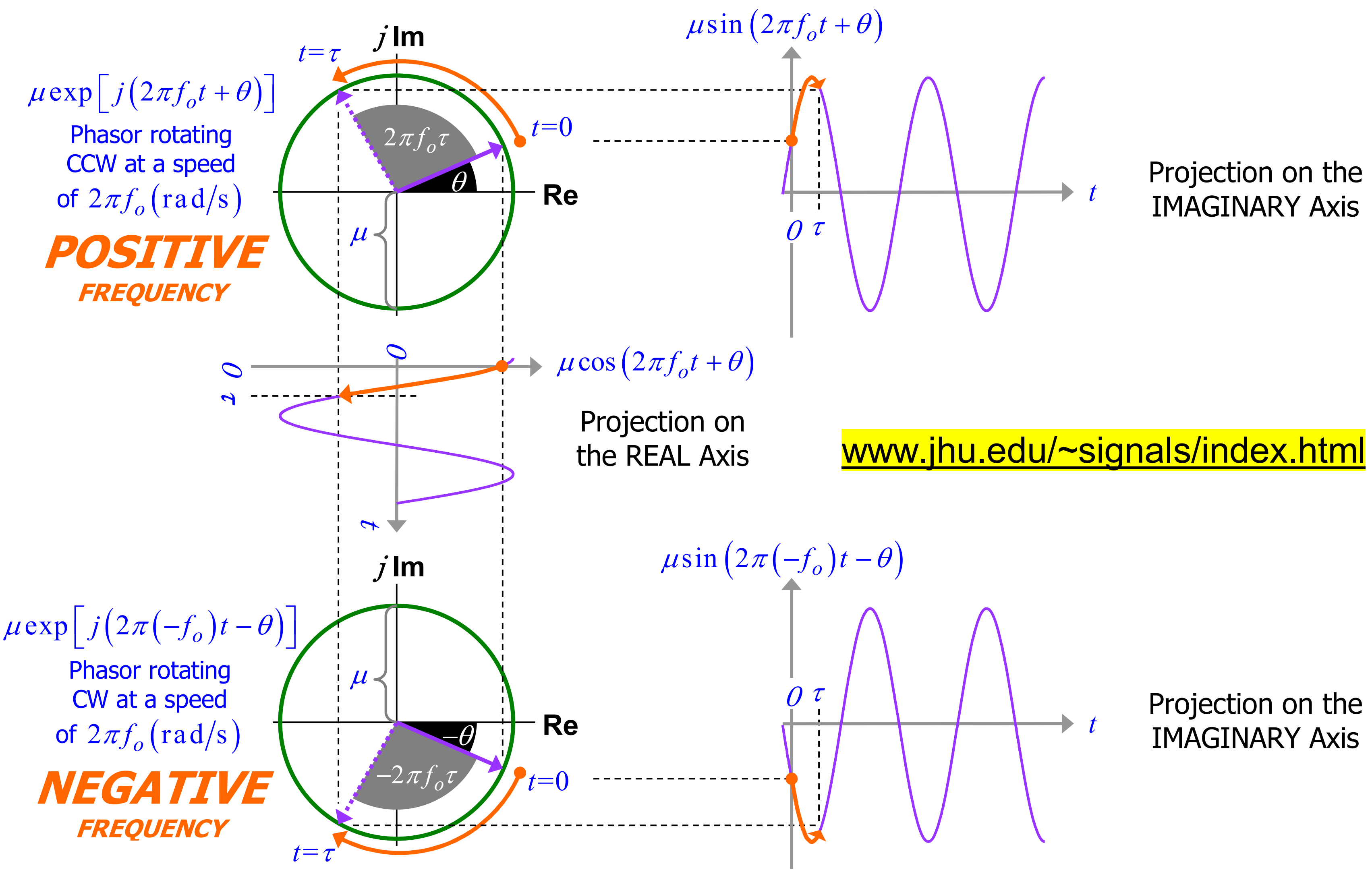
Magnitude Spectrum



Phase Spectrum



2.2.1 **Complex Exponentials and Phasors** (The concept of negative frequency)



2.3 Fourier Series

Unlike sinusoids, the spectra of non-sinusoidal periodic signals such as square wave, sawtooth wave, etc., cannot be determine simply by inspection. The spectra of such signals are derived using a technique called Fourier series.

Fourier series is a way to represent a general periodic signal as the sum of sinusoidal waves. More formally, it decomposes any periodic signal into the sum of (possibly infinite) simple oscillating functions, namely sines and cosines (or, equivalently, complex exponentials).

2.3.1 Complex Exponential Fourier Series

- Any bounded *periodic signal*, $x_p(t)$, can be represented by a sum of *harmonically related* complex sinusoids:

$$x_p(t) = \underbrace{\sum_{k=-\infty}^{\infty} c_k \exp\left(j2\pi kt/T_p\right)}_{\textbf{Fourier series expansion}} \tag{2.1a}$$

  where $1/T_p$ is the fundamental frequency and k/T_p is the k^{th} harmonic frequency of $x_p(t)$.

- c_k are called *Fourier series coefficients* of $x_p(t)$. They constitute the *discrete-frequency spectrum* of $x_p(t)$.

- Given $x_p(t)$, how do we determine the k^{th} Fourier series coefficient, c_k?

  To determine c_k, we multiply $x_p(t)$ by $\exp\left(-j2\pi kt/T_p\right)$ and integrate the product over any one period:

$$\int_{t_o}^{t_o+T_p} x_p(t)\exp\left(-j2\pi kt/T_p\right)dt = \int_{t_o}^{t_o+T_p} \exp\left(-j2\pi kt/T_p\right)\sum_{m=-\infty}^{\infty} c_m \exp\left(j2\pi m\,t/T_p\right)dt$$

$$= \sum_{m=-\infty}^{\infty} c_m \int_{t_o}^{t_o+T_p} \exp\left[-j2\pi(k-m)t/T_p\right]dt$$

$$= \sum_{m=-\infty}^{\infty} c_m \left[\frac{\exp\left[-j2\pi(k-m)t/T_p\right]}{-j2\pi(k-m)/T_p}\right]_{t_o}^{t_o+T_p}$$

$$= \sum_{m=-\infty}^{\infty} c_m \underbrace{\left(\frac{\exp\left[-j2\pi(k-m)t_o/T_p\right]\left[\exp\left[-j2\pi(k-m)\right]-1\right]}{-j2\pi(k-m)/T_p}\right)}_{=\begin{cases} T_p; & m=k \\ 0; & m\neq k\end{cases}}$$

$$= c_k T_p$$

This yields:

$$\therefore c_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\exp\left(-j2\pi kt/T_p\,t\right)dt, \qquad k = 0, \pm 1, \pm 2, \cdots \tag{2.1b}$$

2.3.2 Trigonometric Fourier Series

- The Fourier series expansion of $x_p(t)$ in (2.1a) can also be expressed in terms of cosine and sine functions:

$$\begin{aligned}
x_p(t) &= \sum_{k=-\infty}^{\infty} c_k \exp\left(j2\pi kt/T_p\right) \\
&= \sum_{k=-\infty}^{-1} c_k \exp\left(j2\pi kt/T_p\right) + c_0 + \sum_{k=1}^{\infty} c_k \exp\left(j2\pi kt/T_p\right) \\
&= c_0 + \sum_{k=1}^{\infty}\left[c_{-k}\exp\left(-j2\pi kt/T_p\right) + c_k \exp\left(j2\pi kt/T_p\right)\right] \\
&= c_0 + \sum_{k=1}^{\infty}\left[c_{-k}\cos\left(2\pi kt/T_p\right) - jc_{-k}\sin\left(2\pi kt/T_p\right) + c_k\cos\left(2\pi kt/T_p\right) + jc_k\sin\left(2\pi kt/T_p\right)\right] \\
&= c_0 + \sum_{k=1}^{\infty}\left[\left(c_k + c_{-k}\right)\cos\left(2\pi kt/T_p\right) + j\left(c_k - c_{-k}\right)\sin\left(2\pi kt/T_p\right)\right] \\
&= a_0 + 2\sum_{k=1}^{\infty}\left[a_k\cos\left(2\pi kt/T_p\right) + b_k\sin\left(2\pi kt/T_p\right)\right]
\end{aligned} \tag{2.2a}$$

where

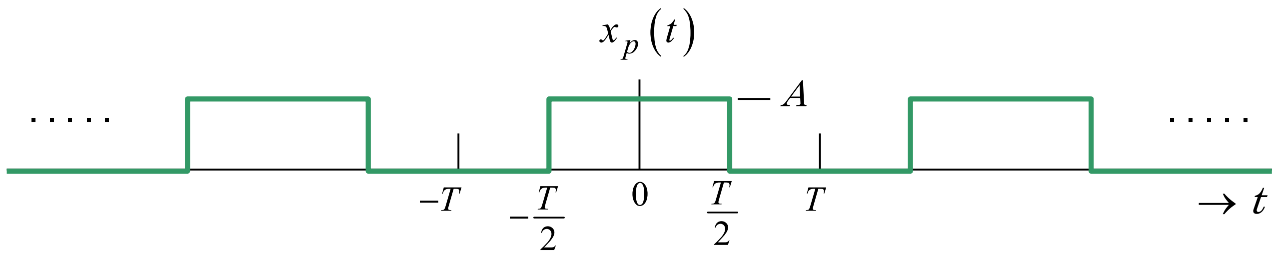
$$\begin{aligned}
a_k &= \frac{c_{-k}+c_k}{2} = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\cos\left(2\pi kt/T_p\right)dt; \qquad k \ge 0 \\
b_k &= \frac{c_{-k}-c_k}{j2} = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\sin\left(2\pi kt/T_p\right)dt; \qquad k > 0
\end{aligned} \tag{2.2b}$$

Table 2-1

SUMMARY	
Fourier Series (*complex* **exp** kernel) a.k.a. **Complex Exponential Fourier Series**	Fourier Analysis (forward transform) $$c_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\exp\left(-j2\pi kt/T_p\right)dt, \qquad k = 0, \pm 1, \pm 2, \cdots$$ Fourier Synthesis (inverse transform) $$x_p(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(j2\pi kt/T_p\right)$$
Fourier Series (**cos & sin** kernels) a.k.a. **Trigonometric Fourier Series**	Fourier Analysis (forward transform) $$a_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\cos\left(2\pi kt/T_p\right)dt; \qquad k \geq 0$$ $$b_k = \frac{1}{T_p}\int_{t_o}^{t_o+T_p} x_p(t)\sin\left(2\pi kt/T_p\right)dt; \qquad k > 0$$ Fourier Synthesis (inverse transform) $$x_p(t) = a_0 + 2\sum_{k=1}^{\infty}\left[a_k\cos\left(2\pi kt/T_p\right) + b_k\sin\left(2\pi kt/T_p\right)\right]$$

Example 2-2:

Spectrum of a square wave, $x_p(t)$.



Period = $T_p = 2T$ *, Pulse Width =* T

The complex exponential Fourier series expansion of $x_p(t)$ *is given by*

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k \exp\left(j2\pi \frac{k}{T_p} t \right); \quad T_p = 2T$$

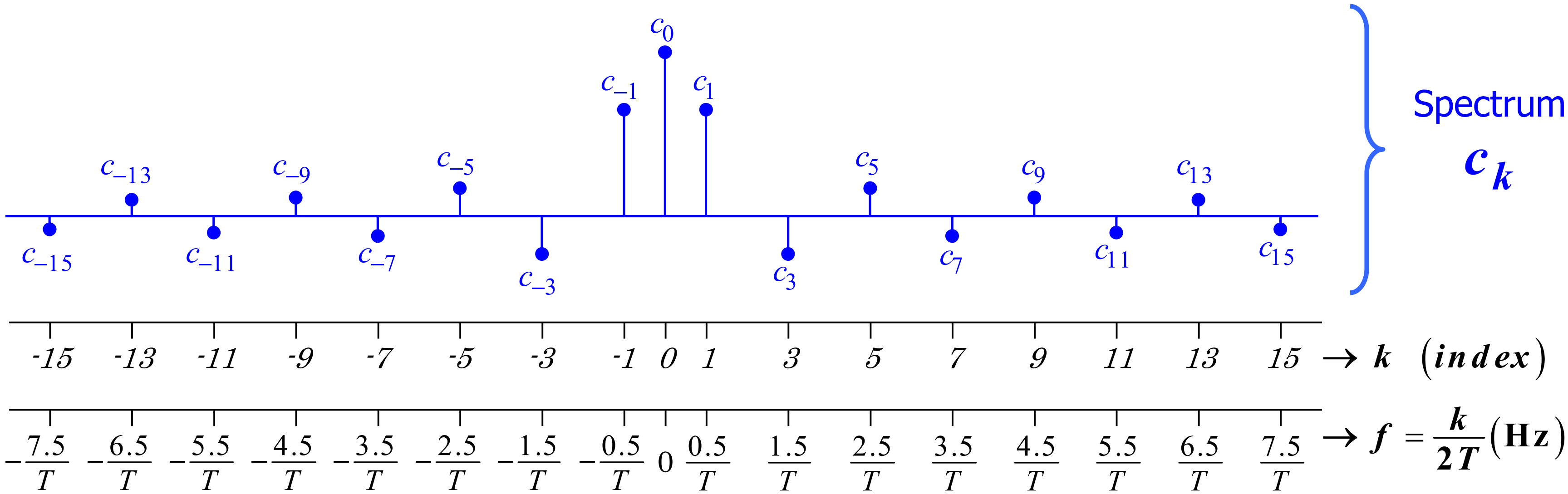
where

$$c_k = \frac{1}{2T} \int_{-T}^{T} x_p(t) \exp\left(-j2\pi \frac{k}{2T} t \right) dt = \frac{A}{2T} \int_{-0.5T}^{0.5T} \exp\left(-j\pi \frac{k}{T} t \right) dt = \frac{A}{2T} \left[\frac{\exp(-j\pi k\, t/T)}{-j\pi\, k/T} \right]_{-0.5T}^{0.5T}$$

$$= \frac{A}{2T} \left[\frac{\exp(j0.5\pi k)}{j\pi\, k/T} - \frac{\exp(-j0.5\pi k)}{j\pi\, k/T} \right] = \frac{A}{2} \left[\frac{\sin(0.5\pi k)}{0.5\pi k} \right] = \frac{A}{2} \operatorname{sinc}\left(\frac{k}{2} \right)$$

Note that $c_k = c_{-k}$*, and* $c_k = 0$ *when* $k = \pm 2, \pm 4, \pm 6, \cdots$.

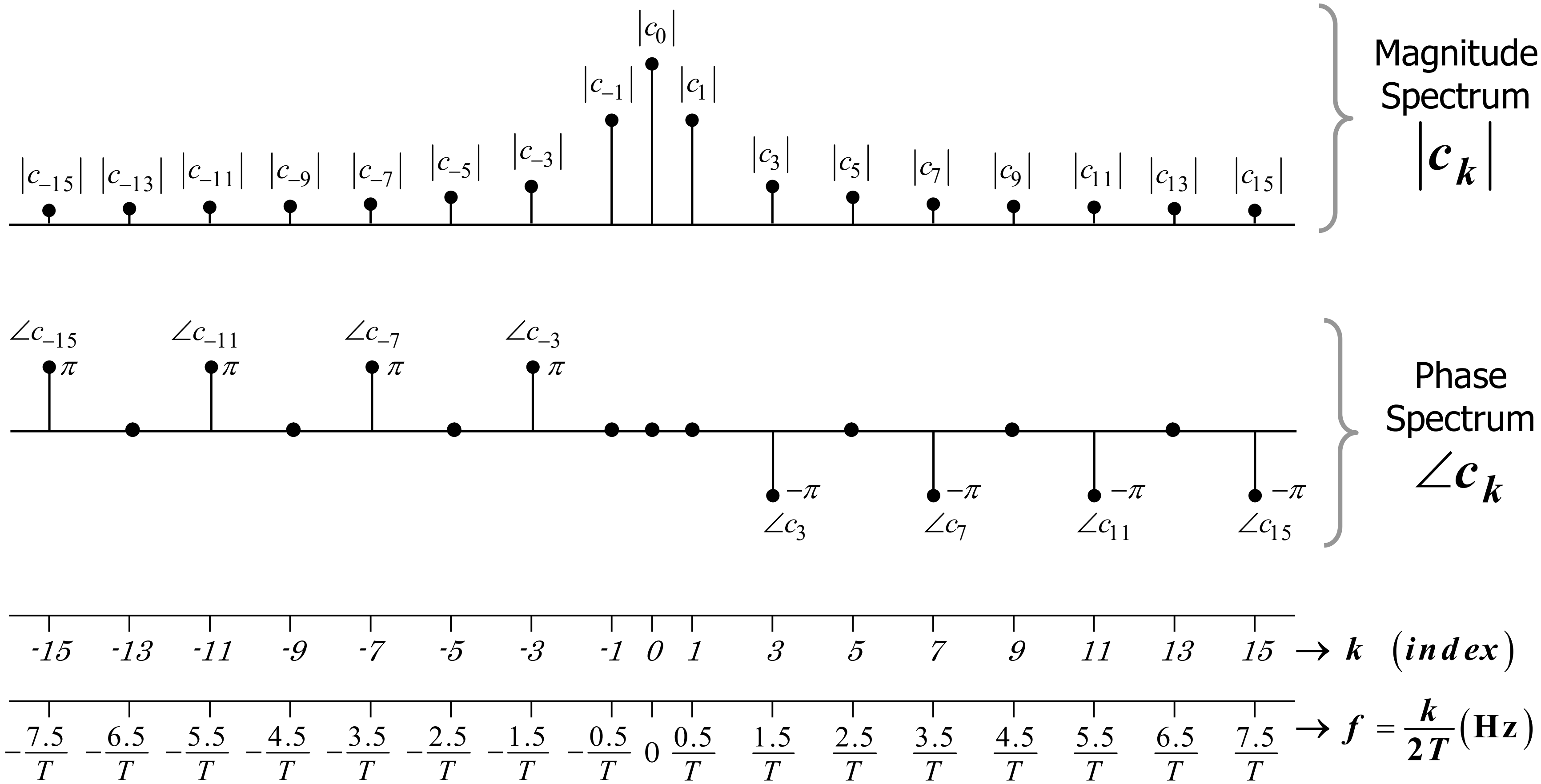
Since c_k is real, the spectrum can be depicted by a ***single*** *spectral plot:*



We may also write c_k in terms of its magnitude and phase:

$$c_k = \frac{A}{2}\operatorname{sinc}\left(\frac{k}{2}\right) = |c_k|\exp\left(j\angle c_k\right) \quad \text{where} \quad \begin{cases} |c_k| = \dfrac{A}{2}\left|\operatorname{sinc}\left(\dfrac{k}{2}\right)\right| \\[2ex] \angle c_k = \begin{cases} 0; & \text{if } c_k \ge 0 \\ \pm\pi; & \text{if } c_k < 0 \end{cases} \end{cases}$$

where the corresponding plots are shown below:



Now, let us examine the trigonometric Fourier series expansion of $x_p(t)$, noting that $T_p = 2T$ and $c_k = c_{-k}$. Applying (2.2b), we get

$$a_k = \frac{c_{-k} + c_k}{2} = c_k \quad \textit{and} \quad b_k = \frac{c_{-k} - c_k}{j2} = 0$$

which, when substituted into (2.2a), yields

$$x_p(t) = c_0 + 2\sum_{k=1}^{\infty} c_k \cos\left(\frac{\pi k t}{T}\right).$$

Here,

$$\left.\begin{array}{lll} c_0 & : & \textit{DC component of } x_p(t) \\ 2c_1 \cos(\pi t/T) & : & \textit{Fundamental frequency component } x_p(t) \\ 2c_k \cos(\pi k t/T) & : & k^{th}\textit{ - harmonic } x_p(t); \quad k > 1 \end{array}\right\} \textit{ in which } c_k = \frac{A}{2}\mathrm{sinc}\left(\frac{k}{2}\right).$$

Note that $[c_0 = 0.5A]$ and $[c_k = 0;\ k = \pm 2, \pm 4, \pm 6, \cdots]$. The latter implies that $x_p(t)$ has no even harmonics.

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Example 2-3:

Consider the signal

$$\boldsymbol{x(t) = (1+j)e^{-j6t} + 3je^{-j4t} + 4 - 3je^{j4t} + (1-j)e^{j6t}}.$$

Show whether or not $x(t)$ is real and periodic. If $x(t)$ is periodic, find its complex exponential Fourier series coefficients, c_k, and sketch its magnitude and phase spectra.

$\boldsymbol{x(t)}$ ***is REAL:***

Reason: Except for a constant term, $x(t)$ is composed purely of complex sinusoids that come in conjugate pairs. This allows $x(t)$ to be re-written as

$$\begin{aligned} x(t) &= (1+j)e^{-j6t} + 3je^{-j4t} + 4 - 3je^{j4t} + (1-j)e^{j6t} \\ &= e^{j0.25\pi}e^{-j6t} + 3e^{j0.5\pi}e^{-j4t} + 4 + 3e^{-j0.5\pi}e^{j4t} + e^{-j0.25\pi}e^{j6t} \\ &= 4 + 3\left[e^{-j(4t-0.5\pi)} + e^{j(4t-0.5\pi)}\right] + \left[e^{-j(6t-0.25\pi)} + e^{j(6t-0.25\pi)}\right] \\ &= 4 + 6\cos(4t-0.5\pi) + 2\cos(6t-0.25\pi) \end{aligned}$$

which indicates that $x(t)$ is real.

$x(t)$ is PERIODIC

Reason: *The sinusoidal components are harmonics, i.e. their frequencies have a highest common factors (HCF), which is the fundamental frequency of* $x(t)$. *The HCF in this case is*

$$HCF\{-6,-4,4,6\}=2,$$

which indicates that $x(t)$ *is periodic with a fundamental frequency of* 2 rad/s.

Complex Exponential Fourier series expansion of $x(t)$:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2kt} \\ &= \cdots + c_{-3}e^{-j6t} + c_{-2}e^{-j4t} + c_{-1}e^{-j2t} + c_0 + c_1 e^{j2t} + c_2 e^{j4t} + c_3 e^{j6t} + \cdots \end{aligned}$$

Comparing this with

$$\begin{aligned} x(t) &= (1+j)e^{-j6t} + 3je^{-j4t} + 10 - 3je^{j4t} + (1-j)e^{j6t} \\ &= e^{j0.25\pi}e^{-j6t} + 3e^{j0.5\pi}e^{-j4t} + 4 + 3e^{-j0.5\pi}e^{j4t} + e^{-j0.25\pi}e^{j6t} \end{aligned}$$

we conclude that

$$c_{-3} = e^{j0.25\pi},\quad c_{-2} = 3e^{j0.5\pi},\quad c_0 = 4,\quad c_2 = 3e^{-j0.5\pi},\quad c_3 = e^{-j0.25\pi}$$

$$c_k = 0;\quad |k| \neq 0, 2, 3$$

Magnitude and Phase Spectra of $x(t)$: