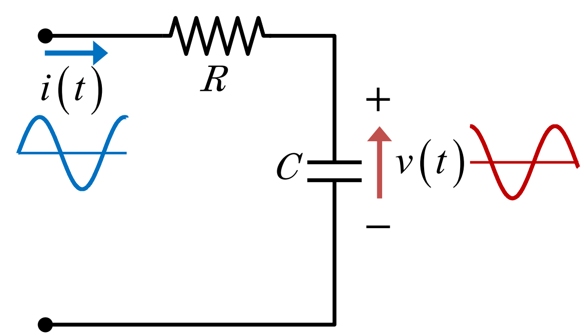
1. Signals and Classification of Signals

In this chapter, we introduce the notion of signals and examine the way in which signals are classified into various categories in accordance with their properties. We also define several important basic signals that are essential to our studies.

1.1 Signals

- Signals can manifest in many forms such as electrical voltage or current, radio wave, infrared and ultraviolet rays, light wave, sound wave, mechanical pressure, etc.
- In signal studies, a signal is a function representing a physical quantity that conveys information about the behavior or nature of a phenomenon.
- Mathematically, a signal is represented as a function of an independent variable t.
    - Usually t represents time and a signal is denoted by $x(t)$. In this case, $x(t)$ is the ***time-domain*** representation of the signal.
- For example, the voltage $v(t)$ may represent the output signal and the current $i(t)$ may represent the input signal of the RC circuit shown on the right.



1.1.1 Classification of Signals

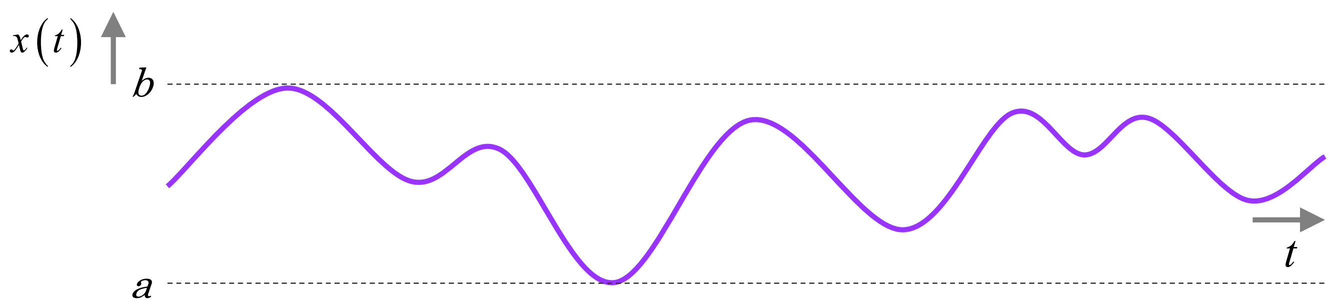
A. *Analog and Digital signals*

Continuous-time signal:

- A signal $x(t)$ is a **continuous-time signal** if t is a continuous variable.
- $x(t)$ is usually depicted as a **waveform**.

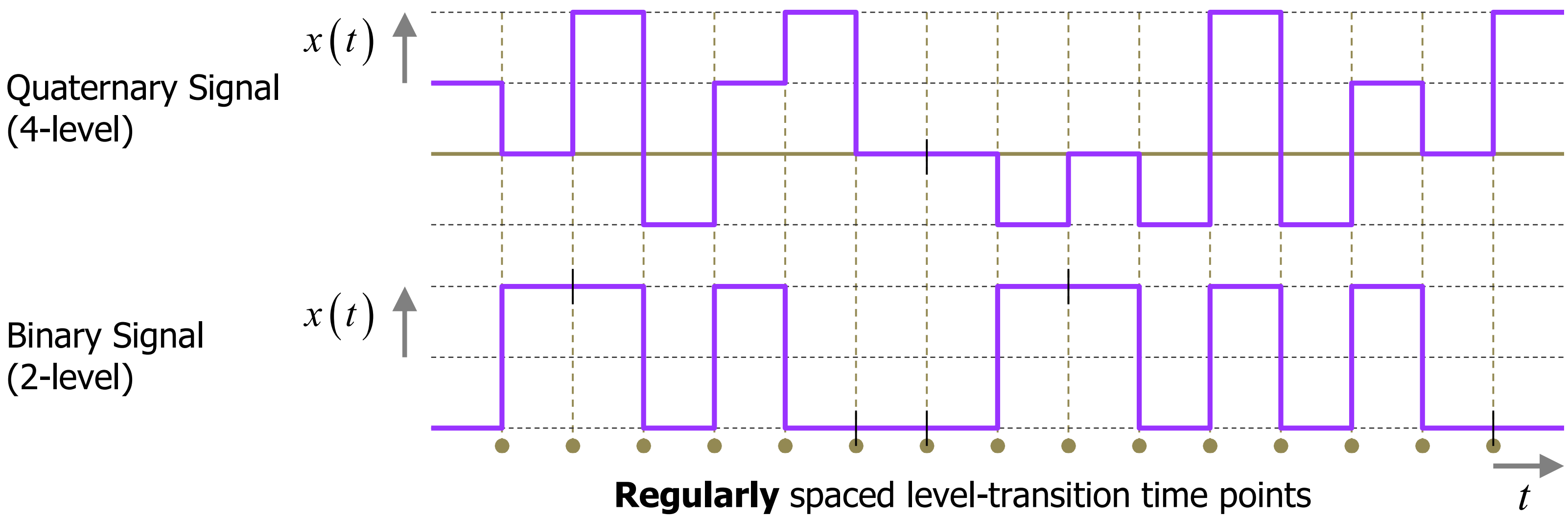
ANALOG SIGNAL - A **continuous-time signal** that can take on any value in the continuous interval (a,b), where a may be $-\infty$ and b may be $+\infty$.

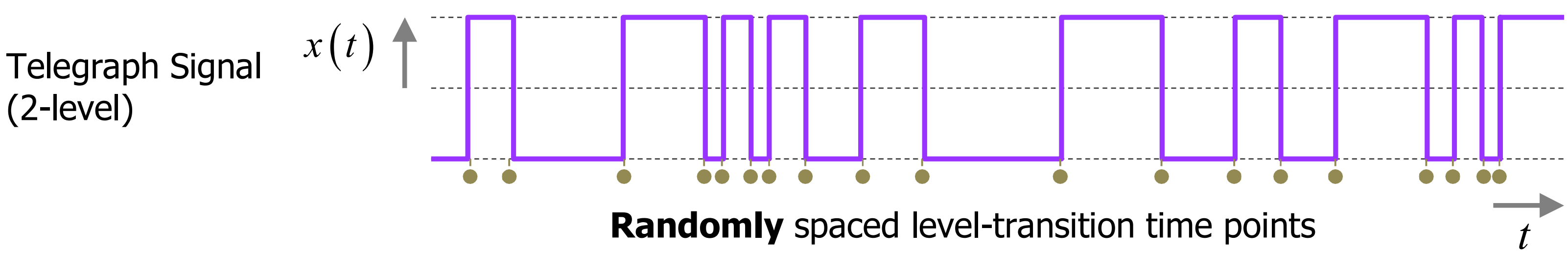
Graphical representation:



DIGITAL SIGNAL - A **continuous-time signal** that can take on only a finite number of distinct levels.

Graphical representation:





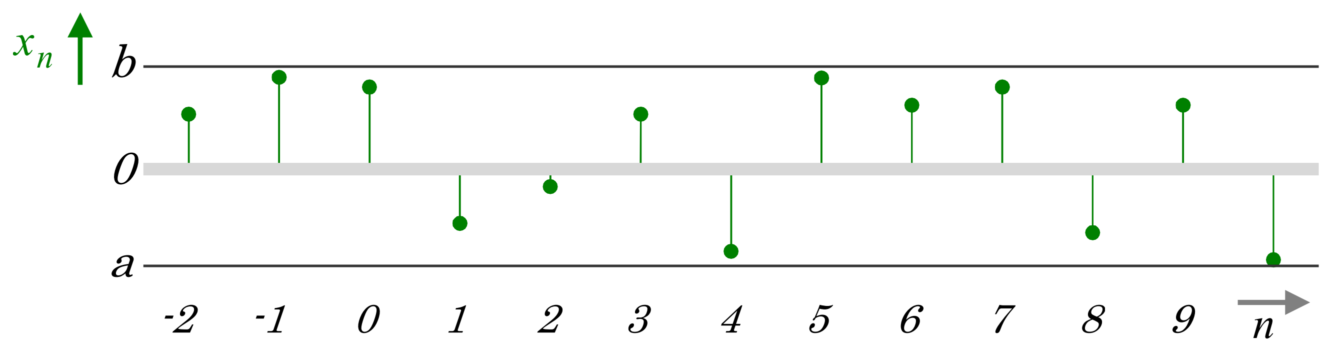
Discrete-time signal:

- A signal is discrete-time if it is defined only at discrete time points.
- A discrete-time signal is usually denoted by x_n, where n is an integer, and depicted as a **sequence of numbers** such as

$$\left\{\cdots\cdots\ x_{-1}, x_0, x_1,\ \cdots\cdots\ , x_n,\ \cdots\cdots\right\}$$

- Discrete-time signals may evolve naturally, for instance the daily closing stock market average which occurs only at the close of each day, or obtained by **sampling** a **continuous-time signal** $x(t)$ such as $x_n = x(t_n)$ where t_n are discrete time points.
- In general, a discrete-time signal can take on any value in the continuous interval (a,b), where a may be $-\infty$ and b may be $+\infty$.

Graphical representation:

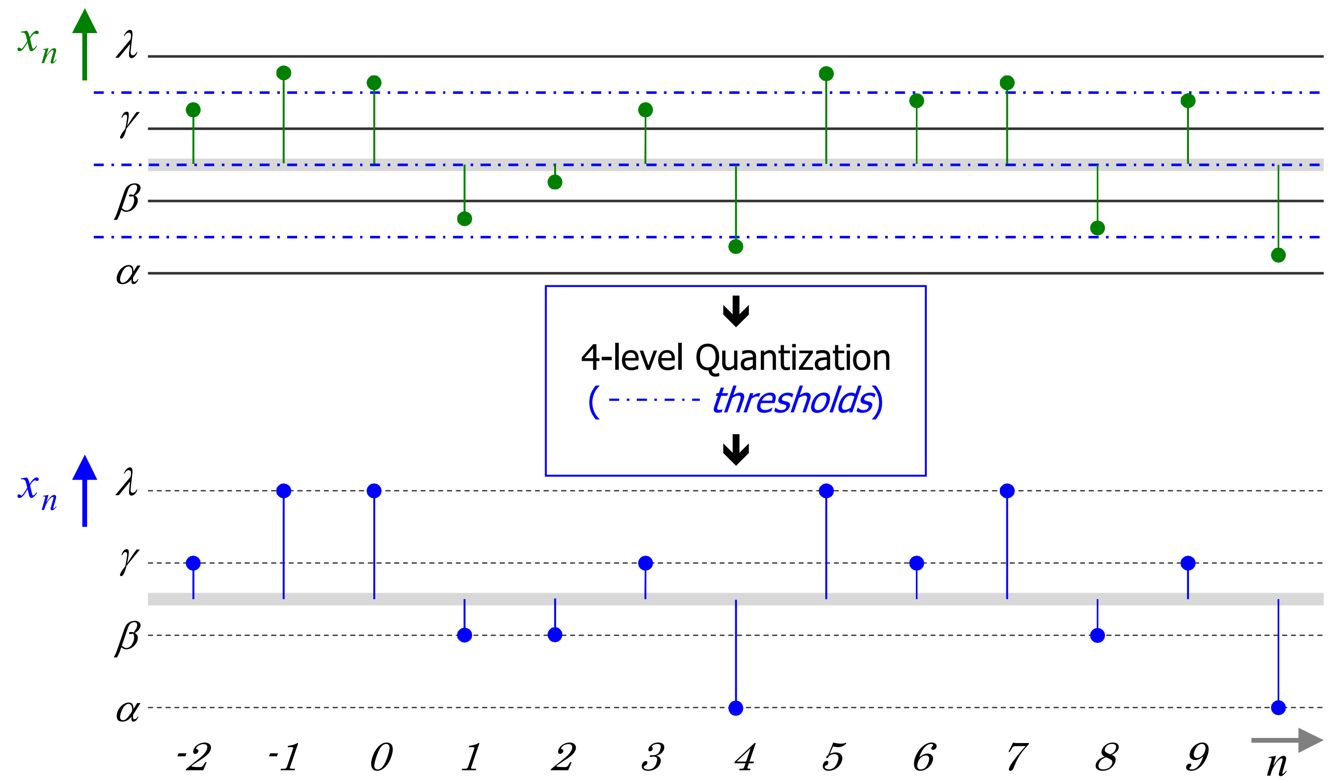


DIGITAL SEQUENCE - A **discrete-time signal** that can take on only a finite number of distinct levels.

Usually obtained by **quantizing** a **discrete-time signal**.

Graphical representation: (*Example - Quaternary (4-level) sequence*)

Discrete-time sequence (*without quantization*)



Quaternary sequence (4-level)

$$\{\cdots \gamma, \lambda, \underset{\substack{\uparrow \\ n=0}}{\lambda}, \beta, \beta, \gamma, \alpha, \lambda, \gamma, \lambda, \beta, \gamma, \alpha, \cdots\}$$

B. *Periodic and Aperiodic signals*

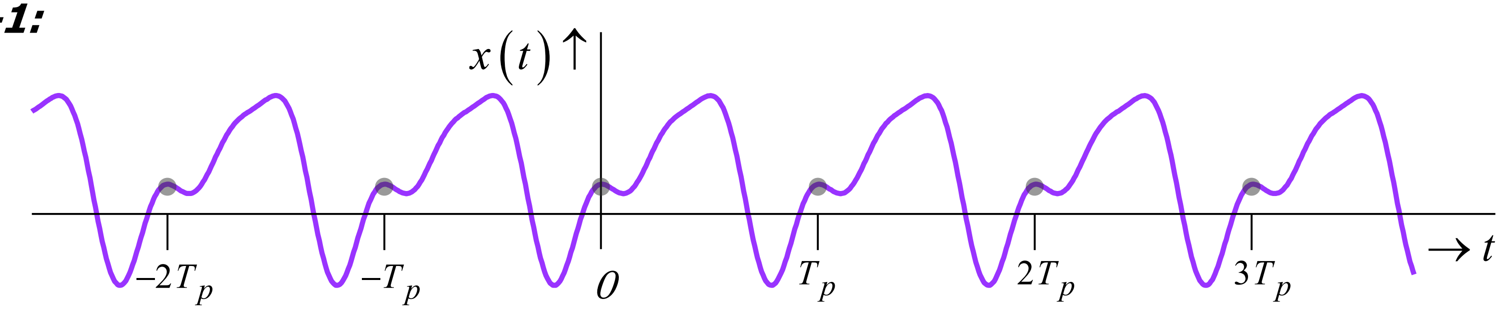
- A signal $x(t)$ is said to be **periodic** if there is a non-zero positive value, T, satisfying

$$x(t) = x(t+T); \quad \forall t. \tag{1.1}$$

- The smallest value of T which satisfies (1.1) is called the "**fundamental period**", or simply "**period**" of $x(t)$.

- The reciprocal of the "fundamental period" is called the "fundamental frequency" of $x(t)$.

Example 1-1:



This signal satisfies $x(t) = x(t+T)$ *for* $T = T_p, 2T_p, \cdots\cdots$

Period (sec): $\boldsymbol{T_p}$ *(shortest repetition interval)*

Fundamental frequency (Hz): $\boldsymbol{f_p = 1/T_p}$

- Any signal which is not periodic is called **aperiodic**.

C. *Real and Complex signals*

- A **complex signal** $x(t)$ is a signal that can be expressed in the form of $a(t)+jb(t)$, where $a(t)$ and $b(t)$ are real signals and j is the imaginary unit, which satisfies the equation $j^2=-1$.

- Cartesian (or rectangular) form of $x(t)$:

$$\boldsymbol{x(t)=a(t)+jb(t)} \tag{1.2a}$$

where

$$a(t)=\mathrm{Re}\left[x(t)\right] \cdots\cdots \textit{Real part of } x(t)$$

$$b(t)=\mathrm{Im}\left[x(t)\right] \cdots\cdots \textit{Imaginary part of } x(t)$$

- Polar form of $x(t)$:

$$\boldsymbol{x(t)=r(t)\exp\left[j\theta(t)\right]} \tag{1.2b}$$

where

$$r(t)=\left|x(t)\right| \cdots\cdots\cdots\cdots \textit{Magnitude of } x(t)$$

$$\theta(t)=\angle x(t) \cdots\cdots\cdots\cdots \textit{Phase Angle of } x(t)$$

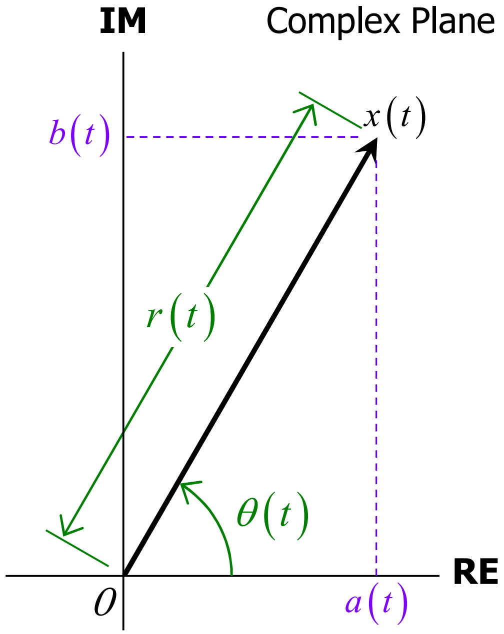
■ Relationship between Cartesian and Polar forms of $x(t)$

Applying **Euler's formula**,

$$\exp(j\theta) = \cos(\theta) + j\sin(\theta),$$

to (1.2b), we get

$$\begin{aligned} x(t) &= r(t)\exp\left[j\theta(t)\right] \\ &= r(t)\cos\left[\theta(t)\right] + jr(t)\sin\left[\theta(t)\right] \end{aligned}$$



Phasor Diagram

Comparing this with (1.2a), we get

$$a(t) = r(t)\cos\left[\theta(t)\right]$$

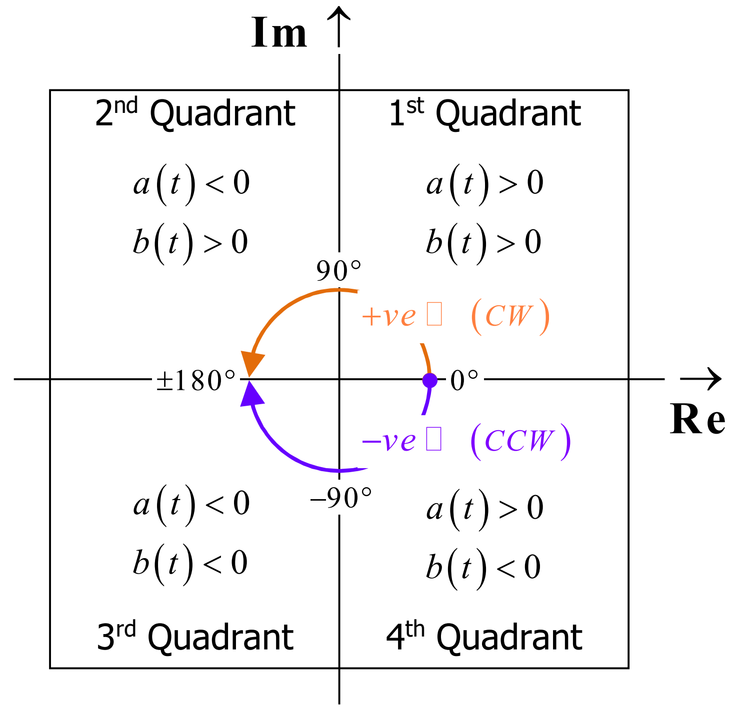
$$b(t) = r(t)\sin\left[\theta(t)\right]$$

or

$$r(t) = \sqrt{a^2(t) + b^2(t)}$$

$$\theta(t) = \tan^{-1}\left(\frac{b(t)}{a(t)}\right)$$

Computing $\theta(t) = \tan^{-1}\left(\frac{b(t)}{a(t)}\right)$ ➔



- The **conjugate** of a complex signal $x(t)$ is denoted by $x^*(t)$ and is obtained by negating the imaginary part of $x(t)$. This is equivalent to replacing $b(t)$ by $-b(t)$, or $\theta(t)$ by $-\theta(t)$

- A signal $x(t)$ is a **real signal** if its value is a real number for all values of t. This is a special case of the complex signal where $[b(t) = 0]$ or $[\theta(t) = \pm n\pi = \pm n180°]$.

Example 1-2:

Write $z(t) = (1-j)\exp(j2\pi t)$ in Cartesian and polar forms. Calculate $\angle z(0.5)$ in each case?

In Cartesian form:

$$z(t) = (1-j)\big(\cos(2\pi t) + j\sin(2\pi t)\big) \cdots\cdots \quad \textit{applying Euler's formula}$$

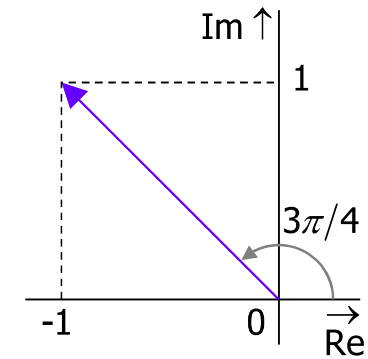
$$= \cos(2\pi t) + j\sin(2\pi t) - j\cos(2\pi t) + \sin(2\pi t)$$

$$= \underbrace{\sin(2\pi t) + \cos(2\pi t)}_{a(t)=\mathrm{Re}[z(t)]} + j\underbrace{\big[\sin(2\pi t) - \cos(2\pi t)\big]}_{b(t)=\mathrm{Im}[z(t)]}$$

$$z(0.5) = -1 + j$$

$$\angle z(0.5) = \tan^{-1}\left(\frac{b(0.5)}{a(0.5)}\right) = \tan^{-1}\left(\frac{1}{-1}\right) = 3\pi/4$$

$$\left[\text{Note the ambiguity: } \tan^{-1}(-1) = -\pi/4 \text{ or } 3\pi/4\right]$$



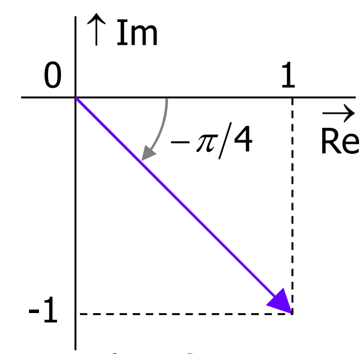
Plot of $-1+j$

In polar form:

$$z(t) = \underbrace{\sqrt{2}\exp\left(-j\frac{\pi}{4}\right)}_{1-j}\exp(j2\pi t) = \underset{r(t)=|z(t)|}{\sqrt{2}}\exp\left(j\underset{\theta(t)=\angle z(t)}{\left(2\pi t - \frac{\pi}{4}\right)}\right)$$

$$z(0.5) = \sqrt{2}\exp(j3\pi/4)$$

$$\angle z(0.5) = \pi - \pi/4 = 3\pi/4$$



Plot of $1-j$

D. *Energy and Power signals*

- The total energy, ***E***, of a signal $x(t)$ is defined as

$$\boldsymbol{E} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt. \tag{1.3a}$$

$x(t)$ is said to be an **energy** signal if and only if

$$0 < \boldsymbol{E} < \infty. \tag{1.3b}$$

- The average power, ***P***, of $x(t)$ is defined as

$$\boldsymbol{P} = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt \tag{1.4a}$$

$x(t)$ is said to be a **power** signal if and only if

$$0 < \boldsymbol{P} < \infty. \tag{1.4b}$$

- Combining (1.3a) and (1.4a) we have $\left[\boldsymbol{P} = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 \, dt \to \frac{1}{2 \cdot \infty} \int_{-\infty}^{\infty} |x(t)|^2 \, dt \to \frac{\boldsymbol{E}}{2 \cdot \infty}\right]$, with which (1.3b) and (1.4b) implies that:

$$\begin{cases} \textbf{Energy} \text{ signals have } \textit{\textbf{zero average power}}, \text{ because } \boldsymbol{E = \textit{finite}} \text{ implies } \boldsymbol{P = 0} \\ \textbf{Power} \text{ signals have } \textit{\textbf{infinite total energy}}, \text{ because } \boldsymbol{P = \textit{finite}} \text{ implies } \boldsymbol{E = \infty} \end{cases}.$$

- Signals that satisfy neither (1.3b) nor (1.4b) are referred to as <u>neither</u> energy <u>nor</u> power signals.

Example 1-5:

I. $\quad x(t)=\begin{cases}\exp(-\alpha t); & t\ge 0\\ 0; & t<0\end{cases};\quad \alpha>0$

Total energy: $\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_0^{\infty}\exp(-2\alpha t)\,dt=\left[\frac{\exp(-2\alpha t)}{-2\alpha}\right]_0^{\infty}=\frac{1}{2\alpha}$

Finite total energy ➔ Zero average power ➔ $x(t)$ *is an **energy signal***

II. $\quad x(t)=\begin{cases}\alpha t; & t\ge 0\\ 0; & t<0\end{cases};\quad \alpha\ne 0$

Total energy: $\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_0^{\infty}\alpha^2t^2\;dt=\left[\frac{\alpha^2t^3}{3}\right]_0^{\infty}=\infty$

Average power: $\lim_{\tau\to\infty}\frac{1}{2\tau}\int_{-\tau}^{\tau}|x(t)|^2\,dt=\lim_{\tau\to\infty}\frac{1}{2\tau}\int_0^{\tau}\alpha^2t^2dt=\lim_{\tau\to\infty}\frac{1}{2\tau}\left[\frac{\alpha^2t^3}{3}\right]_0^{\tau}=\lim_{\tau\to\infty}\frac{\alpha^2\tau^2}{6}=\infty$

Infinite total energy and average power ➔ $x(t)$ *is neither an energy nor a power signal.*

III. $\boldsymbol{x(t) = \alpha \cos\left(2\pi \frac{t}{T}\right)}$

$x(t)$ *is a periodic signal with period T* (try proving it). *Since* $x(t)$ *is periodic, its power averaged over a one-period interval is the same as that averaged over an infinite time interval.*

Average Power:

$$\begin{cases} \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2\, dt = \frac{1}{T}\int_0^T |x(t)|^2\, dt = \frac{1}{T}\int_0^T \alpha^2 \cos^2\left(2\pi \frac{t}{T}\right)\, dt \\ = \frac{\alpha^2}{2T}\int_0^T 1 + \cos\left(4\pi\frac{t}{T}\right)\, dt \\ = \frac{\alpha^2}{2T}\left[t + \frac{\sin(4\pi t/T)}{4\pi/T}\right]_0^T \\ = \frac{\alpha^2}{2T}\left[T + \frac{\sin(4\pi)}{4\pi/T} - 0 - \frac{\sin(0)}{4\pi/T}\right] = \frac{\alpha^2}{2} \end{cases}$$

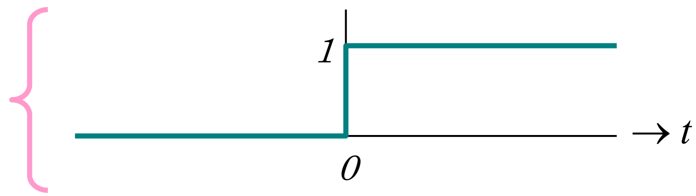
Finite average power ➔ *Infinite total energy* ➔ $x(t)$ *is an* ***power signal***

NOTE: All bounded periodic signals are power signals.

1.2 Basic Signals

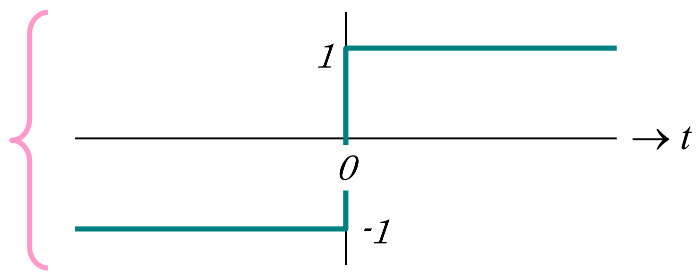
A. *The Unit Step function*

$$u(t) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases} \quad (1.5)$$



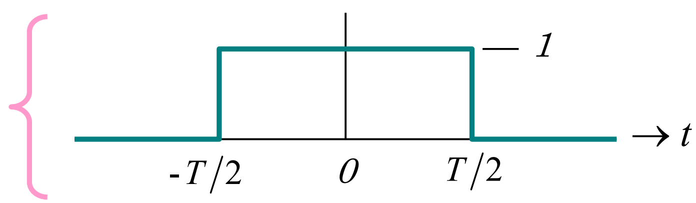
B. *The Sign (or Signum) function*

$$\begin{aligned} \mathrm{sgn}(t) &= \begin{cases} +1; & t \geq 0 \\ -1; & t < 0 \end{cases} \\ &= 2u(t) - 1 \end{aligned} \quad (1.6)$$



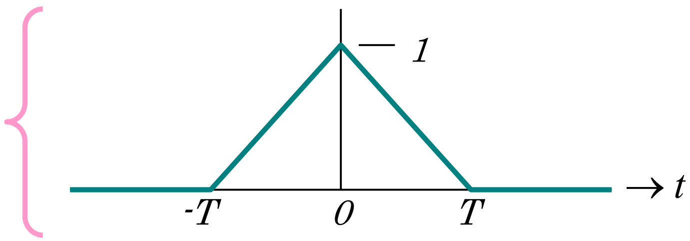
C. *The Rectangle function*

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1; & -T/2 \leq t < T/2 \\ 0; & elsewhere \end{cases} \quad (1.7)$$



D. *The Triangle function*

$$\operatorname{tri}\left(\frac{t}{T}\right)=\begin{cases}1-|t|/T\,; & |t|\le T\\ 0; & |t|>T\end{cases}$$

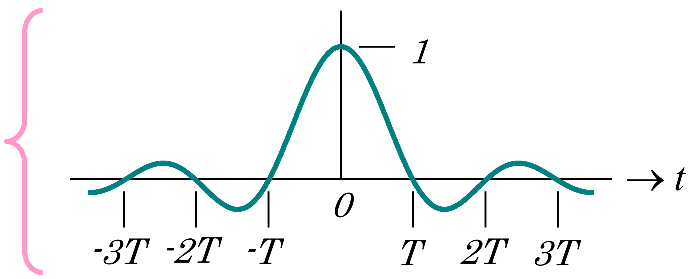


(1.8)

E. *The Sinc (Sine Cardinal) function*

$$\operatorname{sinc}\left(\frac{t}{T}\right)=\begin{cases}\dfrac{\sin(\pi t/T)}{\pi t/T}; & t\ne 0\\ 1; & t=0\end{cases}$$

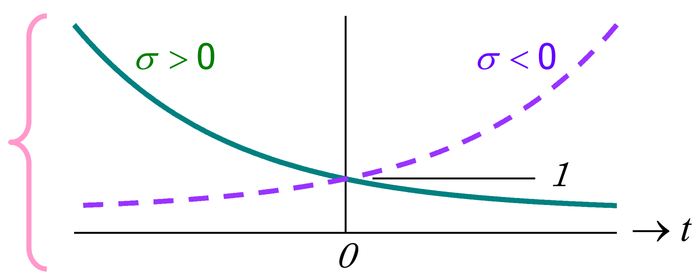
Note: $\operatorname{sinc}(x)=\begin{cases}0; & x=\text{integer}\ne 0\\ 1; & x=0\end{cases}$



(1.9)

F. *Real Exponential signal*

$$x(t)=\exp(-\sigma t)$$

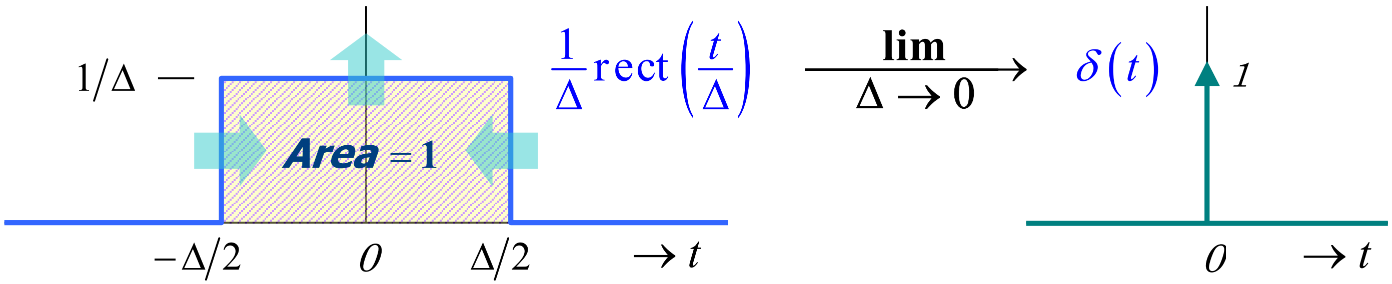


(1.10)

G. *The Unit Impulse (or Dirac-δ) function*

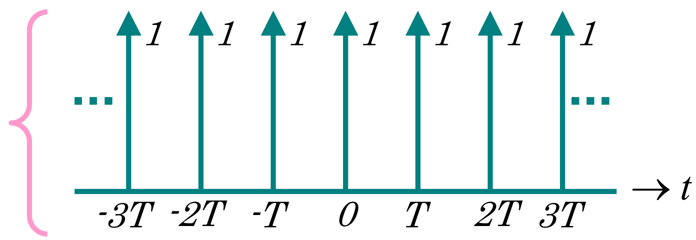
$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\varepsilon}^{\varepsilon} \delta(t)\, dt = 1; \quad \forall \varepsilon > 0 \tag{1.11}$$

We may view the unit impulse as a limiting case of a rectangle pulse which has a unit area that is independent of its pulse width:



H. *The Dirac Comb function*

$$\xi_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{1.12}$$



I. *Sinusoidal Signals*

Sinusoidal signals (or sinusoids) is a collective term for a general class of periodic signals of the form:

- $x(t) = \mu\cos(\omega_o t + \phi) = \frac{\mu}{2}\left[\exp\left[j(\omega_o t + \phi)\right] + \exp\left[-j(\omega_o t + \phi)\right]\right]$ $\begin{cases} \text{REAL SINUSOID} \\ \text{a.k.a. COSINE} \end{cases}$ (1.13)

- $x(t) = \mu\sin(\omega_o t + \phi) = \frac{\mu}{j2}\left[\exp\left[j(\omega_o t + \phi)\right] - \exp\left[-j(\omega_o t + \phi)\right]\right]$ $\begin{cases} \text{REAL SINUSOID} \\ \text{a.k.a. SINE} \end{cases}$ (1.14)

- $x(t) = \mu\exp\left[j(\omega_o t + \phi)\right] = \mu\left[\cos(\omega_o t + \phi) + j\sin(\omega_o t + \phi)\right]$ $\begin{cases} \text{COMPLEX SINUSOID} \\ \text{a.k.a.} \begin{cases} \text{COMPLEX} \\ \text{EXPONENTIAL} \end{cases} \end{cases}$ (1.15)

where

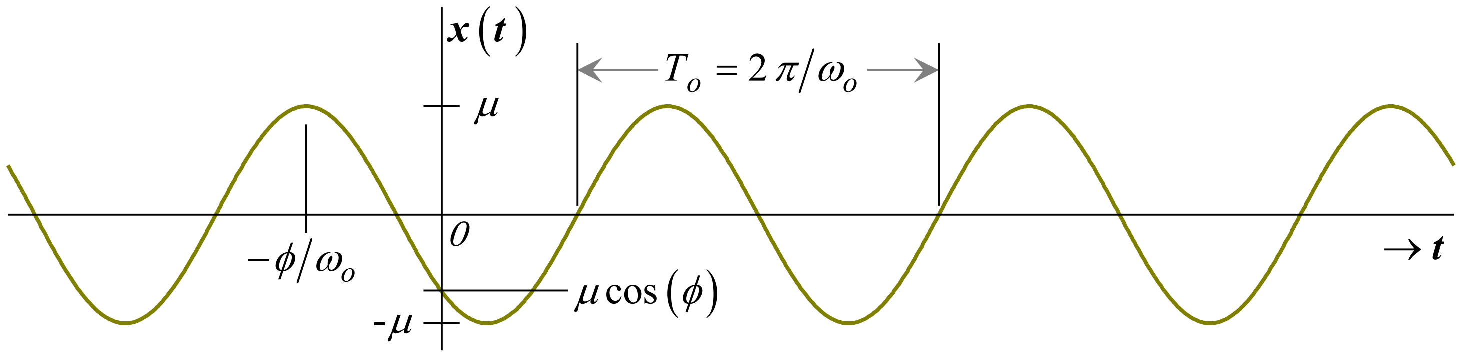
$\mu(>0)$: magnitude (or amplitude)
ω_o : angular frequency (rad/s)
ϕ : phase (radians)
$\omega_o t + \phi$: instantaneous phase (radians)

It is also common to replace ω_o by $2\pi f_o$, where f_o is the ***cyclic frequency*** (in Hz).

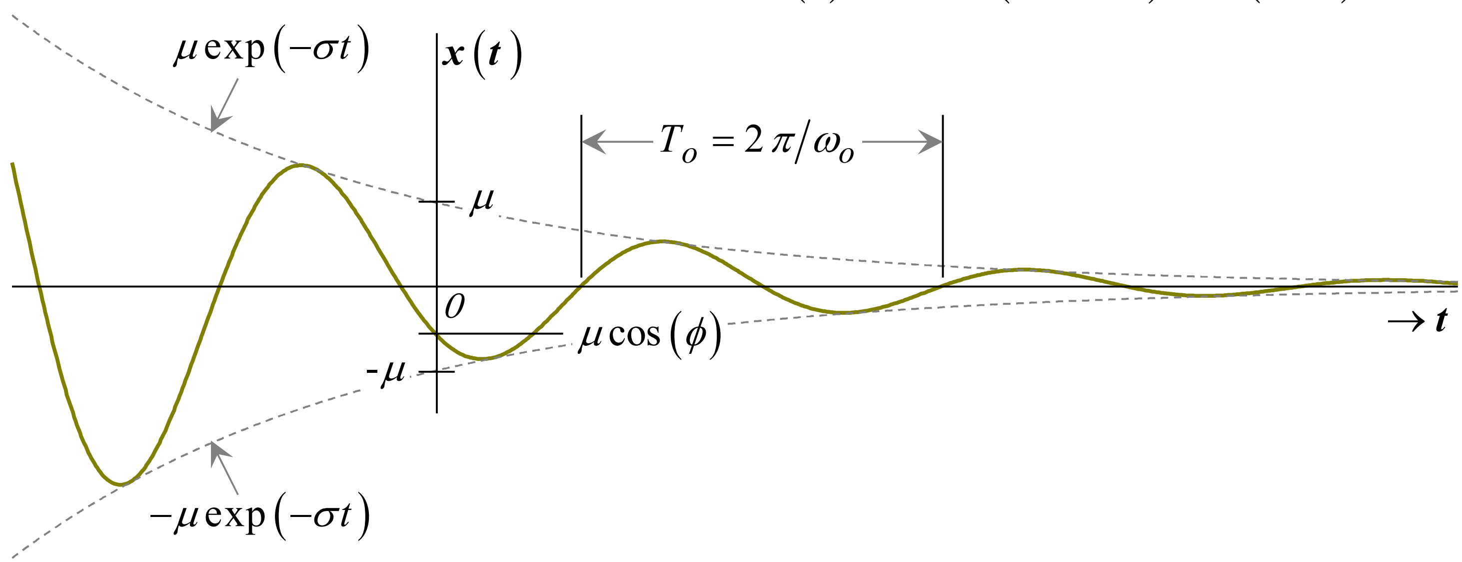
The fundamental period, T_o (in seconds), is given by $T_o = \frac{2\pi}{\omega_o} = \frac{1}{f_o}$.

Example 1-6:

Plot of a real sinusoid: $x(t) = \mu\cos(\omega_o t + \phi) = \mu\cos\left(\omega_o\left(t + \dfrac{\phi}{\omega_o}\right)\right)$



Sketch of an exponentially decaying real sinusoid: $x(t) = \mu\cos(\omega_o t + \phi)\exp(-\sigma t); \quad \sigma > 0$



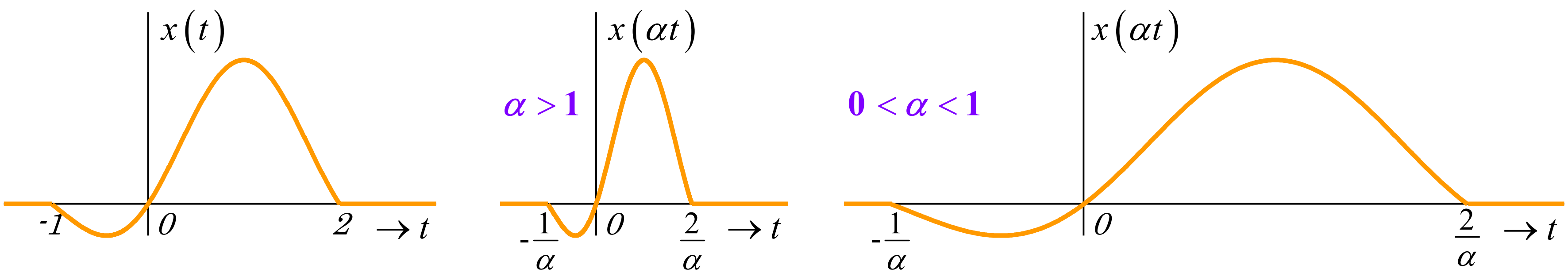
1.3 Time-Scaling, -Reversal and -Shifting of Signals

A. *Time-scaling*

Time-scaling of a signal $x(t)$ is effected by replacing the time variable t by αt, where α is a positive real number.

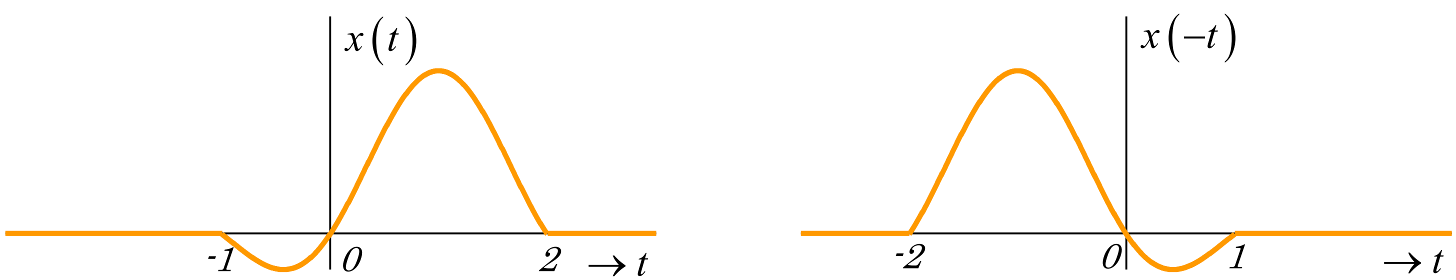
$0 < \alpha < 1$: uniform **expansion** of $x(t)$ along the time axis

$\alpha > 1$: uniform **contraction** of $x(t)$ along the time axis



B. *Time-reversal*

Time-reversal of a signal $x(t)$ is effected by replacing the time variable t by $-t$.

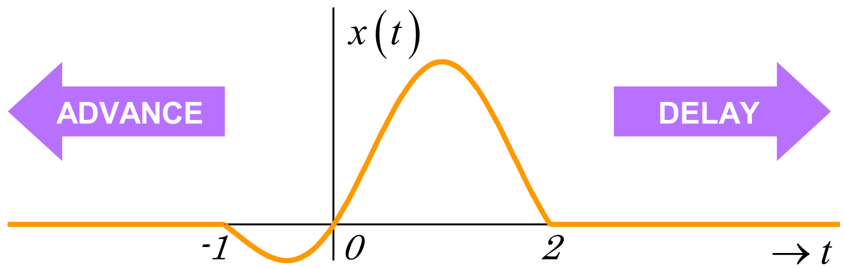


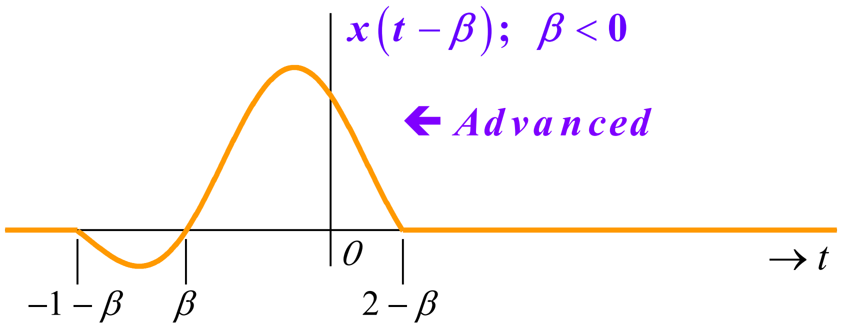
C. *Time-shifting*

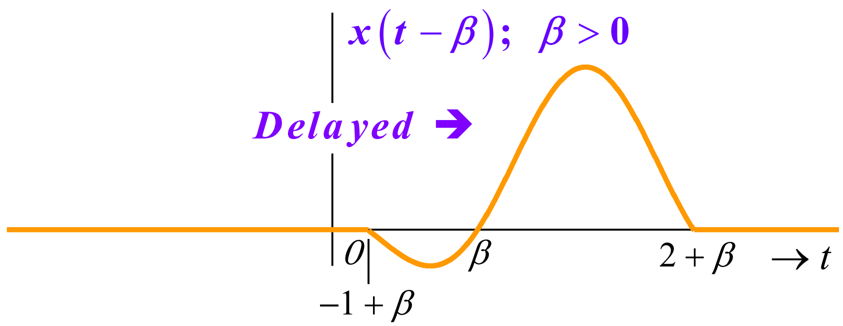
Time-shifting of a signal $x(t)$ is effected by replacing the time variable t by $(t-\beta)$, where β is a real number.

$\beta > 0$: **Delaying** $x(t)$ by β unit of time

$\beta < 0$: **Advancing** $x(t)$ by β unit of time







Example 1-7:

Given $w(t)=\begin{cases} t; & -2\le t<4 \\ 0; & \text{elsewhere} \end{cases}$.

Sketch $\begin{cases} x(t)=w(t-1) \\ y(t)=x(-t) \\ z(t)=y\left(\frac{2}{3}t\right) \end{cases}$.

Express $z(t)$ ***in terms of*** $w(t)$.

$$\begin{aligned} z(t) &= y\left(\frac{2}{3}t\right) \\ &= x\left(-\frac{2}{3}t\right) \\ &= w\left(-\frac{2}{3}t-1\right) \end{aligned}$$