6. Linear Time-Invariant Systems

An important class of continuous-time linear, time-invariant (LTI) systems consists of systems represented by linear constant-coefficient differential equations that have the general form

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \tag{6.1}$$

where $x(t)$ is the input, $y(t)$ is the output, and a_n, b_m are ***real constants*** with $a_N \neq 0$. N, which is the highest derivative of $y(t)$, is the ORDER of the system.

Continuous-time LTI systems that satisfy (6.1) are very common and happen very often in reality; they include electrical circuits composed of resistors, inductors, and capacitors and mechanical systems composed of masses, springs, and dashpots.

In this chapter, we first introduce the notions of impulse response and frequency response of LTI systems in the most general sense and show that they are a Fourier transform pair. Then, based on the class of LTI systems described by (6.1), we introduce the notion of system transfer function to aid our discussion on straight-line Bode plots, which are a simple but accurate method for graphing the system frequency response.

6.1 Impulse Response

The ***impulse response***, $h(t)$, of a continuous-time LTI system is defined to be the response of the system when the input is a unit impulse, $\delta(t)$, that is

$$h(t) = \mathbf{T}\left[\delta(t)\right]. \tag{6.2}$$

Now, suppose the input is an arbitrary signal $x(t)$. From the replication property of $\delta(t)$ in (3.21), $x(t)$ can be expressed as

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau. \tag{6.3}$$

Since the system is linear, the response $y(t)$ of the system to an arbitrary input $x(t)$ can be expressed as

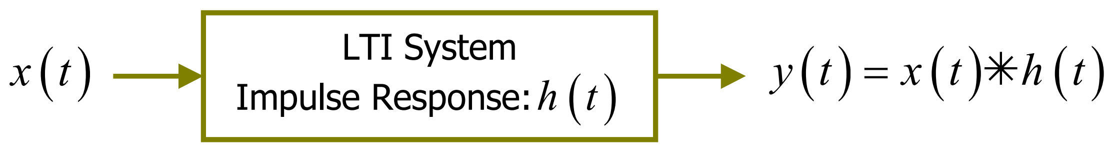
$$\begin{aligned} y(t) &= \mathbf{T}\left[x(t)\right] = \mathbf{T}\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right] \\ &= \int_{-\infty}^{\infty} x(\tau)\,\mathbf{T}\left[\delta(t-\tau)\right]d\tau \end{aligned}. \tag{6.4}$$

Since the system is also time-invariant, from (6.2), we have

$$h(t-\tau) = \mathbf{T}\left[\delta(t-\tau)\right]. \tag{6.5}$$

Substituting (6.5) into (6.4), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t). \tag{6.6}$$

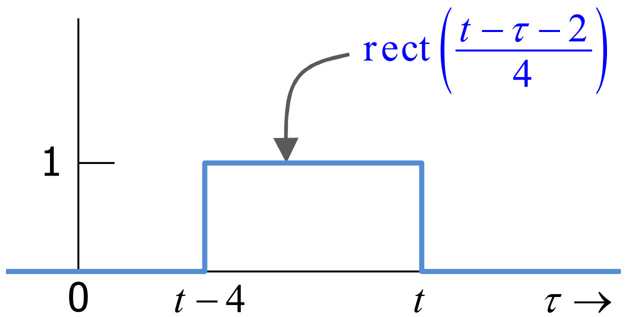


Equation (6.6) shows that **a continuous-time LTI system is completely characterized in the time-domain by its impulse response, $h(t)$, where the output, $y(t)$, of the system to an arbitrary input, $x(t)$, is given by the convolution of $x(t)$ and $h(t)$.**

Example 6-1:

A sinusoid $x(t)=0.25\cos(0.125\pi t)$ is applied at the input of LTI system, which has an impulse response given by $h(t)=0.5\,\mathrm{rect}\left(\frac{t-2}{4}\right)$. Find the system output $y(t)$ using time-domain analysis.

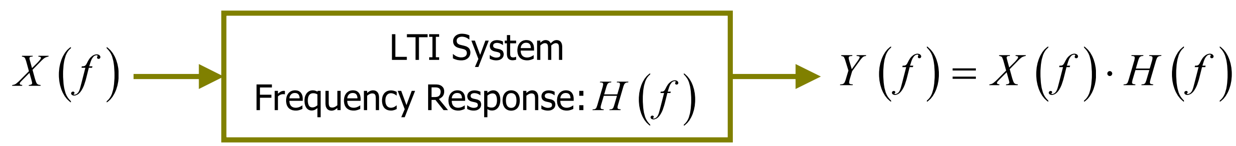
$$
\begin{aligned}
y(t) &= x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\
&= \int_{-\infty}^{\infty} 0.25\cos(0.125\pi\tau)\cdot 0.5\,\mathrm{rect}\left(\frac{t-\tau-2}{4}\right)d\tau \\
&= 0.125\int_{t-4}^{t}\cos(0.125\pi\tau)\,d\tau \\
&= 0.125\left[\frac{\sin(0.125\pi\tau)}{0.125\pi}\right]_{t-4}^{t} \\
&= \frac{1}{\pi}\left[\sin(0.125\pi t)-\sin(0.125\pi t-0.5\pi)\right] \\
&= \frac{1}{\pi}\left[\sin(0.125\pi t)+\cos(0.125\pi t)\right]
\end{aligned}
$$



6.2 Frequency Response

The frequency-domain relationship between the input and output of a LTI system is obtained by applying the Fourier transform to (6.6). This results in,

$$Y(f) = \underbrace{\Im\{x(t) * h(t)\} = X(f) \cdot H(f)}_{\substack{convolution\ property\ of\ the \\ Fourier\ transform\ \text{- see}\ (3.9)}} \tag{6.7}$$



where $X(f) = \Im\{x(t)\}$ and $Y(f) = \Im\{y(t)\}$ are Fourier transforms (or spectra) of the system input and output, respectively, and

$$\begin{aligned} H(f) &= \Im\{h(t)\} \\ &= |H(f)| \exp(j\angle H(f)) \end{aligned} \tag{6.8}$$

is the Fourier transform of the system impulse response. In (6.8), $\boldsymbol{H(f)}$ is called the ***frequency response*** of the LTI system, where $|\boldsymbol{H(f)}|$ and $\angle \boldsymbol{H(f)}$ are the corresponding ***magnitude response*** and ***phase response***, respectively.

Frequency response is an intrinsic property of LTI systems as it characterizes how sinusoidal signals are altered in going through the system. To see this, let the input signal of the LTI system be a complex sinusoid with amplitude A, frequency $f_o\,(Hz)$ and phase $\psi\,(rad/s)$, i.e.,

$$x(t) = Ae^{j(2\pi f_o t+\psi)}. \tag{6.9}$$

The Fourier transform of $x(t)$ is given by

$$X(f) = Ae^{j\psi}\delta(f-f_o). \tag{6.10}$$

Applying (6.7), the Fourier transform of the output, $y(t)$, is then given by

$$\begin{aligned} Y(f) &= X(f)\cdot H(f) \\ &= \underbrace{Ae^{j\psi}\delta(f-f_o)\cdot H(f) = Ae^{j\psi}\delta(f-f_o)\cdot H(f_o)}_{\text{sampling property of Dirac } \delta\text{-function}} \\ &= Ae^{j\psi}\delta(f-f_o)\cdot\left|H(f_o)\right|e^{j\angle H(f_o)} \\ &= A\left|H(f_o)\right|e^{j(\psi+\angle H(f_o))}\delta(f-f_o) \end{aligned}. \tag{6.11}$$

The inverse Fourier transform of (6.11) yields

$$y(t) = \Im^{-1}\{Y(f)\} = A\left|H(f_o)\right|e^{j(2\pi f_o t+\psi+\angle H(f_o))}. \tag{6.12}$$

Equations (6.9) and (6.12) show that when a complex sinusoid of frequency f_o is propagated through a LTI system with frequency response $H(f)$, its amplitude is scaled by $|H(f_o)|$ and phase is shifted by $\angle H(f_o)$. This applies to real sinusoids in the same way, as illustrated below.

$$\left.\begin{array}{l} x(t) = A\boldsymbol{e}^{j(2\pi f_o t+\psi)} \longrightarrow \boxed{H(f)} \longrightarrow y(t) = A|H(f_o)|\boldsymbol{e}^{j(2\pi f_o t+\psi+\angle H(f_o))} \\ x(t) = A\mathbf{cos}(2\pi f_o t+\psi) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = A|H(f_o)|\mathbf{cos}(2\pi f_o t+\psi+\angle H(f_o)) \\ x(t) = A\mathbf{sin}(2\pi f_o t+\psi) \longrightarrow \boxed{H(f)} \longrightarrow y(t) = A|H(f_o)|\mathbf{sin}(2\pi f_o t+\psi+\angle H(f_o)) \end{array}\right\} \qquad (6.13)$$

Sinusoidal Response in $f - domain$

6.2.1 Notation for Expressing Fourier Transfrom in ω-domain

In system studies, frequency functions are usually expressed as functions of $j\omega$ instead of ω or f, where $j^2 = -1$ and $\omega = 2\pi f$. In the discussion to follow, we shall adopt the following notation:

$$\begin{array}{l} \boldsymbol{f-domain}: \Phi(f) = \Im\{\phi(t)\} \quad \begin{cases} Original\ notation\ for\ the \\ Fourier\ transform\ of\ \phi(t) \end{cases} \\ \boldsymbol{\omega-domain}: \tilde{\Phi}(j\omega) = \Phi(f)\Big|_{f=\omega/(2\pi)} = \Phi\left(\dfrac{\omega}{2\pi}\right) \end{array}. \qquad (6.14)$$

Based on (6.14), we write the $\omega - domain$ equivalent of (6.7), (6.8) and (6.13) as:

$$\tilde{Y}(j\omega) = \tilde{X}(j\omega) \cdot \tilde{H}(j\omega) \tag{6.15}$$

$$\tilde{X}(j\omega) \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ \text{Frequency Response: } \tilde{H}(j\omega)\end{array}} \longrightarrow \tilde{Y}(j\omega) = \tilde{X}(j\omega) \cdot \tilde{H}(j\omega)$$

$$\tilde{H}(j\omega) = \left|\tilde{H}(j\omega)\right| \exp\left(j\angle H(j\omega)\right) \tag{6.16}$$

$$\left.\begin{array}{l}
x(t) = A\boldsymbol{e}^{j(\omega_o t + \psi)} \longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t) = A\left|\tilde{H}(j\omega_o)\right|\boldsymbol{e}^{j\left(\omega_o t + \psi + \angle\tilde{H}(j\omega_o)\right)} \\
x(t) = A\,\mathbf{cos}(\omega_o t + \psi) \longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t) = A\left|\tilde{H}(j\omega_o)\right|\mathbf{cos}\left(\omega_o t + \psi + \angle\tilde{H}(j\omega_o)\right) \\
x(t) = A\,\mathbf{sin}(\omega_o t + \psi) \longrightarrow \boxed{\tilde{H}(j\omega)} \longrightarrow y(t) = A\left|\tilde{H}(j\omega_o)\right|\mathbf{sin}\left(\omega_o t + \psi + \angle\tilde{H}(j\omega_o)\right)
\end{array}\right\} \tag{6.17}$$

Sinusoidal Response in $\omega - domain$

Example 6-2:

Repeat Example 6-1 using the Sinusoidal Response approach.

$$H(f) = \Im\{h(t)\} = \Im\left\{0.5\,\mathrm{rect}\left(\frac{t-2}{4}\right)\right\} = 2\,\mathrm{sinc}(4f)\exp(-j4\pi f)$$

$x(t) = 0.25\cos(0.125\pi t)$ $\cdots$ *sinusoid of frequency* 0.0625 Hz

$$H(0.0625) = 2\,\mathrm{sinc}(0.25)\exp(-j0.25\pi) = \frac{4\sqrt{2}}{\pi}\exp(-j0.25\pi)$$

$$\rightarrow \begin{cases} |H(0.0625)| = 4\sqrt{2}/\pi \\ \angle H(0.0625) = -\pi/4 \end{cases}$$

$$y(t) = |H(0.0625)| \cdot 0.25\cos(0.125\pi t + \angle H(0.0625)) = \frac{4\sqrt{2}}{\pi} \cdot 0.25\cos\left(0.125\pi t - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{1}{\sqrt{2}}\cos(0.125\pi t) + \frac{1}{\sqrt{2}}\sin(0.125\pi t)\right] = \frac{1}{\pi}\left[\cos(0.125\pi t) + \sin(0.125\pi t)\right]$$

This problem may also be worked out in the $\omega - domain$ *.* $\cdots\cdots\cdots\cdots\cdots$ $Try\ it\ !$

6.3 Bode Diagram

- A Bode diagram is used to depict the frequency response, $\tilde{H}(j\omega)$, of a LTI system.

- It consists of two plots:

  - The **Magnitude Plot** is a plot of $\left|\tilde{H}(j\omega)\right|_{dB} = 20\log_{10}\left|\tilde{H}(j\omega)\right|$ dB

  - The **Phase Plot** is a plot of $\angle\tilde{H}(j\omega)$ in **degrees**.

Aside: The *Magnitude*, when expressed in dB, is frequently referred to as ***Gain*** since dB is a relative measure. For instance, if v_i and v_o are the input and output of a system, respectively, then we say that the system has a gain of $20\log_{10}\left(\left|v_o/v_i\right|\right)$ dB. Therefore, $20\log_{10}\left(\left|v_o\right|\right)$ is the system gain measured with reference to a ***unit input***, i.e. $\left|v_i\right| = 1$.

- These two plots are normally plotted on a semilog$x$ graph paper where the abscissa (or x-axis) is logarithmically scaled.

- Bode diagrams visualize the frequency response only for positive frequencies. This suffices for real systems since $\left|\tilde{H}(j\omega)\right|$ and $\angle\tilde{H}(j\omega)$ are even and odd functions of ω, respectively.

- Bode diagrams are also asymptotically approximated by straight-lines, and the results are referred to as ***straight-line Bode plots***.

To facilitate our discussion on the construction of straight-line Bode plots for the class of LTI systems represented by (6.1), we first introduce the notion of transfer function.

6.3.1 Transfer Function

The f-$domain$ Fourier transform of (6.1) is given by

$$\sum_{n=0}^{N} a_n \left(j2\pi f\right)^n Y\left(f\right) = \sum_{m=0}^{M} b_m \left(j2\pi f\right)^m X\left(f\right). \tag{6.18}$$

By substituting $f = \dfrac{\omega}{2\pi}$ into (6.18) and applying the notation defined in (6.14), we obtain the ω-$domain$ Fourier transform of (6.1) as

$$\sum_{n=0}^{N} a_n \left(j\omega\right)^n \tilde{Y}\left(j\omega\right) = \sum_{m=0}^{M} b_m \left(j\omega\right)^m \tilde{X}\left(j\omega\right) \tag{6.19}$$

from which the system frequency response is obtained as

$$\tilde{H}\left(j\omega\right) = \frac{\tilde{Y}\left(j\omega\right)}{\tilde{X}\left(j\omega\right)} = \sum_{m=0}^{M} b_m \left(j\omega\right)^m \Bigg/ \sum_{n=0}^{N} a_n \left(j\omega\right)^n . \tag{6.20}$$

Now, suppose we let $\omega = s/j$ in (6.20). This results in

$$\tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \left(\sum_{m=0}^{M} b_m s^m \Big/ \sum_{n=0}^{N} a_n s^n \right) = K' \frac{(s+z_1)(s+z_2)\cdots(s+z_M)}{(s+p_1)(s+p_2)\cdots(s+p_N)}, \tag{6.21}$$

where $K' = \dfrac{b_0}{a_0} \dfrac{p_1 p_2 \cdots p_N}{z_1 z_2 \cdots z_M}$.

In (6.21), the $(-z_m)$ and $(-p_n)$ are roots of $\sum_{m=0}^{M} b_m s^m$ and $\sum_{n=0}^{N} a_n s^n$, respectively. Since the b_m are real, z_m must be either real or complex conjugate paired. Similarly, p_n must be either real or complex conjugate paired because the a_n are real.

It is worthwhile to note that (6.21) can also be obtained by direct application of the Laplace transform to (6.1), where $\tilde{X}(s)$ and $\tilde{Y}(s)$ are Laplace transforms of $x(t)$ and $y(t)$, respectively.

In the context of Laplace transform:

$$\tilde{\boldsymbol{H}}(\boldsymbol{s}) = \frac{\tilde{Y}(s)}{\tilde{X}(s)},$$ by definition, is the ***transfer function*** of the LTI system

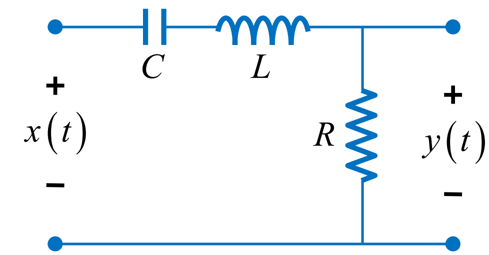
$$\left(-z_m\right) \text{ are called system } \boldsymbol{zeros} \quad \because \tilde{H}(s)\Big|_{s=-z_m} = 0; \qquad m = 1, 2, \cdots, M$$

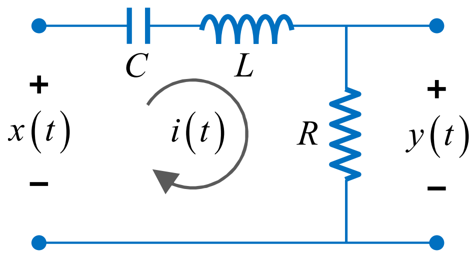
$$\left(-p_n\right) \text{ are called system } \boldsymbol{poles} \quad \because \tilde{H}(s)\Big|_{s=-p_n} = \infty; \qquad n = 1, 2, \cdots, N$$

and $\left(N - M\right)$ is called "***pole excess***" of $\tilde{H}(s)$.

Example 6-3:

> ***Derive the differential equation model, frequency response, transfer function, pole(s) and zero(s) of the system shown on the right, where $x(t)$ and $y(t)$ are the system input and output, respectively.***





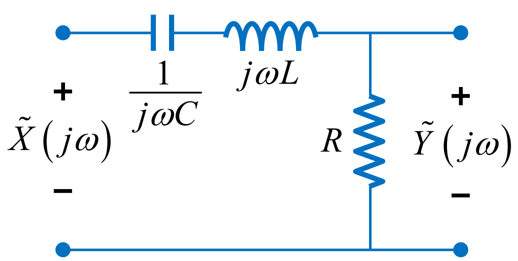
Time-domain Circuit

$$i(t) = \frac{y(t)}{R}$$

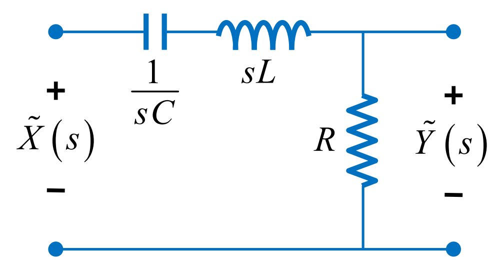
$$x(t) - y(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt + L\frac{di(t)}{dt} = \frac{1}{RC}\int_{-\infty}^{t} y(t)\,dt + \frac{L}{R}\frac{dy(t)}{dt}$$

$$\frac{dx(t)}{dt} - \frac{dy(t)}{dt} = \frac{y(t)}{RC} + \frac{L}{R}\frac{d^2y(t)}{dt^2}$$

$$\frac{L}{R}\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + \frac{y(t)}{RC} = \frac{dx(t)}{dt}$$



Frequency-domain Circuit



s-domain Circuit

$$\tilde{H}(j\omega) = \frac{\tilde{Y}(j\omega)}{\tilde{X}(j\omega)} = \frac{R}{\dfrac{1}{j\omega C} + j\omega L + R} \quad \leftarrow \text{by voltage division}$$

$$= \frac{j\omega RC}{\left(1-\omega^2 LC\right) + j\omega RC}$$

$$\tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{R}{sL + \dfrac{1}{sC} + R} \quad \leftarrow \text{by voltage division}$$

$$= \frac{sRC}{\left(1+s^2 LC\right) + sRC}$$

$$Poles: \ -p_1, -p_2 = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC}; \qquad Zero: \ -z_1 = 0$$

6.3.2 Construction of Straight-line Bode Plots

The construction of straight-line Bode plots begins with the transfer function. To simplify the construction process, it is essential to express (6.21) in a suitable form for each of the following cases: *Systems with Integrators*, *Systems with Differentiators*, and *Systems without Integrator and Differentiator*.

- **dc Gain**

  The "dc gain" of a system is defined as the system response to a zero frequency input. Hence, the "dc gain" of a system is given by $\tilde{H}(0)$ since zero frequency implies $\omega = 0$ or $s = 0$.

- **N^th^-order System with ℓ differentiators**

Rewrite (6.21) as:

$$\tilde{H}(s) = \underset{\substack{\uparrow \\ \text{cascade of} \\ \ell \text{ differentiators}}}{K_d s^{\ell}} \cdot \underbrace{\frac{(s/z_1+1)(s/z_2+1)\cdots(s/z_{M-\ell}+1)}{(s/p_1+1)(s/p_2+1)\cdots(s/p_N+1)}}_{\text{system with unity dc gain}}; \quad \begin{cases} z_m \neq 0 & \forall m \in [1, M-\ell] \\ p_n \neq 0 & \forall n \in [1, N] \end{cases} \tag{6.22}$$

where K_d is the combined gain of the ℓ cascaded differentiators. In this case, the dc gain of the system is $H(0)=0$.

- **N^th^-order System with ℓ integrators**

Rewrite (**6.21**) as:

$$\tilde{H}(s) = \underset{\substack{\uparrow \\ \text{cascade of} \\ \ell \text{ integrators}}}{\frac{K_i}{s^{\ell}}} \cdot \underbrace{\frac{(s/z_1+1)(s/z_2+1)\cdots(s/z_M+1)}{(s/p_1+1)(s/p_2+1)\cdots(s/p_{N-\ell}+1)}}_{\text{system with unity dc gain}}; \quad \begin{cases} z_m \neq 0 & \forall m \in [1, M] \\ p_n \neq 0 & \forall n \in [1, N-\ell] \end{cases} \tag{6.23}$$

where K_i is the combined gain of the ℓ cascaded integrators. In this case, the dc gain of the system is $H(0)=\infty$.

- **N^th^-order System without integrator or differentiator:**

  Rewrite (6.24) as:

$$\tilde{H}(s) = K_{dc} \cdot \underbrace{\frac{(s/z_1+1)(s/z_2+1)\cdots(s/z_M+1)}{(s/p_1+1)(s/p_2+1)\cdots(s/p_N+1)}}_{\text{system with unity dc gain}}; \quad \begin{cases} z_m \neq 0 & \forall m \in [1,M] \\ p_n \neq 0 & \forall n \in [1,N] \end{cases} \tag{6.24}$$

  where $K_{dc} = H(0)$ is the dc gain of the system.

We will first discuss the construction of straight-line Bode plots for the following basic systems

i. $\tilde{H}(s) = K_{dc}$ ································· dc Gain (constant)

ii. $\tilde{H}(s) = K_d s$ ························ differentiator with gain K_d

iii. $\tilde{H}(s) = K_i/s$ ························· integrator with gain K_i

iv. $\tilde{H}(s) = s/z_m + 1$ ··············· zero factor with unity dc gain

v. $\tilde{H}(s) = \dfrac{1}{s/p_n + 1}$ ··············· pole factor with unity dc gain

vi. $\tilde{H}(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}; \quad 0 \leq \zeta \leq 1$ ······ $\begin{cases} \text{\textasciitilde\ Underdamped} \\ \text{2nd-order factor} \\ \text{with unity dc gain} \end{cases}$

and then show how the straight-line Bode plots for higher order systems can be obtained by straightforward summation of the plots obtained for their constituent basic systems.

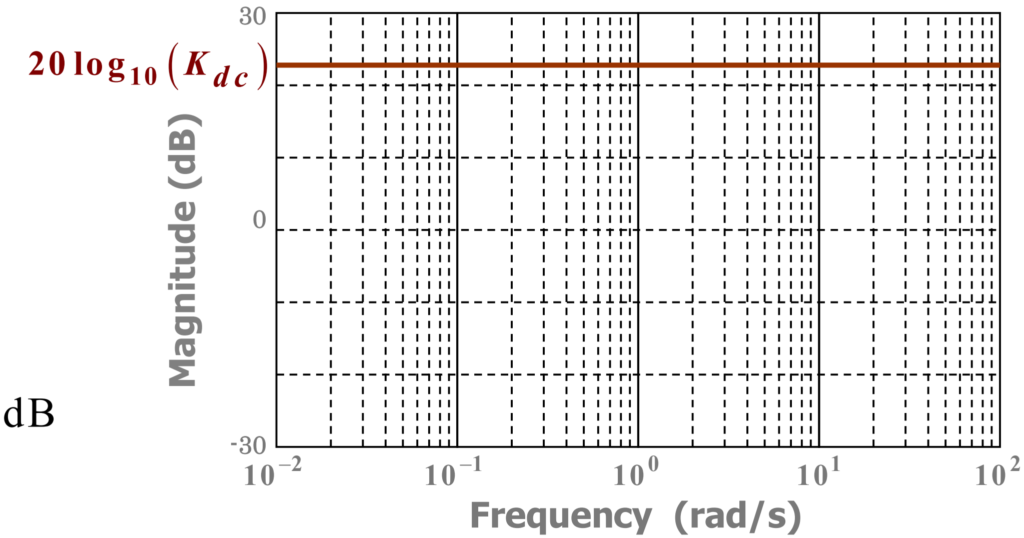
i. Bode plots of a dc Gain: $\tilde{H}(s) = K_{dc}$

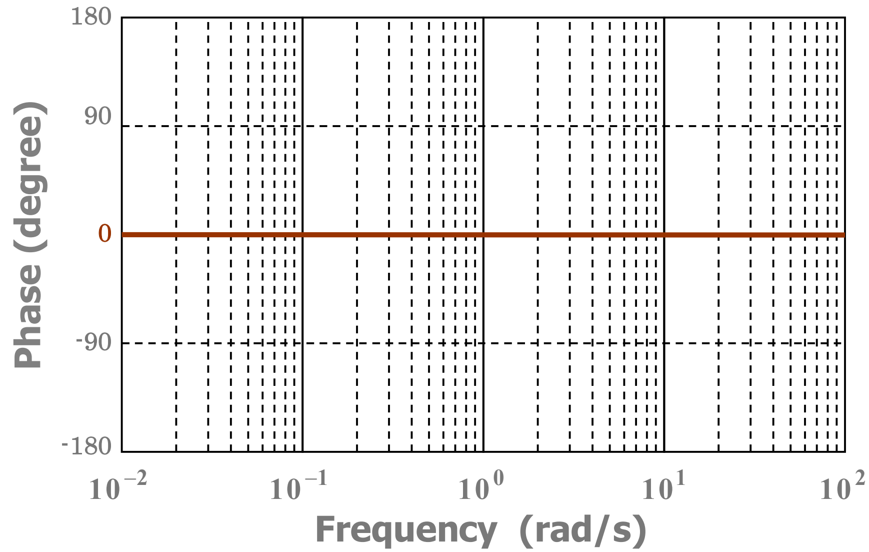
Frequency Response : $\tilde{H}(j\omega) = K_{dc}$

Magnitude Response : $\begin{cases} \left|\tilde{H}(j\omega)\right| = K_{dc} \\ \left|\tilde{H}(j\omega)\right|_{dB} = 20\log_{10}(K_{dc}) \text{ dB} \end{cases}$

Phase Response : $\angle\tilde{H}(j\omega) = 0°$

The magnitude and phase responses are both straight lines with zero gradient.





ii. Bode plots of Differentiator: $\tilde{H}(s) = K_d s$

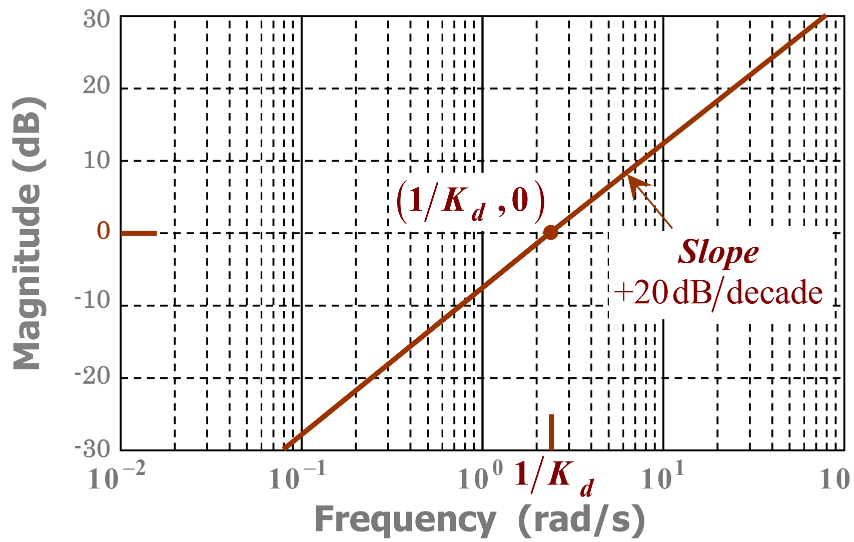
Frequency Response : $\tilde{H}(j\omega) = jK_d\omega$

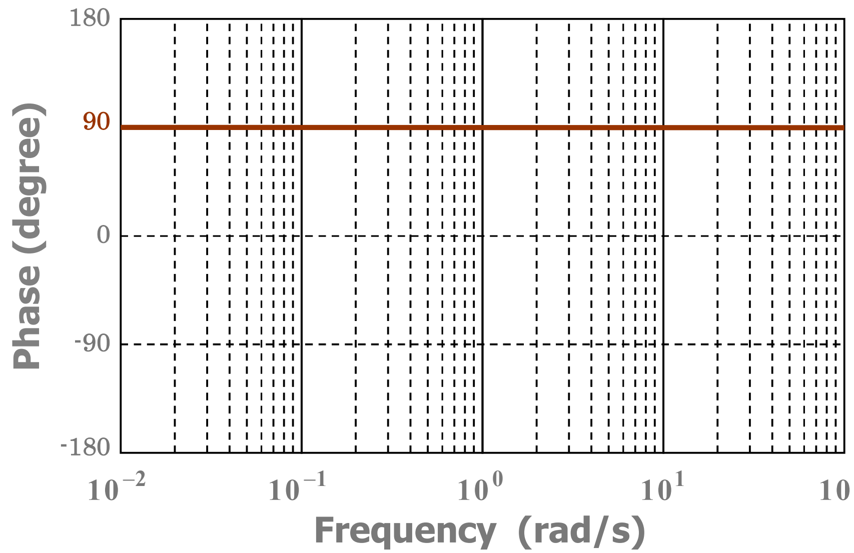
Magnitude Response : $\begin{cases} \left|\tilde{H}(j\omega)\right| = K_d\omega \\ \left|\tilde{H}(j\omega)\right|_{dB} = 20\log_{10}(K_d) \\ \qquad\qquad\qquad \underbrace{+20\log_{10}(\omega)}_{20\ \mathrm{dB/decade}}\ \mathrm{dB} \end{cases}$

The magnitude response is a straight line with slope $20\,\mathrm{dB/decade}$. At $\omega = \frac{1}{K_d}\,\mathrm{rad/s}$, its value is $0\ \mathrm{dB}$.

Phase Response : $\begin{cases} \angle\tilde{H}(j\omega) = \tan^{-1}\left(\frac{K_d\omega}{0}\right) \\ \qquad = 90° \end{cases}$

The phase response is a straight line with zero gradient. Its value is $90°$.

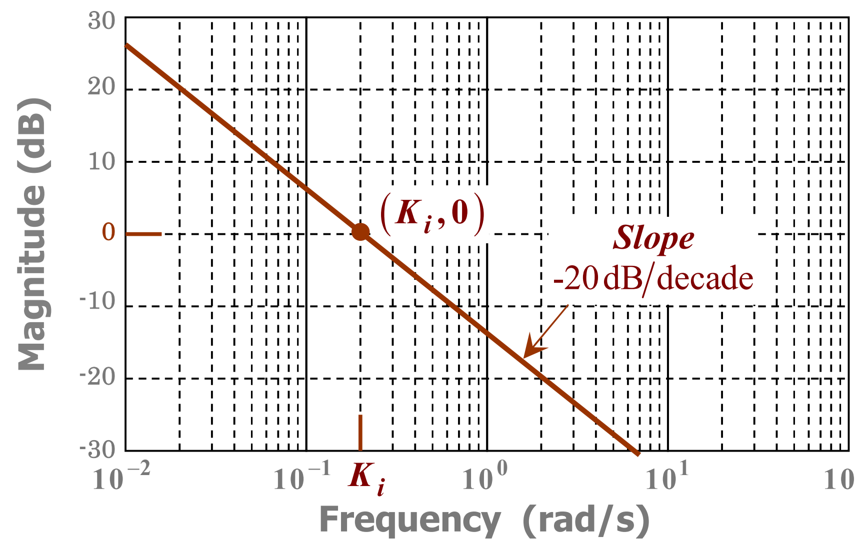




iii. Bode plots of Integrator: $\tilde{H}(s) = \dfrac{K_i}{s}$

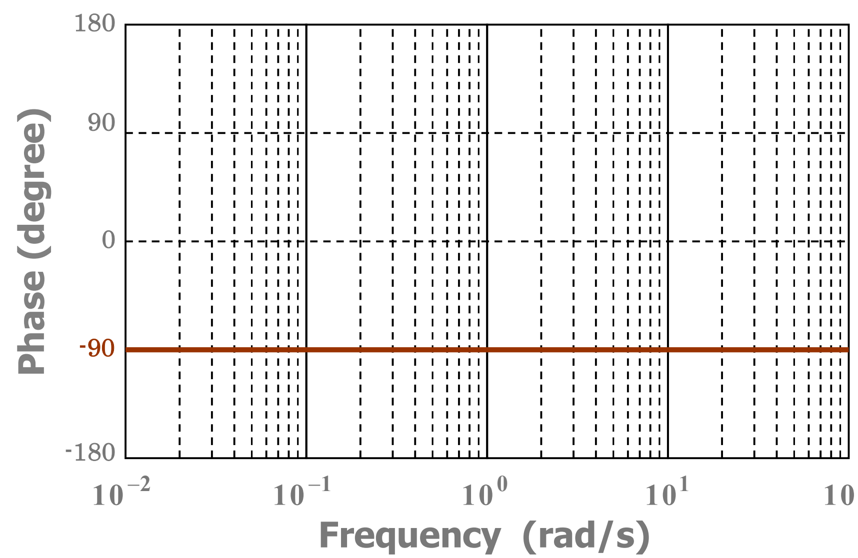
Frequency Response : $\tilde{H}(j\omega) = \dfrac{K_i}{j\omega}$

Magnitude Response :
$$\begin{cases} \left|\tilde{H}(j\omega)\right| = K_i/\omega \\ \left|\tilde{H}(j\omega)\right|_{dB} = 20\log_{10}(K_i) \underbrace{-20\log_{10}(\omega)}_{\text{-20 dB/decade}} \text{ dB} \end{cases}$$



The magnitude response is a straight line with slope $-20\,\mathrm{dB/decade}$. At $\omega = K_i$ rad/s, its value is 0 dB.

Phase Response :
$$\begin{cases} \angle\tilde{H}(j\omega) = -\tan^{-1}\left(\dfrac{\omega/K_i}{0}\right) \\ = -90^\circ \end{cases}$$



The phase response is a straight line with zero gradient. Its value is -90°.

iv. Bode plots of zero factor: $\tilde{H}(s) = \dfrac{s}{z_m} + 1$

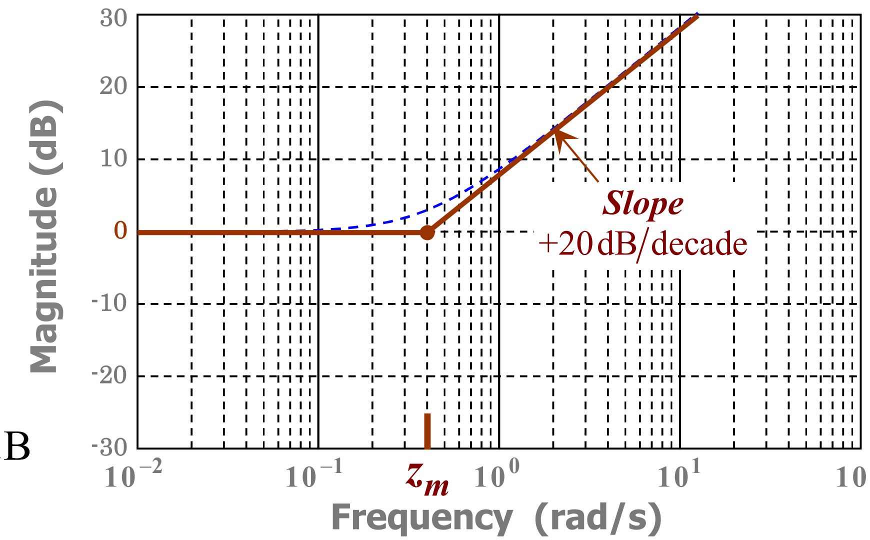
Frequency Response : $\tilde{H}(j\omega) = j\dfrac{\omega}{z_m} + 1$

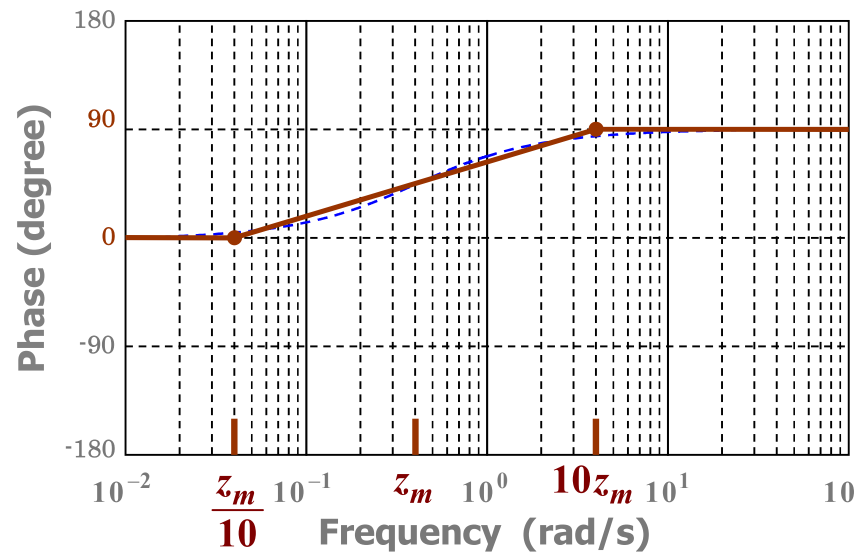
Magnitude Response :
$$\begin{cases} \left|\tilde{H}(j\omega)\right| = \sqrt{\omega^2/z_m^2 + 1} \\ \left|\tilde{H}(j\omega)\right|_{dB} = 20\log_{10}\left(\sqrt{\omega^2/z_m^2 + 1}\right) \text{ dB} \\ \omega << z_m : \left|\tilde{H}(j\omega)\right|_{dB} \to 0 \\ \omega >> z_m : \left|\tilde{H}(j\omega)\right|_{dB} \to \underbrace{20\log_{10}\left(\dfrac{\omega}{z_m}\right)}_{20 \text{ dB/decade}} \end{cases}$$

3dB Corner Frequency : $\omega = z_m$ rad/s

LO-Frequency Phase : $\lim\limits_{\omega \to 0} \tan^{-1}\left(\dfrac{\omega}{z_m}\right) = 0°$

HI-Frequency Phase : $\lim\limits_{\omega \to \infty} \tan^{-1}\left(\dfrac{\omega}{z_m}\right) = 90°$





v. Bode plots of pole factor: $\tilde{H}(s)=\left(\dfrac{s}{p_n}+1\right)^{-1}$

Frequency Response : $\tilde{H}(j\omega)=\left(\dfrac{j\omega}{p_n}+1\right)^{-1}=\dfrac{1}{j\omega/p_n+1}$

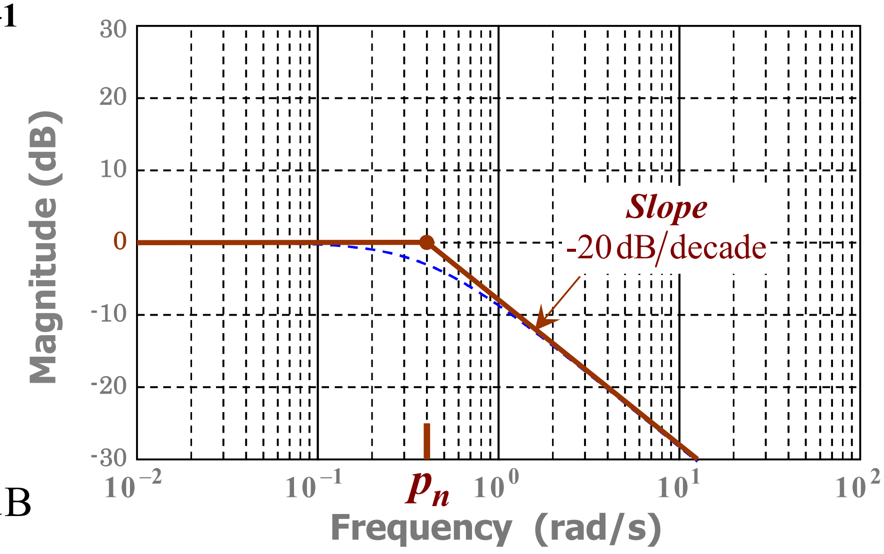
Magnitude Response :

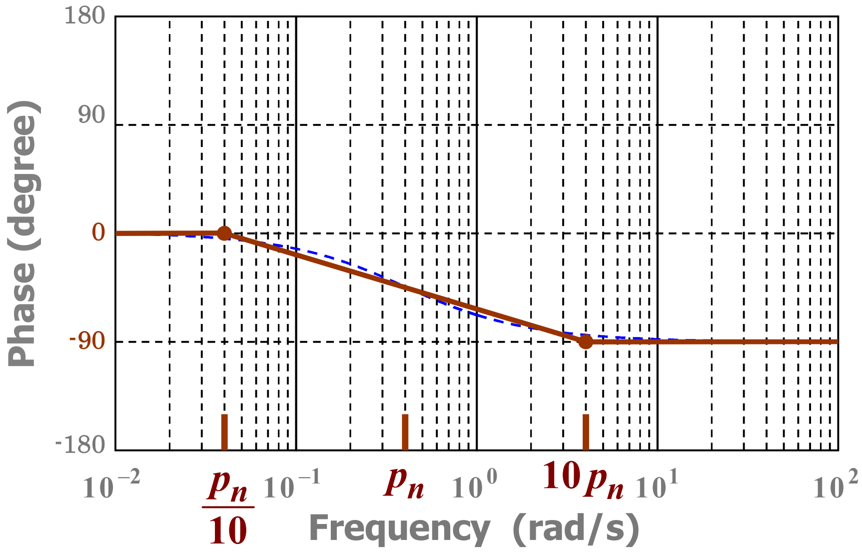
$$\begin{cases} \left|\tilde{H}(j\omega)\right| = \dfrac{1}{\sqrt{\omega^2/p_n^2+1}} \\ \left|\tilde{H}(j\omega)\right|_{dB} = -20\log_{10}\left(\sqrt{\omega^2/p_n^2+1}\right)\ \text{dB} \\ \omega \ll p_n : \left|\tilde{H}(j\omega)\right|_{dB} \to 0 \\ \omega \gg p_n : \left|\tilde{H}(j\omega)\right|_{dB} \to \underbrace{-20\log_{10}\left(\dfrac{\omega}{p_n}\right)}_{\text{-20 dB/decade}} \end{cases}$$

3dB Corner Frequency : $\omega = p_n$ rad/s

LO-Frequency Phase : $\lim\limits_{\omega\to 0} -\tan^{-1}\left(\dfrac{\omega}{p_n}\right)=0^\circ$

HI-Frequency Phase : $\lim\limits_{\omega\to \infty} -\tan^{-1}\left(\dfrac{\omega}{p_n}\right)=-90^\circ$





vi. Bode plots of underdamped 2nd-order factor: $\tilde{H}(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2};\quad 0\le\zeta\le 1$

$$\tilde{H}(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{(s/p_1+1)(s/p_2+1)} \quad \text{where} \quad \begin{cases} p_1=\omega_n\zeta+\omega_n\sqrt{\zeta^2-1} \\ p_2=\omega_n\zeta-\omega_n\sqrt{\zeta^2-1} \end{cases}$$

ζ is called the ***damping ratio***, and ω_n is called the ***undamped natural frequency*** of the system.

When $\zeta>1$, the system is said to be ***overdamped***. In this case, p_1 and p_2 are real and distinct, and $\tilde{H}(s)$ is a cascade of two pole factors. The Bode striaght-line plots of $\tilde{H}(s)$ can be derived from the Bode plots of $\dfrac{1}{s/p_1+1}$ and $\dfrac{1}{s/p_2+1}$ (see Example 6-4).

When $\zeta=1$, the system is said to be ***critically damped***. In this case, p_1 and p_2 are real and equal, and $\tilde{H}(s)$ is a cascade of two identical pole factors. Like the overdamped case, the straight-line Bode plots of $\tilde{H}(s)$ can be derived from the Bode plots of $\dfrac{1}{s/p_1+1}$ and $\dfrac{1}{s/p_2+1}$ with $p_1=p_2=p$.

When $0<\zeta<1$, the system is said to be ***underdamped***. When $\zeta=0$, the system is said to be ***undamped***. In both cases, p_1 and p_2 are a complex conjugate pole pair. The straight-line Bode plots in this case are constructed with $\zeta=1$.

With $\zeta = 1$, we have $p_1 = p_2 = \omega_n$ and

$$\tilde{H}(s) = \left(\frac{1}{s/\omega_n + 1}\right)\left(\frac{1}{s/\omega_n + 1}\right)$$

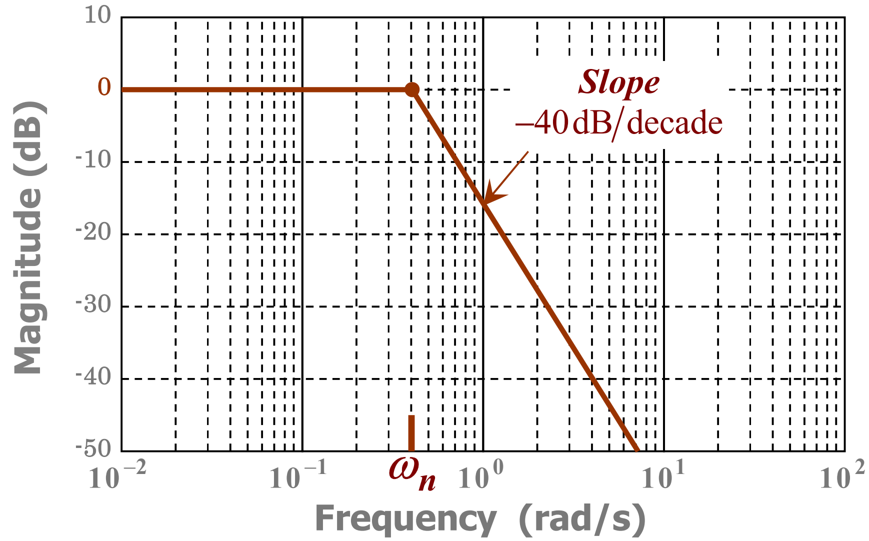
Frequency Response : $\tilde{H}(j\omega) = \left(\frac{1}{j\omega/\omega_n + 1}\right)\left(\frac{1}{j\omega/\omega_n + 1}\right)$

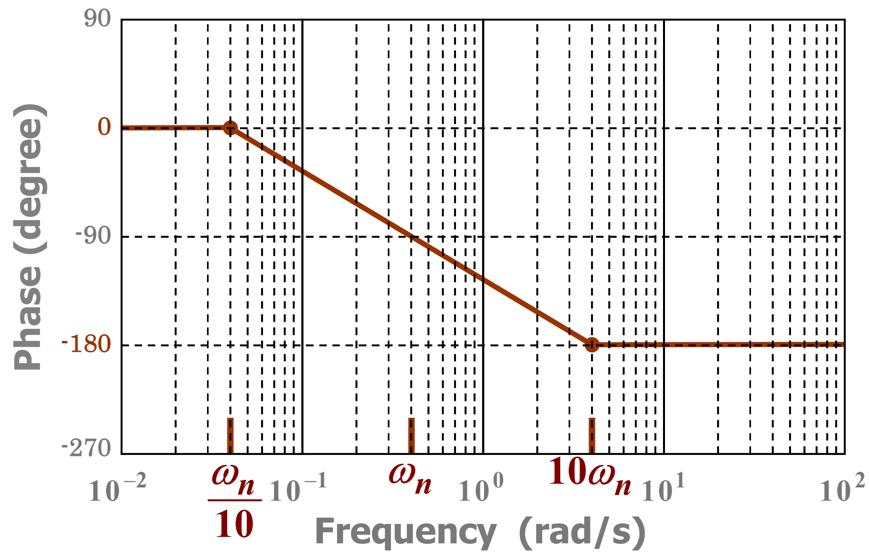
Magnitude Response :
$$\begin{cases} \left|\tilde{H}(j\omega)\right|_{dB} = 20\log_{10}\left[\left|\dfrac{1}{\sqrt{\omega^2/\omega_n^2 + 1}}\right|^2\right] \\ \qquad = -40\log_{10}\left(\sqrt{\omega^2/\omega_n^2 + 1}\right) \text{ dB} \\ \omega << \omega_n : \left|\tilde{H}(j\omega)\right|_{dB} \to 0 \\ \omega >> \omega_n : \left|\tilde{H}(j\omega)\right|_{dB} \to \underbrace{-40\log_{10}\left(\dfrac{\omega}{\omega_n}\right)}_{\text{-40 dB/decade}} \end{cases}$$

6dB Corner Frequency : $\omega = \omega_n$ rad/s

LO-Frequency Phase : $\lim\limits_{\omega \to 0} -2\tan^{-1}\left(\frac{\omega}{\omega_n}\right) = 0^\circ$

HI-Frequency Phase : $\lim\limits_{\omega \to \infty} -2\tan^{-1}\left(\frac{\omega}{\omega_n}\right) = -180^\circ$





Example 6-4:

Draw the straight-line Bode plots for the 2nd-order system: $\tilde{H}(s)=\dfrac{10}{(10s+1)(s+1)}$

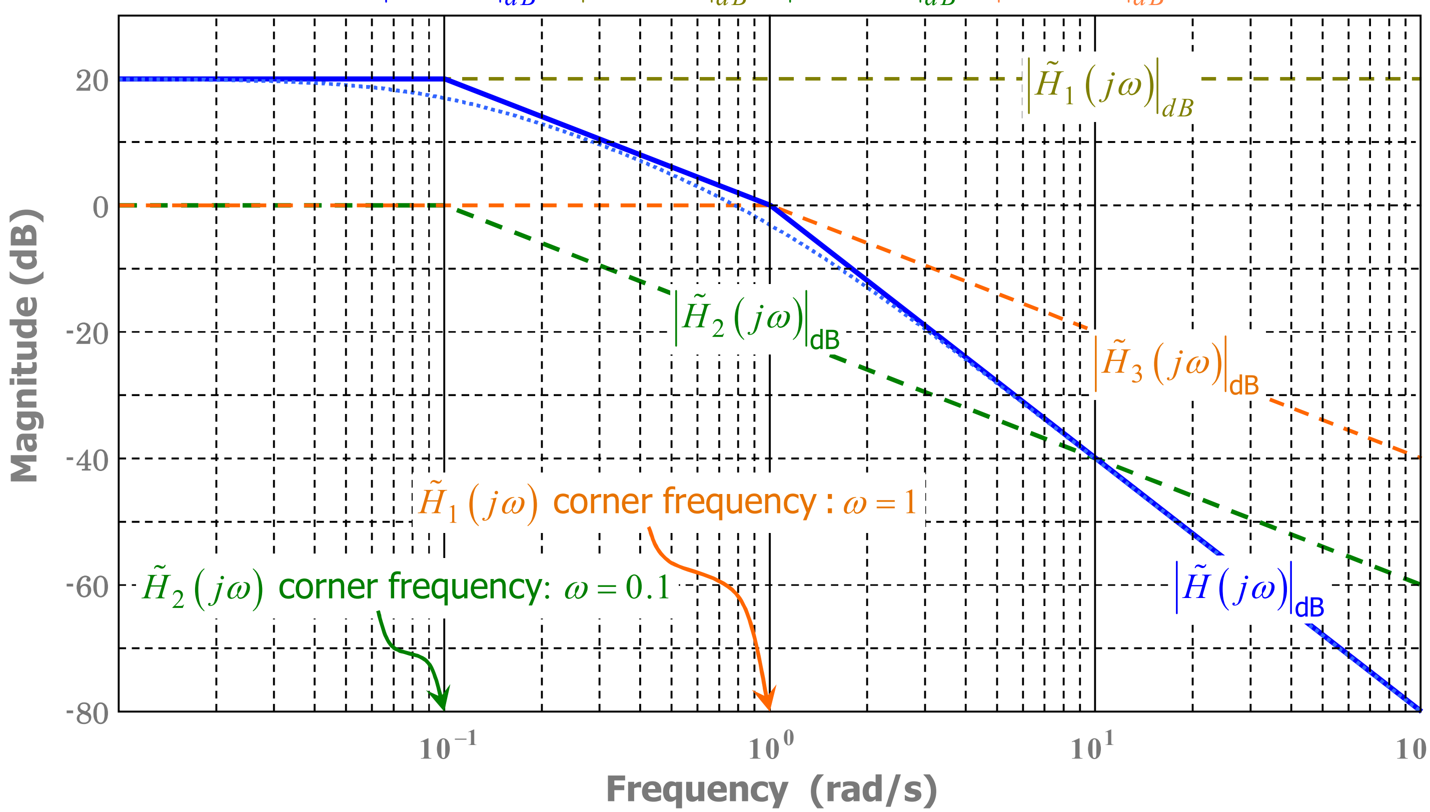
$$\tilde{H}(j\omega)=10\left(\frac{1}{j10\omega+1}\right)\left(\frac{1}{j\omega+1}\right)=10\left(\frac{1}{\frac{j\omega}{0.1}+1}\right)\left(\frac{1}{\frac{j\omega}{1}+1}\right)=\tilde{H}_1(j\omega)\tilde{H}_2(j\omega)\tilde{H}_3(j\omega)$$

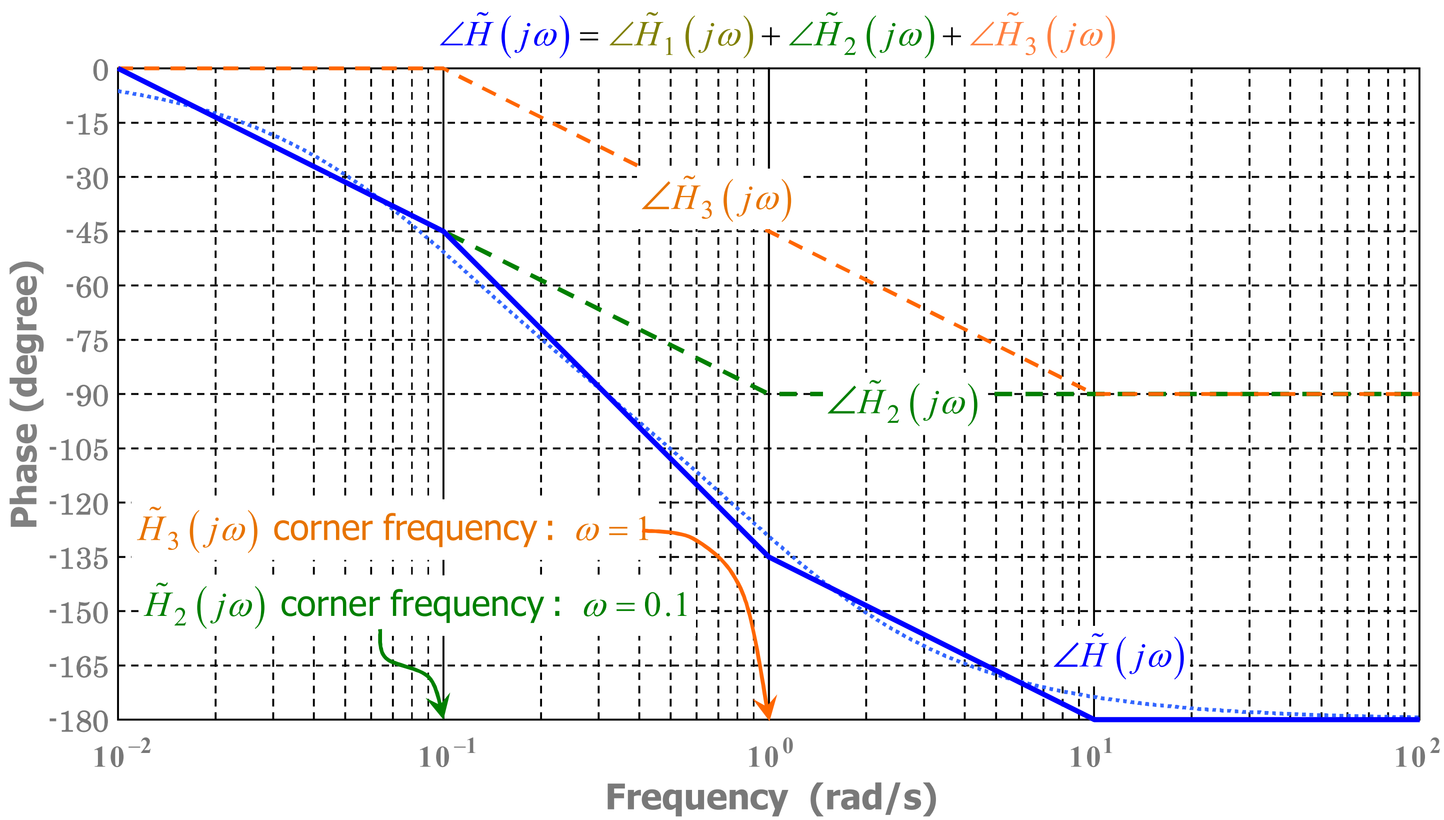
$$\tilde{H}_1(j\omega)=10$$

$$\tilde{H}_2(j\omega)=\frac{1}{\frac{j\omega}{0.1}+1}$$

$$\tilde{H}_3(j\omega)=\frac{1}{j\omega+1}$$

$$\left|\tilde{H}(j\omega)\right|_{dB}=\left|\tilde{H}_1(j\omega)\right|_{dB}+\left|\tilde{H}_2(j\omega)\right|_{dB}+\left|\tilde{H}_3(j\omega)\right|_{dB}$$




$\angle\tilde{H}(j\omega) = \angle\tilde{H}_1(j\omega) + \angle\tilde{H}_2(j\omega) + \angle\tilde{H}_3(j\omega)$
$\angle\tilde{H}_3(j\omega)$
$\angle\tilde{H}_2(j\omega)$
$\angle\tilde{H}(j\omega)$
$\tilde{H}_3(j\omega)$ corner frequency : $\omega = 1$
$\tilde{H}_2(j\omega)$ corner frequency : $\omega = 0.1$
Phase (degree)
Frequency (rad/s)
0
-15
-30
-45
-60
-75
-90
-105
-120
-135
-150
-165
-180
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

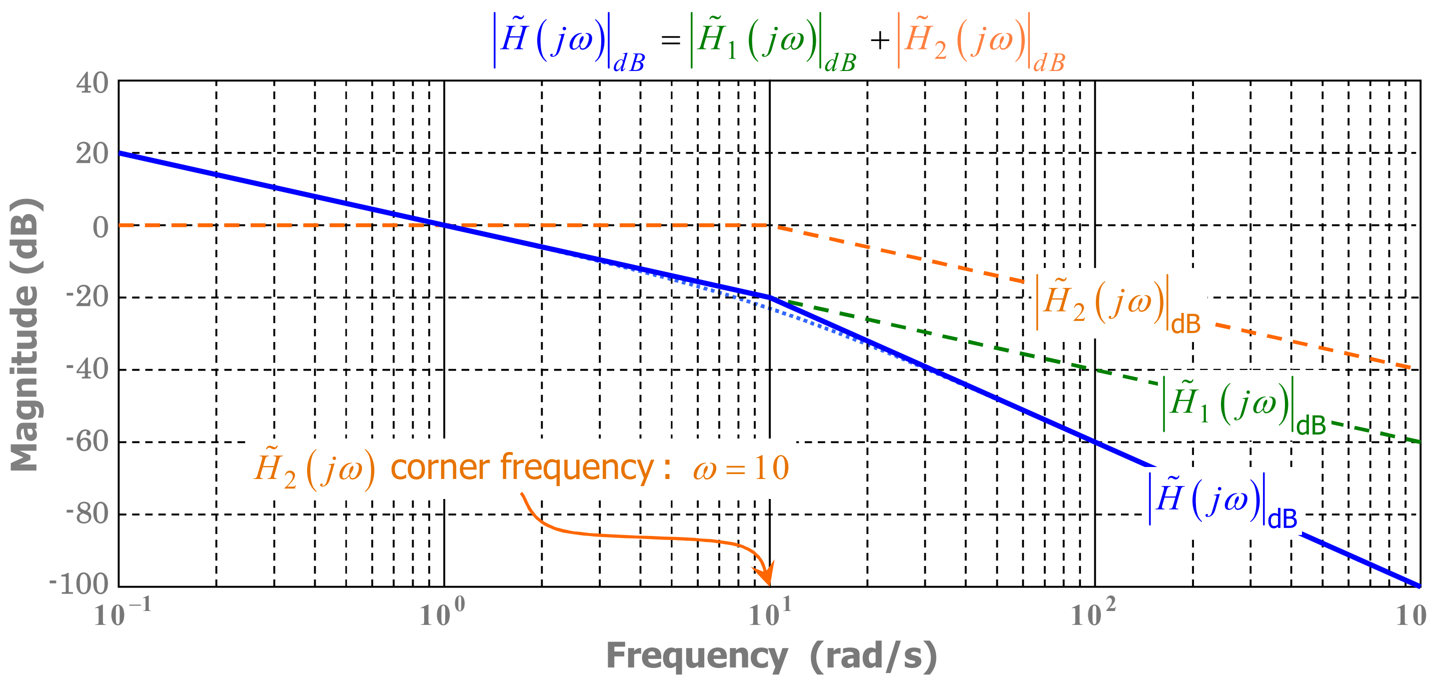
Example 6-5

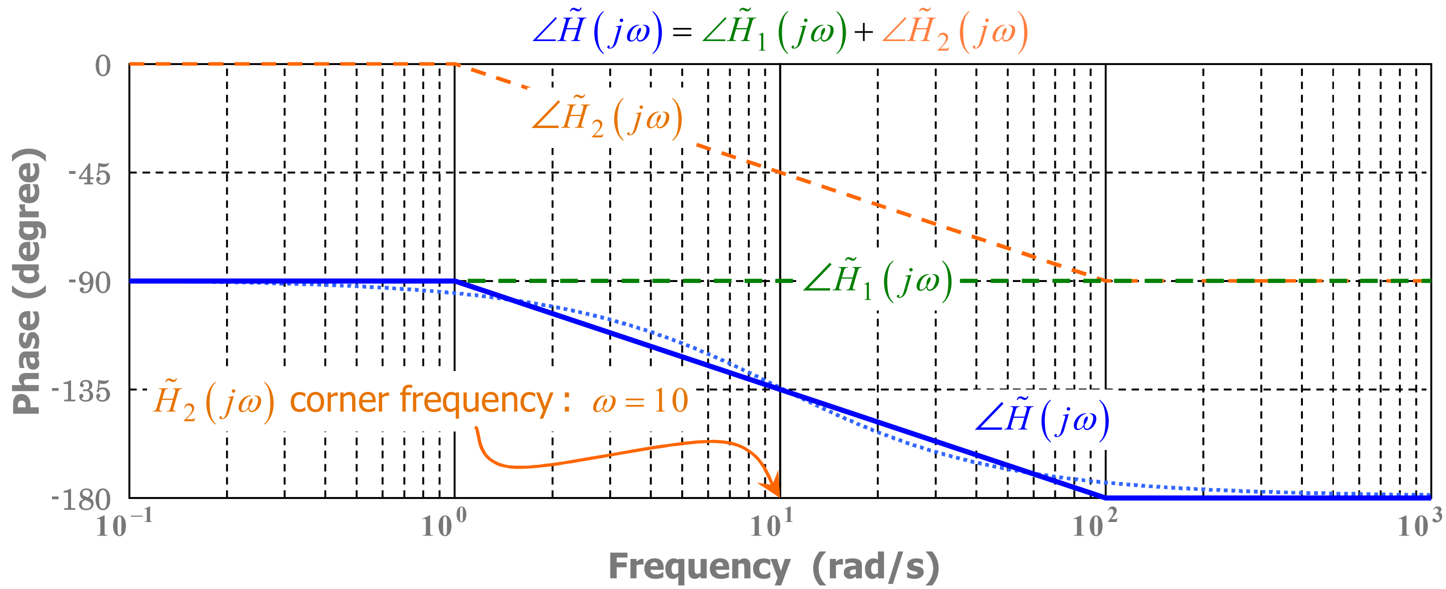
Draw the straight-line Bode plots for the 2nd-order system: $\tilde{H}(s)=\dfrac{1}{s(0.1s+1)}$

$$\tilde{H}(j\omega)=\frac{1}{j\omega}\left(\frac{1}{j0.1\omega+1}\right)=\frac{1}{j\omega}\left(\frac{1}{\frac{j\omega}{10}+1}\right)=\tilde{H}_1(j\omega)\tilde{H}_2(j\omega)$$

$$\tilde{H}_1(j\omega)=\frac{1}{j\omega}$$

$$\tilde{H}_2(j\omega)=\frac{1}{\frac{j\omega}{10}+1}$$





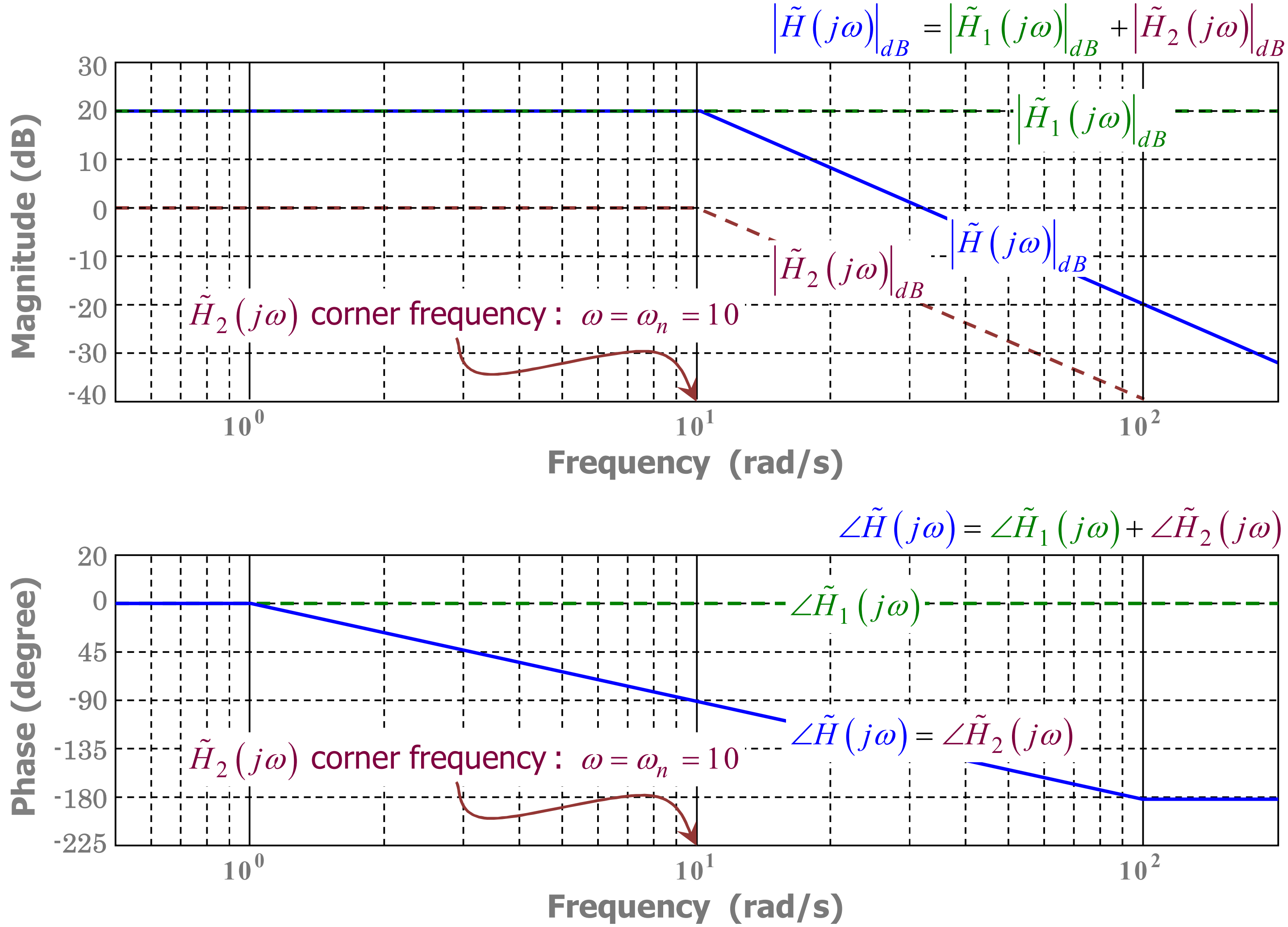
Example 6-6:

Draw the Bode straight - line plots for the 2nd - order system : $\tilde{H}(s) = \dfrac{1000}{s^2 + 10s + 100}$

Comparing $\tilde{H}(s)$ *with the standard form* $\dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$, *we note that* $\tilde{H}(s)$ *is an underdamped 2nd-order system with* $\zeta = 0.5$, $\omega_n = 10$ *and* $K = 10$.

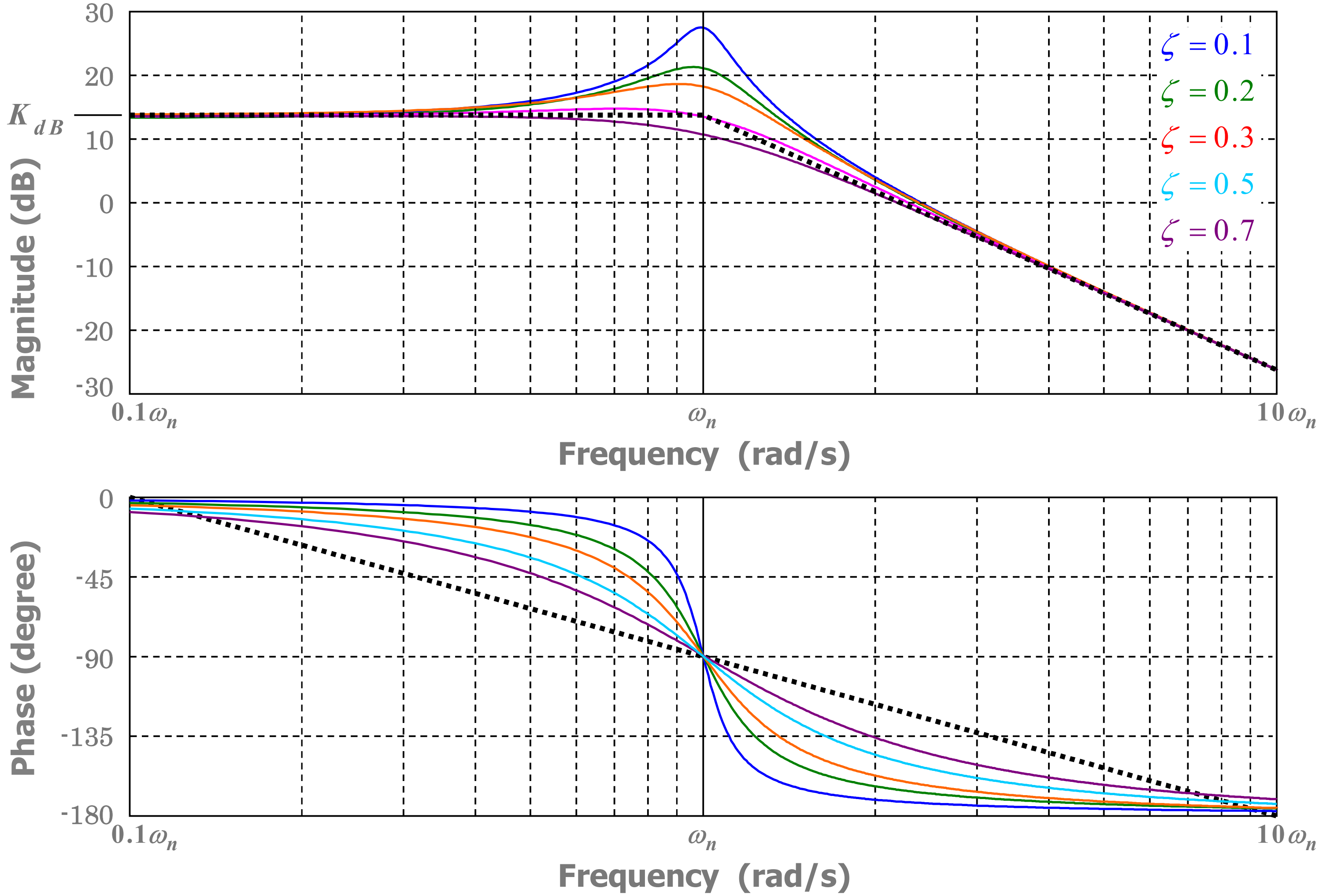
$$\tilde{H}(j\omega) = 10\underbrace{\left[\frac{100}{s^2 + 10s + 100}\right]_{s=j\omega}}_{\text{2nd-order factor}} = \tilde{H}_1(j\omega)\tilde{H}_2(j\omega)$$

$$\tilde{H}_1(j\omega) = 10 \quad \text{and} \quad \tilde{H}_2(j\omega) = \left[\frac{100}{s^2 + 10s + 100}\right]_{s=j\omega}$$



6.3.3 Resonance in Second Order Systems

The Bode plots for different values of damping ratio, ζ, is shown below for $\tilde{H}(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$.



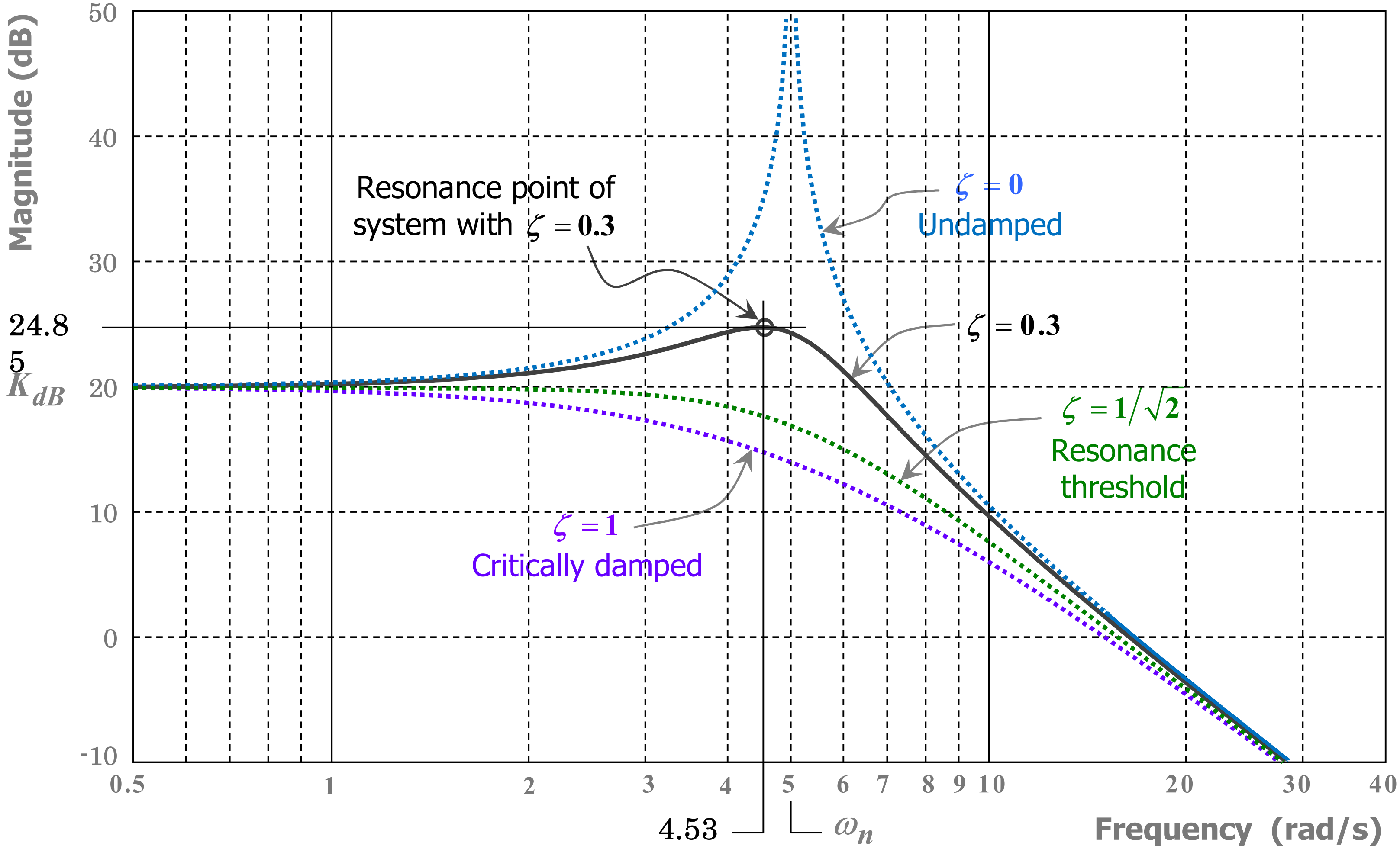
Note that the magnitude response has a 'hump' around $\omega = \omega_n$. The 'hump' is more pronounce when ζ is small and disappears when ζ **rises above** a certain value. This 'hump' is associated with the phenomenon of resonance. The magnitude of the 'hump' is called "**resonance peak**" and the frequency at which it occurs is called "**resonance frequency**".

Stating without proof:

$$\left.\begin{array}{l} \text{Resonance frequency}: \ \omega_r = \omega_n\sqrt{1-2\zeta^2} \\ \text{Resonance peak}: \ \left|\tilde{H}(\omega_r)\right| = \dfrac{K}{2\zeta\sqrt{1-\zeta^2}} \end{array}\right\} \begin{array}{l} \text{Valid only for } \zeta < \dfrac{1}{\sqrt{2}}. \text{ There} \\ \text{is no resonance when } \zeta \geq \dfrac{1}{\sqrt{2}} \end{array} \tag{6.25}$$

Note from (6.28) that $\lim\limits_{\zeta\to 0}\omega_r = \omega_n$ and $\lim\limits_{\zeta\to o}\left|\tilde{H}(\omega_r)\right| = \infty$.

The resonance point of $\tilde{H}(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ $\left(\textit{with } \boldsymbol{K} = \mathbf{10},\ \boldsymbol{\omega_n} = \mathbf{5} \textit{ and } \zeta = \mathbf{0.3}\right)$ is shown in the below Bode magnitude plot of $\tilde{H}(s)$ where, from (6.28), $\boldsymbol{\omega_r} \approx \mathbf{4.53}$ and $\left|\tilde{\boldsymbol{H}}(\boldsymbol{j}\boldsymbol{\omega_r})\right|_{\boldsymbol{dB}} \approx \mathbf{24.85}$.


Magnitude (dB)
50
40
30
24.85
K_{dB} 20
10
0
-10
0.5
1
2
3
4
5
6
7
8
9
10
20
30
40
Frequency (rad/s)
4.53
ω_n
Resonance point of system with $\zeta = 0.3$
$\zeta = 0$
Undamped
$\zeta = 0.3$
$\zeta = 1/\sqrt{2}$
Resonance threshold
$\zeta = 1$
Critically damped

6.3.4 Asymptotic Behavior of Straight-line Bode Plots

- **Asymptotic <u>PHASE</u> of Phase Plot** $\left[\angle\tilde{\boldsymbol{H}}(j\omega)\right]$

High frequency:

$$\lim_{\omega\to\infty}\angle\tilde{H}(j\omega)=\underbrace{\left[\text{Number of POLES - Number of ZEROS}\right]}_{\textit{Inclusive of Integrators and Differentiators}}\times\left(-90^\circ\right) \tag{6.26a}$$

Low frequency:

$$\lim_{\omega\to 0}\angle\tilde{H}(j\omega)=\begin{bmatrix}\text{Number of Integrators} & \\ & \text{- Number of Differentiators}\end{bmatrix}\times\left(-90^\circ\right) \tag{6.26b}$$

- **Asymptotic <u>SLOPE</u> of Magnitude Plot** $\left[\left|\tilde{\boldsymbol{H}}(j\omega)\right|_{\boldsymbol{dB}}\right]$

High frequency:

$$\lim_{\omega\to\infty}\left[\textit{slope of }\left|\tilde{H}(j\omega)\right|_{dB}\right]=\underbrace{\begin{bmatrix}\text{Number of POLES} & \\ & \text{- Number of ZEROS}\end{bmatrix}}_{\textit{Includes Integrators and Differentiators}}\times\left(-20\ \text{dB/decade}\right) \tag{6.27a}$$

Low frequency:

$$\lim_{\omega\to 0}\left[\textit{slope of }\left|\tilde{H}(j\omega)\right|_{dB}\right]=\begin{bmatrix}\text{Number of Integrators} & \\ & \text{- Number of Differentiators}\end{bmatrix}\times\left(-20\ \text{dB/decade}\right) \tag{6.27b}$$

Example 6-7:

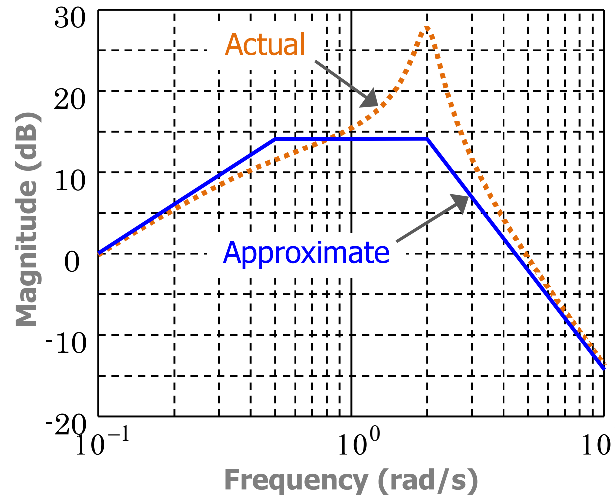
The Bode magnitude plot of a system with the following transfer function

$$\tilde{H}(s) = \frac{K(s+a)}{(s/b+1)\left(s^2 + 2\varsigma\omega_n s + \omega_n^2\right)}$$

is shown on the right.

(a) ***Identify the values of*** a, b, ω_n ***and*** K.

(b) ***What value will the high frequency asymptote of the Bode phase plot for*** $\tilde{H}(s)$ ***converge to?***



Bode magnitude plot

(a) *Rewrite the transfer function as:* $\tilde{H}(s) = \frac{K}{\omega_n^2}(s+a)\cdot\frac{1}{\frac{s}{b}+1}\cdot\frac{\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$.

The Bode magnitude plot has the following features:

- *The existence of a low frequency asymptote of 20 dB/decade indicates the presence of a differentiator* $K_d s$. *Hence,* $a = 0$.
- *The low frequency asymptote passes through the point* $(0.1,\ 0\mathrm{dB})$ *indicates that the derivative gain* $K_d = K/\omega_n^2 = 1/0.1$ *or* $\left(K = 10\omega_n^2\right)$ $\cdots$ ✦.

- *At* $\omega = 0.5\,\mathrm{rad/s}$*, slope changes by* $-20\,\mathrm{dB/decade}$*, which indicates the presence of a pole factor* $\left(\frac{s}{0.5}+1\right)^{-1}$*. Therefore,* $b = 0.5$*.*

- *At* $\omega = 2\,\mathrm{rad/s}$*, slope changes by* $-40\,\mathrm{dB/decade}$*, which indicates the presence of either two repeated pole factors or an underdamped 2nd-order factor. The existence of a resonant peak indicates the latter* $\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)^{-1}$ *with* $\omega_n = 2\,\mathrm{rad/s}$*.*

- *Substituting* $\omega_n = 2\,\mathrm{rad/s}$ *into ✦, we get* $K = 40$*.*

- $\therefore \tilde{H}(s) = \dfrac{20s}{(s+0.5)\left(s^2 + 4\zeta s + 4\right)}$

(b) $\tilde{H}(s)$ *has a pole excess of 2. Hence, the high frequency asymptote of the Bode phase plot will converge to* $2 \times (-90°) = -180°$*.*