National University of Singapore
Department of Electrical & Computer Engineering

EE2023 Signals & Systems Revision Notes

Functions and Transformations

The simplest functions are expressions which map one variable to another and they can be visualized via graphs or plots. Transformations are operations which you apply to change the basic characteristics of the original function. In this section, we reveal some of the transformations which are important for the EE2023 module. We will only consider functions which map real variables into real function values.

1 What are functions?

In mathematics, functions are expressions which relate one variable (independent variable) to another variable (dependent variable). In general, we express a function as $y = f(x)$ where y is the dependent variable and x is the independent variable. f is the mapping function which takes x to y or it takes x-values and change them to y-values through a function called f. x and y are just generic or dummy variables and we can give them different names if we prefer. For example, instead of x, we can have t and instead of y we use s, and f can be v. Then we end up with $s = v(t)$. Often t is used to denote time.

The function f can take many different forms. You have come across this in your high school or Polytechnic mathematics syllabus. Some forms are as follows :

- f is a <u>linear function</u> : $y = f(x) = mx + c$, where m and c are constants. It is linear because if you plot y against x, you get a straight line on a plot. It turns out that m is the gradient or slope of this straight line while c is the intercept on the y-axis.
- f is a <u>quadratic or parabolic function</u> : $y = f(x) = ax^2 + bx + c$ where a, b, and c are con-

stants. In fact, this f is a polynomial function with the highest degree of 2, which comes from the x^2. A polynomial with the highest degree of 2 is called a quadratic function. When you plot y against x, it takes the shape of a parabola, and hence a quadratic function is also often referred to as a parabolic function.

- f may also be a general polynomial function with the highest degree $n \geq 3$. For example

$$y = f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_0$$

where $n \geq 3$ and $a_i,\ i = 0, 1, 2, \ldots n$ are constants. These a_i's are often referred to as the coefficients of $f(x)$ and in this case, there are a total of $(n+1)$ coefficients. When $n = 3$, we call f a cubic function. We usually do not have a name for the function when $n \geq 4$ but we refer to them as n^{th}-order polynomials.

As you can imagine, the higher the value of n or the higher the order of the polynomial function f, the more complex the plot will look. When $n = 1$, f is a linear function and the graph is a straight line. When $n = 2$, f is a quadratic or parabolic function and the graph will be a parabola. When $n = 3$, you get a cubic graph.

Exercise 1. *Use excel or other software to plot the functions below with y against x.*

(a) $y = 2x + 3$

(b) $y = x^2$

(c) $y = x^2 + x + 1$

(d) $y = x^2 + 4x + 4$

(e) $y = x^3 + 2x^2 + x + 1$

In each of the graphs above, observe where the graph cuts the x-axis. What is the value of y when the graph cuts the x-axis? What do you call these points where the graph cuts the x-axis? For each of the functions (a) to (e) above, how many zero crossings are there? Why are there graphs which do not cut the x-axis?

- f is a trigonometric function. These are functions which contain terms like sine and cosine. Examples of trigonometric functions are $y(x) = \sin(2x)$, $s(t) = \cos(4t)$, $v(x) = \sin(x) + \tan(2x)$, etc.

Exercise 2. *Plot the following functions :*

(a) $v(t) = 5\sin(2\pi t)$. *What is the maximum value of* $v(t)$*? Check out the zero crossings of* $v(t)$ *and figure out the role of* 2π.

(b) $s(t) = \tan(t)$. *What is the distinctive feature of this tangent function?*

(c) $y(t) = \tan^{-1}(t)$. *What does the plot look like if you compare to* s *in part (b) above?*

(d) $s_1(t) = \tan(2t)$. *Compare* $s_1(t)$ *to* $s(t)$ *in part (b). What can you conclude?*

What are the distinctive features of trigonometric functions? Take the example of $v(t) = 5\sin(2\pi t)$. In this example, we think of t as time in seconds.

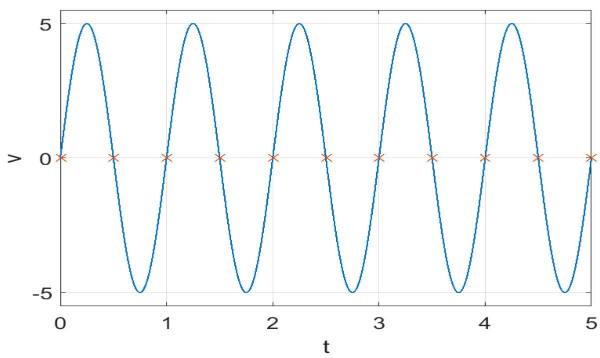

Fig. 1: Sine function

The features of the graph are as follows :

- There is a periodic variation referred to, in this case, as a sinusoidal variation. The signal, v, is thus a periodic signal.
- The repeating function lasts for 1 sec. We call this interval (which repeats) the fundamental period of the function. Hence this period, which we denote as T, is $T = 1$ sec. In other words, one cycle of the function lasts 1 sec.

- Since there is periodicity with period $T = 1$ sec, then the cyclical frequency of the sinusoid is $f = 1/T = 1$ cycle/sec or 1 Hertz (Hz). Hence Hertz is the unit which is equivalent to "number of cycles per unit time".
- In angular terms, 1 cycle is equivalent to 2π radians. Hence the angular frequency is $\omega = 2\pi f = \frac{2\pi}{T} = 2\pi$ rad/sec since $T = 1$. Hence the 2π which you see in $v(t) = 5\sin(2\pi t)$ is the angular frequency of the sinusoid in rad/sec and 5 is the maximum amplitude of the sinusoid. Its frequency is also equivalent to 1 cycle/sec or 1 Hz. So every Hz is equivalent to 2π rad/s.

For another sinusoidal function such as $y(t) = 2\cos(5t)$, the angular frequency of $y(t)$ is 5 rad/sec or $\frac{5}{2\pi}$ Hz. We call $y(t)$ a sinusoidal signal of frequency 5 rad/s and amplitude 2. In general, a pure sinusoidal signal can be written as $y(t) = A\sin(\omega t) = A\cos(\omega t - 0.5\pi)$ since

$$A\cos(\omega t - 0.5\pi) = A\cos(\omega t)\cos 0.5\pi + A\sin(\omega t)\sin 0.5\pi = A\sin(\omega t).$$

What this tells us is that sine and cosine functions are interchangeable because one function can always be written in terms of another by simply adding a phase shift. In this example, a sine function is equivalent to a cosine function with a phase shift of -0.5π. It is important to note that both functions (sine and cosine) have the same frequency of ω rad/s and amplitude of A. The only difference is a phase shift.

Exercise 3. *What happens if you have another signal $s(t) = 2\sin(2\pi t) + 3\cos(4\pi t)$? How many frequency components are in $s(t)$? What are the frequencies of these components? What are their respective amplitudes? What does $s(t)$ look like?*

Exercise 4. *What happens if you have another signal $s(t) = 2\sin(2\pi t + \pi/4) + 3\cos(4\pi t + \pi/8)$? How many frequency components are in $s(t)$? What are the frequencies of these components? How do the phases $\pi/4$ and $\pi/8$ change the signal $s(t)$? Explain your answer by plotting or sketching the signal $s(t)$, first by plotting or sketching their individual components. How does $s(t)$ change if you change the phase values?*

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2 What is a Function Transformation?

In signal processing, signals or functions often go through some form of transformation. In general, transformation means to change from one form to another. In mathematics, you can change a function by making "transformations". The graph can be transformed so that it is taller, shorter, fatter or thinner. These are some of the simple forms of transformations that you will find in EE2023 which will end up having implications on the frequency components in a signal.

In this section, we will introduce how some of these changes can be made. Suppose we are working with a function $y = f(t)$. Then you define another function y' which is a transformation of y. What simple transformations can you make to transform y to y'?

(a) Scaling. $y' = \alpha y = \alpha f(t)$ where α is a positive or negative real constant.

- It is obvious that if $\alpha > 1$, say $\alpha = 2$, then $y' = 2y = 2f(t)$ which means that y' is twice as large as y. In other words, y' is larger than y by two times.
- If $y' = 0.5y = 0.5f(t)$, then y' is only half the size of y.
- If $y' = -y = -f(t)$, then every y' is the negative of y and therefore y' is a "flip" of y over the t-axis.
- If $y' = -0.5y = -0.5f(t)$, then y' not only flips y over the t-axis but is also only half the magnitude of y.

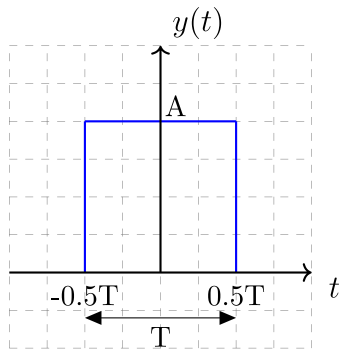
For a simple linear transformation like this ie $y' = \alpha y$, y' is only a "scaling"of y. This means that if $|\alpha| \geq 1$, then $|y'| \geq |y|$, if $|\alpha| \leq 1$, then $|y'| \leq y$ where $|.|$ here refers to the magnitude. Hence this form of transformation only makes the original function taller or shorter or in mathematical terms, the magnitude is either bigger or smaller.

(b) Translation. Translation happens when you move the graph horizontally from one place to another without changing its shape or size. How do you achieve this in mathematics? A translation happens when you make changes to the argument, t. This is achieved with $y' = f(t - \alpha)$.

– Translation or "move" to the right along the t-axis. The graph of y moves to the right along the t-axis if $\alpha > 0$. For example if $\alpha = 5$ sec, then the shape of y' is exactly the same as y, but it is displaced to the right of the original y by 5 sec.

To illustrate the idea of translation, let us introduce a new function which is commonly used in signals. This is the rectangular pulse function defined as follows :

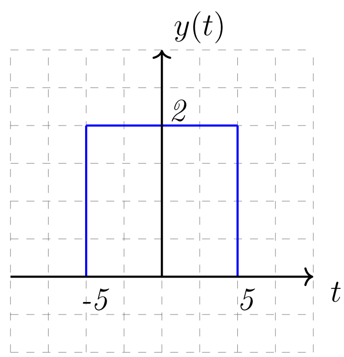
$$y(t) = A \text{ rect}\left(\frac{t}{T}\right) = \begin{cases} A & |t| \leq 0.5T \\ 0 & t > 0.5T \end{cases} \quad \Longrightarrow$$



Notice that A is the amplitude of the rectangular pulse which lasts from $t = -0.5T$ to $0.5T$ and hence the width of the pulse is T sec. This signal will be used to illustrate translations.

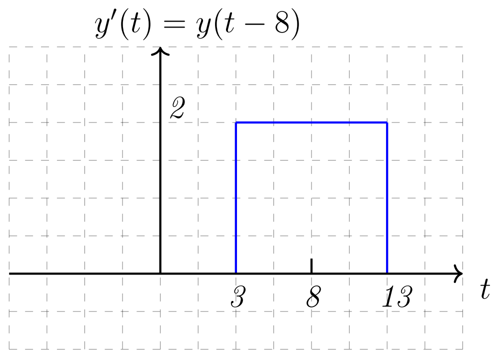
Example 1. *Suppose we have a rectangular pulse $y(t)$ with amplitude 2 and duration 10 sec. Then this signal can be written as :*

$$y(t) = 2 \text{ rect}\left(\frac{t}{10}\right) = \begin{cases} 2 & |t| \leq 5 \\ 0 & t > 5 \end{cases} \quad \Longrightarrow$$



Let's transform y such that the original function is moved to the right by 8 sec? You do so by changing the variable t to $(t-8)$ in $y(t)$. Hence,

$$y'(t) = y(t-8) = 2 \text{ rect}\left(\frac{t-8}{10}\right) \quad \Longrightarrow$$



Notice that the width and height of $y'(t)$ is the same as the original $y(t)$. The tranformed $y'(t)$ is shifted to the right or translated to $t = 8$ and $y'(t)$ is now centred at $t = 8$. This shift to the right also means that the signal has been delayed by 8 sec.

In general, to delay or advance a signal by T_0, the transformation is simple. You do so by replacing every t in the $y(t)$ by $(t \mp T_0)$ where $T_0 > 0$. Hence $y(t - T_0)$ is a delayed signal (later in time), while $y(t + T_0)$ is an advanced signal ie a shift to the left (earlier in time).

Exercise 5. *How do you shift the original signal $y(t)$ to the left of the y-axis, instead of to the right?*

Exercise 6. *How do you scale $y(t)$ so that it is 5 times bigger than the original $y(t)$?*

Exercise 7. *If you have a linear function $y = 2x + 1$, how do you shift the whole line to the right by 3 units? Write down the transformed function.*

(c) Expansion and Compression. This is about making the graph "fatter" (expand) or "skinnier" (compress). How do you achieve this mathematically?

Let's use the same rectangular pulse function from part (b). How can we make it skinnier or fatter by α times?

You do so by replacing t in $y(t)$ by αt. Hence :

$$y(t) = A \text{ rect}\left(\frac{t}{T}\right) \quad \overset{transform}{\Longrightarrow} \quad y'(t) = y(\alpha t) = A \text{ rect}\left(\frac{\alpha t}{T}\right)$$

If $\alpha = 2$, then the function $y'(t)$ becomes half the width of the original function, $y(t)$. If $\alpha = 3$, then $y'(t)$ is one third the width of $y(t)$. When $\alpha = 0.5$, then $y'(t)$ is twice as wide as $y(t)$. In other words, you get the following effect :

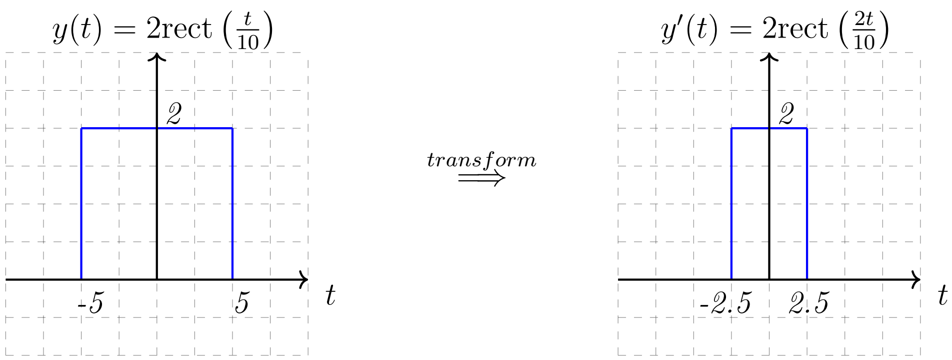
If $\alpha > 1$, your transformed graph is skinnier.

If $0 < \alpha < 1$, your transformed graph is fatter.

Example 2. *Let's transform $y(t)$ by replacing t by $2t$. Hence*

$$y(t) = 2\text{rect}\left(\frac{t}{10}\right) \quad \overset{transform}{\Longrightarrow} \quad y'(t) = y(2t) = 2\text{rect}\left(\frac{2t}{10}\right)$$

What does $y'(t)$ look like now?



You have now learned 3 kinds of transformations which you will encounter in EE2023. These are : scaling, translating and expansion / compression. You should now be able to combine these three ideas to solve the following problems.

Exercise 8. *Consider a triangular pulse defined as follows :*

$$s(t) = A\ \text{tri}\left(\frac{t}{T}\right) = \begin{cases} A(1-|t|) & |t| \le T \\ 0 & t > T \end{cases}$$

Sketch $s(t)$. How is this different from the rectangular pulse in the examples above? What is the width of the triangular pulse, compared to the rectangular pulse? What is the maximum amplitude of $s(t)$?

Exercise 9. *For $s(t)$ above, proceed to write the new signal $s'(t)$ such that the following transformations are achieved.*

(a) *$s'(t)$ is twice the height of $s(t)$.*

(b) *$s'(t)$ is twice the width of $s(t)$.*

(c) *$s'(t)$ is half the width and height of $s(t)$.*

(d) *$s'(t)$ is the same signal as $s(t)$ but delayed by 5 sec.*

(e) *$s'(t)$ is the same height as $s(t)$ but is half the width and delayed by 2 sec.*

(f) *What does $s(-t)$ look like?*

(g) *What does $-s(t)$ look like?*

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3 Addition, Subtraction & Multiplication of Functions

We often have to manipulate signals to create new signals or to change the characteristics of the signals. In this set of notes, signals and functions are taken to mean the same thing. Typically when the function is a function of time, then we can call it a signal. For example, if you have a data logger that logs the voltage from a sensor, then you get a signal which changes with time. It is also a function which changes with time. Hence signals and functions are used interchangeably. The study of functions come from mathematics while the study of signals come from Engineering. Hope this helps you to have some clarity.

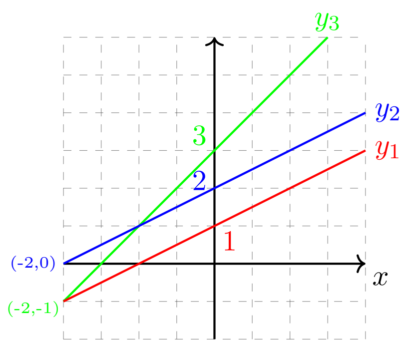
Addition, subtraction and multiplication of functions work the same way as you will add, subtract and multiply any two numbers arithmetically, except that instead of just dealing with numbers, you are dealing with functions. Mathematically, it is straightforward but the visualization of the resulting functions can be quite challenging.

Suppose you have 2 linear functions, $y_1 = x + 1$ and $y_2 = x + 2$. What will $y_3 = y_1 + y_2$ look like graphically? Mathematically, it is quite clear that

$$y_3 = y_1 + y_2 = (x + 1) + (x + 2) = 2x + 3$$

and hence $y_3 = 2x + 3$ which everyone knows how to sketch and what it looks like because it is a straight line with a slope or gradient of 2 and an intercept of 3 on the y-axis. However, it is also good to understand how the addition of the 2 functions can be visualized graphically. The example below illustrates this.

$$\begin{aligned} y_3 &= y_1 + y_2 \\ &= 2x + 3 \quad \Longrightarrow \end{aligned}$$

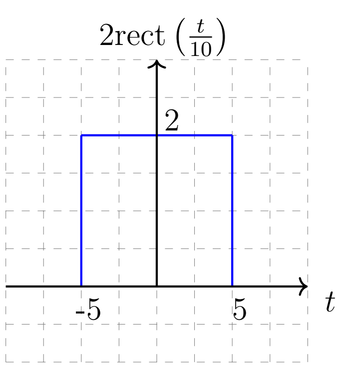


pay attention to how the blue and red lines add to form the green line

This addition of functions can be quite challenging when you are dealing with more complex functions or signals. For example, what does

$$y(t) = 2\text{rect}\left(\frac{t}{10}\right) + \text{tri}\left(\frac{t}{5}\right)$$

look like, where the rect(.) and tri(.) are the rectangle and triangle functions which you have encountered before?



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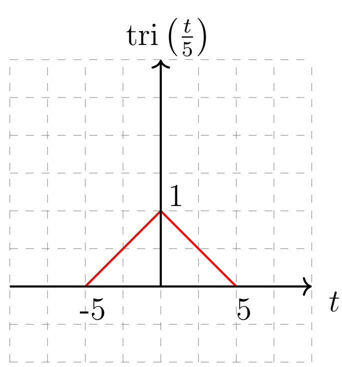

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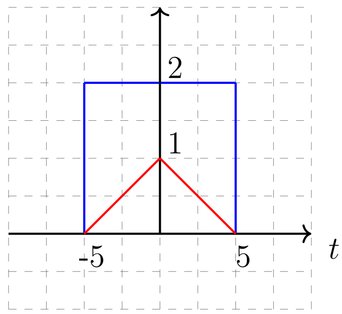
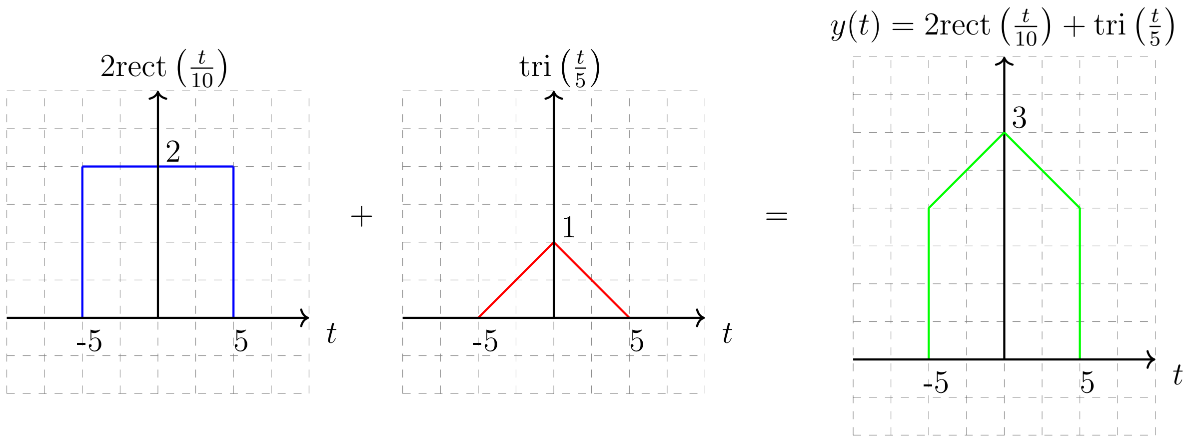
Plotting the two functions on the same graph, can you see how they add?

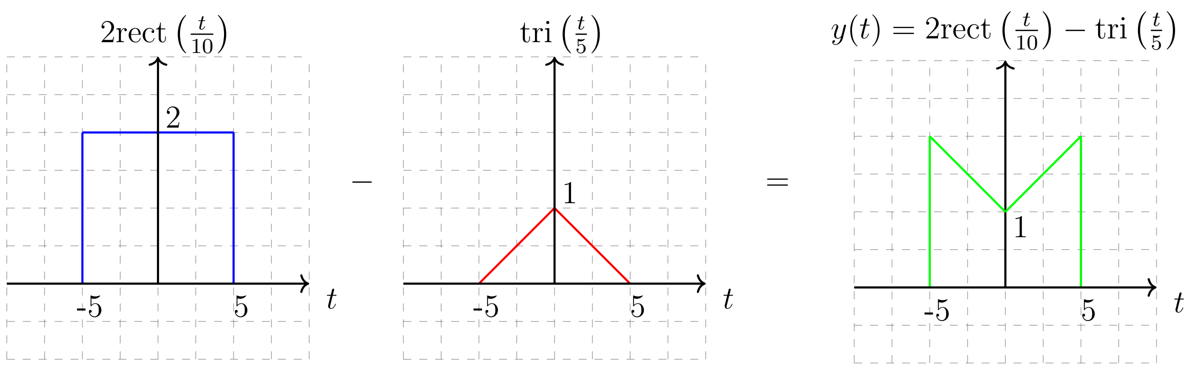




What happens if you subtract one function from the other? What does

$$y(t) = 2\text{rect}\left(\frac{t}{10}\right) - \text{tri}\left(\frac{t}{5}\right)$$

look like?



You can see how adding and subtracting can be done. In the same way, multiplications are dealt with the same way, though it is more challenging. Take the example of the linear functions again. Suppose we form a new function $y_4 = y_1 \times y_2$. What will y_4 look like? In this case, it is not neat to look at point for point multiplication because the resulting function will not be another straight line.

$$\begin{aligned} y_4 &= (x+1)\times(x+2) \\ &= x^2+3x+2 \end{aligned}$$

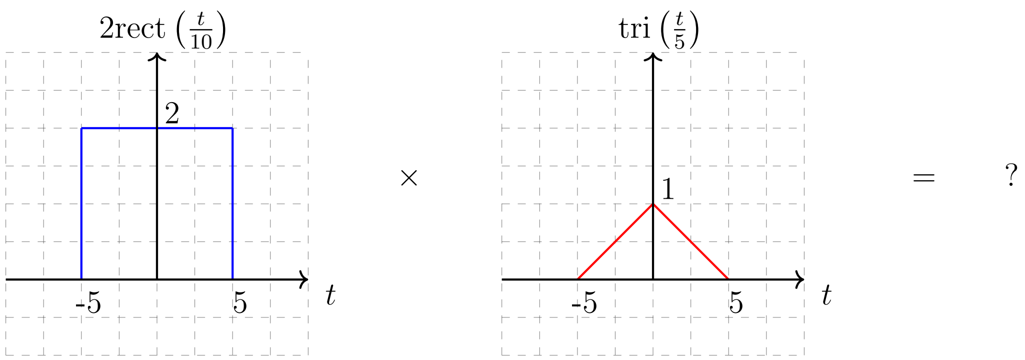
y_4 is now a 2nd order polynomial or a quadratic function and you should be able to visualize what the parabola looks like.

There are however situations when it is also easy to visualize multiplications, particularly when the functions or signals are bounded ie does not extend to infinity along the x-axis.

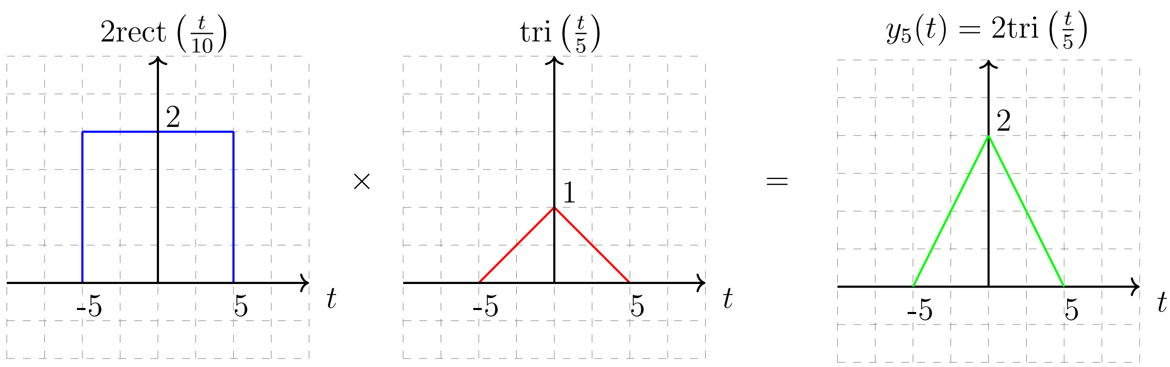
Take the example of

$$y_5(t) = 2\text{rect}\left(\frac{t}{10}\right)\times\text{tri}\left(\frac{t}{5}\right).$$

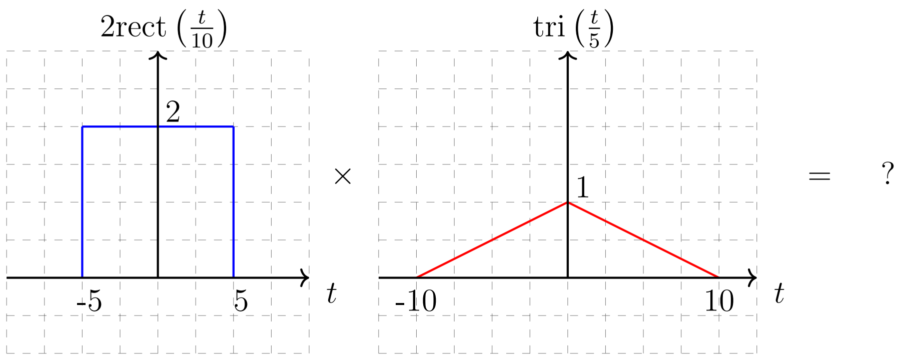
What does $y_5(t)$ look like?

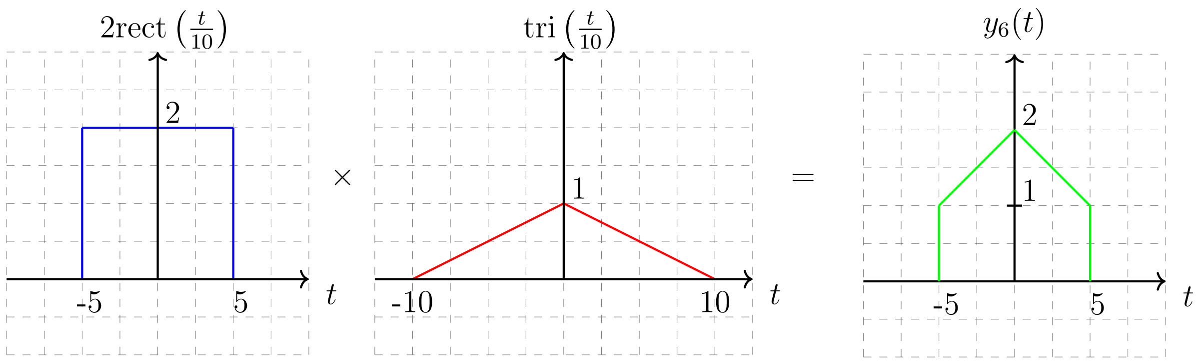


It should be easy to see that $y_5(t) = 2\text{tri}\left(\frac{t}{5}\right)$ because when $|t|>5$, the rectangle function is zero. When $|t|<5$, the rectangle function is 2. Hence,



What about $y_6(t) = 2\text{rect}\left(\frac{t}{10}\right)\times\text{tri}\left(\frac{t}{10}\right)$?





Notice that $y_6(t) = 0$ for $|t| > 5$. This is because the rectangle function is zero outside $|t| > 5$. Hence $y_6(t)$ for $5 < |t| < 10$ is reduced to zero. For $|t| < 5$, ie $-5 < t < 5$, the triangle function is multiplied by 2 which is the value of the rectangle function in this time interval.

Exercise 10. *Sketch the following functions.*

(a) $y(t) = 5\text{rect}\left(\frac{t}{12}\right) + 2\text{rect}\left(\frac{t}{10}\right)$

(b) $y(t) = 5\text{rect}\left(\frac{t}{12}\right) - 2\text{rect}\left(\frac{t}{10}\right)$

(c) $y(t) = 6\text{tri}\left(\frac{t}{12}\right) + 3\text{tri}\left(\frac{t}{6}\right)$

(d) $y(t) = 3\cos(2\pi t)\text{rect}(2t)$

(e) $y(t) = 3\cos(2\pi t)\text{rect}(2t) + \text{rect}\left(\frac{t}{4}\right)$

(f) $y(t) = 3\dfrac{\sin(2\pi t)}{(2\pi t)}$ *for* $-2.5 \leq t \leq 2.5$.