# CG2023 ASSIGNMENT 3 (Fourier Transform)

1. The Fourier transform of signal $x(t) = e^{-\alpha t}u(t)$ is $X(f) = \frac{1}{\alpha + j2\pi f}$.

(a) Find the Fourier transform $Y(f)$ of signal $y(t) = e^{\alpha t}u(-t)$ using the Fourier transform properties.

By time scaling property of FT with $a = -1$, we have $x(-t) \leftrightarrow X(-f)$. $y(t) = x(-t) \leftrightarrow$ $Y(f) = X(-f) = \frac{1}{\alpha - j2\pi f}$.

(b) Find the Fourier transform $Z(f)$ of the signal $z(t) = e^{-|t|}$.

$z(t) = x(t) + y(t)$ with $\alpha = 1$. By linearity property, $Z(f) = \frac{1}{1+j2\pi f} + \frac{1}{1-j2\pi f} = \frac{2}{1+4\pi^2 f^2}$.

2. The waveform of signal $w(t)$ is shown in Figure 3.1.

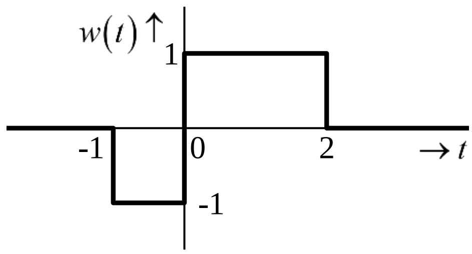


Figure 3.1

(a) Find the expression of $w(t)$ in terms of rectangular pulses.

$w(t) = \text{rect}\left(\frac{t-1}{2}\right) - \text{rect}(t + 0.5)$

(b) Find the Fourier transform $W(f)$ of $w(t)$.

Since $\text{rect}\left(\frac{t}{T}\right) \leftrightarrow T\text{sinc}(fT)$ and by time-shifting property of FT,
$W(f) = 2\text{sinc}(2f)e^{-j2\pi f} - \text{sinc}(f)e^{j\pi f}$.

(c) For the signal $y(t) = W(t)$, find $Y(f)$.

By the duality property, if $x(t) \leftrightarrow X(f)$, then $X(t) \leftrightarrow x(-f)$.
So $Y(f) = w(-f) = \text{rect}\left(\frac{f+1}{2}\right) - \text{rect}(f - 0.5)$.

3. The signal spectrum $X(f)$ is shown in Figure 3.2.

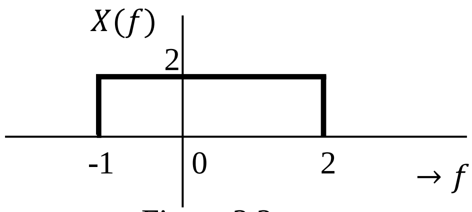


Figure 3.2

(a) Find the time domain waveform $x(t)$ of $X(f)$.

$X(f) = 2\text{rect}\left(\frac{f-0.5}{3}\right)$. By the FT pair $\text{rect}\left(\frac{t}{T}\right) \leftrightarrow T\text{sinc}(fT)$

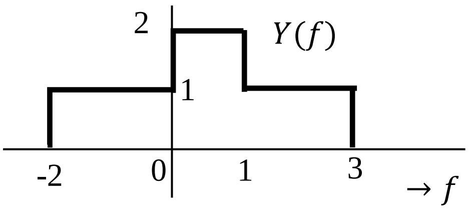
and the duality property of FT, $T\text{sinc}(tT) \leftrightarrow \text{rect}\left(\frac{-f}{T}\right) = \text{rect}\left(\frac{f}{T}\right)$. So $2\text{rect}\left(\frac{f}{3}\right) \leftrightarrow 6\text{sinc}(3t)$.

By frequency shifting property of FT, $x(t) = 6\text{sinc}(3t)e^{j\pi t}$.

(b) Sketch $Y(f)$ where $y(t) = x(t)\cos(2\pi t)$

$\cos(2\pi t) \leftrightarrow \frac{1}{2}[\delta(f-1) + \delta(f+1)]$ and

$Y(f) = X(f) * \frac{1}{2}[\delta(f-1) + \delta(f+1)] = \frac{1}{2}[X(f-1) + X(f+1)]$




(c) Find the Fourier transform $Z(f)$ of the new signal $z(t) = x(t)\cos(40\pi t) + jx(t)\cos(40\pi t)$.

$z(t) = (1+j)x(t)\cos(40\pi t) = \frac{(1+j)}{2}x(t)\left[e^{j40\pi t} + e^{-j40\pi t}\right]$.

By the frequency shifting property, $Z(f) = \frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}X(f-20) + \frac{\sqrt{2}}{2}e^{j\frac{\pi}{4}}X(f+20)$.

4. Given the Fourier transform pair $\delta(t) \leftrightarrow 1$. Find the Fourier transform of the following signals using the Fourier transform properties.

(a) The unit step function $u(t) = \int_{-\infty}^{t}\delta(\tau)d\tau$ .

Since $\delta(t) \leftrightarrow 1$ and $u(t) = \int_{-\infty}^{t}\delta(\tau)d\tau$, by the integration property of FT,

$U(f) = \frac{1}{j2\pi f} \times 1 + \frac{1}{2} \times 1 \times \delta(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$.

(b) The signum function $\mathrm{sgn}(t) = 2u(t) - 1$.

By duality property, the FT for the time domain constant 1 is $\delta(-f) = \delta(f)$.

By linearity property of FT, the FT of sgn($t$) is $2U(f) - \delta(f) = \frac{1}{j\pi f} + \delta(f) - \delta(f) = \frac{1}{j\pi f}$.

(c) The function $h(t) = \frac{1}{\pi t}$.

Based on the result in (b), we have $\mathrm{sgn}(t) \leftrightarrow \frac{1}{j\pi f}$. By duality property, we have the FT pair

$\frac{1}{j\pi t} \leftrightarrow sgn(-f)$. So $H(f) = -j\mathrm{sgn}(f)$.

5. The signal $x(t)$ is shown in Figure 3.5. It is made of two half-cycles of two sinusoids.

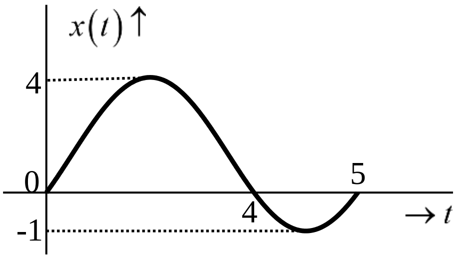



Figure 3.5

(a) Express $x(t)$ using sinusoids and rectangular functions.

The two half-cycle sinusoids' frequencies are $1/8$ Hz and $1/2$ Hz, respectively.

$$x(t) = 4\sin\left(\frac{\pi}{4}t\right) \times \mathrm{rect}\left(\frac{t-2}{4}\right) - \sin(\pi t) \times \mathrm{rect}(t-4.5).$$

(b) Find the Fourier transform $X(f)$ of $x(t)$.

Since $\sin(2\pi f_0 t) \leftrightarrow \frac{1}{2j}[\delta(f-f_0) - \delta(f+f_0)]$ and $\mathrm{rect}\left(\frac{t}{T}\right) \leftrightarrow T\mathrm{sinc}(fT)$, we have

$4\sin\left(\frac{\pi}{4}t\right) \leftrightarrow \frac{2}{j}[\delta(f-1/8) - \delta(f+1/8)]$, $\sin(\pi t) \leftrightarrow \frac{1}{2j}[\delta(f-1/2) - \delta(f+1/2)]$.

$\mathrm{rect}\left(\frac{t-2}{4}\right) \leftrightarrow 4\mathrm{sinc}(4f)e^{-j4\pi f}$ and $\mathrm{rect}(t-4.5) \leftrightarrow \mathrm{sinc}(f)e^{-j9\pi f}$ by time-shifting property of FT.

$$X(f) = \mathcal{F}\left[4\sin\left(\frac{\pi}{4}t\right)\right] * \mathcal{F}\left[\mathrm{rect}\left(\frac{t-2}{4}\right)\right] - \mathcal{F}[\sin(\pi t)] * \mathcal{F}[\mathrm{rect}(t-4.5)]$$

$$= \frac{2}{j}[\delta(f-1/8) - \delta(f+1/8)] * 4\mathrm{sinc}(4f)e^{-j4\pi f} - \frac{1}{2j}[\delta(f-1/2) - \delta(f+1/2)] * \mathrm{sinc}(f)e^{-j9\pi f}$$

$$= \frac{8}{j}\left[\mathrm{sinc}\left(4f-\frac{1}{2}\right)e^{-j4\pi\left(f-\frac{1}{8}\right)} - \mathrm{sinc}\left(4f+\frac{1}{2}\right)e^{-j4\pi\left(f+\frac{1}{8}\right)}\right]$$

$$-\frac{1}{2j}\left[\mathrm{sinc}\left(f-\frac{1}{2}\right)e^{-j9\pi\left(f-\frac{1}{2}\right)} - \mathrm{sinc}\left(f+\frac{1}{2}\right)e^{-j9\pi\left(f+\frac{1}{2}\right)}\right]$$

$$= 8\left[\mathrm{sinc}\left(4f-\frac{1}{2}\right) + \mathrm{sinc}\left(4f+\frac{1}{2}\right)\right]e^{-j4\pi f} - \frac{1}{2}\left[\mathrm{sinc}\left(f-\frac{1}{2}\right) + \mathrm{sinc}\left(f+\frac{1}{2}\right)\right]e^{-j9\pi f}$$

(c) Let $z(t) = \sum_{n=-\infty}^{\infty} x(t-10n)$. Find the Fourier series of $z(t)$.

$z(t) = x(t) * \sum_{n=-\infty}^{\infty}\delta(t-10n)$. $\sum_{n=-\infty}^{\infty}\delta(t-10n) \leftrightarrow \frac{1}{10}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{10}\right)$.

$$Z(f) = X(f) \times \frac{1}{10}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{10}\right) = \frac{1}{10}\sum_{k=-\infty}^{\infty}X(f)\times\delta\left(f-\frac{k}{10}\right)$$

$$= \sum_{k=-\infty}^{\infty}\frac{1}{10}X\left(\frac{k}{10}\right)\times\delta\left(f-\frac{k}{10}\right).$$

For $k \in Z$, the Fourier series of $z(t)$ are

$$c_k = \frac{1}{10}X\left(\frac{k}{10}\right)$$

$= \frac{4}{5}\left[\mathrm{sinc}\left(\frac{2}{5}k - \frac{1}{2}\right) + \mathrm{sinc}\left(\frac{2}{5}k + \frac{1}{2}\right)\right] e^{-j2\pi k/5} - \frac{1}{20}\left[\mathrm{sinc}\left(\frac{k}{10} - \frac{1}{2}\right) + \mathrm{sinc}\left(\frac{k}{10} + \frac{1}{2}\right)\right] e^{-j9\pi k/10}$ .

6. Let $x(t) = \mathrm{sinc}(t)$ and $y(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - 2n)$. Find the Fourier transform $Y(f)$ of $y(t)$.

By the duality property of FT, $\mathcal{F}[x(t)] = \mathrm{rect}(f)$. $\mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t - 2n)] \leftrightarrow \frac{1}{2} \times \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{2})$.

$Y(f) = \mathcal{F}[x(t)] * \mathcal{F}[\sum_{n=-\infty}^{\infty} \delta(t - 2n)]$

$= \mathrm{rect}(f) * \frac{1}{2} \times \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{2}) \; = \frac{1}{2} \times \sum_{k=-\infty}^{\infty} \mathrm{rect}(f - \frac{k}{2})$.