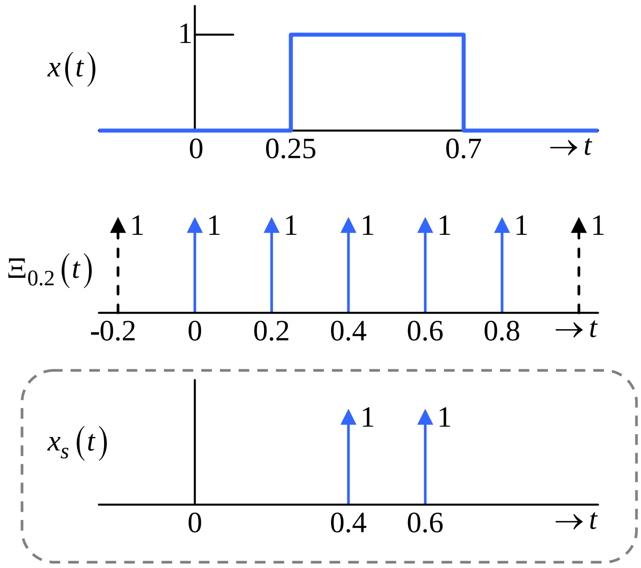
CG2023 TUTORIAL 5 (SOLUTIONS)

Solution to Q.1

$$x_s(t) = \underbrace{\mathrm{rect}\left(\frac{t-0.475}{0.45}\right)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-0.2n)}_{\Xi_{0.2}(t)}$$

$$= \delta(t-0.4) + \delta(t-0.6)$$



$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f \varsigma)$$

$$X_s(f) = \exp(-j2\pi f 0.4) + \exp(-j2\pi f 0.6)$$

$$= \begin{cases} \cos(0.8\pi f) - j\sin(0.8\pi f) + \\ \quad \cos(1.2\pi f) - j\sin(1.2\pi f) \end{cases}$$

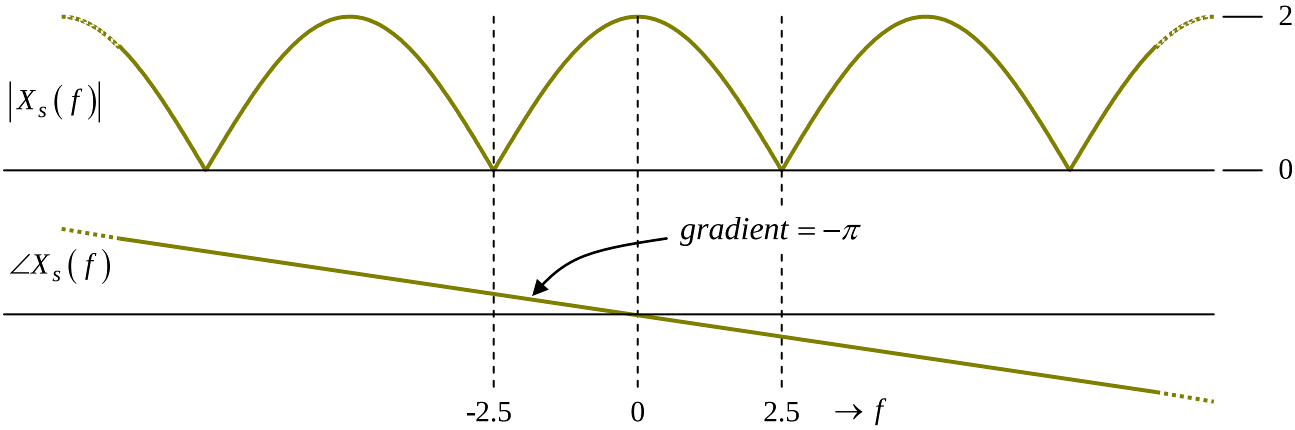
$$= \begin{cases} \left[\cos(0.8\pi f) + \cos(1.2\pi f)\right] - \\ \quad j\left[\sin(0.8\pi f) + \sin(1.2\pi f)\right] \end{cases}$$

$$= 2\cos(\pi f)\cos(0.2\pi f) - j2\sin(\pi f)\cos(0.2\pi f)$$

$$|X_s(f)|^2 = 4\cos^2(\pi f)\cos^2(0.2\pi f) + 4\sin^2(\pi f)\cos^2(0.2\pi f)$$

$$= 4\cos^2(0.2\pi f)$$

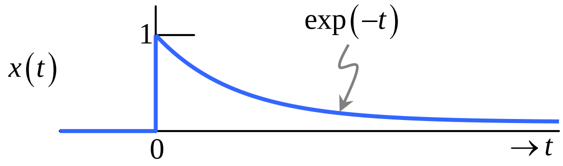
- $|X_s(f)| = 2|\cos(0.2\pi f)|$
- $\angle X_s(f) = -\tan^{-1}\left(\frac{2\sin(\pi f)\cos(0.2\pi f)}{2\cos(\pi f)\cos(0.2\pi f)}\right) = -\tan^{-1}(\tan(\pi f)) = -\pi f$



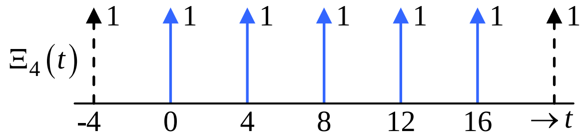
Solution to Q.2

$$x_s(t) = \underbrace{\exp(-t)u(t)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-4n)}_{\Xi_4(t)}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\delta(t-4n)$$



$$\Im\{\delta(t-\varsigma)\} = \exp(-j2\pi f \varsigma)$$

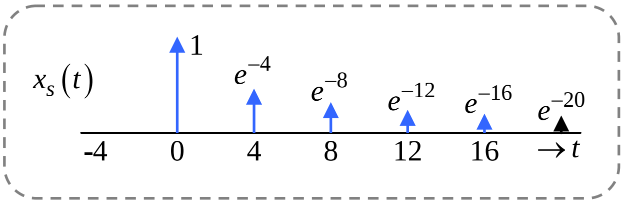


$$X_s(f) = \sum_{n=0}^{\infty} \exp(-4n)\Im\{\delta(t-4n)\}$$

$$= \sum_{n=0}^{\infty} \exp(-4n)\exp(-j2\pi f 4n)$$

$$= \underbrace{\sum_{n=0}^{\infty} \exp(-4n(j2\pi f + 1))}_{\text{Geometric Series}}$$

$$= \frac{1}{1-\exp(-4(j2\pi f + 1))} = \frac{1}{1-\exp(-4)\exp(-j8\pi f)}$$



To show that $X_s(f)$ has a period of 0.25, all we need to show is $X_s(f+0.25) = X_s(f)$ as follows:

$$X_s(f+0.25) = \frac{1}{1-\exp(-4)\exp(-j8\pi(f+0.25))}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)\underbrace{\exp(-j2\pi)}_{1}}$$

$$= \frac{1}{1-\exp(-4)\exp(-j8\pi f)} = X_s(f)$$

Note: When a signal is sampled in the time domain, the spectrum of the sampled signal is periodic with a period equal to the sampling frequency, which in this case is 0.25 Hz.

Solution to Q.3

(a) Rewrite $x(t)=\sum_{n=-\infty}^{\infty} g(t-nT)$ in convolution form:

$$x(t)=g(t)\ast\sum_{n=-\infty}^{\infty}\delta(t-nT) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \ (\ast)$$

Applying the 'Convolution' property of the Fourier ransform to $(\ast)$:

$$X(f)=G(f)\cdot\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right) \quad \cdots\cdots\cdots\cdots \ (\ast\ast)$$

Conclusion: $X(f)$ may be obtained by sampling $G(f)/T$ in the frequency-domain at regular spacings of $1/T$ Hz.

(b) Relationship between c_k and $G(f)$

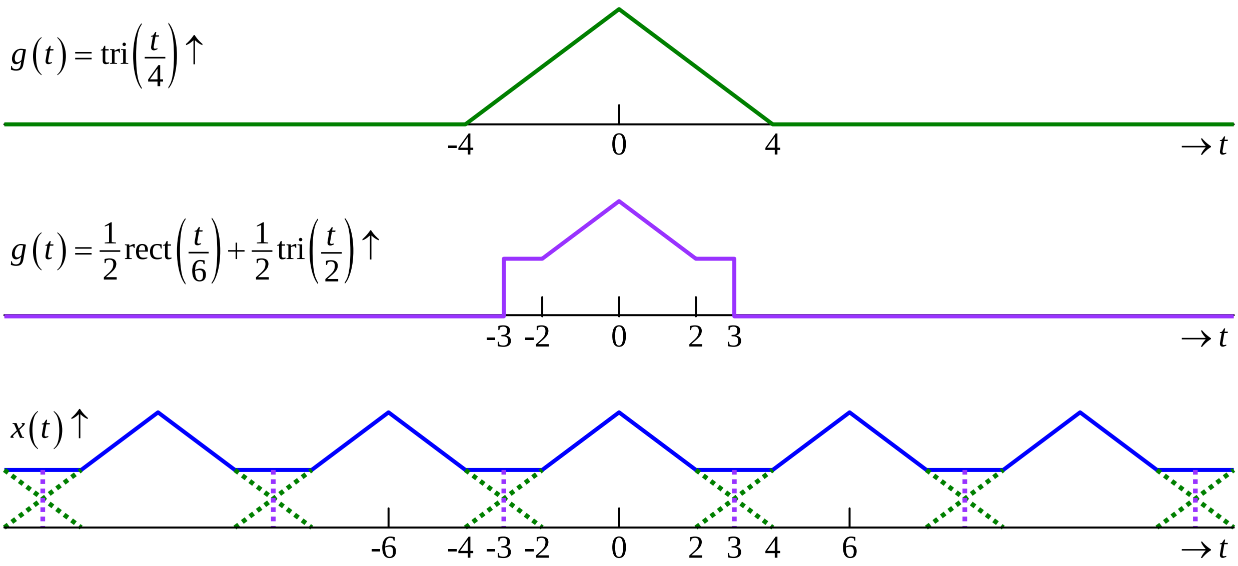
Rewrite $(\ast\ast)$ as: $X(f)=\sum_{k=-\infty}^{\infty}\frac{1}{T}G\left(\frac{k}{T}\right)\delta\left(f-\frac{k}{T}\right)$

In terms of c_k: $X(f)=\sum_{k=-\infty}^{\infty}c_k\delta\left(f-\frac{k}{T}\right)$

Clearly, $c_k=\frac{1}{T}G(f)\Big|_{f=k/T}=\frac{1}{T}G\left(\frac{k}{T}\right)$

(c) Uniqueness of a generating function

The generating function of a periodic signal is NOT unique. For example, both $g(t)=\mathrm{tri}\left(\frac{t}{4}\right)$ and $g(t)=\frac{1}{2}\mathrm{rect}\left(\frac{t}{6}\right)+\frac{1}{2}\mathrm{tri}\left(\frac{t}{2}\right)$ will generate the same periodic signal $x(t)=\sum_{n=-\infty}^{\infty}\mathrm{g}(t-6n)$.



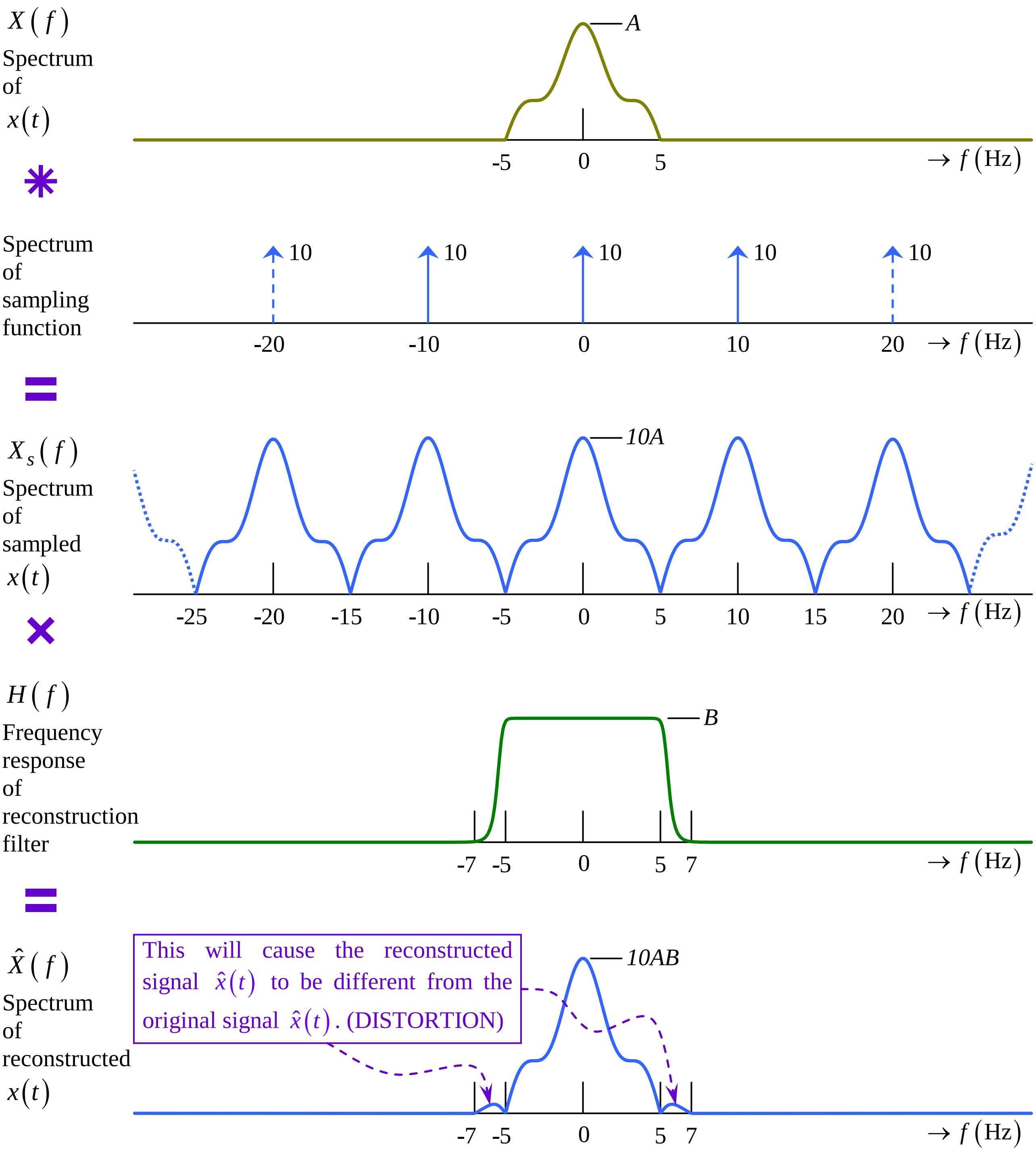
Solution to Q.4

- Nyquist sampling frequency: $2 \times 5 = 10$ Hz

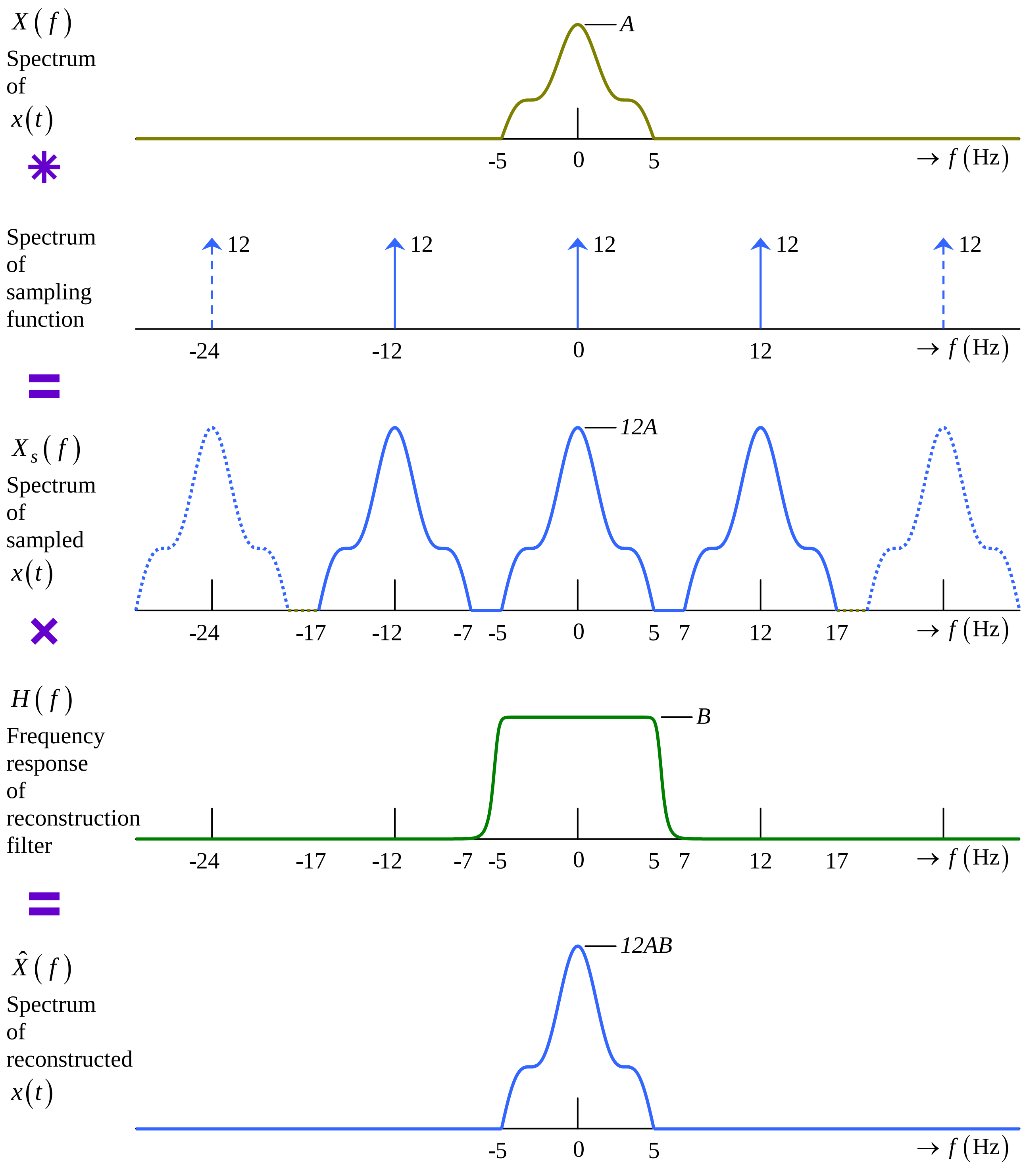
- Recommended sampling frequency: 12 Hz

  The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.

Sampling frequency: **10 Hz (Nyquist Rate)**



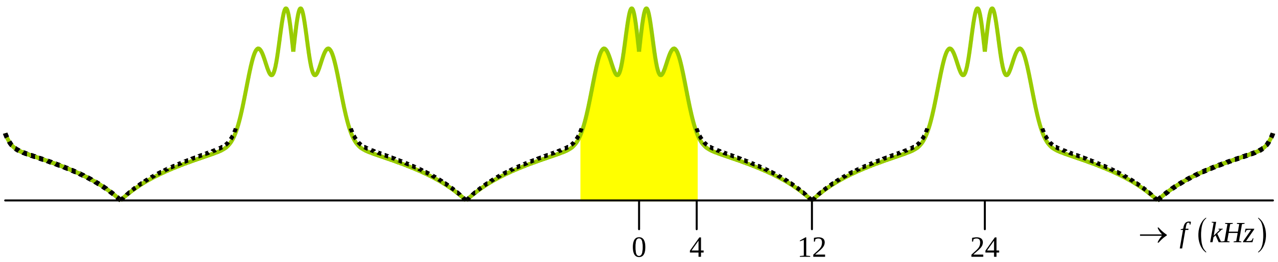
***Sampling frequency:* 12 Hz**



Remarks: Sampling frequencies higher that 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.

Solution to Q.5

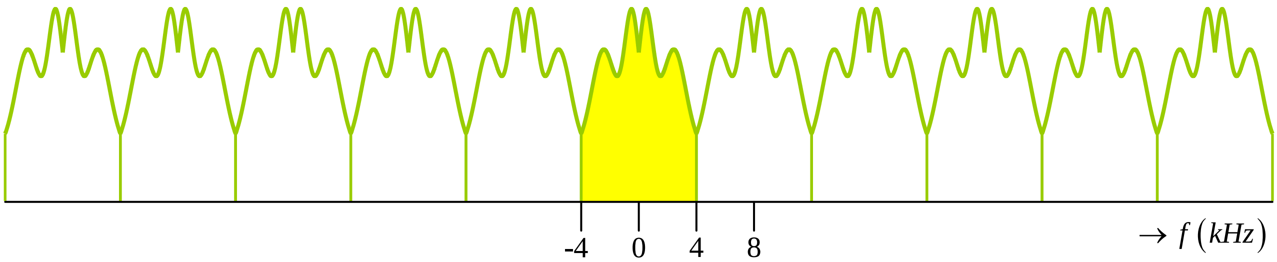
Situation A: $\left(\begin{array}{c}\textbf{\textit{No anti-aliasing lowpass filter}} \\ \textbf{\textit{and no frequency aliasing}}\end{array}\right)$: $\left\{\textbf{Sampling frequency} = 2 \times 12 = 24\ \textbf{\textit{kHz}}\right.$



Advantage: No anti-aliasing LPF needed.

Disadvantage: Sampling frequency is higher than necessary to preserve the *TQ-Band*.

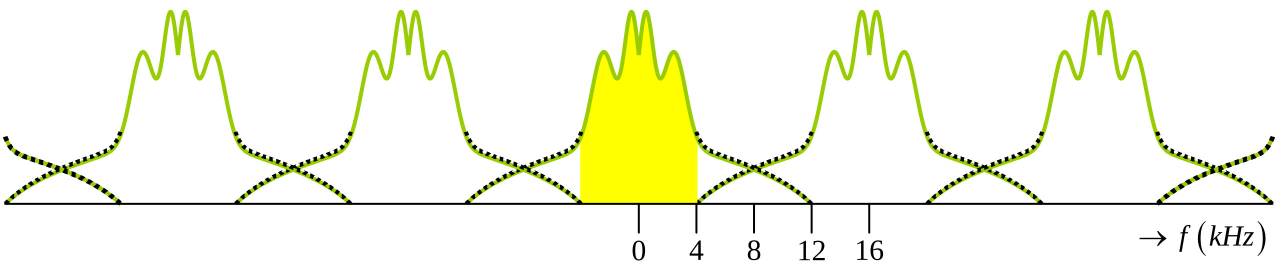
Situation B: ***Lowest sampling frequency***: $\left\{\begin{array}{l}\textbf{Ideal anti-aliasing filter of bandwidth 4 } \textbf{\textit{kHz}} \\ \textbf{Sampling frequency} = 2 \times 4 = 8\ \textbf{\textit{kHz}}\end{array}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band.*

Disadvantage: Require anti-aliasing LPF with sharp cutoff.

Situation C: ***No anti-aliasing lowpass filter***: $\left\{\textbf{Sampling frequency} = 2 \times \left(\frac{12+4}{2}\right) = 16\ \textbf{\textit{kHz}}\right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

Disadvantage: Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.

Solution to S.1

$f_S = 2 \times 20 = 40\ kHz$

$$\left[x_S(t) = x(t) \cdot \sum_n \delta\left(t - \frac{n}{40000}\right) \right] \rightleftarrows \left[\begin{aligned} X_S(f) &= X(f) * 40000 \sum_k \delta(f - 40000k) \\ &= 40000 \sum_k X(f - 40000k) \end{aligned} \right]$$

Specifying $H(f) = \frac{1}{40000} \cdot \mathrm{rect}\left(\frac{f}{40000}\right)$ will lead to $X_s(f)H(f) = X(f)$.

Solution to S.2

(a)
$$\begin{cases} x_S(t) = \sum_{n=-5}^{5} x(5n)\delta(t - 5n) \\ \qquad\qquad\qquad\qquad \uparrow \\ \qquad\qquad\qquad \text{sampling period} \\ \therefore \text{ Sampling frequency: } f_s = \frac{1}{5} = 0.2\ Hz \end{cases}$$

(b) Perfect reconstruction is of $x(t)$ from $x_s(t)$ only possible if $x(t)$ is bandlimited to $f_s/2 = 0.1$, i.e. $X(f) = 0;\ f > 0.1$.

Since $X(f) = \Im^{-1}\{\mathrm{tri}(t)\} = \mathrm{sinc}^2(f)$ has infinite frequency extent, perfect reconstruction is not possible due to frequency aliasing.

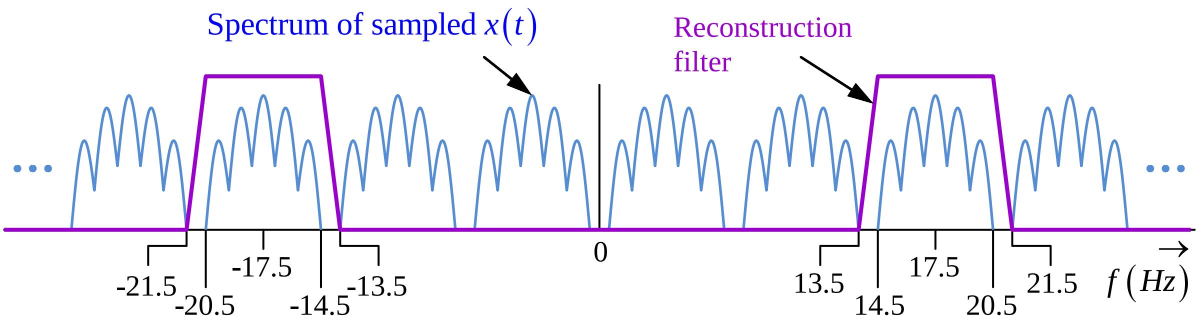
Solution to S.3

(a) Nyquist Sampling Frequency: $f_s = 2 \times 20.5 = 41\ Hz$

(b) Center frequency: $f_c = 17.5\ Hz$ Bandwidth: $B = 6\ Hz$

Possible sampling frequencies : $f_s = 2f_c/k;\quad k = 1,\ 2,\ \cdots,\ \lfloor 2f_c/B \rfloor$

Lowest sampling frequency : $f_s = \frac{2f_c}{\lfloor 2f_c/B \rfloor} = \frac{35}{5} = 7\ Hz$



Reconstruction Filter Specs : $\begin{cases} \text{Ideal passband: } 14.5 < |f| < 20.5 \\ \text{Ideal stopband: } |f| < 13.5 \text{ or } |f| > 21.5 \end{cases}$