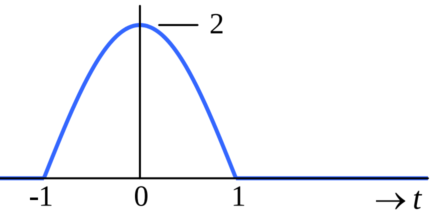
EE2023 TUTORIAL 3 (SOLUTIONS)

Solution to Q.1

$$x(t)=\begin{cases}2\cos(0.5\pi t); & |t|\le 1\\ 0; & |t|>1\end{cases}$$



(a)

Method 1: By applying direct Fourier transform:

$$X(f)=\int_{-\infty}^{\infty}x(t)\exp(-j2\pi ft)\,dt=\int_{-1}^{1}2\cos(0.5\pi t)\exp(-j2\pi ft)\,dt$$

$$=2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\cos(2\pi ft)}_{even\ \ function\ of\ \ t}dt-\cancel{j2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\sin(2\pi ft)}_{odd\ \ function\ of\ \ t}dt}$$

$$=4\int_{0}^{1}\cos(0.5\pi t)\cos(2\pi ft)\,dt$$

$$=2\int_{0}^{1}\cos\big((2\pi f-0.5\pi)t\big)+\cos\big((2\pi f+0.5\pi)t\big)\,dt$$

$$=2\left[\frac{\sin\big((2\pi f-0.5\pi)t\big)}{2\pi f-0.5\pi}+\frac{\sin\big((2\pi f+0.5\pi)t\big)}{2\pi f+0.5\pi}\right]_{0}^{1}$$

$$=2\left(\frac{\sin(2\pi f-0.5\pi)}{2\pi f-0.5\pi}+\frac{\sin(2\pi f+0.5\pi)}{2\pi f+0.5\pi}\right)$$

$$=\frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5}+\frac{\cos(2\pi f)}{2f+0.5}\right)=\frac{2\cos(2\pi f)}{\pi\left(0.25-4f^{2}\right)}$$

Method 2: By applying Fourier transform properties:

$$x(t)=2\cos(0.5\pi t)\cdot\mathrm{rect}(0.5t)$$
$$\Im\{2\cos(0.5\pi t)\}=\delta(f-0.25)+\delta(f+0.25)$$
$$\Im\{\mathrm{rect}(0.5t)\}=2\mathrm{sinc}(2f)$$

Applying the 'Multiplication in time-domain' property of the Fourier transform

$$\left[x(t)=\underbrace{2\cos(0.5\pi t)\cdot\mathrm{rect}(0.5t)}_{\substack{\text{Multiplication in}\\ \text{time-domain}}}\right]\rightleftarrows\left[X(f)=\underbrace{\Im\{2\cos(0.5\pi t)\}\ast\Im\{\mathrm{rect}(0.5t)\}}_{\substack{\text{Convolution in}\\ \text{frequency-domain}}}\right]$$

we get

$$X(f)=\left[\delta(f-0.25)+\delta(f+0.25)\right]\ast 2\mathrm{sinc}(2f)$$

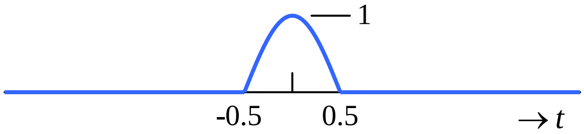
$$=2\mathrm{sinc}(2f-0.5)+2\mathrm{sinc}(2f+0.5)$$

$$=2\left(\frac{\sin(2\pi f-0.5\pi)}{\pi(2f-0.5)}+\frac{\sin(2\pi f+0.5\pi)}{\pi(2f+0.5)}\right)\cdots\cdots\text{ Same result obtained by }\textbf{Method 1}$$

(b)

From Part (a): $X(f) = \dfrac{2\cos(2\pi f)}{\pi\left(0.25 - 4f^2\right)}$

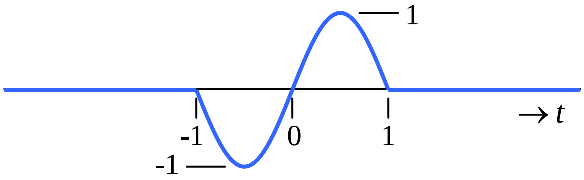
$$\tilde{y}(t) = 0.5x(2t)$$



Applying the ***scaling property***:

$$\tilde{Y}(f) = 0.5\left[\frac{1}{2}X\left(\frac{f}{2}\right)\right]$$
$$= \frac{1}{4}X\left(\frac{f}{2}\right) \qquad \cdots\cdots\cdots\cdots (*)$$

$$y(t) = \tilde{y}(t-0.5) - \tilde{y}(t+0.5)$$



Applying the ***time-shifting property***:

$$Y(f) = \tilde{Y}(f)\exp\left(-j2\pi f\left(\frac{1}{2}\right)\right) - \tilde{Y}(f)\exp\left(j2\pi f\left(\frac{1}{2}\right)\right) \qquad \cdots\cdots\cdots (**)$$

Substituting $(*)$ into $(**)$:

$$\begin{cases} Y(f) = \dfrac{1}{4}X\left(\dfrac{f}{2}\right)\exp(-j\pi f) - \dfrac{1}{4}X\left(\dfrac{f}{2}\right)\exp(j\pi f) \\ \quad = -j\dfrac{1}{2}X\left(\dfrac{f}{2}\right)\sin(\pi f) \\ \quad = \dfrac{1}{j2}\left[\dfrac{2\cos(\pi f)}{\pi\left(0.25 - f^2\right)}\right]\sin(\pi f) \\ \quad = \dfrac{1}{j2}\left[\dfrac{\sin(2\pi f)}{\pi\left(0.25 - f^2\right)}\right] \end{cases}$$

Solution to Q.2

(a)

Fig.Q.2(a)(I) is a plot of $u(t-\gamma)$ against t:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \rightarrow \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & t\ge \gamma\\ 0; & t<\gamma\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } t \text{ while}\\ \text{treating } \gamma \text{ as a parameter}}}$$

Fig.Q.2(a)(II) is a plot of $u(t-\gamma)$ against γ:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right] \rightarrow \left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right] \rightarrow \underbrace{\left[u(t-\gamma)=\begin{cases}1; & \gamma\le t\\ 0; & \gamma>t\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } \gamma \text{ while}\\ \text{treating } t \text{ as a parameter}}}$$

On the γ-axis , since $x(\gamma)=x(\gamma)u(t-\gamma)$ in the integration interval $(-\infty,t]$, we have

$$\int_{-\infty}^{t} x(\gamma)\,d\gamma = \underbrace{\int_{-\infty}^{t} x(\gamma)u(t-\gamma)\,d\gamma = \int_{-\infty}^{\infty} x(\gamma)u(t-\gamma)\,d\gamma}_{\because\ u(t-\gamma)=0 \text{ when } \gamma>t} = x(t)*u(t)$$

(b)

$$\cos(t)u(t)*u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)\,d\gamma=\begin{cases}\int_0^t \cos(\gamma)\,d\gamma; & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\begin{cases}\sin(t); & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\sin(t)u(t)$$

(c)

Using the forward Fourier transform equation, it is straightforward to derive the Fourier transform pair:

$$\operatorname{rect}\left(\frac{t}{\alpha}\right) \rightleftarrows \alpha\cdot\operatorname{sinc}(\alpha f) \quad \cdots\cdots\cdots \quad (*)$$

Applying the 'Duality' property of the Fourier transform to $(*)$:

$$\alpha\cdot\operatorname{sinc}(\alpha t) \rightleftarrows \operatorname{rect}\left(\frac{f}{\alpha}\right) \quad \cdots\cdots \quad (**)$$

Taking the limit $\alpha\rightarrow\infty$ on both sides of $(**)$:

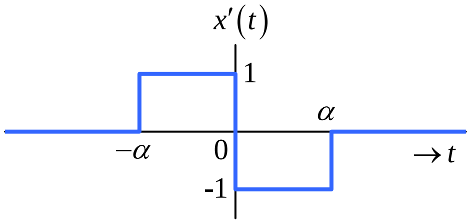
$$\lim_{\alpha\to\infty}\alpha\cdot\operatorname{sinc}(\alpha t) \rightleftarrows \lim_{\alpha\to\infty}\operatorname{rect}\left(\frac{f}{\alpha}\right)=1$$

Hence, $\boldsymbol{\lim_{\alpha\to\infty}\alpha\cdot\operatorname{sinc}(\alpha t)=\Im^{-1}\{1\}=\delta(t)}$

Solution to Q.3

Spectrum of $x'(t)=\dfrac{dx(t)}{dt}$:

$$x'(t)=\frac{dx(t)}{dt}=\mathrm{rect}\left(\frac{t+0.5\alpha}{\alpha}\right)-\mathrm{rect}\left(\frac{t-0.5\alpha}{\alpha}\right)$$



Applying the 'Linearity' property of the Fourier transform:

$$\Im\{x'(t)\}=\Im\left\{\mathrm{rect}\left(\frac{t+0.5\alpha}{\alpha}\right)\right\}-\Im\left\{\mathrm{rect}\left(\frac{t-0.5\alpha}{\alpha}\right)\right\}$$

$$\Im\left\{\mathrm{rect}\left(\frac{t}{\alpha}\right)\right\}=\alpha\cdot\mathrm{sinc}(\alpha f)$$

Applying the 'Time-shifting' property of the Fourier transform:

$$\begin{aligned}\Im\{x'(t)\}&=\alpha\cdot\mathrm{sinc}(\alpha f)\left[\exp(j\pi\alpha f)-\exp(-j\pi\alpha f)\right]\\&=\alpha\cdot\mathrm{sinc}(\alpha f)\left(j2\sin(\pi\alpha f)\right)\\&=j2\pi f\alpha^2\cdot\frac{\sin(\pi\alpha f)}{\pi\alpha f}\cdot\frac{\sin(\pi\alpha f)}{\pi\alpha f}\\&=j2\pi f\alpha^2\mathrm{sinc}^2(\alpha f)\end{aligned}$$

Spectrum of $x(t)$:

$$\Im\{x(t)\}=\Im\left\{\int_{-\infty}^{t}x'(\tau)d\tau\right\}\quad\cdots\quad\text{Noting: }\int_{-\infty}^{\infty}x'dt=0$$

Applying the 'Integration' property of the Fourier transform:

$$\begin{aligned}\Im\{x(t)\}&=\frac{1}{j2\pi f}\Im\{x'(t)\}\\&=\frac{1}{j2\pi f}\cdot j2\pi f\alpha^2\mathrm{sinc}^2(\alpha f)\\&=\alpha^2\mathrm{sinc}^2(\alpha f)\end{aligned}$$

Expressing $x(t)$ as a function of $\mathrm{rect}(\cdot)$:

$$\begin{aligned}\Im\{x(t)\}&=\alpha^2\mathrm{sinc}^2(\alpha f)=\alpha\mathrm{sinc}(\alpha f)\cdot\alpha\mathrm{sinc}(\alpha f)\\&=\Im\left\{\mathrm{rect}\left(\frac{t}{\alpha}\right)\right\}\cdot\Im\left\{\mathrm{rect}\left(\frac{t}{\alpha}\right)\right\}\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ (*)\end{aligned}$$

Applying the 'Convolution' property of the Fourier transform:

$$\Im\left\{\mathrm{rect}\left(\frac{t}{\alpha}\right)\ast\mathrm{rect}\left(\frac{t}{\alpha}\right)\right\}=\Im\left\{\mathrm{rect}\left(\frac{t}{\alpha}\right)\right\}\cdot\Im\left\{\mathrm{rect}\left(\frac{t}{\alpha}\right)\right\}\qquad\cdots\cdots\ (**)$$

Comparing $(*)$ and $(**)$, we have $x(t)=\mathrm{rect}\left(\dfrac{t}{\alpha}\right)\ast\mathrm{rect}\left(\dfrac{t}{\alpha}\right)$

Solution to Q.4

Given: $X(f)=\exp(-\alpha|f|);\quad \alpha>0$

(a) ***Energy Spectral Density of*** $\boldsymbol{x(t)}$***:***

$$E_x(f)=|X(f)|^2=\exp(-2\alpha|f|)$$

Energy of $\boldsymbol{x(t)}$ ***contained within a bandwidth of*** B***:***

$$E_B=\int_{-B}^{B}E_x(f)\,df=2\int_0^B\exp(-2\alpha f)\,df=2\left[\frac{\exp(-2\alpha f)}{-2\alpha}\right]_0^B=\frac{1}{\alpha}\left[1-\exp(-2\alpha B)\right]$$

Total energy of $\boldsymbol{x(t)}$***:***

$$E=\underbrace{\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_{-\infty}^{\infty}|X(f)|^2\,df}_{\text{Rayleigh Energy Theorem}}=\int_{-\infty}^{\infty}E_x(f)\,df=E_B\big|_{B=\infty}=\frac{1}{\alpha}$$

99% energy containment bandwidth, $\boldsymbol{B_{99}}$***, of*** $\boldsymbol{x(t)}$***:***

$$\left[\underbrace{\frac{1}{\alpha}\left[1-\exp(-2\alpha B_{99})\right]}_{E_{B_{99}}}=0.99E=\frac{0.99}{\alpha}\right]\rightarrow\ \exp(2\alpha B_{99})=100$$

$$\rightarrow\ B_{99}=\frac{1}{\alpha}\ln(10)\ \text{Hz}$$

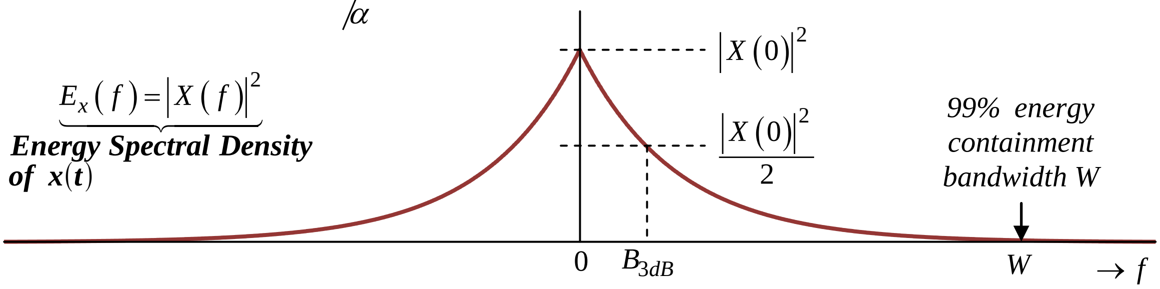
(b) ***3dB bandwidth,*** $\boldsymbol{B_{3dB}}$***, of*** $\boldsymbol{x(t)}$***:***

By definition, $|X(B_{3dB})|=\dfrac{|X(0)|}{\sqrt{2}}$.

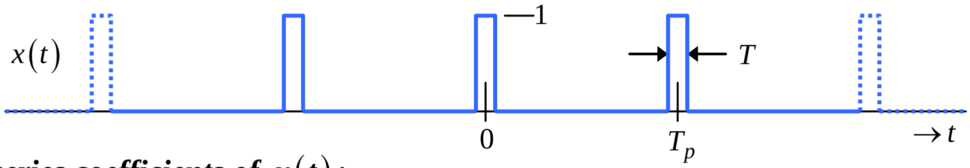
$$\text{Solving: }\left\{\begin{array}{l}|X(f)|=\exp(-\alpha|f|)\\ \left.\begin{array}{l}|X(B_{3dB})|=\exp(-\alpha B_{3dB})\\ |X(0)|=1\end{array}\right\}\ \rightarrow\ \exp(-\alpha B_{3dB})=\dfrac{1}{\sqrt{2}}\\ \qquad\qquad\qquad\qquad\quad\ \rightarrow\ B_{3dB}=\dfrac{1}{2\alpha}\ln(2)\ \text{Hz}\end{array}\right.$$

Percent energy contained within the 3dB bandwidth:

$$\frac{E_{B_{3dB}}}{E}\times100=\frac{\cancel{1/\alpha}\left[1-\exp\left(-\cancel{2\alpha}\dfrac{\ln(2)}{\cancel{2\alpha}}\right)\right]}{\cancel{1/\alpha}}\times100=50\%.$$



Solution to Q.5



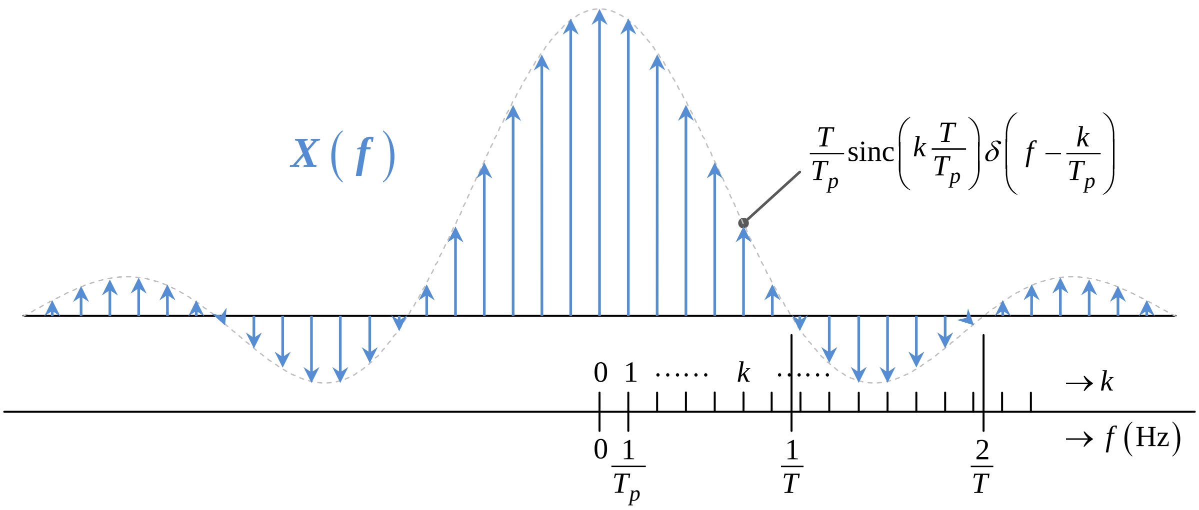
(a) ***Fourier series coefficients of $x(t)$:***

$$X_k = \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T} x(t)\exp\left(-j2\pi kt/T_p\right)dt = \frac{1}{T_p}\int_{-0.5T}^{0.5T} \exp\left(-j2\pi kt/T_p\right)dt$$

$$= \frac{1}{T_p}\left[\frac{\exp\left(-j2\pi kt/T_p\right)}{-j2\pi k/T_p}\right]_{-0.5T}^{0.5T} = \frac{T}{T_p}\left[\frac{\sin\left(\pi kT/T_p\right)}{\pi kT/T_p}\right] = \frac{T}{T_p}\mathrm{sinc}\left(k\frac{T}{T_p}\right)$$

Continuous-frequency spectrum (or Fourier transform) of $x(t)$:

$$X(f) = \sum_{k=-\infty}^{\infty} X_k \delta\left(f-\frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T}{T_p}\mathrm{sinc}\left(k\frac{T}{T_p}\right)\delta\left(f-\frac{k}{T_p}\right)$$



(b) ***Power Spectral Density of $x(t)$:***

$$P_x(f) = \sum_{k=-\infty}^{\infty} \left|X_k\right|^2 \delta\left(f-\frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T^2}{T_p^2}\mathrm{sinc}^2\left(k\frac{T}{T_p}\right)\delta\left(f-\frac{k}{T_p}\right)$$

Average power of $x(t)$:

$$P = \underbrace{\int_{-\infty}^{\infty} P_x(f)\,df = \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T}\left|x(t)\right|^2 dt}_{\text{Parseval Power Theorem}} = \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = \frac{T}{T_p}$$

99% power containment bandwidth, B_{99}, of $x(t)$:

$$B_{99} = \frac{K}{T_p}(\mathrm{Hz}) \quad \cdots \quad \left(\begin{array}{l} \text{where } K \text{ satisfies } \displaystyle\sum_{k=-K}^{K}\left|X_k\right|^2 \ge 0.99P > \sum_{k=-(K-1)}^{(K-1)}\left|X_k\right|^2 \\ \text{in which } \left|X_k\right|^2 = \dfrac{T^2}{T_p^2}\mathrm{sinc}^2\left(k\dfrac{T}{T_p}\right) \text{ and } P = \dfrac{T}{T_p}. \end{array}\right).$$

(c) ***Average power of*** $\boldsymbol{y(t)}$**:**

$$
\begin{aligned}
P &= \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T} |y(t)|^2\, dt \\
&= \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T} |x(t)|^2\, \mu^2 \cos^2(2\pi f_c t)\, dt \\
&= 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} \left[1+\cos(4\pi f_c t)\right] dt \\
&= 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt + 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} \cos(4\pi f_c t)\, dt \\
&\cdots\cdots \text{ since } T >> \frac{1}{f_c} \\
&\simeq 0.5\mu^2 \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = 0.5\mu^2 \frac{T}{T_p}
\end{aligned}
$$

$\left(\begin{array}{l} \textit{Assuming that } \mu \textit{ cannot be changed, the laser pointer} \\ \textit{output power can only be controlled by changing the} \\ \textit{duty cycle } T/T_p \textit{ of the control signal} \end{array}\right) \left(\begin{array}{l} \textit{Allows estimation} \\ \textit{of the battery life} \end{array}\right)$