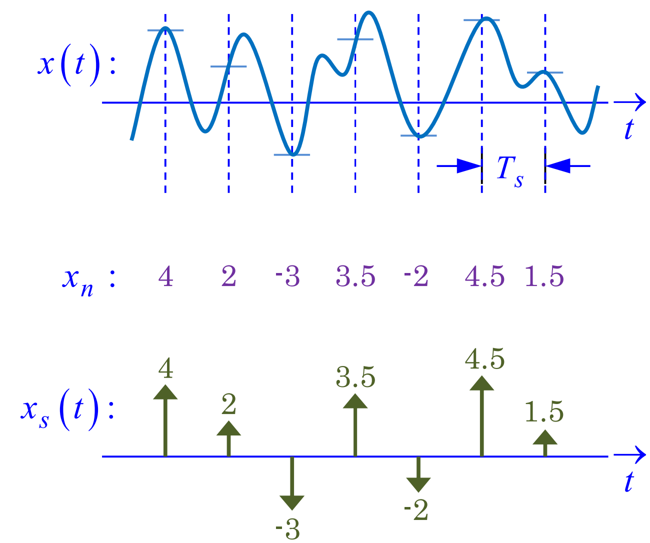
7. Sampling Theorem



Sampling is a process for converting an analog signal into a discrete-time sequence, which can be digitally stored and processed. This is what is done in CD, DVD and Blue Ray recording, and it is also done to your voice signal for transmission over a radio channel by your digital cell phone.

In practice, when we sample an analog signal $x(t)$, we capture the value of the signal at every T_s seconds to produce a discrete-time sequence, $x_n = x(nT_s)$.

Now suppose we sample $x(t)$ by multiplying it with an impulse train to form

$$x_s(t) = \underbrace{x(t)\sum_n \delta(t - nT_s)}_{\textbf{impulse sampling}} = \sum_n x(nT_s)\delta(t - nT_s) = \sum_n x_n \delta(t - nT_s), \tag{7.1}$$

which is a continuous-time signal. It can be shown that the spectrum of x_n is equal to the spectrum (or Fourier transform) of $x_s(t)$. We shall make use of this equivalence to examine the mechanism of signal sampling and reconstruction in the *continuous-time* and *-frequency* domains, which leads to the Nyquist sampling theorem. To aid our discussion on signal reconstruction, we begin by introducing the notion of idealized filters.

7.1 Idealized Filters

Electronic filters are analog circuits which perform the function of removing unwanted frequency components from a signal and/or enhancing wanted ones.

For continuous-time LTI systems, the spectrum of the output signal is obtained by multiplying the spectrum of the input signal with the frequency response of the system. Thus, filters are essentially systems that exhibit some sort of frequency-selective behavior.

The band of frequencies passed by a filter is referred to as the **pass-band**, and the band of frequencies rejected by a filter is called the **stop-band**.

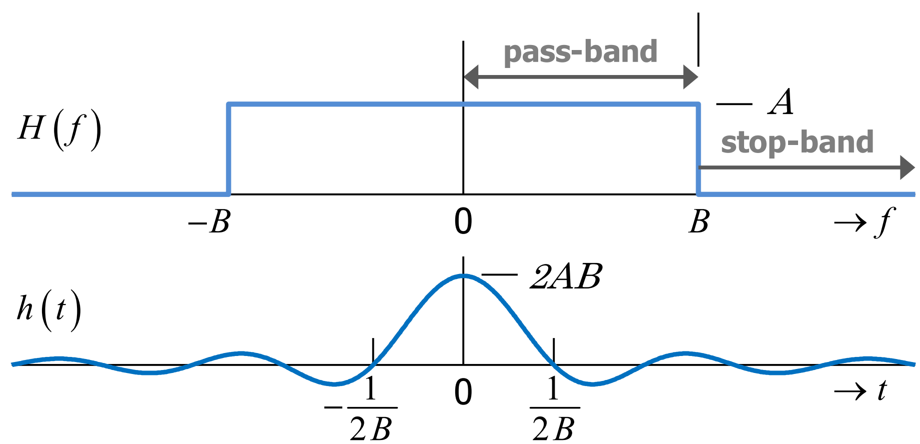
An **idealized filter**, also known colloquially as a "**brick-wall filter**", is one that has full transmission in the pass-band, and complete attenuation in the stop-band, with abrupt transitions. Two such filters are described below.

- **Ideal Low-Pass Filter (LPF)**

$$\text{Frequency response}: \; H(f) = A\,\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$\text{Impulse response}: \; h(t) = 2AB\,\mathrm{sinc}(2Bt)$$

$$\text{Cutoff frequency} = \text{Bandwidth} = B$$



- **Ideal Band-Pass Filter (BPF)**

Frequency response :

$$H(f) = A\left[\text{rect}\left(\frac{f+f_o}{B}\right) + \text{rect}\left(\frac{f-f_o}{B}\right)\right]$$

Impulse response :

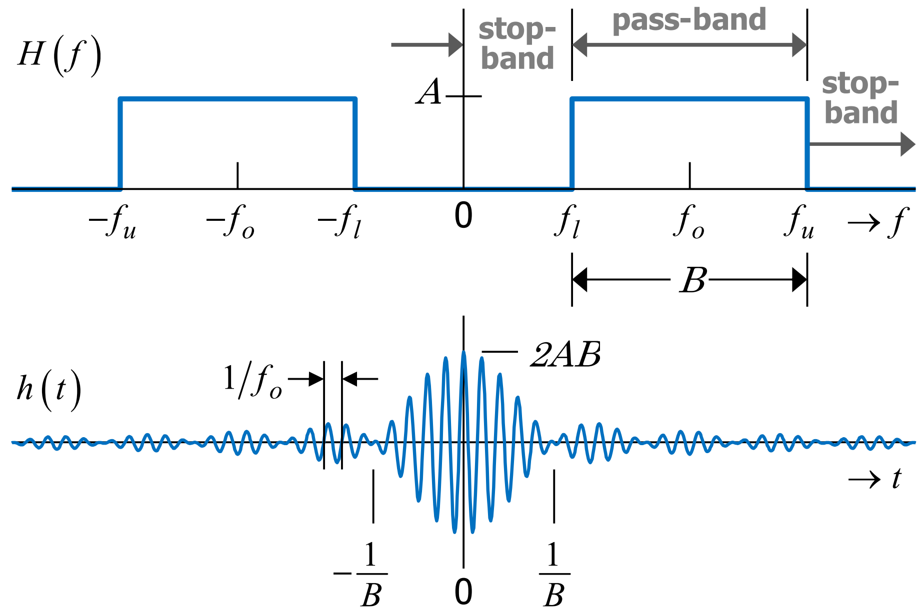
$$h(t) = 2AB\,\text{sinc}(Bt)\cos(2\pi f_o t)$$

$Upper\ Cutoff\ frequency = f_u$

$Lower\ Cutoff\ frequency = f_l$

$Center\ frequency = f_o = 0.5(f_u + f_l)$

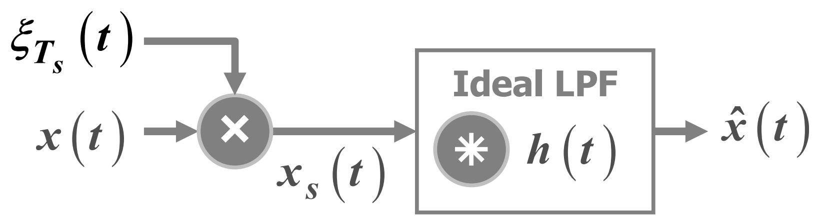
$Bandwidth = B = f_u - f_l$

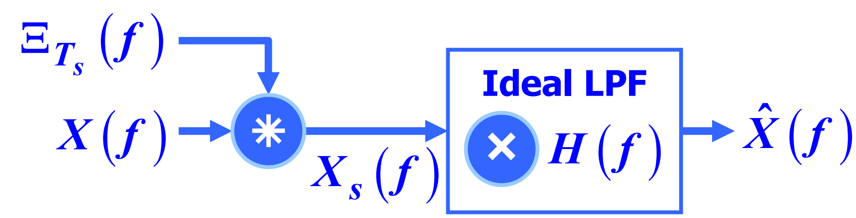


Ideal filters can be realized mathematically by multiplication in the frequency domain or, equivalently, convolution in the time domain. Real-time filters can only approximate this ideal since the filter is non-causal and has an infinite delay. One such real-time filter is the Butterworth filter, which is discussed in Chapter 8.

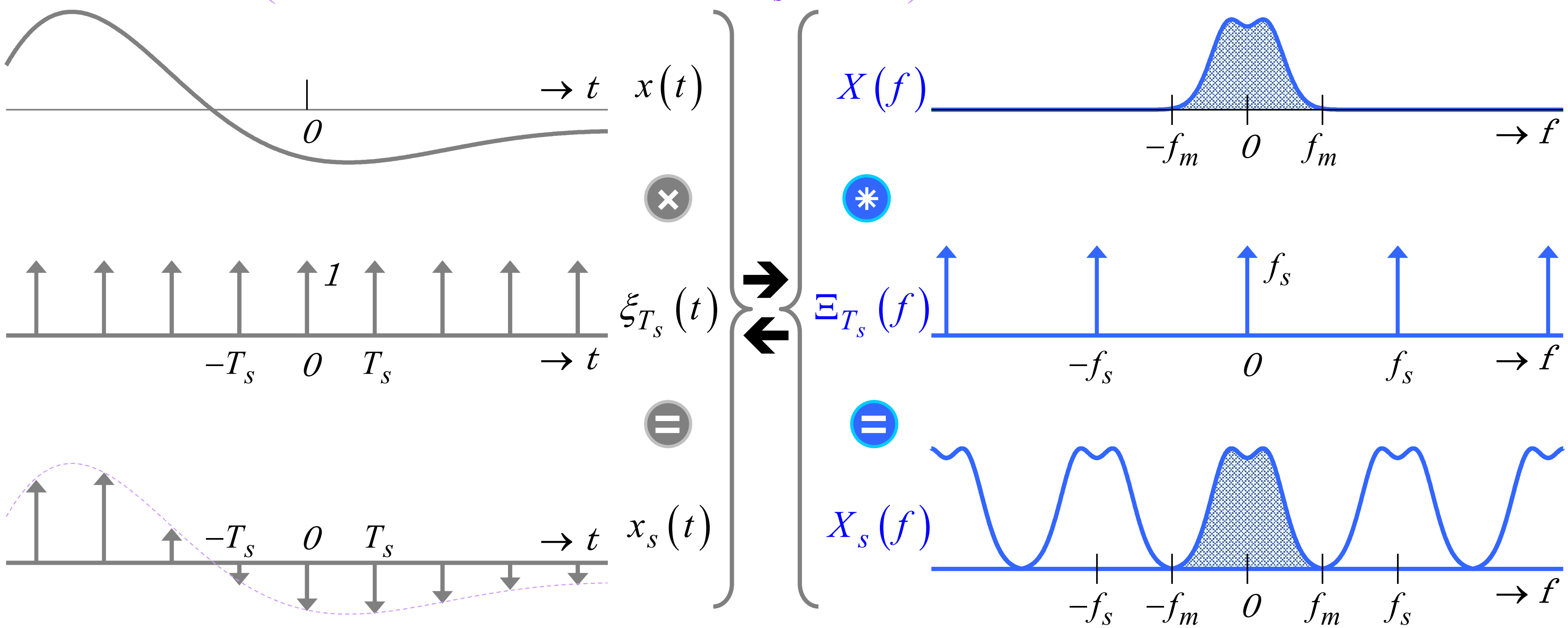
Ideal filters are commonly found in conceptual demonstrations or proofs, such as the sampling theorem where ideal LPF and BPF are used to demonstrate the perfect reconstruction of the original analog signal from its sampled form.

7.2 Continuous-time Sampling and Reconstruction of Signals

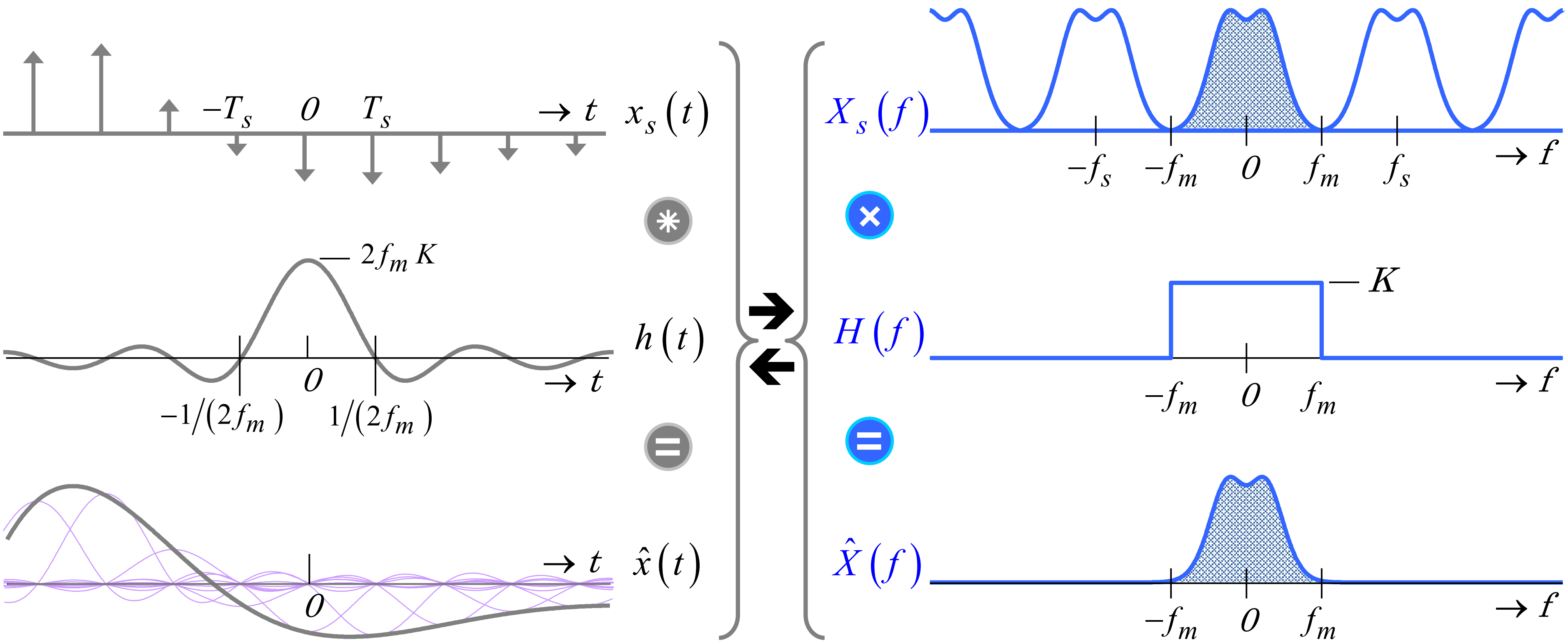




- ***Sampling*** (*Sampling frequency* $f_s = \dfrac{1}{T_s} = 2f_m$)



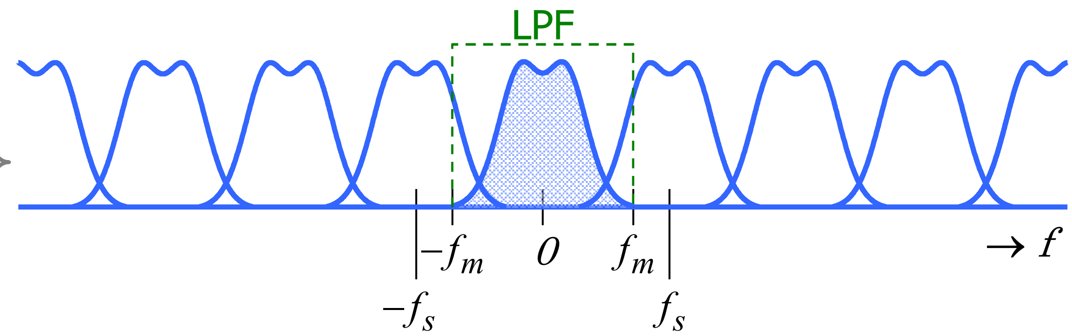
- ***Reconstruction***



What if sampling frequency $f_s = \dfrac{1}{T_s} \neq 2f_m$ **?**

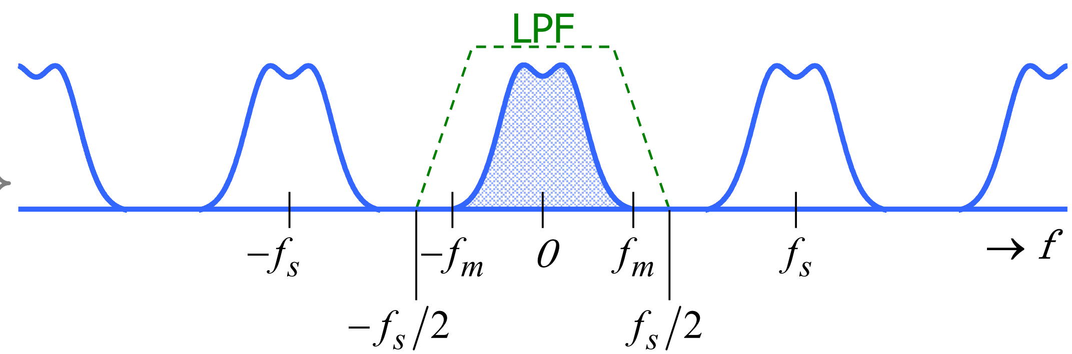
Sampling frequency $f_s = \dfrac{1}{T_s} < 2f_m$

- Spectral images overlap. This is called ***frequency aliasing.***
- Perfect reconstruction is not possible.



Sampling frequency $f_s = \dfrac{1}{T_s} > 2f_m$

- Gaps appear between spectral images due to ***oversampling.***
- Perfect reconstruction is possible.
- Relaxes LPF design.



Conditions for ***perfect reconstruction*** :

$$\left\{\begin{array}{l}\left(\begin{array}{ll} x(t) \text{ must be bandlimited} & : X(f) = 0; \quad |f| > f_m \\ \text{Sampling rate of } x(t) & : f_s \geq 2f_m \end{array}\right) \\ \textit{Reconstruction LPF}: \quad \left(\begin{array}{l} |H(f)| = \begin{cases} K; & |f| < f_m \\ 0; & |f| > f_m \end{cases} \\ \angle H(f) \quad \cdots\cdots \quad Linear \end{array}\right) \end{array}\right\} \leftarrow \text{This leads to the Nyquist Sampling Theorem}$$

Nyquist Sampling Theorem

1. A band-limited signal of finite energy, which has no frequency components higher than W Hz, may be completely described by specifying the values of the signal at instants of time separated by $\frac{1}{2W}$ secs.

2. A band-limited signal of finite energy, which has no frequency components higher than W Hz, may be completely recovered from a knowledge of its samples taken at the rate of $2W$ samples/sec.

'$2W$' is called the Nyquist Sampling Frequency or Nyquist Rate.

Example 7-1:

A signal $x(t)=\mathrm{sinc}^2(2t)$ is sampled at 8 Hz to produce the sampled signal $x_s(t)$. Sketch the spectra of $x(t)$ and $x_s(t)$. Can $x(t)$ be perfectly reconstructed from $x_s(t)$ using an ideal low-pass filter? If "yes", specify the filter. What is the Nyquist sampling frequency for $x(t)$?

From Fourier transform table:

$$\Im\left\{A\,\mathrm{tri}(t/T)\right\} = AT\,\mathrm{sinc}^2(fT).$$

Letting $T=2$ and $A=0.5$, we get

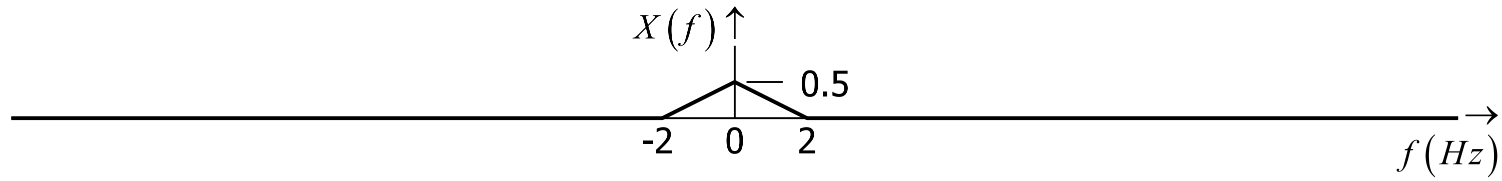
$$\Im\left\{0.5\,\mathrm{tri}(t/2)\right\} = \mathrm{sinc}^2(2f).$$

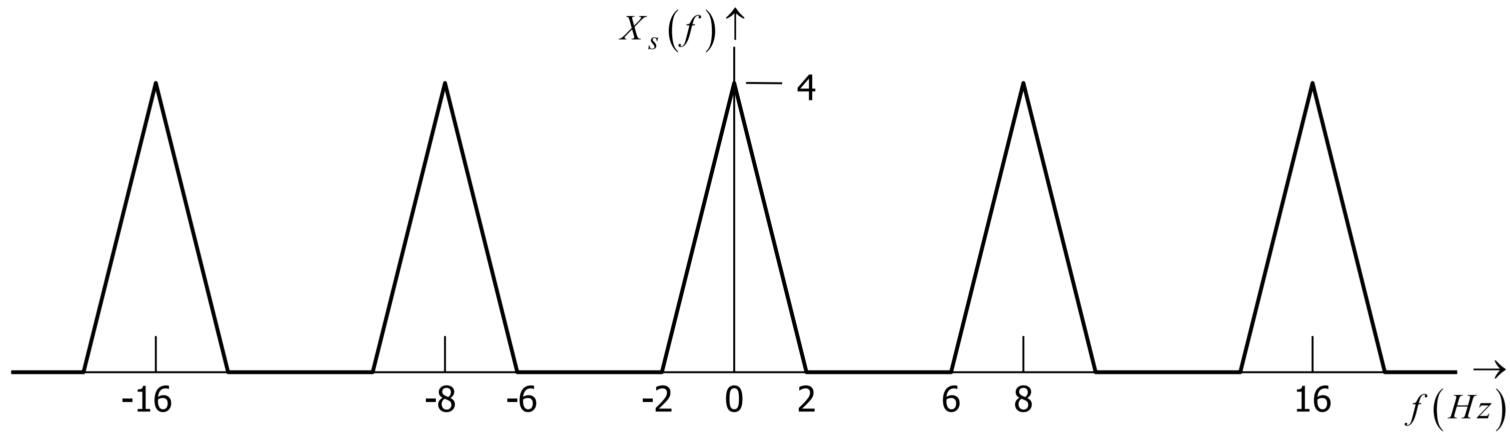
Applying the Duality property of the Fourier transform,

$$X(f) = \Im\left\{\mathrm{sinc}^2(2t)\right\} = 0.5\,\mathrm{tri}(f/2).$$

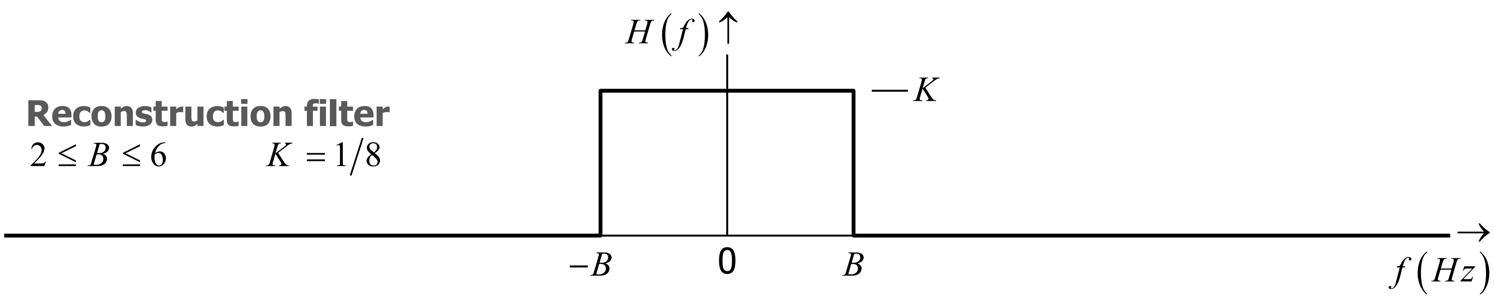
The sampled signal and its spectrum are, respectively, given by

$$\left(\begin{aligned} x_s(t) &= x(t)\sum_{n=-\infty}^{\infty}\delta(t-n/8) \\ &= \mathrm{sinc}^2(2t)\sum_{n=-\infty}^{\infty}\delta(t-n/8) \end{aligned}\right) \quad \textit{and} \quad \left(\begin{aligned} X_s(f) &= X(f) * \sum_{n=-\infty}^{\infty}\delta(t-n/8) \\ &= 0.5\,\mathrm{tri}(f/2) * 8\sum_{k=-\infty}^{\infty}\delta(f-8k) \end{aligned}\right)$$





Yes, $x(t)$ *can be perfectly reconstructed from* $x_s(t)$ *using an ideal low-pass filter.*

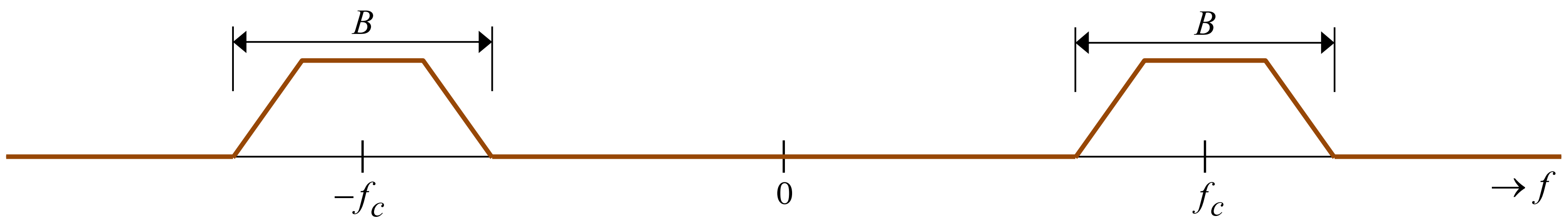


Nyquist sampling frequency for $x(t)$ *is* $2 \times 2 = 4\ Hz$

7.3 Sampling Band-limited Bandpass Signal below Nyquist Rate

If $x(t)$ is a band-limited bandpass signal, it is possible to sample $x(t)$ below Nyquist rate and yet achieve perfect reconstruction using a bandpass reconstruction filter.

Suppose the spectrum $X(f)$ of $x(t)$ is as shown below:



- **Nyquist rate**: $f_s \geq 2f_c + B$ (7.2)

- Is it possible to sample a bandpass signal below Nyquist rate and yet achieve perfect signal reconstruction? **The answer is YES, if $\boldsymbol{f_c \geq B}$.**

  There are two possible scenarios.

  (a) Overlapping spectral images : $\underbrace{\boldsymbol{f_s} = \frac{2f_c}{k}; \quad k = 1,\ 2,\ \cdots,\ \left\lfloor \frac{2f_c}{B} \right\rfloor}$ (7.3a)

  Assumption: The +ve and -ve frequency bands are symmetric about f_c and $-f_c$, respectively

  (b) Un-aliased spectral images : $\frac{2f_c + B}{k+1} \leq \boldsymbol{f_s} \leq \frac{2f_c - B}{k}; \quad k = 1,\ 2,\ \cdots,\ \left\lfloor \frac{2f_c - B}{2B} \right\rfloor$ (7.3b)

  where $\lfloor\ \rfloor$ denotes integer floor.

Example 7-2(a): *Lowest sampling frequency with* ***overlapping spectral images***

A signal has center frequency $f_c = 24\,(\text{Hz})$ and bandwidth $B = 8\,(\text{Hz})$. What are the possible sampling frequencies (below Nyquist rate) that allow for perfect signal reconstruction?

Solution:

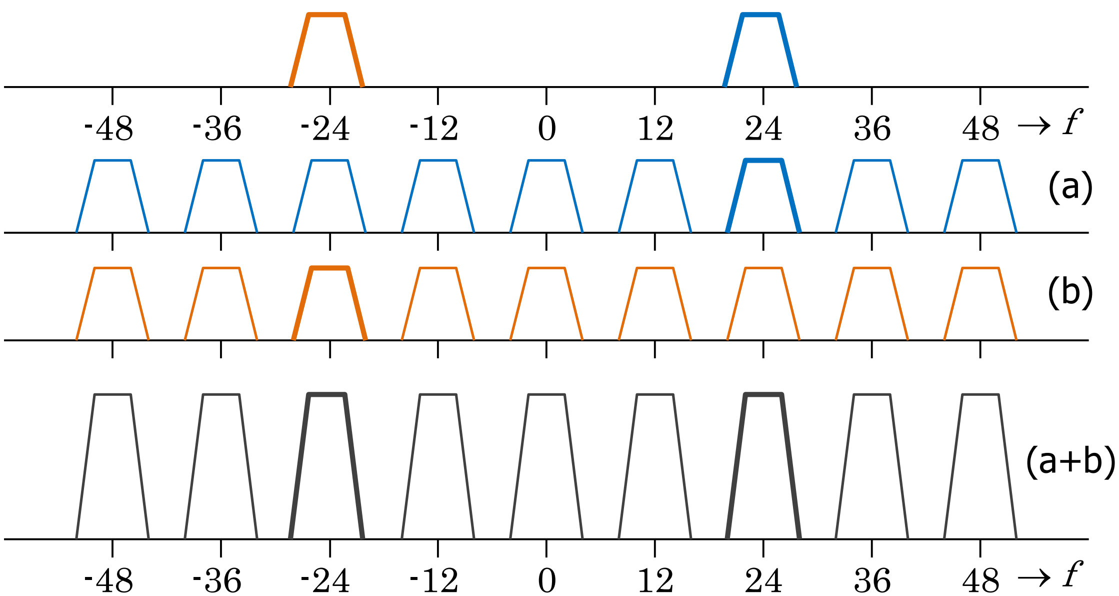
Nyquist rate: $(24+4)\times 2 = 56\,(\text{Hz})$.

From (7.3a): $\boldsymbol{f_s} = \dfrac{2\times 24}{k}; \quad k = 1,\ 2,\ \cdots,\ \left\lfloor \dfrac{2\times 24}{8} \right\rfloor$

$$\boldsymbol{f_s} = \frac{48}{k}; \quad k = 1,\ 2,\ \cdots,\ 6$$

Possible sampling frequencies : $\boldsymbol{f_s} = 48,\ 24,\ 16,\ 12,\ 9.6,\ 8\ Hz$

Lowest sampling frequency : $\boldsymbol{f_s} = 8\ Hz$



Spectrum of ORIGINAL SIGNAL

Images due to the +ve frequency part of the original spectrum

Images due to the –ve frequency part of the original spectrum

Spectrum of SAMPLED SIGNAL $\boldsymbol{f_s = 12\ Hz}$

Example 7-2(b): *Lowest sampling frequency with **un-aliased spectral images***

A signal has center frequency $f_c = 24\,(\text{Hz})$ *and bandwidth* $B = 8\,(\text{Hz})$. *What are the possible sampling frequencies (below Nyquist rate) that allow for perfect signal reconstruction?*

Solution:

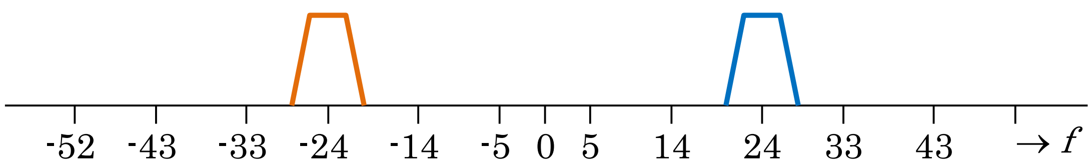
Nyquist rate: $(24+4)\times 2 = 56\,(\text{Hz})$.

From (7.3b): $\dfrac{2\times 24+8}{k+1} \le \boldsymbol{f_s} \le \dfrac{2\times 24-8}{k}; \quad k = 1,\ 2,\ \cdots,\ \left\lfloor \dfrac{2\times 24-8}{2\times 8} \right\rfloor$

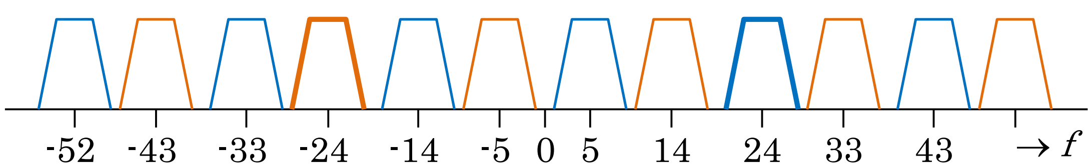
$$\frac{56}{k+1} \le \boldsymbol{f_s} \le \frac{40}{k}; \quad k = 1,\ 2$$

Possible sampling frequencies : $\overbrace{28 \le \boldsymbol{f_s} \le 40}^{\text{with } k=1}$ or $\overbrace{18.67 \le \boldsymbol{f_s} \le 20}^{\text{with } k=2}\ Hz$

Lowest sampling frequency : $\boldsymbol{f_s} = 18.67\ Hz$



Spectrum of ORIGINAL SIGNAL



Spectrum of SAMPLED SIGNAL
$\boldsymbol{f_s = 19\ Hz}$

Example 7-2(c): Reconstruction filter

Specify an ideal filter for reconstructing the $x(t)$ in Examples 7-2(a) & (b) from its sampled versions.

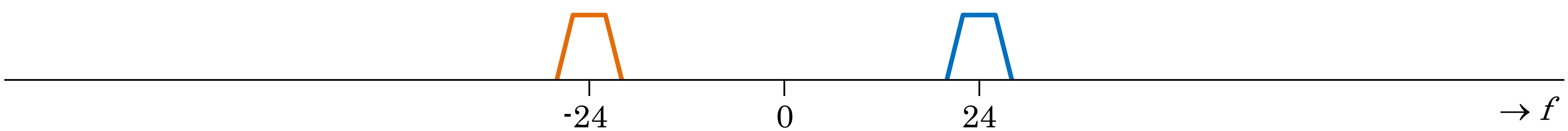
Solution:

The ideal filter should be bandpass and have the following specifications:

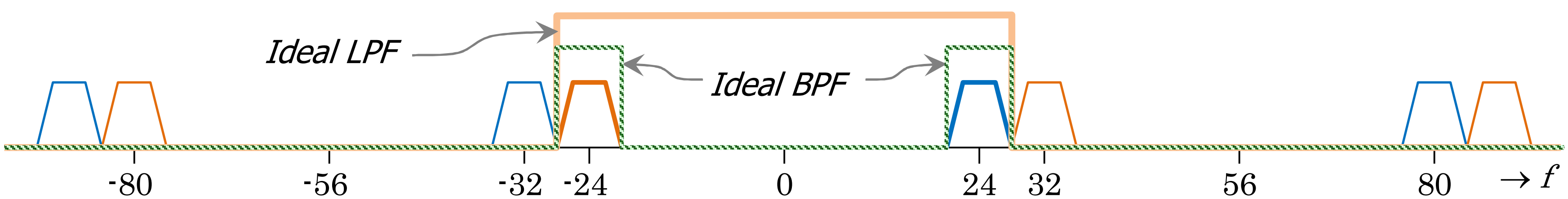
$$|H(f)| = \begin{cases} 1; & 20 < |f| < 28 \\ 0; & \text{otherwise} \end{cases} \quad \text{and} \quad \angle H(f) \cdots \mathit{Linear}$$

Remarks:

When a bandpass signal is sampled at Nyquist rate, an ideal lowpass filter may also be used to reconstruct the signal from its samples. However, this will require higher sampling rate and also lead to a significant reduction in the signal-to-noise ratio at the filter output, and is thus not usually practised.



Spectrum of ORIGINAL SIGNAL



Spectrum of SAMPLED SIGNAL $f_s = 56\ Hz\ (Nyquist\ Rate)$

Derivation of the sampling frequency for band-limited bandpass signal below Nyquist frequency