

National University of Singapore  
School of Computing  
CS1010X: Programming Methodology  
Semester II, 2019/2020

**Mission 3**  
**More Than Thrice**

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**Due: 23 February 2020, 23:59**

## Required Files

- mission03-template.py

## Background

One of the things that makes Python different from other common programming languages is the ability to operate with *higher-order* functions, namely, functions that manipulate and generate other functions.

The function `square` may be applied to a float (or int) and will return another float (or int). We indicate this with the notation:

$$\text{sq} : \text{float}|\text{int} \rightarrow \text{float}|\text{int}$$

If  $f$  and  $g$  are functions of type  $\text{float}|\text{int} \rightarrow \text{float}|\text{int}$ , then we may *compose* them:

```
def compose(f, g):  
    return lambda x: f(g(x))
```

For example, `compose(sq, log)` is a function of type  $\text{float} \rightarrow \text{float}$  that returns the square of the logarithm of its argument, while `compose(log, sq)` returns the logarithm of the square of its argument:

```
>>> from math import *  
>>> def sq(x): return x**2  
>>> sq(log(2))  
0.4804530139182014  
>>> compose(sq, log)(2)  
0.4804530139182014  
>>> log(sq(2))  
1.3862943611198906  
>>> compose(log, sq)(2)  
1.3862943611198906
```

As we have used it above, the function `compose` takes as arguments two functions of type  $F = \text{float}|\text{int} \rightarrow \text{float}|\text{int}$ , and returns another such function. We indicate this with the notation:

$$\text{compose} : (F, F) \rightarrow F$$

Just as squaring a number multiplies the number by itself, thrice of a function composes the function three times. That is, `thrice(f)(n)` will return the same result as `f(f(f(n)))`:

```
>>> def thrice(f): return compose(compose(f, f), f)
>>> thrice(sq)(3)
6561
>>> sq(sq(sq(3)))
6561
```

As used above, `thrice` is of type  $(F \rightarrow F)$ . That is, it takes as input a function of type  $F$  and returns the same kind of function. But `thrice` will actually work for other kinds of input functions. It is enough for the input function  $F$  to have a type of the form  $T \rightarrow T$  (instead of  $F = \text{float}|\text{int} \rightarrow \text{float}|\text{int}$ ), where  $T$  may be any type. So more generally, we can write

$$\text{thrice} : (T \rightarrow T) \rightarrow (T \rightarrow T)$$

Composition, like multiplication, may be iterated. Consider the following:

$$\text{repeated} : ((T \rightarrow T), \text{int}) \rightarrow (T \rightarrow T)$$

Example:

```
>>> def identity(x): return x
>>> def repeated(f, n):
    if n == 0:
        return identity
    else:
        return compose(f, repeated(f, n-1))

>>> repeated(sin, 5)(3.1)
0.041532801333692235
>>> sin(sin(sin(sin(sin(3.1))))))
0.041532801333692235
```

This mission consists of **two** tasks.

## Task 1: Thrice (5 marks)

- The type of `thrice` is of the form  $(T' \rightarrow T')$  (where  $T'$  happens to equal  $(T \rightarrow T)$ ), so we can legitimately use `thrice` as an input to `thrice`! For what value of  $n$  will `thrice(thrice(f))(0)` return the same value<sup>1</sup> as `repeated(f, n)(0)`?
- See if you can now predict what will happen when the following expressions are evaluated. Briefly explain what goes on in each case.

Note: Function `add1` is defined as follows:

```
def add1(x): return x + 1
```

---

<sup>1</sup>“Sameness” of function values is a sticky issue which we don’t want to get into here. We can avoid it by assuming that  $f$  is bound to a value of type  $F$ , so evaluation of `thrice(thrice(f))(0)` will return a number.

- (i) `thrice(thrice)(add1)(6)`
- (ii) `thrice(thrice)(identity)(compose)`
- (iii) `thrice(thrice)(sq)(1)`
- (iv) `thrice(thrice)(sq)(2)`.

## Task 2: Combine them together! (5 marks)

Higher order functions can be used to implement other functions as well. Consider the following higher order function called `combine`:

```
def combine(f, op, n):
    result = f(0)
    for i in range(n):
        result = op(result, f(i))
    return result
```

(a) Let's define the `smiley_sum`  $S(t)$  as follows:

```
S(1) = 1
S(2) = 4 + 1 + 4 = 9
S(3) = 9 + 4 + 1 + 4 + 9 = 27
S(4) = 16 + 9 + 4 + 1 + 4 + 9 + 16 = 59
S(5) = 25 + 16 + 9 + 4 + 1 + 4 + 9 + 16 + 25 = 109
```

If we look closer, we can actually define `smiley_sum` in terms of `combine`!

```
def smiley_sum(t):
    def f(x):
        ...

    def op(x, y):
        ...

    n = ...

    # Do not modify this return statement
    return combine(f, op, n)
```

Fill in the appropriate implementations for `f` and `op`. You are reminded to test your code.

**Reminder:** You are not allowed to modify the return statement!

(b) Your friend who attended the lecture on higher order functions challenges you to define a function that computes the  $n$ -th Fibonacci number using the function `combine`.

Recall the definition for Fibonacci numbers:

```
def fib(n):
    if n == 0 or n == 1:
```

```
        return n
    else:
        return fib(n-1) + fib(n-2)
```

This is his challenge:

```
def new_fib(n):
    def f(x):
        ...

    def op(x, y):
        ...

    # Do not modify this return statement
    return combine(f, op, n+1)
```

Are you able to answer his challenge? If yes, provide a working implementation. If no, explain why.

**Reminder:** You are not allowed to modify the return statement!