CS1231(S) Tutorial 8: Relations

National University of Singapore

2020/21 Semester 1

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.

D1. Let A be a nonempty set.

(a) Why is $\varnothing$ a relation on A?

(b) Is $\varnothing$ reflexive as a relation on A?

(c) Is $\varnothing$ symmetric as a relation on A?

(d) Is $\varnothing$ transitive as a relation on A?

(e) Is $\varnothing$ antisymmetric as a relation on A?

D2. Do you agree with the claim and its proof below? Why?

Claim. If R is a symmetric and transitive relation on a set A, then R is reflexive.

Proof.

1. Let $x \in A$. The aim is to show $x \mathrel{R} x$.
2. Take $y \in A$ such that $x \mathrel{R} y$.
3. Then $y \mathrel{R} x$ by line 2, as R is symmetric.
4. So $x \mathrel{R} x$ by line 2 and line 3, as R is transitive.

D3. For each of the following relations, either prove that it is an equivalence relation or prove that it is not an equivalence relation.

(a) R is the relation on $\mathbb{Z}$ such that $x \mathrel{R} y$ if and only if $x + y$ is even for all $x, y \in \mathbb{Z}$.

(b) R is the relation on $\mathbb{Z}$ such that $x \mathrel{R} y$ if and only if $x + y$ is odd for all $x, y \in \mathbb{Z}$.

(c) R is the relation on $\mathbb{Q} \setminus \{0\}$ such that $x \mathrel{R} y$ if and only if $x/y \in \mathbb{Z}$ for all $x, y \in \mathbb{Q} \setminus \{0\}$.

D4. Draw Hasse diagrams for

(a) the partial order $\{(a, b), (b, c), (a, c), (a, d)\}$ on $\{a, b, c, d\}$; and

(b) the subset relation $\subseteq$ on $\mathcal{P}\{a, b, c\}$,

where a, b, c, d are mutually distinct.

Background

Definition. Let R be a relation from a set A to a set B. Define the relation R^{-1} from B to A by setting

$$y \mathrel{R^{-1}} x \quad \Leftrightarrow \quad x \mathrel{R} y$$

for each $y \in B$ and each $x \in A$.

Tutorial questions

1. Let $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 8, 10, 12, 14\}$. Define a relation R from A to B by setting
$$x \mathrel{R} y \quad \Leftrightarrow \quad x \text{ is prime and } x \mid y$$
for each $x \in A$ and each $y \in B$. Write down the sets R and R^{-1} in roster notation. Do not use ellipses $(\ldots)$ in your answers.

2. Let R be a relation on a set A. Show that R is symmetric if and only if $R = R^{-1}$.

3. For each of the following relations on $\mathbb{Q}$, determine if it is (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) an equivalence relation.

   (a) R is defined by setting $x \mathrel{R} y$ if and only if $xy \geqslant 0$ for all $x, y \in \mathbb{Q}$.

   (b) S is defined by setting $x \mathrel{S} y$ if and only if $xy > 0$ for all $x, y \in \mathbb{Q}$.

   (c) T is defined by setting $x \mathrel{T} y$ if and only if $|x - y| \leqslant 2$ for all $x, y \in \mathbb{Q}$.

4. Define a relation R on $\mathbb{Q}$ as follows: for all $x, y \in \mathbb{Q}$,
$$x \mathrel{R} y \quad \Leftrightarrow \quad x - y \in \mathbb{Z}.$$

   (a) Show that R is an equivalence relation.

   (b) Find an element a in the equivalence class $[\frac{37}{7}]$ that satisfies $0 \leqslant a < 1$.

   (c) Devise a general method to find, for each given equivalence class $[x]$, where $x \in \mathbb{Q}$, an element $a \in [x]$ such that $0 \leqslant a < 1$. Justify your answer.
   (Hint: you may find the Division Theorem helpful.)

5. Let A, B be nonempty sets and f be a surjection $A \to B$. Show that $\mathscr{C}$ is a partition on A, where
$$\mathscr{C} = \big\{\{x \in A : f(x) = y\} : y \in B\big\}.$$

6. Consider the "divides" relation on each of the following sets of integers. For each of these, draw a Hasse diagram, find all largest, smallest, maximal and minimal elements, and a linearization.

   (a) $A = \{1, 2, 4, 5, 10, 15, 20\}$.

   (b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$.

7. **Definition.** Let $\preccurlyeq$ be a partial order on a set P, and $a, b \in P$.

   - We say a, b are *comparable* if $a \preccurlyeq b$ or $b \preccurlyeq a$.
   - We say a, b are *compatible* if there exists $c \in P$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

   (a) Is it true that, in all partially ordered sets, any two comparable elements are compatible? Justify your answer.

   (b) Is it true that, in all partially ordered sets, any two compatible elements are comparable? Justify your answer.