CS1231/CS1231S: Discrete Structures

Tutorial #9: Counting and Probability I

(Week 11: 26 – 30 October 2020)

I. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. A box contains three blue balls and seven white balls. One ball is drawn, its colour recorded, and it is returned to the box. Then another ball is drawn and its colour is recorded as well.

(a) What is the probability that the first ball is blue and the second is white?
(b) What is the probability that both balls drawn are white?
(c) What is the probability that the second ball drawn is blue?

D2. Calculate

(a) the probability that a randomly chosen positive three-digit integer is a multiple of 6.
(b) the probability that a randomly chosen positive four-digit integer is a multiple of 7.

D3. Assuming that all years have 365 days and all birthdays occur with equal probability. What is the smallest value for n so that in any randomly chosen group of n people, the probability that two or more persons having the same birthday is at least 50%?

Write out the equation to solve for n and write a program to compute n.

(This is the well-known *birthday problem*, whose solution is counter-intuitive but true.)

II. Tutorial Questions

1. In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

   (We will use possibility tree to solve this problem for now. In the next tutorial, we will approach this problem using combination.)

2. (Past year's exam question.)
   The figure on the right shows a combination lock with 40 positions.

   To open the lock, you rotate to a number in a clockwise direction, then to a second number in the counterclockwise direction, and finally to a third number in the clockwise direction. If consecutive numbers in the combination cannot be the same, how many combinations of three-number codes are there?

3. There are 789 CS students in SoC. Among them, 672 are taking CS1231S, 629 are taking CS1101S, 153 are taking MA1101R, 608 are taking CS1231S and CS1101S, 87 are taking CS1231S and MA1101R, 53 are taking CS1101S and MA1101R, and 46 are taking all three modules.

   How many CS students are not taking any of these three modules?

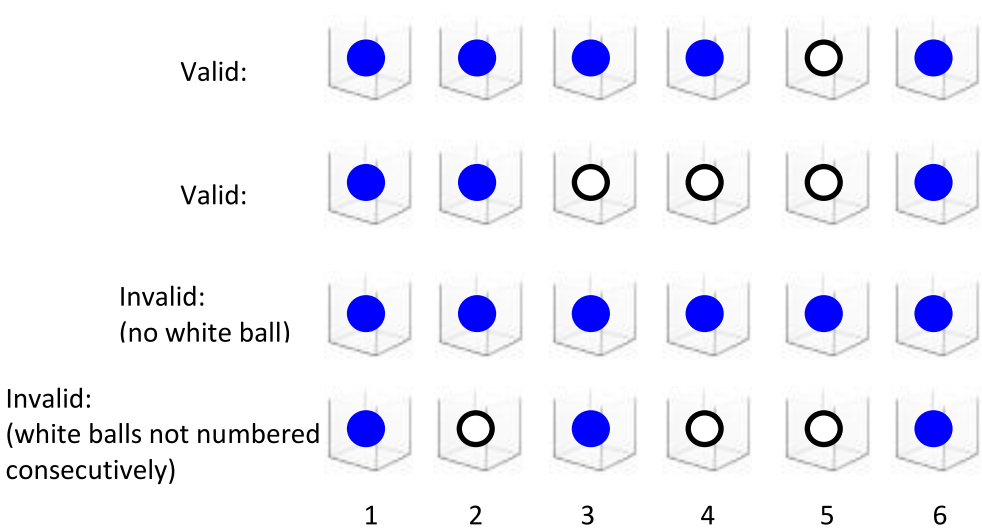
4. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position?

   An integer k is in the correct position if it is at the k^{th} position in the permutation. For example, the permutation 3, $\underline{2}$, 4, 1, $\underline{5}$ has integers 2 and 5 in their correct positions, and the permutation 12, 1, $\underline{3}$, 9, 10, 8, $\underline{7}$, 6, 2, 4, $\underline{11}$, 5 has integers 3, 7, and 11 in their correct positions. Integers that are in their correct positions are underlined for illustration.

5. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   (For this tutorial, use sum of a sequence to solve this problem. In the next tutorial, we will revisit this problem using a different approach.)

   Some examples for $n = 6$ are shown below for your reference.

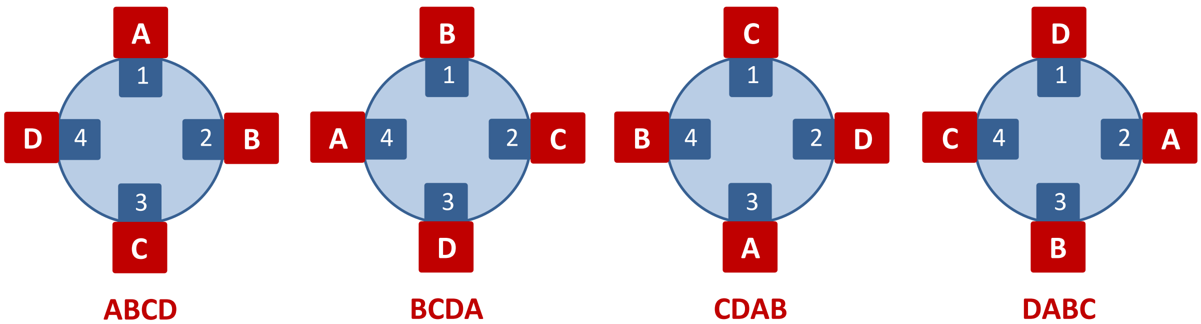


6. In Tutorial #4 D1, you are asked to write down all possible functions $\{a, b, c\} \rightarrow \{1,2\}$.

   How many possible functions $f: A \rightarrow B$ are there if $|A| = n$ and $|B| = k$?

7. We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

   What if we seat them around a circular table? Examine the figure below.



   The four seating arrangements (clockwise from top) *ABCD*, *BCDA*, *CDAB* and *DABC* are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

   This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutations. Hence, there are $\frac{4!}{4}$ or 3! ways of circular permutations for 4 persons. In general, the number of circular permutations of n objects is $(n-1)!$

   Answer the following questions:

   a. In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

   b. In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

   c. In how many ways can $n - 1$ people sit around a circular table with n chairs?

8. (Past year's exam question.)
   Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.

9. Let $S = \{3,4,5,6,7,8,9,10,11,12\}$. What is the smallest number of integers you must choose from S such that two of them sum to 15? In other words, what is the smallest $n \in \mathbb{Z}_{n\geq 2}$ such that for all subsets X of S where $|X| = n$, there exists two distinct elements $x, y \in X$ such that $x + y = 15$.

10. (Past year's exam question.)
    In a city, houses are randomly assigned distinct numbers between 1 and 50 inclusive. What is the minimum number of houses to ensure that there are 5 houses numbered consecutively?

    For example, the number of houses cannot be 10 because we can choose to number the houses 1, 8, 9, 15, 18, 21, 22, 23, 24, 32, hence no 5 houses are numbered consecutively.

    To receive full credit, you must define the pigeons and pigeonholes in your answer.