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Q2. simplify $p \wedge (\sim p \rightarrow r \wedge q) \wedge \sim(q \rightarrow \sim p)$

set up parantheses so as not to confuse order of operation:

$\sim$ first $\quad \wedge \ \vee$ equal $\quad \longrightarrow$ last

$p \wedge ((\sim p) \rightarrow (r \wedge q)) \wedge \sim(q \rightarrow (\sim p))$

$\equiv p \wedge (\sim(\sim p) \vee (r \wedge q)) \wedge \sim(q \rightarrow (\sim p))$ by implication law

$\equiv p \wedge (p \vee (r \wedge q)) \wedge \sim(q \rightarrow (\sim p))$ by double negative law

$\equiv p \wedge (p \vee (r \wedge q)) \wedge \sim((\sim q) \vee (\sim p))$ by implication law

$\equiv p \wedge (p \vee (r \wedge q)) \wedge (\sim(\sim q) \wedge \sim(\sim p))$ by De Morgan's law

$\equiv p \wedge (p \vee (r \wedge q)) \wedge (q \wedge \sim(\sim p))$ by double negative law

$\equiv p \wedge (p \vee (r \wedge q)) \wedge (q \wedge p)$ by double negative law

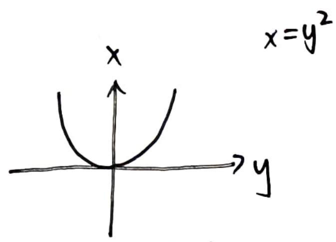
$\equiv p \wedge (q \wedge p)$ by absorption law

$\equiv p \wedge (p \wedge q)$ by commutative law

$\equiv (p \wedge p) \wedge q$ by associative law

$\equiv p \wedge q$ // by idempotent law

Q3. a) $\forall x \in \mathbb{R} \ (\underline{x < 0} \leftrightarrow \forall y \in \mathbb{R} \ (x \neq y^2))$



b) $P(x) = (x \neq 1 \wedge \forall y, z \in \mathbb{N} \ (x = yz \rightarrow (y = 1 \vee y = x)))$, $\forall x \in \mathbb{N}$

If $P(x)$ is true, x is an integer greater than $\underline{1}$ and is only divisible by integers 1 or itself

definition of a prime number

$\underline{\text{x is prime}}$

Q4. To show $((p \vee q \vee r) \wedge (\sim p \to s) \wedge (\sim q \to s)) \to (r \to s)$ is not a tautology, we can prove $\exists\, p, q, r, s$ such that the above statement is False

Proof (by construction)

1. For a conditional statement $a \to b$ to be False, a has to be True and b has to be False

2. Hence, for $((p \vee q \vee r) \wedge (\sim p \to s) \wedge (\sim q \to s)) \to (r \to s)$ to be False $(p \vee q \vee r) \wedge (\sim p \to s) \wedge (\sim q \to s)$ is True

and $(r \to s)$ is False

2.1 For $(r \to s)$ to be False, r is True and s is False (from step 1)

2.2 For $(p \vee q \vee r) \wedge (\sim p \to s) \wedge (\sim q \to s)$ to be True, each statement has to be True

2.3 Since s is False, for $(\sim p \to s)$ and $(\sim p \to s)$ to be True, $\sim p$ and $\sim q$ have to be False $(F \to F \equiv T)$

2.4 p is True and q is True

2.5 $p \vee q \vee r$ is also True

3. Hence, $\exists\, p, q, r, s$ where p is True, q is True, r is True and s is False such that $((p \vee q \vee r) \wedge (\sim p \to s) \wedge (\sim q \to s)) \to (r \to s)$ is False and hence is not a tautology.

Q5. a) Dueet's argument might be false if there exist another sequence $\alpha \in S$ such that $\alpha \neq b\rho + c\sigma$ for some $b, c \in F$

b) Dueet's argument might be true if ω is the only sequence in S.

a) For every odd natural number there is a different natural number such that their sum is even

$$\forall x\ (\sim Even(x) \longrightarrow \exists y \text{ such that } (x \neq y \wedge Even(x+y)))$$

b) The sum of any two prime numbers except the prime number 2 is even

$$\forall x, y\ (Prime(x) \wedge Prime(y) \wedge x \neq 2 \wedge y \neq 2 \longrightarrow Even(x+y))$$

Q7. Proof (by contraposition)

1. Want to prove : let a be a rational number and b be an irrational number if ab is rational, then $a = 0$

2. $\exists p, q \in \mathbb{Z}$ such that $ab = \frac{p}{q}$, $q \neq 0$ (definition of rational number)

3. $\exists r, s \in \mathbb{Z}$ such that $a = \frac{r}{s}$, $s \neq 0$ (definition of rational number)

4. Thus, $b = \frac{p}{q} \cdot \frac{1}{a} = \frac{p}{q} \cdot \frac{s}{r} = \frac{ps}{qr}$ (by basic algebra)

5. ps is an integer by closure of integers under multiplication

6. qr is an integer by closure of integers under multiplication

7. If $\exists m, n \in \mathbb{Z}$ such that $b = \frac{m}{n}$ and $n \neq 0$, then b is a rational number (definition of rational number)

7.1 b is irrational, hence not a rational number

7.2 Then there does not exist $m, n \in \mathbb{Z}$ such that $b = \frac{m}{n}$ or $n = 0$ (contrapositive of 7)

7.3. There exists $m = ps$ and $n = qr$ integers such that $b = \frac{m}{n}$

7.4 Therefore, $n = 0$ (by elimination)

8. $n = qr = 0$

9. since $q \neq 0$ (step 2), $r = 0$ (multiplication by zero)

10. $a = \frac{r}{s}$ (step 3), $a = 0$ (divide 0 by integer)

11. Therefore, by contraposition, the original statement is True.

Q8. $\forall n \in \mathbb{Z}$ (n^2+n is even)

Proof (division into cases)

1. n is either odd or even

2. Case 1 : n is even

   2.1 Then $\exists k \in \mathbb{Z}$ such that $n = 2k$ (definition of even)

   2.2 Then $n^2 = 4k^2$ (by basic algebra)

   2.3 $n^2+n = 4k^2+2k$
   $= 2(2k^2+k)$ (by basic algebra)

   2.4 $2k^2+k$ is an integer by closure of integers under multiplication and addition

   2.5 $n^2+n = 2(2k^2+k)$ is even (definition of even)

3. Case 2 : n is odd

   3.1 Then $\exists m \in \mathbb{Z}$ such that $n = 2m+1$ (definition of odd)

   3.2 Then $n^2 = 4m^2+4m+1$ (by basic algebra)

   3.3 $n^2+n = 4m^2+4m+1+2m+1$
   $= 4m^2+6m+2$
   $= 2(2m^2+3m+1)$ (by basic algebra)

   3.4 $2m^2+3m+1$ is an integer by closure of integers under multiplication and addition

   3.5 $n^2+n = 2(2m^2+3m+1)$ is even (definition of even)

4. In all cases, n^2+n is even, therefore original statement is True //

Q9. Proof (by method of exhaustion)

$\Rightarrow$

$-2 \in A \to 4 \in B$

$-1 \in A \to 1 \in B$

$0 \in A \to 0 \in B$

$1 \in A \to 1 \in B$

$2 \in A \to 4 \in B$

$\Leftarrow$

$0 \in B \to 0 \in A$

$1 \in B \to 1 \in A$, $\to -1 \in A$

$4 \in B \to 2 \in A$, $\to -2 \in A$

statement is true in both directions via exhausting all elements in both sets.

Q10. a) $D_1 = \{1\}$ $D_5 = \{1, 5\}$

$D_2 = \{1, 2\}$ $D_6 = \{1, 2, 3, 6\}$

$D_3 = \{1, 3\}$ $D_7 = \{1, 7\}$

$D_4 = \{1, 2, 4\}$

b) $\bigcup_{k=1}^{7} D_k = \{1, 2, 3, 4, 5, 6, 7\}$

$\bigcap_{k=1}^{7} D_k = \{1\}$

c) $\{n \in \mathbb{Z}_{\geq 0} : n \in D_k \text{ for some } k \in \mathbb{Z}_{\geq 0}\} = \{0, 1, 2, \ldots\}$

$\{n \in \mathbb{Z}_{\geq 0} : n \in D_k \text{ for all } k \in \mathbb{Z}_{\geq 0}\} = \{1\}$

Q11. a) Proof (direct proof)

1. Suppose $P(A \cup B) \subseteq P(A) \cup P(B)$ for sets A, B
   1.1 Let $X = A \cup B$
   1.2 Then $X \in P(A \cup B)$ (by definition of power set)
   1.3 Then $X \in P(A) \cup P(B)$ as $P(A \cup B) \subseteq P(A) \cup P(B)$ (step 1)
   1.4 Then $X \in P(A)$ or $X \in P(B)$ (by definition of $\cup$)
   1.5 Then $X \subseteq A$ or $X \subseteq B$ (by definition of power set)
   1.6 Then $A \cup B \subseteq A$ or $A \cup B \subseteq B$ (from 1.1)
   1.7 ($A \cup B \subseteq A$)
      1.7.1 Let $z \in B$
      1.7.2 Then $z \in B$ or $z \in A$ (by definition of $\vee$)
      1.7.3 So $z \in A \cup B$ (by definition of $\cup$)
      1.7.4 This implies $z \in A$ as $A \cup B \subseteq A$ (from 1.7)
      1.7.5 Therefore $B \subseteq A$ (by definition of subset)
   1.8 ($A \cup B \subseteq B$)
      1.8.1 Let $z \in A$
      1.8.2 Then $z \in A$ or $z \in B$ (by definition of $\vee$)
      1.8.3 So $z \in A \cup B$ (by definition of $\cup$)
      1.8.4 This implies $z \in B$ as $A \cup B \subseteq B$ (from 1.8)
      1.8.5 Therefore $A \subseteq B$ (by definition of subset)

1.9 Lines 1.7.5 and 1.8.5 imply $A \subseteq B$ or $B \subseteq A$ (equivalents of 1.6)

2. Therefore, if $P(A \cup B) \subseteq P(A) \cup P(B)$, then either $A \subseteq B$ or $B \subseteq A$ //

b) Let $A = \{\diamondsuit\}$
$B = \{\heartsuit\}$

$P(\{\diamondsuit, \heartsuit\}) = \{\emptyset, \{\diamondsuit\}, \{\heartsuit\}, \{\diamondsuit, \heartsuit\}\}$

$P(\{\diamondsuit\}) = \{\emptyset, \{\diamondsuit\}\}$

$P(\{\heartsuit\}) = \{\emptyset, \{\heartsuit\}\}$

Therefore $P(A \cup B) \not\subseteq P(A) \cup P(B)$

For any sets A and B, the only sets in their powersets $P(A)$ or $P(B)$ are subsets of either A or B.

However, a set in $P(A \cup B)$ can contain elements from both A and B.

Hence, a choice of A and B where each A and B has 1 element not in the other set will work.

Actually, this is just the contrapositive of part a) $A \not\subseteq B$ and $B \not\subseteq A$