CS1231(S) Tutorial 3: Sets
Solutions

National University of Singapore

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Sometimes there is more than one correct answer.

1. Which of the following are true? Which of them are false?

   (a) $\varnothing \in \varnothing$.

   (b) $\varnothing \subseteq \varnothing$.

   (c) $\varnothing \in \{\varnothing\}$.

   (d) $\varnothing \subseteq \{\varnothing\}$.

   (e) $\{\varnothing, 1\} = \{1\}$.

   (f) $1 \in \{\{1, 2\}, \{2, 3\}, 4\}$.

   (g) $\{1, 2\} \subseteq \{3, 2, 1\}$.

   (h) $\{3, 3, 2\} \subsetneq \{3, 2, 1\}$.

   Solution. F, T, T, T, F, F, T, T.

2. Let $A = \{1, \{1, 2\}, 2, \{1, 2\}\}$. Find $|A|$.

   Solution. $|A| = 3$.

3. Let $A = \{0, 1, 4, 5, 6, 9\}$ and $B = \{0, 2, 4, 6, 8\}$. Find $|A|$, $|B|$, $|A \cap B|$, and $|A \cup B|$.

   Solution. Note that $A \cap B = \{0, 4, 6\}$ and $A \cup B = \{0, 1, 2, 4, 5, 6, 8, 9\}$. So

   $$|A| = 6, \quad |B| = 5, \quad |A \cap B| = 3, \quad \text{and} \quad |A \cup B| = 8.$$

4. Let $A = \{2n + 1 : n \in \mathbb{Z}\}$ and $B = \{2n - 1 : n \in \mathbb{Z}\}$. Is $A = B$? Prove that your answer is correct.

   Solution. Yes, as shown below.

   1. $(\subseteq)$
      1.1. Let $a \in A$.
      1.2. Use the definition of A to find $n \in \mathbb{Z}$ such that $a = 2n + 1$.
      1.3. Then $a = 2(n + 1) - 1$.
      1.4. As $n \in \mathbb{Z}$, we know $n + 1 \in \mathbb{Z}$.
      1.5. So $a \in B$ by the definition of B.
   2. $(\supseteq)$
      2.1. Let $b \in B$.
      2.2. Use the definition of B to find $n \in \mathbb{Z}$ such that $a = 2n - 1$.
      2.3. Then $b = 2(n - 1) + 1$.
      2.4. As $n \in \mathbb{Z}$, we know $n - 1 \in \mathbb{Z}$.
      2.5. So $b \in A$ by the definition of A.
   3. Hence $A = B$ by the definition of set equality. $\square$

5. Let $A = \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 5\}$ and $B = \{x \in \mathbb{R} : 2 \leqslant x \leqslant 5\}$. Is $A = B$? Prove that your answer is correct.

   Solution. No, as shown below.

1. $3.14 \in \mathbb{R}$ and $2 \leqslant 3.14 \leqslant 5$.
2. So $3.14 \in B$ by the definition of B.
3. $3.14 \notin \mathbb{Z}$.
4. So $3.14 \notin A$ by the definition of A.
5. Lines 2 and 4 imply $A \neq B$ by the definition of set equality.

6. Let $U = \{5, 6, 7, \dots, 12\}$ and $M_k = \{n \in \mathbb{Z} : n = km \text{ for some } m \in \mathbb{Z}\}$ for each $k \in \mathbb{Z}$. Find:

(a) $\{n \in U : n \text{ is even}\}$;
(b) $\{n \in U : n = m^2 \text{ for some } m \in \mathbb{Z}\}$;
(c) $\{-5, -4, -3, \dots, 5\} \setminus \{1, 2, 3, \dots, 10\}$;
(d) $\overline{\{5, 7, 9\} \cup \{9, 11\}}$, where U is considered the universal set;
(e) $\big\{(x, y) \in \{1, 3, 5\} \times \{2, 4\} : x + y \geqslant 6\big\}$;
(f) $\mathcal{P}(\{2, 4\})$.

Solution.

(a) $\{6, 8, 10, 12\}$.
(b) $\{9\}$.
(c) $\{-5, -4, -3, -2, -1, 0\}$.
(d) $\overline{\{5, 7, 9\} \cup \{9, 11\}} = \overline{\{5, 7, 9, 11\}} = \{6, 8, 10, 12\}$ when U is considered the universal set.
(e) $\{(3, 4), (5, 2), (5, 4)\}$.
(f) $\{\varnothing, \{2\}, \{4\}, \{2, 4\}\}$.

7. Show that for all sets A, B, C,

$$A \cap (B \setminus C) = (A \cap B) \setminus C.$$

Solution.

1.	$A \cap (B \setminus C) = \{x : x \in A \text{ and } x \in B \setminus C\}$	by the definition of $\cap$;
2.	$= \{x : x \in A \text{ and } (x \in B \text{ and } x \notin C)\}$	by the definition of $\setminus$;
3.	$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \notin C\}$	as "and" is associative;
4.	$= \{x : x \in A \cap B \text{ and } x \notin C\}$	by the definition of $\cap$;
5.	$= (A \cap B) \setminus C$	by the definition of $\setminus$. $\square$

8. (2009/10 Semester 2 exam question B) Prove that for all sets A and B,

$$(A \cup \overline{B}) \cap (\overline{A} \cup B) = (A \cap B) \cup (\overline{A} \cap \overline{B}).$$

Solution. (Note that we no longer need to apply the set identities as strictly as we did in the logic part of the module.)

1.	$(A \cup \overline{B}) \cap (\overline{A} \cup B)$	
2.	$= \big((A \cup \overline{B}) \cap \overline{A}\big) \cup \big((A \cup \overline{B}) \cap B\big)$	as $\cap$ distributes over $\cup$;
3.	$= \big((A \cap \overline{A}) \cup (\overline{B} \cap \overline{A})\big) \cup \big((A \cap B) \cup (\overline{B} \cap B)\big)$	as $\cap$ distributes over $\cup$;
4.	$= (\varnothing \cup (\overline{B} \cap \overline{A})) \cup ((A \cap B) \cup \varnothing)$	by the Complement Law;
5.	$= (\overline{B} \cap \overline{A}) \cup (A \cap B)$	by the Identity Law;
6.	$= (A \cap B) \cup (\overline{A} \cap \overline{B})$	by the Commutative Laws.

One may alternatively use the element method or the truth-table method.

9. Let A, B be sets. Show that $A \subseteq B$ if and only if $A \cup B = B$.

   Solution.

   1. ("Only if")
      1.1. Suppose $A \subseteq B$.
      1.2. ("$A \cup B \subseteq B$")
         1.2.1. Let $z \in A \cup B$.
         1.2.2. Then $z \in A$ or $z \in B$ by the definition of $\cup$.
         1.2.3. Case 1: suppose $z \in A$.
            1.2.3.1. Then $z \in B$ as $A \subseteq B$ from line 1.1.
         1.2.4. Case 2: suppose $z \in B$.
            1.2.4.1. Then $z \in B$.
         1.2.5. In either case, we have $z \in B$.
      1.3. ("$A \cup B \supseteq B$")
         1.3.1. Let $z \in B$.
         1.3.2. Then $z \in A$ or $z \in B$ by the definition of "or".
         1.3.3. So $z \in A \cup B$ by the definition of $\cup$.
      1.4. Lines 1.3 and 1.2 imply $A \cup B = B$ by the definition of set equality.
   2. ("If")
      2.1. Suppose $A \cup B = B$.
      2.2. We prove $A \subseteq B$ as follows.
         2.2.1. Let $z \in A$.
         2.2.2. Then $z \in A$ or $z \in B$ by the definition of "or".
         2.2.3. So $z \in A \cup B$ by the definition of $\cup$.
         2.2.4. This implies $z \in B$ as $A \cup B = B$ by line 2.1. $\square$

10. For sets A and B, define $A \oplus B = (A \setminus B) \cup (B \setminus A)$.

    (a) Let $A = \{1, 4, 9, 16\}$ and $B = \{2, 4, 6, 8, 10, 12, 14, 16\}$. Find $A \oplus B$.

    (b) Show that for all sets A, B,

    $$A \oplus B = (A \cup B) \setminus (A \cap B).$$

    Solution.

    (a) $A \setminus B = \{1, 9\}$ and $B \setminus A = \{2, 6, 8, 10, 12, 14\}$. So $A \oplus B = \{1, 2, 6, 8, 9, 10, 12, 14\}$.

    (b) Compare the following truth tables.

$z \in A$	$z \in B$	$z \in A \setminus B$	$z \in B \setminus A$	$z \in A \oplus B$
T	T	F	F	F
T	F	T	F	T
F	T	F	T	T
F	F	F	F	F

$z \in A$	$z \in B$	$z \in A \cup B$	$z \in A \cap B$	$z \in (A \cup B) \setminus (A \cap B)$
T	T	T	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

    Since the last columns of the two tables are the same, we conclude that $A \oplus B = (A \cup B) \setminus (A \cap B)$. $\square$

    Instead of the truth tables above, one may prove this using the set identities. Here U denotes the universal set.

1. $A \oplus B$
2. $= (A \setminus B) \cup (B \setminus A)$ by the definition of $\oplus$;
3. $= (A \cap \overline{B}) \cup (B \cap \overline{A})$ by the Set Difference Law;
4. $= \big((A \cap \overline{B}) \cup B\big) \cap \big((A \cap \overline{B}) \cup \overline{A}\big)$ by the Distributive Law;
5. $= (A \cup B) \cap (\overline{B} \cup B) \cap (A \cup \overline{A}) \cap (\overline{B} \cup \overline{A})$ by the Distributive Law;
6. $= (A \cup B) \cap U \cap U \cap (\overline{B} \cup \overline{A})$ by the Complement Law;
7. $= (A \cup B) \cap U \cap U \cap (\overline{B \cap A})$ by De Morgan's Law;
8. $= (A \cup B) \cap (\overline{B \cap A})$ by the Identity Law;
9. $= (A \cup B) \cap (\overline{A \cap B})$ by the Commutative Law;
10. $= (A \cup B) \setminus (A \cap B)$ by the Set Difference Law. □

Alternatively, one may also use the element method.

11. (2015/16 Semester 1 exam question 16(a)) Denote by $|x|$ the absolute value of the integer x, i.e.,

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0; \\ -x, & \text{if } x < 0. \end{cases}$$

Given the set $S = \{-9, -6, -1, 3, 5, 8\}$, for each of the following statements, state whether it is true or false, with explanation.

(a) $\exists z \in S\ \ \forall x, y \in S\ \ z > |x - y|$.

(b) $\exists z \in S\ \ \forall x, y \in S\ \ z < |x - y|$.

Solution.

(a) This statement is false, as shown below.
   1. It suffices to show $\forall z \in S\ \ \exists x, y \in S\ \ z \leqslant |x - y|$.
   2. Take any $z \in S$.
   3. Let $x = 8$ and $y = -9$.
   4. Then $x, y \in S$ and $|x - y| = |8 - (-9)| = 17 > 8 = \max S \geqslant z$. □

(b) This statement is true, as shown below.
   1. $-1 \in S$.
   2. $|x - y| \geqslant 0 > -1$ for all $x, y \in S$. □

12. For sets $A_m, A_{m+1}, \ldots, A_n$, define

$$\bigcup_{i=m}^{n} A_i = A_m \cup A_{m+1} \cup \cdots \cup A_n \quad \text{and} \quad \bigcap_{i=m}^{n} A_i = A_m \cap A_{m+1} \cap \cdots \cap A_n.$$

(a) Let $A_i = \{x \in \mathbb{Z} : x \geqslant i\}$ for each $i \in \mathbb{Z}$. Write down $\bigcup_{i=2}^{5} A_i$ and $\bigcap_{i=2}^{5} A_i$ in roster notation.

(b) Let $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_\ell$ be sets such that

$$\bigcup_{i=1}^{k} B_i \subseteq \bigcap_{j=1}^{\ell} C_j.$$

Show that $B_i \subseteq C_j$ for all $i \in \{1, 2, \ldots, k\}$ and all $j \in \{1, 2, \ldots, \ell\}$.

Solution.

(a) $\bigcup_{i=2}^{5} A_i = \{2, 3, 4, \ldots\}$ and $\bigcap_{i=2}^{5} A_i = \{5, 6, 7, \ldots\}$.

(b) 1. Let $B_1, B_2, \ldots, B_k, C_1, C_2, \ldots, C_\ell$ be sets such that $\bigcup_{i=1}^k B_i \subseteq \bigcap_{j=1}^\ell C_j$.

2. 2.1. Let $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, \ell\}$.
   2.2. Take any $z \in B_i$.
   2.3. Then $z \in B_1$ or $z \in B_2$ or $\ldots$ or $z \in B_k$ by the definition of "or", as $i \in \{1, 2, \ldots, k\}$.
   2.4. So $z \in B_1 \cup B_2 \cup \cdots \cup B_k = \bigcup_{i=1}^k B_i$ by the definition of $\cup$ and $\bigcup$.
   2.5. Hence $z \in \bigcap_{j=1}^\ell C_j = C_1 \cap C_2 \cap \cdots \cap C_\ell$ as $\bigcup_{i=1}^k B_i \subseteq \bigcap_{j=1}^\ell C_j$ by line 1.
   2.6. Thus $z \in C_1$ and $z \in C_2$ and $\ldots$ and $z \in C_\ell$ by the definition of $\cap$.
   2.7. In particular, we know $z \in C_j$ as $j \in \{1, 2, \ldots, \ell\}$.

3. So $B_i \subseteq C_j$ for any $i \in \{1, 2, \ldots, k\}$ and any $j \in \{1, 2, \ldots, \ell\}$. $\square$