CHAPTER 3 INDUCTION

SECTION 3.1 MATHEMATICAL INDUCTION

Mathematical induction is used to prove statements that asserts that

$P(n)$ is true for all $n \in \mathbb{Z}^+$ where $P(n)$ is a propositional function.

It is an extremely important proof technique.

PRINCIPLE OF MATHEMATICAL INDUCTION

To prove $\forall n \in \mathbb{Z}^+(P(n))$ where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: Verify that $P(1)$ is true.

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To prove $\forall n \in \mathbb{Z}^+(P(n))$ where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: Verify that $P(1)$ is true.

INDUCTIVE STEP: Show that

$$\forall k \in \mathbb{Z}^+(P(k) \to P(k+1))$$

is true.

To complete the inductive step, we assume that $P(k)$ is true, (known as the **INDUCTION HYPOTHESIS**), and

prove that $P(k+1)$ is true.

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prove that $P(k+1)$ is true.

(It may seem circular and thus requires some clarification. We are not asserting that $P(k)$ is true for all k here.

What we are saying is that under the hypothesis that $P(k)$ is true,

we can prove that $P(k+1)$ is true.)

What we do here is the following.

$$\begin{aligned} P(1) &\quad \text{(Base step)} \\ P(1) \Rightarrow P(2) & \\ P(2) \Rightarrow P(3) & \\ P(3) \Rightarrow P(4) & \\ \cdots & \end{aligned}$$

Eventually, we get $P(5), P(6), \ldots$.

$\boxed{\textbf{EXAMPLE}}$

- Prove that $\forall n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

PROOF: Let $P(n)$ be

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Base step: P(1) is true since

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$$

Inductive step: Assume that $P(k)$ is true, where $k \geq 1$, i.e.,

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

Then $P(k+1)$ is true since

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$$

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\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^{k} i\right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Then $P(k+1)$ is true since

$$\begin{aligned}\sum_{i=1}^{k+1} i &= \Big(\sum_{i=1}^{k} i\Big) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)((k+1)+1)}{2}.\end{aligned}$$

Thus $P(n)$ is true for all $n \in \mathbb{Z}^+$ by mathematical induction.

- Prove that $n < 2^n$ for all $n \in \mathbb{Z}^+$.

PROOF: Let $P(n)$ be the proposition that $n < 2^n$.

Base step: $P(1)$ is true since $1 < 2^1$.

Inductive step: Assume that $P(k)$ is true. From $P(k)$ we have

$$k < 2^k.$$

Add 1 to both sides and we have

$$k + 1 < 2^k + 1$$

Inductive step: Assume that $P(k)$ is true. From $P(k)$ we have

$$k < 2^k.$$

Add 1 to both sides and we have

$$\begin{aligned} k + 1 &< 2^k + 1 \\ &< 2^k + 2^k = 2^{k+1} \end{aligned}$$

and hence $P(k+1)$ is true.

Therefore by mathematical induction $n < 2^n$ for all $n \in \mathbb{Z}^+$.

- The **HARMONIC NUMBERS** H_j, $j \in \mathbb{Z}^+$, are defined by

$$H_j = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{j}.$$

Prove that

$$H_{2^n} \geq 1 + \frac{n}{2} \quad \text{for all } n \in \mathbb{Z}_{\geq 0}.$$

PROOF: Let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

Base step: $P(0)$ is true since $H_{2^0} = \frac{1}{1} \geq 1 + \frac{0}{2}$.

Inductive step: Assume that $P(k)$ is true. Then

$$\begin{aligned}
H_{2^{k+1}} = &\left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^k}\right) \\
&+ \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\right)
\end{aligned}$$

Inductive step: Assume that $P(k)$ is true. Then

$$
\begin{aligned}
H_{2^{k+1}} &= \Big(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^k}\Big) \\
&\qquad\qquad + \Big(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\Big) \\
&= H_{2^k} + \Big(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\Big)
\end{aligned}
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Inductive step: Assume that $P(k)$ is true. Then

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&\qquad + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\right) \\
&= H_{2^k} + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\right) \\
&\geq \left(1 + \frac{k}{2}\right) + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\right)
\end{aligned}$$

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&= H_{2^k} + \Big(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\Big) \\
&\geq \Big(1 + \frac{k}{2}\Big) + \Big(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}\Big) \\
&\geq \Big(1 + \frac{k}{2}\Big) + \Big(\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}\Big) \\
&= \Big(1 + \frac{k}{2}\Big) + 2^k \cdot \frac{1}{2^{k+1}}
\end{aligned}$$

Inductive step: Assume that $P(k)$ is true. Then

$$\begin{aligned}
H_{2^{k+1}} &= \Big(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^k}\Big) \\
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&= \Big(1 + \frac{k}{2}\Big) + 2^k \cdot \frac{1}{2^{k+1}} \\
&= 1 + \frac{k+1}{2}
\end{aligned}$$

Thus $P(k+1)$ is true and the result follows by mathematical induction.

THEOREM:

NUMBER OF SUBSETS OF A FINITE SET

A set with n elements has 2^n subsets.

PROOF: Let $Q(n)$ be the above proposition.

Base step: When $n = 0$, the set concerned is $\emptyset$ which has only one subset.

Thus $Q(0)$ is true.

Inductive step: Assume that $Q(0), \ldots, Q(k)$ are true.

Let X be any set with $k+1$ elements.

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Let X be any set with $k+1$ elements.

Take a particular element $a \in X$.

Then $Y = X - \{a\}$ is a set with k elements. By the induction hypothesis,

$$|P(Y)| = 2^k.$$

Subsets of X can be divided into two types:

(i) Those that do not contain a.

These are precisely the subsets of Y and there are 2^k subsets of this type.

(ii) Those that contain a.

If the element a is deleted, they become subsets of Y. Thus each corresponds to a subset of Y.

Therefore there are also 2^k subsets of this type.

Thus
$$|P(X)| = 2^k + 2^k = 2^{k+1}.$$
Hence $Q(k+1)$ is true.

The result then follows by the principle of mathematical induction.

SUM OF GP

For all integers $n \in \mathbb{Z}_{\geq 0}$, and all real numbers $r \neq 1$:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

PROOF: When $n = 0$, l.h.s $= 1$ and r.h.s $= \frac{r-1}{r-1} = 1$. Thus the formula is true when $n = 0$.

Assume that the formula is true for $n = 0, 1, \ldots, k$. Thus $\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1}$. Then

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1} = \frac{r^{k+1} - 1}{r - 1} + r^{k+1} = \frac{r^{k+2} - 1}{r - 1}.$$

Thus the formula is also true at $k + 1$.

By the principle of mathematical, the formula is true.

- Prove that for any integer $n \geq 1$, if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be covered by an L-tromino.

(An L-tromino is an L-shape formed by 3 squares of the checkerboard.)

SOLN: Let $P(n)$ be the given statement.

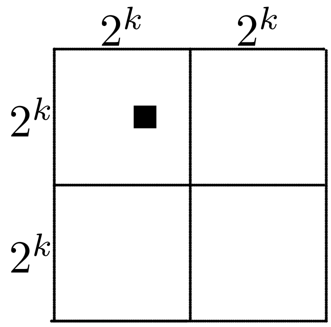
Basis step. $P(1)$ is true since the board is itself an L-tromino.

Assume that $P(k)$ is true.

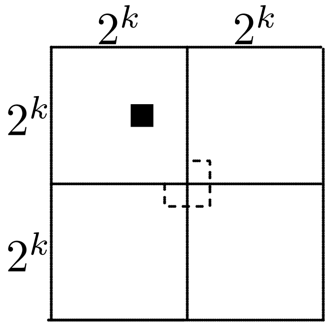
Consider a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed.

Divide the checkerboard in 4 equal quadrants so that each quadrant is a $2^k \times 2^k$ board.

Without loss of generality, assume that the removed square is from the first quadrant.



Now remove a tromino from the centre of the board.



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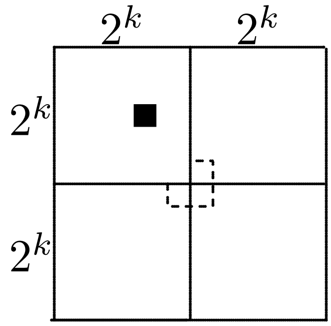

Now we are left with four $2^k \times 2^k$ checkerboards, each with a square removed.

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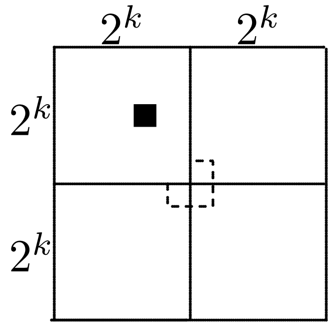

Now we are left with four $2^k \times 2^k$ checkerboards, each with a square removed. Thus by the induction hypothesis, each can be covered by trominoes.

Hence the $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed can be so covered as well.

SECTION 3.2 STRONG MATHEMATICALINDUCTION

STRONG MATHEMATICAL INDUCTION

To prove $\forall n \in \mathbb{Z}^+(P(n))$ where $P(n)$ is a propositional function, we complete two steps:

BASE STEP:

Verify that $P(1), ..., P(m)$ are all true. (i.e. for the first few values of n, $P(n)$ is true.)

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BASE STEP:

Verify that $P(1), ..., P(m)$ are all true. (i.e. for the first few values of n, $P(n)$ is true.)

INDUCTIVE STEP:

Show that $\forall k \geq m(P(1) \wedge P(2) \wedge \cdots \wedge P(k) \to P(k+1))$ is true.

To complete the inductive step, we assume that

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It may seem circular and thus requires some clarification. We are not asserting that $P(k)$ is true for all k here. What we are saying is that under the hypothesis that $P(1), \ldots, P(k)$ are true, we can prove that $P(k+1)$ is true.

What we do here is the following.

$$\begin{aligned} P(1), P(2) &\quad \text{(Base step when } m = 2\text{)} \\ P(1) \wedge P(2) \Rightarrow P(3) & \\ P(1) \wedge P(2) \wedge P(3) \Rightarrow P(4) & \\ \cdots & \end{aligned}$$

Eventually, we get $P(5), P(6), \ldots$.

REMARK

Difference between proving by normal induction and by strong induction:

• For normal induction, only $P(k)$ is assumed when proving $P(k+1)$.

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• For normal induction, only $P(k)$ is assumed when proving $P(k+1)$.

• For strong induction, we may assume $P(1), \ldots, P(k)$ when proving $P(k+1)$.

• Usually proving by strong induction is easier, since we can assume more information when trying to prove $P(k+1)$ in the inductive step.

EXAMPLE

- Suppose that $h_0, h_1, \ldots$ is a sequence defined as follows:

$$h_0 = 1, h_1 = 2, h_2 = 3$$

and

$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \quad \text{for} \quad k \geq 3$$

Prove that $h_n \leq 3^n$ for all $n \geq 0$.

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PROOF: Let $P(n)$ be the proposition that $h_n \leq 3^n$.

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PROOF: Let $P(n)$ be the proposition that $h_n \leq 3^n$.

Base step: Note that $h_n \leq 3^n$ for $n = 0, 1, 2$.

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Inductive step: Now assume that it's true for all $n = 0, 1, 2, \ldots k$, where $k \geq 2$. Then

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Base step: Note that $h_n \leq 3^n$ for $n = 0, 1, 2$.

Inductive step: Now assume that it's true for all $n = 0, 1, 2, \ldots k$, where $k \geq 2$. Then

$$\begin{aligned} h_{k+1} &= h_k + h_{k-1} + h_{k-2} \\ &\leq 3^k + 3^{k-1} + 3^{k-2} \\ &\leq 3 \times 3^k \\ &= 3^{k+1}. \end{aligned}$$

Hence the result holds for $n = k + 1$ and the proof is complete.

- **FIBONACCI NUMBERS** $F_0, F_1, \ldots$ are defined by

$$F_0 = 0, F_1 = 1$$
$$F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 1$$

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5, \ldots$

- Prove that for $n \geq 3$,

$$F_n > \alpha^{n-2} \quad \text{where } \alpha = (1+\sqrt{5})/2$$

SOLN: Let $P(n)$ be $F_n > \alpha^{n-2}$, $n \geq 3$.

Base step:

Since $F_3 = 2 > \alpha$, $F_4 = 3 > \alpha^2$,

$P(3), P(4)$ are true.

Induction step:

Suppose $P(n)$ is true, i.e., $F_n > \alpha^{n-2}$ for $n = 3, \ldots, k$.

First note that

$$\alpha^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{3+\sqrt{5}}{2} = 1 + \alpha.$$

Now

$$\begin{aligned} F_{k+1} &= F_k + F_{k-1} \\ &> \alpha^{k-2} + \alpha^{k-3} \\ &= \alpha^{k-3}(\alpha + 1) = \alpha^{k-1} \end{aligned}$$

Hence $P(k+1)$ is true.

THEOREM: (Well-Ordering Principle)

Every nonempty subset of $\mathbb{Z}_{\geq 0}$ has a least element.

REMARK

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• The set $(0, 1)$ does not have a least element. For if $a \in (0, 1)$ is a least element, then $a/2 \in (0, 1)$ and is $< a$. This gives a contradiction.

PROOF:

The theorem says

$$\forall X \subseteq \mathbb{Z}_{\geq 0}, \quad X \text{ is non-empty} \Rightarrow X \text{ has a least element.}$$

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Its contrapositive is

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Assume $X \subseteq \mathbb{Z}_{\geq 0}$ has no least element. We want to prove that $\forall n \in \mathbb{Z}^+, n \notin X$.

Let $P(n)$ be $n \notin X$.

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Base step: $0 \notin X$. Otherwise, 0 would be the least element of X, contradicting the assumption "X has no least element". Therefore, $P(0)$ is true.

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Inductive step: Suppose $P(0), \ldots, P(k)$ are all true. That is, $j \notin X$ for all $j = 0, ..., k$.

Then we must have $k+1 \notin X$. Otherwise, $k+1$ would be the least element of X, contradicting the assumption "X has no least element". Therefore, $P(k+1)$ is true.

By strong mathematical induction, $\forall n, P(n)$ (i.e. $\forall n, n \notin X$). Hence, X is empty.

SECTION 3.3 RECURSIVELY DEFINED SEQUENCES

Consider the following sequences:

- $2,\ 9,\ 16,\ 23,\ 30, \ldots$
- $1,\ 2,\ 4,\ 8,\ 16, \ldots$
- $2,\ 3,\ 6,\ 18,\ 108, \ldots$

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Observe that if the sequence is denoted $a_1, a_2, a_3, \ldots$, then three sequences above satisfies:

- $a_{n+1} = a_n + 7$ for all $n \in \mathbb{Z}^+$;

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- $a_{n+1} = a_n + 7$ for all $n \in \mathbb{Z}^+$;
- $a_{n+1} = 2a_n$ for all $n \in \mathbb{Z}^+$;
- $a_{n+1} = a_n \times a_{n-1}$ for all $n \in \mathbb{Z}_{\geq 2}$.

These are **recursively defined** sequences, where, other than the first few terms, each successive term depends on the previous terms in such a sequence.

EXAMPLE

- The Fibonacci sequence $F_0, F_1, F_2, F_3, \ldots$ is defined recursively.

Sets can also be defined recursively.

$\boxed{\textbf{EXAMPLE}}$

• The set E of all positive even integers can be defined recursively as follows.

Base step $2 \in E$.

Recursive step If $x \in E$, then $x + 2 \in E$.

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The next step yields $3 + 6 = 9 \in S$ and $6 + 6 = 12 \in S$.

Let A be the set of multiples of 3.

Claim: We want to prove that $A = S$.

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Then $3(k+1) = 3k + 3 \in S$ since $3k \in S$ and $3 \in S$. Thus $P(k+1)$ is true as well.

By mathematical induction, $3n \in S$ for all n.

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This is true since x and y being multiply of 3 implies that $x+y$ is a multiple of 3.

Existence and Uniqueness of Recursively defined Sequences

THEOREM:

Let $m \in \mathbb{Z}^+$, $a_0, a_1, \ldots, a_{m-1} \in A$ and $f : A^m \to A$ be a function. Then there is a unique infinite sequence $x_0, x_1, \ldots$ defined by

BASE STEP $x_0 = a_0, x_1 = a_1, \ldots x_{m-1} = a_{m-1}$

INDUCTIVE STEP $x_n = f(x_{n-1}, x_{n-2}, \ldots, x_{n-m})$, for all $n \geq m$.

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$n_1 \neq 0, \ldots m-1$, because $x_0 = a_0, x_1 = a_1, \ldots x_{m-1} = a_{m-1}$ are defined in the base step. Therefore, $n_1 \geq m$.

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