CS1231/CS1231S: Discrete Structures

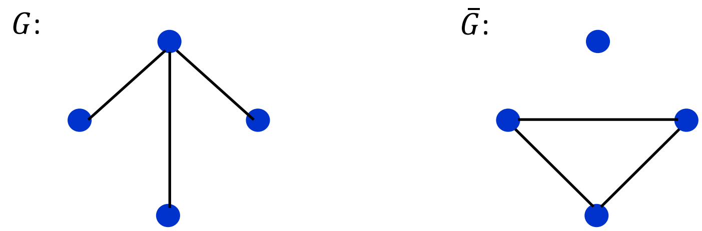
Tutorial #11: Graphs and Trees

Answers

II. Definitions

Definition 1. If G is a simple graph, the *complement* of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

Definition 2. A *self-complementary* graph is isomorphic with its complement.

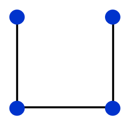
Definition 3. A simple circuit (cycle) of length three is called a *triangle*.

III. Tutorial Questions

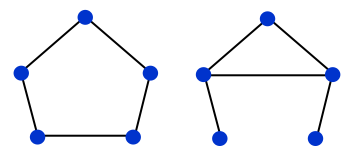
1. Draw all self-complementary graphs with (a) four vertices; (b) five vertices.

Answers:

(a)



(b)



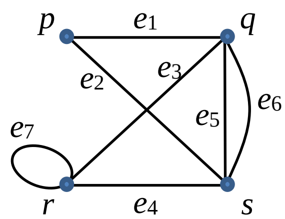
2. Show that every simple graph with at least two vertices has two vertices of the same degree.

   (This is similar to the popular puzzle: "Prove that at a party with at least two persons, there are two people who know the same number of people".)

Answer:

1. Take any simple graph G with n ($n \geq 2$) vertices.
2. Case 1: If G has no vertex with degree $n - 1$, then
   - 2.1 The vertex degrees in G lie in the range $[0..n-2]$.
3. Case 2: If G has a vertex v with degree $n - 1$, then
   - 3.1 Firstly v does not have degree 0 (since $n - 1 > 0$).
   - 3.2 Also, as v is connected to every other vertex, no other vertex can have degree 0 too.
   - 3.3 Hence, the vertex degrees in G lie in the range $[1..n-1]$.
4. In all cases, there are at most $n - 1$ possible vertex degrees in G which has n vertices.
5. Therefore, at least two vertices in G must have the same degree (by the Pigeonhole Principle).

3. Given the graph shown below:



(a) Write the adjacency matrix A for the graph. Let the rows and columns be p, q, r and s.

(b) Find A^2 and A^3.

(c) How many walks of length 2 are there from p to q? From s to itself? List out all the walks.

(d) How many walks of length 3 are there from r to s? From s to p? List out all the walks.

Answers:

(a)
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

(b)
$$A^2 = \begin{bmatrix} 2 & \mathbf{2} & 2 & 2 \\ 2 & 6 & 3 & 2 \\ 2 & 3 & 3 & 3 \\ 2 & 2 & 3 & \mathbf{6} \end{bmatrix} \qquad A^3 = \begin{bmatrix} 4 & 8 & 6 & 8 \\ 8 & 9 & 11 & 17 \\ 6 & 11 & 9 & \mathbf{11} \\ \mathbf{8} & 17 & 11 & 9 \end{bmatrix}$$

(c) There are **2** walks of length 2 from p to q: $< pe_2se_5q >$ and $< pe_2se_6q >$.

There are **6** walks of length 2 from s to itself:
$< se_2pe_2s >, < se_4re_4s >, < se_5qe_5s >, < se_6qe_6s >, < se_5qe_6s >, < se_6qe_5s >$.

(d) There are **11** walks of length 3 from r to s:

2 ways via q: $< re_3qe_1pe_2s >, < re_3qe_3re_4s >$.

3 ways via r: $< re_7re_7re_4s >, < re_7re_3qe_5s >, < re_7re_3qe_6s >$.

6 ways via s: $< re_4se_2pe_2s >, < re_4se_4re_4s >, < re_4se_5qe_5s >,$
$< re_4se_6qe_6s >, < re_4se_5qe_6s >, < re_4se_6qe_5s >$.

There are **8** walks of length 3 from s to p:

2 ways via p: $< se_2pe_2se_2p >, < se_2pe_1qe_1p >$.

4 ways via q:
$< se_5qe_5se_2p >, < se_5qe_6se_2p >, < se_6qe_5se_2p >, < se_6qe_6se_2p >$.

2 ways via r: $< se_4re_4se_2p >, < se_4re_3qe_1p >$.

4. Prove that for any simple graph G with six vertices, G or its complementary graph $\bar{G}$ contains a triangle.

Answer:

1. Take any simple graph G with 6 vertices. (Figure 4a shows an example.)

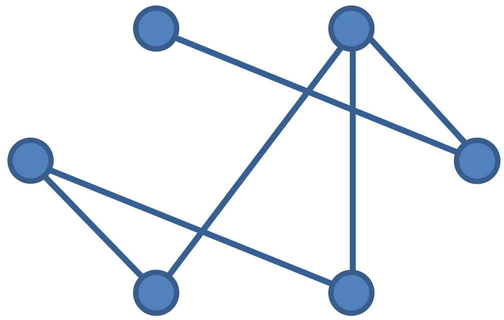


Figure 4a

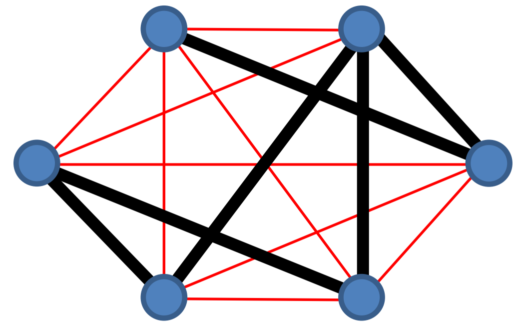


Figure 4b

2. Draw a black edge between adjacent vertices in G and draw a red edge between non-adjacent vertices in G.

   (In Figure 4b, the thick black lines are the original edges in G while the thin red lines are the edges in $\bar{G}$.)

3. Call this graph G'. Now G' is a complete graph with every edge either black or red, and we want to prove that it has a black triangle or a red triangle.

4. Let v be an arbitrary vertex of G'.

   4.1 There are 5 edges incident to v, which are either black or red.

   4.2 Therefore, (at least) 3 of these 5 edges are of the same colour c, by the Generalized Pigeonhole Principle. (In Figure 4c, 3 edges incident to v are red.)

   4.3 For these 3 edges that are of the same colour c, call the vertices at the other end of these edges u_1, u_2 and u_3.

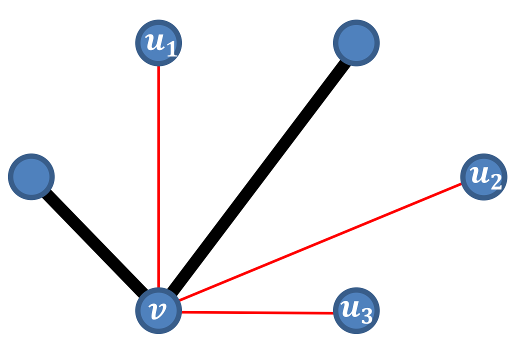


Figure 4c

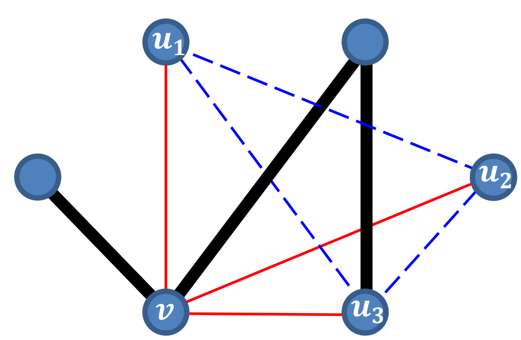


Figure 4d

4.4 Case 1: If there is an edge of colour c between any two of u_1, u_2 and u_3, then that edge forms a triangle of colour c with the two edges coming from v.

In Figure 4d, the 3 dashed blue lines are the edges between u_1, u_2 and u_3. In this example, two of them – (u_1, u_3) and (u_2, u_3) – are of colour c (red). We need just one of them to be of colour c; let's pick (u_1, u_3). This means there is a triangle of colour c with v, u_1 and u_3.

4.5 Case 2: If there are no edges of colour c between any pair of u_1, u_2 and u_3, then the edges between these 3 vertices form a triangle of colour opposite to c.

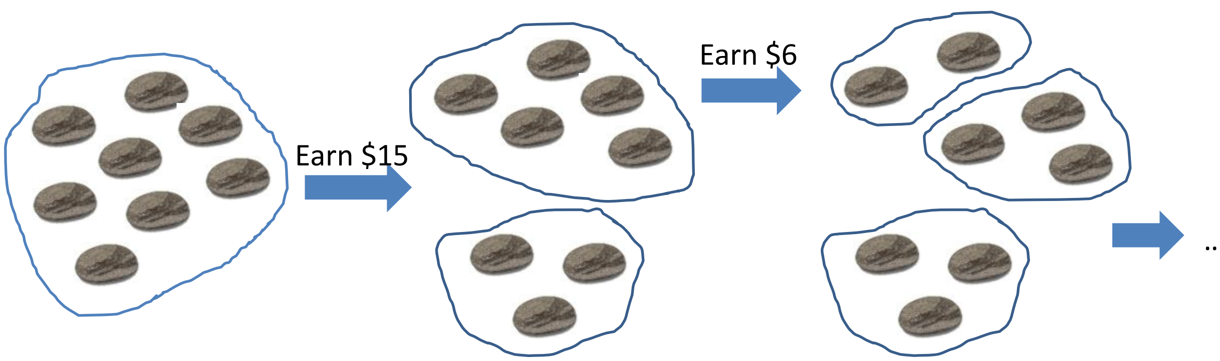
5. In all cases, there is a triangle of the same colour.

(You may look up Ramsey's theorem for the generalization of this problem, though it is beyond the scope of CS1231/CS1231S.)

5. (AY2017/18 Semester 1 Exam Question)
Suppose you are given a pile of stones. At each step, you can separate a pile of k stones into two piles of k_1 and k_2 stones. (Obviously, $k_1 + k_2 = k$.) On doing this, you earn \$$(k_1 \times k_2)$.

What is the maximum amount of money you can earn at the end if you start with a pile of n stones? Explain your answer.

The diagram below illustrates the (incomplete) process of separating a pile of 8 stones.



Answer:

One can earn at most \$$n(n-1)/2$.

Represent each stone as a vertex in a graph. Two vertices are connected with an edge if the stones they represent are in the same pile. In the beginning, we have a graph of n vertices with $n(n-1)/2$ edges as it is a K_n complete graph.

Separating a pile of k stones into two piles of k_1 and k_2 stones corresponds to removing some edges from the graph. The number of edges removed is exactly $k_1 \times k_2$.

The maximum amount of money one can earn is when the stones are separated into piles with single stone, that is, when all the edges are removed. Since the number of edges in a complete graph K_n is $n(n-1)/2$, this is the maximum amount one can earn.

6. How many edges are there in a forest with v vertices and k components?

Answer:

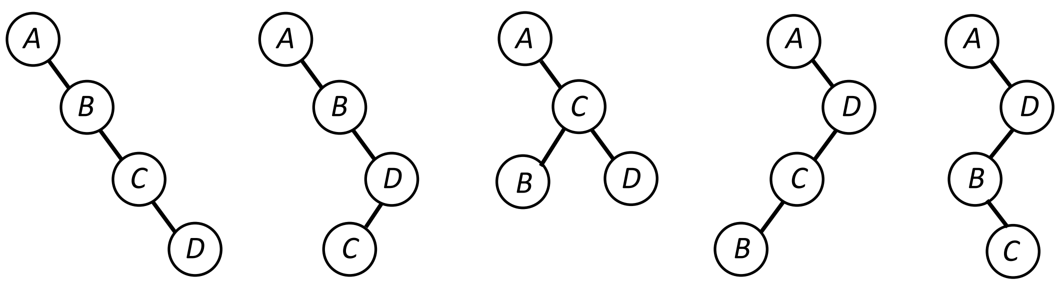
1. Let the number of vertices in the i^{th} component of the forest be v_i.
2. Each component of a forest is a tree, so the number of edges in the i^{th} component is $v_i - 1$, by Theorem 10.5.2.
3. Hence, the total number of edges is

$$\sum_{i=1}^{k}(v_i - 1) = \sum_{i=1}^{k} v_i - \sum_{i=1}^{k} 1 = v - k.$$

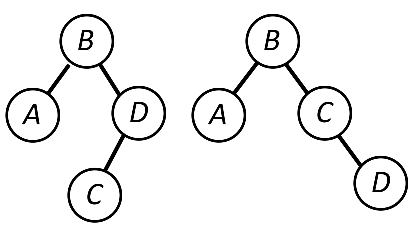
7. How many possible binary trees with 4 vertices *A*, *B*, *C* and *D* have this in-order traversal: *A B C D*? Draw them.

Answer: 14 possible binary trees.

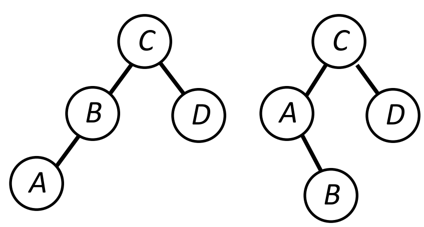
Rooted at *A*: 5



Rooted at *B*: 2



Rooted at *C*: 2

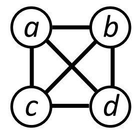


Rooted at *D*: 5 (not shown)

8. Consider a simple connected graph G with n vertices.

(a) If G is a complete graph, the total number of spanning trees in G is n^{n-2}. (You may check out *Cayley's formula*, though it is not in the scope of CS1231/CS1231S.)

Show all the spanning trees in the complete graph K_4 below:



(b) If G is not a complete graph, we may follow the following steps to calculate the number of spanning trees in G:

1. Let A be the adjacency matrix of $G = (V, E)$ where $V = \{v_1, v_2, \cdots, v_n\}$.
2. Let D be the diagonal matrix with its diagonal elements d_{ii} = degree of v_i, for $1 \le i \le n$.
3. Let $M = D - A$.
4. Let M' be the submatrix obtained by removing row i and column i from M, where $1 \le i \le n$.
5. Number of spanning trees is the determinant of M', denoted by det(M') or $|M'|$.

The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of a 3×3 matrix is

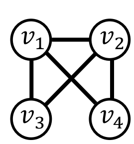
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

The determinant of a 4×4 matrix is

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}.$$

(You may generalize it to $n \times n$ matrix. See Leibnix formula.)

For example, given this graph G:



The matrixes A, D, M and M' (by removing row 1 and column 1 from M), are shown below:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix};\ D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix};\ M = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

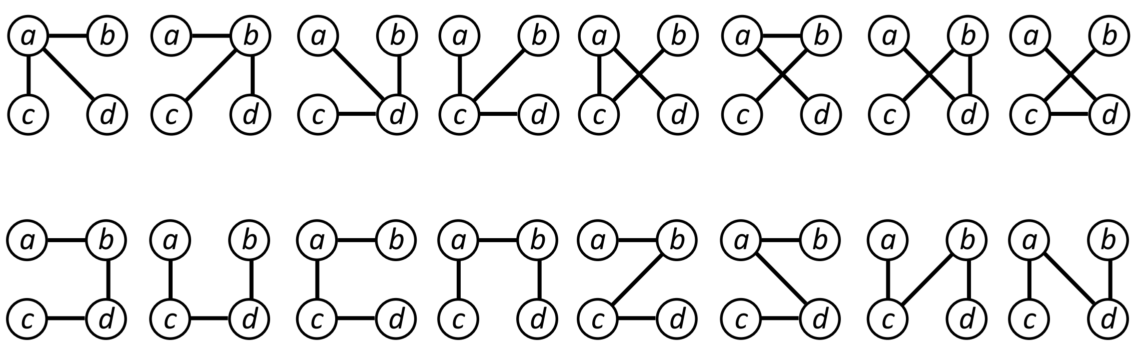
$$M' = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}; \quad \det(M') = 3(4) + 1(-2) - 1(2) = 8.$$

Therefore, there are 8 spanning trees in G.

Using the above method, verify that there are 125 spanning trees in K_5, which is 5^{5-2}.

Answer:

(a) There are $4^{4-2} = 16$ spanning trees in K_4.



(b) Matrix M' for K_5:

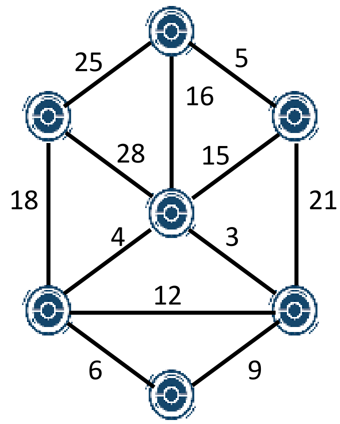
$$M' = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$$

$$\det(M') = 4\begin{vmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{vmatrix} + \begin{vmatrix} -1 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{vmatrix} - \begin{vmatrix} -1 & 4 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & 4 \end{vmatrix} + \begin{vmatrix} -1 & 4 & -1 \\ -1 & -1 & 4 \\ -1 & -1 & -1 \end{vmatrix}$$

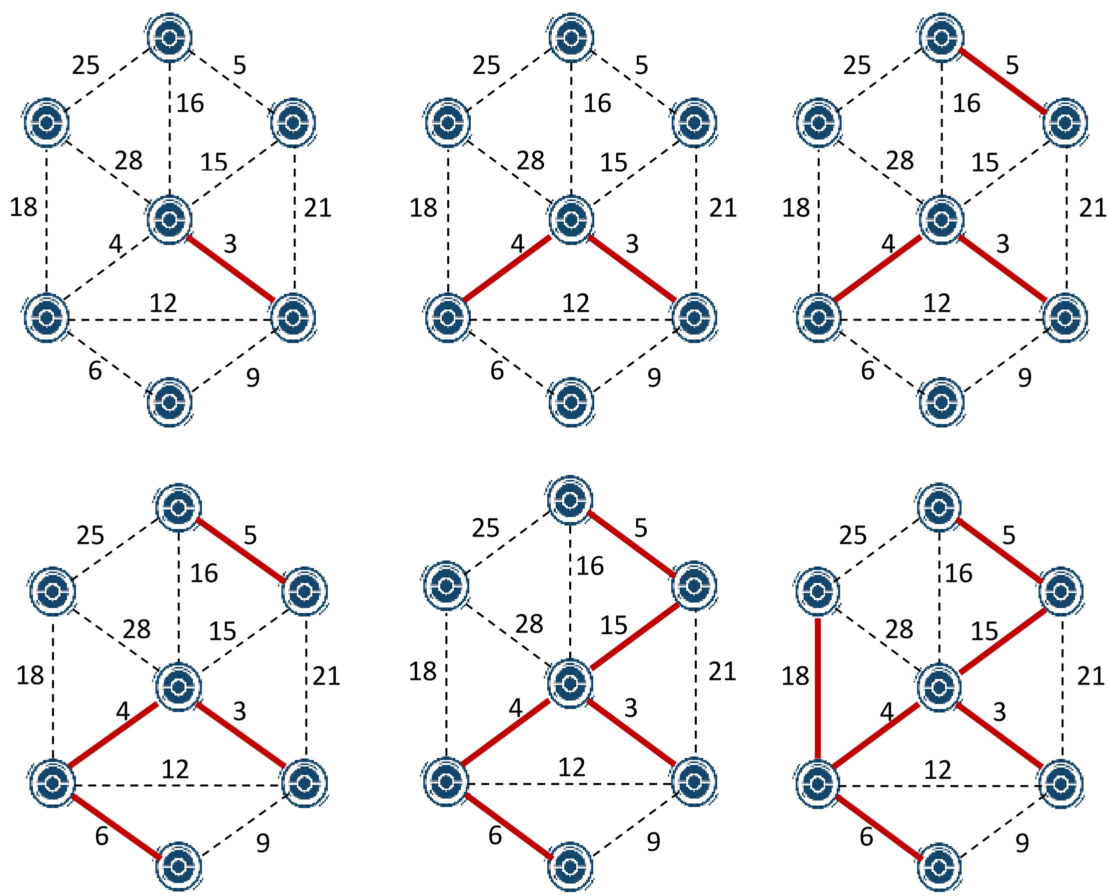
$$= 4(50) + (-25) - (25) + (-25) = 125.$$

9. (AY2016/17 Semester 1 Exam Question)
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

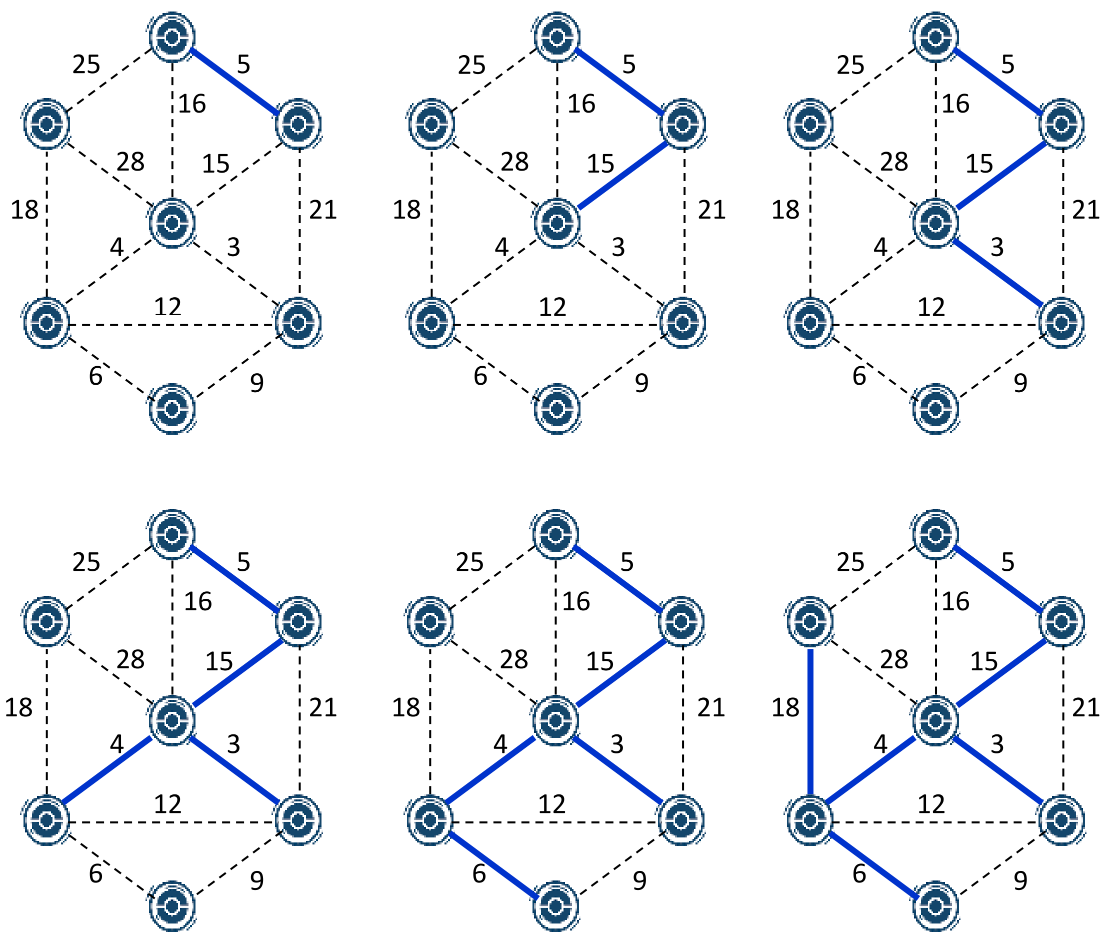
Indicate the order of the edges inserted into the MST in your answer.



Answer:
Kruskal's algorithm



Prim’s algorithm



10. Construct the binary tree given the following in-order and pre-order traversals of the tree:

In-order: I A D J N H B E K O F L G C M

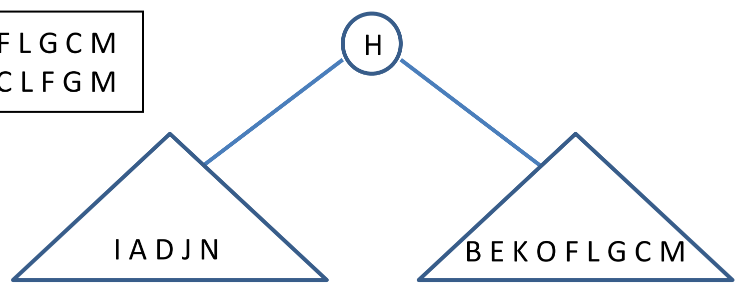
Pre-order: H N A I J D O B K E C L F G M

Draw diagrams to trace the steps of your construction.

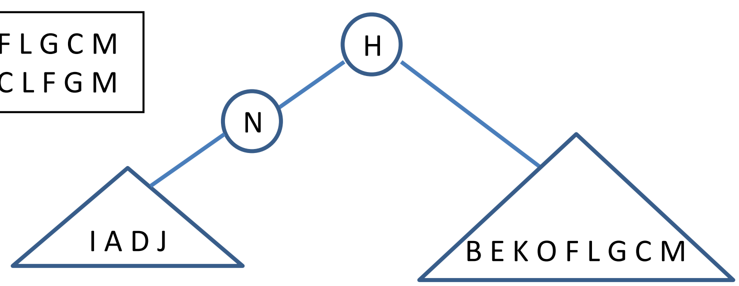
Answer:

- The first letter in the pre-order traversal must be the root of the tree. In our example, it is "H".
- All letters appearing before "H" in the in-order traversal must belong to the left subtree of "H", and all letters appearing after "H" in the in-order traversal must belong to the right subtree of "H".
- Recursively work on the left subtree and right subtree of "H".
- The steps are illustrated below.

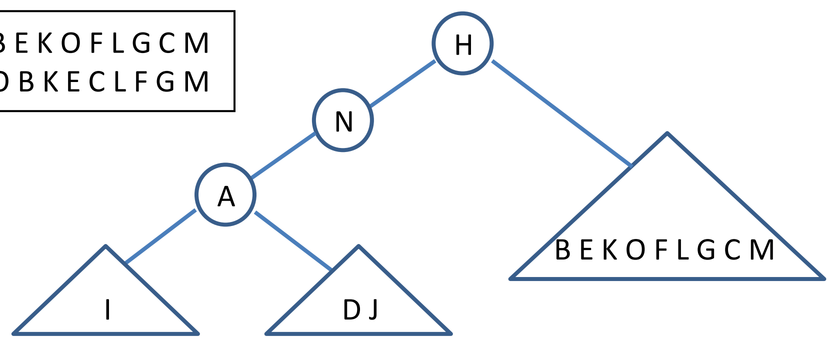
In-order: I A D J N **H** B E K O F L G C M
Pre-order: **H** N A I J D O B K E C L F G M



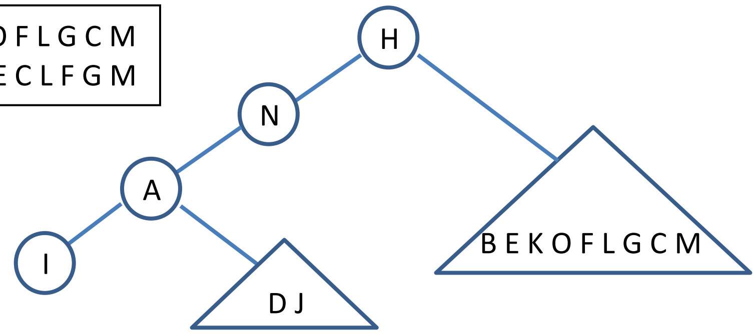
In-order: I A D J **N** **H** B E K O F L G C M
Pre-order: **H** **N** A I J D O B K E C L F G M



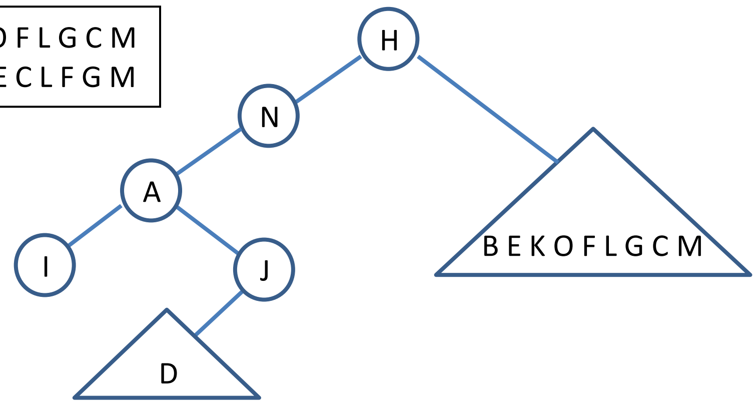
In-order: I **A** D J **N** **H** B E K O F L G C M
Pre-order: **H** **N** **A** I J D O B K E C L F G M



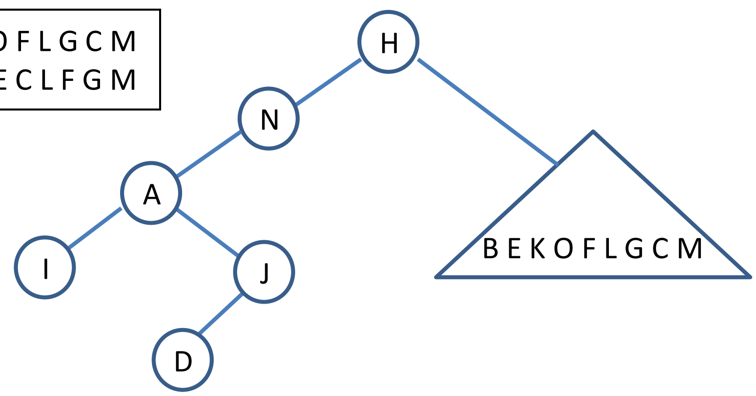
In-order: **I A** D J **N H** B E K O F L G C M
Pre-order: **H N A I** J D O B K E C L F G M



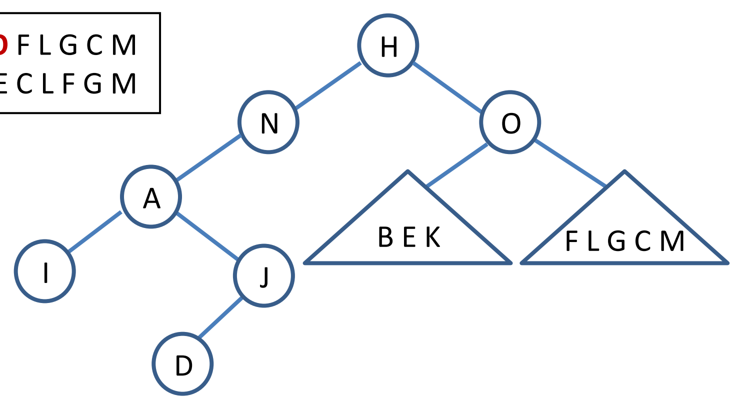
In-order: **I A** D **J N H** B E K O F L G C M
Pre-order: **H N A I J** D O B K E C L F G M



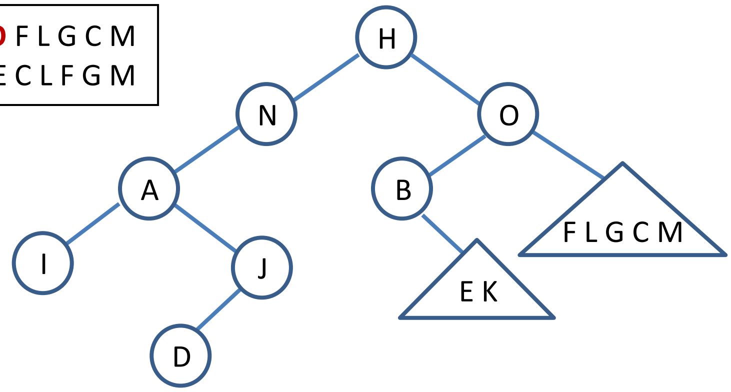
In-order: **I A D J N H** B E K O F L G C M
Pre-order: **H N A I J D** O B K E C L F G M



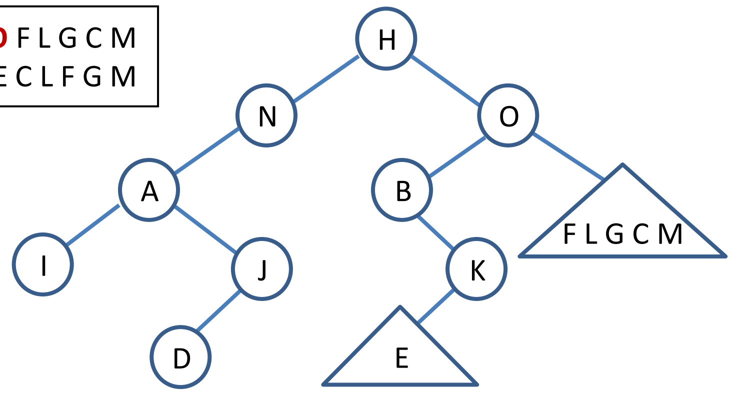
In-order: **I A D J N H** B E K **O** F L G C M
Pre-order: **H N A I J D O** B K E C L F G M



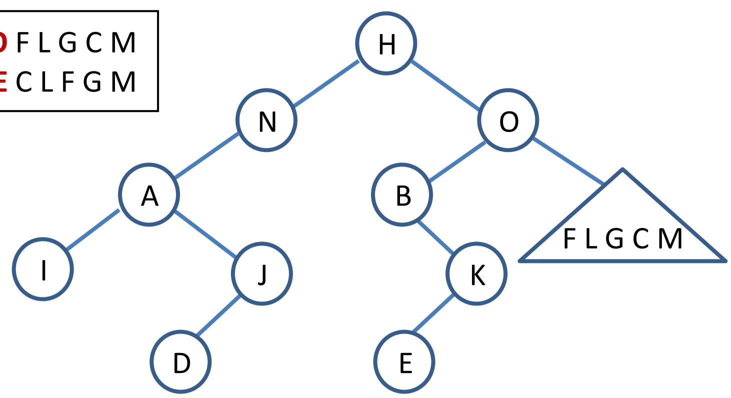
In-order: **I A D J N H B** E K **O** F L G C M
Pre-order: **H N A I J D O B** K E C L F G M



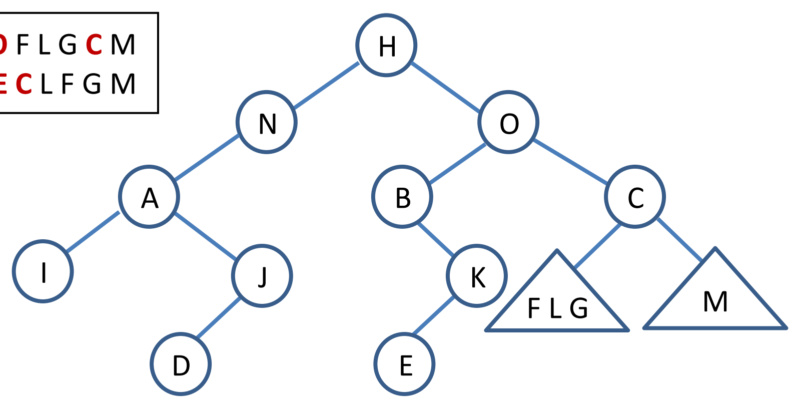
In-order: **I A D J N H B** E **K O** F L G C M
Pre-order: **H N A I J D O B K** E C L F G M



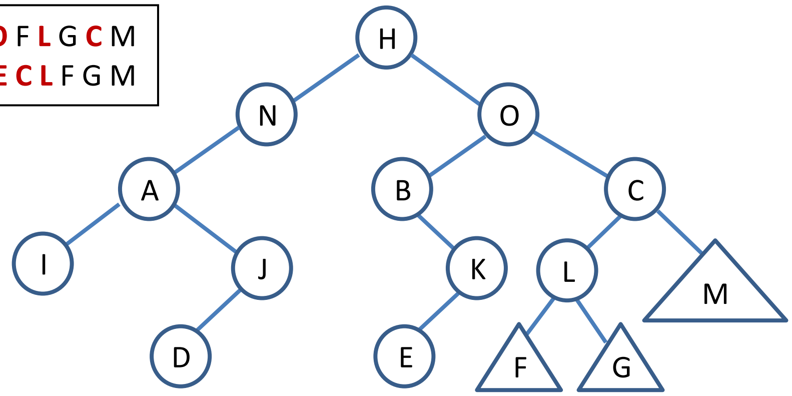
In-order: **I A D J N H B E K O** F L G C M
Pre-order: **H N A I J D O B K E** C L F G M



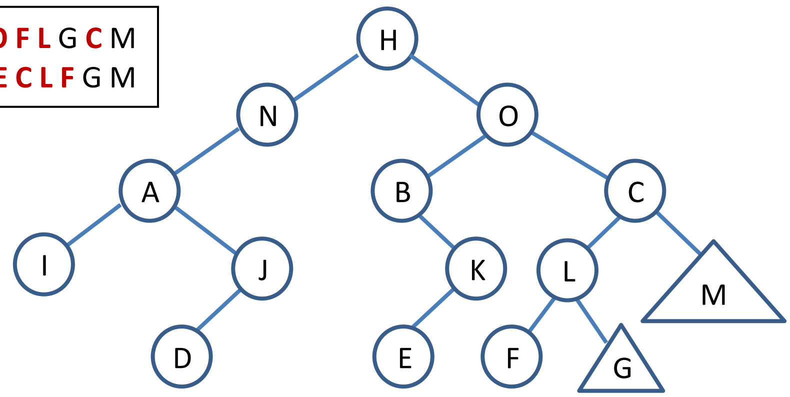
In-order: **I A D J N H B E K O** F L G **C** M
Pre-order: **H N A I J D O B K E C** L F G M



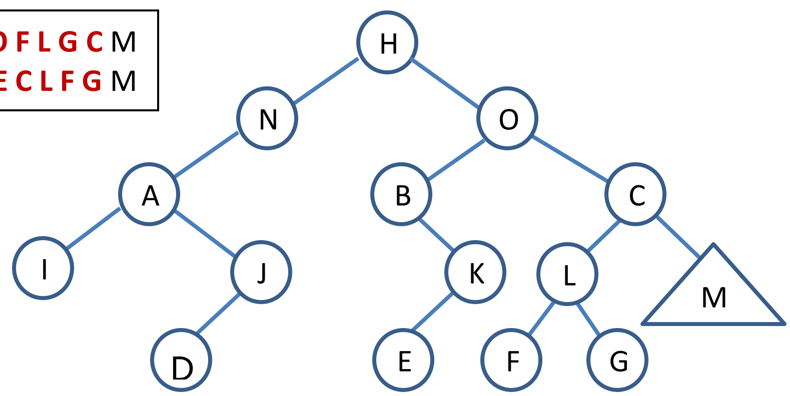
In-order: **I A D J N H B E K O** F **L** G **C** M
Pre-order: **H N A I J D O B K E C L** F G M



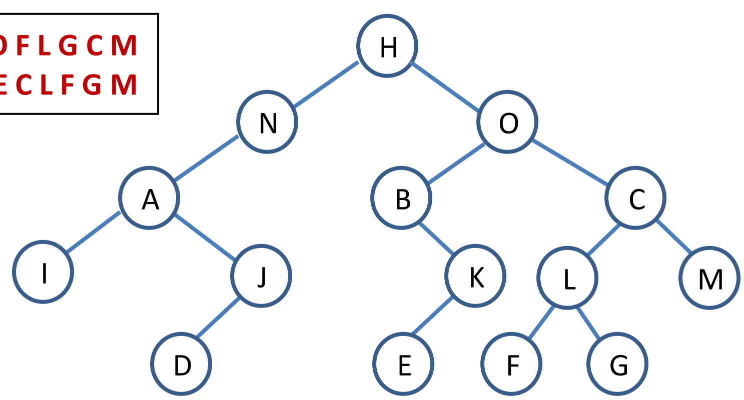
In-order: **I A D J N H B E K O F L** G **C** M
Pre-order: **H N A I J D O B K E C L F** G M

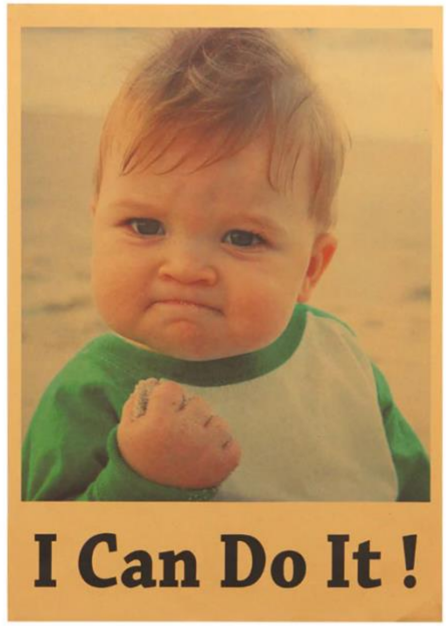


In-order: **I A D J N H B E K O F L G C** M
Pre-order: **H N A I J D O B K E C L F G** M



In-order: **I A D J N H B E K O F L G C M**
Pre-order: **H N A I J D O B K E C L F G M**




I Can Do It !


GOOD LUCK
IN YOUR EXAMS
knock 'em out with your GENIUS!