CS1231/CS1231S: Discrete Structures

Tutorial #10: Counting and Probability II

Answers

I. Discussion Questions

You are strongly encouraged to discuss D1 – D3 on LumiNUS forum. No answers will be provided.

D1. Suppose a random sample of 2 lightbulbs is selected from a group of 8 bulbs in which 3 are defective, what is the expected value of the number of defective bulbs in the sample? Let X represent of the number of defective bulbs that occur on a given trial, where $X = 0,1,2$. Find $E[X]$.

Answer:

$P(X = 0) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56}$; $P(X = 1) = \frac{5}{8} \times \frac{3}{7} + \frac{3}{8} \times \frac{5}{7} = \frac{30}{56}$; $P(X = 2) = \frac{3}{8} \times \frac{2}{7} = \frac{6}{56}$

$E[X] = \left(0 \times \frac{20}{56}\right) + \left(1 \times \frac{30}{56}\right) + \left(2 \times \frac{6}{56}\right) = \frac{42}{56} = \frac{3}{4} = \mathbf{0.75}$.

D2. How many **injective functions** are there from a set A with m elements to a set B with n elements, where $m \leq n$?

Answer:

First element in A has n choices for its image in B, second element in A has $(n - 1)$ choices, etc, Therefore, the number of injective functions is

$$n(n-1)(n-2)\cdots(n-m+1) = \boldsymbol{P(n,m)}.$$

D3. How many **surjective functions** are there from a 5-element set A to a 3-element set B?

Answer:

The total number of functions is 3^5, since each element in A has 3 choices for its image in B.

There are 2 cases of non-surjective functions, as follows.

Case 1: One of the 3 elements in B has no pre-images.
There are 3 ways to choose the element in B that has no pre-images.
For the remaining two elements in B, there are $2^5 - 2$ functions (why)?
Therefore, there are $3 \times (2^5 - 2) = 90$ functions.

Case 2: Two of the 3 elements in B has no pre-images.
There are 3 ways to choose the 2 elements in B that have no pre-images, and the element that has pre-images must have all the pre-images in A.

Therefore, there are $3^5 - 90 - 3 = \mathbf{150}$ surjective functions.

II. Tutorial Questions

1. Your organization has 6 designers, 12 business consultants, and 20 programmers. How many possible teams of 5 members can you have if:

   a. The team is made up completely of programmers.

   b. The team must have at least 1 programmer.

   c. The team must have at least 2 programmers, at least 1 designer and at least 1 business consultant.

Answers:

(a) $\binom{20}{5} = \mathbf{15,504}$ teams.

(b) Total number of possible teams: $\binom{6+12+20}{5} = 501{,}942$.

For teams with no programmers, the teams consist entirely of designers or consultants. Hence, number of possible teams: $\binom{6+12}{5} = 8{,}568$.

Therefore, number of team with at least one programmer: $501{,}942 - 8{,}568 = \mathbf{493,374}$.

(c) There are 3 possibilities:

1. Number of teams with 3 programmers, 1 designer and 1 consultant:

$$\binom{20}{3}\binom{6}{1}\binom{12}{1} = 1140 \times 6 \times 12 = 82{,}080.$$

2. Number of teams with 2 programmers, 2 designers and 1 consultant:

$$\binom{20}{2}\binom{6}{2}\binom{12}{1} = 190 \times 15 \times 12 = 34{,}200.$$

3. Number of teams with 2 programmers, 1 designer and 2 consultants:

$$\binom{20}{2}\binom{6}{1}\binom{12}{2} = 190 \times 6 \times 66 = 75{,}240.$$

Therefore, total number ways: $82{,}080 + 34{,}200 + 75{,}240 = 191{,}520$.

2. You are the Director of Research at your company and you have \$25m to spend. There are 15 projects that require funding. Funding amounts are in units of \$1m, though projects may not necessarily receive funding (i.e. they get \$0).

   a. How many ways can you fund the 15 projects?

   b. The Chief Executive Officer insists that you must provide exactly \$3m for one particular project, and at least \$2m for each of five other particular projects. How many ways can you fund the 15 projects?

   Answers:

   (a) Multiset problem with $n = 15, r = 25$.

$$\binom{25 + 15 - 1}{25} = \mathbf{15,084,504,396}\ ways$$

   (b) Exactly \$3m to 1 project, at least \$2m for 5 projects. So 14 projects left to fund with \$12m left. Multiset problem with $n = 14, r = 12$.

$$\binom{12 + 14 - 1}{12} = \mathbf{5,200,300}\ ways$$

3. Think of a set with $m + n$ elements as composed of two parts, one with m elements and the other with n elements. Give a **combinatorial argument** to show that

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} \qquad \text{... (A)}$$

   where $m, n \in \mathbb{Z}^+$, $r \le m$ and $r \le n$.

   Call the above equation (A). Using equation (A), prove that for all integers $n \ge 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

   Answer:

   Selecting r elements from $m + n$ elements can be seen as dividing into the cases of selecting k elements from the part that contains the m elements and the remaining $(r - k)$ elements from the part that contains the n elements, for $0 \le k \le r$. Hence equation (A).

   Let $m = n = r$, then equation (A) becomes

$$\binom{2r}{r} = \binom{r}{0}\binom{r}{r} + \binom{r}{1}\binom{r}{r-1} + \cdots + \binom{r}{r}\binom{r}{0}$$

   However, as $\binom{n}{r} = \binom{n}{n-r}$ from example 8 in Lecture #11, the above is equivalent to

$$\binom{2r}{r} = \binom{r}{0}\binom{r}{0} + \binom{r}{1}\binom{r}{1} + \cdots + \binom{r}{r}\binom{r}{r} = \binom{r}{0}^2 + \binom{r}{1}^2 + \cdots + \binom{r}{r}^2.$$

4. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

Answer:

Recall the Binomial Theorem:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b^1 + \binom{n}{2} a^{n-2}b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

We have $a = 2x^2, b = \frac{1}{x}, n = 9$.

The general term in the expansion of $(a+b)^n$ is given by:

$$\binom{n}{r} a^{n-r} b^r = \binom{9}{r} (2x^2)^{9-r} \left(\frac{1}{x}\right)^r = \binom{9}{r} 2^{9-r} \cdot x^{18-2r} \cdot x^{-r} = \binom{9}{r} 2^{9-r} \cdot x^{18-3r}$$

For this term to be independent of x, we must have $18 - 3r = 0$, or $r = 6$.

Therefore, the term independent of x is

$$\binom{9}{6} 2^{9-6} = 84 \times 2^3 = \mathbf{672}$$

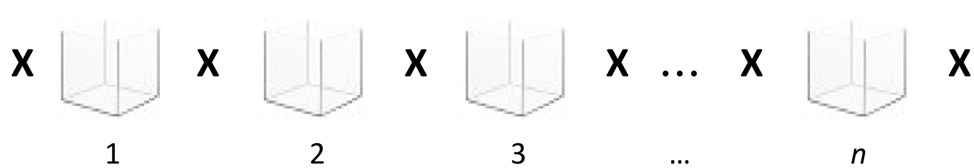
5. Let's revisit Question 5 of Tutorial #8:

Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

Last week, the answer given was

For k $(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways. Therefore, total number of ways is $\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Now, let's use another approach to solve this problem. Draw crosses on the side of the boxes as shown below. How do you use these crosses?



Answer:

The task is similar to choosing two out of the $n+1$ crosses to mark the start and end of the consecutively numbered boxes that contain white balls.

This is $\binom{n+1}{2}$, which is also equal to $n(n+1)/2$.

6. You meet a hustler on the street who lets you toss 3 separate coins once each for $2. If you get 3 heads, you win $10. If you get 2 heads (not in a row), you win $5, if you get 2 heads (in a row), you win $1. Otherwise you win nothing. If you play this game many, many times, how much would you win overall per game?

Answer:

HHH: $0.5^3 = 0.125$
HTH: $0.5^3 = 0.125$
HHT or THH: $2 \times (0.5^3) = 0.25$
TTT, TTH, THT, or HTT: 4 x (0.5^3) = 0.5

Expected winning per game:
$0.125 \times \$(10-2) + 0.125 \times \$(5-2) + 0.25 \times \$(1-2)$ + 0.5 x $(0 – 2)
$= \$1 + \$0.375 - \$0.25 - \$1 = \$\mathbf{0.125}$

7. The hustler now has two loaded coins with probability of 0.7 of getting tails, and one fair coin with probability of 0.5 of getting heads or tails. Is there a particular arrangement of coins (e.g. FLL, where F=fair and L=loaded) that he should use to maximize his profits? Explain why your choice works.

Answer:

Possible arrangements of coins: FLL, LFL, LLF. We can now build a table of the possible probabilities for each of the winning outcomes:

	F L L	L F L	L L F
HHH (1)	0.5 × 0.3 × 0.3 = 0.045	0.3 × 0.5 × 0.3 = 0.045	0.3 × 0.3 × 0.5 = 0.045
HTH (2)	0.5 × 0.7 × 0.3 = 0.105	0.3 × 0.5 × 0.3 = 0.045	0.3 × 0.7 × 0.5 = 0.105
HHT (3)	0.5 × 0.3 × 0.7 = 0.105	0.3 × 0.5 × 0.7 = 0.105	0.3 × 0.3 × 0.5 = 0.045
THH (4)	0.5 × 0.3 × 0.3 = 0.045	0.7 × 0.5 × 0.3 = 0.105	0.7 × 0.3 × 0.5 = 0.105
TTT (5)	0.5 x 0.7 x 0.7 = 0.245	0.7 x 0.5 x 0.7 = 0.245	0.7 x 0.7 x 0.5 = 0.245
TTH (6)	0.5 x 0.7 x 0.3 = 0.105	0.7 x 0.5 x 0.3 = 0.105	0.7 x 0.7 x 0.5 = 0.245
THT (7)	0.5 x 0.3 x 0.7 = 0.105	0.7 x 0.5 x 0.7 = 0.245	0.7 x 0.3 x 0.5 = 0.105
HTT (8)	0.5 x 0.7 x 0.7 = 0.245	0.3 x 0.5 x 0.7 = 0.105	0.3 x 0.7 x 0.5 = 0.105
Winning (1) × -8 + (2) × -3 + [(3) + (4)] × 1 + ((5) + (6) + (7) + (8)) x 2	$0.875	**$1.115**	$0.875

He should use **L F L**. We want to maximize (3) to (8) because it reduces the overall winning (since the bettor actually loses a dollar for (3) and (4) and $2 for (5) to (8)). Doing L F L maximizes these two because the tail in both cases end up in the “loaded” position giving a large probability of 0.7.

8. One urn contains 10 red balls and 25 green balls, and a second urn contains 22 red balls and 15 green balls. A ball is chosen as follows: First an urn is selected by tossing a loaded coin with probability 0.4 of landing heads up and probability of 0.6 of landing tails up. If the coin lands heads up, the first urn is chosen; otherwise, the second urn is chosen. Then a ball is picked at random from the chosen urn.

   Write your answers correct to three significant figures.

   (a) What is the probability that the chosen ball is green?

   (b) If the chosen ball is green, what is the probability that it was picked from the first urn?

Answers:

(a) $\frac{4}{10} \cdot \frac{25}{35} + \frac{6}{10} \cdot \frac{15}{37} = \frac{137}{259} = 52.9\%.$

Probability that the chosen ball is green is **52.9%**.

(b) Let G be the event that the chosen ball is green, U_1 the event that the ball came from the first urn, and U_2 the event that the ball came from the second urn.

$P(U_1) = 0.4, P(U_2) = 0.6, P(G|U_1) = \frac{25}{35}, P(G|U_2) = \frac{15}{37}.$

By Bayes' Theorem,

$$P(U_1|G) = \frac{P(G|U_1) \cdot P(U_1)}{P(G|U_1) \cdot P(U_1) + (G|U_2) \cdot P(U_2)}$$

$$= \frac{\left(\frac{25}{35}\right) \times 0.4}{\left(\frac{25}{35}\right) \times 0.4 + \left(\frac{15}{37}\right) \times 0.6} = \frac{74}{137} = 54.0\%$$

Therefore, the probability that the green ball is chosen from the first urn is **54.0%**.

Alternatively, using the result from part (a),

$$P(U_1|G) = \frac{P(U_1 \cap G)}{P(G)} = \frac{0.4 \times \frac{25}{35}}{\frac{137}{259}} = \frac{74}{137} = 54.0\%$$

9. (AY2015/16 Semester 1 exam question)

Let $A = \{1, 2, 3, 4\}$. Since each element of $P(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($P(S)$ denotes the powerset of S.)

Assuming each relation in $P(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

Can you generalize your answer to any set A with n elements?

Answers:

(a) 1/16

(b) 1/64

For the general case, let $|A| = n$. In general, we need to consider all possible n^2 pairs. Thus the total number of possible combinations, i.e. the number of elements in $P(A \times A)$, is 2^{n^2}.

(a) To count the number of reflexive relations, note that all the pairs (a, a) for all $a \in A$ must be in the relation, so these n pairs are fixed. We are then free to choose to include or not any of the $(n^2 - n)$ remaining pairs. This gives us:

$$\frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}$$

(b) For symmetric relations, if some pair (a, b) is in the relation, then (b, a) must also be in the relation. Hence, either (a, b) and (b, a) are both included in the relation, or both are not. This gives us $\frac{n^2+n}{2}$ pairs to choose to include or not, giving us:

$$\frac{2^{\frac{n^2+n}{2}}}{2^{n^2}} = \frac{1}{2^{\frac{n^2-n}{2}}}$$

10. Let's revisit Question 1 of Tutorial #8:

> In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

The solution given last week uses a possibility tree to depict the 15 ways. Now, let's approach this problem using combination.

Let us define a function $W(a, b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win. Hence,

$$W(a,b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0; \\ W(a, b-1) + W(a-1, b), & \text{if a} > 0 \text{ and } b > 0. \end{cases}$$

We may express $W(a, b)$ as a simple combination formula as follows:

$$W(a,b) = \binom{a+b}{a}.$$

Verify the above.

Now, we denote the function $T(n, k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k ($k \le n$) games.

Derive a simple combination formula for $T(n, k)$ (hint: relate function T to function W), and hence solve $T(4, 2)$ which is the problem in question 1 of tutorial #8.

Answer:

To verify that $W(a,b) = \binom{a+b}{a}$, it suffices to verify that the function $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ such that $(a, b) \mapsto \binom{a+b}{b}$ satisfies the recursive definition of $W(a, b)$.

Pick any $a, b \in \mathbb{Z}_{\geqslant 0}$.

Case 1: $a = 0$, then $\binom{a+b}{a} = \binom{b}{0} = 1 = W(a, b)$.

Case 2: $b = 0$, then $\binom{a+b}{a} = \binom{a}{a} = 1 = W(a, b)$.

Case 3: $a > 0$ and $b > 0$, then

$$\binom{a+b}{a} = \binom{a+b-1}{a-1} + \binom{a+b-1}{a} \text{ (by Pascal's formula: } \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}\text{)}$$
$$= \binom{a+(b-1)}{a} + \binom{(a-1)+b}{a-1} = W(a, b-1) + W(a-1, b) = W(a, b)$$

We have verified that $W(a,b) = \binom{a+b}{a}$ for all cases.

Now, we can see that

$$T(n,k) = W(n-k, n) = \binom{2n-k}{n-k} = \binom{2n-k}{n}.$$

Hence, $T(4,2) = \binom{6}{4} = 15$.