CS1231/CS1231S: Discrete Structures

Tutorial #9: Counting and Probability I

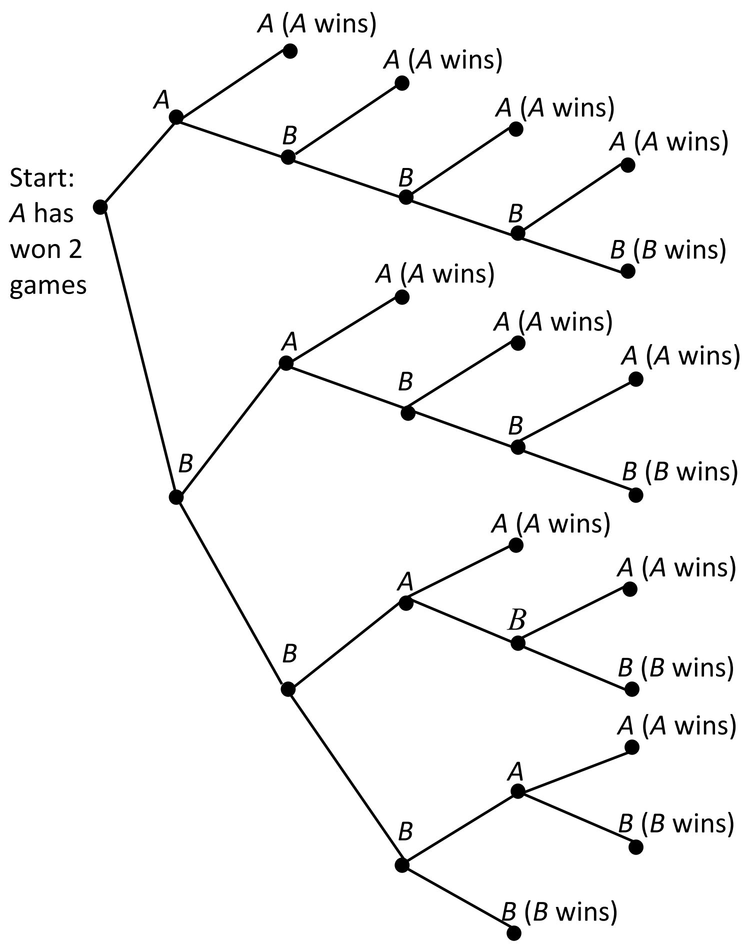
Answers

II. Tutorial Questions

1. In a certain tournament, the first team to win four games wins the tournament. Suppose there are two teams A and B, and team A wins the first two games. How many ways can the tournament be completed?

   (We will use possibility tree to solve this problem for now. In the next tutorial, we will approach this problem using combination.)

   Answer:
   15 ways.



The following are the 15 ways, excluding the first two games which are won by team A:

(1) *AA*, (2) *ABA*, (3) *ABBA*, (4) *ABBBA*, (5) *ABBBB*, (6) *BAA*, (7) *BABA*, (8) *BABBA*, (9) *BABBB*, (10) *BBAA*, (11) *BBABA*, (12) *BBABB*, (13) *BBBAA*, (15) *BBBAB*, (15) *BBBB*.

2. (Past year's exam question.)
The figure on the right shows a combination lock with 40 positions.

To open the lock, you rotate to a number in a clockwise direction, then to a second number in the counterclockwise direction, and finally to a third number in the clockwise direction. If consecutive numbers in the combination cannot be the same, how many combinations of three-number codes are there?

Answer: $40 \times 39 \times 39 = 60840$

3. There are 789 CS students in SoC. Among them, 672 are taking CS1231S, 629 are taking CS1101S, 153 are taking MA1101R, 608 are taking CS1231S and CS1101S, 87 are taking CS1231S and MA1101R, 53 are taking CS1101S and MA1101R, and 46 are taking all three modules.

How many CS students are not taking any of these three modules?

Answer:

1. Let A, B and C be the sets of CS students taking CS1231S, CS1101S and MA1101R respectively.

Tutors: I've changed the notation from $N(S)$ to $|S|$ to be consistent with Lawrence's.

2. By the inclusion/exclusion rule (theorem 9.3.3),
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 672 + 629 + 153 - 608 - 87 - 53 + 46 = 752.$$
3. $|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{(A \cup B \cup C)}| = |U| - |A \cup B \cup C| = 789 - 752 = 37.$
4. There are 37 CS students who are not taking any of these three modules.

4. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position?

An integer k is in the correct position if it is at the k^{th} position in the permutation. For example, the permutation 3, $\underline{2}$, 4, 1, $\underline{5}$ has integers 2 and 5 in their correct positions, and the permutation 12, 1, $\underline{3}$, 9, 10, 8, $\underline{7}$, 6, 2, 4, $\underline{11}$, 5 has integers 3, 7, and 11 in their correct positions. Integers that are in their correct positions are underlined for illustration.

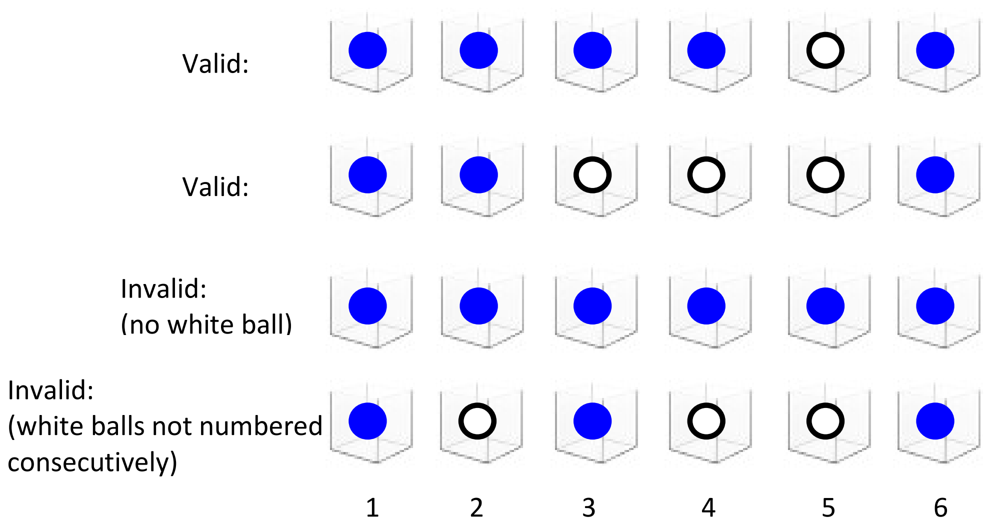
Answer:

1. Let $|P_k|$ be the number of permutations with integer k in its correct position.
2. $|P_1| = |P_2| = |P_3| = (n-1)!$
3. $|P_1 \cap P_2| = |P_2 \cap P_3| = |P_1 \cap P_3| == (n-2)!$
4. $|P_1 \cap P_2 \cap P_3| = (n-3)!$
5. By the inclusion/exclusion rule (theorem 9.3.3),
$|P_1 \cup P_2 \cup P_3| = 3(n-1)! - 3(n-2)! + (n-3)! = (3n^2 - 12n + 13)(n-3)!$

5. Given n boxes numbered 1 to n, each box is to be filled with either a white ball or a blue ball such that at least one box contains a white ball and boxes containing white balls must be consecutively numbered. What is the total number of ways this can be done?

   (For this tutorial, use sum of a sequence to solve this problem. In the next tutorial, we will revisit this problem using a different approach.)

   Some examples for $n = 6$ are shown below for your reference.



   Answer:

   1. For k $(1 \leq k \leq n)$ consecutively numbered boxes that contain white balls, there are $n - k + 1$ ways.
   2. Therefore, total number of ways is

$$\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

6. In Tutorial #4 D1, you are asked to write down all possible functions $\{a, b, c\} \rightarrow \{1,2\}$.

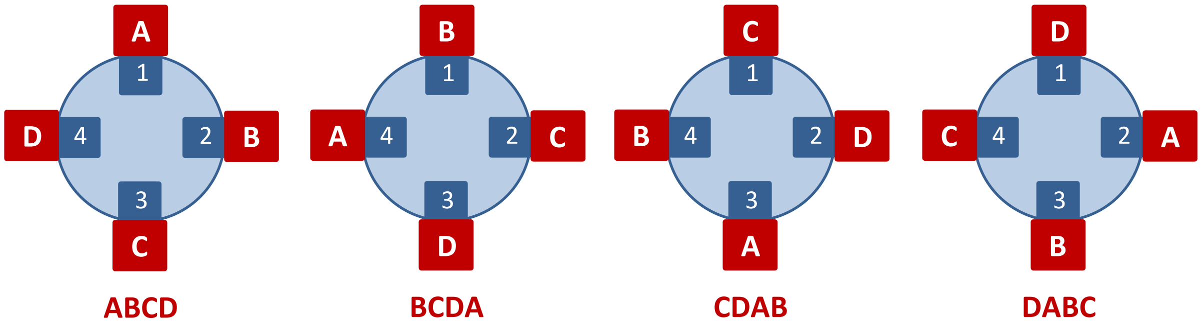
   How many possible functions $f: A \rightarrow B$ are there if $|A| = n$ and $|B| = k$?

   Answer:

   Each of the n elements in A must be mapped to one of the k elements in B.
   Therefore, there are k^n possible functions f.

7 We have learned that the number of permutations of n distinct objects is $n!$, but that is on a straight line. If we seat four guests Anna, Barbie, Chris and Dorcas on chairs on a straight line they can be seated in 4! or 24 ways.

What if we seat them around a circular table? Examine the figure below.



The four seating arrangements (clockwise from top) *ABCD*, *BCDA*, *CDAB* and *DABC* are just a single permutation, as in each arrangement the persons on the left and on the right of each guest are still the same persons. Hence, these four arrangements are considered as one permutation.

This is known as *circular permutation*. The number of linear permutations of 4 persons is four times its number of circular permutations. Hence, there are $\frac{4!}{4}$ or 3! ways of circular permutations for 4 persons. In general, the number of circular permutations of n objects is $(n-1)!$

Answer the following questions:

a. In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

b. In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

c. In how many ways can $n-1$ people sit around a circular table with n chairs?

Answers:

a. 8 boys can be seated in a circle in 7! ways. There are 8 spaces between the boys, which can be occupied by 4 girls in *P(8,4)* ways. Hence, total number of ways = 7! × P(8,4) = 5040 × 1680 = 8467200.

b. There are 5! = 120 ways for 6 people to sit around a circular table. There are 2 × 4! = 48 ways for Eric and Freddy to sit together. Therefore, the answer is 120 – 48 = 72 ways.

c. Treat the empty chair as just another person, therefore there are $(n-1)!$ ways to seat $n-1$ people around a table with n chairs.

8. (Past year's exam question.)
Prove that if you randomly put 51 points inside a unit square, there are always three points that can be covered by a circle of radius 1/7.

Answer:

1. Divide the unit square into 25 equal smaller squares of side 1/5 each.
2. Then at least one of these small squares would contain at least three points. (Otherwise, if every square contains two points or less, the total number of points is no more than $2 \times 25 = 50$, which contradicts our assumptions that there are 51 points – Generalised PHP.)
3. Now, the circle circumvented around the small square with the three points inside also contains these three points and has radius

$$\sqrt{\left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^2} = \sqrt{\frac{2}{100}} = \sqrt{\frac{1}{50}} < \sqrt{\frac{1}{49}} = \frac{1}{7}$$

Note to tutors: A common answer given by students is to divide the unit square into 16 smaller squares of side ¼ each. This is wrong. Although the area of each small square, 0.0625, is smaller than the area of a circle of radius 1/7 which is 0.0641, the small square is not entirely enclosed by the circle, because 1/8 < 1/7.

9. Let $S = \{3,4,5,6,7,8,9,10,11,12\}$. What is the smallest number of integers you must choose from S such that two of them sum to 15? In other words, what is the smallest $n \in \mathbb{Z}_{n\geq 2}$ such that for all subsets X of S where $|X| = n$, there exists two distinct elements $x, y \in X$ such that $x + y = 15$.

Answer:

1. Partition the set S into the following 5 subsets, where each subset contains a pair of integers that sum to 15: $\{3,12\}, \{4,11\}, \{5,10\}, \{6,9\}, \{7,8\}$.
2. For any $n \leq 5$, we can choose n elements such that each element belongs to a different subset. Then we won't be able to find two elements among them that sum to 15.
3. However, if more than 5 integers are chosen from set S, 2 of them must be from the same subset according to PHP.
4. Therefore, the smallest n is 6.

10. (Past year's exam question.)
In a city, houses are randomly assigned distinct numbers between 1 and 50 inclusive. What is the minimum number of houses to ensure that there are 5 houses numbered consecutively?

For example, the number of houses cannot be 10 because we can choose to number the houses 1, 8, 9, 15, 18, 21, 22, 23, 24, 32, hence no 5 houses are numbered consecutively.

To receive full credit, you must define the pigeons and pigeonholes in your answer.

Answer:

Split the numbers into 10 pigeonholes: 1-5, 6-10, 11-15, …, 46-50. Therefore, there must be at least $10 \times 4 + 1 = 41$ houses (pigeons).

If there are 40 houses, then you may label the houses 1-4, 6-9, 11-14, 16-19, 21-24, 26-29, 31-34, 36-39, 41-44, and 45-49.