CS1231/CS1231S: Discrete Structures

Tutorial #11: Graphs and Trees

(Week 13: 9 – 13 November 2020)

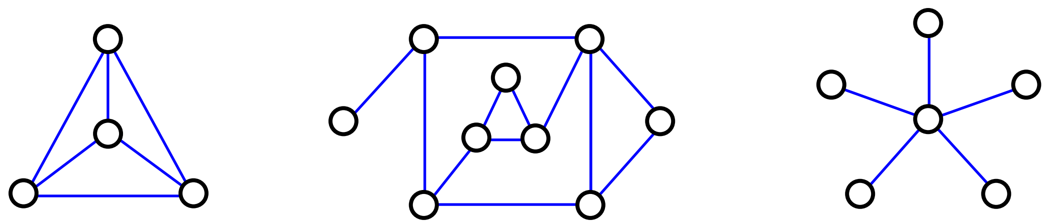
I. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1. For any simple connected graph with n $(n > 0)$ vertices, what is the minimum and maximum number of edges the graph may have?

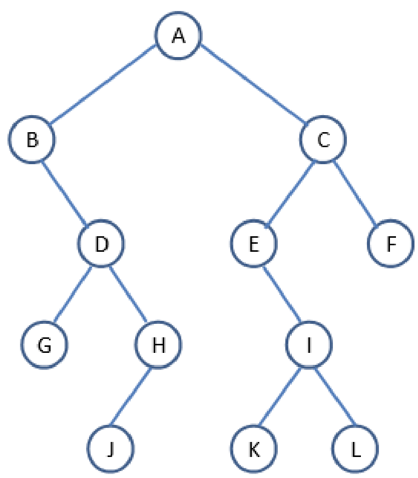
D2. (AY2016/17 Semester 1 Exam Question)
How many simple graphs on 3 vertices are there? In general, how many simple graphs on n $(n > 1)$ vertices are there?

D3. Verify Euler's formula $f = e - v + 2$ on the planar graphs below.



(a) (b) (c)

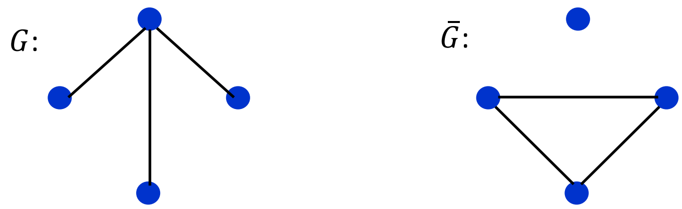
D4. Given the following binary tree, write the pre-order, in-order, and post-order traversals of its vertices.



II. Definitions

> **Definition 1.** If G is a simple graph, the *complement* of G, denoted $\bar{G}$, is obtained as follows: the vertex set of $\bar{G}$ is identical to the vertex set of G. However, two distinct vertices v and w of $\bar{G}$ are connected by an edge if and only if v and w are not connected by an edge in G.

The figure below shows a graph G and its complement $\bar{G}$.



A graph G and its complement $\bar{G}$.

> **Definition 2.** A *self-complementary* graph is isomorphic with its complement.

> **Definition 3.** A simple circuit (cycle) of length three is called a *triangle*.

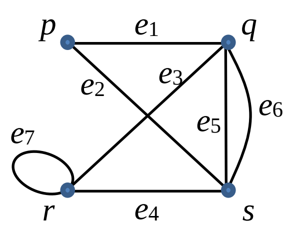
III. Tutorial Questions

1. Draw all self-complementary graphs with (a) four vertices; (b) five vertices.

2. Show that every simple graph with at least two vertices has two vertices of the same degree.

   (This is similar to the popular puzzle: "Prove that at a party with at least two persons, there are two people who know the same number of people".)

3. Given the graph shown below:



   (a) Write the adjacency matrix A for the graph. Let the rows and columns be p, q, r and s.

   (b) Find A^2 and A^3.

   (c) How many walks of length 2 are there from p to q? From s to itself? List out all the walks.

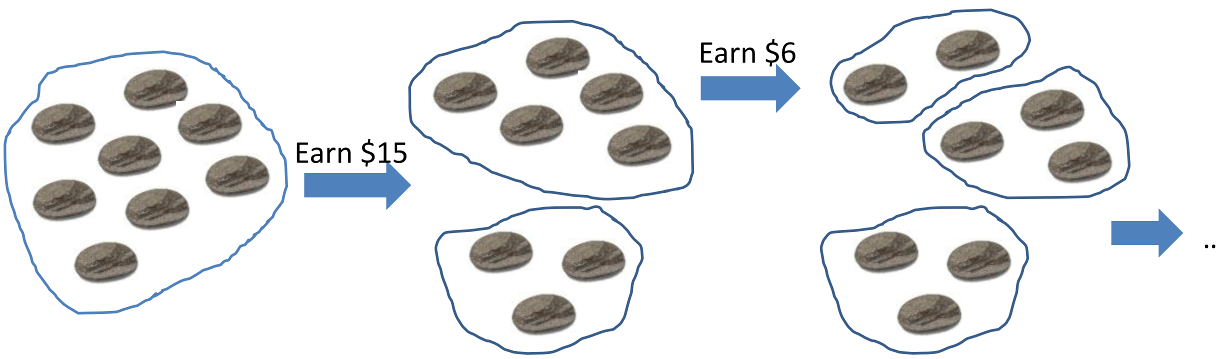
   (d) How many walks of length 3 are there from r to s? From s to p? List out all the walks.

4. Prove that for any simple graph G with six vertices, G or its complementary graph $\bar{G}$ contains a triangle.

5. (AY2017/18 Semester 1 Exam Question)
Suppose you are given a pile of stones. At each step, you can separate a pile of k stones into two piles of k_1 and k_2 stones. (Obviously, $k_1 + k_2 = k$.) On doing this, you earn $\$(k_1 \times k_2)$.

   What is the maximum amount of money you can earn at the end if you start with a pile of n stones? Explain your answer.

   The diagram below illustrates the (incomplete) process of separating a pile of 8 stones.



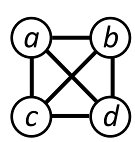
6. How many edges are there in a forest with v vertices and k components?

7. How many possible binary trees with 4 vertices *A*, *B*, *C* and *D* have this in-order traversal: *A B C D*? Draw them.

8. Consider a simple connected graph G with n vertices.

   (a) If G is a complete graph, the total number of spanning trees in G is n^{n-2}. (You may check out *Cayley's formula*, though it is not in the scope of CS1231/CS1231S.)

   Show all the spanning trees in the complete graph K_4 below:



   (b) If G is not a complete graph, we may follow the following steps to calculate the number of spanning trees in G:

      1. Let A be the adjacency matrix of $G = (V, E)$ where $V = \{v_1, v_2, \cdots, v_n\}$.
      2. Let D be the diagonal matrix with its diagonal elements d_{ii} = degree of v_i, for $1 \leq i \leq n$.

3. Let $M = D - A$.

4. Let M' be the submatrix obtained by removing row i and column i from M, where $1 \leq i \leq n$.

5. Number of spanning trees is the determinant of M', denoted by det(M') or $|M'|$.

The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The determinant of a 3×3 matrix is

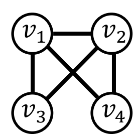
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

The determinant of a 4×4 matrix is

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a\begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b\begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c\begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d\begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}.$$

(You may generalize it to $n \times n$ matrix. See Leibnix formula.)

For example, given this graph G:



The matrixes A, D, M and M' (by removing row 1 and column 1 from M), are shown below:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix};\ D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix};\ M = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$$

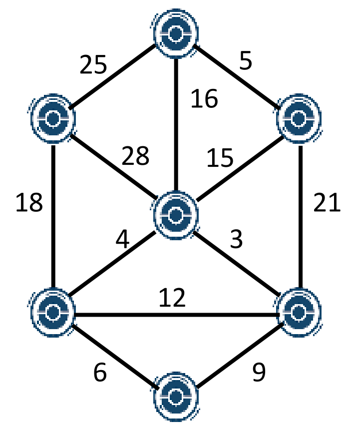
$$M' = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix};\quad \det(M') = 3(4) + 1(-2) - 1(2) = 8.$$

Therefore, there are 8 spanning trees in G.

Using the above method, verify that there are 125 spanning trees in K_5, which is 5^{5-2}.

9. (AY2016/17 Semester 1 Exam Question)
The figure below shows a graph where the vertices are Pokestops. Using either Kruskal's algorithm or Prim's algorithm, find its minimum spanning tree (MST). If you use Prim's algorithm, you must start with the top-most vertex.

Indicate the order of the edges inserted into the MST in your answer.



10. Construct the binary tree given the following in-order and pre-order traversals of the tree:

In-order: I A D J N H B E K O F L G C M

Pre-order: H N A I J D O B K E C L F G M

Draw diagrams to trace the steps of your construction.