

CS1231/CS1231S: Discrete Structures
Tutorial #2: Logic of Quantified Statements (Predicate Logic)
(Week 4: 31 August – 4 September 2020)

1. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1 An elementary definition in number theory is the following on divisibility:

For integers d and n , $d|n$ if and only if $n = kd$ for some integer k .

(Here, $d|n$ means “ d divides n ”, and d is called a *divisor* or *factor*.)

(a) State the above definition symbolically.

(b) Does 2 divide $2\sqrt{2}$?

D2. Explain why English can be ambiguous at times using the following sentence:

“Every boy loves a girl.”

Interpret the above sentence in two ways and write the quantified statement for each of them.

D3. The following table shows when the quantified statements are true and when they are false.

Statement	True when...	False when...
$\forall x P(x)$	$P(x)$ is true for every x .	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x .

Complete the table below for mixed quantifiers.

Statement	True when...	False when...
$\forall x \forall y P(x, y)$		
$\forall x \exists y P(x, y)$		
$\exists x \forall y P(x, y)$		
$\exists x \exists y P(x, y)$		

- D4. Let the domain of discourse D be the set of all students at NUS, and let $M(s)$ be “ s is a Math major”, $C(s)$ be “ s is a Computer Science major” and $E(s)$ be “ s is an Engineering major”. Express each of the following statements using quantifiers, variables and the predicates $M(s)$, $C(s)$ and $E(s)$.

Part (a) has been done for you.

- (a) Every Computer Science major is an Engineering major.

Answer: $\forall s \in D (C(s) \rightarrow E(s))$.

Discuss: Why is the following answer wrong?

Wrong answer: $\forall s \in D (C(s) \wedge E(s))$.

Can you give an example to show the difference between these two answers?

- (b) No Computer Science major are Engineering majors.
(c) Some Computer Science major are not Math majors.
(d) If a student is not a Math major, then the student is either a Computer Science major or an Engineering major, but not both.

2. Additional Notes

Note that “logic of quantified statements” (chapter 3) is commonly known as “predicate logic”, as opposed to “propositional logic” in chapter 2.

We picked up some frequently asked questions and created this *Additional Notes* section to include some materials not covered in lecture that might be of interest to you.

Equivalent expressions: The following quantified statements are equivalent. We use the shorter notation on the left.

- $\forall x \in D, P(x) \equiv \forall x((x \in D) \rightarrow P(x))$
- $\exists x \in D, P(x) \equiv \exists x((x \in D) \wedge P(x))$

Well-formed formula (wff)

- **true** and **false** are wffs.
- A proposition variable is a wff.
- A predicate name followed by a list of variables (eg: $P(x)$, $Q(x, y)$), which is called an *atomic formula*, is a wff.
- If A , B and C are wffs, then so are $\sim A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.
- If x is a variable and A is a wff, then so are $\forall x A$ and $\exists x A$.

Bound variables and Scope of quantifiers

- When a quantifier is used on a variable x in a predicate statement, we say that x is *bound*. If no quantifier is used on a variable, we say that the variable is *free*.

Examples: In the statement $\forall x \exists y P(x, y)$, both x and y are bound. In the statement $\forall x P(x, y)$, x is bound but y is free.

- The scope of a quantifier is the range in the formula where the quantifier “engages in”. It is put right after the quantifier, often in parentheses (eg: $\forall x (P(x))$).

Sometimes, when parentheses are not present (eg: $\forall x P(x)$), the scope is understood to be the smallest wff following the quantification.

- For example, in $\exists x P(x, y)$, the variable x is bound while y is free. In $\forall x (\exists y P(x, y) \vee Q(x, y))$, x and the red y in $P(x, y)$ are bound but the blue y in $Q(x, y)$ is free, because the scope of $\exists y$ is $P(x, y)$, whereas the scope of $\forall x$ is $(\exists y P(x, y) \vee Q(x, y))$. If you want to change the blue y to red y , you need to add parentheses: $\forall x (\exists y (P(x, y) \vee Q(x, y)))$. Then, the outermost pair of parentheses may be removed, i.e.: $\forall x \exists y (P(x, y) \vee Q(x, y))$.

3. Common Mistakes

- Using commas (,) in place of appropriate connectives, for example, $\forall x P(x), Q(x)$ where it should be $\forall x (P(x) \wedge Q(x))$.
- Treating predicates as functions returning some value. Eg: given the following predicates
 - $Loves(x, y)$: x loves y
 - $Reindeer(x)$: x is a reindeer

Some students wrote $Loves(x, Reindeer(y))$ in part of their answers. Since $Reindeer(y)$ is a predicate, its value is either true or false. So, the above is akin to writing $Loves(x, true)$ or $Loves(x, false)$ which does not make sense! The correct way is to use the appropriate connectives.

4. Tutorial Questions

1. For each of the following statements, write its **converse**, **inverse** and **contrapositive**. Indicate which among the statement, its converse, its inverse, and its contrapositive are true and which are false. Give a counterexample for each that is false. Proof not required if it is true.
 - a. $\forall n \in \mathbb{Z} (6|n \rightarrow 2|n \wedge 3|n)$. (Note: " $d|n$ " is as defined in D1.)
 - b. $\forall r \in \mathbb{R} (r > 3 \rightarrow r^2 > 9)$.
 - c. $\forall p, q$ that are statements, $((p \rightarrow q) \rightarrow \sim p)$.
2. Given the predicate $Loves(x, y)$ which means " x loves y ", translate the following English sentences into predicate logic statements. For this question, you may leave out the domain of discourse in your statements (i.e. you do not need to specify what domain a variable belongs to, such as $\forall x \in D$; just write $\forall x$).
 - a. Everybody loves someone else.
 - b. Nobody except John loves Mary. (or: Only John loves Mary; nobody else loves Mary.)
3. Prove or disprove the following statement:

$\forall a, b, c \in \mathbb{Z}$, if $a - b$ is even and $a - c$ is even, then $b - c$ is even.

(We have covered closure of integers under addition and multiplication in tutorial #1. Integers are also closed under subtraction, and you can cite this for this question.)

4. Some of the arguments below are valid, others are invalid. State which are valid and which are invalid and write out the reasons.
- All honest people pay their taxes.
Darth is not honest.
 \therefore Darth does not pay his taxes.
 - For every student x , if x studies CS1231, then x is good at logic.
Tarik studies CS1231.
 \therefore Tarik is good at logic.
 - Sum of any two rational numbers is rational.
The sum $r + s$ is rational.
 \therefore r and s are both rational.

5. Let V be the set of all visitors to Universal Studios Singapore on a certain day, $T(v)$ be “ v took the Transformers ride”, $G(v)$ be “ v took the Battlestar Galactica ride”, $E(v)$ be “ v visited the Ancient Egypt”, and $W(v)$ be “ v watched the Water World show”.

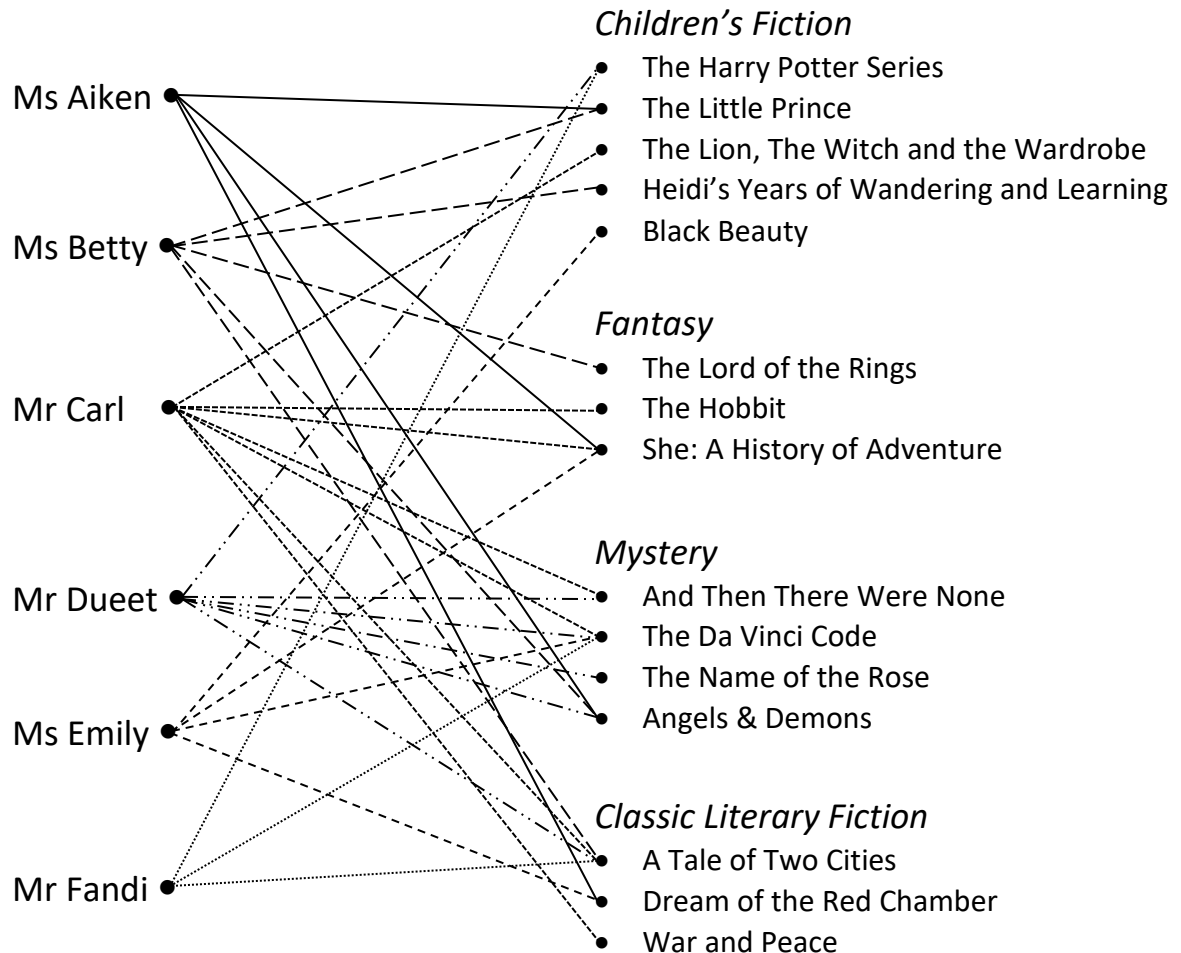
Express each of the following statements using quantifiers, variables, and the predicates $T(v)$, $G(v)$, $E(v)$ and $W(v)$. The statements are not related to one another. Part (a) has been done for you.

- a. Every visitor watched the Water World show.

Answer for (a): $\forall v \in V (W(v))$.
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- Every visitor who took the Battlestar Galactica ride also took the Transformers ride.
- There is a visitor who took both the Transformers ride and the Battlestar Galactica ride.
- No visitor who visited the Ancient Egypt watched the Water World show.
- Some visitors who took the Transformers ride also visited the Ancient Egypt but some (who took the Transformers ride) did not (visit the Ancient Egypt).

6. Refer to the figure below, which shows six readers and four book genres (Children's Fiction, Fantasy, Mystery and Classic Literary Fiction) with selected book titles of each genre. A line is drawn between a reader and a book title if, and only if, that reader reads that book. For example, Ms Aiken reads "The Little Prince", but Mr Dueet does not read "Black Beauty".



Let R , G , and T be the sets of readers, genres and titles respectively. The following predicates are also given:

- $Male(x)$: x is male (you may assume that a person who is not a male is a female)
- $Read(x, y)$: x reads y (for indicating that a reader reads a certain title)
- $Belong(x, y)$: x belongs to y (for indicating that a title is under a certain genre)

For each of the following, indicate whether the statement is true or false, and optionally, write the quantified statement symbolically.

- a. Some title is read by all the female readers.
- b. Every reader reads some title in every genre.
- c. Some reader reads all titles of some genre.
- d. There is some genre for which some reader does not read any of its titles.

7. Given the following argument:

1. If an object is above all the triangles, then it is above all the blue objects.
 2. If an object is not above all the gray objects, then it is not a square.
 3. Every black object is a square.
 4. Every object that is above all the gray objects is above all the triangles.
- \therefore If an object is black, then it is above all the blue object.

- a. Reorder the premises in the argument to show that the conclusion follows as a valid consequence from the premises, by applying universal transitivity (Lecture 3 slide 96). (Hint: It may be helpful to rewrite the statements in if-then form and replace some statements by their contrapositives.)

You may use self-explanatory predicate names such as *Triangle(x)*, *Square(x)*, etc.

- b. Rewrite your answer in part (a) using predicates and quantified statements.

8. Let P and Q be predicates. Prove that:

- a. $(\forall x \in D P(x)) \wedge (\forall x \in D Q(x))$ is true if and only if $\forall x \in D (P(x) \wedge Q(x))$ is true.
- b. $(\exists x \in D P(x)) \wedge (\exists x \in D Q(x))$ and $\exists x \in D (P(x) \wedge Q(x))$ are not equivalent.

9. Consider the statement: $\forall x, y \in \mathbb{R} (x > y \rightarrow x^2 > y^2)$.

- a. Prove that the statement is false by studying the negation of the statement.
- b. What is wrong with this proof:

“Let $x = -1$ and $y = 2$. Then $x^2 = 1, y^2 = 4$ and $x^2 \not> y^2$, so that statement is false.”