CS1231(S) Tutorial 4: Functions

National University of Singapore

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Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.

D1. Write down all possible functions $\{a, b, c\} \to \{1, 2\}$.

D2. (a) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is neither injective nor surjective.

(b) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is injective but not surjective.

(c) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is surjective but not injective.

(d) Define a function $\mathbb{Z} \to \mathbb{Z}^+$ that is both injective and surjective.

You may use anything to define these functions, but you must give a precise definition.

D3. Find the values of the following:

(a) $\lfloor 12.31 \rfloor$;

(b) $\lceil 12.31 \rceil$;

(c) $\lfloor -12.31 \rfloor$;

(d) $\lceil -12.31 \rceil$;

(e) $\lfloor \sqrt{2} \rfloor + \lfloor \sqrt{2} + \frac{1}{2} \rfloor$;

(f) $\lfloor 2\sqrt{2} \rfloor$;

(g) $\lfloor \sqrt{2} \rfloor + \lfloor \sqrt{2} + \frac{1}{3} \rfloor + \lfloor \sqrt{2} + \frac{2}{3} \rfloor$;

(h) $\lfloor 3\sqrt{2} \rfloor$.

Tutorial questions

1. Which of the following formulas define a function $f\colon \mathbb{Q} \to \mathbb{Q}$?

   (a) $f(n) = \pm n$.

   (b) $f(n) = 2\sqrt{n}$.

   (c) $f(n) = \frac{1}{n^2+1}$.

   (d) $f(n) = \lfloor \sin n \rfloor$.

2. Let U be a set and $A \subseteq U$ such that $\varnothing \neq A \neq U$. Define the function $\chi\colon U \to \mathbb{Z}$ by setting, for all $x \in U$,
$$\chi(x) = \begin{cases} 0, & \text{if } x \notin A; \\ 1, & \text{if } x \in A. \end{cases}$$
   Find the domain, the codomain, and the image of χ.

3. Which of the functions defined in the following are injective? Which are surjective? Prove that your answers are correct. If a function defined below is both injective and surjective, then find a formula for the inverse of the function. Here we denote by Bool the set $\{\textbf{true}, \textbf{false}\}$.
$$f\colon \mathbb{Q} \to \mathbb{Q}; \quad x \mapsto 12x + 31, \qquad g\colon \mathrm{Bool}^2 \to \mathrm{Bool}; \quad (p, q) \mapsto p \wedge {\sim}q, \qquad h\colon \mathrm{Bool}^2 \to \mathrm{Bool}^2; \quad (p, q) \mapsto (p \wedge q, p \vee q),$$

$$\begin{aligned} k\colon \mathbb{Z} &\to \mathbb{Z}; \\ x &\mapsto \begin{cases} x, & \text{if } x \text{ is even;} \\ 2x-1, & \text{if } x \text{ is odd.} \end{cases} \end{aligned}$$

4. Let $f\colon B \to C$.
   (a) Suppose f is injective. Show that $g \circ f$ is injective whenever g is an injective function with domain C.
   (b) Suppose we have a function g with domain C such that $g \circ f$ is injective. Show that f is injective.

5. Let $f\colon B \to C$.
   (a) Suppose f is surjective. Show that $f \circ h$ is surjective whenever h is a surjective function with codomain B.
   (b) Suppose we have a function h with codomain B such that $f \circ h$ is surjective. Show that f is surjective.

6. Let $A = \{1, 2, 3\}$. The *order* of a bijection $f\colon A \to A$ is defined to be the least $n \in \mathbb{Z}^+$ such that
$$\underbrace{f \circ f \circ \ldots \circ f}_{n\text{-many } f\text{'s}} = \mathrm{id}_A.$$
   Define functions $g, h\colon A \to A$ by setting, for all $x \in A$,
$$g(x) = \begin{cases} 1, & \text{if } x = 2; \\ 2, & \text{if } x = 1; \\ x, & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} 2, & \text{if } x = 3; \\ 3, & \text{if } x = 2; \\ x, & \text{otherwise.} \end{cases}$$
   Find the orders of g, h, $g \circ h$, and $h \circ g$.

7. Let A, B, C be sets. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for all bijections $f\colon A \to B$ and all bijections $g\colon B \to C$.

8. Fix sets A, B. Define the *graph* of a function $f\colon A \to B$ to be
$$\{(x, y) \in A \times B : y = f(x)\}.$$
   (a) Assuming $A \neq \varnothing$, find a subset $S \subseteq A \times B$ that cannot be the graph of any function $A \to B$.
   (b) Show that a subset $S \subseteq A \times B$ is the graph of a function $A \to B$ if and only if
$$\forall x \in A \quad \exists! y \in B \quad (x, y) \in S.$$

9. Let $f\colon A \to B$ be a function. Let $X \subseteq A$ and $Y \subseteq B$.
   (a) Compare the sets X and $f^{-1}(f(X))$. Is one always a subset of the other? Justify your answer.
   (b) Compare the sets Y and $f(f^{-1}(Y))$. Is one always a subset of the other? Justify your answer.