

4. Methods of Proof

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4. Methods of Proof

4.1 Direct Proof and Counterexample

- Definitions: even and odd numbers; prime and composite.
- Proving existential statements by constructive proof.
- Disproving universal statements by counterexample.
- Proving universal statements by exhaustion.
- Proving universal statements by generalizing from the generic particular.

4.2 Proofs on Rational Numbers

- Every integer is a rational number.
- Sum of any two rational numbers is rational.

4.3 Proofs on Divisibility

- Positive divisor of a positive integer; divisors of 1; transitivity of divisibility.

4.4 Indirect Proof

- Proof by contradiction; proof by contraposition.

Reference: Epp's Chapter 4 Elementary Number Theory and Methods of Proof

4.1 Definitions

4.1.1. Definitions

Assumptions

- In this text we assume a familiarity with the laws of basic algebra, which are listed in Appendix A.
- We also use the three properties of equality: For all objects A , B , and C , (1) $A = A$, (2) if $A = B$ then $B = A$, and (3) if $A = B$ and $B = C$, then $A = C$.
- In addition, we assume that there is no integer between 0 and 1 and that the set of all integers is closed under addition, subtraction, and multiplication. This means that sums, differences, and products of integers are integers.
- Of course, most quotients of integers are not integers. For example, $3 \div 2$, which equals $3/2$, is not an integer, and $3 \div 0$ is not even a number.

Recall from Lecture #2:

Definitions: Even and Odd Integers

An integer n is **even** if, and only if, n equals twice some integer.

An integer n is **odd** if, and only if, n equals twice some integer plus 1.

Symbolically, if n is an integer, then

n is even $\iff \exists$ an integer k such that $n = 2k$.

n is odd $\iff \exists$ an integer k such that $n = 2k + 1$.

Definitions: Prime and Composite

An integer n is **prime** iff $n > 1$ and for all positive integers r and s , if $n = rs$, then either r or s equals n .

An integer n is **composite** iff $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$.

In symbols:

n is prime: $\forall r, s \in \mathbb{Z}^+$, if $n = rs$ then either $r = 1$ and $s = n$ or $r = n$ and $s = 1$.

n is composite: $\exists r, s \in \mathbb{Z}^+$ s.t. $n = rs$ and $1 < r < n$ and $1 < s < n$.



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Given the following predicate:

$$P(x) = (x \neq 1) \wedge \forall y, z (x = yz \rightarrow ((y = x) \vee (y = 1)))$$

and that the domain of x , y and z is \mathbb{Z}^+ , what is $P(x)$?

- A. $P(x)$ is true iff x is a prime number.
- B. $P(x)$ is true iff x is a number other than 1.
- C. $P(x)$ is always true irrespective of the value of x .
- D. $P(x)$ is true if x has exactly two factors other than 1 and x .
- E. None of the above.

4.1.2. Proving Existential Statements by Constructive Proof

An existential statement:

$$\exists x \in D \text{ s.t. } Q(x)$$

is true iff $Q(x)$ is true for at least one x in D .

To prove such statement, we may use **constructive proofs of existence**:

- Find an x in D that makes $Q(x)$ true; or
- Give a set of directions for finding such an x .

Example #1

- a. Prove that there exists an even integer n that can be written in two ways as a sum of two prime numbers.
- b. Suppose r and s are integers. Prove that there is an integer k such that $22r + 18s = 2k$.

a. Let $n = 10$. Then $10 = 5 + 5 = 3 + 7$, where 3, 5 and 7 are all prime numbers.

Note that the question does not say that the two prime numbers must be distinct.

b. Let $k = 11r + 9s$. Then k is an integer because it is a sum of products of integers (by closure property); and $2k = 2(11r + 9s) = 22r + 18s$ (by distributive law).

4.1.3. Disproving Universal Statements by Counterexample

Given an universal (conditional) statement:

$$\forall x \in D, P(x) \rightarrow Q(x).$$

Showing **this statement is false** is **equivalent** to showing that **its negation is true**.

The negation of the above statement is an existential statement:

$$\exists x \in D, P(x) \wedge \sim Q(x).$$

Disproving Universal Statements: Counterexample

To prove that an existential statement is true, we use an example (constructive proof), which is called the **counterexample** for the original universal conditional statement.

Disproof by Counterexample

To disprove a statement of the form

$$\forall x \in D, P(x) \rightarrow Q(x),$$

Find a value of x in D for which the hypothesis $P(x)$ is true but the conclusion $Q(x)$ is false.

Such an x is called a **counterexample**.

Disproving Universal Statements: Counterexample

Example #2: Disprove the following statement

$$\forall a, b \in \mathbb{R}, \text{ if } a^2 = b^2 \text{ then } a = b.$$

Counterexample: Let $a = 1$ and $b = -1$. Then $a^2 = 1^2 = 1$ and $b^2 = (-1)^2 = 1$ and so $a^2 = b^2$. But $a \neq b$.

4.1.4. Proving Universal Statements by Exhaustion

Given an universal conditional statement:

$$\forall x \in D, P(x) \rightarrow Q(x).$$

When D is finite or when only a finite number of elements satisfy $P(x)$, we may prove the statement by the **method of exhaustion**.

Proving Universal Statements: Exhaustion

Example #3: Prove the following statement

$\forall n \in \mathbb{Z}$, if n is even and $4 \leq n \leq 26$, then n can be written as a sum of two primes.

Proof (by method of exhaustion):

▪ $4 = 2 + 2$

▪ $6 = 3 + 3$

▪ $8 = 3 + 5$

▪ $10 = 5 + 5$

▪ $12 = 5 + 7$

▪ $14 = 11 + 3$

▪ $16 = 5 + 11$

▪ $18 = 7 + 11$

▪ $20 = 7 + 13$

▪ $22 = 5 + 17$

▪ $24 = 5 + 19$

▪ $26 = 7 + 19$

4.1.5. Proving Universal Statements by Generalizing from the Generic Particular

The most powerful technique for proving a universal statement is one that works regardless of the size of the domain (possibly infinite) over which the statement is quantified.

Generalizing from the Generic Particular

To show that every element of a set satisfies a certain property, suppose x is a *particular* but *arbitrarily chosen* element of the set, and show that x satisfies the property.

Example #4: Prove that the sum of any two even integers is even.

Proof:

1. Let m and n be two particular but arbitrarily chosen even integers.

1.1 Then $m = 2r$ and $n = 2s$ for some integers r and s
(by definition of even number)

1.2 $m + n = 2r + 2s = 2(r + s)$ (by basic algebra)

1.3 $2(r + s)$ is an integer (by closure under integer addition and multiplication) and an even number (by definition of even number)

1.4 Hence $m + n$ is an even number.

2. Therefore the sum of any two even integers is even.

4.2 Proofs on Rational Numbers

Definition

4.2.1. Definition

In this section, we will apply proof techniques we have learned on rational numbers.

Definition: Rational Numbers

A real number r is **rational** if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator.

A real number that is not rational is **irrational**.

$$r \text{ is rational} \iff \exists \text{ integers } a \text{ and } b \text{ such that } r = \frac{a}{b} \text{ and } b \neq 0.$$

4.2.2. Every Integer is a Rational Number

Theorem 4.2.1 (5th: 4.3.1)

Every integer is a rational number.

Proof:

1. Let a be a particular but arbitrarily chosen integer.
 - 1.1 Then $a = \frac{a}{1}$ which is in the form $\frac{a}{b}$ where a and $b (= 1)$ are integers.
 - 1.2 Hence a is a rational number.
2. Therefore every integer is a rational number.

4.2.3. The Sum of Any Two Rational Numbers is Rational

Theorem 4.2.2 (5th: 4.3.2)

The sum of any two rational numbers is rational.

Proof:

1. Let r and s be two particular but arbitrarily chosen rational numbers.
 - 1.1 Then $r = \frac{a}{b}$ and $s = \frac{c}{d}$ for some integers a, b, c, d with $b \neq 0$ and $c \neq 0$ (by definition of rational number).
 - 1.2 Then $r + s = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ (by basic algebra).
 - 1.3 Since $ad + bc$ and bd are integers (by closure under integer addition and multiplication) and $bd \neq 0$, so $r + s$ is rational.
2. Therefore the sum of any two rational numbers is rational.

Recall from Lecture #2:

Corollary

A result that is a **simple deduction** from a theorem.

Example:

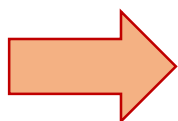
- (Chapter 4)

Theorem 4.2.2 (5th: 4.3.2) The sum of any two rational numbers is rational

Corollary 4.2.3 (5th: 4.3.3) The double of a rational number is rational.

Theorem 4.2.2 (5th: 4.3.2)

The sum of any two rational numbers is rational.



Corollary 4.2.3 (5th: 4.2.3)

The double of a rational number is rational.

4.3 Proofs on Divisibility

Definition

4.3.1. Definition

Recall from Lecture #2:

Definition: Divisibility

If n and d are integers and $d \neq 0$, then

n is **divisible** by d iff n equals d times some integer.

We use the notation $d \mid n$ to mean “ d divides n ”. Symbolically, if $n, d \in \mathbb{Z}$ and $d \neq 0$:

$$d \mid n \iff \exists k \in \mathbb{Z} \text{ such that } n = dk.$$

4.3.2. Theorems

Theorem 4.3.1 (5th: 4.4.1) A Positive Divisor of a Positive Integer

For all positive integers a and b , if $a \mid b$, then $a \leq b$.

Proof (direct proof):

1. Let a and b be two positive integers and $a \mid b$.
 - 1.1 Then there exists an integer k such that $b = ak$ (by definition of divisibility).
 - 1.2 Since both a and b are positive integers, k is positive, i.e. $k \geq 1$.
 - 1.3 Therefore $a \leq ak = b$.
2. Therefore for all positive integers a and b , if $a \mid b$, then $a \leq b$.

Theorem 4.3.2 (5th: 4.4.2) Divisors of 1

The only divisors of 1 are 1 and -1.

Proof (by division into cases):

1. Suppose m is any integer that divides 1.
 - 1.1 Then there exists an integer k such that $1 = mk$ (by definition of divisibility).
 - 1.2 Since mk is positive, either both m and k are positive, or both negative.
 - 1.3 Case 1: Both m and k are positive.
 - 1.3.1 Since $m \mid 1$, $m \leq 1$ (by Theorem 4.3.1).
 - 1.3.2 Then $m = 1$.
 - 1.4 Case 2: Both m and k are negative.
 - 1.4.1 Then $-m$ is a positive integer divisor of 1, i.e. $-m \mid 1$.
 - 1.4.2 By the same reasoning in 1.3.1 and 1.3.2, $-m = 1$, or $m = -1$.
2. Therefore the only divisors of 1 are 1 and -1.

Theorems: Transitivity of Divisibility

Theorem 4.3.3 (5th: 4.4.3) Transitivity of Divisibility

For all integers a, b and c , if $a \mid b$ and $b \mid c$, then $a \mid c$.

Proof:

1. Suppose a, b, c are integers s.t. $a \mid b$ and $b \mid c$.
 - 1.1 Then $b = ar$ and $c = bs$ for some integers r and s . (by definition of divisibility)
 - 1.2 Then $c = bs = (ar)s$ (by substitution) $= a(rs)$ (associativity)
 - 1.3 Let $k = rs$, then k is an integer (by closure property) and $c = ak$.
2. Therefore $a \mid c$.

4.4 Indirect Proof

4.4.1. Indirect Proof: Proof by Contradiction

Sometimes when a direct proof is hard to derive, we can try indirect proof.

Example: Theorem 4.7.1 (5th: 4.8.1) $\sqrt{2}$ is irrational.

Proof by Contradiction

1. Suppose the statement to be proved, S , is false. That is, the negation of the statement, $\sim S$, is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement S is true.

Theorem 4.6.1 (5th: 4.7.1)

There is no greatest integer.

Proof (by contradiction):

1. Suppose not, i.e. there is a greatest integer.
 - 1.1 Let call this greatest integer g , and $g > n$ for all integers n .
 - 1.2 Let $G = g + 1$.
 - 1.3 Now, G is an integer (closure of integers under addition) and $G > g$.
 - 1.4 Hence, g is not the greatest integer \rightarrow contradicting 1.1.
2. Therefore there is no greatest integer.

4.4.2. Indirect Proof: Proof by Contraposition

Recall: Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Proof by Contraposition

1. Statement to be proved: $\forall x \in D, P(x) \rightarrow Q(x)$.
2. Rewrite the statement into its contrapositive form:
$$\forall x \in D, \sim Q(x) \rightarrow \sim P(x).$$
3. Prove the contrapositive statement by a direct proof.
 - 3.1 Suppose x is an (particular but arbitrarily chosen) element of D s.t. $Q(x)$ is false.
 - 3.2 Show that $P(x)$ is false.
4. Therefore, the original statement $\forall x \in D, P(x) \rightarrow Q(x)$ is true.

Proposition 4.6.4 (5th: 4.7.4)

For all integers n , if n^2 is even then n is even.

Proof (by contraposition):

1. Contrapositive statement:

For all integers n , if n is odd then n^2 is odd.

2. Let n be an arbitrarily chosen odd number.

2.1 Then $n = 2k + 1$ for some integer k (definition of odd number).

2.2 Then $n^2 = (2k + 1)(2k + 1) = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

2.3 Let $m = 2k^2 + 2k$. Now, m is an integer (closure property) and $n^2 = 2m + 1$.

2.4 So n^2 is odd.

3. Therefore, for all integers n , if n^2 is even then n is even.

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