CS1231(S) Tutorial 5: Mathematical Induction

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More challenging questions are indicated by an asterisk (*). When asked to prove a statement by induction, one may use regular or Strong Mathematical Induction.

Terminology

Definition 8.1.1. Let $n, d \in \mathbb{Z}$. Then d is said to *divide* n if

$$n = dk \quad \text{for some } k \in \mathbb{Z}.$$

We write $d \mid n$ for "d divides n", and $d \nmid n$ for "d does not divide n".

Questions for discussion on the LumiNUS Forum

Answers to these questions will not be provided.

D1. Prove by induction that for all $n \in \mathbb{Z}_{\geqslant 0}$,

$$1 \times 2^1 + 2 \times 2^2 + \cdots + n \times 2^n + (n+1) \times 2^{n+1} = n2^{n+2} + 2.$$

D2. Prove by induction that 6 divides $7^n - 1$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

D3. What is wrong (if any) with the following proof that $2^n = 1$ for all $n \in \mathbb{Z}_{\geqslant 0}$?

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition " $2^n = 1$ ".
2. (Base step) $P(0)$ is true because $2^0 = 1$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1) \ldots, P(k)$ are true, i.e., that
   $$2^0 = 2^1 = \cdots = 2^k = 1.$$
   3.2. Then $\quad 2^{k+1} = \dfrac{2^k \times 2^k}{2^{k-1}}$
   3.3. $\quad = \dfrac{1 \times 1}{1}$ by the induction hypothesis;
   3.4. $\quad = 1.$
   3.5. Thus $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true by Strong MI.

D4. Abelard (a twelfth-century Parisian logician) and Eloise (the niece of a canon of Notre Dame) are playing games. Each game has a fixed length, say $n \in \mathbb{Z}_{\geqslant 0}$. In the game, the players take turns to play a move, starting with Eloise. A play of the game thus looks like

$$(x_1, x_2, \ldots, x_n),$$

where $x_1, x_3, \ldots$ are the moves by Eloise, and $x_2, x_4, \ldots$ are the moves by Abelard. When a player plays a move x_i, she/he is able to see all the previous moves $x_1, x_2, \ldots, x_{i-1}$

in the game. The rules of the game, set out before the game begins, consist of a set R: Eloise wins if and only if the play of the game $(x_1, x_2, \ldots, x_n)$ is an element of R. There is no draw.

Show by induction on n that no matter what n and R are, one of the players can guarantee a win.

D5. Peter needs to climb a flight of stairs of n steps, where $n \in \mathbb{Z}_{\geqslant 1}$. He can go up 1 or 2 steps with each stride. Let s_n be the number of ways in which Peter can climb n steps. (So $s_2 = 2$ for example, since he can climb 2 steps in 1 stride going up 2 steps, or in 2 strides each going up 1 step.)

(a) Express s_n in terms of $s_1, s_2, \ldots, s_{n-1}$.
(b) What is the sequence $s_1, s_2, \ldots$?

Tutorial questions

1. Prove by induction that for all $n \in \mathbb{Z}_{\geqslant 1}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1).$$

2. Let $x \in \mathbb{R}_{\geqslant -1}$. Prove by induction that $1 + nx \leqslant (1+x)^n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

3. Prove by induction that 3 divides $n^3 + 11n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

4. Let a be an odd integer. Prove by induction that 2^{n+2} divides $a^{2^n} - 1$ for all $n \in \mathbb{Z}_{\geqslant 1}$. (Note that $a^{b^c} = a^{(b^c)}$ by convention.)

5.* Prove by induction that

$$\forall n \in \mathbb{Z}_{\geqslant 8}\ \exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n = 3x + 5y).$$

(As a consequence, any integer-valued transaction over 8 dollars can be carried out using only 3-dollar and 5-dollar coins.)

6.* Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall n \in \mathbb{Z}_{\geqslant 1}\ \exists \ell \in \mathbb{Z}_{\geqslant 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}\ (i_1 < i_2 < \cdots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}).$$

(Hint: think in terms of binary representations.)

Definition 7.2.2. The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by setting

$$F_0 = 0 \quad \text{and} \quad F_1 = 1 \quad \text{and} \quad F_{n+2} = F_{n+1} + F_n$$

for each $n \in \mathbb{Z}_{\geqslant 0}$.

7. Show that $F_{n+4} = 3F_{n+2} - F_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

8. Show by induction that $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

9. Let $a_0, a_1, a_2, \ldots$ be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. Prove by induction that $a_n < 3^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

10. Define a set S recursively as follows.

(a) $2 \in S$. (base clause)
(b) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)
(c) Membership for S can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $0, 6, 15, 16, 36$ are in S? Which are not?