# CHAPTER 2 SETS, FUNCTIONS

## SECTION 2.1 SETS

**DEFINITION:**

A **SET** is an unordered collection of objects.

The objects of a set are the **ELEMENTS** or **MEMBERS** of the set.

$1, 2, \cdots, 10 \in \mathbb{Z} \leftarrow$ set of integers

**REMARK**

$1 \in \{1, 2\}$ $3 \notin \{1, 2\}$

- **NOTATIONS** $x \in A$: object $x$ is a member of the set $A$.

  $x \notin A$: object $x$ is not a member of the set $A$.

  $x_1, \ldots, x_n \in A$: $x_1, \ldots, x_n$ are members of $A$.

- One way to describe a set is to lists its members within a pair of braces.

  $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. $\neq (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$

  The set of positive odd integers less than 10: $\{1, 3, 5, 7, 9\}$.

  $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$

**SOME IMPORTANT SETS**

- $\mathbb{R}$: real nos. $0, 1, 2, -1, -2, \frac{1}{2}, -\frac{1}{3}, \pi, -\sqrt{2}$
- $\mathbb{Q}$: rational nos. $\frac{1}{2}$ $-\frac{1}{3}$ $5 = \frac{5}{1}$ $\sqrt{2}, \pi$ not rational
- $\mathbb{Z}$: integers.

  $0, 1, -1, 2, -2, \ldots$

0 positive? No
negative? No

0 even? yes.
odd? no.

- Positive nos. are $> 0$.
- Negative nos. are $< 0$.
- Nonnegative nos. are $\geq 0$.

- $\mathbb{R}^+$: pos. real nos.
- $\mathbb{R}^-$: neg. real nos.
- $\mathbb{R}_{\geq 0}$: nonneg. real nos.

$\mathbb{Z}_{\geqslant 2} = \{2, 3, 4, 5, \cdots\}$

$\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geq 0}$ are similarly defined.

$\mathbb{N} = \{0, 2, 1, 3, 5, 4, \dots\}$

- $\mathbb{N}$: natural nos. (In this module, $\mathbb{N} = \mathbb{Z}_{\geq 0}$)
- $\mathbb{C}$: complex nos.

  $\mathbb{C}$

$\underline{\underline{\mathbb{N}}} = \{0, | 1, | 2, | 3 \dots\}$

1

2

3

• Sometimes the $\ldots$ is used to represent elements that are understood. For example, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Z}^+ = \{1, 2, \ldots\}$

$\mathbb{N} = \{0, 1, 2, \ldots\}$

- A set can also be defined by listing its properties:

The set of positive even numbers less than 100:

Domain

$$\{x \in \mathbb{Z}^+ \mid x/2 \in \mathbb{Z}^+, x < 100\}.$$

$\{x \in \mathbb{Z}^+ \mid x/2 \in \mathbb{Z}^+, x/3 \in \mathbb{Z}^+, x/4 \in \mathbb{Z}^+, x < 100\}$

You can also use $\{\ldots : \ldots\}$ instead of $\{\ldots \mid \ldots\}$.

For example, $\{x \in \mathbb{Z}^+ \mid x/2 \in \mathbb{Z}^+, x < 100\}$ can also be written as

$$\{x \in \mathbb{Z}^+ : x/2 \in \mathbb{Z}^+, x < 100\}.$$

$\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} = \{x \mid x \in \mathbb{R}, x > 0\}$,

$\mathbb{Z}_{\geq 0} = \{x \in \mathbb{Z} \mid x \geq 0\}$, etc.

wrong notation

× $\{x, x \in \mathbb{R}, x > 0\}$

× $\{x \in \mathbb{R}, x > 0\}$

Notation

$\{\text{domain} \mid \text{prop}\}$

$x \in \mathbb{Z}$

more general

$\{x \mid x \in \mathbb{Z}, \text{prop}\}$

$\boxed{\times \{x > 0 \mid x \in \mathbb{R}\}}$

$\mathbb{Z}$ is one elet in $\{\mathbb{Z}, \mathbb{N}, \mathbb{Q}\}$

- Members of a set can themselves be sets.

Thus $\{\mathbb{Z}, \mathbb{N}, \mathbb{Q}\}$ is a set with 3 elements which are also sets.

$\mathbb{Z}$ infinite set

$(-1)^2 = 1^2$, $(-3)^2 = 3^2$, $(-5)^2 = 5^2$

**EXAMPLE**

sets

The set of all integers that are squares of an odd integer.

odd integers $= \{1, -1, 3, -3, \ldots\}$

**SOLN 1:** $\{1^2, 3^2, 5^2, \ldots\}$.

$= \{1^2, (-1)^2, 3^2, (-3)^2, \ldots\}$

**SOLN 2:** $\{x \mid x$ is the square of an odd integer$\}$. ~~$\{x$ is an odd integer $\mid$~~ elet in the set

prop.

**SOLN 3:** $\{x^2 \mid x$ is an odd integer $\}$.

not a sentence $\rightarrow$ $x^2$

$= \{$ odd integers $\}$

elet in the set

$\{\boxed{x}$ odd integer $\mid \underbrace{x^2 < 10}_{\text{must be a sentence.}}\}$

$= \{1, -1, 3, -3\}$ must be a sentence.

**SET EQUALITY**

The sets $A$ and $B$ are **EQUAL** if they have the same members. We write $A = B$. Thus

$$A = B \quad \text{iff} \quad \forall x(x \in A \leftrightarrow x \in B).$$

WANT TO prove $\underline{A = B}$

$x \in A \Rightarrow \dots$ | $x \in B \Rightarrow \dots$

Any elt $\Rightarrow \dots$ | $\Rightarrow \dots$

$\Rightarrow x \in B$ | $\Rightarrow x \in A$

$x \in A \rightarrow x \in B$ | $x \in B \rightarrow x \in A$

v.s. numbers $a = b$

$\underline{a \le b}$ and $\underline{b \le a}$

**Order, Repetition Do Not Matter**

For example $\{1,3,7\} = \{7,1,3\} = \boxed{\{7,1,1,1,3,3,1,1\}}$.

only 3 elets in it.

$\boxed{\textbf{EXAMPLE}}$

Show that $A = B$ where

$$A = \{x \in \mathbb{Z} \mid x^8 - 1 = 0\}, \quad B = \{x \in \mathbb{Z} \mid x^4 - 1 = 0\}.$$

**PROOF:**

1. We need to show

$$x \in A \Rightarrow x \in B \quad \text{and} \quad x \in B \Rightarrow x \in A.$$

Same meaning in a proof. $\left\{ \begin{array}{l} \Rightarrow \quad \therefore \\ \rightarrow \end{array} \right.$ not same as "="

2. We have

$$\Rightarrow x^4 - 1 = 0 \quad (\text{by def of } B)$$

$$\begin{aligned}
(\text{Step 2.1})\ x \in B &\Rightarrow x^4 = 1 && (\text{by } 2.2) \\
&\Rightarrow x^8 = 1 \Rightarrow x^8 - 1 = 0 && (\text{by } 2.3)\ (\text{by } 2.4) \\
&\Rightarrow x \in A && (\text{by } 2.5 \text{ and def of } A)
\end{aligned}$$

$(2.1)\ \boxed{x \in B} \Rightarrow x^4 - 1 = 0 \quad (\text{by def of } B)$

$\Rightarrow \cancel{x^8 - 1 = x^4 - 1}\ (\text{by} \cdots)$

$\Rightarrow \boxed{x \in A}$

e.g.

$$x > 2 \Rightarrow x > 1$$

"$\neq$"

$a^2 - b^2 = (a-b)(a+b)$

$x^8 - 1$

$= (x^4)^2 - 1^2 = (x^4-1)(x^4+1) = 0$

3. $$\begin{aligned} (3.1)\ x \in A &\Rightarrow^{(3.2)} x^8 - 1 = (x^4-1)\underbrace{(x^4+1)}_{\geq 1} = 0 \quad \text{(by def of A)} \\ &\Rightarrow^{(3.3)} \underline{x^4 - 1 = 0} \text{ (by 3.2)} \quad \neq 0 \ (\because x \in \mathbb{R}) \\ &\Rightarrow^{(3.4)} \underline{x \in B.} \quad \text{(by def of B)} \end{aligned}$$

**DEFINITION:**

Let $A$, $B$ be sets. The set $A$ is a **SUBSET** of the set $B$ if every element of $A$ is an element of $B$. $\boxed{\forall x \, (x \in A \to x \in B)}$

We write

$$\boxed{A \subseteq B.}$$

$$A = B \iff \overbrace{\forall x \, (x \in A \to x \in B)}^{A \subseteq B} \wedge \underbrace{\forall x \, (x \in B \to x \in A)}_{B \subseteq A}$$

$A \nsubseteq B$ iff $\sim(\forall x\,(x \in A \to x \in B))$

$A \subseteq B$ iff $\forall x\,(x \in A \to x \in B)$

Clearly, $A$ is not a subset of $B$ if it has an element that is not an element of $B$, i.e.,

$\exists x \sim (x \in A \to x \in B)$

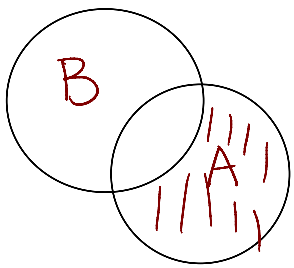
$$A \not\subseteq B \quad \text{iff} \quad \exists x((x \in A) \wedge (x \notin B))$$

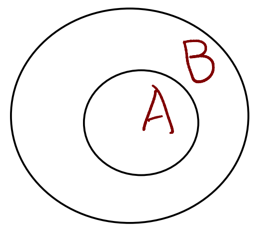
$\exists x\,(x \in A \wedge x \notin B)$

For example $\mathbb{Z} \subseteq \mathbb{Q}$, and $\mathbb{Q} \subseteq \mathbb{R}$. That is, $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

$p \to q \equiv \sim p \vee q$

$\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$




$A \subseteq B$

$A \nsubseteq B$

$\exists x\,(x \in A \wedge x \notin B)$

$\mathbb{Z} \subseteq \mathbb{Z}$ ✓

**DEFINITION:**

The set $A$ is a **PROPER SUBSET** of the set $B$ if $A \subseteq B$ and $A \neq B$.

We write $A \subsetneq B$.

$A \subseteq B \wedge A \neq B$

$\mathbb{Z}$ is not a proper subset of $\mathbb{Z}$

**DEFINITION:**

**THE UNIVERSAL SET** is the set that consists of all the objects under discussion and is usually denoted by $U$.

In different contexts, we have different universal sets.

The set that has no members are called the **THE EMPTY SET** or **NULL SET**, and is denoted by $\emptyset$ or $\{\ \}$.

A set with a single element is called a **SINGLETON SET**. $\{a\}$

$\{x\}$ $\{0\}$

$A \subseteq B \quad \forall x \ (x \in A \to x \in B)$

**THEOREM:**

For every set $S$, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$

Proof: (i) We need to prove $\forall x, \underbrace{\overset{F}{\boxed{x \in \emptyset}} \to x \in S}_{T}$.

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| F | * | T |

This is *vacuously true* since $x \in \emptyset$ is always false.

(ii) is left as exercise.

$\emptyset$ no elet in it

$1 \in \emptyset \quad F$

$x \in \emptyset \quad F$

(ii) $\forall x \quad \underline{x \in S} \to x \in S$

$T \to T$

$F \to F$ } T

"$\phi$" empty box

$\neq$

**REMARK** $\{\phi\} = \{$empty box$\}$ has only one elet in it

Note that $\{\emptyset\}$ is **not** empty.

called "$\phi$" (empty box)

It is a singleton set whose element is the empty set.

Similarly, $\{\{1\}, 2\}$ is **not** $\{1, 2\}$.

$\{1,2\}$ has 2 elets: 1, 2

$\{\{1\}, 2\}$ has 2 elements: $\{1\}$ and 2.

$\{1\} \in \{\{1\}, 2\}$

$1 \notin \{\{1\}, 2\}$

## DISTINCTION BETWEEN $\in$ AND $\subseteq$

$\boxed{\{2\}} \notin \{\{1\}, \boxed{2}\}$

The following expressions are correct:

$\boxed{2} \in \{1, \boxed{2}, 3\}$; $\boxed{\{2\}} \in \{\{1\}, \boxed{\{2\}}\}$ v.s. $\boxed{2} \notin \{\{1\}, \boxed{\{2\}}\}$

$\{\boxed{2}\} \subseteq \{1, \boxed{2}, 3\}$ $\{\boxed{1}, \boxed{2}\} \not\subseteq \{\boxed{1}, \{1, 2\}\}$

v.s.

$\{\boxed{\{2\}}\} \subseteq \{\{1\}, \boxed{\{2\}}\}$. $\{\boxed{\{1, 2\}}\} \subseteq \{1, \boxed{\{1, 2\}}\}$

$\boxed{2} \notin \{1, \boxed{\{2\}}\}$

$\boxed{\{2\}} \in \{1, 2, \boxed{\{2\}}\}$

$\{\boxed{2}\} \subseteq \{1, \boxed{2}, \{2\}\}$

The following expressions are incorrect:

$$\boxed{\{2\}} \notin \{1, \boxed{2}, 3\}$$

not a set $\rightarrow$ $\cancel{2} \subseteq \{1, 2, 3\}$ set $\subseteq$ set ~~number $\subseteq$ set~~

$$\{\boxed{2}\} \not\subseteq \{\{1\}, \boxed{\{2\}}\}.$$

**DEFINITION:**

Let $S$ be a set. $|\{1,2\}| = 2$ $|\{\{1,2,2\}\}| = 1$

If there are exactly $n$ different elements in the set, we say that $S$ is a **FINITE SET** and that $n$ is its **CARDINALITY**.

We write $|S| = n$. $|\underbrace{\{1,2,\cancel{2}\}}_{\text{How many elets.}}| = 2$

**EXAMPLE**

- $|A| = 50$ where $A = \{x \in \mathbb{N} \mid x < 100, x \text{ odd}\}$.
- $|\emptyset| = 0$.

**DEFINITION:**

Let $A$ be a set.

The **POWER SET** of $A$, written $P(A)$, is the set of all subsets of $A$.

$P(A) = \{ \text{ list out all subsets of } A \}$

**EXAMPLE** $|\phi| = 0$ $|P(\phi)| = 1 = 2^0$

- $P(\emptyset) = \{\emptyset\}$. $P(\phi) = \{ \phi \}$

$|\{1\}| = 1$

- $P(\{1\}) = \{\emptyset, \{1\}\}$ has $2 = 2^1$ elets $P(\{1\}) = \{ \phi, \{1\} \}$
- $P(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. has $4 = 2^2$ elets

$P(\{1,2\}) = \{ \phi, \underbrace{\{1\}, \{2\}}_{1 \text{ elets}}, \{1,2\} \}$

$|\{1,2\}| = 2$

Later, we'll prove the following theorem which explains the term "power set".

**THEOREM:** If $|S| = n$, then $|P(S)| = 2^n$.

$|\phi| = 0$ $\quad |\{\phi\}| = 1$ ($\because$ $\{\phi\}$ has 1-elet in it and the elet is $\phi$)

$|\{1,2\}| = 2$ $\quad |\{\{\phi\}\}| = 1$

$|\{\phi, 1, 2\}| = 3$

$\phi \subseteq$ every set

"$\phi \in$ every set" ✗ $\quad \phi \in \{\phi, 1, 2\}$ ✓

$\phi \in \{1, 2\}$ ✗

$\phi = \{\ \}$ ✓ $\neq \underbrace{\{\phi\}}$

not empty it has $\phi \in \{\phi\}$

## CARTESIAN PRODUCTS

**DEFINITION:**

order does not matter for sets. $\{1, 2\} = \{2, 1\}$

Let $n \in \mathbb{N}$.

The **ORDERED $n$-TUPLE**,

order matters! $(1, 2) \neq (2, 1)$

$$(x_1, \ldots, x_n)$$

ordered $\neq \{1, 2\}$

is the ordered collection that has

$x_1$ as the first element, $x_2$ as the second element, ...,

and $x_n$ as the $n^{\text{th}}$ element.

Two ordered $n$-tuples $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ are equal if

$(1^2, 2^2, 3^2)$

$$x_1 = y_1,\ \ldots,\ x_n = y_n.$$

$= (1, 4, 9)$

$(1, 2)$

$\neq (4, 9, 1)$

An **ORDERED PAIR** is an ordered 2-tuple, and an **ORDERED TRIPLE** an ordered 3-tuple.

- Do not confuse $(x_1, \ldots, x_n)$ with $\{x_1, \ldots, x_n\}$.
- $(1, 2) \neq (2, 1)$, $(3, (-2)^2, .5) = (\sqrt{9}, 4, .5)$.

$(1, 1, 2) \neq (2, 1, 1) \neq (1, 2)$

$\sqrt{9} \stackrel{?}{=} \pm 3$

$\sqrt{9} = 3$

$\pm\sqrt{9} = \pm 3$

$(\sqrt{9}, -\sqrt{9}) = (3, -3)$

$\{1, -1\} \neq (1, -1)$

$\{1, 2, 3\} = \{3, 2, 1\}$

$\{1, 2, 3\} \neq (1, 2, 3)$

$\{1, 2, 3\} \neq (3, 2, 1)$

**DEFINITION:**

The **CARTESIAN PRODUCT** of a set $A$ and a set $B$, written $A \times B$, (read "$A$ cross $B$"), is the set of all ordered pairs $(x, y)$ where $x \in A$, $y \in B$. Thus

$\mathbb{R} \times \mathbb{R}$

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

The **CARTESIAN PRODUCT** of the sets $A_1, \dots, A_n$ is

$$A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}.$$

If $A_1 = \ldots = A_n = A$, then

$$A_1 \times \cdots \times A_n = A^n.$$

$\boxed{\textbf{EXAMPLE}}$

$$\{0, 1, x\} \times \{a, b\} \quad = \{(\triangle, \circ)\}$$
$$= \{(0, a), (0, b), (1, a), (1, b), (x, a), (x, b)\}$$

$$= \{(0, a), (0, b)$$
$$(1, a), (1, b)$$
$$(x, a), (x, b) \}$$

$$\{0, 1, x\} \times \{a, b\}$$
$$\neq$$
$$\{a, b\} \times \{0, 1, x\}$$
$$= \{(a, 0), (a, 1), (a, x)$$
$$(b, 0), (b, 1), (b, x)\}$$

$$
\begin{aligned}
\{0,1\} \times \{0,1\} \times \{x,y\} &= \{ (\triangle, \bigcirc, \square) \} \\
&= \{(0,0,x),(0,0,y),(0,1,x),(0,1,y), \\
&\quad (1,0,x),(1,0,y),(1,1,x),(1,1,y)\}
\end{aligned}
$$

$\boxed{\textbf{THEOREM: } |A \times B| = |A| \times |B|.}$

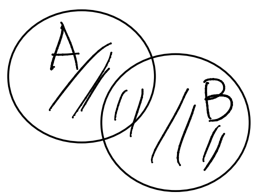
## SECTION 2.2 SET OPERATIONS

**DEFINITION:**

Let $A$, $B$ be subsets of a universal set $U$.

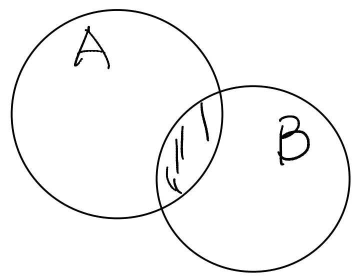
The **UNION** of $A$ and $B$, written $A \cup B$, is the set that contains elements that are in $A$ or in $B$ or in both, i.e.,

$$A \cup B = \{x \mid \underbrace{(x \in A) \vee (x \in B)}\}.$$



2. The **INTERSECTION** of $A$ and $B$, written $A \cap B$, is the set that contains elements that are in both $A$ and $B$, i.e.,

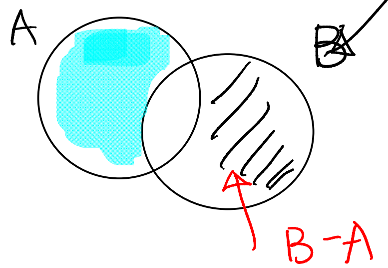
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$



3. The **COMPLIMENT** of $A$ in $B$ (**DIFFERENCE** of $B$ with $A$), written $B - A$ or $B \setminus A$, is the set that contains elements that are in $B$ but not in $A$, i.e.,

$$B - A = B \setminus A = \{x \mid (x \in B) \wedge (x \notin A)\}.$$

minus

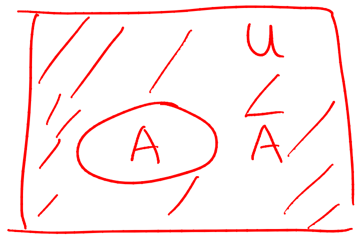



$A - B \neq B - A$

complement of B in A

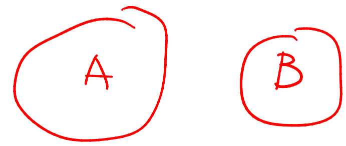
4. The **COMPLEMENT** of $A$ is the set $\overline{A} = U - A$, i.e.,

$$\overline{A} = \{x \mid x \notin A\}$$



5. Two sets $A$ and $B$ are **DISJOINT** if

$$A \cap B = \emptyset.$$

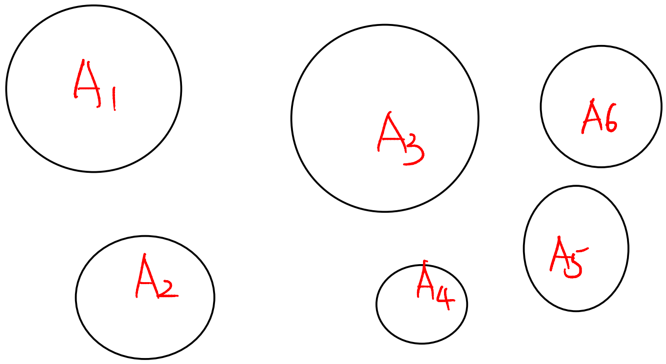


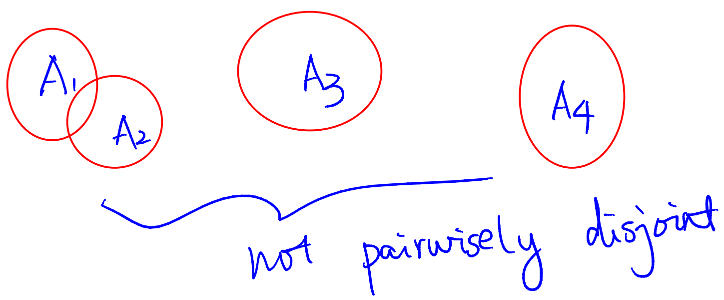

$A = \{1, 2\}$ $B = \{4, 5\}$

$A, B$ disjoint

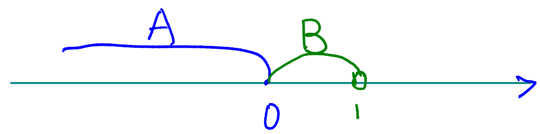
6. Sets $A_1, \ldots, A_n$ are **MUTUALLY or PAIRWISE DISJOINT** if

$$A_i \cap A_j = \emptyset \quad \forall i \neq j$$



$\boxed{\textbf{EXAMPLE}}$




Let $U = \mathbb{R}$,

$A = \{x \mid x \leq 0\} = (-\infty, 0]$, $\quad B = \{x \mid 0 \leq x < 1\} = [0, 1)$.

Then

$$\begin{aligned} A \cup B &= \{x \mid (x \leq 0) \vee (0 \leq x < 1)\} \\ &= \{x \mid x < 1\} = (-\infty, 1) \end{aligned}$$

$$\begin{aligned} A \cap B &= \{x \mid (x \leq 0) \wedge (0 \leq x < 1)\} \\ &= \{0\} \end{aligned}$$

$\sim(x \geq 0 \wedge x < 1)$

$$\overline{B} = \{x \mid \sim (0 \leq x < 1)\}$$

$\sim(x \geq 0) \vee \sim(x < 1)$

$$= \{x \mid (x < 0) \vee (x \geq 1)\}$$

De Morgan's Law

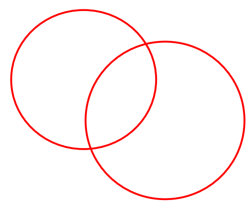
$$= (-\infty, 0) \cup [1, \infty)$$

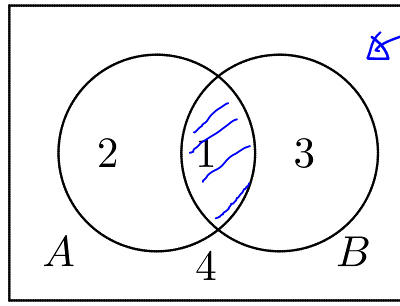
cannot be used to show '⊆' for power set

for prob solving

**VENN DIAGRAMS**

The relation between 2 or 3 sets can be visualized effectively with a Venn diagram.






If you use Venn Diag for question (with 2 sets) then use this one "only"

Region "1"

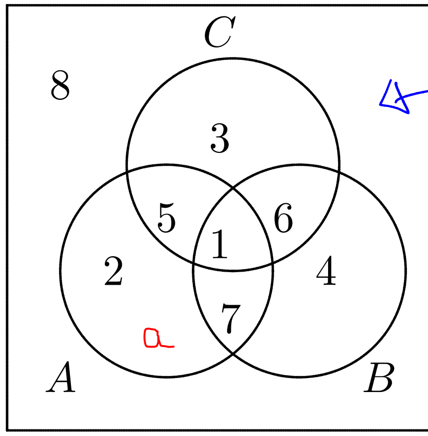
put together

$A = 1 + 2,\ B = 1 + 3,\ A \cup B = 1 + 2 + 3.$

$A \cap B = 1,\ A - B = 2,\ B - A = 3.$

$\overline{A \cup B} = 4,\ \overline{A} = 3 + 4,\ \overline{B} = 2 + 4,$

If no intersection of A&B then say $1 = 7 = \phi$




standard venn diag for 3 sets

Region

$A = 1 + 2 + 5 + 7, \ B = 1 + 4 + 6 + 7,$

$A \cap B = 1 + 7, \ A \cap B \cap C = 1$

$\overline{A \cup B \cup C} = 8, \text{ etc.}$

$\overline{A \cup B \cup C} = 8$

$= \bar{A} \cap \bar{B} \cap \bar{C}$

$= (3+4+6+8) \cap (2+3+5+8)$

$\cap (2+4+7+8) = 8$

X nonstandard Venn Diagram.