CHAPTER 2 SETS, FUNCTIONS

SECTION 2.1 SETS

DEFINITION:

A **SET** is an unordered collection of objects.

The objects of a set are the **ELEMENTS** or **MEMBERS** of the set.

REMARK

- **NOTATIONS** $x \in A$: object x is a member of the set A.

  $x \notin A$: object x is not a member of the set A.

  $x_1, \ldots, x_n \in A$: $x_1, \ldots, x_n$ are members of A.

- One way to describe a set is to lists its members within a pair of braces.

  $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

  The set of positive odd integers less than 10: $\{1, 3, 5, 7, 9\}$.

SOME IMPORTANT SETS

- $\mathbb{R}$: real nos.
- $\mathbb{Q}$: rational nos.
- $\mathbb{Z}$: integers.
- Positive nos. are > 0.
- Negative nos. are < 0.
- Nonnegative nos. are ≥ 0.
- $\mathbb{R}^+$: pos. real nos.
- $\mathbb{R}^-$: neg. real nos.
- $\mathbb{R}_{\geq 0}$: nonneg. real nos.

  $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{Z}_{\geq 0}$ $\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{Q}_{\geq 0}$ are similarly defined.

- $\mathbb{N}$: natural nos. (In this module, $\mathbb{N} = \mathbb{Z}_{\geq 0}$)
- $\mathbb{C}$: complex nos.
- Sometimes the $\ldots$ is used to represent elements that are understood. For example,

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \quad \mathbb{Z}^+ = \{1, 2, \ldots\} \quad \mathbb{N} = \{0, 1, 2, \ldots\}$$

- A set can also be defined by listing its properties:

  The set of positive even numbers less than 100 $= \{x \in \mathbb{Z}^+ \mid x/2 \in \mathbb{Z}^+, x < 100\}$.

  You can also use $\{\ldots : \ldots\}$ instead of $\{\ldots \mid \ldots\}$.

  For example, $\{x \in \mathbb{Z}^+ \mid x/2 \in \mathbb{Z}^+, x < 100\}$ can also be written as

$$\{x \in \mathbb{Z}^+ : x/2 \in \mathbb{Z}^+, x < 100\}.$$

  $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\} = \{x \mid x \in \mathbb{R}, x > 0\}$, $\mathbb{Z}_{\geq 0} = \{x \in \mathbb{Z} \mid x \geq 0\}$, etc.

• Members of a set can themselves be sets. Thus $\{\mathbb{Z}, \mathbb{N}, \mathbb{Q}\}$ is a set with 3 elements which are also sets.

EXAMPLE

The set of all integers that are squares of an odd integer.

SOLN 1: $\{1^2, 3^2, 5^2, \ldots\}$.

SOLN 2: $\{x \mid x \text{ is the square of an odd integer}\}$.

SOLN 3: $\{x^2 \mid x \text{ is an odd integer }\}$.

SET EQUALITY

The sets A and B are **EQUAL** if they have the same members. We write $A = B$. Thus

$$A = B \quad \text{iff} \quad \forall x(x \in A \leftrightarrow x \in B).$$

Note: When we write $p \Leftrightarrow q$, it means that the proposition $p \leftrightarrow q$ is a true proposition. In other words, $\forall x(x \in A \Leftrightarrow x \in B)$ means that $\forall x(x \in A \Leftrightarrow x \in B)$ is a true proposition. Likewise for $p \Rightarrow q$, which means $p \rightarrow q$ is a true proposition.

Order, Repetition Do Not Matter

For example $\{1, 3, 7\} = \{7, 1, 3\} = \{7, 1, 1, 1, 3, 3, 1, 1\}$.

EXAMPLE

Show that $A = B$ where

$$A = \{x \in \mathbb{Z} \mid x^8 - 1 = 0\}, \quad B = \{x \in \mathbb{Z} \mid x^4 - 1 = 0\}.$$

PROOF:

We need to show

$$x \in A \Rightarrow x \in B \quad \text{and} \quad x \in B \Rightarrow x \in A.$$

We have

$$x \in B \Rightarrow x^4 = 1 \Rightarrow x^8 = 1 \Rightarrow x \in A$$

and

$$x \in A \Rightarrow x^8 - 1 = (x^4 - 1)(x^4 + 1) = 0 \Rightarrow x^4 - 1 = 0 \Rightarrow x \in B.$$

Thus $A = B$.

DEFINITION:

Let A, B be sets. The set A is a **SUBSET** of the set B if every element of A is an element of B.

We write

$$A \subseteq B.$$

Clearly,

- A is a subset of B if

$$x \in A \Rightarrow x \in B$$

- A is not a subset of B if it has an element that is not an element of B, i.e.,

$$A \nsubseteq B \quad \text{iff} \quad \exists x((x \in A) \wedge (x \notin B))$$

For example $\mathbb{Z} \subseteq \mathbb{Q}$, and $\mathbb{Q} \subseteq \mathbb{R}$. That is, $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

DEFINITION:

The set A is a **PROPER SUBSET** of the set B if $A \subseteq B$ and $A \neq B$.

We write $A \subsetneq B$.

DEFINITION:

THE UNIVERSAL SET is the set that consists of all the objects under discussion and is usually denoted by U.

In different contexts, we have different universal sets.

The set that has no members are called the **THE EMPTY SET** or **NULL SET**, and is denoted by $\emptyset$ or $\{\ \}$.

A set with a single element is called a **SINGLETON SET**.

THEOREM:

For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$

Proof: (i) We need to prove $\forall x, x \in \emptyset \rightarrow x \in S$ but this is *vacuously true* since $x \in \emptyset$ is always false. (ii) is left as exercise.

REMARK

Note that $\{\emptyset\}$ is **not** empty. It is a singleton set whose element is the empty set.

Similarly, $\{\{1\}, 2\}$ is **not** $\{1, 2\}$. $\{\{1\}, 2\}$ has 2 elements: $\{1\}$ and 2.

DISTINCTION BETWEEN $\in$ AND $\subseteq$

The following expressions are correct:

$$2 \in \{1,2,3\}; \quad \{2\} \in \{\{1\},\{2\}\}; \quad \{2\} \subseteq \{1,2,3\}, \quad \{\{2\}\} \subseteq \{\{1\},\{2\}\}.$$

The following expressions are incorrect:

$$\{2\} \in \{1,2,3\}, \quad 2 \subseteq \{1,2,3\}, \quad \{2\} \subseteq \{\{1\},\{2\}\}.$$

DEFINITION:

Let S be a set. If there are exactly n elements in the set, we say that S is a **FINITE SET** and that n is its **CARDINALITY**. We write $|S| = n$.

EXAMPLE

- $|A| = 50$ where $A = \{x \in \mathbb{N} \mid x < 100, x \text{ odd}\}$.
- $|\emptyset| = 0$.

DEFINITION:

Let A be a set. The **POWER SET** of A, written $P(A)$, is the set of all subsets of A.

EXAMPLE

- $P(\emptyset) = \{\emptyset\}$.
- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $P(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$.

Later, we'll prove the following theorem which explains the term "power set".

THEOREM: If $|S| = n$, then $|P(S)| = 2^n$.

CARTESIAN PRODUCTS

DEFINITION:

Let $n \in \mathbb{N}$. The **ORDERED n-TUPLE**, $(x_1, \dots, x_n)$ is the ordered collection that has x_1 as the first element, x_2 as the second element, ..., and x_n as the n^{th} element.

Two ordered n-tuples $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ are equal if

$$x_1 = y_1, \ \dots, \ x_n = y_n.$$

An **ORDERED PAIR** is an ordered 2-tuple, and an **ORDERED TRIPLE** an ordered 3-tuple.

- Do not confuse $(x_1, \dots, x_n)$ with $\{x_1, \dots, x_n\}$.
- $(1,2) \neq (2,1)$, $(3, (-2)^2, .5) = (\sqrt{9}, 4, .5)$.

DEFINITION:

The **CARTESIAN PRODUCT** of a set A and a set B, written $A \times B$, (read "A cross B"), is the set of all ordered pairs (x, y) where $x \in A$, $y \in B$. Thus

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

The **CARTESIAN PRODUCT** of the sets $A_1, \dots, A_n$ is

$$A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}.$$

If $A_1 = \ldots = A_n = A$, then

$$A_1 \times \cdots \times A_n = A^n.$$

EXAMPLE

$$\{0,1,x\} \times \{a,b\} = \{(0,a), (0,b), (1,a), (1,b), (x,a), (x,b)\}$$
$$\{0,1\} \times \{0,1\} \times \{x,y\} = \{(0,0,x), (0,0,y), (0,1,x), (0,1,y), (1,0,x), (1,0,y), (1,1,x), (1,1,y)\}$$

THEOREM: $|A \times B| = |A| \times |B|$.

SECTION 2.2 SET OPERATIONS

DEFINITION:

Let A, B be subsets of a universal set U.

1. The **UNION** of A and B, written $A \cup B$, is the set that contains elements that are in A or in B or in both, i.e.,
$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$

2. The **INTERSECTION** of A and B, written $A \cap B$, is the set that contains elements that are in both A and B, i.e.,
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$

3. The **COMPLIMENT** of A in B (**DIFFERENCE** of B with A), written $B - A$ or $B \setminus A$, is the set that contains elements that are in B but not in A, i.e.,
$$B - A = B \setminus A = \{x \mid (x \in B) \wedge (x \notin A)\}.$$

4. The **COMPLEMENT** of A, written $\overline{A}$ or A^c, is the set $\overline{A} = A^c = U - A$, i.e., $\overline{A} = A^c = \{x \in U \mid x \notin A\}$.

5. Two sets A and B are **DISJOINT** if $A \cap B = \emptyset$.

6. Sets $A_1, \ldots, A_n$ are **MUTUALLY or PAIRWISE DISJOINT** if $A_i \cap A_j = \emptyset \; \forall i \neq j$.

EXAMPLE

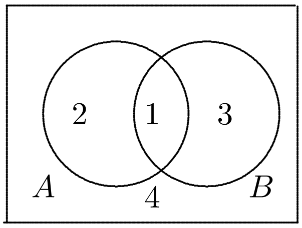
Let $U = \mathbb{R}$ and $A = \{x \mid x \leq 0\} = (-\infty, 0]$, $B = \{x \mid 0 \leq x < 1\} = [0, 1)$. Then

$$\begin{aligned} A \cup B &= \{x \mid (x \leq 0) \vee (0 \leq x < 1)\} = \{x \mid x < 1\} = (-\infty, 1) \\ A \cap B &= \{x \mid (x \leq 0) \wedge (0 \leq x < 1)\} = \{0\} \\ \overline{B} &= \{x \mid \sim (0 \leq x < 1)\} = \{x \mid (x < 0) \vee (x \geq 1)\} = (-\infty, 0) \cup [1, \infty) \end{aligned}$$

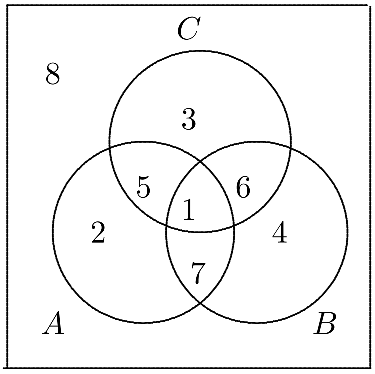
VENN DIAGRAMS

The relation between 2 or 3 sets can be visualized effectively with a Venn diagram.

Draw a rectangle to represent a set U. Draw circles to represent various subsets of U. The diagram below shows 2 and 3 sets.



$A = 1 + 2,\ B = 1 + 3,\ A \cup B = 1 + 2 + 3$
$A \cap B = 1,\ A - B = 2,\ B - A = 3$
$\overline{A \cup B} = 4,\ \overline{A} = 3 + 4,\ \overline{B} = 2 + 4$



$A = 1 + 2 + 5 + 7,\ B = 1 + 4 + 6 + 7$
$C = 1 + 3 + 5 + 6$
$A \cap B = 1 + 7,\ A \cap B \cap C = 1$
$\overline{A \cup B \cup C} = 8$, etc.

SET IDENTITIES

IDENTITY LAWS: $A \cup \emptyset = A$
$A \cap U = A.$

UNIVERSAL BOUND LAWS: $A \cup U = U$
$A \cap \emptyset = \emptyset$

IDEMPOTENT LAWS: $A \cup A = A$
$A \cap A = A$

DOUBLE COMPLEMENTATION LAWS:
$\overline{(\overline{A})} = A$

COMMUTATIVE LAWS: $A \cup B = B \cup A$
$A \cap B = B \cap A$

ASSOCIATIVE LAWS: $(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$

DISTRIBUTIVE LAWS: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

DE MORGAN'S LAWS: $\overline{A \cup B} = \overline{A} \cap \overline{B}$
$\overline{A \cap B} = \overline{A} \cup \overline{B}$

ABSORPTION LAWS: $A \cup (A \cap B) = A$

	$A \cap (A \cup B) = A$
COMPLEMENT LAWS:	$A \cup \overline{A} = U, \quad A \cap \overline{A} = \emptyset$
SET DIFFERENCE LAWS:	$\overline{\emptyset} = U, \quad \overline{U} = \emptyset$
	$A \setminus B = A \cap \overline{B}$

We show how to prove one of the above identities in different ways. The others can be proved in the same way.

- Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

SOLN 1: For any x,

$$
\begin{aligned}
& x \in \overline{A \cap B} \\
\Rightarrow\ & x \notin A \cap B \\
\Rightarrow\ & \sim (x \in A \cap B) \\
\Rightarrow\ & \sim (x \in A \wedge x \in B) \\
\Rightarrow\ & x \notin A \vee x \notin B \\
\Rightarrow\ & x \in \overline{A} \vee x \in \overline{B} \\
\Rightarrow\ & x \in \overline{A} \cup \overline{B}.
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
& x \in \overline{A} \cup \overline{B} \\
\Rightarrow\ & x \in \overline{A} \vee x \in \overline{B} \\
\Rightarrow\ & x \notin A \vee x \notin B \\
\Rightarrow\ & \sim (x \in A \wedge x \in B) \\
\Rightarrow\ & \sim (x \in A \cap B) \\
\Rightarrow\ & x \in \overline{A \cap B}
\end{aligned}
$$

Alternatively, it's not hard to see that the steps can actually be reversed and so we can write

For any x,

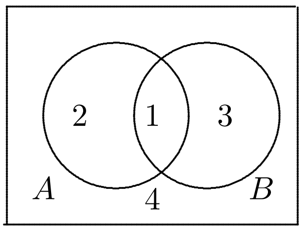
$$\begin{aligned}
& x \in \overline{A \cap B} \\
\Leftrightarrow \quad & x \notin A \cap B \\
\Leftrightarrow \quad & \sim (x \in A \cap B) \\
\Leftrightarrow \quad & \sim (x \in A \wedge x \in B) \\
\Leftrightarrow \quad & x \notin A \vee x \notin B \\
\Leftrightarrow \quad & x \in \overline{A} \vee x \in \overline{B} \\
\Leftrightarrow \quad & x \in \overline{A} \cup \overline{B}.
\end{aligned}$$

SOLN 2: (TRUTH TABLE)

$x \in A$	$x \in B$	$x \in \overline{A}$	$x \in \overline{B}$	$x \in A \cap B$	$x \in \overline{A \cap B}$	$x \in \overline{A} \cup \overline{B}$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The table is constructed as follows. Take an arbitrary element $x \in U$. There are 4 cases. The first case is $x \in A$ and $x \in B$. This correspondence to the first row which indicates the membership of x is the other sets. The other remaining three cases are $x \in A$ and $x \notin B$, $x \notin A$ and $x \in B$, $x \notin A$ and $x \notin B$. Since the columns corresponding to $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ (the last two columns) are identical, the two sets are equal.

SOLN 3: (VENN DIAGRAM)



$$\overline{A} = 2 + 4, \ \overline{B} = 2 + 4, \ \overline{A} \cup \overline{B} = 2 + 3 + 4$$

$$A \cap B = 1, \ \overline{A \cap B} = 2 + 3 + 4$$

$$\therefore \overline{A} \cup \overline{B} = \overline{A \cap B}$$

EXAMPLE

- Show $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

SOLN:

$$\begin{aligned} \overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) \\ &= \overline{A} \cap (\overline{B} \cup \overline{C}) \\ &= (\overline{B} \cup \overline{C}) \cap \overline{A} \\ &= (\overline{C} \cup \overline{B}) \cap \overline{A} \end{aligned}$$

- Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

SOLN: Consider $x \in A \cap (B \cup C)$. Then

$$\begin{aligned} & (x \in A) \quad \text{and} \quad (x \in B \cup C). \\ \therefore \quad & (x \in A) \quad \text{and} \quad (x \in B \text{ or } x \in C) \\ \therefore \quad & (x \in A \text{ and } x \in B) \quad \text{or} \quad (x \in A \text{ and } x \in C) \\ \therefore \quad & x \in A \cap B \quad \text{or} \quad x \in A \cap C \\ \therefore \quad & x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Conversely, consider

$$x \in (A \cap B) \cup (A \cap C).$$

We have two cases. Case 1: $x \in A \cap B$.

$$\begin{aligned} \therefore \quad & x \in A \quad \text{and} \quad x \in B \\ \therefore \quad & x \in A \quad \text{and} \quad x \in B \cup C \\ \therefore \quad & x \in A \cap (B \cup C) \end{aligned}$$

Case 2: $x \in A \cap C$.

$$\begin{aligned} \therefore \quad & x \in A \quad \text{and} \quad x \in C \\ \therefore \quad & x \in A \quad \text{and} \quad x \in B \cup C \\ \therefore \quad & x \in A \cap (B \cup C) \end{aligned}$$

Since both cases lead to $x \in A \cap (B \cup C)$, we conclude

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C).$$

Thus the proof is complete.

- Is the following true?

$$\forall \quad \text{sets} \quad A, B, C, \big((A \setminus B) \cup (B \setminus C) = A \setminus C\big).$$

SOLN: A Venn diagram suggests that it is false. It also suggest a counter example: Let $A = \{1,2\}$, $B = \{2,3,4\}$, $C = \{4,5\}$. Then lhs $= \{1,2,3\}$, and rhs $= \{1,2\}$.

REMARK

- Representation of union, intersection, and Cartesian product of the sets $A_1, \ldots, A_n$.

  Union: $A_1 \cup A_2 \ldots \cup A_n = \cup_{i=1}^n A_i$.

  Intersection: $A_1 \cap A_2 \ldots \cap A_n = \cap_{i=1}^n A_i$.

  Cartesian product: $A_1 \times A_2 \ldots \times A_n = \Pi_{i=1}^n A_i$.

- Recall that UNIVERSAL SET U is the set that consists of all the objects under discussion. U can be different in different context.

  How about the EMPTY SET $\emptyset$? Is it the same or different in different context?

SOLN: Recall that $\emptyset$ is the set that has no members. Suppose there were 2 $\emptyset$'s, say $\emptyset_1$ and $\emptyset_2$. Then we have

$$\forall x (x \in \emptyset_1 \Rightarrow x \in \emptyset_2)$$

and

$$\forall x (x \in \emptyset_2 \Rightarrow x \in \emptyset_1).$$

By the definition of set equation, $\emptyset_1 = \emptyset_2$. $\therefore$ there is only one $\emptyset$ in all context.

SECTION 2.3 FUNCTIONS

DEFINITION:

Let A, B be nonempty sets. A **FUNCTION** f from A to B, $f : A \to B$, is an assignment of **exactly one element** of B to each element of A. For each $a \in A$, if b is the unique element in B assigned to a, we write $f(a) = b$ or $f : a \mapsto b$.

The set A is called the **DOMAIN** and the set B is called the **CO-DOMAIN**.

If $f(a) = b$, we say that b is the **IMAGE** or **VALUE** of a and a is a **PREIMAGE** of b. (a has exactly one "value" or "image" but the element b may have any number, including 0, of preimages.)

The set of all values of f is called its **RANGE** or **IMAGE**. Thus the range (image) is the set

$$f(A) = \{b \in B \mid \exists a \in A(b = f(a))\}.$$

We also use the shorthand $f(A) = \{f(a) \mid a \in A\}$.

Note the difference between the arrows $\to$ and $\mapsto$ in the definition of f. When $f(a)$ can be written down as a closed formula in terms of a, we can replace the line $a \mapsto f(a)$ by the explicit formula for $f(a)$. For example, we can write

$$\begin{aligned} f : A &\to B \\ a &\mapsto a^2 + 1 \end{aligned}$$

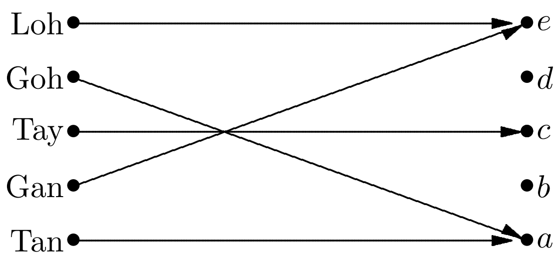
or write

$$\begin{aligned} f : A \to B \\ f(a) = a^2 + 1 \end{aligned}.$$

However, it is wrong to write

$$\begin{aligned} f : A \to B \\ f(a) \mapsto a^2 + 1 \end{aligned}$$

Functions can be specified in many different ways. Sometimes we explicitly state the assignments using a diagram shown below, or by mean of a formula such as $f(x) = x + 1$. Sometimes we also use a computer program to specify a function. Here the domain would be all the possible inputs and the images would be the corresponding outputs.



$A = \{\text{Loh, Goh, Tay, Gan, Tan}\}$, $f(A) = \{a, c, e\}$

The above is called an "Arrow Diagram".

EXAMPLE

Consider $f : X \to Y$ with $f(x) = y$ if $x^2 + y^2 = 1$.

• If $X = Y = \mathbb{R}$, then f is not a function since the element 2 in the domain does not have an image.

• If $X = [-1, 1]$, $Y = \mathbb{R}$, then f is still not a function even though every element in X has an image. The reason is that $0 \in X$ corresponds to two elements, ± 1, in Y.

• If $X = [-1, 1]$ and $Y = [0, \infty)$, then f is a function. The image is $[0, 1]$ and every element $y \neq 1$, in the image has two preimages $\pm\sqrt{1 - y^2}$.

TERMINOLOGY

We say "the function f is **well-defined**" if f is a function; we say "the function f is **not well-defined**" if f is not a function (a contradiction of terms).

EXAMPLE

• Let S be the set of all bit strings. Define $f : S \to \mathbb{Z}$ by

$$\forall a \in S, \quad f(a) = \quad \text{number of 0's in a} \quad .$$

Then f is a function (the function f is well-defined). Its range (image) is $\mathbb{Z}_{\geq 0}$.

• Let S_n the set of all bit strings of length n. Define $H : S_n \times S_n \to \mathbb{Z}$ by

$$H(a, b) = \quad \text{number of places in which } a,\ b \text{ are different} \quad .$$

For example, when $n = 4$, then $H(1101, 0011) = 3$. H is a function (the function H is well-defined). Its range (image) is $\{0, 1, \ldots, n\}$.

• Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Then f is a function (the function f is well-defined). Its range (image) is $\mathbb{R}_{\geq 0}$. In fact, $f(x) = |x|$, the absolute value of x.

• Define $f : \mathbb{Q} \to \mathbb{Z}$ by $f(m/n) = m$, where $m, n \in \mathbb{Z}$. This is not a function (the function f is not well-defined) because the rational number $1/2$ can have many different values:

$$f(1/2) = 1, f(2/4) = 2, \quad \text{etc}$$

• Consider the SORT programme that sorts any finite sequence of real numbers in increasing order. This can be considered a function whose domain is the set of finite sequences of real numbers. The range (image) of SORT is then the set of nondecreasing sequences. For example, the image of $(1, 2, 3, 3, 2, 1)$ is $(1, 1, 2, 2, 3, 3)$.

• A **sequence** (or more accurately, an infinite sequence) is a function whose domain is $\mathbb{Z}^+$: $f : \mathbb{Z}^+ \to B$ as an infinite tuple

$$(f(1), f(2), f(3), \ldots) = (f(n))_{n \in \mathbb{Z}^+}.$$

B is the set of codomain. $f(1), f(2), f(3), \ldots \in B$.

For example, a function $f : \mathbb{Z}^+ \to \mathbb{R}$ is a **real** sequence, and a function $g : \mathbb{Z}^+ \to \mathbb{Z}$ is a sequence of **integers**.

DEFINITION:

Let f, g be functions from A to $\mathbb{R}$. Then $f + g$ and fg are also functions from A to $\mathbb{R}$ defined by

$$(f + g)(x) = f(x) + g(x), \qquad (fg)(x) = f(x)g(x).$$

EXAMPLE

• Let f, g be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x) = x^2$ and $g(x) = x + x^3$. Then $f + g$ and fg are functions defined by $(f + g)(x) = f(x) + g(x) = x^2 + x + x^3$ and $(fg)(x) = f(x)g(x) = x^2(x + x^3) = x^3 + x^5$.

ONE-TO-ONE & ONTO FUNCTIONS

DEFINITION:

A function $f : X \to Y$ is **ONE-TO-ONE** or **INJECTIVE** iff

$$\forall a, b \in X, \quad f(a) = f(b) \Rightarrow a = b$$

REMARK

- f is one-to-one if every element in the codomain has at most one preimage.
- f is one-to-one if every element in the image has exactly one preimage.
- f is one-to-one if distinct elements of the domain have distinct images.
- f is one-to-one if every "horizontal" line intersects its graph in at most one point.
- f is **NOT** 1-1 if $\exists a \neq b, \quad f(a) = f(b)$.
- f is 1-1 if $\forall a, b,\ a \neq b \Rightarrow f(a) \neq f(b)$.

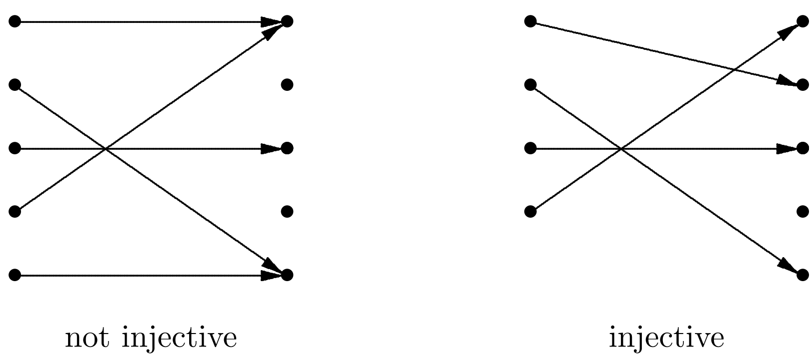
EXAMPLE

Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by $f(x) = 4x - 1$ and $g(x) = x^2$.

Then f is 1-1 because

$$f(a) = f(b) \Rightarrow 4a - 1 = 4b - 1 \Rightarrow a = b$$

However, g is not 1-1 because $g(2) = g(-2) = 4$.



DEFINITION:

Let $A, B \subseteq \mathbb{R}$. A function $f : A \to B$ is said to be **INCREASING** if $(x > y) \to f(x) \geq f(y)$.

The function is **STRICTLY INCREASING** if $(x > y) \to f(x) > f(y)$.

DECREASING and **STRICTLY DECREASING** functions are defined similarly.

REMARK

It follows easily the definition that a strictly increasing or strictly decreasing function is 1-1 as $x \neq y$ will imply that $f(x) \neq f(y)$.

EXAMPLE

Let $f(x) = x^2$. Then f is not injective if the domain is $\mathbb{R}$ since $f(-2) = f(2)$. However, if the domain is $\mathbb{R}_{\geq 0}$, then the function is strictly increasing since $x > y > 0$ implies hat $x^2 > y^2$, i.e., $f(x) > f(y)$. Thus in the case, f is 1-1.

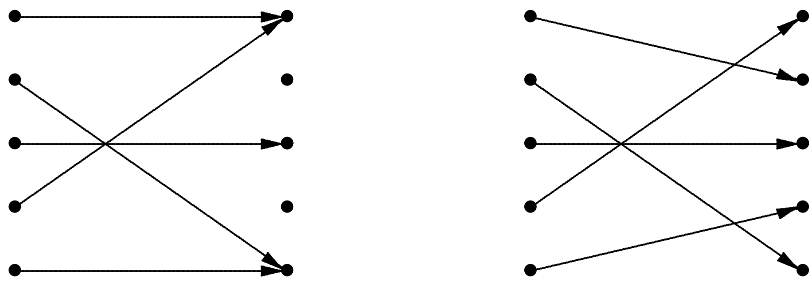
DEFINITION:

A function $f : X \to Y$ is **ONTO** or **SURJECTIVE** if

$$\forall y \in Y \exists x \in X (f(x) = y).$$

REMARK

- f is onto if its image is equal to its codomain.
- f is onto if the “horizontal” line through a point in its codomain intersects its graph.
- f is **NOT** onto if $\exists y \in Y$ with no preimage.



not onto   onto, also 1-1

EXAMPLE

Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{N}$ by $f(x) = 4x - 1$ and $g(n) = n^2$.

Then f is onto because

$$\forall y \in \mathbb{R}, \quad \text{if } x = (y+1)/4, \text{ then} \quad f(x) = y.$$

However, g is not onto as 2 has no preimage.

DEFINITION:

The function f is a **BIJECTION** if it is both 1-1 and onto.

EXAMPLE

- $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 4x - 1$ is a bijection.
- Let A be a set. The **IDENTITY FUNCTION** on A, $i_A : A \to A$, where $i_A(x) = x$ for all $x \in A$, is s bijection.

INVERSE FUNCTIONS

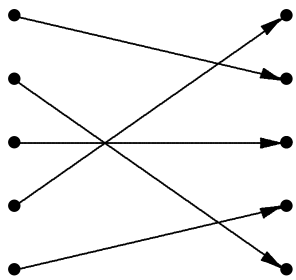
THEOREM:

Let $f : X \to Y$ be a bijection. Then there is a function $g : Y \to X$ defined as follows:

$$\forall y \in Y, g(y) = x \Leftrightarrow f(x) = y.$$

REMARK

In terms of the arrow diagram, it says that if you reverse the arrows, you still get a function. The following is an example.



A bijection. (*Reversing the arrows still yields a function*).

Proof: For each $y \in Y$, since f is a bijection, y has a unique preimage x. This preimage then becomes the (unique) image of y under g. Therefore g is a function.

DEFINITION:

The function g in the above theorem is called the **INVERSE FUNCTION** for f and is denoted as f^{-1}.

Note: Do not confuse f^{-1} with $1/f$. The latter is the function that assigns to every x, the value $1/f(x)$ and is defined only when $f(x) \neq 0$ for all x.

EXAMPLE

- The inverse of the identity function on A is itself, i.e., $i_A^{-1} = i_A$.
- Find the inverse of the function $f : \mathbb{Z} \to \mathbb{Z}$, where $f(x) = x + 1$ for all $x \in \mathbb{Z}$, if it exists.

SOLN: We first prove that it is a bijection so that the inverse exists.

f is 1-1: Let $f(a) = f(b)$. Then $a + 1 = b + 1$ and therefore $a = b$. Thus f is 1-1.

f is onto: Let $y \in \mathbb{Z}$. We need to find an x such that $f(x) = y$. This means $f(x) = x+1 = y$ which gives $x = y - 1$. Thus $x = y - 1$ is a preimage of y. Hence f is onto.

Thus f is a bijection and its inverse exists.

From the "onto" proof, we see that $f^{-1}(y) = y - 1$.

DEFINITION:

Let $f : X \to Y$, and $g : Y \to Z$ be functions. Define the **COMPOSITION FUNCTION** $g \circ f : X \to Z$ as follows:

$$\forall x \in X, \quad g \circ f(x) = g(f(x)).$$

EXAMPLE

- $f : \mathbb{Z} \to \mathbb{Z}$, $g : \mathbb{Z} \to \mathbb{Z}$ are defined by $f(n) = n + 1$, $g(n) = n^2$. Then

$$g \circ f(n) = g(f(n)) = g(n+1) = (n+1)^2.$$
$$f \circ g(n) = f(g(n)) = f(n^2) = n^2 + 1$$

REMARK

We see that $g \circ f \neq f \circ g$.

DEFINITION:

Two functions f and g are **EQUAL**, denoted $f = g$, if and only if:

the domains of f and g are equal;

the codomains of f and g are equal;

$f(x) = g(x)$ for all x in the domain of f (= domain of g).

- Let $f : X \to Y$ be a function. Then

$$f \circ i_X(x) = f(i_X(x)) = f(x) \quad \text{and} \quad i_Y \circ f(x) = i_Y(f(x)) = f(x)$$

Therefore

$$f \circ i_X = i_Y \circ f.$$

- Let $f : X \to Y$ be a bijection. Then it has an inverse function f^{-1} and

$$f \circ f^{-1}(y) = f(f^{-1}(y)) = f(x) = y$$
$$f^{-1} \circ f(x) = f^{-1}(f(x)) = f^{-1}(y) = x$$

Thus

$$f \circ f^{-1} = i_Y \quad \text{and} \quad f^{-1} \circ f = i_X.$$

IMAGES and PREIMAGES

Let $f : A \to B$ be a function.

- For $a \in A$, recall

  if $f(a) = b$, then b is the **IMAGE** of a under f, and that a is a **PREIMAGE** of b under f.

  the range of f is

$$\{f(x) \mid x \in A\}.$$

DEFINITION:

Let $X \subseteq A$ and $Y \subseteq B$. Then

$$f(X) = \{f(x) \mid x \in X\} = \{b \in B \mid \exists x \in X, f(x) = b\};$$
$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}.$$

We call $f(X)$ the set of **IMAGE** of X under f, and $f^{-1}(Y)$ the set of **PREIMAGE** of Y under f.

EXAMPLE

Define $f : \mathbb{Z} \to \mathbb{Z}$ by setting $f(x) = x^2$ for every $x \in \mathbb{Z}$.

- If $X = \{-1, 0, 1\}$, then

$$f(X) = \{f(-1), f(0), f(1)\} = \{1, 0, 1\} = \{0, 1\}$$

- If $Y = \{0, 1, 2\}$, then

$$f^{-1}(Y) = \{0, -1, 1\}$$

REMARK

- Let $x \in B$ and $Y \subseteq B$. Note the difference of $f^{-1}(x)$ and $f^{-1}(Y)$:

$f^{-1}(x)$ is the inverse function of f. To have an inverse function, f must be a bijection.

$f^{-1}(Y)$ is the set of preimage of Y under f. To have a preimage of Y, f does not have to be bijective.

- For $a \in A$ and $Y \subseteq B$,
$$a \in f^{-1}(Y) \quad \Leftrightarrow \quad f(a) \in Y.$$

- If $X \neq \emptyset$, then $f(X) \neq \emptyset$.

  If $Y \neq \emptyset$, then $f^{-1}(Y)$ may and may not be $\emptyset$. Can you give examples?

- If $X' \subseteq X$, then $f(X') \subseteq f(X)$.

  If $Y' \subseteq Y$, then $f^{-1}(Y') \subseteq f^{-1}(Y)$.

ASSOCIATIVITY of COMPOSITION of FUNCTIONS

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be functions. Then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Proof.

1. Both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have domain A.
2. Both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have codomain D.
3. For all $a \in A$,
$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a)));$$
$$((h \circ g) \circ f)(a) = (h \circ g)(f(a)) = h(g(f(a))).$$

Thus, $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$.

REMARK

- We may thus write $h \circ g \circ f$ without ambiguity.
- For $f : A \to A$ and $n \in \mathbb{Z}^+$, we write f^n for
$$\underbrace{f \circ f \circ \ldots \circ f}_{n}.$$
- We further define f^0 to be i_A (so that $f^0(a) = a$ for all $a \in A$) by convention.

EXAMPLE

Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

Then $f^n(x) = x^{(2^n)}$.

FLOOR AND CEILING FUNCTIONS

DEFINITION:

The **THE FLOOR** of $x \in \mathbb{R}$, written $\lfloor x \rfloor$, is the largest integer $\leq x$.

The **THE CEILING** of $x \in \mathbb{R}$, written $\lceil x \rceil$, is the smallest integer $\geq x$.

REMARK

- Thus, when $n \in \mathbb{Z}$,

$$\lfloor x \rfloor = n \quad \text{iff} \quad n \leq x < n+1$$

$$\lceil x \rceil = n \quad \text{iff} \quad n-1 < x \leq n$$

- You **ROUND DOWN** to get the floor and **ROUND UP** to get the ceiling.

EXAMPLE

- $\lfloor -3 \rfloor = \lceil -3 \rceil = -3$, $\quad \lfloor -2.7 \rfloor = -3$, $\quad \lfloor 0 \rfloor = 0$, $\quad \lfloor 4.979 \rfloor = 4$, $\quad \lceil -2.7 \rceil = -2$.
- $\forall x \in \mathbb{R}, \lfloor x \rfloor \leq x \leq \lceil x \rceil$. Equalities hold if and only if x is an integer.

PROOF: The result follows from that the fact that if $x \in \mathbb{Z}$, $\lfloor x \rfloor = x = \lceil x \rceil$; and if $x \notin \mathbb{Z}$, then $\exists n \in \mathbb{Z}$ with $n < x < n+1$. Then $\lfloor x \rfloor = n < x < n+1 = \lceil x \rceil$.

- Prove or disprove that for all real numbers x and y, $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.

SOLN: The statement is false and a counter example is $x = y = .7$.

- For all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$, $\lfloor x+m \rfloor = \lfloor x \rfloor + m$.

PROOF: Let $\lfloor x \rfloor = n$. We need to show that $\lfloor x+m \rfloor = n+m$. We have

$$\lfloor x \rfloor = n \Rightarrow n \leq x < n+1 \Rightarrow n+m \leq x+m < n+m+1 \Rightarrow \lfloor x+m \rfloor = n+m.$$

- For all $x \in \mathbb{R}$, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

PROOF: Suppose $\lfloor x \rfloor = n$. Then $n \leq x < n+1$. (If you simply multiply by 2, you get $2n \leq 2x < 2n+2$ and you won't be able to determine the value of $\lfloor 2x \rfloor$.)

Case (i) $n \le x < n + \frac{1}{2}$. Then $2n \le 2x < 2n + 1$ and

$$n + \frac{1}{2} \le x + \frac{1}{2} < n + 1 \quad \Rightarrow \quad n \le x + \frac{1}{2} < n + 1.$$

Therefore $\lfloor 2x \rfloor = 2n$, $\lfloor x + \frac{1}{2} \rfloor = n$ and $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

Case (ii) $n + \frac{1}{2} \le x < n + 1$. Then $2n + 1 \le 2x < 2n + 2$ and $n + 1 \le x + \frac{1}{2} < n + \frac{3}{2} < n + 2$. Therefore $\lfloor 2x \rfloor = 2n + 1$, $\lfloor x + \frac{1}{2} \rfloor = n + 1$ and $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

SECTION 2.4 CARDINALITY

Earlier we define the cardinality of a finite set as the number of the elements in the set. Thus if $A = \{1, 2, 4, 6\}$, then $|A| = 4$. But what is the cardinality of an infinite set? For example, how do you compare the cardinality of $\mathbb{Q}$ and $\mathbb{Z}$? We shall now use the idea of 1-1 correspondence, or bijective function, to extend this to define the cardinality of infinite sets.

The intuitive idea is this. If there are 100 seats in a cinema $S = \{s_1, s_2, \ldots, s_{100}\}$ and the audience is $A = \{a_1, a_2, \ldots, a_{100}\}$, then we know that every seat is taken, i.e., there is a 1-1 correspondence between the seats and the audience and $|A| = |S|$.

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_{100} \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ s_1 & s_2 & s_3 & \cdots & s_{100} \end{array}$$

Now imagine that the cinema has an infinite number of seats $S = \{s_1, s_2, \ldots\}$. Suppose the members of the audience hold the tickets with numbers 1, 2, …, i.e., $A_1 = \{a_1, a_2, \ldots\}$. Then everybody will have a seat, i.e., there is still 1-1 correspondence. We can say that $|A_1| = |S|$.

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_n & \cdots \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \\ s_1 & s_2 & s_3 & \cdots & s_n & \cdots \end{array}$$

What happens if an additional person walks in with ticket number 0? Then $A_2 = \{a_0, a_1, a_2, \ldots\} = A_1 \cup \{a_0\}$. The solution is very simple: ask every body to move to the next seat, i.e., the holder of ticket number n will now take seat number $n + 1$. There is still a 1-1 correspondence and $|A_2| = |S|$.

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \cdots & a_{n-1} & \cdots \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \\ s_1 & s_2 & s_3 & \cdots & s_n & \cdots \end{array}$$

What happens if the theater double sells the tickets, i.e., 2 tickets of the same number were sold? Here $A_3 = \{a_1, b_1, a_2, b_2, \ldots\}$. There is still a solution. Just ask the holders of ticket number n to take the seats numbered $2n-1$ and $2n$. Then again everyone will have a seat. Thus there is a 1-1 correspondence and we can claim that $|A_3| = |S|$.

$$\begin{array}{cccccccc} a_1 & b_1 & a_2 & b_2 & \ldots & a_n & b_n & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \ldots & \downarrow & \downarrow & \ldots \\ s_1 & s_2 & s_3 & s_4 & \ldots & s_{2n-1} & s_{2n} & \ldots \end{array}$$

In all the cases discuss, there is a 1-1 correspondence between the set of seats and the set of audience and we say that they have the same cardinality.

DEFINITION: Let A and B be any sets. A has the **SAME CARDINALITY** as B if there is a bijection $f : A \to B$ and we write $|A| = |B|$.

DEFINITION:

A set is **COUNTABLE** if it is finite or has the same cardinality as $\mathbb{N}$. A set is **UNCOUNTABLE** if it is not countable

REMARK

It follows from the definition that when an infinite set S is countable, each of its elements is associated with an element of $\mathbb{N}$. If we denote the element that corresponds to i as a_i, then

$$S = \{a_1, a_2, a_3, \ldots\}.$$

Thus we conclude a set is countable iff its elements can be arranged in a sequence.

EXAMPLE

- The set of odd positive integers A is countable.

PROOF: The A is countable since its elements can be arranged in the sequence $1, 3, 5, \ldots$.

- The set of even integers, $2\mathbb{Z}$, is countable.

PROOF: The elements can be arranged as:

$$0, \quad 2, \quad -2, \quad 4, \quad -4, \quad 6, \quad -6, \quad \ldots.$$

• $\mathbb{Z}$ is countable.

PROOF: We can arrange the integers as the sequence:

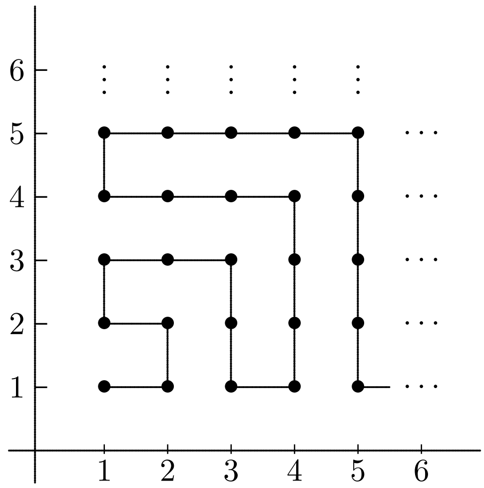
$$0, \quad 1, \quad -1, \quad 2, \quad -2, \quad 3, \quad -3, \quad \ldots.$$

• If $A \subseteq B$ and B countable, then so is A.

PROOF: We first arrange the elements of B in a sequence. Then delete the elements that do not belong to A. What is left is a sequence of the elements of A.

• $\mathbb{N} \times \mathbb{N}$ is countable.

PROOF: The elements of $\mathbb{N} \times \mathbb{N}$ exactly the points in the coordinate plane whose coordinates are both positive integers. These points can be arranged in a sequences as shown.



• In general, if A, B are both countable, then $A \times B$ is countable.

• $\mathbb{Q}$ is countable.

SOLN: Each $\frac{a}{b} \in \mathbb{Q}$, $\gcd(a, b) = 1$, $b \geq 1$, can be regarded as an ordered pair (a, b). Thus $Q \subseteq \mathbb{Z} \times \mathbb{Z}$ and is thus countable.

THEOREM (CANTOR): The set $(0, 1)$ is uncountable.

PROOF: We shall prove by contradiction. Suppose that the set is countable, i.e., $(0, 1) = \{b_1, b_2, \ldots\}$. Then the decimal representations of these numbers can be written in a sequence as follows:

$$
\begin{aligned}
b_1 &= 0\cdot\, \underline{a_{11}}\, a_{12}\, a_{13}\, a_{14}\, a_{15}\, a_{16}\, a_{17} \dots \\
b_2 &= 0\cdot\, a_{21}\, \underline{a_{22}}\, a_{23}\, a_{24}\, a_{25}\, a_{26}\, a_{27} \dots \\
b_3 &= 0\cdot\, a_{31}\, a_{32}\, \underline{a_{33}}\, a_{34}\, a_{35}\, a_{35}\, a_{37} \dots \\
b_4 &= 0\cdot\, a_{41}\, a_{42}\, a_{43}\, \underline{a_{44}}\, a_{45}\, a_{45}\, a_{47} \dots \\
b_5 &= 0\cdot\, a_{51}\, a_{52}\, a_{53}\, a_{54}\, \underline{a_{55}}\, a_{55}\, a_{57} \dots \\
b_6 &= 0\cdot\, a_{61}\, a_{62}\, a_{63}\, a_{64}\, a_{65}\, \underline{a_{66}}\, a_{67} \dots \\
&\vdots
\end{aligned}
$$

We shall construct a number between 0 and 1 that is not in the sequence. Let $d = 0.d_1 d_2 d_3 \dots$ where

$$d_n = \begin{cases} 4 & \text{if } a_{nn} \neq 4 \\ 5 & \text{if } a_{nn} = 4. \end{cases}$$

We see that for each n, d is different from b_n in the n^{th} decimal position. Thus $d \neq b_n$ for all n. So d is not a number in the sequence, but d is a number between 0 and 1 and this gives rise to a contradiction.

THEOREM (CANTOR-BERNSTEIN)

Let $f : A \to B$ and $g : B \to A$ be injective functions. Then there exists a bijective function $h : A \to B$.