# CS1231(S) Tutorial 5: Mathematical Induction Solutions

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1. Prove by induction that for all $n \in \mathbb{Z}_{\geqslant 1}$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1).$$

*Solution.*

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition

$$\text{“ } 1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}\, n(n+1)(2n+1) \text{ ”}.$$

2. (Base step) $P(1)$ is true because $1^2 = 1 = \frac{1}{6} \times 1 \times (1+1) \times (2 \times 1 + 1)$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that

$$\text{“ } 1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}\, k(k+1)(2k+1) \text{ ”}.$$

   3.2. Then $1^2 + 2^2 + \cdots + k^2 + (k+1)^2$
   3.3. $= \frac{1}{6}\, k(k+1)(2k+1) + (k+1)^2$ by the induction hypothesis;
   3.4. $= \frac{1}{6}(k+1)\big(k(2k+1) + 6(k+1)\big)$
   3.5. $= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$
   3.6. $= \frac{1}{6}(k+1)(k+2)(2k+3)$
   3.7. $= \frac{1}{6}(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)$.
   3.8. Thus $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI. $\square$

2. Let $x \in \mathbb{R}_{\geqslant -1}$. Prove by induction that $1 + nx \leqslant (1+x)^n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

*Solution.*

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition “ $1 + nx \leqslant (1+x)^n$ ”.
2. (Base step) $P(1)$ is true because $1 + 1x = 1 + x = (1+x)^1$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that $1 + kx \leqslant (1+x)^k$.
   3.2. Then $(1+x)^{k+1}$
   3.3. $= (1+x)^k(1+x)$
   3.4. $\geqslant (1+kx)(1+x)$ by the induction hypothesis, as $1 + x \geqslant 0$;
   3.5. $= 1 + (k+1)x + kx^2$
   3.6. $\geqslant 1 + (k+1)x$ as $k \geqslant 1$ and $x^2 \geqslant 0$.

3.7. Thus $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI. $\square$

3. Prove by induction that 3 divides $n^3 + 11n$ for all $n \in \mathbb{Z}_{\geqslant 1}$.

*Solution.*

1. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition " 3 divides $n^3 + 11n$ ".
2. (Base step) $P(1)$ is true because $1^3 + 11 \times 1 = 12 = 3 \times 4$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that 3 divides $k^3 + 11k$.
   3.2. Use the definition of "divides" to find $\ell \in \mathbb{Z}$ such that $k^3 + 11k = 3\ell$.
   3.3. Then $(k+1)^3 + 11(k+1)$
   3.4. $= (k^3 + 3k^2 + 3k + 1) + (11k + 11)$
   3.5. $= (k^3 + 11k) + 3(k^2 + k + 4)$
   3.6. $= 3\ell + 3(k^2 + k + 4)$ by line 3.2;
   3.7. $= 3(\ell + k^2 + k + 4)$ where $\ell + k^2 + k + 4 \in \mathbb{Z}$.
   3.8. Thus $P(k+1)$ is true by the definition of "divides".
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}$ $P(n)$ is true by MI. $\square$

4. Let $a$ be an odd integer. Prove by induction that $2^{n+2}$ divides $a^{2^n} - 1$ for all $n \in \mathbb{Z}_{\geqslant 1}$. (Note that $a^{b^c} = a^{(b^c)}$ by convention.)

*Solution.*

1. Use the definition of "odd' to find $\ell \in \mathbb{Z}$ such that $a = 2\ell + 1$.
2. For each $n \in \mathbb{Z}_{\geqslant 1}$, let $P(n)$ be the proposition " $2^{n+2}$ divides $a^{2^n} - 1$ ".
3. (Base step)
   3.1. Note $a^{2^1} - 1 = a^2 - 1$
   3.2. $= (a-1)(a+1)$
   3.3. $= (2\ell + 1 - 1)(2\ell + 1 + 1)$ by line 1;
   3.4. $= 4\ell(\ell + 1)$.
   3.5. Case 1: $\ell$ is odd.
      3.5.1. Use the definition of "odd" to find $m \in \mathbb{Z}$ such that $\ell = 2m + 1$.
      3.5.2. Then $a^{2^1} - 1 = 4\ell(\ell + 1)$ by lines 3.1–3.4;
      3.5.3. $= 4(2m+1)((2m+1)+1)$ by the choice of $m$ on line 3.5.1;
      3.5.4. $= 8(2m+1)(m+1)$ where $(2m+1)(m+1) \in \mathbb{Z}$.
      3.5.5. So $2^{1+2}$ divides $a^{2^1} - 1$ as $8 = 2^{1+2}$.
   3.6. Case 2: $\ell$ is even.
      3.6.1. Use the definition of "even" to find $m \in \mathbb{Z}$ such that $\ell = 2m$.
      3.6.2. Then $a^{2^1} - 1 = 4\ell(\ell + 1)$ by lines 3.1–3.4;
      3.6.3. $= 4(2m)(2m+1)$ by the choice of $m$ on line 3.6.1;
      3.6.4. $= 8m(2m+1)$ where $m(2m+1) \in \mathbb{Z}$.
      3.6.5. So $2^{1+2}$ divides $a^{2^1} - 1$ as $8 = 2^{1+2}$.
   3.7. Since $\ell$ is either odd or even, we conclude that $2^{1+2}$ divides $a^{2^1} - 1$ in all cases.
   3.8. So $P(1)$ is true.
4. (Induction step)
   4.1. Let $k \in \mathbb{Z}_{\geqslant 1}$ such that $P(k)$ is true, i.e., that $2^{k+2}$ divides $a^{2^k} - 1$.
   4.2. Use the definition of "divides" to find $m \in \mathbb{Z}$ such that $a^{2^k} - 1 = 2^{k+2}m$.
   4.3. Then $a^{2^{k+1}} - 1 = a^{2^k \times 2} - 1$
   4.4. $= (a^{2^k})^2 - 1$
   4.5. $= (a^{2^k} - 1)(a^{2^k} + 1)$
   4.6. $= (2^{k+2}m)\big((2^{k+2}m + 1) + 1\big)$ by the choice of $m$;
   4.7. $= 2^{k+3}m(2^{k+1}m + 1)$ where $m(2^{k+1}m + 1) \in \mathbb{Z}$.

4.8. Thus $P(k+1)$ is true by the definition of "divides".

5. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true by MI. $\square$

5. Prove by induction that

$$\forall n \in \mathbb{Z}_{\geqslant 8}\ \exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n = 3x + 5y).$$

(As a consequence, any integer-valued transaction over 8 dollars can be carried out using only 3-dollar and 5-dollar coins.)

*Solution.*

1. For each $n \in \mathbb{Z}_{\geqslant 8}$, let $P(n)$ be the proposition " $\exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n = 3x + 5y)$ ".
2. (Base step) $P(8)$ is true because $8 = 3 \times 1 + 5 \times 1$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 8}$ such that $P(k)$ is true.
   3.2. Find $x, y \in \mathbb{Z}_{\geqslant 0}$ such that $k = 3x + 5y$.
   3.3. Case 1: $y > 0$.
      3.3.1. Then $k + 1 = (3x + 5y) + 1$ by the choice of $x, y$;
      3.3.2. $= 3(x+2) + 5(y-1)$ where $x + 2 \in \mathbb{Z}_{\geqslant 0}$.
      3.3.3. As $y > 0$, we know $y - 1 \in \mathbb{Z}_{\geqslant 0}$.
      3.3.4. So $P(k+1)$ is true.
   3.4. Case 2: $y = 0$.
      3.4.1. Then $k = 3x + 3 \times 0 = 3x$
      3.4.2. $\therefore\ x = k/3 \geqslant 8/3$ as $k \geqslant 8$;
      3.4.3. $\therefore\ x \geqslant \lceil 8/3 \rceil = 3$ as $x \in \mathbb{Z}$.
      3.4.4. Thus $k + 1 = 3x + 1 = 3(x-3) + 5 \times 2$, where $x - 3 \in \mathbb{Z}_{\geqslant 0}$.
      3.4.5. So $P(k+1)$ is true.
   3.5. Thus $P(k+1)$ is true in all cases.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 1}\ P(n)$ is true by MI. $\square$

*Alternative solution.*

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition " $\exists x, y \in \mathbb{Z}_{\geqslant 0}\ (n + 8 = 3x + 5y)$ ".
2. (Base step)
   2.1. $P(0)$ is true because $0 + 8 = 8 = 3 \times 1 + 5 \times 1$.
   2.2. $P(1)$ is true because $1 + 8 = 9 = 3 \times 3 + 5 \times 0$.
   2.3. $P(2)$ is true because $2 + 8 = 10 = 3 \times 0 + 5 \times 2$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+2)$ is true.
   3.2. Apply $P(k)$ to find $x, y \in \mathbb{Z}_{\geqslant 0}$ such that $k + 8 = 3x + 5y$.
   3.3. Then $(k+3) + 8 = (k+8) + 3$
   3.4. $= (3x + 5y) + 3$ by the choice of $x, y$;
   3.5. $= 3(x+1) + 5y$ where $x + 1, y \in \mathbb{Z}_{\geqslant 0}$.
   3.6. Thus $P(k+3)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}\ P(n)$ is true by Strong MI. $\square$

6. Prove by induction that every positive integer can be written as a sum of *distinct* non-negative integer powers of 2, i.e.,

$$\forall n \in \mathbb{Z}_{\geqslant 1}\ \exists \ell \in \mathbb{Z}_{\geqslant 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}\ (i_1 < i_2 < \cdots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}).$$

(Hint: think in terms of binary representations.)

*Solution.*

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition

$$\text{“}\ \exists \ell \in \mathbb{Z}_{\geqslant 1}\ \exists i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}\ (i_1 < i_2 < \cdots < i_\ell \wedge n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell})\ \text{”}.$$

2. (Base step) $P(1)$ is true because $1 = 2^0$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k)$ is true.
   3.2. Find $m \in \mathbb{Z}$ such that $k+1 = 2m$ or $k+1 = 2m+1$. This is possible because $k+1$ is either odd or even.
   3.3. Note $\quad 2m \leqslant k+1 \quad$ as $k+1 = 2m$ or $k+1 = 2m+1$;
   3.4. $\qquad\qquad \leqslant k+k \quad$ as $k \geqslant 1$;
   3.5. $\qquad\qquad = 2k$.
   3.6. So $m \leqslant k$.
   3.7. Also $\qquad 2m+1 \geqslant k+1 \quad$ as $k+1 = 2m$ or $k+1 = 2m+1$;
   3.8. $\qquad \therefore \qquad 2m \geqslant k \geqslant 0$
   3.9. $\qquad \therefore \qquad m \geqslant 0$.
   3.10. By lines 3.6 and 3.9, we know that $P(m)$ is true by the induction hypothesis.
   3.11. Apply $P(m)$ to find $\ell \in \mathbb{Z}_{\geqslant 1}$ and $i_1, i_2, \ldots, i_\ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$i_1 < i_2 < \cdots < i_\ell \quad \text{and} \quad m = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}.$$

   3.12. Case 1: $k+1 = 2m$.
      3.12.1. Then $\quad k+1 = 2m$
      3.12.2. $\qquad\qquad = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}) \qquad$ by the choice of $i_1, i_2, \ldots, i_\ell$;
      3.12.3. $\qquad\qquad = 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_\ell+1}$.
      3.12.4. Also $i_1+1 < i_2+1 < \cdots < i_\ell+1$ as $i_1 < i_2 < \cdots < i_\ell$.
      3.12.5. So $P(k+1)$ is true.
   3.13. Case 2: $k+1 = 2m+1$.
      3.13.1. Then $\quad k+1 = 2m+1$
      3.13.2. $\qquad\qquad = 2(2^{i_1} + 2^{i_2} + \cdots + 2^{i_\ell}) + 1 \qquad$ by the choice of $i_1, i_2, \ldots, i_\ell$;
      3.13.3. $\qquad\qquad = 2^0 + 2^{i_1+1} + 2^{i_2+1} + \cdots + 2^{i_\ell+1}$.
      3.13.4. Also $0 < i_1+1 < i_2+1 < \cdots < i_\ell+1$ as $0 \leqslant i_1 < i_2 < \cdots < i_\ell$.
      3.13.5. So $P(k+1)$ is true.
   3.14. Thus $P(k+1)$ is true in any case.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}$ $P(n)$ is true by Strong MI. $\square$

7. Show that $F_{n+4} = 3F_{n+2} - F_n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

   *Solution.*

   1. $F_{n+4} = F_{n+3} + F_{n+2} \qquad$ by the definition of $F_{n+4}$;
   2. $\qquad = (F_{n+2} + F_{n+1}) + F_{n+2} \qquad$ by the definition of $F_{n+3}$;
   3. $\qquad = 2F_{n+2} + F_{n+1}$
   4. $\qquad = 3F_{n+2} - F_{n+2} + F_{n+1}$
   5. $\qquad = 3F_{n+2} - (F_{n+1} + F_n) + F_{n+1} \qquad$ by the definition of $F_{n+2}$;
   6. $\qquad = 3F_{n+2} - F_n$.

8. Show by induction that $F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n$ for every $n \in \mathbb{Z}_{\geqslant 0}$.

   *Solution.*

   1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition

$$\text{“ } F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n \text{ ”}.$$

   2. (Base step)
      2.1. Since $F_0 = 0$ and $F_1 = 1$,

$$F_{0+1}^2 - F_{0+1}F_0 - F_0^2 = 1^2 - 1 \times 0 - 0^2 = 1 = (-1)^0.$$

      2.2. So $P(0)$ is true.

3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(k)$ is true, i.e., that

$$F_{k+1}^2 - F_{k+1}F_k - F_k^2 = (-1)^k.$$

   3.2. Then $F_{(k+1)+1}^2 - F_{(k+1)+1}F_{k+1} - F_{k+1}^2$
   3.3. $= F_{k+2}^2 - F_{k+2}F_{k+1} - F_{k+1}^2$
   3.4. $= (F_{k+1} + F_k)^2 - (F_{k+1} + F_k)F_{k+1} - F_{k+1}^2$ by the definition of $F_{k+2}$;
   3.5. $= F_{k+1}^2 + 2F_{k+1}F_k + F_k^2 - F_{k+1}^2 - F_kF_{k+1} - F_{k+1}^2$
   3.6. $= -(F_{k+1}^2 - F_{k+1}F_k - F_k)$
   3.7. $= -(-1)^k$ by the induction hypothesis;
   3.8. $= (-1)^{k+1}$.
   3.9. Thus $P(k+1)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}$ $P(n)$ is true by MI. □

9. Let $a_0, a_1, a_2, \ldots$ be the sequence satisfying

$$a_0 = 0, \quad a_1 = 2, \quad a_2 = 7, \quad \text{and} \quad a_{n+3} = a_{n+2} + a_{n+1} + a_n$$

for all $n \in \mathbb{Z}_{\geqslant 0}$. Prove by induction that $a_n < 3^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

*Solution.*

1. For each $n \in \mathbb{Z}_{\geqslant 0}$, let $P(n)$ be the proposition " $a_n < 3^n$ ".
2. (Base step)
   2.1. $P(0)$ is true because $a_0 = 0 < 1 = 3^0$.
   2.2. $P(1)$ is true because $a_1 = 2 < 3 = 3^1$.
   2.3. $P(2)$ is true because $a_2 = 7 < 9 = 3^2$.
3. (Induction step)
   3.1. Let $k \in \mathbb{Z}_{\geqslant 0}$ such that $P(0), P(1), \ldots, P(k+2)$ are true.
   3.2. $P(k), P(k+1), P(k+2)$ are true means

$$a_k < 3^k \quad \text{and} \quad a_{k+1} < 3^{k+1} \quad \text{and} \quad a_{k+2} < 3^{k+2}.$$

   3.3. Then $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ by the definition of $a_{k+3}$;
   3.4. $< 3^{k+2} + 3^{k+1} + 3^k$ by the induction hypothesis;
   3.5. $< 3^{k+2} + 3^{k+2} + 3^{k+2}$
   3.6. $= 3 \times 3^{k+2} = 3^{k+3}$.
   3.7. Thus $P(k+3)$ is true.
4. Hence $\forall n \in \mathbb{Z}_{\geqslant 0}$ $P(n)$ is true by Strong MI. □

10. Define a set $S$ recursively as follows.

   (a) $2 \in S$. (base clause)
   (b) If $x \in S$, then $3x \in S$ and $x^2 \in S$. (recursion clause)
   (c) Membership for $S$ can always be demonstrated by (finitely many) successive applications of clauses above. (minimality clause)

Which of the numbers $0, 6, 15, 16, 36$ are in $S$? Which are not?

*Solution.*

**Structural induction over $S$.** To prove that $\forall n \in S$ $P(n)$ is true, where each $P(n)$ is a proposition, it suffices to:

**(base step)** show that $P(2)$ is true; and

**(induction step)** show that $\forall x \in S \; \big(P(x) \Rightarrow P(3x) \wedge P(x^2)\big)$ is true.

- We know $0 \notin S$ because all $x \in S$ satisfy $x \geqslant 2$, as one can show by structural induction over $S$.
- $$\begin{array}{rrl} & 2 \in S & \text{by the base clause.} \\ \therefore & 6 \in S & \text{by the recursion clause with } x = 2 \text{ and the previous line.} \\ \therefore & 36 \in S & \text{by the recursion clause with } x = 6 \text{ and the previous line.} \end{array}$$
- $$\begin{array}{rrl} & 2 \in S & \text{by the base clause.} \\ \therefore & 4 \in S & \text{by the recursion clause with } x = 2 \text{ and the previous line.} \\ \therefore & 16 \in S & \text{by the recursion clause with } x = 4 \text{ and the previous line.} \end{array}$$
- We know $15 \notin S$ because no $x \in S$ is odd, as one can show by structural induction over $S$.