CS1231/CS1231S: Discrete Structures
Tutorial #2: Logic of Quantified Statements (Predicate Logic)
Answers

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1. Discussion Questions

These are meant for you to discuss on the LumiNUS Forum. No answers will be provided.

D1 An elementary definition in number theory is the following on divisibility:

For integers d and n, $d|n$ if and only if $n = kd$ for some integer k.

(Here, $d|n$ means "d divides n", and d is called a *divisor* or *factor*.)

(a) State the above definition symbolically.

(b) Does 2 divide $2\sqrt{2}$?

Answers:

(a) $(\forall d, n \in \mathbb{Z}\ d|n) \leftrightarrow (\exists k \in \mathbb{Z}\ n = kd)$

(b) This is an irrelevant question since $2\sqrt{2}$ is not even an integer. It does not satisfy the condition that d and n are integers.

D2. Explain why English can be ambiguous at times using the following sentence:

"Every boy loves a girl."

Interpret the above sentence in two ways and write the quantified statement for each of them.

Answer:

Let B be the set of boys and G the set of girls, and predicate $Loves(x, y)$ be x loves y.

$\forall b \in B, \exists g \in G, Loves(b, g)$ … John loves Mary, Ali loves Wati, etc. (Question: must the girls be unique?)

$\exists g \in G, \forall b \in B, Loves(b, g)$ … There is this girl who is loved by all the boys!

D3. The following table shows when the quantified statements are true and when they are false.

Statement	True when…	False when…
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Complete the table below for mixed quantifiers.

Statement	True when…	False when…
$\forall x \forall y\, P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair x, y for which $P(x, y)$ is false.
$\forall x \exists y\, P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y\, P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y\, P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

D4. Let the domain of discourse D be the set of all students at NUS, and let $M(s)$ be "s is a Math major", $C(s)$ be "s is a Computer Science major" and $E(s)$ be "s is an Engineering major". Express each of the following statements using quantifiers, variables and the predicates $M(s)$, $C(s)$ and $E(s)$.

Part (a) has been done for you.

(a) Every Computer Science major is an Engineering major.

Answer: $\forall s \in D\ (C(s) \rightarrow E(s))$.

Discuss: Why is the following answer wrong?

Wrong answer: $\forall s \in D\ (C(s) \land E(s))$.

Can you give an example to show the difference between these two answers?

(b) No Computer Science major are Engineering majors.

(c) Some Computer Science major are not Math majors.

(d) If a student is not a Math major, then the student is either a Computer Science major or an Engineering major, but not both.

2. Additional Notes

Note that "logic of quantified statements" (chapter 3) is commonly known as "predicate logic", as opposed to "propositional logic" in chapter 2.

We picked up some frequently asked questions and created this *Additional Notes* section to include some materials not covered in lecture that might be of interest to you.

Equivalent expressions: The following quantified statements are equivalent. We use the shorter notation on the left.

- $\forall x \in D, P(x) \equiv \forall x\big((x \in D) \to P(x)\big)$
- $\exists x \in D, P(x) \equiv \exists x\big((x \in D) \wedge P(x)\big)$

Well-formed formula (wff)

- **true** and **false** are wffs.
- A proposition variable is a wff.
- A predicate name followed by a list of variables (eg: $P(x)$, $Q(x, y)$), which is called an *atomic formula*, is a wff.
- If A, B and C are wffs, then so are $\sim A$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ and $(A \leftrightarrow B)$.
- If x is a variable and A is a wff, then so are $\forall x\ A$ and $\exists x\ A$.

Bound variables and Scope of quantifiers

- When a quantifier is used on a variable x in a predicate statement, we say that x is *bound*. If no quantifier is used on a variable, we say that the variable is *free*.

  Examples: In the statement $\forall x\, \exists y\, P(x, y)$, both x and y are bound. In the statement $\forall x\, P(x, y)$, x is bound but y is free.

- The scope of a quantifier is the range in the formula where the quantifier "engages in". It is put right after the quantifier, often in parentheses (eg: $\forall x\ \big(P(x)\big)$).

  Sometimes, when parentheses are not present (eg: $\forall x\, P(x)$), the scope is understood to be the smallest wff following the quantification.

- For example, in $\exists x\, P(x, y)$, the variable x is bound while y is free. In $\forall x\big(\exists y\, P(x, y) \vee Q(x, y)\big)$, x and the red y in $P(x, y)$ are bound but the blue y in $Q(x, y)$ is free, because the scope of $\exists y$ is $P(x, y)$, whereas the scope of $\forall x$ is $\big(\exists y\, P(x, y) \vee Q(x, y)\big)$. If you want to change the blue y to red y, you need to add parentheses: $\forall x\left(\exists y\ \big(P(x, y) \vee Q(x, y)\big)\right)$. Then, the outermost pair of parentheses may be removed, i.e.: $\forall x\, \exists y\ \big(P(x, y) \vee Q(x, y)\big)$.

3. Common Mistakes

- Using commas (,) in place of appropriate connectives, for example, $\forall x\ P(x), Q(x)$ where it should be $\forall x\ \big(P(x) \wedge Q(x)\big)$.
- Treating predicates as functions returning some value. Eg: given the following predicates
    - $Loves(x, y)$: x loves y
    - $Reindeer(x)$: x is a reindeer

  Some students wrote $Loves\big(x, Reindeer(y)\big)$ in part of their answers. Since $Reindeer(y)$ is a predicate, its value is either true or false. So, the above is akin to writing $Loves(x, true)$ or $Loves(x, false)$ which does not make sense! The correct way is to use the appropriate connectives.

4. Tutorial Questions

1. For each of the following statements, write its **converse**, **inverse** and **contrapositive**. Indicate which among the statement, its converse, its inverse, and its contrapositive are true and which are false. Give a counterexample for each that is false. Proof not required if it is true.

   a. $\forall n \in \mathbb{Z}\ (6|n \rightarrow 2|n \wedge 3|n)$. (Note: "$d|n$" is as defined in D1.)
   b. $\forall r \in \mathbb{R}\ (r > 3 \rightarrow r^2 > 9)$.
   c. $\forall p, q$ that are statements, $\big((p \rightarrow q) \rightarrow {\sim}p\big)$.

   To tutors: This is a straight-forward question to start off the tutorial. Don't spend too much time on this though.

   Answers:

   a. Statement: $\forall n \in \mathbb{Z}\ (6|n \rightarrow 2|n \wedge 3|n)$. (True)
   Converse: $\forall n \in \mathbb{Z}\ (2|n \wedge 3|n \rightarrow 6|n)$. (True)
   Inverse: $\forall n \in \mathbb{Z}\ (6 \nmid n \rightarrow 2 \nmid n \vee 3 \nmid n)$. (True)
   Contrapositive: $\forall n \in \mathbb{Z}\ (2 \nmid n \vee 3 \nmid n \rightarrow 6 \nmid n)$. (True)

   To tutors: Some students may write ${\sim}(6|n)$ instead. Let them know they can use $\nmid$.

   b. Statement: $\forall r \in \mathbb{R}\ (r > 3 \rightarrow r^2 > 9)$. (True)
   Converse: $\forall r \in \mathbb{R}\ (r^2 > 9 \rightarrow r > 3)$. (False)
   Inverse: $\forall r \in \mathbb{R}\ (r \leq 3 \rightarrow r^2 \leq 9)$. (False)
   Contrapositive: $\forall r \in \mathbb{R}\ (r^2 \leq 9 \rightarrow r \leq 3)$. (True)
   Counterexample for converse and inverse statements: Let $r = -5$.

   c. Statement: $\forall p, q$ that are statements, $\big((p \rightarrow q) \rightarrow {\sim}p\big)$. (False)
   Converse: $\forall p, q$ that are statements, $\big({\sim}p \rightarrow (p \rightarrow q)\big)$. (True)
   Inverse: $\forall p, q$ that are statements, $({\sim}(p \rightarrow q) \rightarrow p)$. (True)
   Contrapositive: $\forall p, q$ that are statements, $\big(p \rightarrow {\sim}(p \rightarrow q)\big)$. (False)
   Counterexample for original statement and contrapositive: Let both p and q be true.

2. Given the predicate $Loves(x, y)$ which means "x loves y", translate the following English sentences into predicate logic statements. For this question, you may leave out the domain of discourse in your statements (i.e. you do not need to specify what domain a variable belongs to, such as $\forall x \in D$; just write $\forall x$).

   a. Everybody loves someone else.

   b. Nobody except John loves Mary. (or: Only John loves Mary; nobody else loves Mary.)

Answers:

To tutors: This question is to show students they can use operators like = and $\neq$, instead of creating unnecessary predicates like $Equal(p, q)$.

a. $\forall p\ \exists q\ \left((p \neq q) \wedge Loves(p, q)\right)$

b. $Loves(John, Mary) \wedge \forall x\left((x \neq John) \rightarrow {\sim}Loves(x, Mary)\right)$

To tutors: Some students may not be aware that they can use John and Mary directly, instead of creating predicates like IsJohn(x) and IsMary(x).

3. Prove or disprove the following statement:

$$\forall a, b, c \in \mathbb{Z}, \text{ if } a - b \text{ is even and } a - c \text{ is even, then } b - c \text{ is even.}$$

(We have covered closure of integers under addition and multiplication in tutorial #1. Integers are also closed under subtraction, and you can cite this for this question.)

Answer:

Proof (direct proof)

1. Take any integers a, b, c.
2. Suppose $a - b$ is even and $a - c$ is even.
   - 2.1 There is an integer s s.t. $a - b = 2s$ (by definition of even numbers).
   - 2.2 Similarly, there is an integer t s.t. $a - c = 2t$ (by definition of even numbers).
   - 2.3 Then $b - c = (a - c) - (a - b) = 2t - 2s = 2(t - s)$ (by basic algebra).
   - 2.4 Let $p = t - s$.
     Now, $b - c = 2p$ where p is an integer (by closure of integers under subtraction).
3. Therefore, $b - c$ is even (by definition of even numbers). *

*: We may omit step 2.4 and skip to step 3 as $b - c = 2(t - s)$ is in the form of an even number since $t - s$ is an integer by closure under subtraction.

4. Some of the arguments below are valid, others are invalid. State which are valid and which are invalid and write out the reasons.

   a. All honest people pay their taxes.
      Darth is not honest.
      ∴ Darth does not pay his taxes.

   b. For every student x, if x studies CS1231, then x is good at logic.
      Tarik studies CS1231.
      ∴ Tarik is good at logic.

   c. Sum of any two rational numbers is rational.
      The sum $r + s$ is rational.
      ∴ r and s are both rational.

Answers:

a. Invalid; inverse error.

   $\forall x\ (Honest(x) \rightarrow PayTaxes(x))$

   $\sim Honest(Darth)$

   $\therefore \sim PayTaxes(Darth)$

b. Valid by universal modus ponens.

   $\forall x\ (TakesCS1231(x) \rightarrow GoodAtLogic(x))$

   $TakesCS1231(Tarik)$

   $\therefore GoodAtLogic(Tarik)$

c. Invalid; converse error.

   $\forall x, y\ (Rational(x) \wedge Rational(y) \rightarrow Rational(x + y))$

   $Rational(r + s)$

   $\therefore Rational(r) \wedge Rational(s)$

5. Let V be the set of all visitors to Universal Studios Singapore on a certain day, $T(v)$ be "v took the Transformers ride", $G(v)$ be "v took the Battlestar Galactica ride", $E(v)$ be "v visited the Ancient Egypt", and $W(v)$ be "v watched the Water World show".

   Express each of the following statements using quantifiers, variables, and the predicates $T(v)$, $G(v)$, $E(v)$ and $W(v)$. The statements are not related to one another. Part (a) has been done for you.

   a. Every visitor watched the Water World show.

Answer for (a): $\forall v \in V\ \big(W(v)\big)$.

   b. Every visitor who took the Battlestar Galactica ride also took the Transformers ride.

   c. There is a visitor who took both the Transformers ride and the Battlestar Galactica ride.

   d. No visitor who visited the Ancient Egypt watched the Water World show.

   e. Some visitors who took the Transformers ride also visited the Ancient Egypt but some (who took the Transformers ride) did not (visit the Ancient Egypt).

Answers

b. $\forall v \in V\ \big(G(v) \rightarrow T(v)\big)$

c. $\exists v \in V\ \big(T(v) \wedge G(v)\big)$

d. $\forall v \in V\ \big(E(v) \rightarrow \sim W(v)\big)$
   Alternatively: $\forall v \in V\ \big(\sim E(v) \vee \sim W(v)\big)$

e. $\Big(\exists v \in V\ \big(T(v) \wedge E(v)\big)\Big) \wedge \Big(\exists u \in V\ \big(T(u) \wedge \sim E(u)\big)\Big)$

   Note that the above is a conjunction (AND) of two existential clauses. Each existential variable v or u has a scope only within its own clause. Therefore, there is no ambiguity to use the same variable in both clauses, as shown below:

   $\Big(\exists v \in V\ \big(T(v) \wedge E(v)\big)\Big) \wedge \Big(\exists v \in V\ \big(T(v) \wedge \sim E(v)\big)\Big)$

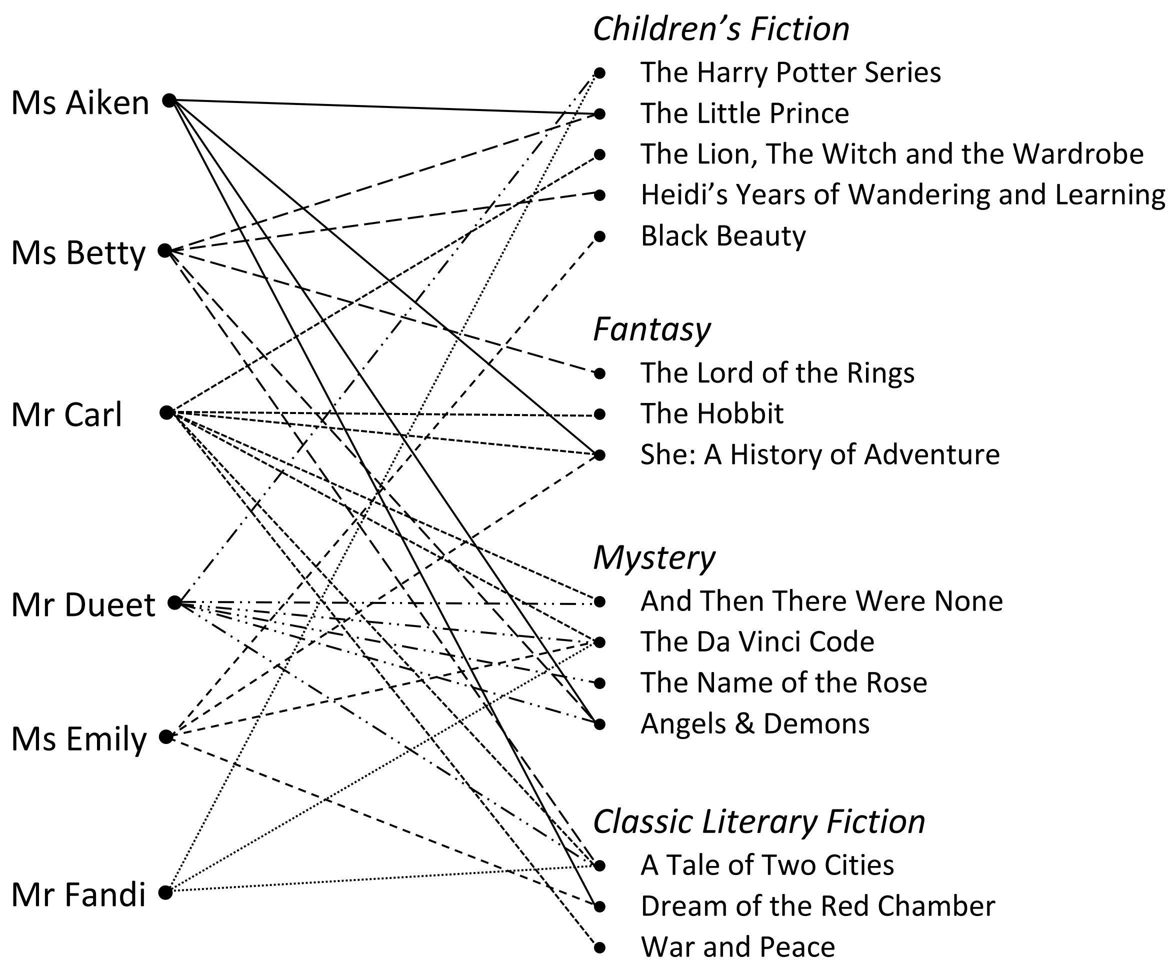
   Nevertheless, you may use different variable names to avoid confusion.

   Also, note that it is incorrect to write

   $$\exists v \in V\ \Big(\big(T(v) \wedge E(v)\big) \wedge \big(T(v) \wedge \sim E(v)\big)\Big)$$

   because the scope of v here covers both $\big(T(v) \wedge E(v)\big)$ as well as $\big(T(v) \wedge \sim E(v)\big)$. This means that if there exists a visitor who took the Transformers Ride and visited Ancient Egypt, the same visitor took the Transformers Ride but did not visit Ancient Egypt! The statement becomes false (by applying commutativity and associativity of conjunction).

6. Refer to the figure below, which shows six readers and four book genres (Children's Fiction, Fantasy, Mystery and Classic Literary Fiction) with selected book titles of each genre. A line is drawn between a reader and a book title if, and only if, that reader reads that book. For example, Ms Aiken reads "The Little Prince", but Mr Dueet does not read "Black Beauty".



Let R, G, and T be the sets of readers, genres and titles respectively. The following predicates are also given:

- $Male(x)$: x is male (you may assume that a person who is not a male is a female)
- $Read(x, y)$: x reads y (for indicating that a reader reads a certain title)
- $Belong(x, y)$: x belongs to y (for indicating that a title is under a certain genre)

For each of the following, indicate whether the statement is true or false, and optionally, write the quantified statement symbolically.

a. Some title is read by all the female readers.
b. Every reader reads some title in every genre.
c. Some reader reads all titles of some genre.
d. There is some genre for which some reader does not read any of its titles.

Answers:

a. **False.** None of the titles is read by all three female readers.

$\exists t \in T, \forall r \in R\ (\sim Male(r) \rightarrow Read(r,t))$

b. **False.** Mr Dueet doesn't read any Fantasy title.

$\forall r \in R, \forall g \in G, \exists t \in T\ (Read(r,t) \wedge Belong(t,g))$

c. **True.** Mr Dueet reads all the Mystery titles.

$\exists r \in R, \exists g \in G \left(\forall t \in T\ (Belong(t,g) \rightarrow Read(r,t))\right)$

d. **True.** None of the Fantasy titles is read by Dueet (and Fandi).

$\exists g \in G, \exists r \in R\ \left(\forall t \in T\ (Belong(t,g) \rightarrow \sim Read(r,t))\right)$

7. Given the following argument:

> 1. If an object is above all the triangles, then it is above all the blue objects.
> 2. If an object is not above all the gray objects, then it is not a square.
> 3. Every black object is a square.
> 4. Every object that is above all the gray objects is above all the triangles.
>
> $\therefore$ If an object is black, then it is above all the blue object.

a. Reorder the premises in the argument to show that the conclusion follows as a valid consequence from the premises, by applying universal transitivity (Lecture 3 slide 96). (Hint: It may be helpful to rewrite the statements in if-then form and replace some statements by their contrapositives.)

   You may use self-explanatory predicate names such as $Triangle(x)$, $Square(x)$, etc.

b. Rewrite your answer in part (a) using predicates and quantified statements.

Answers:

a.

3. If an object is black, then it is a square.
2. (Contrapositive form) If an object is a square, then it is above all the gray objects.
4. If an object is above all the gray objects, then it is above all the triangles.
1. If an object is above all the triangles, then it is above all the blue objects.

$\therefore$ If an object is black, then it is above all the blue objects.

b.

Let O, the domain, be the set of objects.

3. $\forall x \in O, \{Black(x) \rightarrow Square(x)\}$.
2. (Contrapositive form) $\forall x \in O, \big\{Square(x) \rightarrow \{\forall y \in O[Gray(y) \rightarrow Above(x, y)]\}\big\}$.
4. $\forall x \in O, \big\{\{\forall y \in O, [Gray(y) \rightarrow Above(x, y)]\} \rightarrow \{\forall z \in O, [Triangle(z) \rightarrow Above(x, z)]\}\big\}$.
1. $\forall x \in O, \big\{\{\forall z \in O, [Triangle(z) \rightarrow Above(x, z)]\} \rightarrow \{\forall w \in O, [Blue(w) \rightarrow Above(x, w)]\}\big\}$.

$\therefore \forall x \in O, \big\{Black(x) \rightarrow \{\forall w \in O, [Blue(w) \rightarrow Above(x, w)]\}\big\}$.

8. Let P and Q be predicates. Prove that:

a. $\big(\forall x \in D\ P(x)\big) \wedge \big(\forall x \in D\ Q(x)\big)$ is true if and only if $\forall x \in D\ \big(P(x) \wedge Q(x)\big)$ is true.

b. $\big(\exists x \in D\ P(x)\big) \wedge \big(\exists x \in D\ Q(x)\big)$ and $\exists x \in D\ \big(P(x) \wedge Q(x)\big)$ are not equivalent.

Answers:

a. To prove an "if and only if" statement, you need to prove both directions.

($\Rightarrow$) 1. Suppose $\big(\forall x \in D\ P(x)\big) \wedge \big(\forall x \in D\ Q(x)\big)$ is true.

2. Consider any $a \in D$.

   2.1 Since $\forall x \in D\ P(x)$ is true, we have $P(a)$ is true. (universal instantiation)

   2.2 Similarly, $Q(a)$ is true.

   2.3 Therefore, $P(a) \wedge Q(a)$ is true for any $a \in D$.

3. Therefore, $\forall x \in D\ \big(P(x) \wedge Q(x)\big)$ is true.

($\Leftarrow$) 1. Suppose $\forall x \in D\ \big(P(x) \wedge Q(x)\big)$ is true.

2. Consider any $a \in D$.

   2.1 Then $P(a) \wedge Q(a)$ is true.

   2.2 So, $P(a)$ is true and $Q(a)$ is true.

   2.3 Since $P(a)$ is true for any $a \in D$, we have $\forall x \in D\ P(x)$ is true.

   2.4 Similarly, since $Q(a)$ is true for any $a \in D$, we have $\forall x \in D\ Q(x)$ is true.

3. Therefore, $\big(\forall x \in D\ P(x)\big) \wedge \big(\forall x \in D\ Q(x)\big)$ is true.

b. To claim that $\big(\exists x \in D\ P(x)\big) \wedge \big(\exists x \in D\ Q(x)\big)$ and $\exists x \in D\ \big(P(x) \wedge Q(x)\big)$ are equivalent is to claim that they have the same truth values for any, D, P and Q, i.e. there is an implicit universal quantification over D, P and Q. To prove inequivalence, it therefore suffices to give a counterexample. There are many possible counterexamples. Here's one:

   Let $D = \mathbb{N}$, $P(x)$ be "$x^2 = 0$" and $Q(x)$ be "$x^2 = 1$".

   Then $(\exists x \in \mathbb{N}\ x^2 = 0) \wedge (\exists x \in \mathbb{N}\ x^2 = 1)$ is true, but $\exists x \in \mathbb{N}\ (x^2 = 0 \wedge x^2 = 1)$ is false.

9. Consider the statement: $\forall x, y \in \mathbb{R}\ (x > y \rightarrow x^2 > y^2)$.

   a. Prove that the statement is false by studying the negation of the statement.

   b. What is wrong with this proof:

      "Let $x = -1$ and $y = 2$. Then $x^2 = 1, y^2 = 4$ and $x^2 \ngtr y^2$, so that statement is false."

Answers:

a. $\sim\big(\forall x, y \in \mathbb{R}\ (x > y \rightarrow x^2 > y^2)\big) \equiv \exists x, y \in \mathbb{R}\ (x > y \wedge x^2 \leq y^2)$ which is true (example: $x = 1, y = -2$), so the original statement is false.

b. The counterexample $x = -1$ and $y = 2$ is **irrelevant** since it does not satisfy $x > y$.

   Counterexamples have to satisfy the hypothesis ($x > y$ in this case).