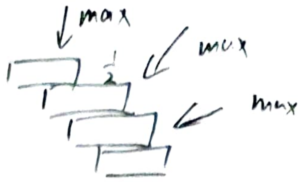
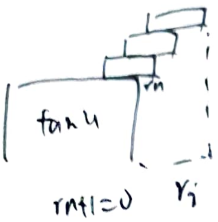
lecture 13 : Asymptotics

greedy algorithm



find cg

Let r_i = amount ith block hangs over ~~bottom block~~ the edge



find r_1

the centre of mass C_k must lie on the (k+1)st block, or it will tip over

table = (n+1)

For greedy stacking, $C_k = r_{k+1}$ → edge of the kth block.

recursive formula.

centre of mass of kth block by itself $r_k - \frac{1}{2}$

$$C_k = \frac{\overset{\text{weight}}{(k-1)}C_{k-1} + 1\cdot\left(r_k - \frac{1}{2}\right)}{k-1 \quad +1} = \frac{(k-1)\overset{C_{k-1}}{C_{k-1}} + r_k - \frac{1}{2}}{k}$$

$$= C_{k-1} - \frac{1}{2k}$$

$$r_k - r_{k+1} = \frac{1}{2k}$$

how much kth block sticks out

$$r_1 - r_2 = \frac{1}{2}$$

$$r_2 - r_3 = \frac{1}{4}$$

$$\underset{0}{\overset{}{r_1 - r_{n+1}}} = \frac{1}{2} + \frac{1}{4} \cdots \frac{1}{2n} = \sum_{i=1}^{n} \frac{1}{2i} = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{i}$$

harmonic sum

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$ → find integration bound.

$H_{1,000,000} = 14.3927\ldots$

$H_1 = 1$

$H_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$H_4 = 25/12 > 2$ → sticks over edge

Asymptotic notation

how a function grows in a limit

$\boxed{\text{tilde}}$ $f(x) \sim g(x)$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$

bounded upto constant factors

$\boxed{\text{oh, big-oh}}$ $f(x) = O(g(x))$ if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$ finite

$f(x) \in O(g(x))$ set of functions

Theorem: let $f(x) = x$ $g(x) = x^2$: Then $f(x) = O(g(x))$

Pf: $\lim_{x \to \infty} \frac{x}{x^2} = 0 < \infty$ $\square$ //

polynomial < exponential.

Thm: $x^{10} = O(e^x)$? $\lim_{x \to \infty} \frac{x^{10}}{e^x} = 0 < \infty$

matrix multiplication $O(n^3)$

Is $4^x = O(2^x)$ no $4^x = 2^{2x}$ $\lim_{x \to \infty} \frac{2^{2x}}{2^x} = 2^x \to \infty$

Is $10 = O(1)$ yes $|10| < \infty$

$H_n = \ln(n) + \delta + O(\frac{1}{n}) \Rightarrow H_n - \ln(n) - \delta = O(\frac{1}{n})$

$H_n \sim \ln(n) + \delta$

$H_n \sim \ln(n) + 10^6$ is also true ratio is still 1. T ✗

$H_n - \ln(n) \sim \delta$

$\not\sim 10^6$

$f(x) \geq O(g(x))$ is meaningless

lower bound

$\boxed{\text{omega}}$ $f(x) = \Omega(g(x))$ if $\lim_{x \to \infty} \left| \frac{f(x)}{g(x)} \right| > 0$

$f(x) = O(g(x))$ iff $g(x) = \Omega(f(x))$

$\frac{x}{100} = \Omega(100x + 25)$

at least quadratic $T(n) = \Omega(n^2)$

Analysis / Order of Growth

2. $F(n) = \underbrace{\log 2^n}_{O(n)} + \underbrace{\sqrt{n}}_{O(\sqrt{n})} + \underbrace{100000000}_{O(1)}$

$$\lim_{n\to\infty} \left| \frac{\sqrt{n}}{n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{2} n^{-\frac{1}{2}}}{1} \right| = \lim_{n\to\infty} \left| \frac{1}{2n^{\frac{1}{2}}} \right| = 0$$

$\therefore \sqrt{n} = O(n)$

upper bound.

$$\log_2 2^n = n$$

$$\log_{10} 2^n = \frac{\log_2 2^n}{\log_2 10} = \frac{n}{\log_2 10} \approx O(n)$$

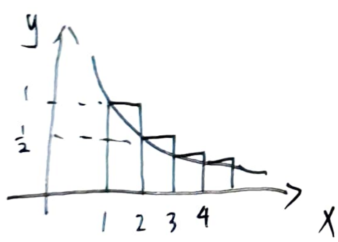
$$F(n) = O(n)$$

3 a) $F(n) = n + \frac{1}{2}n + \frac{1}{3}n + \cdots \frac{n}{n}$

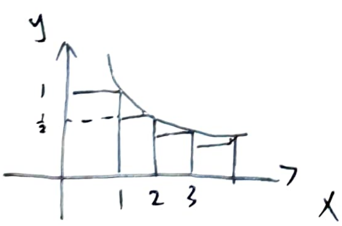
$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right) \rightarrow$ Harmonic series $\sum_{i=1}^{n} \frac{1}{i}$

$H(n)$

$y = \frac{1}{x}$



$$\sum_{i=1}^{n} \frac{1}{i} > \underbrace{\int_1^{n+1} \frac{1}{x}\,dx}_{\text{area under curve}}$$



$$\sum_{i=1}^{n} \frac{1}{i} < \underbrace{H(1)}_{\text{first rectangle}} + \underbrace{\int_1^{n} \frac{1}{x}\,dx}_{\text{area under curve.}}$$

$$\ln(n+1) - \cancel{\ln 1}^{\,0} < \sum_{i=1}^{n} \frac{1}{i} < 1 + \ln(n) - \cancel{\ln 1}^{\,0}$$

$H(n) \sim \ln(n)$

$H(n) \rightarrow \infty$ as $n \rightarrow \infty$

$H(n) = O(\log n)$

$F(n) = O(n \log n)$

h) $G(n) = n + \frac{1}{2}n + \frac{1}{4}n + \cdots 1$ (geometric series)

$$= n\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots \left(\frac{1}{2}\right)^{\log_2 n}\right)$$

$$\frac{n}{2^i} = 1$$

$$n = 2^i$$

$$i = \log_2 n$$

$$S = 1 + x + x^2 + \cdots x^{n-1}$$

$$Sx = x + x^2 + \cdots x^n$$

$$S(x-1) = x^n - 1$$

$$S = \frac{x^n - 1}{x - 1}$$

$$G(n) = n\,\frac{\left(1 - \frac{1}{2}^{\log_2 n + 1}\right)}{1 - \frac{1}{2}}$$

$$2^{-\log_2 n} = 2^{\log_2 \frac{1}{n}} = \frac{1}{n}$$

$$= n\left(\frac{1 - \frac{1}{2n}}{\frac{1}{2}}\right) \to 1$$

$$= O(n)$$