# CS3230: Assignment for Week 4 Solutions

Due: Sunday, 13th Feb 2022, 11:59 pm SGT.

---

1. Let $f(n)$ denote the maximum number of comparisons that Quicksort makes on an array of length $n$. We claim that $f(n) \leq \frac{n(n-1)}{2}$ for all $n$, and prove this by strong induction. The base cases $n = 0$ and $n = 1$ hold since $f(0) = f(1) = 0$. Assume that the statement holds up to $n$; we will prove it for $n+1$. Given an array of length $n+1$, Quicksort performs $n$ comparisons to partition the array into two subarrays of length $i$ and $n-i$ for some $0 \leq i \leq n$, and recurse on the two subarrays. Hence,

$$\begin{aligned} f(n+1) &\leq n + \max_{0\leq i\leq n}(f(i) + f(n-i)) \\ &\leq n + \max_{0\leq i\leq n}\left(\frac{i(i-1)}{2} + \frac{(n-i)(n-i-1)}{2}\right) \\ &= n + \max_{0\leq i\leq n}\left(\frac{n^2+2i(i-n)-n}{2}\right), \end{aligned}$$

where we use the inductive hypothesis for the second inequality. Note that $i(i-n) \leq 0$ for all $0 \leq i \leq n$, so $f(n+1) \leq n + \frac{n^2-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2} = \frac{(n+1)((n+1)-1)}{2}$, completing the induction.

To show that $f(n) \geq \frac{n(n-1)}{2}$, observe that if the array is already sorted, Quicksort performs $n-1$ comparisons in the first step, $n-2$ comparisons in the second step, and so on, so the total number of comparisons is $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$.

Hence, $f(n) = \frac{n(n-1)}{2}$ for all $n$.

2. Each permutation of the numbers results in a sorted array with probability $1/n!$, so the number of iterations follows a geometric distribution with expectation $n!$.

3. By definition of expected value,

$$\mathbb{E}[X] = \sum_{n=1}^{\infty}(n \cdot \Pr[X = n]) = \sum_{n=1}^{\infty}\frac{n}{2^n}.$$

Let $C$ denote the latter sum. We have

$$2C = \sum_{n=1}^{\infty} \frac{2n}{2^n} = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \sum_{n=1}^{\infty} \left( \frac{n-1}{2^{n-1}} + \frac{1}{2^{n-1}} \right) = \sum_{n=0}^{\infty} \frac{n}{2^n} + \sum_{n=0}^{\infty} \frac{1}{2^n} = C + 2,$$

so $C = 2$.

Note that $X$ describes the number of flips of a fair coin until it comes up heads, which is a geometric distribution with probability $p = 1/2$. The calculations above show that its expected value is $1/p = 2$.