CS3230 Design & Analysis of Algorithms **February 5, 2022**

Indicator Random Variables

Prepared by: Christian James Welly (TA)

1.1 Introduction

Indicator Random Variables are simple but very useful tools for Randomised Analysis. You would have seen the use of indicator random variables in Week 5 lecture to prove guarantees pertaining to Universal Hashing and Perfect Hashing. This document explores the concept and several examples of using indicator random variables for analysis.

1.1.1 Definition

Let X be a indicator random variable associated with an event A that occurs with probability p:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases} \tag{1.1}$$

This means that the indicator random variable X takes on the value of 1 with probability of p, and 0 with probability $1-p$ (either event A occurs, or it does not occur). This can also be expressed as $\Pr(X=1) = p$ and $\Pr(X=0) = 1-p$.

1.1.2 Expectation

A very useful property of an indicator random variable is its expectation:

Claim 1.1.1. *Let an X be an indicator random variable associated with an event A that occurs with probability p, then $E[X] = p$*

Proof.

$$\begin{aligned} E[X] &= 1 \cdot \Pr(X=1) + 0 \cdot \Pr(X=0) \qquad \text{[definition of expectation]} \\ &= 1 \cdot p + 0 \cdot (1-p) \\ &= p \end{aligned}$$

□

The simple fact that you can obtain the expectation of an indicator random variable by computing the probability of the associated event occurring is immensely useful, and will be used a lot throughout randomised analysis.

1.2 Worked Examples

We will build up our familiarity of using indicator random variables by going through some worked examples. A common (but not always!) strategy when doing analysis with indicator random variables is as follows:

1. Identify what event you want to count in expectation. Call it X
2. Express X as the sum of indicator random variables: $X = \sum_{i=1}^{n} X_i$. Calculate $\Pr(X_i = 1)$ and you get $E[X_i]$ for free! (By Claim 1.1.1)
3. To compute $E[X]$, apply linearity of expectations which yields $\sum_{i=1}^{n} E[X_i]$, and substitute their probabilities

1.2.1 Four coin flips

Let's say we have a fair coin. We flip the coin 4 times. What is the expected number of heads?

Intuition should tell you that if we have 4 coin flips, we should expect that we get 2 heads. We will now formalise this intuition:

Let X be the random variable representing the total number of heads. We let X_i be the indicator random variable for the event that the i-th coin flip gives a head. Then $X = X_1 + X_2 + X_3 + X_4$.

The intuition of expressing it in this way here is that X counts **all** the occurrences of the coin flip resulting in heads, while the individual X_i counts the **individual** trials. You may note that X_i is an indicator random variable which can only take value either 0 or 1, thus it will "contribute" to X only if the i-th coin lands head.

We also note that $E[X_i] = \Pr(X_i = 1) = \frac{1}{2}$

$$\begin{aligned}
E[X] &= E[X_1 + X_2 + X_3 + X_4] \\
&= E[X_1] + E[X_2] + E[X_3] + E[X_4] && \text{[Linearity of expectation]} \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&= 2
\end{aligned}$$

1.2.2 Generalisation: n coin flips

Instead of flipping our coin 4 times, what if we flip our coin n times? What would be the expected number of heads?

The analysis follows exactly like before! This time, we have n indicator random variables:

Let X be the random variable representing the total number of heads. We let X_i be the indicator random variable for the event that the i-th coin flip gives a head. Then $X = \sum_{i=1}^{n} X_i$. Also note that $\Pr(X_i = 1) = \frac{1}{2}$

$$\begin{aligned}
E[X] &= E[\sum_{i=1}^{n} X_i] \\
&= \sum_{i=1}^{n} E[X_i] && \text{[Linearity of expectation]} \\
&= \sum_{i=1}^{n} \frac{1}{2} \\
&= \frac{n}{2}
\end{aligned}$$

1.2.3 More generalisations: Binomial Distribution!

The binomial distribution is a generalisation where we make n independent experiments, and the probability of success of an experiment is p (for coin flips, it would be $p = \frac{1}{2}$). In a binomial distribution, we would like to count the expected number of successes in this sequence of n experiments.

Hopefully, you can see that the analysis is pretty much the same as before!

Let X be the random variable representing the total number of successes. We let X_i be the indicator random variable for the event that the i-th experiment is a success. Then $X = \sum_{i=1}^{n} X_i$. Also note that $\Pr(X_i = 1) = p$

$$\begin{aligned}
E[X] &= E[\sum_{i=1}^{n} X_i] \\
&= \sum_{i=1}^{n} E[X_i] \qquad \text{[Linearity of expectation]} \\
&= \sum_{i=1}^{n} p \\
&= np
\end{aligned}$$

This simple exercise also gives us a proof that the expectation of a binomial distribution is np. Isn't it neat?

1.2.4 Number of 1s in a 2D array

Although this exercise is a worked example, it is recommended that you try this question on your own first.

Let's say you have a $n \times n$ 2D array A. Every cell has integer values chosen uniformly at random from 1 to n inclusive. What is the expected number of cells that have value of 1?

Let X be the number of cells that have a value of 1. We let $X_{i,j}$ be the indicator random variable for the event that the (i, j)-entry of A has a value of 1. Then $E[X_{i,j}] = \Pr(X_{i,j} = 1) = \frac{1}{n}$

Note that $X = \sum_{j=1}^{n} \sum_{i=1}^{n} X_{i,j}$, since we are counting every cells in the 2D array.

$$\begin{aligned}
E[X] &= E[\sum_{j=1}^{n} \sum_{i=1}^{n} X_{i,j}] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} E[X_{i,j}] \qquad \text{[Linearity of Expectations]} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{1}{n} \\
&= \sum_{j=1}^{n} n \times \frac{1}{n} \\
&= \sum_{j=1}^{n} 1 \\
&= n
\end{aligned}$$

1.3 Exercises

This section contains exercises for you to apply these concepts! Answers (but not workings) are provided at the end.

1.3.1 Coloured Balls

A bag contains 3 balls with different colours: red, green and blue. You will do the following procedure n times:

1. Take a ball from the bag. Note down its colour
2. Place the ball back into the bag

What is the expected number of **blue** balls that you would get at the end of this procedure?

1.3.2 Biased coins!

Let's say we have n **biased** coins. The i-th coin has probability of $\frac{1}{i}$ of flipping heads. What is the expected number of heads after flipping all these n coins one-by-one? You may express your answer in asymptotic notation.

1.3.3 More biased coins!

Let's say we have $\log_2 n$ **biased** coins (you are assured that you will have integer number of coins). The i-th coin has probability of $\frac{1}{2^{i-1}}$ of flipping heads. What is the expected number of heads after flipping all these $\log_2 n$ coins one-by-one?

1.3.4 Randomised Quicksort, again!

In lecture, you have seen an analysis of randomised quicksort using recurrences. Now, show that the expected runtime of randomised quicksort is $\theta(n \lg n)$ by using indicator random variables.

Hint: the runtime of quicksort is dependent on the number of comparisons made. What are you counting here, and how can you decompose them into indicator random variables?

1.4 Exercise Answers

1. Coloured Balls: $\frac{n}{3}$
2. Biased Coins: $\theta(\lg n)$
3. More biased coins: $2(1 - \frac{1}{n})$
4. Randomised Quicksort, again: CLRS section 7.4. provides a detailed analysis. You may wish to consult the textbook.