W06: Hashing

CS3230 AY21/22 Sem 2

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Analysis using Indicator Random Variables

(Recap)

Indicator Random Variables

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Indicator RV is like a form of “counter” for the occurrence of some event.
Let *X* be an indicator RV, where a certain event occurs with probability *p*:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1 - p \end{cases}$$

You may observe that it being 0 or 1 is the way we “count” whether some event occurs

Indicator Random Variables (Expectation)

Let X be an indicator random variable with probability p of the event happening.

$E[X]$

$= 1\,(p) + 0\,(1 - p)$ [Definition of expectation]

$= p$

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

Very useful! Simple (hopefully) to calculate

$$\boxed{E[X] = p}$$

The analysis “pattern” (1 & 2 are “interchangeable”)

Most (but not all) analysis in Randomised Algorithms follow this “pattern”.

1. Identify a Random Variable to “count” what you want (e.g. X. Goal: $E[X]$)

2. Express this RV as a **sum** of random variables (e.g. $X = X_1 + X_2 + \ldots + X_n$)
    a. Calculate the relevant probability for X_1, X_2, ...
    b. Calculate the individual expectation of the “sub”-random variables. ($E[X_1]$, $E[X_2]$, …)

3. Use linearity of expectations on $E[X]$. Then you add up the expectation of the “sub”-random variables (from step 2b)

Universal Hashing

Universal Hashing (Motivation)

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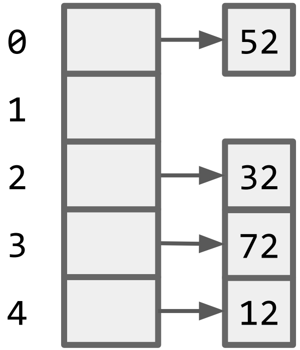
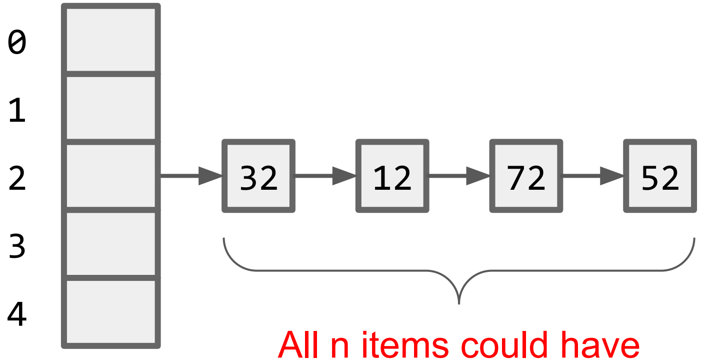


Ideal!



All n items could have hashed to the same bucket!

Universal Hashing (Motivation)

- Suppose your hash table uses a particular hash function *h*
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- We want to “**fight against**” the adversary! Instead of deterministically choosing hash function:

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  - **Randomly** choose from a **set** of hash functions
  - Use that hash function instead -- adversary can’t possibly know for sure what it is!

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- We want to "**fight against"** the adversary! Instead of deterministically choosing hash function:
  - **Randomly** choose from a **set** of hash functions
  - Use that hash function instead -- adversary can't possibly know for sure what it is!
- We want to choose from a **"good set"** of hash functions -- Universal Hash Family is one way to define this "good set"!

Universal Hashing (Definition)

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***universal*** if for all $x \neq y$:

$$\frac{|h \in \mathcal{H} : h(x) = h(y)|}{|\mathcal{H}|} \leq \frac{1}{M}.$$

H = Our set of hash functions

U = All the items in the universe

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Number of hash functions where x and y collide

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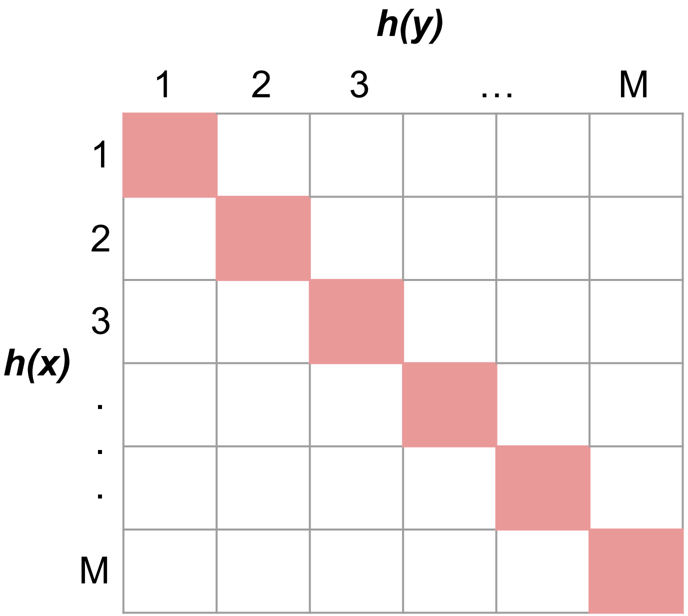
$$\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}.$$

Alternative formulation

Universal Hashing (Illustration)

Universal: $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$

Take **any** different *x*, *y* in *U*. Write the number of hash functions *h* in *H* that lie in each box.



If *H* is universal, sum of values in red cells must be at most ***|H|/M***

Example

<u>Definition</u>: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***universal*** if for all $x \neq y$:

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Pairs: (a, b)

M = 2 (result is 0 or 1)

|H| = 3

	a	b
h_1	0	0
h_2	1	0
h_3	0	1

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LHS:

(a, b) collides for h_1 only

Therefore, since |H| = 3

LHS = ⅓

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Therefore, since $|H| = 3$

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RHS:

1/M = ½

We have $\boxed{⅓} \leq \boxed{½}$, so this set is universal!

Example 2

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***universal*** if for all $x \neq y$:

$$\frac{|h \in \mathcal{H} : h(x) = h(y)|}{|\mathcal{H}|} \leq \frac{1}{M}.$$

Pairs: (a, b), (a, c), (b, c)

M = 2 (result is 0 or 1)

|H| = 3

RHS = ½

	a	b	c
h_1	0	0	1
h_2	1	1	0
h_3	1	0	1

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	a	b	c
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(b, c)

↳ No collision

↳ LHS = 0/3

$0/3 \leq 1/2$

ok!

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(b, c)

↳ No collision

↳ LHS = $0/3$

$0/3 \leq 1/2$

ok!

(a, c)

↳ Collides for h_3

↳ LHS = $1/3$

$1/3 \leq 1/2$

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(b, c)

↳ No collision

↳ LHS = $0/3$

$0/3 \leq 1/2$

ok!

(a, c)

↳ Collides for h_3

↳ LHS = $1/3$

$1/3 \leq 1/2$

ok!

(a, b):

↳ Collides for h_1, h_2

↳ LHS = $2/3$

$\frac{2}{3} > \frac{1}{2}$

Not ok!

Not all pairs satisfy the universality condition!

Therefore not universal

Pairwise Independent Family

Pairwise Independent

H = Our set of hash functions
U = All the items in the universe
M = How many "buckets" in the hash table

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***pairwise-independent*** if for all $x \neq y$ and any two hash values i_1, i_2:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}.$$

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Intuitively:

- Think of M^2 as all possible pairs of i_1 and i_2: e.g. { (0, 0), (0, 1), (1, 0), (1, 1) }

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Relate it to "independence":
Pr(h(x) = i_1) = 1/m
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Pr both = (1/m)(1/m)

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Intuitively:

- Think of M^2 as all possible pairs of i_1 and i_2: e.g. { (0, 0), (0, 1), (1, 0), (1, 1) }
- For all distinct x and y, I can choose i_1 and i_2 to be anything I want
  - And the result "should feel uniformly distributed" (remember, M^2 is all possibilities)

Pairwise Independent (Illustration)

Pairwise-independent:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Take **any** different *x*, *y* in *U*. Write the number of hash functions *h* in *H* that lie in each box.

h(y)

h(x)	1	2	3	…	M
1	$\frac{\lvert H \rvert}{M^2}$	$\frac{\lvert H \rvert}{M^2}$	$\frac{\lvert H \rvert}{M^2}$	…	$\frac{\lvert H \rvert}{M^2}$
2	$\frac{\lvert H \rvert}{M^2}$	$\frac{\lvert H \rvert}{M^2}$	…	…	…
3	$\frac{\lvert H \rvert}{M^2}$	…	…	…	…
⋮	…	…	…	…	$\frac{\lvert H \rvert}{M^2}$
M	$\frac{\lvert H \rvert}{M^2}$	…	…	$\frac{\lvert H \rvert}{M^2}$	$\frac{\lvert H \rvert}{M^2}$

If *H* is pairwise independent, the value in each cell must be the same, i.e. $|H|/M^2$

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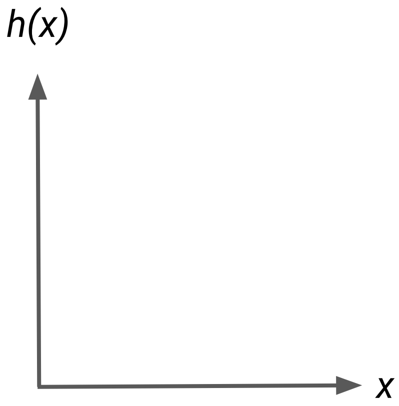

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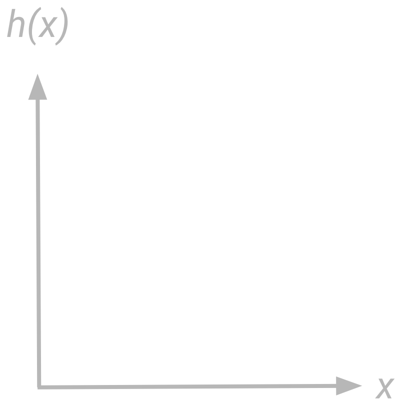

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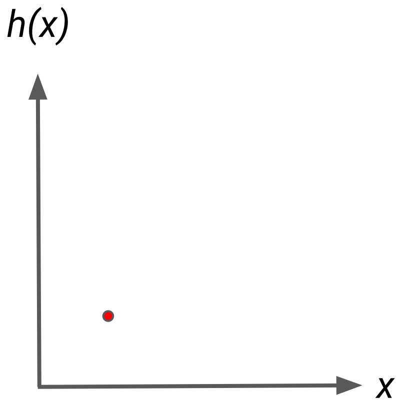

Even after the first point is placed, the second point is still “random”!

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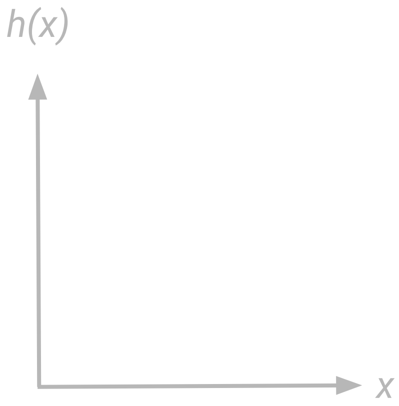

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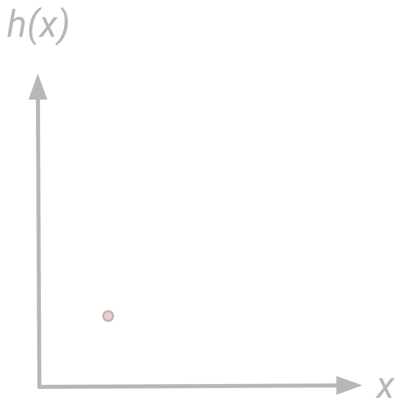

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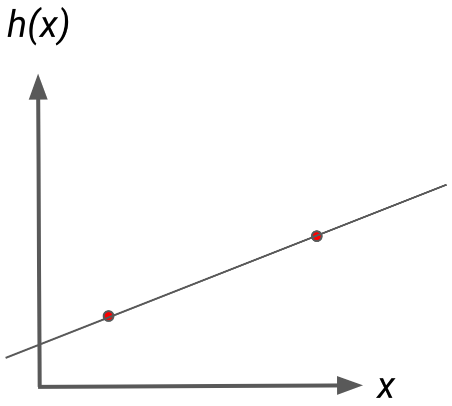

But after the second point, the third point can be determined! Not independent anymore

Pairwise Independent (Example)

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***pairwise-independent*** if for all $x \neq y$ and any two hash values i_1, i_2:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}.$$

$|\mathcal{H}| = 4$

$M = 2$ (result is 0 or 1)

	a	b
h_1	1	0
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	a	b
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h_3	1	1
h_4	0	1

For all i_1, i_2

$$\boxed{\text{LHS: } \Pr[h(a) = i_1, h(b) = i_2] = \frac{1}{|\mathcal{H}|} = \frac{1}{4}}$$

$$\boxed{\text{RHS: } \frac{1}{M^2} = \frac{1}{2^2} = \frac{1}{4}}$$

$\therefore \text{LHS} = \text{RHS} \quad \forall x, y, i_1, i_2$

Note: Definition with Equality

Pairwise-independent:

$$\Pr_{h\sim\mathcal{H}}[h(x)=i_1,h(y)=i_2] \boxed{=} \frac{1}{M^2}$$

$\Longleftrightarrow$

Pairwise-independent:

$$\Pr_{h\sim\mathcal{H}}[h(x)=i_1,h(y)=i_2] \boxed{\leq} \frac{1}{M^2}$$

Tutorial question, earlier version of slides + recording uses ≤ to define pairwise-independent, whereas this version defines it with equality. The two definitions are **equivalent**. However, equality seems to be more common.

Proof Sketch: Consider all M^2 combinations of i_1 and i_2.

1 = Sum of combinations of Probability of hashing to i_1 and i_2 respectively ≤ $M^2 * (1/M^2) = 1$

First equality is due to probability axioms after considering all cases.
For each probability of hashing to i_1 and i_2, it must be exactly $1/M^2$

Question 1: Pairwise Independent → Universal?

Q1

Does Pairwise-Independent family imply Universal family?

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***pairwise-independent*** if for all $x \neq y$ and any two hash values i_1, i_2:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

?

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***universal*** if for all $x \neq y$:

$$\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$$

Question 1 (Answer)

Yes! Pairwise Independent Family is always Universal Hash family

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Yes! Pairwise Independent Family is always Universal Hash family

Proof strategy:

1. Use definition of Universal Hash family
2. Express it similar to Pairwise Independent family
3. Since we assume Pairwise Independent, use its property

Question 1 (Answer)

Pairwise-independent:

$$\Pr_{h\sim\mathcal{H}}[h(x)=i_1, h(y)=i_2] = \frac{1}{M^2}$$

Universal: $\Pr_{h\sim\mathcal{H}}[h(x)=h(y)] \leq \frac{1}{M}$

$$\Pr[h(x)=h(y)]$$

Question 2

One side of Universal Family

Question 1 (Answer)

Pairwise-independent:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Universal: $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$

Question 2

Make it look like pairwise independent

$$\Pr[h(x) = h(y)]$$

$$= \sum_i \Pr[h(x) = i, h(y) = i]$$

e.g. M = 2, {0, 1}

Then

Pr[h(x) = h(y)]

= Pr[h(x) = 0, h(y) = 0]

+ Pr[h(x) = 1, h(y) = 1]

Question 1 (Answer)

Pairwise-independent:

$$\Pr_{h\sim\mathcal{H}}[h(x)=i_1, h(y)=i_2] = \boxed{\frac{1}{M^2}}$$

Universal: $\Pr_{h\sim\mathcal{H}}[h(x)=h(y)] \le \frac{1}{M}$

Question 2

Use property of pairwise independent!

$$\Pr[h(x)=h(y)]$$

$$=\sum_i \Pr[h(x)=i, h(y)=i]$$

$$=\boxed{M}\cdot\boxed{\frac{1}{M^2}}$$

There are M possible hash values

Question 1 (Answer)

Pairwise-independent:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Universal: $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$

Question 2

Proven!

$$\boxed{\Pr[h(x) = h(y)]}$$

$$= \sum_i \Pr[h(x) = i, h(y) = i]$$

$$= M \cdot \frac{1}{M^2} = \boxed{\frac{1}{M}}$$

Question 2: Universal $\to$ Pairwise Independent?

Q2

Does Universal family imply Pairwise-independent family?

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***pairwise-independent*** if for all $x \neq y$ and any two hash values i_1, i_2:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

$$\Uparrow ?$$

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***universal*** if for all $x \neq y$:

$$\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$$

Question 2 (Answer)

Pairwise-independent:
$$\Pr_{h\sim\mathcal{H}}[h(x)=i_1, h(y)=i_2] = \frac{1}{M^2}$$

Universal: $\Pr_{h\sim\mathcal{H}}[h(x)=h(y)] \leq \frac{1}{M}$

No! Come up with counterexample:

	a	b
h_1	0	0
h_2	0	1

Universal

LHS: collision for h_1

: $\frac{1}{2}$

RHS: $\frac{1}{M}$: $\frac{1}{2}$

$\therefore$ LHS $\leq$ RHS

Question 2 (Answer)

Pairwise-independent:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Universal: $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$

No! Come up with counterexample:

	a	b
h_1	0	0
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Universal

LHS: collision for h_1

$: \frac{1}{2}$

RHS: $\frac{1}{M} : \frac{1}{2}$

$\therefore$ LHS $\leq$ RHS

Not Pair-wise Independent

LHS: $\Pr[h(a) = 0, h(b) = 0] = \frac{1}{2}$

RHS: $\frac{1}{M^2} = \frac{1}{2^2} = \frac{1}{4}$

$\therefore$ LHS $\neq$ RHS

$\Rightarrow$ Not pairwise-independent

Pairwise Independent and Universal

- This means that pairwise independent family is a **stronger** notion of hash family!

Pairwise Independent and Universal

- This means that pairwise independent family is a **stronger** notion of hash family!
- Intuitively:
  - In Pairwise Independent Family, you can ***freely*** choose any pair of hash values i_1 *and* i_2
  - In Universal Family, you are ***limited*** to hashes of x and y that collide (equal i_1 *and* i_2)

Pairwise-independent:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Universal: $\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$

Question 3:
Pairwise Independent -- in a particular slot

Question 3

- Now you have a hash function from pairwise independent family
- Hash **N** distinct elements
  - At most, what is the expected number of elements which hashes to a particular **slot *j***?

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***pairwise-independent*** if for all $x \neq y$ and any two hash values i_1, i_2:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}.$$

Question 3

A **pairwise independent** family $\mathcal{H}$ of hash functions mapping $\mathcal{U}$ to $\{1, \dots, M\}$ has the property that for any two distinct universe elements x, y and for any two hash values i_1, i_2:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] \leq \frac{1}{M^2}.$$

Suppose you hash N distinct elements using h randomly drawn from a pairwise independent family. The expected number of elements which hash to slot 1 is at most?

(A) N/M^2
(B) N/M
(C) $N/2M$
(D) $N/4M$

Question 3 (Answer)

Strategy: Indicator Random Variables

1. Identify a Random Variable to “count” what you want (e.g. X. Goal: $E[X]$)
2. Express this RV as a **sum** of random variables (e.g. $X = X_1 + X_2 + \ldots + X_n$)
   a. Calculate the relevant probability for X_1, X_2, ...
   b. Calculate the individual expectation of the “sub”-random variables. ($E[X_1]$, $E[X_2]$, …)
3. Use linearity of expectations on $E[X]$. Then you add up the expectation of the “sub”-random variables (from step 2b)

Question 3 (Answer)

Goal: Expected number items hashing to slot j

Let X be the random variable representing the number of items hashing to slot j

Let X_i be the **indicator random variable** that item i (x_i) hashes to slot j

$X = X_1 + X_2 + \ldots X_N$

Question 3 (Answer)

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Let X be the random variable representing the number of items hashing to slot j

Let X_i be the **indicator random variable** that item i (x_i) hashes to slot j

$X = X_1 + X_2 + \ldots X_N$

Next things to do:

- Calculate $Pr(X_i = 1)$. This is enough to get $E[X_i]$!
- Then we can calculate $E[X]$ easily by linearity of expectations

Question 3 (Computing $Pr(X_i = 1)$)

Pairwise-independent:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Let X_i be the **indicator random variable** that item i (x_i) hashes to slot j

$$\Pr(X_i = 1)$$

Question 3

Getting ready!

Question 3 (Computing $Pr(X_i = 1)$)

Pairwise-independent:

$$\Pr_{h\sim\mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Let X_i be the **indicator random variable** that item i (x_i) hashes to slot j

$$\Pr(X_i = 1) = \Pr(h(x_i) = j)$$

Question 3

This is how we defined the indicator!

Question 3 (Computing $Pr(X_i = 1)$)

Pairwise-independent:

$$\Pr_{h\sim\mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Let X_i be the **indicator random variable** that item i (x_i) hashes to slot j

$$\Pr(X_i = 1) = \Pr(\boxed{h(x_i) = j})$$

$$= \boxed{\sum_{k=0}^{M-1}} \Pr(\boxed{h(x_i) = j}, h(\boxed{y}) = \boxed{k})$$

Question 3

"Make it look like pairwise indep":
k = goes through all M items.
y = item distinct from x_i

Question 3 (Computing $Pr(X_i = 1)$)

Pairwise-independent:

$$\boxed{\Pr_{h\sim\mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}}$$

Let X_i be the **indicator random variable** that item i (x_i) hashes to slot j

$$\Pr(X_i = 1) = \Pr(h(x_i) = j)$$

$$= \sum_{k=0}^{M-1} \boxed{\Pr(h(x_i) = j, h(y) = k)}$$

$$= \sum_{k=0}^{M-1} \boxed{\frac{1}{M^2}}$$

Question 3

Pairwise Independence!

Question 3 (Computing $Pr(X_i = 1)$)

Pairwise-independent:

$$\Pr_{h \sim \mathcal{H}}[h(x) = i_1, h(y) = i_2] = \frac{1}{M^2}$$

Let X_i be the **indicator random variable** that item i (x_i) hashes to slot j

$$\begin{aligned}
\Pr(X_i = 1) &= \Pr(h(x_i) = j) \\
&= \sum_{k=0}^{M-1} \Pr(h(x_i) = j, h(y) = k) \\
&= \boxed{\sum_{k=0}^{M-1}} \frac{1}{M^2} \\
&= \frac{1}{M}
\end{aligned}$$

Question 3

M hash values → M * (1/M^2)

Question 3: Apply linearity!

Goal: Expected number items hashing to slot *j*

Let *X* be the random variable representing the number of items hashing to slot *j*
Let X_i be the **indicator random variable** that item *i* (x_i) hashes to slot *j*

$X = X_1 + X_2 + \dots X_N$

$$E[X] = E[\sum_{i=1}^{\boxed{N}} X_i]$$

Recall: hashing N distinct elements!

$$= \sum_{i=1}^{N} \boxed{E[X_i]}$$

From earlier slide

$$= \sum_{i=1}^{N} \boxed{\frac{1}{M}}$$

$$= \frac{N}{M}$$

Question 4: Same bound for Universal Family?

Question 4

- Now you have a hash function from **universal family instead**
- Hash **N** distinct elements
  - Is the expected number of elements which hashes to a particular **slot *j*** still the same as before? i.e. Still ≤ N/M?

Definition: Suppose $\mathcal{H}$ is a set of hash functions mapping U to $[M]$. We say $\mathcal{H}$ is ***universal*** if for all $x \neq y$:

$$\Pr_{h \sim \mathcal{H}}[h(x) = h(y)] \leq \frac{1}{M}$$

Question 4

The same upper bound as the previous question holds for a hash function drawn from a universal (instead of pairwise independent) family.

True or False?

False

Universal, from Q2 $\left\{ \begin{array}{c|c|c} & a & b \\ \hline h_1 & 0 & 0 \\ \hline h_2 & 0 & 1 \\ \hline & & \end{array} \right.$

$N=2$ (a and b)

$M=2$ (0 and 1)

$\therefore \frac{N}{M}=1$

False

Universal, from Q2 $\left\{ \begin{array}{c|c|c} & a & b \\ \hline h_1 & 0 & 0 \\ \hline h_2 & 0 & 1 \\ \hline \end{array} \right.$

$N = 2$ (a and b)

$M = 2$ (0 and 1)

$\therefore \frac{N}{M} = 1$

Define IRV X_i:

$$X_i \begin{cases} 1 & \text{if } h(i) = 0 \\ 0 & \text{otherwise} \end{cases}$$

Count it if it hashes to slot 0

$Pr[X_a = 1] = 1$ (both hash gives 0)

$Pr[X_b = 1] = \frac{1}{2}$ (only h_1

False

Universal, from Q2

	a	b
h_1	0	0
h_2	0	1

$N=2$ (a and b)

$M=2$ (0 and 1)

$\therefore \frac{N}{M}=1$

Define IRV X_i:

$$X_i \begin{cases} 1 & \text{if } h(i)=0 \\ 0 & \text{otherwise} \end{cases}$$

Count it if it hashes to slot 0

$Pr[X_a=1]=1$ (both hash gives 0)

$Pr[X_b=1]=\frac{1}{2}$ (only h_1

X: total hashing to slot 0

$$\mathbb{E}[X] = \mathbb{E}[X_a] + \mathbb{E}[X_b]$$

$$= 1 + \frac{1}{2}$$

$$= 1.5$$

False

Universal, from Q2

	a	b
h_1	0	0
h_2	0	1

$N=2$ (a and b)

$M=2$ (0 and 1)

$\therefore \frac{N}{M}=1$

Define IRV X_i:

$$X_i \begin{cases} 1 & \text{if } h(i)=0 \\ 0 & \text{otherwise} \end{cases}$$

$Pr[X_a=1]=1$ (both hash gives 0)

$Pr[X_b=1]=\frac{1}{2}$ (only h_1

X: total hashing to slot 0

$$\boxed{\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X_a]+\mathbb{E}[X_b] \\ &= 1 + \frac{1}{2} \\ &= 1.5 \end{aligned}}$$

BUT: $1.5 > 1$

$\mathbb{E}[X] > \frac{N}{M}$

Count it if it hashes to slot 0

- Now you have a hash function from **universal family instead**
- Is the expected number of elements which hashes to a particular **slot *j*** still the same as before? i.e. **Still ≤ N/M**?

The following slides will show another example based on the hash family in lecture. If there is no time, it can be skipped

Lecture Example of Universal Hashing

Suppose U is indexed by u-bit strings, and $M = 2^m$.
For any binary matrix A with m rows and u columns:

$$h_A(x) = Ax \pmod 2$$

Claim: $\{h_A : A \in \{0,1\}^{m \times u}\}$ is universal.

Lecture Example of Universal Hashing

Suppose $\boxed{U}$ is indexed by $\boxed{u}$-bit strings, and $\boxed{M} = 2^{\boxed{m}}$.
For any binary matrix A with m rows and u columns:

$$h_A(x) = Ax \pmod 2$$

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What is U? What is u?
What is M? What is m?

Lecture Example

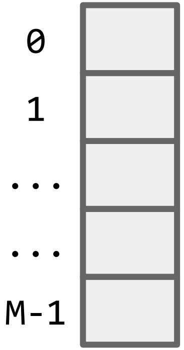
Suppose U is indexed by u-bit strings, and $M = 2^m$.
For any binary matrix A with m rows and u columns:

$$h_A(x) = Ax \pmod 2$$

<u>Claim</u>: $\{h_A : A \in \{0,1\}^{m \times u}\}$ is universal.

U = number of elements in the universe

M = the buckets it is mapping to



Hash table

Lecture Example

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<u>Claim</u>: $\{h_A : A \in \{0,1\}^{m \times u}\}$ is universal.

U = number of elements in the universe

e.g. {0, 1, 2, … 7}

Universe *U*:
{0, 1, 2, 3, 4, 5, 6, 7}

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Universe *U*:
{0, 1, 2, 3, 4, 5, 6, 7}

Index	*element in U*
000	*0*
001	*1*
010	*2*
011	*3*
100	*4*
101	*5*
110	*6*
111	*7*

Lecture Example

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u = θ(log U)

Universe *U*:
{0, 1, 2, 3, 4, 5, 6, 7}

Index	*element in U*
000	*0*
001	*1*
010	*2*
011	*3*
100	*4*
101	*5*
110	*6*
111	*7*

Taken from CS3230 Prerequisites Revision slides

Decimals to Binary

$O(log(x))$ bits to represent!

An integer $x >= 1$, needs $n = \lfloor log_2(x) \rfloor + 1$ bits to represent it

x	Binary Repr.	n
1	1	1
2	10	2
3	11	2
4	100	3
5	101	3
10	1010	4
23	10111	5
63	111111	6
64	1000000	7

Lecture Example

Suppose U is indexed by u-bit strings, and $M = 2^m$.
For any binary matrix A with m rows and u columns:

$$h_A(x) = Ax \pmod 2$$

Claim: $\{h_A : A \in \{0,1\}^{m \times u}\}$ is universal.

M = the number of buckets in the hash table

m = the len of binary representation of M

m=2

idx	M
00	0
01	1
10	2
11	3

Q4: Another Example

U = size of universe, u = len of binary representation of idx to U
M = number of buckets, m = len of binary representation of idx to M

In lecture, there is an example where we took an element in the universe x, and then we use the hash function $h(x) = Ax \pmod 2$, where A is a m *by* u binary matrix. Over the randomly chosen A, this is a Universal Family

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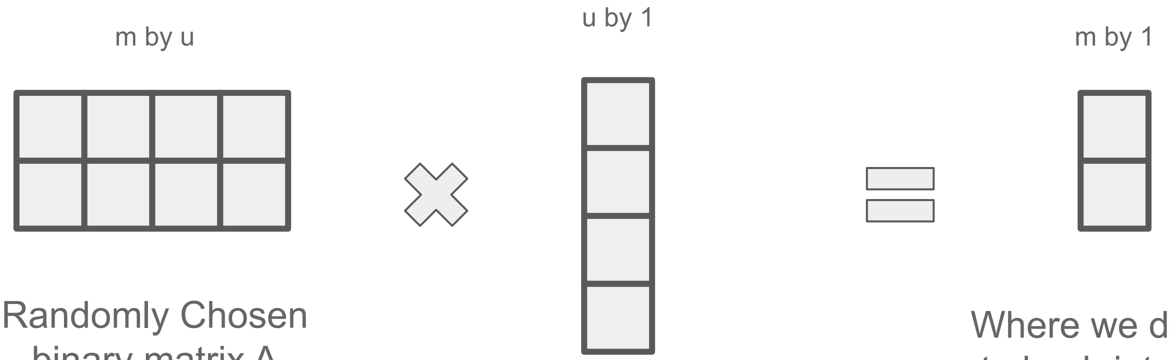


Randomly Chosen binary matrix A

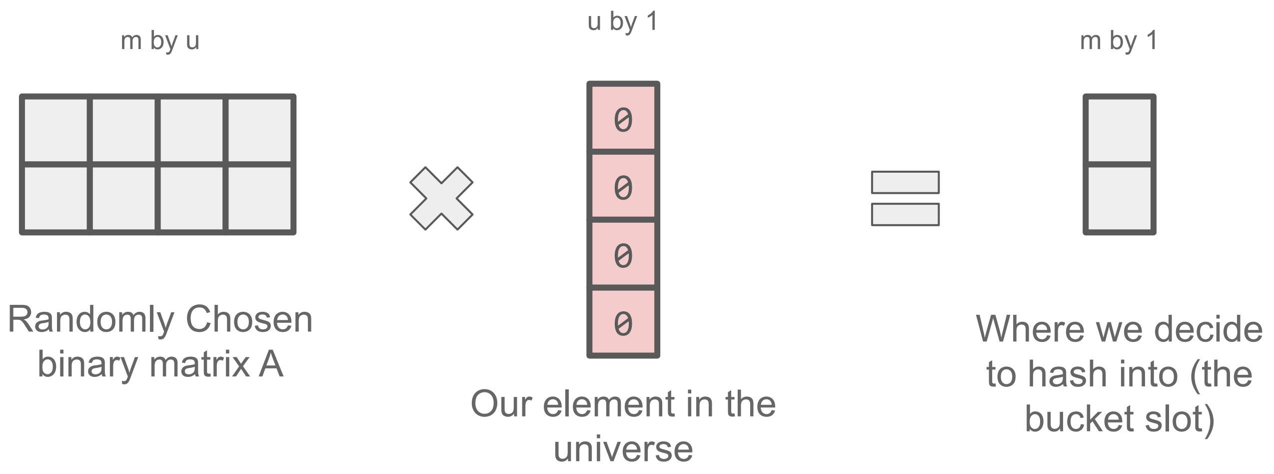
Our element in the universe

Where we decide to hash into (the bucket slot)

Q4: Another Example

U = size of universe, *u* = len of binary representation of idx to U
M = number of buckets, *m* = len of binary representation of idx to M

Let's say x = 0



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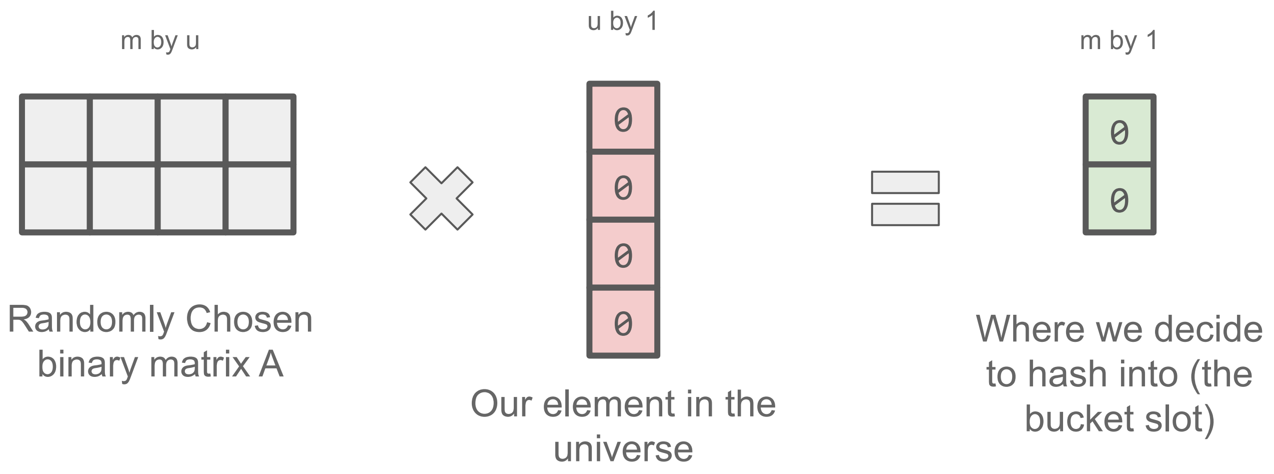
Let's say x = 0. Then the resultant h(x) must be 0 as well, regardless of what A is!



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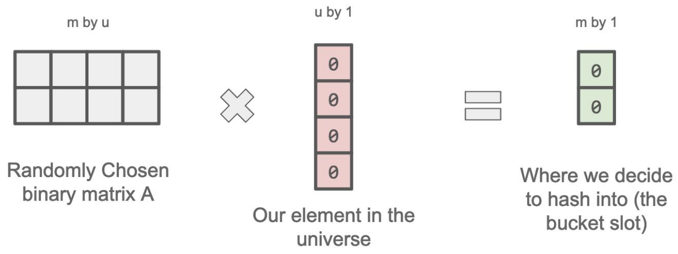
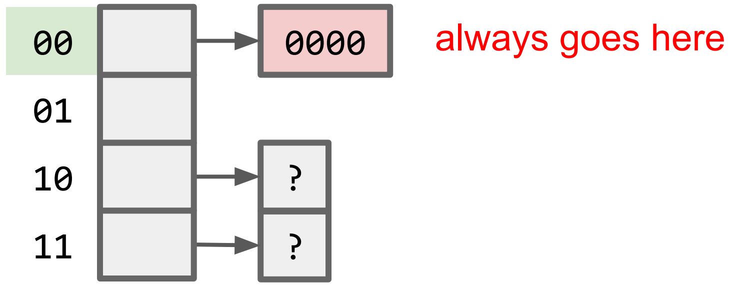
This means *E[mapping to slot 0]* $\geq$ *1* as long as there is an $x = 0$.





Our hash table

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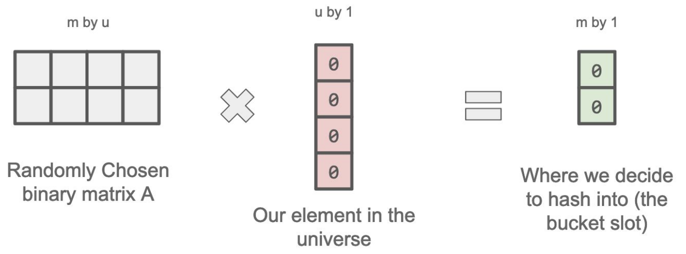
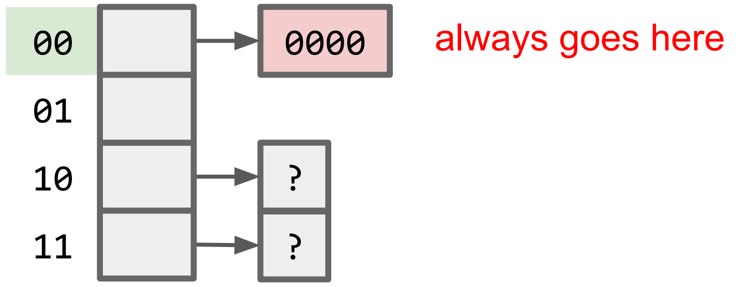
Our goal: Is the expectation ≤ N/M?

Ans to q3





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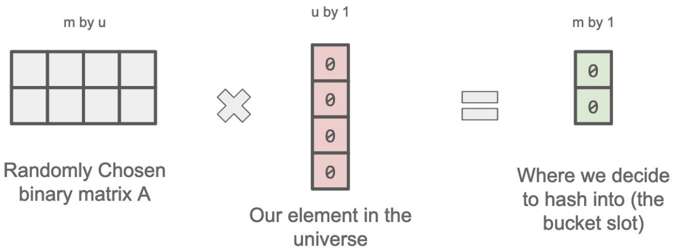
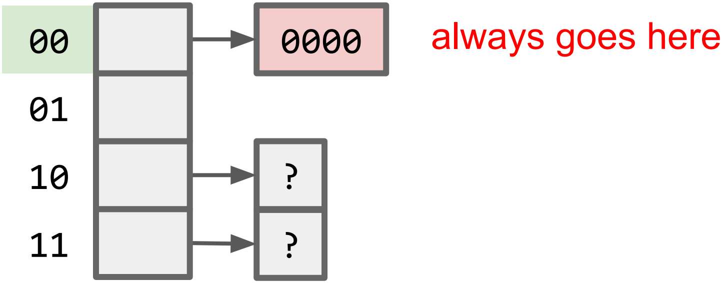
Ans to q3





Our hash table

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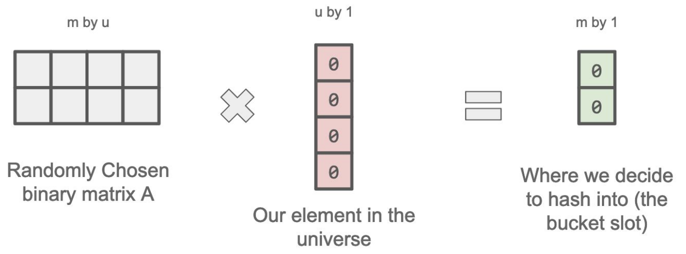
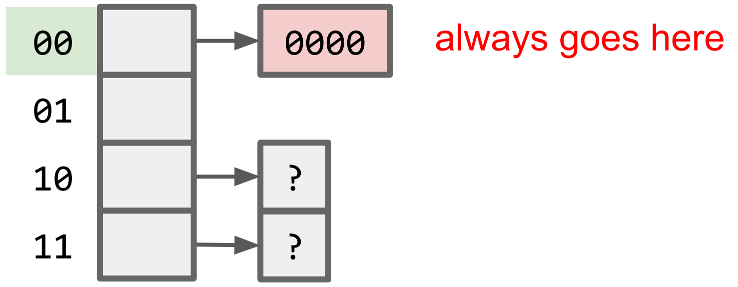
This means *E[mapping to slot 0] ≥ 1* as long as there is an *x = 0*.
Our goal: Is the expectation ≤ N/M? But if M > N, then N/M < 1. Cannot satisfy!

Ans to q3





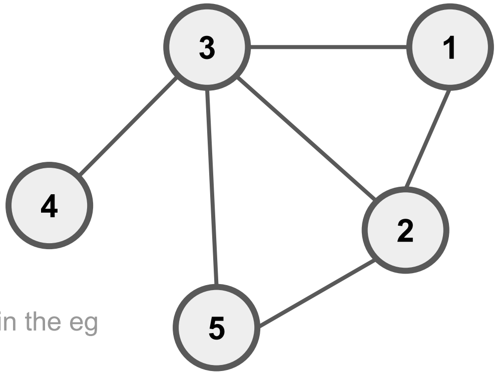
Our hash table

Question 5: Edges across the cut

Question 5

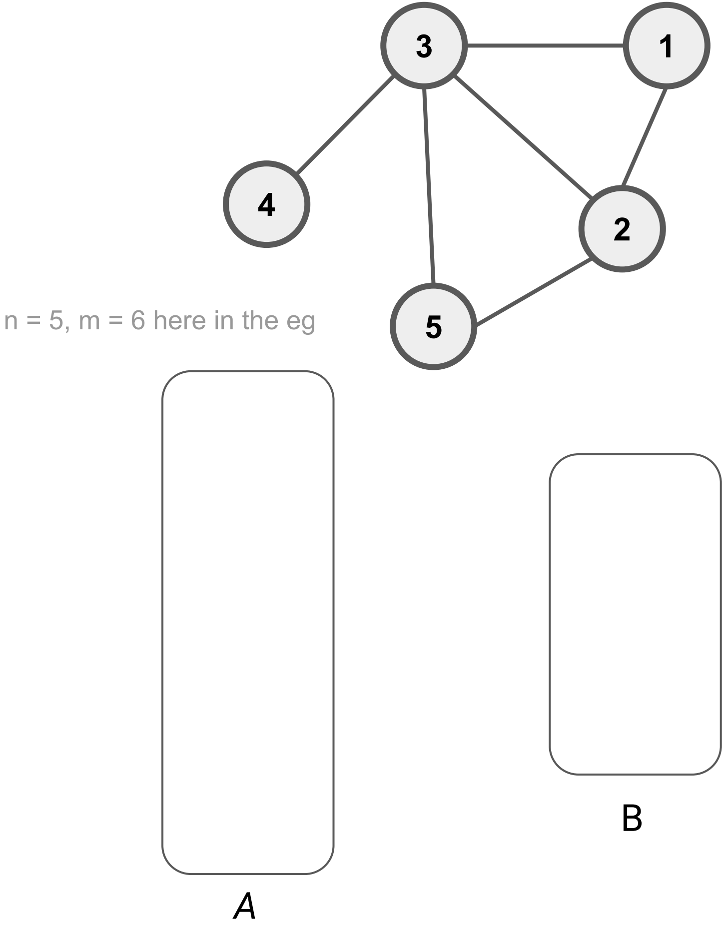
Let *G* be an undirected graph with *n* nodes and *m* edges.

n = 5, m = 6 here in the eg



Question 5

Let G be an undirected graph with n nodes and m edges. Partition the graph into two parts A and B randomly as follows:



Question 5

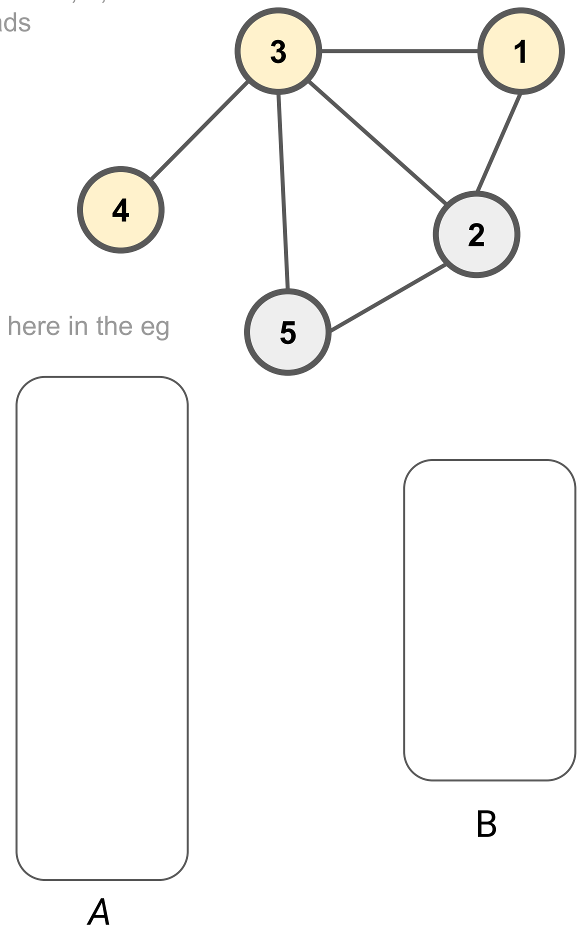
Let G be an undirected graph with n nodes and m edges. Partition the graph into two parts A and B randomly as follows:

For each node v, toss an independent fair coin:

- Heads: Put v in A

Let's say, nodes 1, 3, and 4 gave us heads

n = 5, m = 6 here in the eg

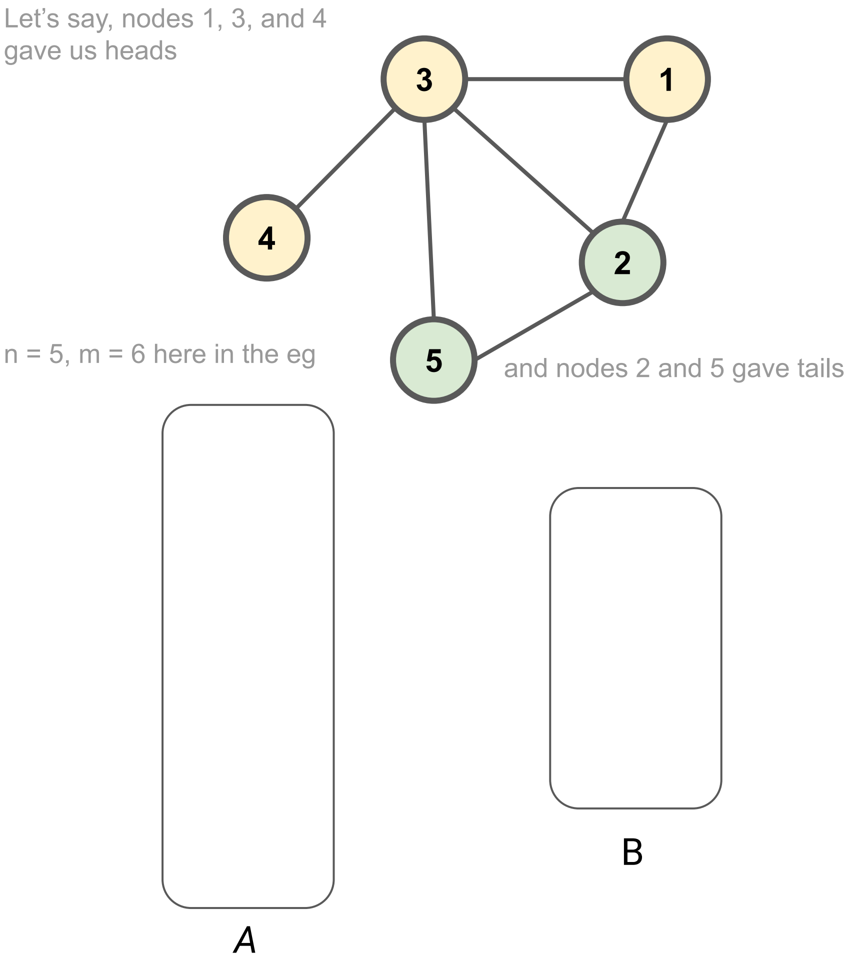


Question 5

Let G be an undirected graph with n nodes and m edges. Partition the graph into two parts A and B randomly as follows:

For each node v, toss an independent fair coin:

- Heads: Put v in A
- Tails: Put v in B



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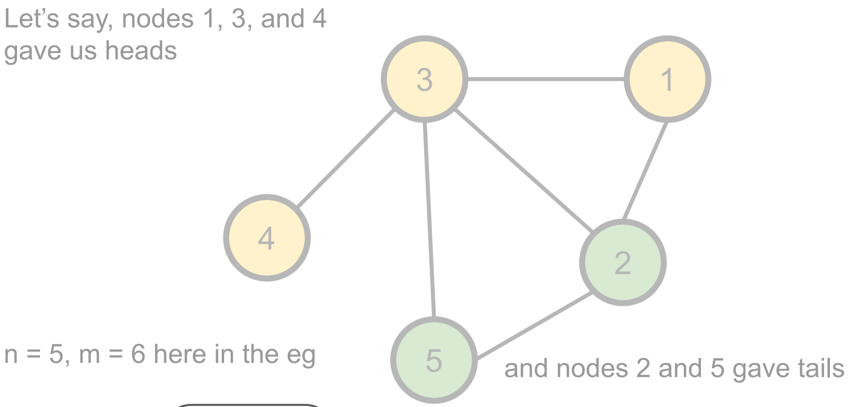
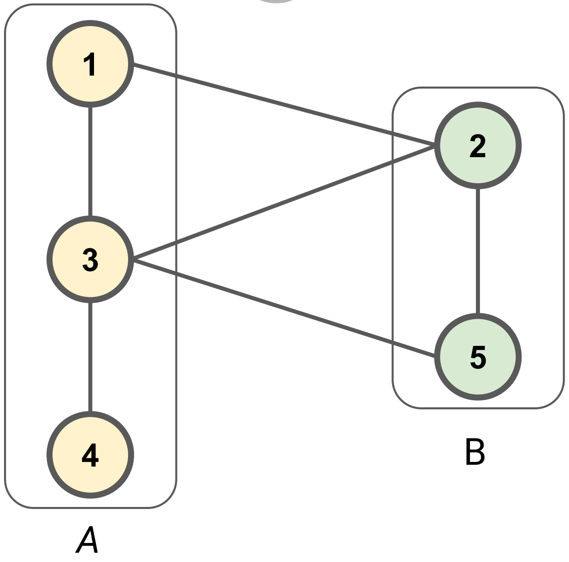




Question 5

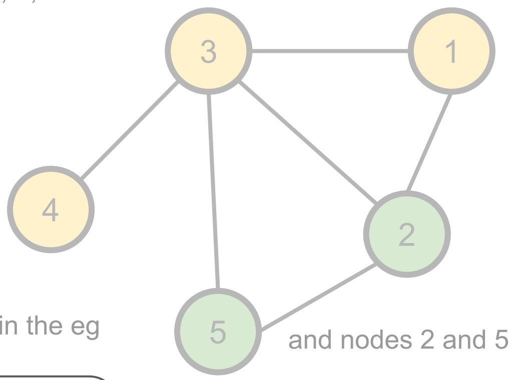
Let G be an undirected graph with n nodes and m edges. Partition the graph into two parts A and B randomly as follows:

For each node v, toss an independent fair coin:

- Heads: Put v in A
- Tails: Put v in B

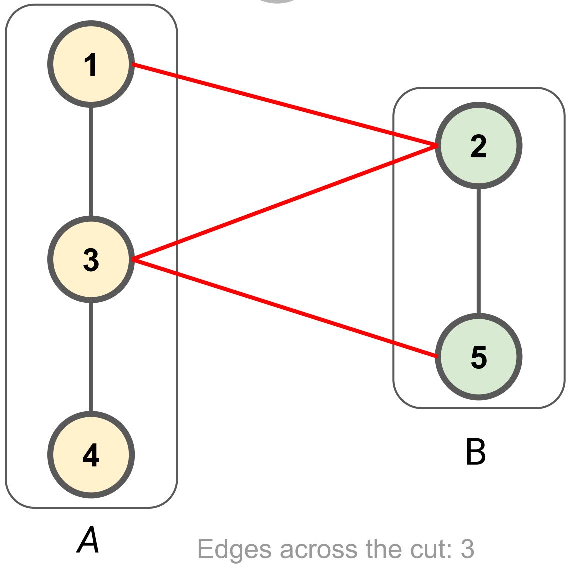
What is the expected number of edges which cross the cut? (One endpoint in A & other in B)

Let's say, nodes 1, 3, and 4 gave us heads



n = 5, m = 6 here in the eg

and nodes 2 and 5 gave tails



Edges across the cut: 3

Question 5 (Answer)

1. Identify a Random Variable to "count" what you want (e.g. X. Goal: $E[X]$)
2. Express this RV as a **sum** of random variables (e.g. $X = X_1 + X_2 + \ldots + X_n$)
   a. Calculate the relevant probability for X_1, X_2, ...
   b. Calculate the individual expectation of the "sub"-random variables. ($E[X_1]$, $E[X_2]$, …)
3. Use linearity of expectations on $E[X]$. Then you add up the expectation of the "sub"-random variables (from step 2b)

Goal: Expected number of edges crossing the cut

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Question 5 (Answer)

Goal: Expected number of edges crossing the cut

Let X be the random variable representing the number of edges crossing the cut

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Goal: Expected number of edges crossing the cut

Let X be the random variable representing the number of edges crossing the cut

For purpose of analysis: label each edge from 1 to m.

Let X_i be the **indicator random variable** that edge i crosses the cut

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$X = X_1 + X_2 + \ldots X_m$

Next things to do:

- Calculate $Pr(X_i = 1)$. This is enough to get $E[X_i]$!
- Then we can calculate $E[X]$ easily by linearity of expectations

Question 5 (Answer)

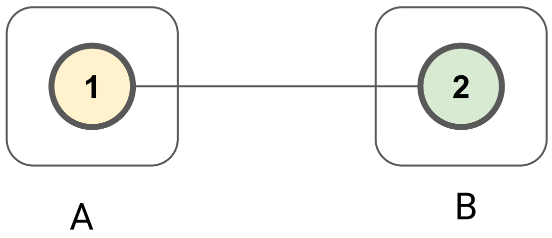
For purpose of analysis: label each edge from 1 to m.
Let X_i be the **indicator random variable** that edge i crosses the cut

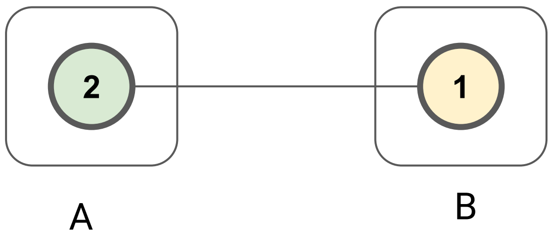
When is $X_i = 1$?

For purpose of analysis: label each edge from *1* to *m*. Let X_i be the **indicator random variable** that edge *i* crosses the cut

Question 5 (Answer)

When is $X_i = 1$? When the two endpoints are in different partitions. 2 cases:





Question 5 (Answer)

For purpose of analysis: label each edge from *1* to *m*.
Let X_i be the **indicator random variable** that edge *i* crosses the cut

When is $X_i = 1$? When the two endpoints are in different partitions. 2 cases:

$Pr(X_i = 1)$
$= (½)(½) + (½)(½)$
$= ¼ + ¼$
$= ½$

First case: 1 goes to A, and 2 goes to B
Second case: 2 goes to A, and 1 goes to B

Implies $E[X_i] = ½$

Let *X* be the random variable representing the number of edges crossing the cut

For purpose of analysis: label each edge from *1* to *m*.
Let X_i be the **indicator random variable** that edge *i* crosses the cut

Question 5 (Answer)

$$E[X] = E[\sum_{j=1}^{m} X_j]$$

Question 5

How we defined X

Question 5 (Answer)

Let *X* be the random variable representing the number of edges crossing the cut

For purpose of analysis: label each edge from *1* to *m*.
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$$E[X] = E[\sum_{j=1}^{m} X_j]$$
$$= \sum_{j=1}^{m} E[X_j]$$

Question 5

Linearity of expectations

Question 5 (Answer)

Let X be the random variable representing the number of edges crossing the cut

For purpose of analysis: label each edge from *1* to *m*.
Let X_i be the **indicator random variable** that edge *i* crosses the cut

$$E[X] = E[\sum_{j=1}^{m} X_j]$$

$$= \sum_{j=1}^{m} E[X_j]$$

$$= \sum_{j=1}^{m} \frac{1}{2}$$

Question 5

Our earlier calculation!

$Pr(X_i = 1)$
$= (½)(½) + (½)(½)$
$= ¼ + ¼$
$= ½$

Implies $E[X_i] = ½$

Question 5 (Answer)

Let *X* be the random variable representing the number of edges crossing the cut

For purpose of analysis: label each edge from *1* to *m*.
Let X_i be the **indicator random variable** that edge *i* crosses the cut

$$
\begin{aligned}
E[X] &= E[\sum_{j=1}^{m} X_j] \\
&= \sum_{j=1}^{m} E[X_j] \\
&= \sum_{j=1}^{m} \frac{1}{2} \\
&= \frac{m}{2}
\end{aligned}
$$

Question 5

Quick mafs