W04: Correctness and Divide & Conquer

CS3230 AY21/22 Sem 2

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- **Initialisation**: Invariant is true before the first iteration of the loop
- **Maintenance**: If invariant true before an iteration, it remains true before the next iteration
- **Termination**: When the algorithm terminates, the invariant gives us a useful property for showing correctness

Question 1: Dijkstra’s Correctness

Dijkstra's Algorithm

G = (V, E) is an undirected graph. With *s* as the start node. **All edges in G are positive weights**.

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def Dijkstra(G, s):
For all node u except s, d(u) = inf
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while R != V:
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Also known as: Relaxation Step

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- Use *d(u)* as an *estimate* -- "Right now, I can get to *d(u)* with this much cost. Can it be better?"
- Repeat:
  - Consider the vertex with *minimum* estimate
  - Add it to our R (results, what we are sure of the distance)
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Question 1

Walkthrough

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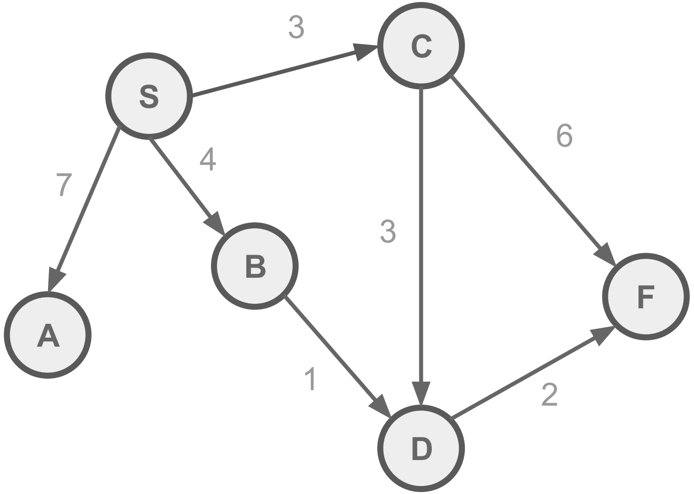

Question 1

Initialisation of d(u)

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def Dijkstra(G, s):
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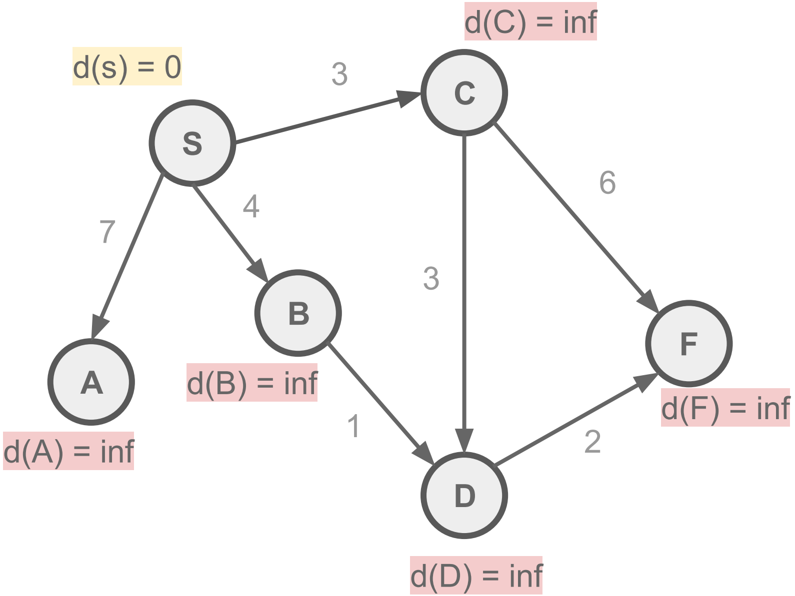

Question 1

From this slide onwards, R = set of red nodes

```
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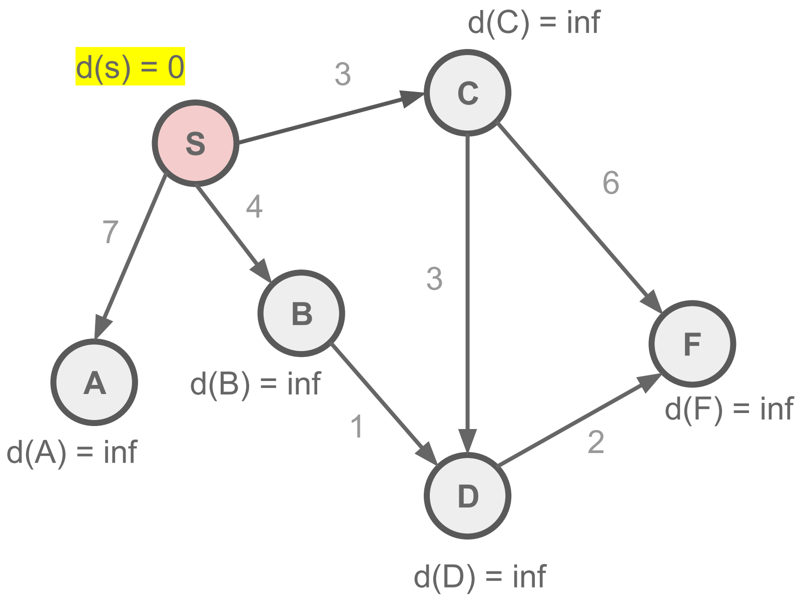

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Relax all the neighbours!

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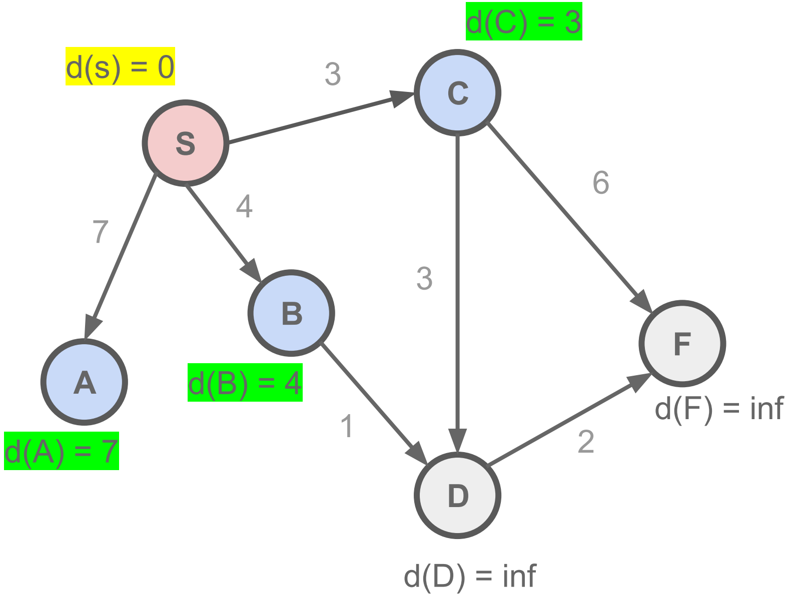
Existing estimate is good

Can do better by going through node u



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Repeat the process!

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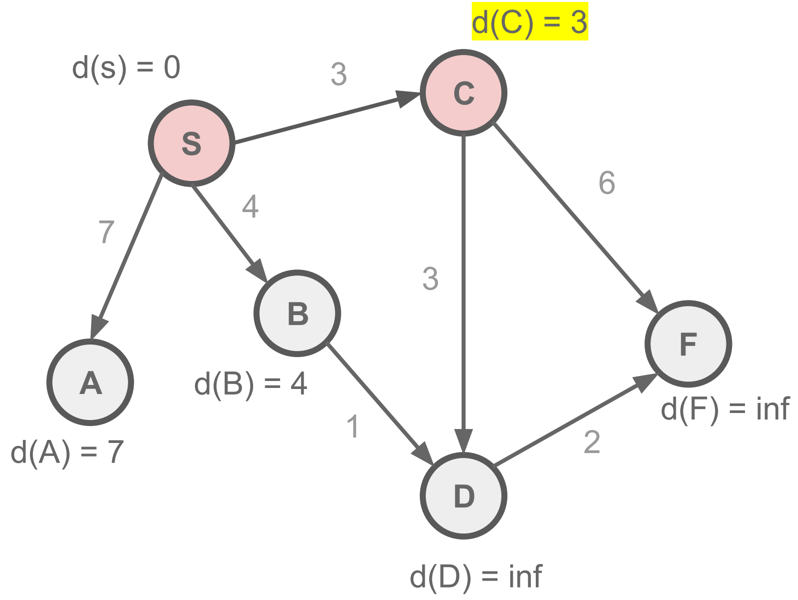

Question 1

Relax neighbours of C - now note that we *think* we can get to D and F with cost 6 and 9 respectively

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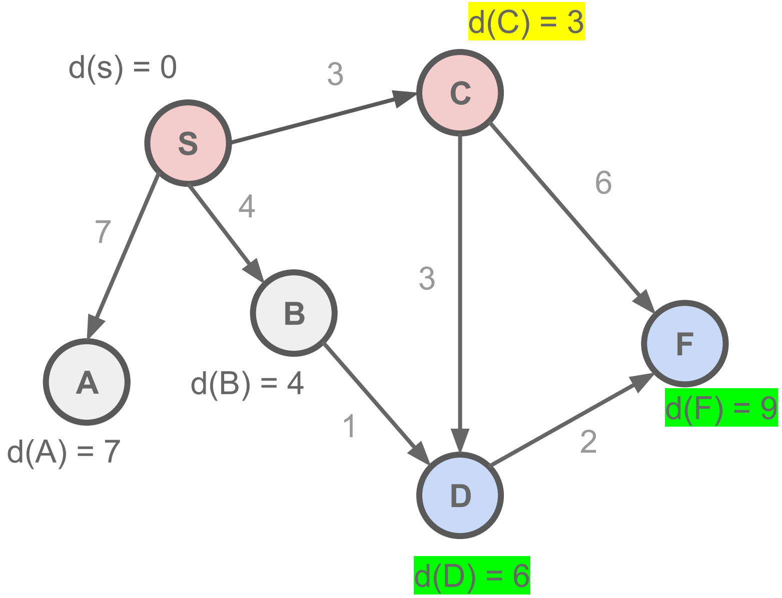

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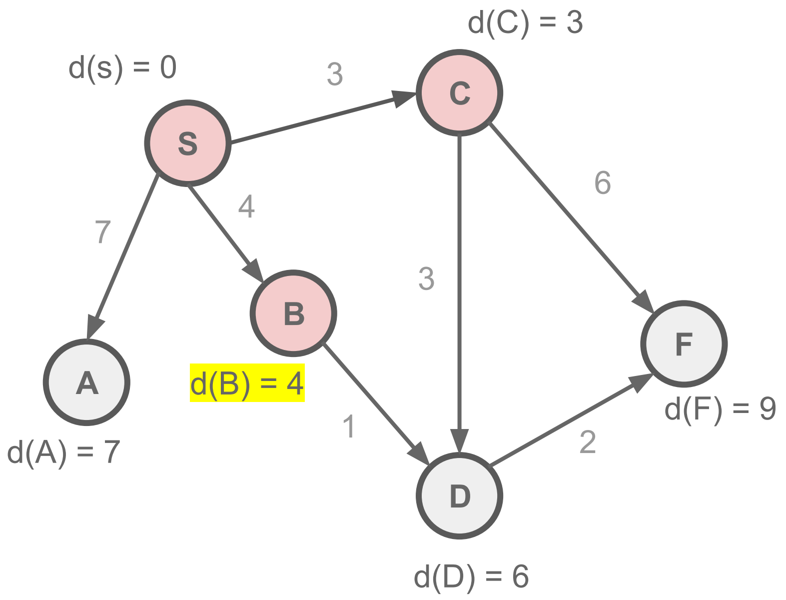

Question 1

IMPORTANT: Now we have a **better** estimate to reach node D. Instead of going through C (prev estimate)

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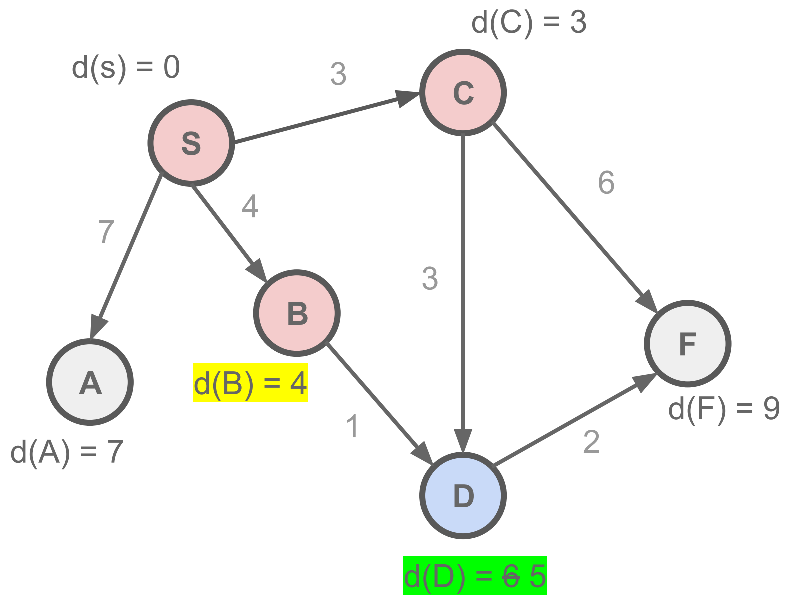

Question 1

Dijkstra(G, s)

1. *For all* $u \in V \setminus \{s\}$, $d(u)=\infty$;
2. $d(s)=0$; $R=\{\}$;
3. *While* $R \neq V$
4.   *pick* $u \notin R$ *with the smallest* $d(u)$
5. $R = R \cup \{u\}$
6.   *for all neighbor* v *of* u,
7. $d(v) = \min\{ d(v), d(u)+w(u,v))\}$

G=(V,E) is an undirected graph.
s is the start node
Assume all edges in G are of positive weights.

What is the invariant for the while loop?

Can you show that this algorithm correctly compute the shortest distance from s to all nodes?

Dijkstra’s Correctness Proof (Intuition)

What happens when the algorithm terminates?

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Invariant idea:

Relate the nodes already in R with "already have the correct shortest path cost"

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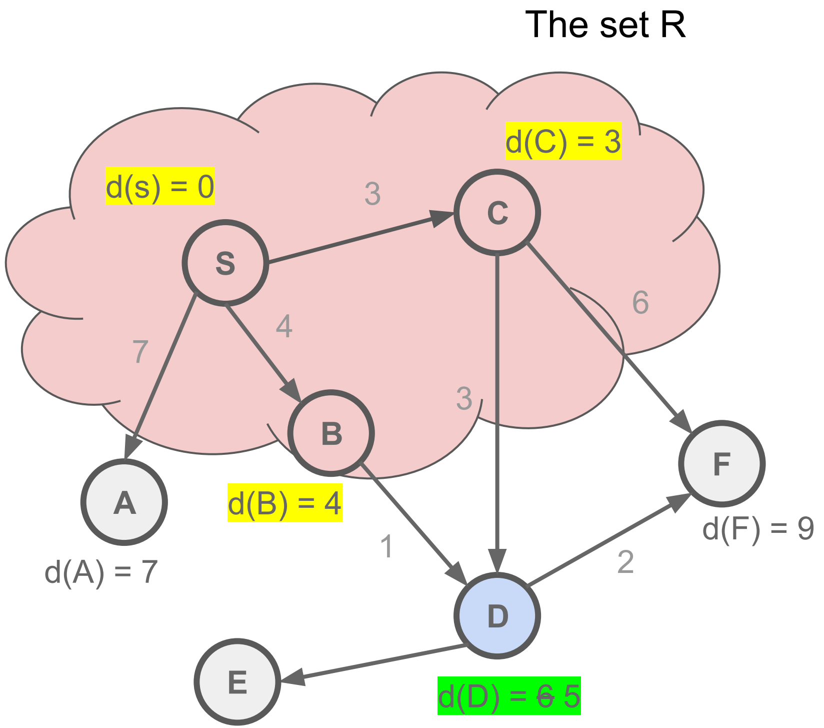
Notice that all the nodes in set R would already have their true shortest path cost set!



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Furthermore, recall that in the algorithm, we simply add the next node with the minimum estimate

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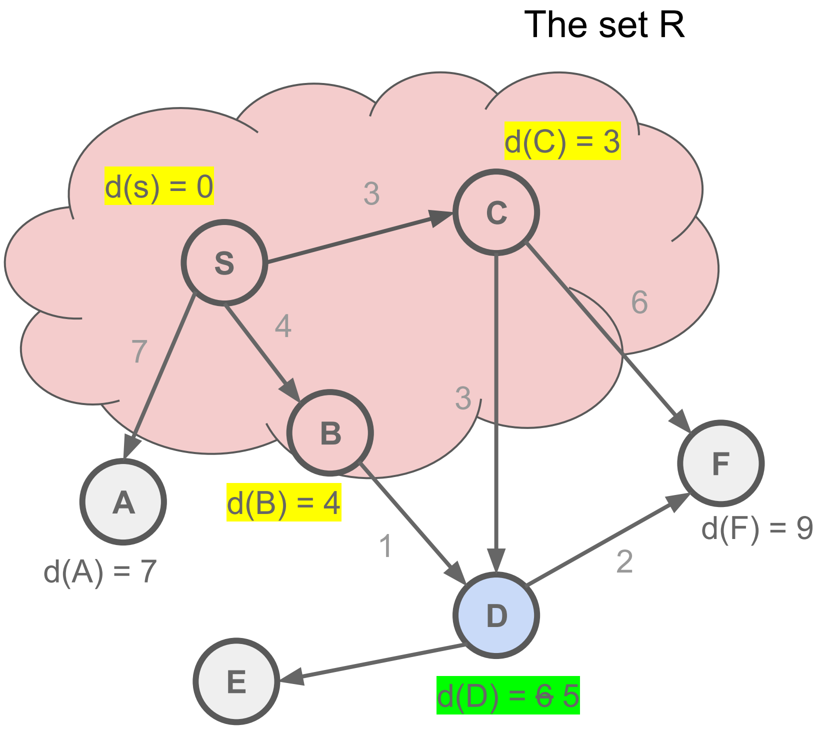
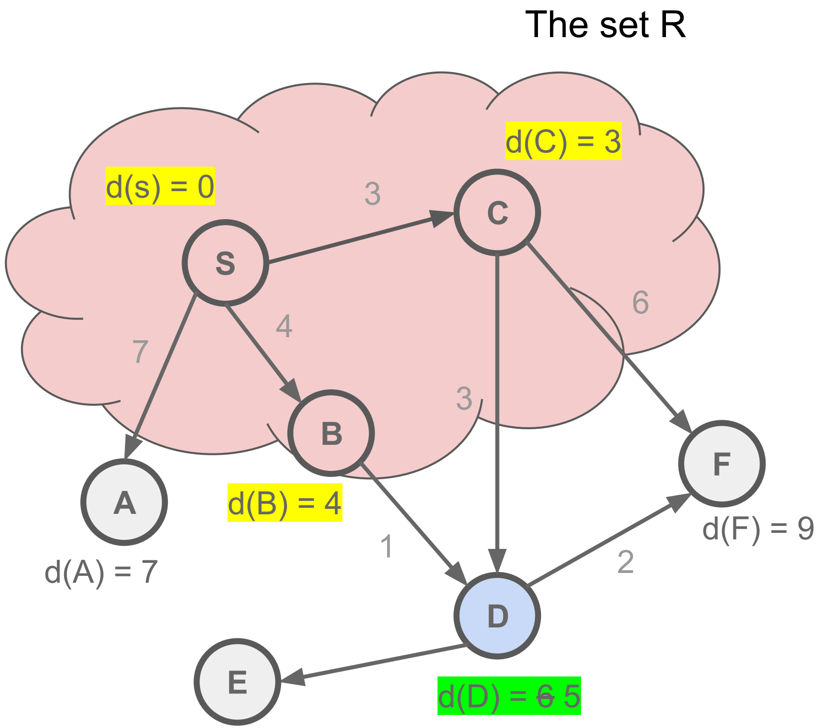


Dijkstra’s Correctness Proof (Formalisation)

Let ***δ(a → b)*** be the (true) shortest distance from a to b in G



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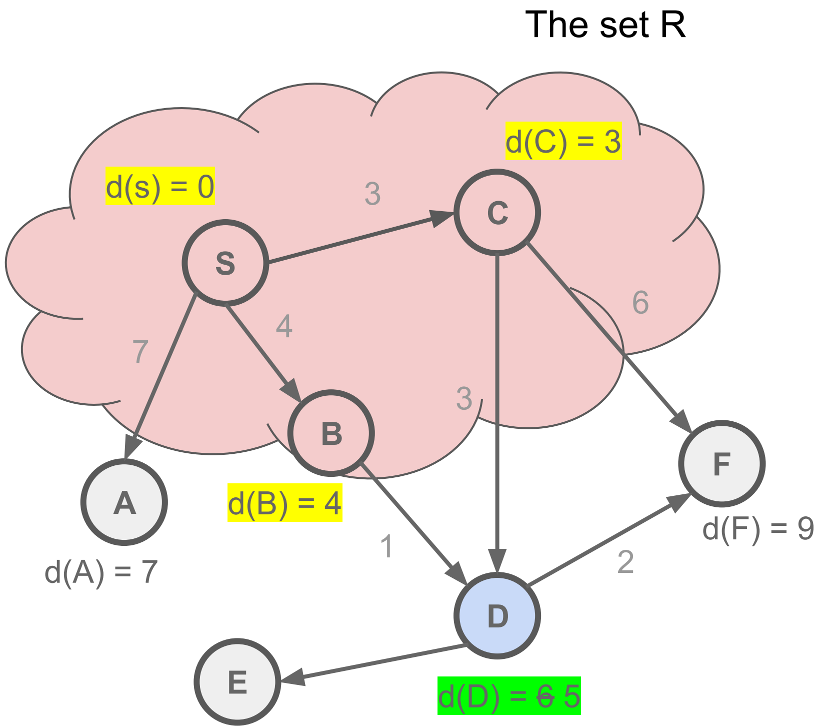
Let $\delta(\boldsymbol{a} \rightarrow \boldsymbol{b})$ be the (true) shortest distance from a to b in G

Invariant 1 (INV1):
For all node u in R, d(u) = $\delta(s \rightarrow u)$ [d(u) is the shortest distance from s to u already]



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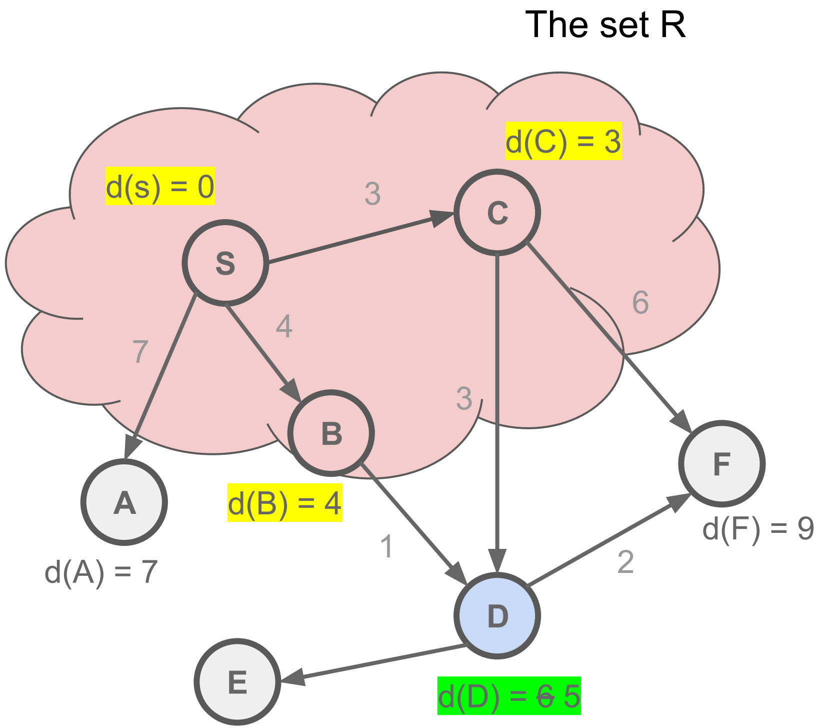
Let ***Δ(X, u)*** be *min{ δ(s → v) + w(v, u) for v in X }*



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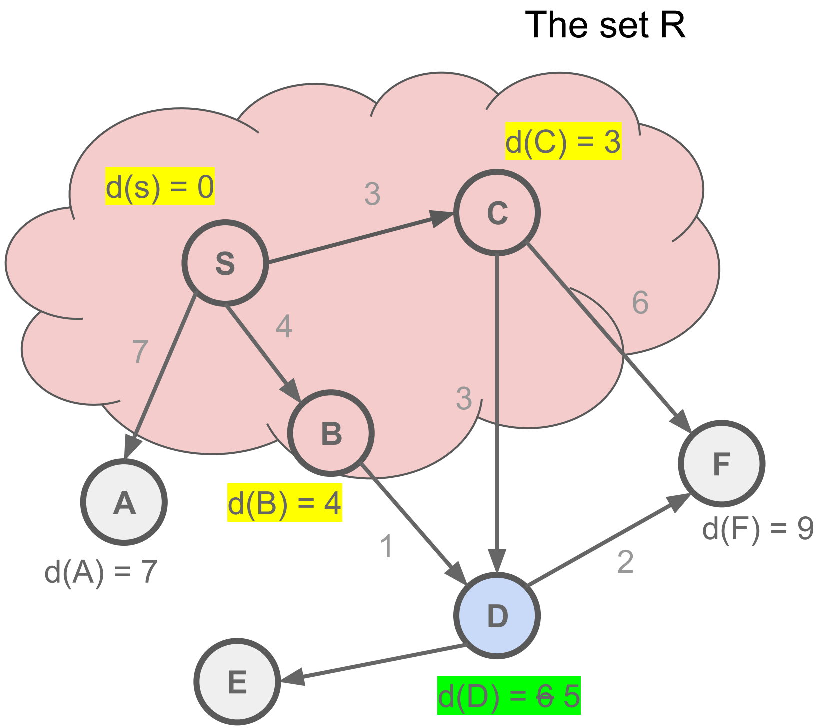
Read it as: “from all the nodes in X to me, take the ***best estimate***”. e.g. node D on the right can either have a distance 6 (through C) or 5 (through B)



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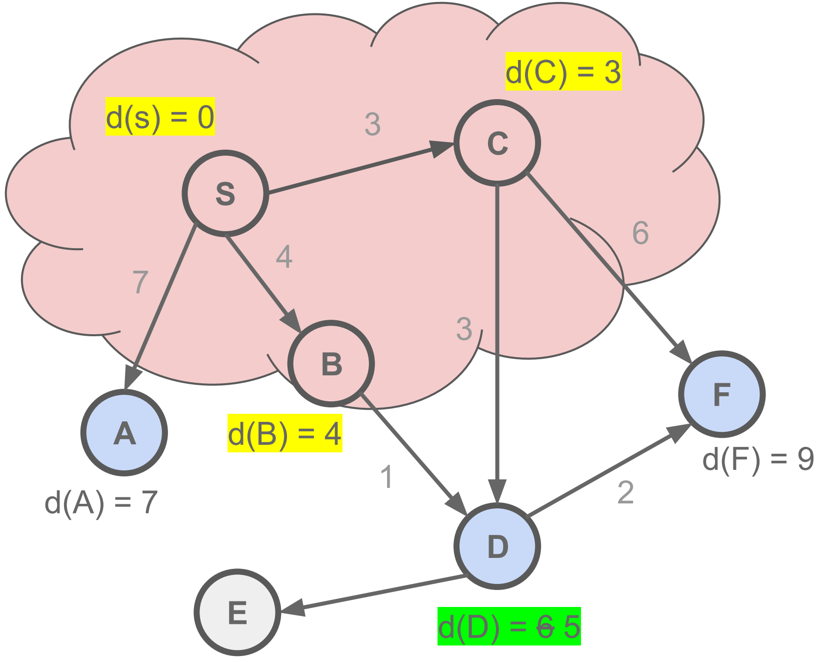
Invariant 2 (INV2):
For every neighbour v of all nodes in R,
d(v) = Δ(R, v)

The set R



e.g. E is not a neighbour of node in R

Dijkstra **Initialisation**

Before the first iteration of the while loop, we set $R = \{\}$.

INV1 and INV2 are trivially true (there is no node in R)

Let $\delta(a \rightarrow b)$ be the (true) shortest distance from a to b in G

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Invariant 2 (INV2):
For every neighbour v of all nodes in R, $d(v) = \Delta(R, v)$

```
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2. d(s) = 0; R = {};
3. while R != V:
```

Dijkstra **Maintenance** (INV1)

In steps 4 and 5, we add the node u with smallest d(u) into R

To maintain INV1, we need to show that this node u that we pick has d(u) = $\delta(s \rightarrow u)$ [Recall what it means to be inside R]

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Proof by Contradiction! Assume that the node u **does not have** d(u) = $\delta(s \to u)$

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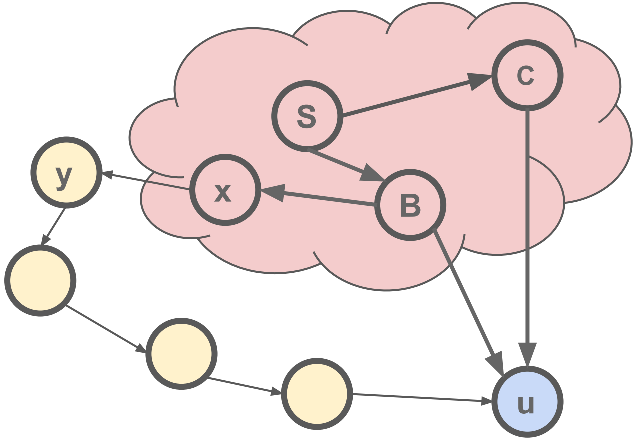
The set R



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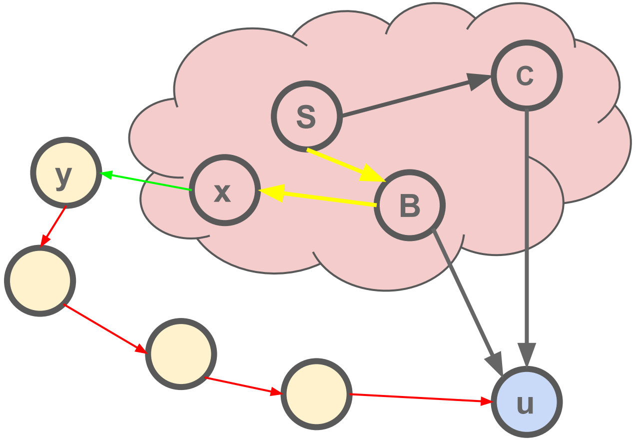
Weight of Q: $\delta(s \rightarrow x)$ + $w(x, y)$ + $\delta(y \rightarrow u)$ < $d(u)$ [by assumption]

The set R



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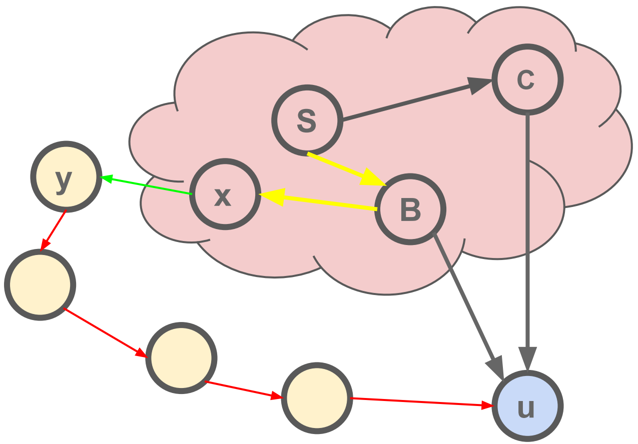
Proof by Contradiction! Assume that the node u **does not have** d(u) = δ (s → u). There is some other path from S to reach u that is shorter (via edge x → y). Call this path Q.

Weight of Q: $\delta(s \to x) + w(x, y) + \delta(y \to u) < d(u)$ [by assumption]

$d(y) = \Delta(R, y) = \min\{ \delta(s \to r) + w(r, y) \text{ for } r \text{ in } R \}$ [by invariant 2]

The set R



Let δ***(a → b)*** be the (true) shortest distance from a to b in G

Let ***Δ(X, u)*** be $\min\{ \delta(s \to v) + w(v, u) \text{ for } v \text{ in } X \}$

Invariant 2 (INV2):
For every neighbour v of all nodes in R,
$d(v) = \Delta(R, v)$

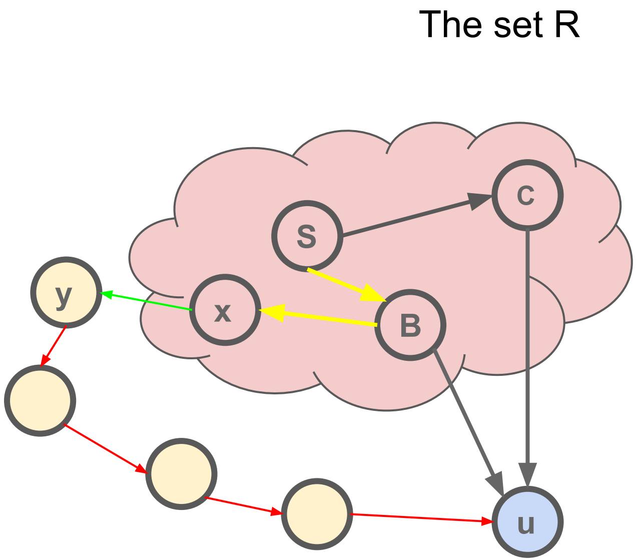
Proof by Contradiction! Assume that the node u **does not have** d(u) = $\delta(s \rightarrow u)$. There is some other path from S to reach u that is shorter (via edge $x \rightarrow y$). Call this path Q.

Weight of Q: $\delta(s \rightarrow x) + w(x, y) + \delta(y \rightarrow u) < d(u)$ [by assumption]

$d(y) = \Delta(R, y) = \min\{ \delta(s \rightarrow r) + w(r, y) \text{ for } r \text{ in } R \}$ [by invariant 2]

$d(y) \leq \delta(s \rightarrow x) + w(x, y)$ [d(y) is the min, so anything else must be ≤]

e.g. d(y) = min {100, 3, 50, 42 }
so d(y) ≤ 100 or d(y) ≤ 3 or d(y) ≤ 50 or d(y) ≤ 42



Let $\delta(a \rightarrow b)$ be the (true) shortest distance from a to b in G

Let $\Delta(X, u)$ be $\min\{ \delta(s \rightarrow v) + w(v, u) \text{ for } v \text{ in } X \}$

Invariant 2 (INV2):
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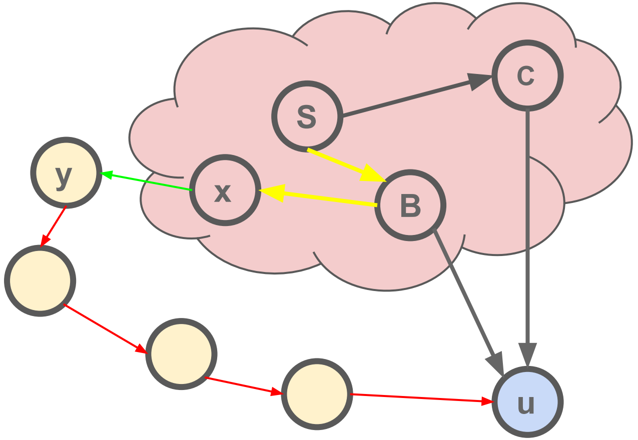
$d(y) \leq \delta(s \rightarrow x) + w(x, y)$ [d(y) is the min, so anything else must be ≤]

$\leq \delta(s \rightarrow x) + w(x, y) + \delta(y \rightarrow u)$ [edges are positive. Just add them in]

It's like saying 100 ≤ 100 + 3

The set R



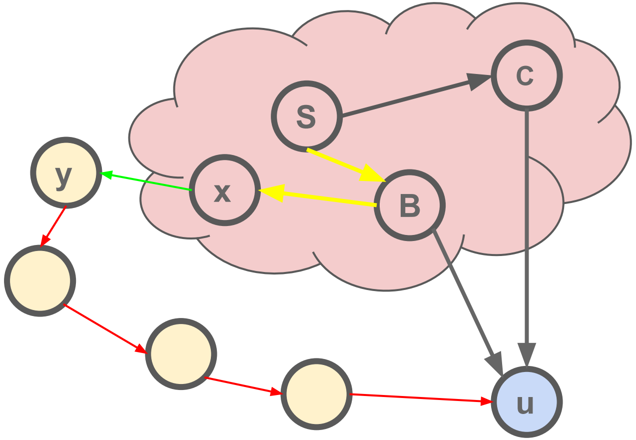
Let $\delta(a \rightarrow b)$ be the (true) shortest distance from a to b in G

Let $\Delta(X, u)$ be $\min\{ \delta(s \rightarrow v) + w(v, u) \text{ for } v \text{ in } X \}$

Invariant 2 (INV2):
For every neighbour v of all nodes in R,
$d(v) = \Delta(R, v)$

Proof by Contradiction! Assume that the node u **does not have** $d(u) = \delta(s \rightarrow u)$. There is some other path from S to reach u that is shorter (via edge $x \rightarrow y$). Call this path Q.

The set R



Weight of Q: $\delta(s \rightarrow x) + w(x, y) + \delta(y \rightarrow u) < d(u)$ [by assumption]

$d(y) = \Delta(R, y) = \min\{ \delta(s \rightarrow r) + w(r, y) \text{ for } r \text{ in } R \}$ [by invariant 2]

$d(y) \leq \delta(s \rightarrow x) + w(x, y)$ [d(y) is the min, so anything else must be ≤]

$\leq \delta(s \rightarrow x) + w(x, y) + \delta(y \rightarrow u)$ [edges are positive. Just add them in]

$< d(u)$

Let $\delta(a \rightarrow b)$ be the (true) shortest distance from a to b in G

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For every neighbour v of all nodes in R,
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$\leq \delta(s \rightarrow x) + w(x, y) + \delta(y \rightarrow u)$ [edges are positive. Just add them in]

$< d(u)$

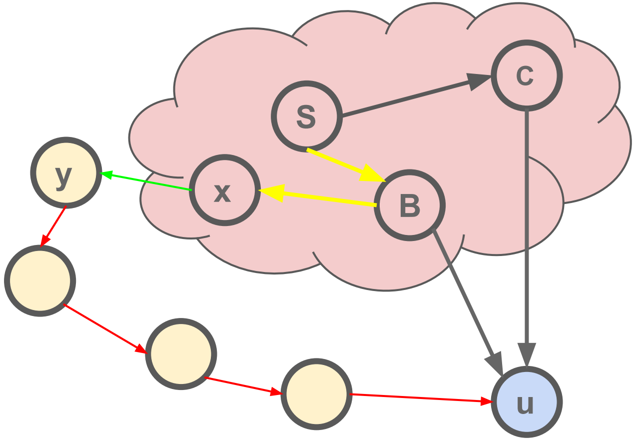
Our algorithm: chooses smallest d(u) → d(u) ≤ d(y)
Our math: d(y) < d(u)

Contradiction!
Thus, d(u) is already the shortest distance ($\delta(s \rightarrow u)$)

```
4.   pick u not in R, with the smallest d(u)

5.   R = R union {u}
```

The set R



Let $\delta(a \rightarrow b)$ be the (true) shortest distance from a to b in G

Let $\Delta(X, u)$ be $\min\{ \delta(s \rightarrow v) + w(v, u) \text{ for } v \text{ in } X \}$

Invariant 2 (INV2):
For every neighbour v of all nodes in R,
$d(v) = \Delta(R, v)$

Maintenance - Aside

You can also derive the contradiction by starting off with the fact that d(y) ≥ d(u) [by our algorithm].

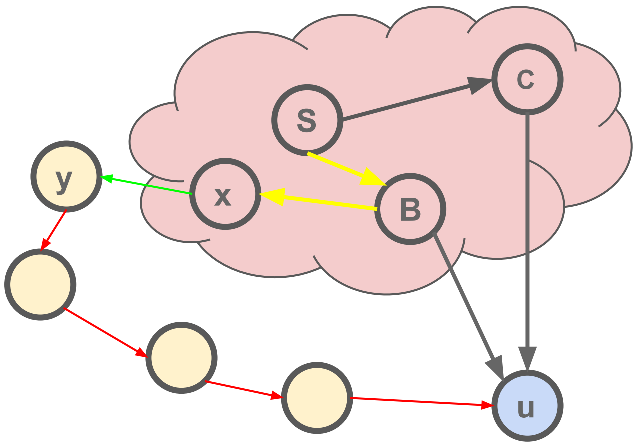
Then show that d(y) + some edges > d(u) [bcs the edge are positive]

This implies that the path via y is longer (contradicting that we assumed there was a shorter path via y)

```
4.   pick u not in R, with the smallest d(u)

5.   R = R union {u}
```

The set R



Let *δ(a → b)* be the (true) shortest distance from a to b in G

Let ***Δ(X, u)*** be *min{ δ(s → v) + w(v, u) for v in X }*

Invariant 2 (INV2):
For every neighbour v of all nodes in R,
d(v) = Δ(R, v)

Dijkstra **Maintenance** (INV2)

To maintain INV2, observe that we **just added** node u. Now we need to make sure all the neighbours of u are correctly updated

Let $\delta(a \rightarrow b)$ be the (true) shortest distance from a to b in G

Invariant 1 (INV1):
For all node u in R, d(u) = $\delta(s \rightarrow u)$ [d(u) is the shortest distance from s to u already]

Let $\Delta(X, u)$ be $\min\{ \delta(s \rightarrow v) + w(v, u)$ *for v in X* $\}$

Invariant 2 (INV2):
For every neighbour v of all nodes in R, $d(v) = \Delta(R, v)$

```
for v in neighbours(u):
  d(v) = min(d(v), d(u) + w(u, v))
```

Dijkstra **Maintenance** (INV2)

To maintain INV2, observe that we **just added** node u. Now we need to make sure all the neighbours of u are correctly updated

But hey! That's exactly our algorithm in steps 6-7

Let $\boldsymbol{\delta(a \rightarrow b)}$ be the (true) shortest distance from a to b in G

Invariant 1 (INV1):
For all node u in R, d(u) = $\delta(s \rightarrow u)$ [d(u) is the shortest distance from s to u already]

Let $\boldsymbol{\Delta(X, u)}$ be $\min\{\, \delta(s \rightarrow v) + w(v, u)$ for v in $X \,\}$

Invariant 2 (INV2):
For every neighbour v of all nodes in R, $d(v) = \Delta(R, v)$

```
6.   for v in neighbours(u):
7.     d(v) = min(d(v), d(u) + w(u, v))
```

Dijkstra **Termination** (INV1)

After the last iteration of the loop, **R** includes all vertices in **G**.

Let $\delta(a \to b)$ be the (true) shortest distance from a to b in G

Invariant 1 (INV1):
For all node u in R, d(u) = $\delta(s \to u)$ [d(u) is the shortest distance from s to u already]

Let $\Delta(X, u)$ be $min\{ \delta(s \to v) + w(v, u)$ for v in $X \}$

Invariant 2 (INV2):
For every neighbour v of all nodes in R, $d(v) = \Delta(R, v)$

```
3. while R != V:
```

Dijkstra **Termination** (INV1)

After the last iteration of the loop, **R** includes all vertices in **G**.

So **d(u)** is the shortest distance from **s** for every node **u** in G (by INV1)

Let *δ(**a** $\rightarrow$ **b**)* be the (true) shortest distance from a to b in G

Invariant 1 (INV1):
For all node u in R, d(u) = *δ(s $\rightarrow$ u)* [d(u) is the shortest distance from s to u already]

Let ***Δ(X, u)*** be *min{ δ(s $\rightarrow$ v) + w(v, u) for v in X }*

Invariant 2 (INV2):
For every neighbour v of all nodes in R, *d(v) = Δ(R, v)*

```
3. while R != V:
```

Question 2 - 2D Peak-finding

Question 2 - 2D Peak-finding

Given a 2D-array *A* of size *m* rows and *n* columns. *A[i][j]* is called a “peak” if it is **greater than or equal to** its adjacent neighbours
Note: just a peak is enough, it does not have to be a global peak!

n = 4

m = 3

0	23	4	7
1	18	18	1
6	15	0	8

0	23	4	7
1	18	18	1
6	15	0	8

0	23	4	7
1	18	18	1
6	15	0	8

Question 2 - Proposed Pseudocode

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 is a peak return p1 else return p2
```

Question 2

Let’s try on this A

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 is a peak return p1 else return p2
```

0	23	4	7	23	4	7
1	18	18	1	18	18	1
6	15	0	8	15	0	8

Question 2

Take the middle column

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 is a peak return p1 else return p2
```

0	23	4	7	23	4	7
1	18	18	1	18	18	1
6	15	0	8	15	0	8

Question 2

8 is the max on this column

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 is a peak return p1 else return p2
```

0	23	4	7	23	4	7
1	18	18	1	18	18	1
6	15	0	8	15	0	8

Question 2

15 is greater than 8. 8 is not a peak!

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 is a peak return p1 else return p2
```

0	23	4	7	23	4	7
1	18	18	1	18	18	1
6	15	0	8	15	0	8

Question 2

Recurse on the right side

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 is a peak return p1 else return p2
```

0	23	4	7	23	4	7
1	18	18	1	18	18	1
6	15	0	8	15	0	8

Question 2

Recurse on the left side

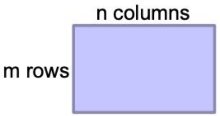
```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 is a peak return p1 else return p2
```

0	23	4	7	23	4	7
1	18	18	1	18	18	1
6	15	0	8	15	0	8

Question 2



Suppose we are given a 2D-array A of size m rows by n columns. An element in the array $A[i][j]$ is called a "peak" if it is **greater than or equal to** its adjacent neighbours (if they exist -- an element is always considered greater than or equal to non-existent elements). For example, the 8 in the middle is a peak in the following array.

$$\begin{array}{cccc} * & * & 5 & * \\ * & 8 & 8 & 3 \\ * & * & 2 & * \end{array}$$

Consider the following algorithm to return any "peak":

```
Find2DPeak(A):
    If A only has a column, return the maximal element of the column
    Otherwise:
        Select the middle column of the A
        Find the maximal element of the column
        If the maximal element is a peak, return that element
        Else
            p1 = Find2DPeak(right half of A excluding middle col)
            p2 = Find2DPeak(left half of A excluding middle col)
            If p1 or p2 is a peak, return either one, otherwise return None
```

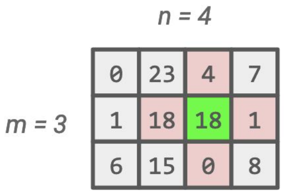
What is the runtime of the algorithm?

- $\Theta(mn)$
- $\Theta(m \lg n)$
- $\Theta(\lg m \lg n)$
- $\Theta(m^2 \lg n)$

Time Complexity - Intuition

- You are dividing the array into two, but you are doing recursion on **both sides**
- You are still checking every columns!

Time Complexity - Method one



Recall: m rows and n columns.

Consider the **recurrence with two parameters**:

Time Complexity - Method one

$n = 4$

$m = 3$

0	23	4	7
1	18	18	1
6	15	0	8

Recall: m rows and n columns.

Consider the **recurrence with two parameters**:

$T(m, 1) = cm$

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)

    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)

        if candidate_peak is peak:
            return candidate_peak
        else:
            p1 = Find2DPeak(A, but only the right side)
            p2 = Find2DPeak(A, but only the left side)
            if p1 or p2 is a peak return p1 else None
```

Time Complexity - Method one

Recall: m rows and n columns.

Consider the **recurrence with two parameters**:

$T(m, 1) = cm$

$T(m, n) = 2T(m, n/2) + cm$

Possible to solve this with recursion tree!
(Exercise)

n = 4

m = 3

0	23	4	7
1	18	18	1
6	15	0	8

```
def Find2DPeak(A):
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            p2 = Find2DPeak(A, but only the left side)
            if p1 or p2 is a peak return p1 else None
```

Time Complexity - Method two

Observation: The time to process every column is the same: *θ(m)*

n = 4

m = 3

0	23	4	7
1	18	18	1
6	15	0	8

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
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Time Complexity - Method two

n = 4

m = 3

0	23	4	7
1	18	18	1
6	15	0	8

Observation: The time to process every column is the same: *θ(m)*

Idea: Count the number of columns processed! Multiply processing time later

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
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            p2 = Find2DPeak(A, but only the left side)
            if p1 or p2 is a peak return p1 else None
```

Time Complexity - Method two

$n = 4$

$m = 3$

0	23	4	7
1	18	18	1
6	15	0	8

Observation: The time to process every column is the same: $\theta(m)$

Idea: Count the number of columns processed! Multiply processing time later

- In the divide step, the middle column is processed, and no column is processed in the combined step - $\theta(1)$
- **Two** subproblems - each **half** the size!

Recurrence:

$C(n) = 2C(n/2) + \theta(1)$

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

Time Complexity - Method two

$n = 4$, $m = 3$

0	23	4	7
1	18	18	1
6	15	0	8

Observation: The time to process every column is the same: *θ(m)*

Idea: Count the number of columns processed! Multiply processing time later

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Recurrence:

$C(n) = 2C(n/2) + \theta(1)$

Master Theorem (Case 1):

$\log_b a = \log_2 2 = 1$

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Time Complexity - Method two

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1	18	18	1
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Observation: The time to process every column is the same: $\theta(m)$

Idea: Count the number of columns processed! Multiply processing time later

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Recurrence:

$C(n) = 2C(n/2) + \theta(1)$

Master Theorem (Case 1):

$\log_b a = \log_2 2 = 1$

$f(n) = \theta(1) = O(n^{1-\varepsilon})$

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.

Time Complexity - Method two

n = 4

m = 3

0	23	4	7
1	18	18	1
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Observation: The time to process every column is the same: *θ(m)*

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Recurrence:

$C(n) = 2C(n/2) + \theta(1)$

Master Theorem (Case 1):

$\log_b a = \log_2 2 = 1$

$f(n) = \theta(1) = O(n^{1-\varepsilon})$

Therefore, $C(n) = \theta(n^1)$

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Time Complexity - Method two

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Observation: The time to process every column is the same: $\theta(m)$

Idea: Count the number of columns processed! Multiply processing time later

- In the divide step, the middle column is processed, and no column is processed in the combined step - $\theta(1)$
- **Two** subproblems - each **half** the size!

Recurrence:

$C(n) = 2C(n/2) + \theta(1)$

Master Theorem (Case 1):

$\log_b a = \log_2 2 = 1$

$f(n) = \theta(1) = O(n^{1-\varepsilon})$

Therefore, $C(n) = \theta(n^1)$

Total time:

$\theta(mn)$

Time Complexity - Method two

Total time of *$\theta(mn)$* is ***no better*** than just iterating the 2D array normally! Does this mean that this algorithm is bad?

Time Complexity - Method two

Total time of $\theta(mn)$ is ***no better*** than just iterating the 2D array normally! Does this mean that this algorithm is bad?

Not exactly -- we can get new ideas from it on how to solve the problem!

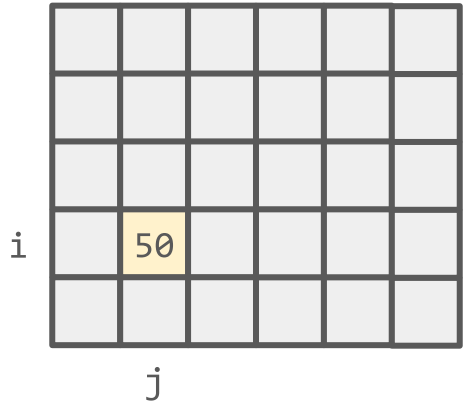
Question 3: Finding useful property for better 2D Peak-finding

Question 3

Let *A[i][j]* be the largest element in column *j*. Assume that *A[i][j + 1] ≥ A[i][j]*. Argue in the subarray B, where B is A but column *j + 1* onwards, any peak that is the largest element in its column is also a peak of the entire array A.

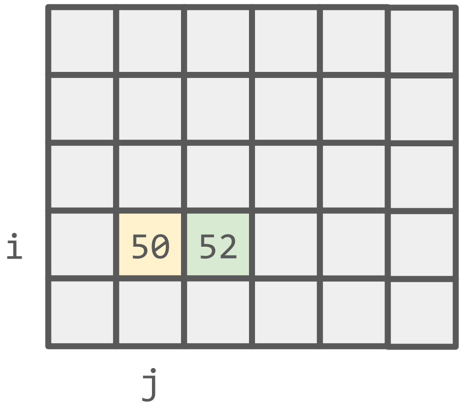
Question 3

Let $A[i][j]$ be the largest element in column j. Assume that $A[i][j + 1] \geq A[i][j]$. Argue in the subarray B, where B is A but column $j + 1$ onwards, any peak that is the largest element in its column is also a peak of the entire array A.



Question 3

Let *A[i][j]* be the largest element in column *j*. Assume that *A[i][j + 1] ≥ A[i][j]*. Argue in the subarray B, where B is A but column *j + 1* onwards, any peak that is the largest element in its column is also a peak of the entire array A.



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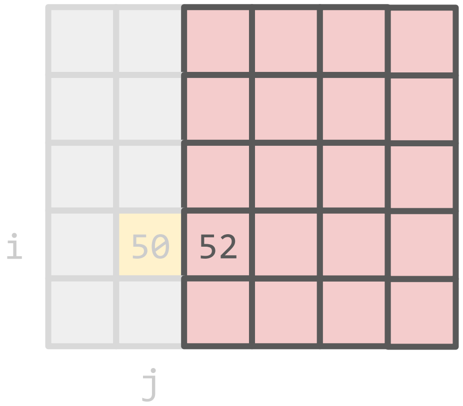
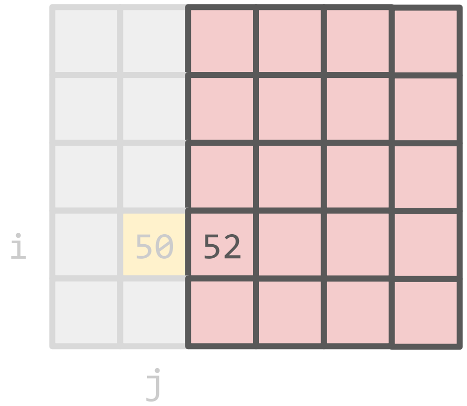


This question was added later on for completeness of q4. Hence it was not in the qn sheet

Question 3b (not in tutorial qn sheet)

Argue that this sub-array will always have a peak.

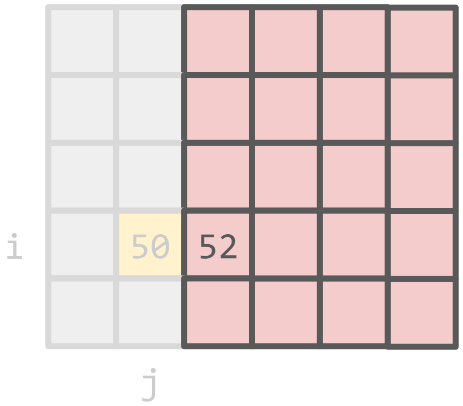


This question was added later on for completeness of q4. Hence it was not in the qn sheet

Question 3b (not in tutorial qn sheet)

Argue that this sub-array will always have a peak:

Any 2D array must have a peak. Simply take the maximum element in the 2D array, which will be a peak in this array.



Question 3

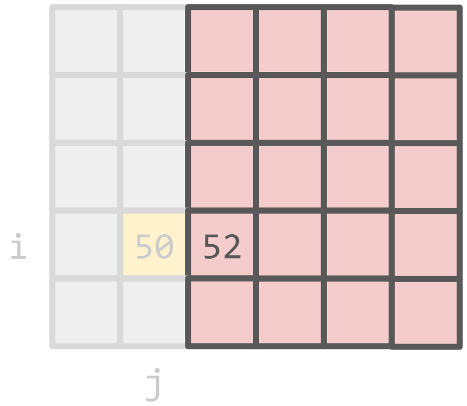
Suppose we are given a 2D-array A of size *m* rows by *n* columns. An element in the array $A[i][j]$ is called a "peak" if it is **greater than or equal to** its adjacent neighbours (if they exist -- an element is always considered greater than or equal to non-existent elements). For example, the 8 in the middle is a peak in the following array.

$$\begin{array}{cccc} * & * & 5 & * \\ * & 8 & 8 & 3 \\ * & * & 2 & * \end{array}$$

Let $A[i][j]$ be the largest element in column j. Assume that $A[i][j+1] \geq A[i][j]$. Argue that any peak in the the subarray $A[1..m][j+1..n]$ that is the largest element in its column is also a peak of the entire array A.

Question 3

Let *A[i][j]* be the largest element in column *j*. Assume that *A[i][j + 1] ≥ A[i][j]*. Argue in the subarray B, where B is A but column *j + 1* onwards, any peak that is the largest element in its column is also a peak of the entire array A.



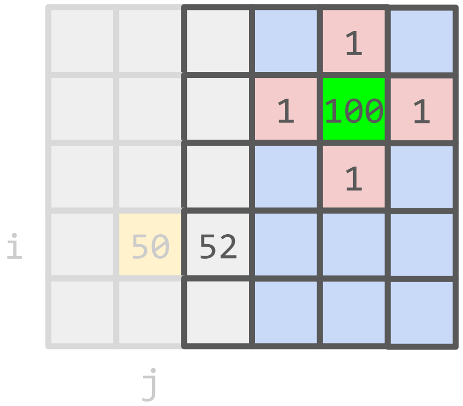
Now back to the original question at hand!

Q3 - Case 1

Let *A[i][j]* be the largest element in column *j*. Assume that *A[i][j + 1] ≥ A[i][j]*. Argue in the subarray B, where B is A but column *j + 1* onwards, any peak that is the largest element in its column is also a peak of the entire array A.

A peak in the columns ***j* + 2** onwards will be a peak in the overall array!

Intuition: Regardless of whether you add back the part you “removed”, whether this is a peak or not is **unaffected**

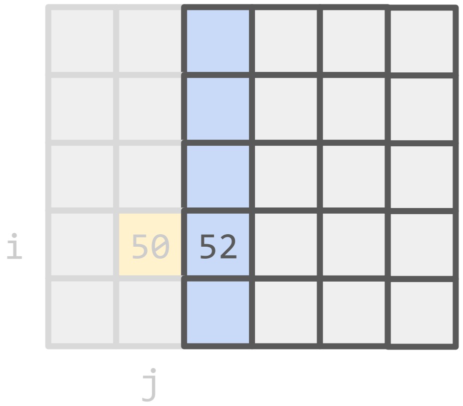


Q3 - Case 2

Let $A[i][j]$ be the largest element in column j. Assume that $A[i][j + 1] \geq A[i][j]$. Argue in the subarray B, where B is A but column $j + 1$ onwards, any peak that is the largest element in its column is also a peak of the entire array A.

A peak in the column ***j* + 1** needs a bit more work

Intuition: If you add back the part of A that you removed, **will that peak still remain a peak**?



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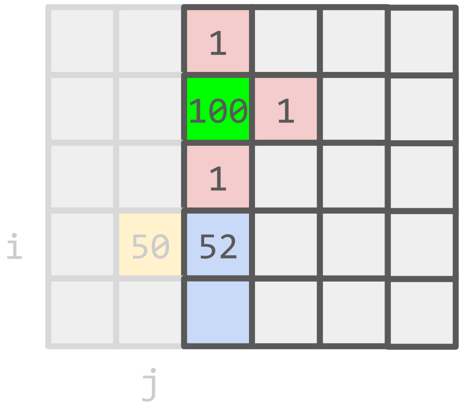
Let's say that the element 100 (note, larger than 52) here is a peak. Within the subarray, it's clearly a peak



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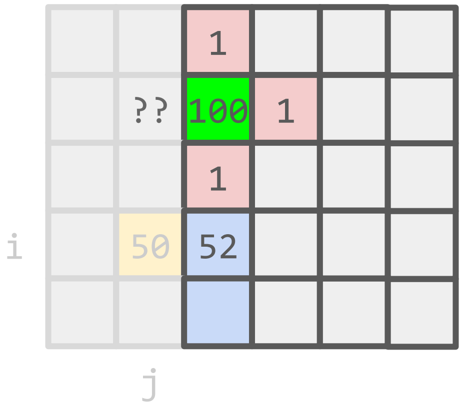
Let's say that the element 100 (note, larger than 52) here is a peak. Within the subarray, it's clearly a peak

But what about after considering the element to its left?? Will 100 stay a peak?



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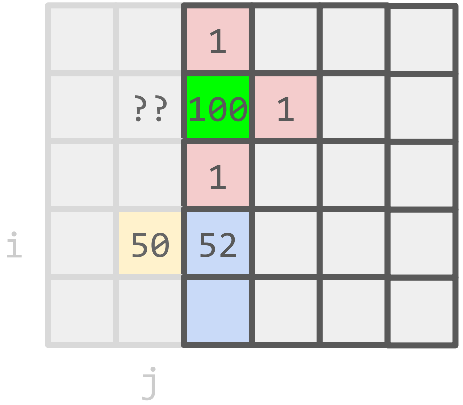
Observe: 50 is the largest element in column j. **So '??' has to be ≤ 50**



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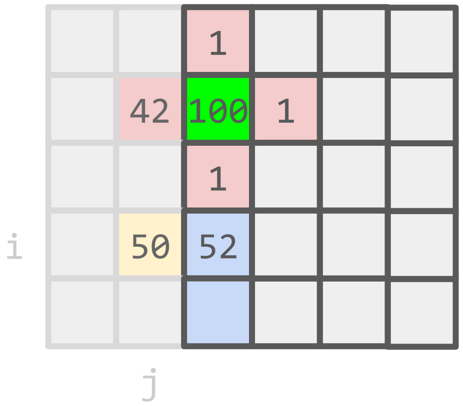
Observe: 50 is the largest element in column j. **So '??' has to be ≤ 50**

Let's say we put 42. Then 100 is **still a peak**, even when we consider the bigger array!



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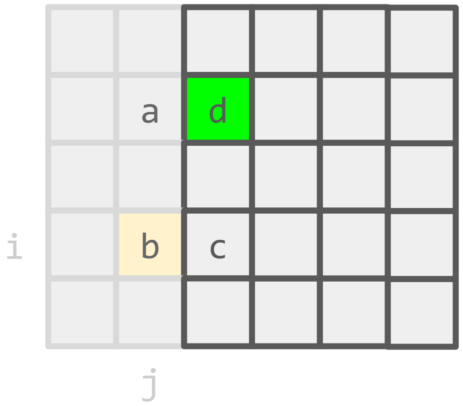
Generalising: Let d be the peak in the subarray.



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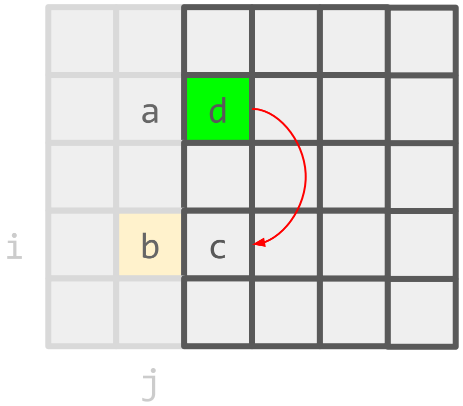

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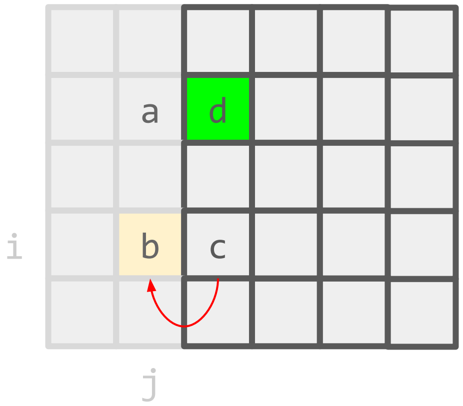

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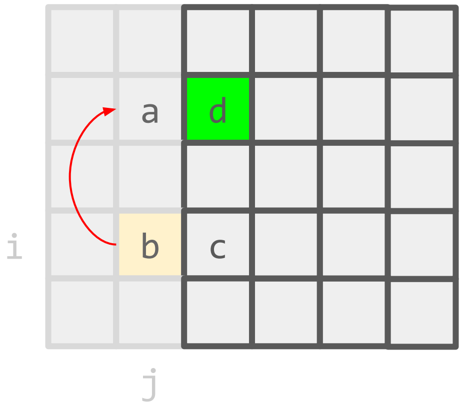
Generalising: Let d be the peak in the subarray.

d
≥ c (the peak should be the largest element in the column)
≥ b (*A[i][j+1] ≥ A[i][j]*)
≥ a (*A[i][j]* is the largest in col j)

Thus, d ≥ a



Question 4: Better 2D Peak-finding

Question 4

Suppose we are given a 2D-array A of size *m* rows by *n* columns. An element in the array $A[i][j]$ is called a "peak" if it is **greater than or equal to** its adjacent neighbours (if they exist -- an element is always considered greater than or equal to non-existent elements). For example, the 8 in the middle is a peak in the following array.

$$\begin{array}{cccc} * & * & 5 & * \\ * & 8 & 8 & 3 \\ * & * & 2 & * \end{array}$$

Using the idea in Question 3, describe an algorithm that is asymptotically faster that the one given in Question 2. What is it's runtime?

Question 4 (Solution)

```
def Find2DPeak(A):
    if num_of_cols(A) == 1:
        return max(A)
    else:
        mid_col = middle column of A
        candidate_peak = max(mid_col)
        if candidate_peak is peak:
            return candidate_peak
        else if candidate_peak <= element to its right:
            return Find2DPeak(A, but only the right side)
        else:
            return Find2DPeak(A, but only the left side)
```

Either side must contain a peak, by qn 3b

Question 4 (Solution)

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def Find2DPeak(A):
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```

If element to its left and to its right are smaller than `candidate_peak`, this case has been accounted for by the check if `candidate_peak is peak`

Question 4 (Solution)

Let *A[i][j]* be the largest element in column *j*. Assume that *A[i][j + 1] ≥ A[i][j]*. Argue in the subarray B, where B is A but column *j + 1* onwards, any peak that is the largest element in its column is also a peak of the entire array A.

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```

Important: return the max in the column! Otherwise, claim in Q3 does not hold

Q4 Analysis

Similar as before, but this time the recurrence for the number of columns processed is: $C(n) = C(n/2) + \theta(1)$

Q4 Analysis

2. $f(n) = \Theta(n^{\log_b a} \lg^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

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$log_b a = log_2 1 = 0$

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$f(n) = \theta(1) = \theta(n^0 (logn)^0) = \theta(1)$ [our choice of k = 0]

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So $C(n) = \theta(n^0 (\log n)^{0+1}) = \theta(\log n)$ [the +1 is because of master theorem]

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So $C(n) = \theta(n^0 (\log n)^{0+1}) = \theta(\log n)$ [the +1 is because of master theorem]

Total running time: $\theta(m\log n)$