CS3230: Assignment for Week 2 Solutions

Due: Sunday, 30th Jan 2022, 11:59 pm SGT.

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	A	B	O	o	Ω	ω	Θ
(a)	$n^3 + 4n$	$(\lg n)^{2022}$	no	no	yes	yes	no
(b)	n^9	1.01^n	yes	yes	no	no	no
(c)	$n^{1.5}$	$n \lg n$	no	no	yes	yes	no
(d)	2^n	3^n	yes	yes	no	no	no
(e)	$\lg(n^4)$	$\lg(n^8)$	yes	no	yes	no	yes
(f)	n^{10}	$n^{\lg n}$	yes	yes	no	no	no

Notes:

(a) Observe that $n^3 + 4n > n^3$, and invoke Lemma 2.2.4(i) of Lecture 2 notes. (In general, logs grow slower than polynomials.)

(b) Invoke Lemma 2.2.4(ii) of Lecture 2 notes. (In general, polynomials grow slower than exponentials.)

(c) $\lim_{n\to\infty} \frac{n^{1.5}}{n \lg n} = \lim_{n\to\infty} \frac{n^{0.5}}{\lg n} = \infty$, following the proof of Lemma 2.2.4(i) of Lecture 2 notes.

(d) $\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$.

(e) $\lim_{n\to\infty} \frac{\lg(n^4)}{\lg(n^8)} = \lim_{n\to\infty} \frac{4 \lg n}{8 \lg n} = \lim_{n\to\infty} \frac{4}{8} = \frac{1}{2}$.

(f) $\lim_{n\to\infty} \frac{n^{10}}{n^{\lg n}} = \lim_{n\to\infty} \frac{1}{n^{\lg n - 10}} = 0$.

2. This is false, for example if $f(n) = 2n$ and $g(n) = n$. Then $\lim_{n\to\infty} \frac{2^{f(n)}}{2^{g(n)}} = \lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} 2^n = \infty$, so $2^{f(n)} = \omega(2^{g(n)})$.

3. The base case holds since $f(2) = 2\lg 2 = 2$. For the inductive step, assume that $f(n) = n\lg n$ for some n, and consider the next power of 2, i.e., $2n$. The given formula tells us that

$$\begin{aligned} f(2n) &= 2f(n) + 2n \\ &= 2n\lg n + 2n \\ &= 2n(\lg n + 1) \\ &= 2n(\lg n + \lg 2) \\ &= 2n\lg(2n), \end{aligned}$$

which means that $f(2n)$ also satisfies the given formula. This completes the induction.