CS3230 Design & Analysis of Algorithms **January 22, 2022**

Proofs by Induction

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1.1 Introduction

Induction is a crucial tool that is widely used in Mathematics and Computer Science. As you will need to make use of it throughout CS3230, here we provide a simple refresher on the topic, and a few example applications. All exercises in these notes are **optional** and intended for individual practice.

1.2 Induction

Induction is a method of proof. Typically, induction is applicable whenever we want to prove some statement S_n that depends on some integer parameter n. To perform induction,

1. Establish that the statement is true for the **base case**.
2. Show that if the statement is true for the n-th case (this is the **induction hypothesis**), then the statement is also true for the $(n+1)$-th case.
3. Conclude that the statement holds for all cases.

Proof of validity. Induction is valid due to the **well-ordering property**. As a refresher for CS1231, the well-ordering property states that any nonempty set of natural numbers contains a least element, i.e. every other element in that set is greater than it.

Suppose that both 1 and 2 described above hold, and define $S \subseteq \mathbb{N}^+$ to be the set containing all n such that the n-th case **does not** hold (we consider $n = 1$ to be the base case). We know that $1 \notin S$. If S is nonempty, define $x \in S$ to be the least element. Then $x - 1 \notin S$ and $x - 1 \in \mathbb{N}^+$, and from 2 we have that the x-th case holds, i.e. $x \notin S$. By contradiction, S is empty.

Example 1. First, lets prove the following well known fact: for all integers $n \geq 1$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1.1}$$

Base case: We first show that this is true for the base case, when $n = 1$.

$$1 = \frac{1 \cdot 2}{2} = 1 \tag{1.2}$$

Inductive step: Assume that the induction hypothesis holds, i.e. the equation holds for n:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1.3}$$

We prove that the equation holds for $n+1$ as follows:

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{(n+1)(n+2)}{2}\end{aligned} \tag{1.4}$$

Conclusion: The equation holds for all $n \geq 1$.

Exercise. Prove that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Notice the power of an inductive argument; this shows that the equation is true for **any** choice of n. This is the one of the strengths of induction as it allows you to generalise your observations. Take note however, that if you want to apply induction, you must already have a hypothesis you wish to prove in mind.

Example 2. Sometimes we require a more complex inductive hypothesis in order to perform induction. Consider the statement that for all $n \geq 8$, $n = 3a + 5b$ for some non-negative integers a and b.

Here we use a modified induction hypothesis: for some $k \geq 3$, $3k-1$, $3k$ and $3k+1$ can all be written in the desired form.

Base case: For $k = 3$, we have $8 = 3 \cdot 1 + 5 \cdot 1$, $9 = 3 \cdot 3$, $10 = 5 \cdot 2$.

Inductive step: Assume that the induction hypothesis holds for k. For each n, let a_n and b_n be defined such that $n = 3a_n + 5b_n$ if n can be represented in this form.

Then we can write:

$$\begin{aligned}3k+2 &= (3k-1)+3 \\ &= (3a_{3k-1} + 5b_{3k-1}) + 3 \text{ (applying the induction hypothesis to (3k - 1))} \\ &= 3(a_{3k-1}+1) + 5b_{3k-1}\end{aligned} \tag{1.5}$$

The working is similar for $3k+3$ and $3k+4$ and is left as an exercise. Therefore $3(k+1)-1$, $3(k+1)$ and $3(k+1)+1$ can all be written in the desired form.

Conclusion: Note that all integers $n \geq 8$ can be written as $3k-1$, $3k$ or $3k+1$ for some $k \geq 3$. By induction, for all $n \geq 8$, $n = 3a + 5b$ for some integers $a, b \geq 0$.

Remark. The previous example can also be thought of as three separate instances of induction on the sequences $\{8, 11, 14, ...\}$, $\{9, 12, 15, ...\}$ and $\{10, 13, ...\}$. Alternatively, it can be proven using strong induction, which will be discussed next.

1.3 Strong induction

Sometimes, (weak) induction is insufficient to prove the statement. A stronger variant of induction, known as strong induction, makes use of the fact that you have proven all cases between the base case and the current case. Your argument template changes slightly in the second step:

1. Establish that the statement is true for some base case.

2. Show that if the statement is true for **all the previous cases, up to the n-th case**, then this implies that the statement is also true for the $(n+1)$th case.
3. Conclude that the statement holds for all cases.

Example 3. Let's use strong induction to prove that all integers greater than 1 are either prime, or have a prime factorisation.

Base case: 2 is a prime number.

Inductive step: For some $n \geq 2$, suppose that all integers between 2 and n inclusive are either prime or have a prime factorisation. There are two cases to consider:

1. The number $n+1$ is prime, in which case it satisfies our hypothesis.
2. The number $n+1$ is not prime. By definition, this means that we can write $n+1 = a \cdot b$ where a and b are positive integers greater than 1 and smaller than $n+1$. This is where strong induction is useful; because our hypothesis says that all integers greater than or equal to 2 (our base case) and smaller than or equal to n are prime, or have a prime factorisation, we can express both a and b as a product of prime numbers, giving us a prime factorisation for $n+1$.

In either case, we have shown that assuming all integers greater than 1 up to n are prime, or have a prime factorisation, then it must be true that $n+1$ is prime, or has a prime factorisation. Thus we can conclude by induction that this statement is true for any integer n greater than 1.

Example 4. Recall that the Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$, and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. We shall show that $F_n \leq 2^{n-2}$ for all $n \geq 2$.

Base case: $F_2 = 1 \leq 1 = 2^{2-2}$, $F_3 = 2 \leq 2 = 2^{3-2}$.

Inductive step: For some $n \geq 3$, suppose that $F_n \leq 2^{n-2}$ and $F_{n-1} \leq 2^{n-3}$. Then

$$F_{n+1} = F_n + F_{n-1} \leq 2^{n-2} + 2^{n-3} < 2^{n-2} + 2^{n-2} = 2^{n-1}. \tag{1.6}$$

By induction, we conclude that $F_n \leq 2^{n-2}$ for all $n \geq 2$.

Remark. Note that in Example 4, we do not require all previous cases to perform the inductive step, and only the previous two cases are used. The proof is still valid even if the induction step assumes $F_k \leq 2^{k-2}$ for all $3 \leq k \leq n$.

Exercise. Adapt the proof of Example 4 to show that $F_n \geq \left(\frac{8}{5}\right)^{n-2}$ for all $n \geq 2$.

1.4 Pitfalls

Here are some common mistakes to avoid when writing proofs by induction.

- **Forgetting the base case.** An induction argument does not work without it!
- **Wrong base case.** In Example 4, it is not sufficient to have F_2 as the base case. This is because $F_3 = F_2 + F_1$, but F_1 does not satisfy the inequality.
- **Proving the converse.** In the inductive step, start with the smaller case(s) to prove the larger case, and not the other way round.
- **Weak induction hypothesis.** When proving Example 3, it is not sufficient to assume only that n satisfies the hypothesis.
- **Missing conclusion.** Remember to include the conclusion as it is an important logical step in the induction process.

1.5 Additional Exercises

1. Show that 3 divides $n^3 + 2n$ whenever n is a non-negative integer.
2. Show that 5 divides $n^5 - n$ whenever n is a non-negative integer.
3. Show that $1/(2n) \leq [1 \cdot 3 \cdot 5 \ldots (2n-1)]/(2 \cdot 4 \ldots 2n)$ whenever n is a positive integer.
4. Show that $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$
5. Let $H_n = 1 + \frac{1}{2} \ldots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$. H_n is known as the nth *harmonic number*. Show that $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$.

1.6 Acknowledgements

These notes and exercises are partially based off of Kenneth H. Rosen's "Discrete Math and its Applications", Chapter 3.2.