Design and Analysis of Algorithms

CS3230

Lecture 7
Amortized Analysis

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Asymptotic notation for multiple parameters

- What does $O(m+n)$ or $O(mn)$ mean?

- For two functions $f(m,n)$ and $g(m,n)$, we say that $f(m,n) = O(g(m,n))$ if there exist constants c, m_0, n_0 such that
$$0 \leq f(m,n) \leq c \cdot g(m,n)$$
for all $m \geq m_0$ or $n \geq n_0$.

- Similar definitions for other asymptotic notations (but not our focus). See Exercise 3.1-8 of CLRS.

Amortized Analysis

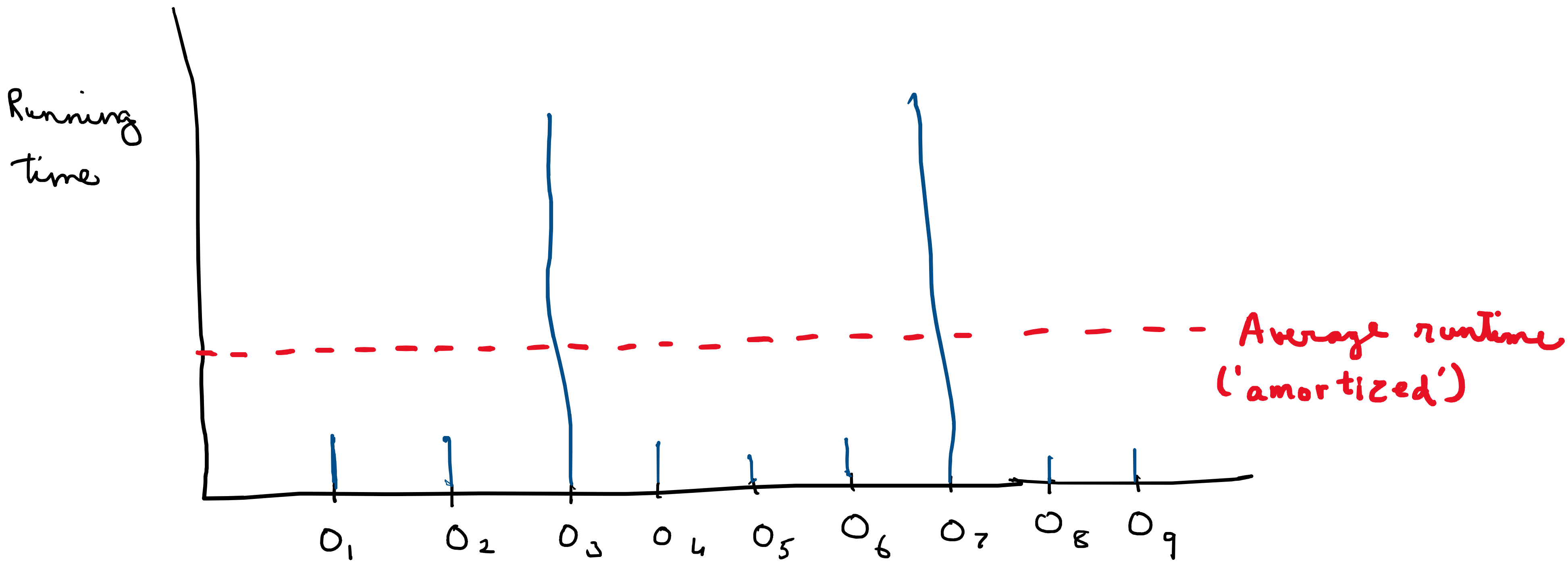
- There is a sequence of n operations $\underbrace{o_1, o_2, \ldots, o_n}$
- Let t(i) be the time complexity of i-th operation o_i
- Let T(n) be the time complexity of all n operations

$$T(n) = \sum_{i=1}^{n} t(i)$$
$$= n\, f(n)$$

f(n) = worst case time complexity of any of the n operations

Could be grossly wrong!

Amortized Analysis



Binary counter

A motivating example for amortized analysis

k-bit Binary Counter: How do we increment?

Ctr	A[4]	A[3]	A[2]	A[1]	A[0]	
0	0	0	0	0	0	
1	0	0	0	0	1	
2	0	0	0	1	0	
3	0	0	0	1	1	
4	0	0	1	0	0	
5	0	0	1	0	1	
6	0	0	1	1	0	
7	0	0	1	1	1	
8	0	1	0	0	0	
9	0	1	0	0	1	
10	0	1	0	1	0	
11	0	1	0	1	1	

Objective: Count total Bit flips (0 → 1, 1 → 0) during n increments

$T(n)$ = total no. of bit flips during n increments

Aim: To get a tight bound on $T(n)$

```
INCREMENT(A)
1.  i ← 0
2.  while i < length[A] and A[i] = 1 do
3.          A[i] ← 0          ⊳ flip 1→0
4.          i ← i + 1
5.  if  i < length[A]
6.      then  A[i] ← 1        ⊳ flip 0→1
```

k-bit Binary Counter: No. of bit flips

Ctr	A[4]	A[3]	A[2]	A[1]	A[0]	*Cost*
0	0	0	0	0	0	*0*
1	0	0	0	0	1	*1*
2	0	0	0	1	0	*2*
3	0	0	0	1	1	*1*
4	0	0	1	0	0	*3*
5	0	0	1	0	1	*1*
6	0	0	1	1	0	*2*
7	0	0	1	1	1	*1*
8	0	1	0	0	0	*4*
9	0	1	0	0	1	*1*
10	0	1	0	1	0	*2*
11	0	1	0	1	1	*1*

Objective: Count total Bit flips (0 → 1, 1 → 0) during n increments

$\boldsymbol{T(n)}$ = total no. of bit flips during $\boldsymbol{n}$ increments

Aim: To get a tight bound on $\boldsymbol{T(n)}$

Attempt 1:

Let $\boldsymbol{t(i)}$ = no. of bit flips during $\boldsymbol{i}$th increment

$$\boldsymbol{T(n) = \sum_{i=1}^{n} t(i)}$$

In the worst case, $\boldsymbol{t(i)} = k$

→ $\boldsymbol{T(n) = O(n\,k)}$

Is this a tight bound?

k-bit Binary Counter: No. of bit flips

Ctr	A[4]	A[3]	A[2]	A[1]	A[0]	*Cost*	*Total Cost*
0	0	0	0	0	**0**	*0*	*0*
1	0	0	0	**0**	**1**	*1*	*1*
2	0	0	0	1	**0**	*2*	*3*
3	0	0	**0**	**1**	**1**	*1*	*4*
4	0	0	1	0	**0**	*3*	*7*
5	0	0	1	**0**	**1**	*1*	*8*
6	0	0	1	1	**0**	*2*	*10*
7	0	**0**	**1**	**1**	**1**	*1*	*11*
8	0	1	0	0	**0**	*4*	*15*
9	0	1	0	**0**	**1**	*1*	*16*
10	0	1	0	1	**0**	*2*	*18*
11	0	1	**0**	**1**	**1**	*1*	*19*

Objective: Count total Bit flips (0 → 1, 1 → 0) during n increments

$T(n)$ = total no. of bit flips during n increments

Aim: To get a tight bound on $T(n)$

Attempt 2:

Let $f(i)$ = no. of times ith bit flips

$$T(n) = \sum_{i=0}^{k-1} f(i)$$

$f(0) = n$
$f(1) = n/2$
$f(2) = n/4$
$f(i) = n/2^i$

Much better than $O(nk)$ since $k \approx \log n$

→ $T(n) = n \sum_{i=0}^{k-1} 2^{-i} < 2n$

Amortized Analysis

- ***Amortized analysis*** is a strategy for analyzing a sequence of operations to show that the average cost per operation is small, even though a single operation within the sequence might be expensive.

- In our last example, average cost per increment = $\frac{T(n)}{n} < 2 = O(1)$

We say **amortized cost** of each increment = $O(1)$

Amortized Analysis

- ***Amortized analysis*** is a strategy for analyzing a sequence of operations to show that the average cost per operation is small, even though a single operation within the sequence might be expensive.

- Note, ***no probability is involved***!

Do not get confused with the average-case analysis

- An amortized analysis *guarantees* the *average performance* of each operation in the *worst case*.

- In our last example, average cost per increment = $\frac{T(n)}{n} < 2 = O(1)$

We say **amortized cost** of each increment = $O(1)$

Another Example: Queues

- Consider a queue with two operations:
  - *INSERT(x): Inserting* one element x and
  - *EMPTY(): Emptying* the queue, implemented by deleting all the elements one by one.
- What is the <u>worst case</u> running time for a sequence of n operations?

- Cost of a single INSERT: $\Theta(1)$
- Cost of a single EMPTY: $\Theta(n)$ (because at most n elements inserted)

Amortized Analysis: Queues

- An EMPTY is a sequence of DELETE’s where each DELETE removes one element from the front of the queue

- Notice: #DELETE’s $\leq$ #INSERT’s

- If there are k INSERT’s in the sequence, sum of cost of all the EMPTY’s is $\leq k$.

- Total cost: $\leq k + k = 2k \leq 2n$. Amortized cost is $O(1)$.

Types of Amortized Analyses

- Three common amortization arguments:
  - *Aggregate* method
  - *Accounting* method
  - *Potential* method

- We have just seen two examples of aggregate analysis.

- The aggregate method, though simple, lacks the precision of the other two methods. In particular, the accounting and potential methods allow a specific ***amortized cost*** to be allocated to each operation.

Accounting (Banker's) Method

- Charge ith operation a fictitious ***amortized cost*** *c(i)*.
- Idea is that the amortized cost *c(i)* is a fixed cost for each operation, while the true cost *t(i)* varies depending on when the operation is called.
- The amortized cost *c(i)* must satisfy:

$$\sum_{i=1}^{n} t(i) \leq \sum_{i=1}^{n} c(i) \text{ for all } n$$

The total amortized cost provides an upper bound on the total true cost.

Accounting (Banker's) Method

- Typically, the fixed amortized cost $c(i)$ will be more than the true cost $t(i)$. Only occasionally, the true cost $t(i)$ will be larger than $c(i)$.

- The extra amount we pay for cheap operations early on can be thought of as **credit** paid in advance for the rare, expensive operations later!

- Analysis must ensure that there's always enough credit to pay for true cost.

- NOTE: Different operations can have different amortized costs.

Amortized Analysis of Queues

- For INSERT, set amortized cost to 2. (True cost is 1.)

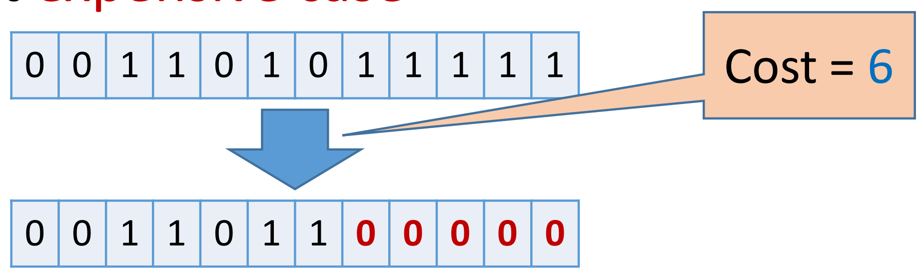
- For EMPTY, set amortized cost to 0. (True cost is size of queue.)

- Whenever an element is inserted, we pay an extra 1. This extra 1 can be used as credit that can pay for deleting it later.

- Total cost is at most 2*#INSERT $\leq 2n$.

Accounting Analysis of Binary Increment

- First identify the most expensive case



Observe, 5 bits are reset to 1 → 0, and only one bit is set to 0 → 1

k-bit Binary Counter: How to increment?

Ctr	A[4]	A[3]	A[2]	A[1]	A[0]	
0	0	0	0	0	0	
1	0	0	0	0	1	
2	0	0	0	1	0	
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8	0	1	0	0	0	
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10	0	1	0	1	0	
11	0	1	0	1	1	

INCREMENT(A)
1. $i \leftarrow 0$
2. **while** $i < length[A]$ **and** $A[i] = 1$ **do**
3. $\quad A[i] \leftarrow 0$ ⊳ *flip* 1→0
4. $\quad i \leftarrow i + 1$
5. **if** $i < length[A]$
6. $\quad$ **then** $A[i] \leftarrow 1$ ⊳ *flip* 0→1

Accounting Analysis of Binary Increment

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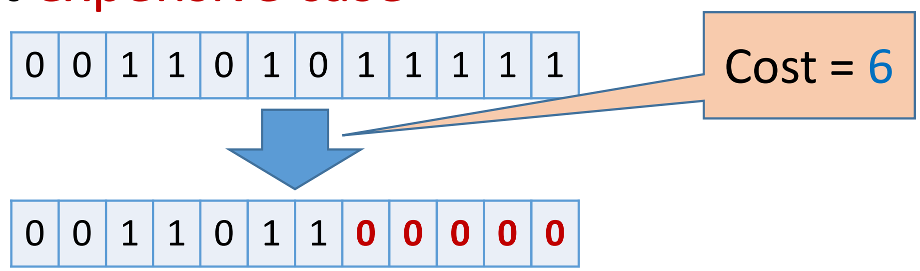


Observe, 5 bits are reset to 1 → 0, and only one bit is set to 0 → 1

Can we do 1 → 0
“free of cost” ?

Accounting Analysis of Binary Increment

- Charge $2 for each 0 → 1

$1 pays for the actual bit setting.
$1 is stored in the bank.

At some point of time, this bit must have been set

- **Observation:** At any point, every 1 bit in the counter has $1 on its bank.
- Use that $1 as credit to pay for resetting 1 → 0. (reset is "*free*")

Example:

0 0 0 1$^{\$1}$ 0 1$^{\$1}$ 0

0 0 0 1$^{\$1}$ 0 1$^{\$1}$ 1$^{\$1}$ Amortized Cost = $2

0 0 0 1$^{\$1}$ 1$^{\$1}$ 0 0 Amortized Cost = $2

Accounting Analysis of Binary Increment

- **Invariant we need to keep:** Bank balance never drops below 0. Thus, the sum of the amortized costs provides an upper bound on the sum of the true costs.

Observation: At any point, every 1 bit in the counter has $1 on its bank.

Claim: After **i** increments, the amount of money in the bank is the number of 1's in the binary representation of **i**.

Accounting Analysis of Binary Increment

- **Claim**: After i increments, the amount of money in the bank is the number of 1's in the binary representation of i.
- **Proof**:
  - Every time we set a bit 0 → 1, we pay \$2.
  - \$1 is used to flip the bit while \$1 is stored in bank.
  - Every time we reset a bit from 1 → 0, we use \$1 from bank to flip the bit.
  - Hence, the amount of money in the bank is the number of 1's in the binary representation of i.

Accounting Analysis of Binary Increment

- Since the number of 1's in the binary representation of i is non-negative, previous slide shows that the bank balance is always non-negative.

- Conclusion:
  - Amortized cost for each increment = 2 = $O(1)$
  - Amortized cost for n increments = $2n = O(n)$
  - Actual cost for n increments = $O(n)$

Actual cost ≤ Amortized cost

Potential Method

$\boldsymbol{\phi}$: Potential function associated with the algorithm/data-structure (as opposed to with specific object in data-structure)

$\boldsymbol{\phi(i)}$: Potential at the end of $\boldsymbol{i}$th operation

$\boldsymbol{c_i}$: Amortized cost of $\boldsymbol{i}$th operation

$\boldsymbol{t_i}$: True cost of $\boldsymbol{i}$th operation

Key relation: $\boldsymbol{c_i = t_i + \phi(i) - \phi(i-1)}$

Adding up the above relation for $\boldsymbol{i = 1, 2, \dots, n}$:

$$\boldsymbol{c_1 + c_2 + \cdots + c_n \geq t_1 + t_2 + \cdots + t_n + \phi(n) - \phi(0)}$$

Potential Method

Key relation: $c_i = t_i + \phi(i) - \phi(i-1)$

$$c_1 + c_2 + \cdots + c_n = t_1 + t_2 + \cdots + t_n + \phi(n) - \phi(0)$$

Typically, we define $\phi(0) = 0$. So as long as $\phi(n) \geq 0$ for all n, the amortized cost is an upper bound of the true cost:

$$c_1 + c_2 + \cdots + c_n \geq t_1 + t_2 + \cdots + t_n$$

Potential Method on Queue

Potential function: $\boldsymbol{\phi(i)}$ = number of elements in the queue after $\boldsymbol{i}$-th operation

We have $\boldsymbol{\phi(0) = 0}$. Moreover, $\boldsymbol{\phi(n) \geq 0}$ for all $\boldsymbol{n}$.

Amortized cost for INSERT:

$$\boldsymbol{c_i = t_i + \phi(i) - \phi(i-1) = 1 + 1 = 2}$$

Amortized cost for EMPTY (suppose there are $\boldsymbol{k}$ elements in the queue):

$$\boldsymbol{c_i = t_i + \phi(i) - \phi(i-1) = k + 0 \ - k = 0}$$

For binary increment, potential function is the number of 1-bits (see CLRS for analysis)

Dynamic table

For Insertion only

How large should a table be?

- **Goal:** Make the table as small as possible, but large enough so that it won't overflow (otherwise becomes inefficient).
- **Problem:** What if we don't know the proper size in advance?

How large should a table be?

- **Goal:** Make the table as small as possible, but large enough so that it won't overflow (or otherwise become inefficient).
- **Problem:** What if we don't know the proper size in advance?

Solution: ***Dynamic tables.***

- **IDEA:** Whenever the table overflows, "grow" it by allocating (via **`malloc`** or **`new`**) a new, larger table. Move all items from the old table into the new one, and free the storage for the old table.

- Dynamic tables are implemented as ArrayList in Java or std:vector in C++.

Some Notations

- n: number of elements in the table.
- **createTable**(k): A system-call that creates a table of size k and returns its pointer.
- **size**(T): the size of table T.
- **copy**(T,T'): copies the contents of table T into T'.
- **free**(T): free the space (return the space to OS) occupied by table T.

A <u>trivial</u> way to perform Insert($\boldsymbol{x}$)

If ($\boldsymbol{n = 0}$)

    $\boldsymbol{T} \leftarrow$ **createTable**($\mathbf{1}$);

Else

// Table is full

    If($\boldsymbol{n} =$ **size**($\boldsymbol{T}$))

    { $\boldsymbol{T'} \leftarrow$ **createTable**($\boldsymbol{n + 1}$);

        copy($\boldsymbol{T}$,$\boldsymbol{T'}$);

        free($\boldsymbol{T}$);

        $\boldsymbol{T} \leftarrow \boldsymbol{T'}$

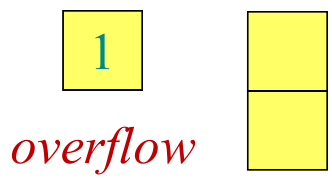
    }

Insert $\boldsymbol{x}$ into $\boldsymbol{T}$;

$\boldsymbol{n} \leftarrow \boldsymbol{n + 1}$;

Example of a Dynamic Table

1. INSERT
2. INSERT



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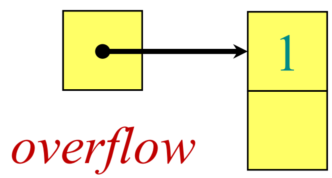

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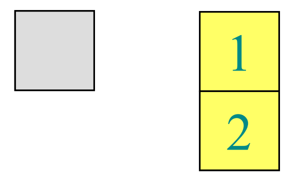

Example of a Dynamic Table

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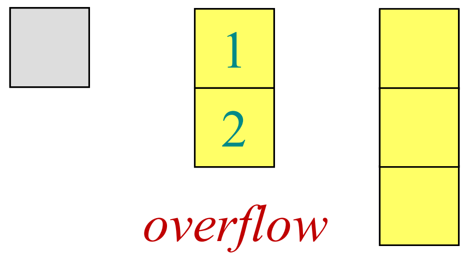

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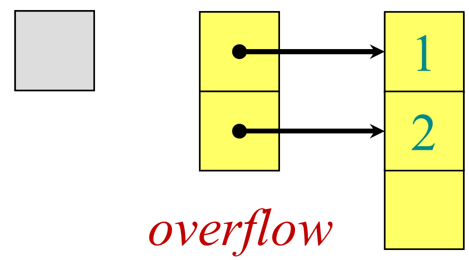

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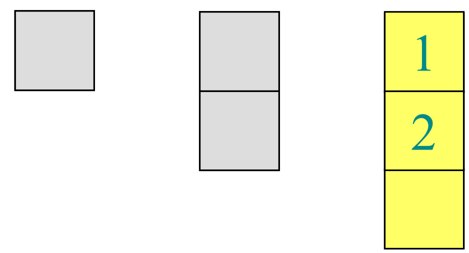

Example of a Dynamic Table

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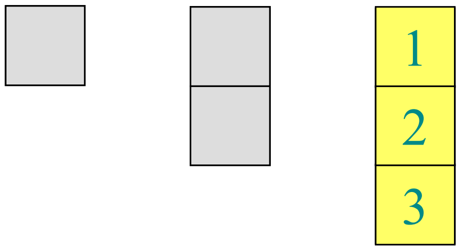


A trivial way to perform Insert(x)

If ($n = 0$)
    T ← createTable(1);
Else
    If(n = size(T))   // Table is full
    { T' ← createTable($n+1$);
        copy(T,T');
        free(T);
        T← T'
    }
Insert x into T;
n← $n+1$;

Idea: Every time table is full, create a new table of **double the size**

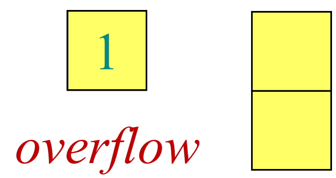
Time complexity of n insertions : $O(n^2)$

Proposed way to perform Insert(x)

```
If (n = 0)
    T ← createTable(1);
Else
    If(n = size(T))    // Table is full
    {   T' ← createTable( 2n);
        copy(T,T');
        free(T);
        T ← T'
    }
Insert x into T;
n ← n + 1;
```

Example of an Efficient Dynamic Table

1. INSERT
2. INSERT



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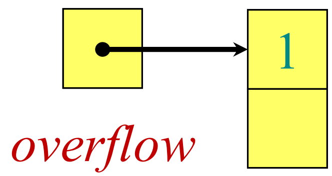

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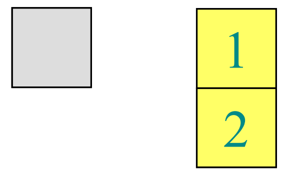

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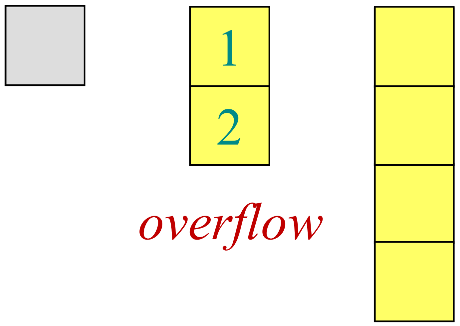

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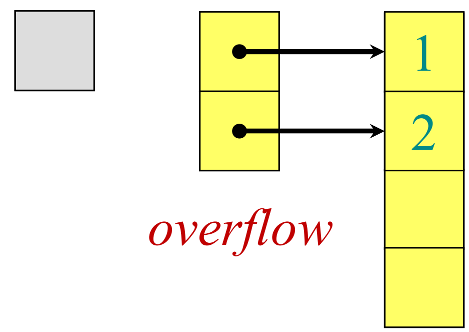

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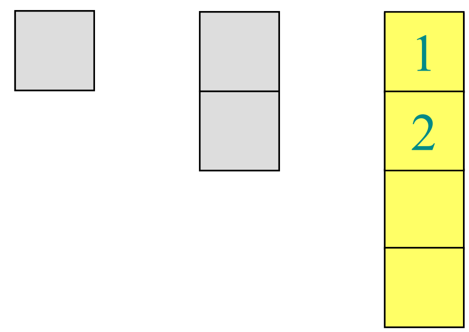

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1. INSERT
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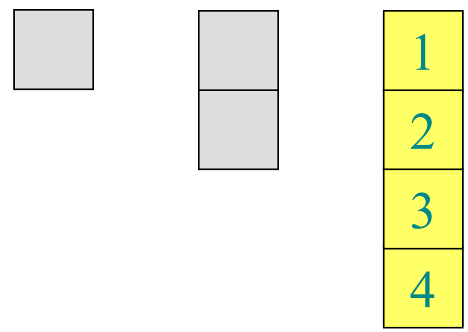
4. INSERT



Example of an Efficient Dynamic Table

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2. INSERT
3. INSERT
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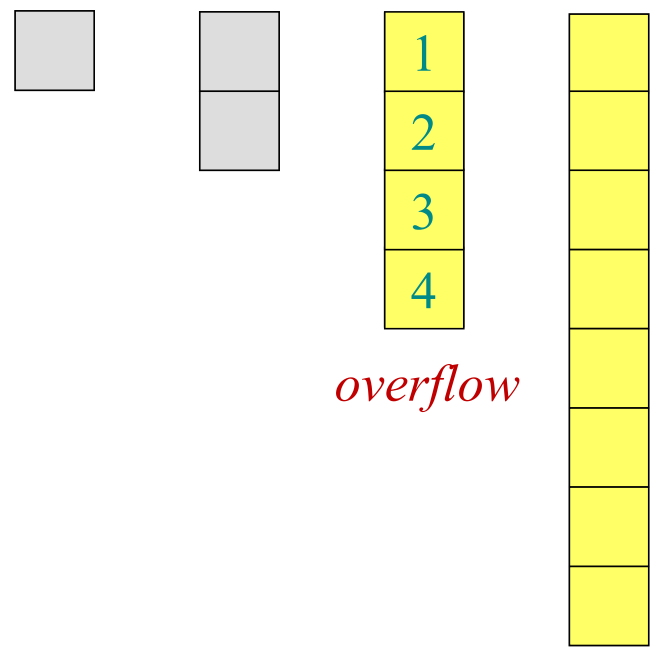
5. INSERT



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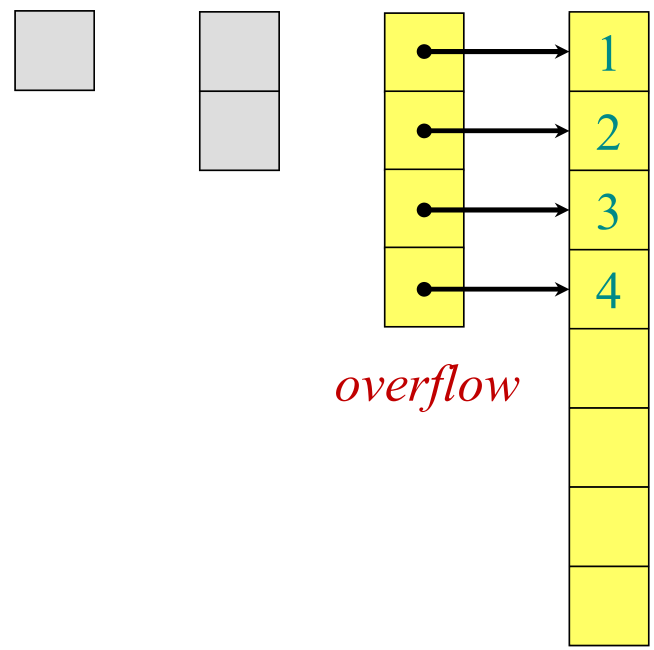

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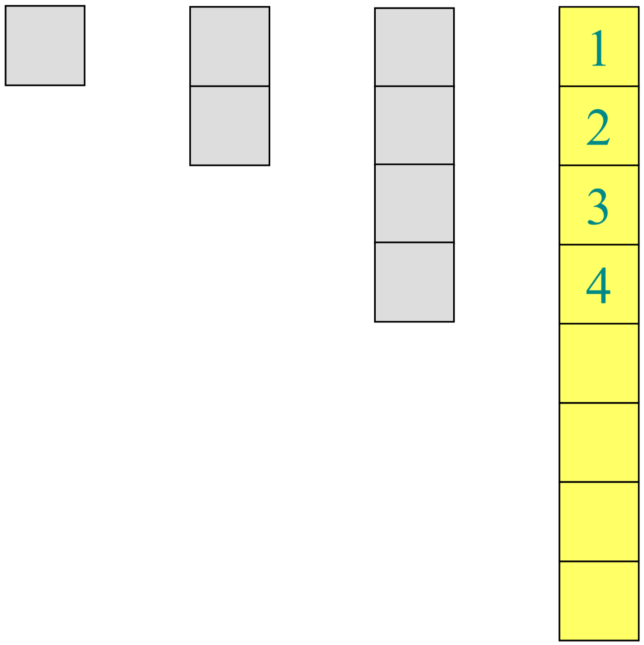
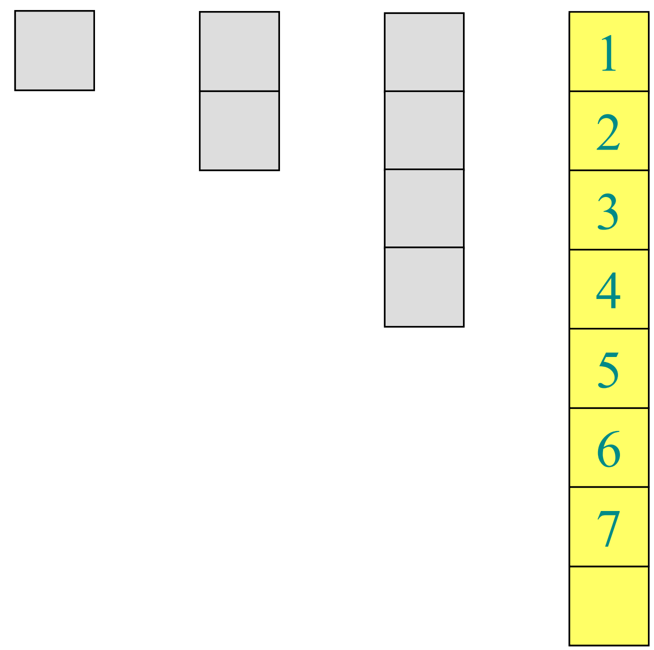


Example of an Efficient Dynamic Table

1. INSERT
2. INSERT
3. INSERT
4. INSERT
5. INSERT
6. INSERT
7. INSERT



Worst-case Analysis

- Consider a sequence of n insertions.

- The worst-case time to execute one insertion is $O(n)$.

- So, the worst-case time for n insertions is $n \cdot O(n) = O(n^2)$.

Is this tight?

Amortized Analysis

- Observe, once the table is full, we create a table of double the size.
- It will take **O(1)** time for next many insertions (filling up empty slots) until overflow happens.
- So the heavy operation (copying the table into new table) will occur only when $\boldsymbol{n-1}$ is a power of $\mathbf{2}$.

Aggregate Method

Let $t(i) =$ the cost of the i th insertion

$$= \begin{cases} i & \text{if } i-1 \text{ is an exact power of } 2, \\ 1 & \text{otherwise.} \end{cases}$$

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$size_i$	**1**	**2**	**4**	**4**	**8**	**8**	**8**	**8**	**16**	**16**
$t(i)$	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
		1	**2**		**4**				**8**	

Cost for i th insert

Cost for copying due to overflow

Aggregate Method

$$= 2^{\lfloor \log(n-1) \rfloor + 1} - 1$$
$$\leq 2(n-1)$$

(Geometric series)

Cost of n insertions $= \sum_{i=1}^{n} t(i)$

$$\leq \text{n} + \boxed{\sum_{j=0}^{\lfloor \log(n-1) \rfloor} 2^j}$$

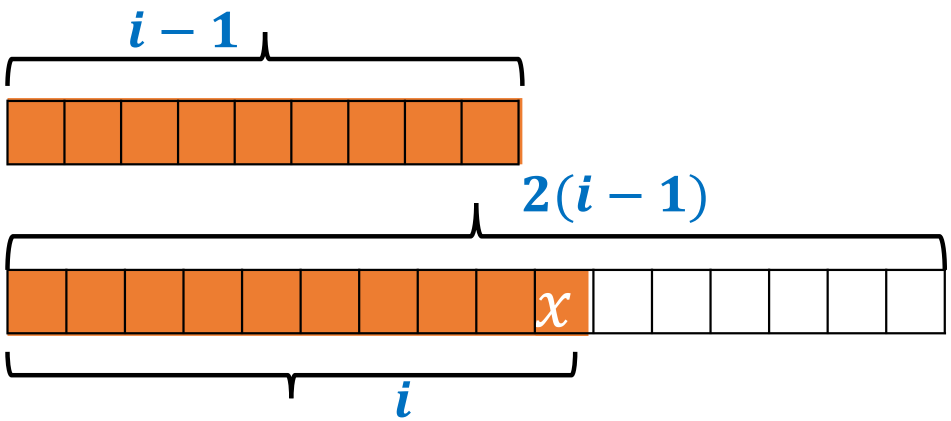
$$\leq 3n$$

Thus, the average cost of each insertion in dynamic table is $=$ O(n)/n $=$ O(1).

Accounting Method

- Charge \$3 for each insertion
  - \$1 for the insertion into the current table
  - \$1 for moving when the table expands
  - \$1 for moving another item (which was moved before) when the table expands
- Suppose the table has size m after an expansion, so it currently holds $m/2$ elements. Consider the next insertion.
  - \$1 to pay for the insertion itself
  - \$1 stored for moving this item when the table expands
  - \$1 stored for moving one of the existing $m/2$ items when the table expands

Potential Method for Insert($\boldsymbol{x}$)



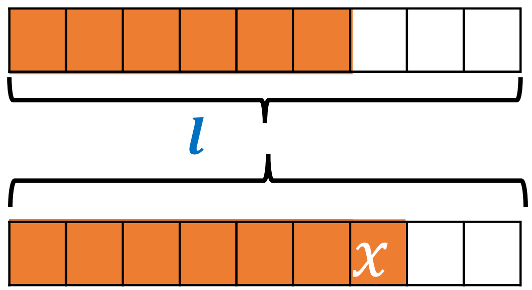
$\phi(i\text{-}1) = i - 1$ Before **Insert**($\boldsymbol{x}$)

$\phi(i) = 2$ After **Insert**($\boldsymbol{x}$)

Operation Insert(x)	Actual Cost	$\Delta\phi_i$	Amortized Cost
Case 1: when table is not full	1		
Case 2: when table is already full	$\boldsymbol{i}$	$\mathbf{3} - \boldsymbol{i}$	$\mathbf{3}$

$$\phi(\boldsymbol{i}) = \mathbf{2}\boldsymbol{i} - \boldsymbol{size}(\boldsymbol{T})$$

Potential Method for Insert(x)



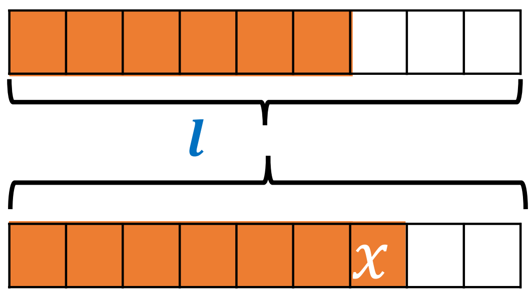
$\phi(i\text{-}1) = 2(i-1) - l$ Before **Insert**(x)

$\phi(i) = 2i - l$ After **Insert**(x)

Operation Insert(x)	Actual Cost	$\Delta\phi_i$	Amortized Cost
Case 1: when table is not full	1	$\mathbf{2}$	$\mathbf{3}$
Case 2: when table is already full	$\boldsymbol{i}$	$\mathbf{3 - \boldsymbol{i}}$	$\mathbf{3}$

$$\phi(i) = 2i - size(T)$$

Potential Method for Insert(x)



$\phi(i\text{-}1) = 2(i-1) - l$ — Before **Insert**(x)

$\phi(i) = 2i - l$ — After **Insert**(x)

Operation Insert(x)	**Actual Cost**	$\Delta\phi_i$	**Amortized Cost**
Case 1: when table is not full	1	**2**	**3**
Case 2: when table is already full	$\boldsymbol{i}$	$\boldsymbol{3-i}$	**3**

Amortized cost of n insertions $= 3n = O(n)$

Actual cost of n insertions $= O(n)$

Conclusion

- Amortized costs can provide a clean abstraction of data-structure performance.
- Amortized analysis can be performed using all 3 methods: aggregate method, accounting method, potential method.
  - But each method has some situations where it is arguably the simplest or most precise.
  - Choice of potential function can be somewhat tricky, and you sometimes need to play around with different options.
- Different schemes may work for assigning amortized costs in the accounting method, or potentials in the potential method, sometimes yielding radically different bounds.

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