Floors and Ceilings in Divide-and-Conquer Recurrences*

William Kuszmaul
Charles E. Leiserson

MIT CSAIL
{kuszmaul,cel}@mit.edu

Abstract

The master theorem is a core tool for algorithm analysis. Many applications use the discrete version of the theorem, in which floors and ceilings may appear within the recursion. Several of the known proofs of the discrete master theorem include substantial errors, however, and other known proofs employ sophisticated mathematics. We present an elementary and approachable proof that applies generally to Akra-Bazzi-style recurrences.

1 Introduction

The ***master theorem*** [7] gives solutions to recurrences of the form $T(n) = aT(n/b)+f(n)$. Over the past thirty years, the master theorem has played an important role in undergraduate algorithms education. It has proven useful both for teaching recurrences and for introducing students to formal proof techniques. The master theorem is also widely used within research papers, allowing for researchers to analyze algorithm running times without needing to perform *ad hoc* analyses.

Most applications of the master theorem involve recursions with floors and ceilings, but people tend to ignore them, because the floors and ceilings really don't make a difference in the asymptotic solution. For example, the running time of merge sort is often expressed as $T(n) = T(n/2) + T(n/2) + \Theta(n)$, whereas the actual recurrence is $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n)$. Indeed, statements of the master theorem tend to include the claim that the theorem holds in the presence of floors and ceilings.

To distinguish the two situations, we call the master theorem without floors and ceilings the ***continuous master theorem***[1] and the master theorem with floors and ceilings the ***discrete master theorem***. When we speak only of the master theorem, we mean the discrete master theorem, but we usually include the term "discrete" in this paper for clarity in distinguishing the two cases.

Several academic works provide proofs and proof sketches of the discrete master theorem. To the best of our knowledge, however, all of these proofs are either incomplete, incorrect, or require sophisticated mathematics.

Contributions. We present an elementary proof of the discrete master theorem assuming the correctness of the continuous master theorem. More generally, we prove that for a large family of recursively defined functions (in particular, Akra-Bazzi-style recurrences [2, 9]), ceilings and floors within the recursive subproblem sizes do not affect the asymptotic growth of the function. The proof is designed both to be formal and to be elementary enough (no calculus, and we try not to skip any steps in our arguments) that it can be appreciated by a mathematically inclined undergraduate. Achieving this combination requires careful and involved arguments (specifically, as we shall see, in the structure and and interplay between Lemmas 3.2 and 3.3).

Prior work. Aho, Hopcroft, and Ullman [1, Theorem 2.1] offered one of the first general treatments of divide-and-conquer recurrences for computer science, giving three cases for recurrences of the form $T(n) = aT(n/b) + cn$. Bentley, Haken, and Saxe [3] introduced the master theorem in modern form, although they only

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[1]This terminology is not meant to convey the implication that either $T(n)$ or $f(n)$ need be continuous, only that their domain is real numbers. But since the term "real master theorem" has an ambiguous meaning, we felt that the abuse of the word "continuous" was justified.

proved the theorem for exact powers of b. Cormen, Leiserson, and Rivest [5, Section 4.3] presented the discrete master theorem, extending Bentley, Haken, and Saxe's earlier treatment to include floors and ceilings, but their proof is at best a sketch, not a rigorous argument, and it leaves key issues unaddressed. These problems have persisted through two subsequent editions [6, 7] with the additional coauthor Stein.

Subsequent work on the master theorem has either focused on generalizations or involved elaborate correctness arguments. Akra and Bazzi [2] give a useful extension to the master theorem to analyze divide-and-conquer style recurrences of the more general form $T(n) = \sum_{i=1}^{k} a_i T(n/b_i) + f(n)$. In unpublished notes, Leighton [9] offers a simplified proof of the Akra-Bazzi result and extends the Akra-Bazzi theorem to apply to a large family of discrete recurrences that includes those with floors and ceilings. Campbell [4], however, points out several technical errors in Leighton's notes and devotes over 300 pages to carefully correct the issues, which demonstrates how difficult and subtle the issues with divide-and-conquer recurrences can be. Drmota and Szpankowski [8] prove a generalization of the discrete master theorem. Their asymptotic analysis combines a wide array of impressive techniques, relying both on Dirichlet series and on machinery from previous work (Tauberian theorems and the Perron-Mellin formula). Roura [11] gives an elegant generalization that includes the discrete case, but the treatment is likewise quite sophisticated. Other generalizations of the master theorem have simply not considered (or only remarked upon) the impact of floors and ceilings [12, 10]. In unpublished work, Yap [13] separates the problem of solving recurrences from the problem of handling floors and ceilings (the approach we follow). He elegantly employs real induction to give a proof of the continuous master theorem, but his argument for the discrete master theorem contains an unfortunate error,[2] resulting in a flawed proof.

In this paper, we follow Yap's lead in separating the problem of solving recurrences from the problem of handling floors and ceilings. We give an elementary formal proof that floors and ceilings have no asymptotic effect on the master theorem and Akra-Bazzi-style recurrences. In other words, solutions to the continuous versions of these recurrences immediately imply asymptotic solutions to the discrete versions.

[2] Using Yap's notation, on page 23 of [13], Yap argues that, by real induction, $G(n, T'(b'_1(n)), \ldots, T'(b'_k(n)))$ is at most $G(n, CT(b'_1(n)), \ldots, CT(b'_k(n)))$. This reasoning is unsound, however, because the inductive hypothesis is that $T'(n) \le CT(n+\Delta(n))$, rather than that $T'(n) \le CT(n)$. Yap's misstep seems to be an artifact of the fact that the latter inductive hypothesis was used (in a separate inductive argument) in its final application of induction on the previous page. Yap's conference version of the paper [14] omits the errant section on floors and ceilings.

2 Preliminaries

We begin with some basic mathematical definitions. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers, $\mathbb{Z}^+ = \{1, 2, \ldots\}$ denote the positive integers, and $\mathbb{R}$ denote the reals.

Recursive functions. This paper considers two classes of recursive functions. The first is ***continuous recurrences*** of the form

$$T(n) = f(n) + \sum_{i=1}^{t} a_i T(n/b_i) \ , \qquad (2.1)$$

where $t \in \mathbb{Z}^+$; the constants $a_1, a_2, \ldots, a_t \in \mathbb{R}$ are strictly positive; the constants $b_1, b_2, \ldots, b_t \in \mathbb{R}$ are strictly greater than 1; and $f(n)$ is a nonnegative function defined on the nonnegative reals. The second is ***discrete recurrences***, which are defined like continuous recurrences, except that each term $T(n/b_i)$ in recurrence (2.1) is replaced with either $T(\lceil n/b_i \rceil)$ or $T(\lfloor n/b_i \rfloor)$. Formally, for discrete recurrences, we partition $\{1, 2, \ldots, t\}$ into two sets S and $\overline{S}$, and consider recurrences of the form

$$T(n) = f(n) + \sum_{i \in S} a_i T(\lfloor n/b_i \rfloor) + \sum_{i \in \overline{S}} a_i T(\lceil n/b_i \rceil) \ .$$

Whereas continuous recurrences are defined over the nonnegative reals, discrete recurrences are defined on $n \in \mathbb{N}$ — meaning that both $T(n)$ and $f(n)$ need only be defined on $n \in \mathbb{N}$ — although we'll generally assume that $f(n)$ is defined on the nonnegative reals.

Formally, we say that a function $T : \mathbb{N} \to \mathbb{R}$ ***satisfies*** a recurrence (such as recurrence (2.1)) if there exists a constant $\widehat{n} > 0$ such that for all $n \ge \widehat{n}$, the recurrence holds. Since the recurrence (2.1) need hold only for $n \ge \widehat{n}$, our results apply when the function $f(n)$ is defined only for $n \ge \widehat{n}$. For convenience, we shall treat the domain of $f(n)$ as being nonnegative reals, although the value of $f(n)$ for $n < \widehat{n}$ is irrelevant, and the value need not even be defined.

Base-case restriction. Whenever discussing a recursive function $T(n)$, we shall also place the following ***base-case restriction*** on $T(n)$. We assume that for any $\widehat{n} > 0$, there exist positive constants c_1, c_2 such that $c_1 \le T(n) \le c_2$ for all $n \le \widehat{n}$. In other words, $T(n)$ always evaluates to a positive constant in any base case. For discrete recurrences, the base-case restriction is equivalent to $T(n)$ being a positive-valued function, because $c_1 = \min\{T(i) \mid 0 \le i \le \widehat{n}\}$ and $\max\{T(i) \mid 0 \le i \le \widehat{n}\}$ are both positive. The same is true for continuous recurrences in which $T(n)$ is a continuous function over $\mathbb{R}$, because positive-valued continuous functions over closed intervals are guaranteed to achieve a positive infimum c_1 and a finite supremum c_2.

Polynomial-growth condition. When studying recurrences, it will be helpful to insist that the function $f(n)$ in (2.1) exhibit well-behaved growth. A function $f(n)$ defined on the nonnegative reals satisfies the ***polynomial-growth condition*** if there exists a constant $\widehat{n} > 0$ such that the following holds: for every constant $\phi \geq 1$, there exists a constant $d > 1$ (depending on ϕ) such that

$$d^{-1} f(n) \leq f(\psi n) \leq d f(n)$$

for all $1 \leq \psi \leq \phi$ and $n \geq \widehat{n}$. Leighton [9] introduced a similar notion, which Campbell [4] later refined to correct several technical issues in Leighton's treatment of recurrences. Our definition follows that of Campbell. When we use the polynomial-growth condition, we shall say so explicitly, but there is another condition that we shall always assume, namely, that $f(n)$ has a finite upper bound on any finite interval.

Asymptotic notation. There are many (not quite equivalent) definitions of asymptotic notations, none of which have a dramatic impact on the results of this paper. We follow the following conventions. We say that $f(n) = O(g(n))$ if there exist constants $\widehat{n} \geq 0$ and $c > 0$ such that $f(n) \leq c g(n)$ for all $n \geq \widehat{n}$. We say that $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$. And we say $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

Other well-definedness properties. We generally assume that the recursive function given is well defined. In addition, we assume properties, such as that the recursion has finite depth, without proof, because the proofs are straightforward induction. Campbell [4] provides a careful treatment of well-definedness issues in divide-and-conquer recurrences.

3 Analyzing discrete recurrences

This section carefully establishes that, for a large family of recursively defined functions, ceilings and floors applied to recursive subproblem sizes do not affect the asymptotic growth of the function. We shall use this result in Section 4 to show that for the master theorem from [7, Sec. 4.5], floors and ceilings do not affect asymptotic growth.

Consider a continuous recurrence $T(n)$ of the form (2.1) and a discrete recurrence $T'(n)$ of the same form, except that we replace each term $T(n/b_i)$ in (2.1) with either $T(\lceil n/b_i \rceil)$ or $T(\lfloor n/b_i \rfloor)$. The main theorem in this section states that $T'(n) = \Theta(T(n))$. In other words, if you drop the floors and ceilings in $T'(n)$ to obtain $T(n)$, then the asymptotic solution to $T(n)$ is also a solution to $T'(n)$.

In order to analyze recurrences with ceilings and floors, we begin by showing that a certain class of products always evaluates to a constant, which will be useful for later lemmas.

Lemma 3.1. *Let $\beta > 1$ be a constant, and let $n \in \mathbb{Z}^+$. Define*

$$L = \prod_{i=1}^{n} \left(1 - \frac{1}{\beta^i + 1}\right)^2 ,$$

and

$$U = \prod_{i=1}^{n} \left(1 + \frac{1}{\beta^i - 1}\right)^2 .$$

Then the asymptotic bounds $L = \Omega(1)$ and $U = O(1)$ hold.

Proof. Since $\beta > 1$ implies that $1/\beta^i < 1/(\beta^1 - 1)$, we have

$$\begin{aligned} 1/L &= \prod_{i=1}^{n} \left(1 + \frac{1}{\beta^i}\right)^2 \\ &< \prod_{i=1}^{n} \left(1 + \frac{1}{\beta^i - 1}\right)^2 \\ &= U . \end{aligned}$$

Thus, it suffices to prove only that $U = O(1)$. Using the fact that $1 + 1/x \leq e^{1/x}$ for $x \neq 0$, we can conclude that

$$\begin{aligned} U &= \prod_{i=1}^{n} \left(1 + \frac{1}{\beta^i - 1}\right)^2 \\ &\leq \prod_{i=1}^{\infty} \left(1 + \frac{1}{\beta^i - 1}\right)^2 \\ &\leq \left(\prod_{i=1}^{\infty} e^{1/(\beta^i - 1)}\right)^2 \\ &= \exp\left(\sum_{i=1}^{\infty} \frac{2}{\beta^i - 1}\right) \\ &= \exp(O(1)) \\ &= O(1) , \end{aligned}$$

as desired. ∎

The next lemma considers a sequence of problem sizes $n_0 > n_1 > n_2 > \cdots > n_k$, where each problem size n_i is obtained by dividing the problem size n_{i-1} by some constant $\beta_i > 1$ and then possibly performing a floor or ceiling. Lemma 3.2 shows that (as long as n_0 is sufficiently large) the net effect of the floors and ceilings on $n_1, n_2, \ldots, n_k$ is only a constant factor, that is, $n_k = \Theta\left(n_0/\prod_{i=1}^k \beta_i\right)$. To simplify the proof of Lemma 3.2, we add an additional condition that the n_i's are each at least some constant (a function of the β_i constants). We shall see in Lemma 3.3 that this assumption can be removed in exchange for simply assuming that n_0 is sufficiently large.

LEMMA 3.2. *Let $\beta > 1$ be a real constant, and for $i = 1, 2, \ldots, k$, let $\beta_i \geq \beta$ be a real constant. Define $B = \prod_{i=1}^k \beta_i$. Then there exists a constant $c \geq 1$, depending only on β, such that the following property holds. For any real numbers $n_0, n_1, \ldots, n_k$ satisfying the constraints that $n_i > \max\{\beta, 1 + 1/(\sqrt{\beta} - 1)\}$ and $\lfloor n_{i-1}/\beta_i \rfloor \leq n_i \leq \lceil n_{i-1}/\beta_i \rceil$, we have*

$$c^{-1/4}(n_0/B) \leq n_k \leq c^{1/4}(n_0/B) \ . \tag{3.2}$$

Proof. For $i = 1, 2, \ldots, k$, define the ***rounding factor***

$$\rho_i = \frac{n_i}{n_{i-1}/\beta_i} \ . \tag{3.3}$$

The rounding factors are defined so that

$$\begin{aligned}
(n_0/B)\prod_{i=1}^k \rho_i &= \frac{n_0 \prod_{i=1}^k \rho_i}{\prod_{i=1}^k \beta_i} \\
&= n_0 \prod_{i=1}^k \frac{\rho_i}{\beta_i} \\
&= n_0 \prod_{i=1}^k \frac{n_i}{n_{i-1}} \\
&= n_k \ ,
\end{aligned}$$

since the product telescopes. In order to establish inequality (3.2), it therefore suffices to show that

$$c^{-1/4} \leq \prod_{i=1}^k \rho_i \leq c^{1/4} \tag{3.4}$$

for some constant $c \geq 1$ (depending only on β).

In order to prove inequality (3.4), we shall employ a simple bound on the ρ_i's. Recall that $\lfloor n_{i-1}/\beta_i \rfloor \leq n_i \leq \lceil n_{i-1}/\beta_i \rceil$, which implies that

$$n_i - 1 \leq n_{i-1}/\beta_i \leq n_i + 1 \ ,$$

since $n_{i-1}/\beta_i - 1 \leq \lfloor n_{i-1}/\beta_i \rfloor$ and $\lceil n_{i-1}/\beta_i \rceil \leq n_{i-1}/\beta_i + 1$. Substituting these bounds for n_{i-1}/β_i into the definition (3.3) of the rounding factors yields

$$\frac{n_i}{n_i + 1} \leq \rho_i \leq \frac{n_i}{n_i - 1} \ ,$$

or equivalently,

$$1 - \frac{1}{n_i + 1} \leq \rho_i \leq 1 + \frac{1}{n_i - 1} \ . \tag{3.5}$$

In the remainder of the proof, we shall establish inequality (3.4) by using inequality (3.5) as a building block.

Partition the half-open interval $(\beta, n_0]$ into regions $R_1, R_2, \ldots, R_{\lceil \log_\beta n_0 \rceil}$, where each $R_j = (\beta^j, \beta^{j+1}] \cap (\beta, n_0]$ (recall that $\beta < n_0$ by assumption). We now show that each region R_j contains at most two n_i's. By the assumption that each n_i satisfies $n_i > 1+1/(\sqrt{\beta}-1)$, the second inequality of (3.5) tells us that

$$\begin{aligned}
\rho_i &\leq 1 + \frac{1}{n_i - 1} \\
&\leq 1 + \frac{1}{1/(\sqrt{\beta} - 1)} \\
&= \sqrt{\beta} \ .
\end{aligned} \tag{3.6}$$

By the definition (3.3) of the rounding factors, we have that for $i = 0, 1, \ldots, k-2$,

$$n_{i+2} = \frac{n_i \rho_{i+1} \rho_{i+2}}{\beta_{i+1} \beta_{i+2}}.$$

By inequality (3.6), it follows that $n_{i+2} \leq n_i/\beta$, and hence n_i and n_{i+2} cannot fall into the same partitioned region. Consequently, each region contains at most two n_i's, and additionally, every n_i falls into some region, since $n_i > \beta$ for all i.

For each $n_i \in R_j$, since $n_i > \beta^i$, inequality (3.5) tells us that

$$1 - \frac{1}{\beta^j + 1} \leq \rho_i \leq 1 + \frac{1}{\beta^j - 1} \ . \tag{3.7}$$

Since each R_j contains at most two n_i's, it follows from the second inequality in (3.7) that

$$\begin{aligned}
\prod_{i=1}^k \rho_i &= \prod_{j=1}^{\lceil \log_\beta n_0 \rceil} \left(\prod_{n_i \in R_j} \rho_i \right) \\
&\leq \prod_{j=1}^{\lceil \log_\beta n_0 \rceil} \left(1 + \frac{1}{\beta^j - 1} \right)^2 .
\end{aligned}$$

Lemma 3.1 therefore provides that $\prod_{i=1}^k \rho_i \leq c^{1/4}$ for some constant c depending only on β, which establishes the second inequality in (3.4). Similarly, since each R_j contains at most two n_i's, it follows from the first inequality in (3.7) that

$$\begin{aligned}\prod_{i=1}^{k}\rho_i &= \prod_{j=1}^{\lceil\log_\beta n_0\rceil}\left(\prod_{n_i\in R_j}\rho_i\right)\\ &\geq \prod_{j=1}^{\lceil\log_\beta n_0\rceil}\left(1-\frac{1}{\beta^j+1}\right)^2 .\end{aligned}$$

By Lemma 3.1, we thus have that $\prod_{i=1}^{k}\rho_i \geq c^{-1/4}$ for some constant c depending only on β, which establishes the first inequality in (3.4), completing the proof. ∎

We now remove from Lemma 3.2 the assumption that $n_i > \max\left\{\beta, 1/(\sqrt{\beta}-1)+1\right\}$. In particular, if c is a sufficiently large constant as a function of $\min_i \beta_i$ and $\max_i \beta_i$, and if n_0 is assumed to be at least cB, then the assumption that $n_0,\ldots,n_k > \max\left\{\beta, 1/(\sqrt{\beta}-1)+1\right\}$ becomes redundant with the other assumptions in Lemma 3.2. (But, as we shall see, this redundancy is only easy to uncover with the help of Lemma 3.2!) Consequently, we can obtain the following lemma.

Lemma 3.3. *Let $\beta_{min}, \beta_{max} > 1$ be real constants, and for $i = 1,2,\ldots,k$ let $\beta_{min} \leq \beta_i \leq \beta_{max}$ be a real constant. Define $B = \prod_{i=1}^{k}\beta_i$. Then there exists a constant $c \geq 1$, depending only on β_{min} and β_{max}, such that the following property holds. For any integers $n_0, n_1, \ldots, n_k$ with $n_0 \geq cB$ and satisfying $\lfloor n_{i-1}/\beta_i\rfloor \leq n_i \leq \lceil n_{i-1}/\beta_i\rceil$ for $i = 1,2,\ldots,k$, we have*

$$c^{-1}(n_0/B) \leq n_k \leq c(n_0/B) \ . \tag{3.8}$$

Proof. Let c be the constant from Lemma 3.2 in the case where $\beta = \beta_{\min}$. (Note that c is a function of $\beta_{\min}$.) Assume without loss of generality that c is a sufficiently large constant so that

$$\sqrt{c} > \max\left\{\frac{1}{\sqrt{\beta_{\min}}-1}+1, \beta_{\min}\right\}, \tag{3.9}$$

and for $i = 1,2,\ldots,k$, that

$$c^{1/4} > 2\beta_i \ . \tag{3.10}$$

As required, the constant c depends only on $\beta_{\min}$ (due to Lemma 3.2 and inequality (3.9)) and on $\beta_{\max}$ (due to inequality (3.10)).

If $n_j \geq \sqrt{c}$ for all j, then by inequality (3.9), the lemma follows from Lemma 3.2. To complete the proof of the lemma, we show that the case $n_j \leq \sqrt{c}$ cannot occur for any j.

Assume for contradiction that $n_j < \sqrt{c}$ for some j, and let j be the smallest such value. (We have that $j \geq 1$, because $n_0 \geq cB > \sqrt{c}$, where the second inequality follows from $c \geq 1$ and $B > 1$.) We can apply Lemma 3.2 to $\beta_1, \beta_2, \ldots, \beta_{j-1}$ and $n_0, n_1, \ldots, n_{j-1}$ and observe that the conditions for the lemma are met:

- $\beta_1, \ldots, \beta_{j-1} \geq \beta$ because $\beta = \min_{i=1}^{k}\beta_i$;
- $n_{j-1} \geq \max\{1/(\sqrt{\beta}-1)+1, \beta\}$ because $n_{j-1} > \sqrt{c}$ and by inequality (3.9);
- $\lfloor n_{i-1}/\beta_i\rfloor \leq n_i \leq \lceil n_{i-1}/\beta_i\rceil$ for $i = 1,2,\ldots,j-1$ by assumption.

If $j-1=0$, we have trivially that $n_0 \geq c^{-1/4}(n_0/B)$, and if $j-1>0$, we have $n_{j-1} \geq c^{-1/4}(n_0/B)$ by Lemma 3.2. Since $n_0 \geq cB$, it follows that $n_{j-1} \geq c^{3/4}$ which is assumed to be a sufficiently large constant. We therefore have that

$$\begin{aligned}n_j &\geq \lfloor n_{j-1}/\beta_j\rfloor\\ &\geq n_{j-1}/(2\beta_j)\\ &> n_{j-1}/c^{1/4} \qquad \text{(by inequality (3.10))}\\ &\geq \sqrt{c}\ ,\end{aligned}$$

contradicting the choice of j. ∎

We now prove the main result of the section.

Theorem 3.4. *Let $a_1, \ldots, a_t > 0$ and $b_1, \ldots, b_t > 1$ be constants, and let $f(n)$ be a nonnegative function defined on the nonnegative reals that satisfies the polynomial-growth condition. Consider a function $T(n)$ defined on the nonnegative reals satisfying the recurrence*

$$T(n) = f(n) + \sum_{i=1}^{t} a_i T(n/b_i) \ . \tag{3.11}$$

For any second function $T'(n)$ defined on $\mathbb{N}$ satisfying recurrence (3.11), except that each $T(n/b_i)$ is replaced either with $T(\lceil n/b_i\rceil)$ or with $T(\lfloor n/b_i\rfloor)$, we have $T'(n) = \Theta(T(n))$.

Proof. The proof compares the contributions of individual recursive subproblems to each of $T(n)$ and $T'(n)$ and then uses Lemma 3.3 to show that each subproblem contributes the same amount, up to constant factors, to $T'(n)$ as it contributes to $T(n)$.

Let c be the constant from Lemma 3.3 in the case where $\min_i \beta_i = \min_{i=1}^{t} b_i$ and $\max_i \beta_i = \max_{i=1}^{t} b_i$. Recall that the constant c in Lemma 3.3 is a function of

$x = \min_i \beta_i$ and $y = \max_i \beta_i$ and that c is monotonically decreasing in x and monotonically increasing in y.

Let $\widehat{n}$ be a sufficiently large constant. (Later, we shall select $\widehat{n}$ to ensure that we only ever evaluate $T(n)$ and $T'(n)$ on values of n that satisfy their respective recurrences, as well as to ensure that we can use the polynomial-growth condition for $f(n)$). Let $p = \max\{\widehat{n},\, c \cdot \max_{i=1}^{t} b_i\}$. Suppose that we evaluate $T(n)$ using $n \le p$ as a base case. That is, we use the following recursive process to evaluate $T(n)$:

- If $n > p$, then we recursively evaluate each of $T(n/b_1), T(n/b_2), \ldots, T(n/b_t)$.[3] Then we return $f(n) + \sum_{i=1}^{t} a_i T(n/b_i)$.[4]
- If $n \le p$, then we simply return $T(n)$.

Let $\mathcal{S}$ denote the set of recursive subproblems in this process. A subproblem $q \in \mathcal{S}$ at depth $j \ge 0$ can be represented by a j-tuple $q = \langle q(1), q(2), \ldots, q(j)\rangle$, where $q(i) \in \{1, 2, \ldots, t\}$ indicates which branch of recursion is taken between the $(i-1)$th and ith levels of recursion. The empty tuple $\langle\rangle$ represents the original depth-0 subproblem, which we shall also refer to as the ***root***. We denote the number j of entries in the tuple q by $|q|$. The ***size*** n_q of a subproblem $q \in \mathcal{S}$ is the number n_q for which q is evaluating $T(n_q)$ — that is,

$$n_q = n / \prod_{i=1}^{j} b_{q(i)} \ .$$

Let $\mathcal{S}_0 = \{q \in \mathcal{S} \mid n_q \le p\}$ be the base-case subproblems in $\mathcal{S}$, and let $\mathcal{S}_1 = \mathcal{S} \setminus \mathcal{S}_0$ denote the non-base-case subproblems. Summing over all subproblems, we have

$$\begin{aligned} T(n) &= \sum_{q\in\mathcal{S}_1} f(n_q) \prod_{i=1}^{|q|} a_{q(i)} \\ &\quad + \sum_{q\in\mathcal{S}_0} T(n_q) \prod_{i=1}^{|q|} a_{q(i)} \ , \end{aligned} \tag{3.12}$$

because each $q \in \mathcal{S}_1$ contributes $f(n_q)\prod_{i=1}^{|q|} a_{q(i)}$ and each $q \in \mathcal{S}_0$ contributes $T(n_q)\prod_{i=1}^{|q|} a_{q(i)}$, where in both cases $\prod_{i=1}^{|q|} a_{q(i)}$ is the product of the multiplicative weights acquired by recursing from the root subproblem to the subproblem q.

[3] Even if $b_i = b_j$ for $i \ne j$, we still evaluate $T(n/b_i)$ and $T(n/b_j)$ separately.

[4] Since $n \ge \widehat{n}$ for a sufficiently large constant $\widehat{n}$, we know that $T(n)$ satisfies recurrence (3.11).

Since $n_q \le p = O(1)$ for each $q \in \mathcal{S}_0$, it follows that $T(n_q) = O(1)$, and thus,

$$\begin{aligned} T(n) &= \sum_{q\in\mathcal{S}_1} f(n_q) \prod_{i=1}^{|q|} a_{q(i)} \\ &\quad + \Theta\left(\sum_{q\in\mathcal{S}_0} \prod_{i=1}^{|q|} a_{q(i)}\right) \ . \end{aligned} \tag{3.13}$$

We now describe how to evaluate $T'(n)$ using the same recursive layout (i.e., the same set $\mathcal{S}$ of recursive subproblems). When computing $T'(n)$, the empty subproblem tuple $q = \langle\rangle$ corresponds to a subproblem of size $n'_q = n$. For each nonroot subproblem $q = \langle q(1), q(2), \ldots, q(j)\rangle$, if $r = \langle q(1), q(2), \ldots, q(j-1)\rangle$ is q's recursive parent, then the subproblem size n'_q satisfies

$$\left\lfloor \frac{n'_r}{b_{q(j)}} \right\rfloor \le n'_q \le \left\lceil \frac{n'_r}{b_{q(j)}} \right\rceil \ ,$$

by the definition of $T'(n)$. Assume for the moment that all of the subproblem sizes n'_q are sufficiently large that $T'(n'_q)$ satisfies its recursive definition (which we shall prove shortly using the fact that $p \ge \widehat{n}$, where $\widehat{n}$ is a sufficiently large constant).

In the same way that equation (3.12) expands $T(n)$, we can expand $T'(n)$ in terms of its subproblem sizes as

$$\begin{aligned} T'(n) &= \sum_{q\in\mathcal{S}_1} f(n'_q) \prod_{i=1}^{|q|} a_{q(i)} \\ &\quad + \sum_{q\in\mathcal{S}_0} T'(n'_q) \prod_{i=1}^{|q|} a_{q(i)} \ . \end{aligned} \tag{3.14}$$

In the evaluation of $T'(n)$ given by equation (3.14), note that the set of base-case subproblems $\mathcal{S}_0$ continues to be determined by the recursive process for computing $T(n)$ (not $T'(n)$). That is, a subproblem q is a base case if $n_q \le p$, which does not necessarily imply that $n'_q \le p$. The fact that we use the same subproblem set $\mathcal{S}$ in both equations (3.13) and (3.14) now allow for us to compare the two equations term by term.

Since $n_q > p$ for all non-base-case subproblems $q \in S_1$, it follows that $n_q > p/\max_i b_i \ge c$ for all $q \in \mathcal{S}$. Lemma 3.3 (with the $b_{q(1)}, \ldots, b_{q(j)}$ corresponding to the $\beta_1, \ldots, \beta_t$, $\beta_{\min} = \min_i b_i$, and $\beta_{\max} = \max_i b_i$) therefore tells us that $n'_q = \Theta(n_q)$ for all $q \in \mathcal{S}$. That is, there exists a constant $\phi > 1$ such that $n'_q \in [\phi^{-1} n_q, \phi n_q]$. Since $n'_q \ge n_q/\phi \ge \widehat{n}/\phi$ and $\widehat{n}$ is taken to be a sufficiently large constant, we are guaranteed that $T'(n'_q)$ satisfies its recursive definition on all subproblem sizes n'_q (which is necessary for the correctness of (3.14)). We also have $f(n'_q) = \Theta(f(n_q))$ for all $q \in \mathcal{S}$, which follows from the polynomial-growth condition for $f(n)$ and the bound $n'_q \in [\phi^{-1} n_q, \phi n_q]$, because $n_q \geq \widehat{n}$ for a sufficiently large constant $\widehat{n}$. For $q \in \mathcal{S}_0$, we can further conclude that $n'_q = \Theta(1)$ and thus that $T'(n'_q) = \Theta(1)$ by the base-case restriction on recursive functions. Substituting these asymptotic bounds into equation (3.14) yields

$$T'(n) = \sum_{q \in \mathcal{S}_1} \Theta(f(n_q)) \prod_{i=1}^{|q|} a_{q(i)} + \Theta\left(\sum_{q \in \mathcal{S}_0} \prod_{i=1}^{|q|} a_{q(i)} \right) . \tag{3.15}$$

Comparing equations (3.13) and (3.15), we conclude that $T'(n) = \Theta(T(n))$, as desired. ∎

We conclude this section with two corollaries.

Corollary 3.5. *Theorem 3.4 continues to hold even if floors and ceilings in the recursive computation of $T'(n)$ are determined in "real time," meaning that two recursive subproblems that are on the same subproblem size m may choose to use different configurations of floors and ceilings.*

Proof. The proof of Theorem 3.4 does not require that the ceilings and floors within the recursion be the same in each recursion instance. That is, for each subproblem instantiated during the recursion, floors and ceilings can be applied arbitrarily and optionally. ∎

For example, for the recurrence $T(n) = T(n/2) + T(n/3) + f(n)$, Corollary 3.5 says that the solution is the same as if in any given subproblem instance during the recursion, $n/2$ and $n/3$ are sometimes rounded up to integers, sometimes rounded down to integers, a mix of up and down, or even sometimes left alone.

Corollary 3.6. *Theorem 3.4 continues to hold even if $T'(n)$ is redefined as follows: $T'(n)$ is any function satisfying recurrence (3.11), except that each $T(n/b_i)$ is replaced with $T(n/b_i + c_i)$ for some constant $c_i \in [-O(1), O(1)]$.*

Proof. The proof of Theorem 3.4 uses only one fact about ceilings and floors: that $\lceil m \rceil \leq m + 1$ and $\lfloor m \rfloor \geq m - 1$ for all m. Thus, the theorem continues to hold if $T'(n)$ is defined so that each $T(n/b_i)$ in the recursion is replaced with $T(n/b_i + c_i)$ for some constant $c_i \in [-1, 1]$. If we define $T''(n) = T'(rn)$ for a positive constant r, then by applying this modified version of the theorem to $T''(n)$, we arrive at the desired corollary: the theorem continues to hold if $T'(n)$ is defined so that each $T(n/b_i)$ in recurrence (3.11) is replaced with $T(n/b_i + c_i)$ for some constant $c_i \in [-r, r]$. ∎

Leighton [9] generalizes Corollary 3.6 even further to perturbing each argument by a function $c_i(n)$, where $c_i(n) \leq n/\log^{1+\varepsilon} n$ for some constant $\varepsilon > 0$. For a detailed proof, see [4].

4 Master theorem with floors and ceilings

In this section, we use Theorem 3.4 to prove the discrete master theorem from the continuous version, whose proof we shall assume. We restate a continuous version of the master theorem from [7], expanding Case 2 along the lines of Exercise 4.6-2. We also weaken the requirement from [7] that $a \geq 1$ and adopt Yap's observation [13] that we can allow $a > 0$.

Theorem 4.1. (Continuous master theorem) *Let $a > 0$ and $b > 1$ be constants, let $f(n)$ be a nonnegative function defined on the positive reals, and let $T(n)$ be defined on the nonnegative reals to satisfy the recurrence*

$$T(n) = aT(n/b) + f(n) .$$

Then $T(n)$ obeys the following asymptotic bounds:

1. *If we have $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.*
2. *If we have $f(n) = \Theta(n^{\log_b a} \lg^k n)$, where k is a nonnegative constant, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.*
3. *If we have $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$ — and $f(n)$ additionally satisfies the regularity condition $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n — then $T(n) = \Theta(f(n))$.*

To prove the discrete master theorem, we shall need the following lemma, which will allow us to invoke Theorem 3.4 during the proof.

Lemma 4.2. *Suppose that a function $f(n) = \Theta(n^c \lg^k n)$, where $c, k \geq 0$ are constants, is defined on the nonnegative reals. Then $f(n)$ satisfies the polynomial-growth condition.*

Proof. Since $f(n) = \Theta(n^c \lg^k n)$, there must exist positive constants r_1, r_2 such that, for all $n > r_1$, we have

$$r_2^{-1} n^c \lg^k n \leq f(n) \leq r_2 n^c \lg^k n . \tag{4.16}$$

To prove the polynomial-growth condition for $f(n)$, we show that for every $\phi > 1$, there exists a constant

$d > 1$ such that $d^{-1}f(n) \le f(\psi n) \le df(n)$ for all $\psi \in [1, \phi]$ and $n > r_1$. Consider $\psi \in [1, \phi]$ and $n > r_1$. By (4.16), we have

$$r_2^{-1}(\psi n)^c \lg^k(\psi n) \le f(\psi n) \le r_2(\psi n)^c \lg^k(\psi n) \ . \tag{4.17}$$

Observe that

$$\begin{aligned}(\psi n)^c \lg^k(\psi n) &= \psi^c n^c \lg^k(\psi n) \\ &= \psi^c n^c (\lg n + \lg \psi)^k, \\ &= \psi^c \left(1 + \frac{\lg \psi}{\lg n}\right)^k n^c \lg^k n,\end{aligned}$$

which lies between $n^c \lg^k n$ and $r_3 n^c \lg^k n$ for $r_3 = \phi^c(1 + \lg \phi)^k$. It follows by (4.17) that $d^{-1}f(n) \le f(\psi n) \le df(n)$ holds for $d = r_2 r_3$. □

We now prove that the master theorem holds for recursions involving floors and ceilings.

Theorem 4.3. (Discrete master theorem) *Let $a > 0$ and $b > 1$ be constants, let $f(n)$ be a nonnegative function defined on the positive reals, and let $T(n)$ be defined on $\mathbb{N}$ to satisfy the recurrence $T(n) = a_1 T(\lfloor n/b \rfloor) + a_2 T(\lceil n/b \rceil) + f(n)$ for nonnegative constants $a_1 + a_2 = a$.*

1. *If we have $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.*
2. *If we have $f(n) = \Theta(n^{\log_b a} \lg^k n)$, where k is a nonnegative constant, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.*
3. *If we have $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$ — and $f(n)$ additionally satisfies the regularity condition $a_1 f(\lfloor n/b \rfloor) + a_2 f(\lceil n/b \rceil) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n — then $T(n) = \Theta(f(n))$.*

Proof. In order to apply Theorem 3.4 to the continuous master theorem, the key hurdle is meeting the requirement that $f(n)$ satisfies the polynomial-growth condition.

We shall make use of the following straightforward fact, which can be proved by induction: replacing $f(n)$ with a "smaller" function $f'(n)$ satisfying $f'(n) \le f(n)$ for all n on which $f(n)$ is defined (respectively, a "larger" function $f'(n)$ satisfying $f'(n) \ge f(n)$ for all n on which $f(n)$ is defined) does not increase (respectively, decrease) $T(n)$.

We first prove the upper bound in Case 1 of the theorem. Suppose that $f(n) = O(n^c)$ for $c < \log_b a$. Then we have $f(n) \le rn^c$ for some $r > 0$ and all sufficiently large n, which in turn implies that $f(n) \le r(n^c + 1)$ for some (possibly different) constant $r > 0$ and all $n \in \mathbb{N}$. Lemma 4.2 implies that $f'(n) = r(n^c + 1)$ satisfies the constraints of Theorem 3.4. Since increasing $f(n)$ to equal $f'(n)$ cannot decrease $T(n)$, it follows that $T(n) = O(n^c)$ by Theorem 4.1.

For the lower bound of Case 1, observe that decreasing $f(n)$ to equal 0 for all n cannot increase $T(n)$. Since $f(n) = 0$ satisfies the polynomial-growth condition, we can apply Theorem 3.4 to Theorem 4.1 to deduce that $T(n) = \Omega(n^{\log_b a})$, which completes the proof of Case 1.

Case 2 of the theorem follows directly from Lemma 4.2, Theorem 3.4, and Case 2 of Theorem 4.1.

Finally, we prove Case 3 of the theorem. Theorem 3.4 does not apply to this case, since the function $f(n)$ may grow so fast that there exists a positive constant ψ for which $f(n) \ne \Theta(f(\psi n))$. Fortunately, Case 3 can be handled by a straightforward *ad hoc* argument. Since $T(n) \ge f(n)$ trivially, it suffices to show that $T(n) = O(f(n))$.

By assumption, there exists a constant $p > 0$ such that for all $n \ge p$, we have

$$a_1 f(\lfloor n/b \rfloor) + a_2 f(\lceil n/b \rceil) \le rf(n) \tag{4.18}$$

and the recurrence for $T(n)$ holds. Since p is a positive constant, there must exist a constant $s \ge 1$ such that for all $n \in \mathbb{Z}^+$ satisfying $n < p$, we have $T(n) \le sf(n)$.

We shall prove by induction that for $q = s/(1-r)$, we have

$$T(n) \le qf(n) \tag{4.19}$$

for all $n \in \mathbb{Z}^+$. We use $n \le p$ as an inductive base case, since $T(n) \le sf(n)$ for $n \le p$. Suppose that $n > p$, and suppose, as an inductive hypothesis, that inequality (4.19) holds for all smaller n. Then we have

$$\begin{aligned}T(n) &= a_1 T(\lfloor n/b \rfloor) + a_2 T(\lceil n/b \rceil) + f(n) \\ &\le a_1 q f(\lfloor n/b \rfloor) + a_2 q f(\lceil n/b \rceil) + f(n)\end{aligned}$$

by the inductive hypothesis. By inequality (4.18), it follows that

$$T(n) \le qrf(n) + f(n) \ .$$

Plugging in $q = s/(1-r)$ yields

$$\begin{aligned}T(n) &\le \left(\frac{sr}{1-r} + 1\right) f(n) \\ &= \left(\frac{sr + 1 - r}{1-r}\right) f(n) \\ &= \left(\frac{s - (1-r)s + 1 - r}{1-r}\right) f(n) \\ &\le \left(\frac{s}{1-r}\right) f(n) \\ &= qf(n) \ ,\end{aligned}$$

which establishes inequality (4.19) and completes the proof. □

5 Conclusion

This paper has possibly been more pedantic than some readers may require, but past proofs of the (discrete) master theorem have contained many imprecisions. There are many subtleties in the foundations of the argument. Our proof is elementary in that it does not rely on calculus and involves little mathematical sophistication beyond basic algebra, although the argument is involved. We have attempted to write the proofs at a level accessible to an advanced undergraduate.

We can suggest one straightforward avenue for future work. Leighton [9] describes a stronger variant of the master theorem in which each term $T(n/b_i)$ can be substituted with $T(n/b_i + h_i(n))$, where $|h_i(n)| \leq n/\lg^{1+\varepsilon} n$ for some constant $\varepsilon > 0$. The results in this paper achieve the same result when $h_i(n) = O(1)$. We believe that the proofs in this paper, hopefully with only little added complexity, can be generalized to handle larger perturbation functions $h_i(n)$.

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