Design and Analysis of Algorithms

National University of Singapore

CS3230

Lecture 11
NP-Completeness

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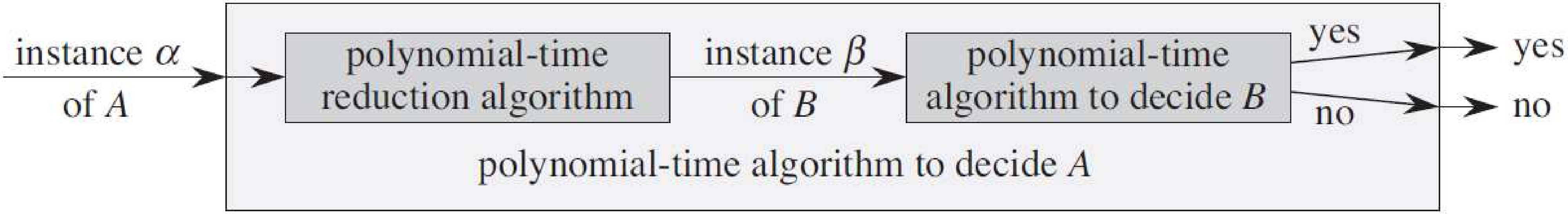
Reductions



Suffices to show:

- Reduction runs in polynomial time
- If α is a YES-instance of A, β is a YES-instance of B
- If β is a YES-instance of B, α is a YES-instance of A

3-SAT

1-SAT is trivial: a formula can be satisfied iff no literal occurs simultaneously with its negation

YES-instance: $(\overline{x_1}) \wedge (x_2) \wedge (x_3) \wedge (\overline{x_4})$

NO-instance: $(\overline{x_1}) \wedge (x_2) \wedge (x_3) \wedge (x_1)$

SAT where each clause contains exactly 3 literals (not necessarily distinct)

2-SAT can be solved in polynomial time (not required to know for this Module)

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Satisfying assignment: $x_1 = \text{True}, x_2 = \text{True}, x_3 = \text{False}, x_4 = \text{True}$

Non-satisfying assignment: $x_1 = \text{True}, x_2 = \text{False}, x_3 = \text{False}, x_4 = \text{False}$

Φ is a YES-instance if and only if it admits at least one satisfying assignment.

3-SAT $\leq_P$ INDEPENDENT-SET

Given an instance Φ of 3-SAT, goal is to construct an instance (G, k) of INDEPENDENT-SET so that G has an independent set of size k iff Φ is satisfiable.

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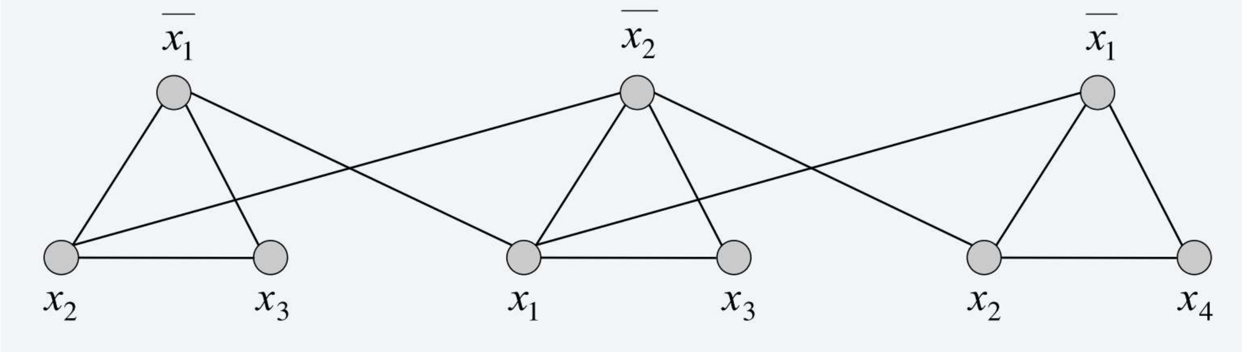
Reduction

- G contains 3 vertices for each clause, one for each literal
- Connect 3 literals in clause in a triangle
- Connect literal to each of its negations
- Set k = number of clauses



$$(\overline{x_1} \vee x_2 \vee x_3)$$
$$\wedge\ (x_1 \vee \overline{x_2} \vee x_3)$$
$$\wedge\ (\overline{x_1} \vee x_2 \vee x_4)$$

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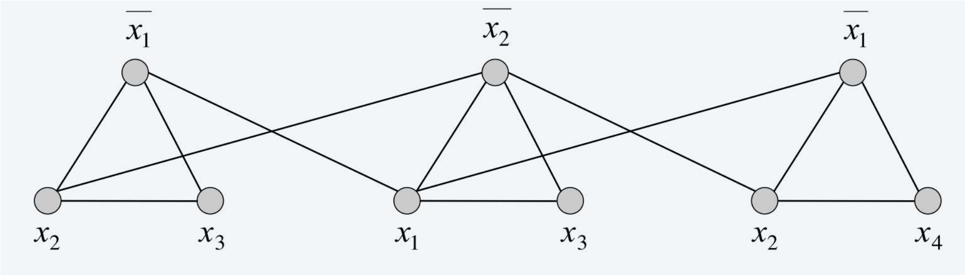

$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Reduction clearly runs in polynomial time.

3-SAT $\leq_P$ INDEPENDENT-SET

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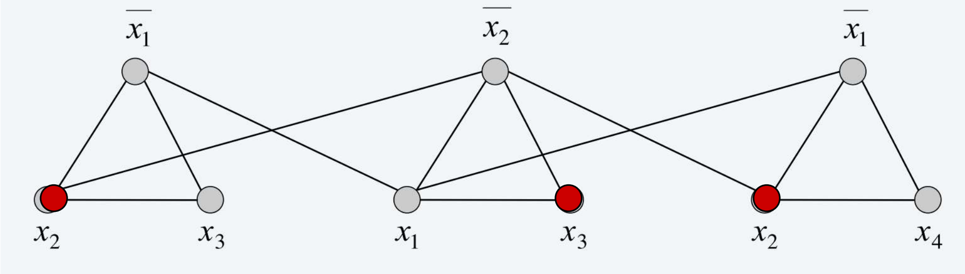


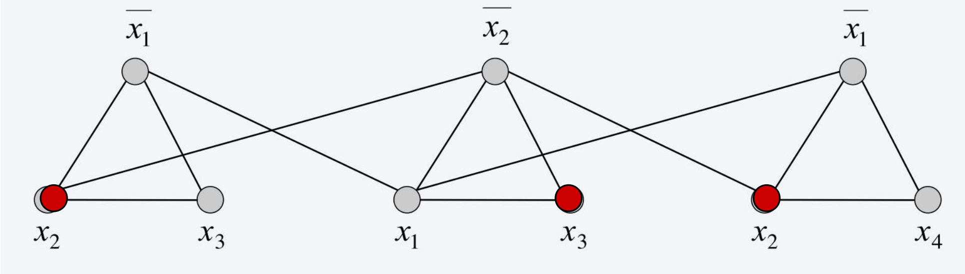
$(\overline{x_1} \vee x_2 \vee x_3)$
$\wedge (x_1 \vee \overline{x_2} \vee x_3)$
$\wedge (\overline{x_1} \vee x_2 \vee x_4)$

Suppose Φ is a YES-instance. Take any satisfying assignment for Φ and select a true literal from each clause. Corresponding k vertices form an independent set in G.

3-SAT $\leq_P$ INDEPENDENT-SET

Reduction

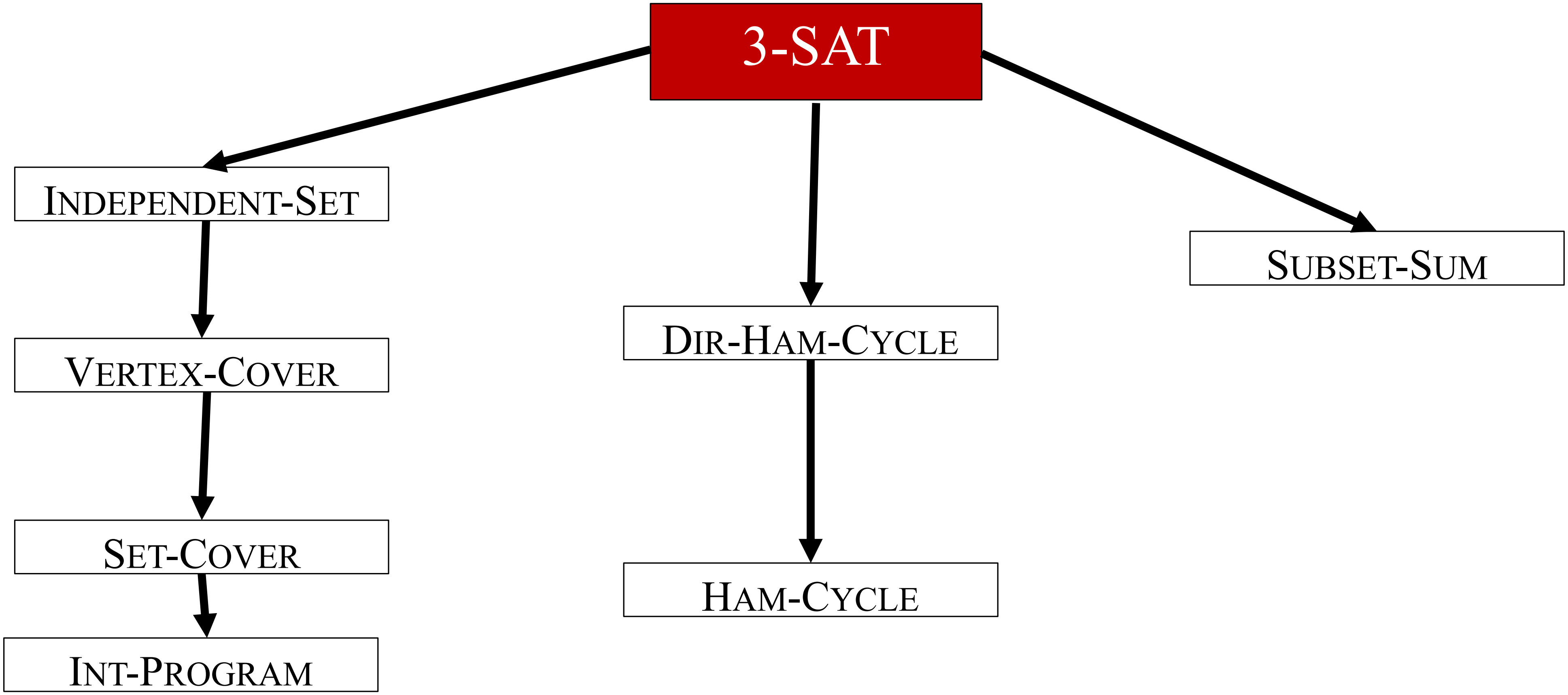
- G contains 3 vertices for each clause, one for each literal
- Connect 3 literals in clause in a triangle
- Connect literal to each of its negations
- Set k = number of clauses



$$(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

Suppose (G, k) is a YES-instance. Let S be the independent set of size k. Each of the k triangles must contain exactly one vertex in S. Set these literals to true, so all clauses satisfied.

Reductions


3-SAT
INDEPENDENT-SET
VERTEX-COVER
SET-COVER
INT-PROGRAM
DIR-HAM-CYCLE
HAM-CYCLE
SUBSET-SUM

What have we shown?

If 3-SAT does not have a poly-time algorithm, then none of these problems have poly-time algorithms!

Why do we care?

- Problems believed to be hard are the basis of secure cryptosystems. For example, credit cards, https internet security, digital signatures, etc.

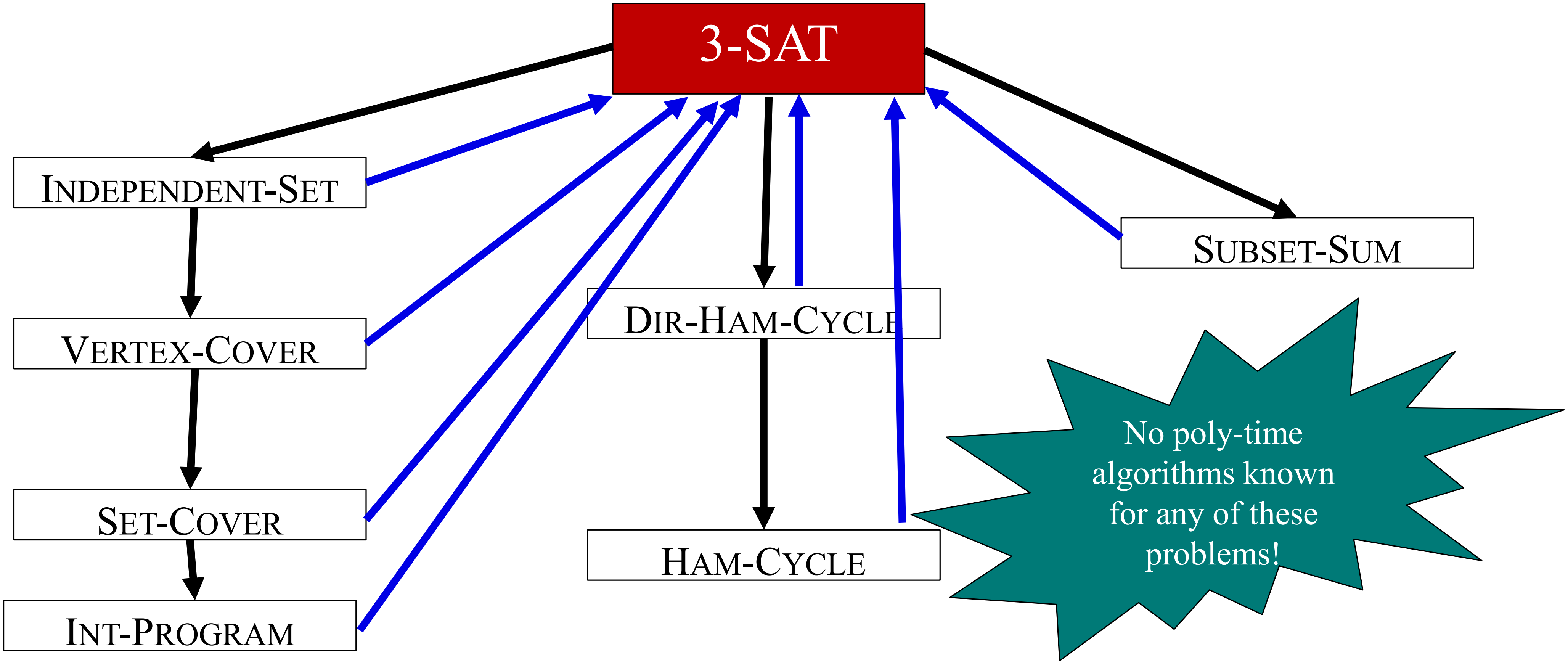
- Entire area of **complexity theory** devoted to comparing the hardness of different problems.

But…

If 3-SAT does not have a poly-time algorithm, then none of these problems have poly-time algorithms!

What if someone finds an $O(n^2)$-time algorithm for 3-SAT tomorrow? Seems dangerous to say that all these problems are hard based on lack of progress on this one specific problem…

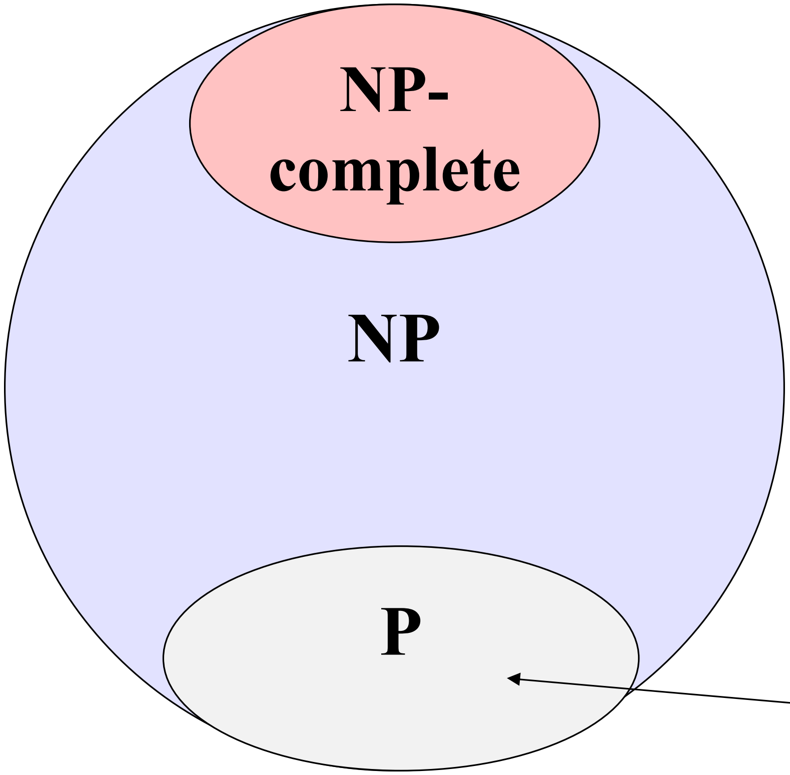
Reductions


3-SAT
INDEPENDENT-SET
VERTEX-COVER
SET-COVER
INT-PROGRAM
DIR-HAM-CYCLE
HAM-CYCLE
SUBSET-SUM
No poly-time algorithms known for any of these problems!

In fact…

- Hundreds of problems, many encountered routinely in applications, are **computationally equivalent** to 3-SAT.
  - A poly-time algorithm for any of these problems implies poly-time algorithm for others.
  - Equivalently, if you believe any of these problems is hard, all of the problems are hard.

Complexity Classes



Problems solvable in (deterministic) poly time

The class NP

NP (short for “non-deterministic polynomial”) is the class of problems for which polynomial-time verifiable **certificates** of YES-instances exist.

Equivalently, this is the class of problems that can be solved in polynomial time by a “non-deterministic Turing machine”, hence the name “non-deterministic polynomial”.

Example: SUBSET-SUM

Recall: In SUBSET-SUM, given a list of integers S and target t, problem is to decide if there is $S' \subseteq S$ that sums up to t.

Certificate is the subset S' itself. Verifier checks whether the sum of elements of S' is t in polynomial time.

Hence, SUBSET-SUM is in NP.

$S = \{2, 4, 5, 7, 8, 10\}$

$t = 19$

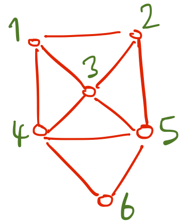
certificate: $S' = \{2, 7, 10\}$

Example: HAM-CYCLE

Recall: In Ham-Cycle, given a graph G, problem is to decide whether there is a simple cycle that visits each vertex once.

Certificate is the cycle itself. Verifier checks in polynomial time whether it is a cycle and visits each vertex once.

Hence, HAM-CYCLE is in NP.

$G =$ 

certificate : $1 \to 3 \to 2 \to 5 \to 6 \to 4 \to 1$

$\mathbf{P} \subseteq \mathbf{NP}$

Any problem in **P** is in **NP**. Why?

The certificate can be anything. The verifier $V(x,\cdot)$ can solve for the instance x by itself and check if it is a YES-instance.

Why NP?

NP captures natural search problems, similar to searching for a needle in a haystack. We know we have the needle when we see it!

> *"NP captures vast domains of computational, scientific, and mathematical endeavors, and seems to roughly delimit what mathematicians and scientists have been aspiring to compute feasibly."* — Christos Papadimitriou

NP-Hard and NP-Complete

A problem A is said to be **NP-Hard** if for every problem B in **NP**:

$$B \leq_P A$$

"at least as hard as every problem in NP"

Problem A is said to be **NP-Complete** if it is in **NP** and also **NP**-Hard.

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The hardest problems in NP

The Cook-Levin Theorem

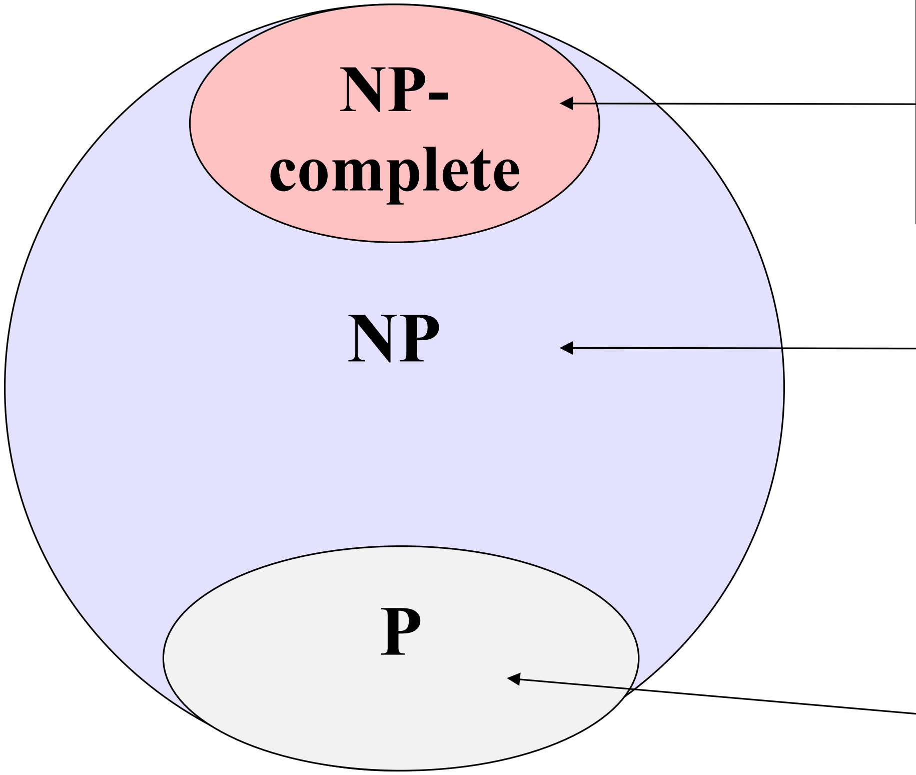
Theorem: *Any* problem in **NP** poly-time reduces to 3-SAT. Hence, 3-SAT is **NP**-hard and **NP**-complete.

The Cook-Levin Theorem

Theorem: *Any* problem in **NP** poly-time reduces to 3-SAT. Hence, 3-SAT is **NP**-hard and **NP**-complete.

Hence, so are HAM-CYCLE, SUBSET-SUM, VERTEX-COVER, INDEPENDENT-SET,…

Complexity Classes



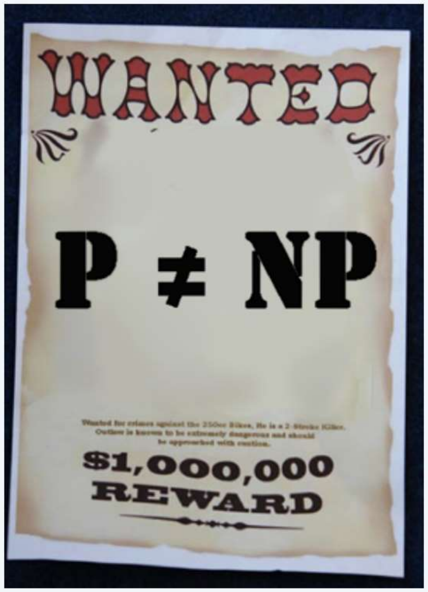
Problems that are computationally equivalent to 3-SAT

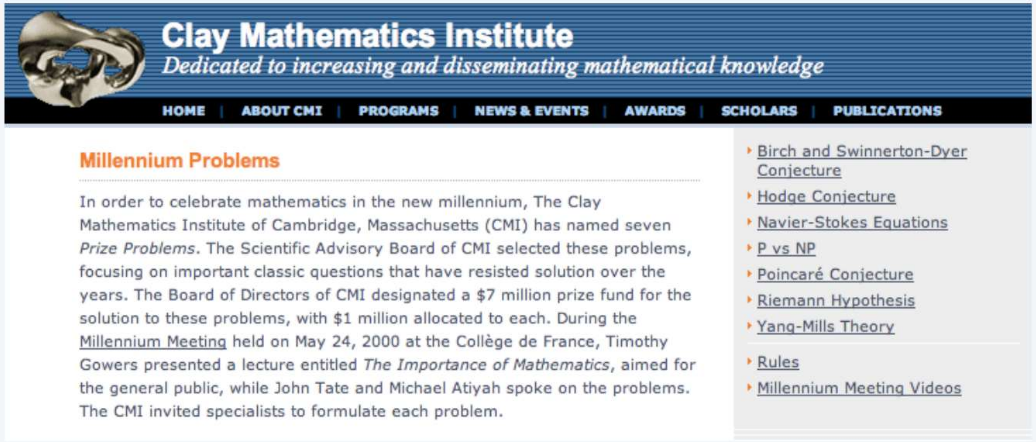
Problems that poly-time reduce to 3-SAT

Problems solvable in (deterministic) poly time

Millennium Prize

$1 million dollars for resolution of **P=NP** or **P≠NP**


WANTED
P ≠ NP
Wanted for crimes against the 250cc Bikes, He is a 2-Stroke Killer.
Outlaw is known to be extremely dangerous and should
be approached with caution.
$1,000,000
REWARD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Millennium Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven *Prize Problems*. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solution to these problems, with $1 million allocated to each. During the Millennium Meeting held on May 24, 2000 at the Collège de France, Timothy Gowers presented a lecture entitled *The Importance of Mathematics*, aimed for the general public, while John Tate and Michael Atiyah spoke on the problems. The CMI invited specialists to formulate each problem.

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

- Rules
- Millennium Meeting Videos

Some Quotes

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"I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (i) It is a legitimate mathematical possibility and (ii) I do not know." — Jack Edmonds 1966

"If I had to bet now, I would bet that P is not equal to NP. I estimate the half-life of this problem at 25–50 more years, but I wouldn't bet on it being solved before 2100. " — Bob Tarjan (2002)

Some Quotes

“ I think that in this respect I am on the loony fringe of the mathematical community: I think (not too strongly!) that P=NP and this will be proved within twenty years. Some years ago, Charles Read and I worked on it quite bit, and we even had a celebratory dinner in a good restaurant before we found an absolutely fatal mistake. ” — Béla Bollobás (2002)

Extent and Impact

- Garey and Johnson’s book, “Computers and Intractability”, includes over 300 **NP-complete** problems and is the #1 cited reference in computer science!
- NP-completeness is used in more than 6,000 publications per year (more than “compiler”, “OS”, “database”)
- Main intellectual export of computer science.

Beyond NP-completeness

- Approximation algorithms: Find an approximate solution in polynomial time.
  - Simple greedy algorithm gives a 2-approximation for VERTEX-COVER.
- Use heuristics and exploit structure in typical inputs.
  - Modern SAT solvers can solve even relatively large instances quite fast.

Remainder of Semester

- Assignment 10 due on April 10
- Assignment 11 due on April 17
- Assignment 12 cancelled!
- Programming assignment 2 (optional) due April 21 (Thu)
- Final exam practice questions to be released tomorrow
  - Review session to go over these questions on April 20 (Wed), 2-4pm on Zoom

Final Exam

- 5-7pm on April 26 (Tuesday)
- Logistics similar to midterm, uploaded to LumiNUS
- Format of the exam also similar to midterm
- From material before the midterm, only material from Lectures 2 and 3 will be tested
- Suggestion: Review assignments, tutorials, and final exam practice questions

Showing NP-completeness

- There will be a question that asks you to show that a certain problem (let’s call it X) is NP-complete.
- What are the steps?
- First, show that X is in NP.
  - A YES-instance has a certificate that can be verified in polynomial time.

Showing NP-completeness

- Second, show that X is NP-hard.
- This is done by giving a polynomial-time reduction **from** another NP-hard problem A. (In the exam, I will tell you which problem A to use.)
- The reduction should convert any instance of A into an instance of X.
  - You are not trying to solve the instance of A directly.
  - The reduction should not depend on whether the instance of A is a YES-instance or a NO-instance.

Showing NP-completeness

- To show that the reduction is valid, you need to show that
  - The reduction runs in polynomial time (if this is clear, you can simply say that it is clear)
  - If the instance of A is a YES-instance, then the instance of X is also a YES-instance
  - If the instance of X is a YES-instance, then the instance of A is also a YES-instance

Showing NP-completeness

- Some tips:
  - If you are not sure how to start a reduction, try something trivial, see where it fails and how you can fix it.
  - Review reductions from lectures, assignments, tutorials.

Input Representation

- Why is the knapsack dynamic programming algorithm **not** a polynomial-time algorithm?
- The algorithm runs in time $O(nW)$.

$O(n + \lg W)$ also correct

- However, the input has size only $\underline{O(n \lg W)}$, because the number W can be represented using $\lg W$ bits.
- The running time of the algorithm is not polynomial in the size of the input.

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That’s all for CS3230!

Please fill in the student feedback survey.

Good luck with the final exam (and beyond) ☺

Acknowledgements

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- The slides are modified from
  - the slides from Prof. Erik D. Demaine and Prof. Charles E. Leiserson
  - the slides from Prof. Lee Wee Sun
  - the slides from Prof. Kevin Wayne
  - the slides from Prof. Arnab Bhattacharyya