CS3230: Assignment for Week 5 Solutions

Due: Sunday, 20th Feb 2022, 11:59 pm SGT.

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1. No. For example, since $3 \equiv 0 \pmod 3$, $h_i(0) = h_i(3)$ for all $i \in \{1,2,3\}$, so 0 and 3 always collide.

2. (a) For each $1 \le i \le n$, let Y_i be an indicator random variable for whether the i-th bin is empty. We have $\mathbb{E}[Y_i] = \left(\frac{n-1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n$ for all i. Let Y be the random variable whose value is the fraction of empty bins. We have $Y = \frac{1}{n} \cdot \sum_{i=1}^n Y_i$. So by linearity of expectation, $\mathbb{E}[Y] = \frac{1}{n} \cdot \sum_{i=1}^n \mathbb{E}[Y_i] = \frac{1}{n} \cdot n \cdot \left(1 - \frac{1}{n}\right)^n = \left(1 - \frac{1}{n}\right)^n$. This converges to $1/e$ as $n \to \infty$.

   (b) For each $1 \le i \le n$, let Z_i be an indicator random variable for whether the ball with label i goes into the bin with label i. We have $\mathbb{E}[Z_i] = 1/n$ for all i. Let Z be the random variable whose value is the number of balls that go into the bin with the same label. We have $Z = \sum_{i=1}^n Z_i$. So by linearity of expectation, $\mathbb{E}[Z] = \sum_{i=1}^n \mathbb{E}[Z_i] = 1$.

3. For each pair $1 \le i < j \le n$, let X_{ij} be an indicator random variable for whether the i-th element is smaller than the j-th element. Because of symmetry, we have $\mathbb{E}[X_{ij}] = 1/2$ for all $i < j$. Let X be the random variable whose value is the number of pairs $i < j$ such that i-th element is smaller than the j-th element. We have $X = \sum_{i<j} X_{ij}$. So by linearity of expectation, $\mathbb{E}[X] = \mathbb{E}[\sum_{1\le i<j\le n} X_{ij}] = \sum_{1\le i<j\le n} \mathbb{E}[X_{ij}] = \sum_{1\le i<j\le n} \frac{1}{2} = \frac{n(n-1)}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}$.