CS3230 Assignment 4

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1. maximum #comparisons deterministic Quicksort makes when sorting n distinct numbers.

let $T(n)$ be the maximum number of comparisons of Quicksort on array of size n

pivot produces subarrays of size j and $n-j-1$ $\quad T(n) = T(j) + T(n-j-1) + n-1$

cost of partition is $n-1$ regardless of permutation of input

In the worst case when input is sorted / reverse sorted, 1 side of partition has no element,

$$T(0) = 0 \qquad T(1) = 0 \qquad T(2) = 1$$

$$\begin{aligned} T(n) &= T(0) + T(n-1) + n-1 \\ &= T(n-2) + n-2 + n-1 \\ &= T(n-3) + n-3 + n-2 + n-1 \\ &= \vdots \\ &= T(1) + 1 + 2 + \cdots n-2 + n-1 \\ &= \frac{(n-1)(n)}{2} \end{aligned}$$

Geometric series & Expectation

$$S = \sum_{i=1}^{\infty} x^i = x + x^2 + x^3 + \cdots$$

$$xS = x^2 + x^3 + \cdots$$

$$S(1-x) = x \Rightarrow S = \frac{x}{1-x}$$

$$T = \sum_{i=1}^{\infty} i x^i = x + 2x^2 + 3x^3 + \cdots$$

$$xT = x^2 + 2x^3 + 3x^4 + \cdots$$

$$T(1-x) = x + x^2 + x^3 + \cdots = S = \frac{x}{1-x}$$

$$T = \frac{x}{(1-x)^2}$$

OR

$$S = \sum_{i=1}^{\infty} x^i = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

differentiate wrt to x

$$\frac{dS}{dx} = \sum_{i=1}^{\infty} i x^{i-1} = 1 + 2x + 3x^2 + \cdots$$

$$\begin{aligned} T &= x \cdot \sum_{i=1}^{\infty} i x^{i-1} = x + 2x^2 + 3x^3 + \cdots \\ &= x \cdot \frac{d}{dx}\left(\frac{x}{1-x}\right) \\ &= x \cdot \frac{d}{dx}\left(\frac{1}{1-x} - 1\right) \\ &= x \cdot -\frac{1}{(1-x)^2}(-1) \\ &= \frac{x}{(1-x)^2} \end{aligned}$$

2. Bogosort?

let X be the number of times the while loop runs.

probability for uniform random permutation to sort array $= \frac{1}{n!}$

X follows geometric distribution with probability $= \frac{1}{n!}$

$$E(X) = 1 \cdot \frac{1}{n!} + 2 \cdot \frac{n!-1}{n!} \cdot \frac{1}{n!} + 3 \cdot \frac{n!-1}{n!} \cdot \frac{n!-1}{n!} \cdot \frac{1}{n!}$$

$$= \sum_{i=1}^{\infty} i \cdot \frac{(n!-1)^{i-1}}{(n!)^i}$$

let $\frac{n!-1}{n!}$ be p

$$= \frac{1}{n!-1} \sum_{i=1}^{\infty} i \left(\frac{n!-1}{n!}\right)^i$$

$$= \frac{1}{n!-1} \sum_{i=1}^{\infty} i p^i$$

see derivation from previous page

$$= \frac{1}{n!-1} \frac{p}{(p-1)^2}$$

$$= \frac{1}{n!-1} \frac{\frac{n!-1}{n!}}{\left(\frac{n!-1}{n!}-1\right)^2} = n!$$

* check $E(X) = \frac{1}{\frac{1}{n!}} = n!$

3.

$$E(X) = \sum_{i=1}^{\infty} \frac{i}{2^i}$$

let $p = \frac{1}{2}$.

$$= \sum_{i=1}^{\infty} i p^i$$

$$= \frac{p}{(p-1)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

* check $E(X) = \frac{1}{\frac{1}{2}} = 2$