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CS3230 Assignment 5

1. For $i \in \{1,2,3\}$ $h_i : \{0,1,2,3,4\} \to \{0,1,2\}$ such that

$$h_i(x) = ix \pmod 3 \quad \text{for all } x \in \{0,1,2,3,4\}$$

	0	1	2	3	4
h_1	0	1	2	0	1
h_2	0	2	1	0	2
h_3	0	0	0	0	0

$M = 3$

For elements 0 and 3, all 3 hash functions cause 0 and 3 to collide.

Since $\frac{3}{3} > \frac{1}{3}$, H is not universal.

Elements 1 and 4 collide for all 3 hash functions as well.

2. a) Let X be the random variable representing the number of empty bins and X_i be the indicator random value such that X_i is 1 if the ith bin is empty and 0 otherwise

$$X = X_1 + X_2 + \dots X_n = \sum_{i=1}^{n} X_i$$

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(\text{bin } i \text{ is empty after } n \text{ balls}) = \left(1 - \frac{1}{n}\right)^n$$

Expected faction of empty bins:

$$E\left[\frac{X}{n}\right] = \frac{1}{n} E[X] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{n} \cdot n \left(1 - \frac{1}{n}\right)^n$$

$$= \left(1 - \frac{1}{n}\right)^n \quad \text{or} \quad \left(\frac{n-1}{n}\right)^n$$

As $n \to \infty$, $\lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$

b) Let X be the random variable representing the number of balls that go in the bin with the same label

Let X_i be the indicator random variable such that $X_i = 1$ if the ith ball goes into the ith bin and 0 otherwise

$$X = X_1 + X_2 + \dots X_n = \sum_{i=1}^{n} X_i$$

$$E[X_i] = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0) = P(\text{ith ball goes into ith bin}) = \frac{1}{n}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$
$$= \sum_{i=1}^{n} \frac{1}{n}$$
$$= n \cdot \frac{1}{n}$$
$$= 1$$

3. Let e_i be the ith smallest element in the array

Expected #pairs of integers such that left < right $= \sum_{\pi} \frac{1}{n!} Q(\pi)$ where $Q(\pi)$ is the # of pairs such that left < right for a particular permutation π

$$= \frac{1}{n!} \left(\sum_{\pi} Q(\pi) \right)$$

pairs such that left < right over all permutations $n!$

For each pair where $i < j$ and e_i is to the left of e_j in the array at fixed positions, there are $(n-2)!$ such pairs across all $n!$ permutations

e.g. $e_1 \, e_2 \, \underbrace{|e_3 \dots e_n|}_{} \rightarrow (n-2)!$ permutations

there are $\binom{n}{2}$ choices of i, j such that $i < j$

and $\binom{n}{2}$ choices of positions to place e_i and e_j such that e_i is left of e_j

$\therefore$ Expectation of #pairs such that left < right $= \dfrac{\binom{n}{2}^2 \cdot (n-2)!}{n!} = \dfrac{\binom{n}{2} \cdot \binom{n}{2}}{n \cdot n-1} = \dfrac{\frac{(n-1)(n)}{2} \cdot \frac{\cancel{(n-1)}\cancel{(n)}}{2}}{\cancel{n} \cdot \cancel{(n-1)}}$

$$= \frac{(n-1)(n)}{4}$$