CS3230: Assignment for Week 5

Due: Sunday, 20th Feb 2022, 11:59 pm SGT.

Please upload PDFs containing your solutions (hand-written & scanned, or typed) by 20th Feb, 11:59 pm to `Assignments/Assignment5/Submissions`. Name the file **Assignment5_SID.pdf**, where SID should be replaced by your student ID.

You may discuss the problems with your classmates or read material online, but you should write up your solutions on your own. Please note the names of your collaborators or online sources in your submission; failure to do so would be considered plagiarism.

Note: For all questions in this assignment except the second question in 2(a), you should justify your answer.

1. (1 point) For $i \in \{1, 2, 3\}$, let $h_i : \{0, 1, 2, 3, 4\} \to \{0, 1, 2\}$ be a function such that

$$h_i(x) = ix \pmod 3$$

   for all $x \in \{0, 1, 2, 3, 4\}$. Is the family $\mathcal{H} = \{h_1, h_2, h_3\}$ universal?

2. (7 points, graded for correctness) Suppose that you throw n balls uniformly and independently at random into n bins.

   (a) (4 points) What is the expected fraction of empty bins? What does this fraction converge to as n goes to infinity? (For the second question, you may write down the answer without justification.)

   (b) (3 points) Suppose that the balls are labelled $1, 2, \ldots, n$ (each label used exactly once) and so are the bins. What is the expected number of balls that go into the bin with the same label?

3. (1 point) Consider an array with a uniform random permutation of n distinct integers. What is the expected number of (unordered) pairs of integers in the array such that the integer to the left is smaller than the integer to the right?