Design and Analysis of Algorithms

CS3230

Lecture 9

Greedy Algorithms

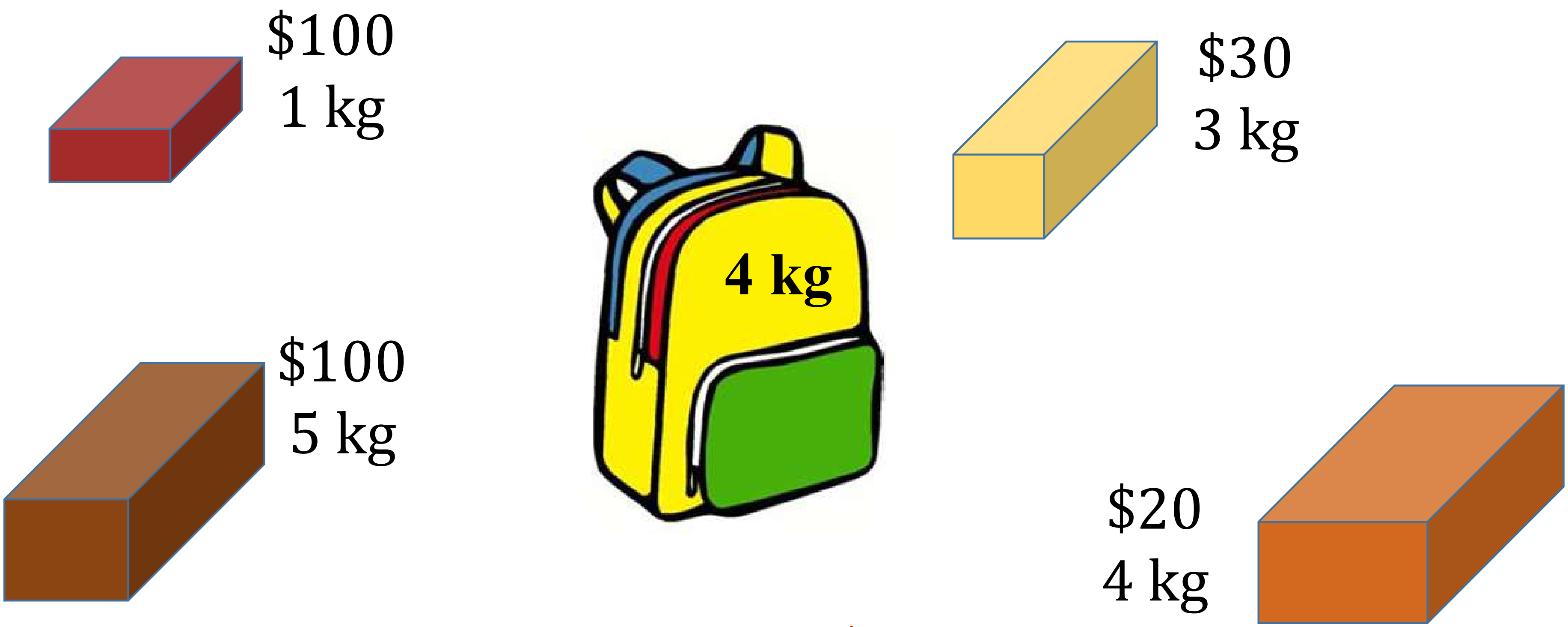
Warut Suksompong

Today: Greedy Algorithms

A very general technique, like divide-and-conquer and dynamic programming

> Technique is to recast the problem so that <u>only one subproblem</u> needs to be solved at each step. Beats divide-and-conquer and dynamic programming, **when it works**.

Fractional Knapsack



taking 3 kg of this would give value

$\frac{3}{5} \cdot (\$100) = \$60$

Fractional Knapsack

Input:

$(w_1, v_1), (w_2, v_2), \ldots, (w_n, v_n)$ and W

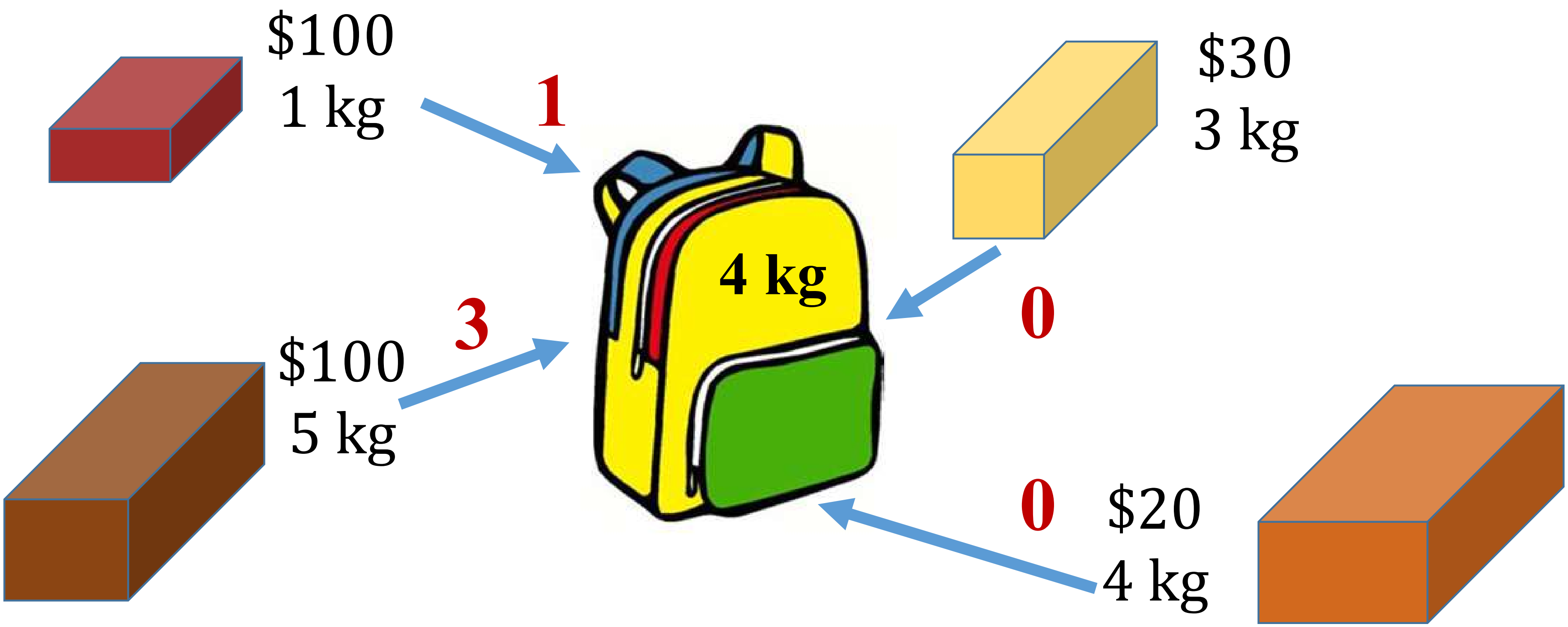
Output:

fraction of item i taken

Weights $x_1, \ldots, x_n$ that maximize $\sum_i v_i \cdot \frac{x_i}{w_i}$ subject to:

$$\sum_i x_i \leq W \text{ and } 0 \leq x_j \leq w_j \text{ for all } j \in \{1,2, \ldots, n\}.$$

Fractional Knapsack



Optimal Substructure

If we remove $\boldsymbol{w}$ **kgs of one item** $\boldsymbol{j}$ from the optimal knapsack, then the remaining load must be the optimal knapsack weighing at most $\boldsymbol{W - w}$ **kgs** that one can take from the $\boldsymbol{n - 1}$ **original items and** $w_j - \boldsymbol{w}$ **kgs of item** $\boldsymbol{j}$.

Optimal Substructure: Proof

cut-and-paste argument

- Let X be the value of the optimal knapsack. Suppose that the remaining load after removing w kgs of item j was not the optimal knapsack weighing at most $W - w$ kgs that one can take from the $n - 1$ original items and $w_j - w$ kgs of item j.

- This means that there is a knapsack of value $> X - v_j \cdot \frac{w}{w_j}$ with weight $\leq W - w$ kgs among the $n - 1$ other items and $w_j - w$ kgs of item j.

- Combining with w kgs of item j gives knapsack of value $> X$ and weight at most W for original input. **Contradiction!**

Dynamic Programming?

In integral knapsack problem, we used the optimal substructure to formulate DP for deciding whether to add item j.

But in this case, we can do better….

Make a guess!

Suppose we do not know anything about algorithm design and would like to solve the fractional knapsack problem in the real life. What strategy will you use?

1. Will first take the item with **maximum value**, then the item with **second maximum value** and **so on** until exceed the weight (the last chosen item could be fractional)

2. Will first take the item with **minimum weight**, then the item with **second minimum weight** and **so on** until exceed the weight (the last chosen item could be fractional)

correct answer

3. Will first take the item with **maximum value-per-weight ratio**, then the item with **second maximum value-per-weight ratio** and **so on** until exceed the weight (the last chosen item could be fractional)

Greedy-choice Property

Let $\boldsymbol{j^*}$ be an item with the **maximum value/kg**, $\boldsymbol{v_j/w_j}$. Then, there exists an optimal knapsack containing $\boldsymbol{min(w_{j^*}, W)}$ **kgs** of item $\boldsymbol{j^*}$.

If $w_{j^*} \leq W$: w_{j^*}

take entire item j^*

If $w_{j^*} > W$: W

take weight W of item j^*

Greedy-choice Property: Why?

Claim: Let j^* be an item with the maximum value/kg, v_j/w_j. Then, there exists an optimal knapsack containing $\min(w_{j^*}, W)$ kgs of item j^*.

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- Suppose an optimal knapsack contains x_1 kgs of item 1, x_2 kgs of item 2, ..., x_n kgs of item n such that:

$$x_1 + x_2 + \cdots + x_n = \min(w_{j^*}, W)$$

Greedy-choice Property: Why?

Claim: Let j^* be an item with the maximum value/kg, v_j/w_j. Then, there exists an optimal knapsack containing $\min\left(w_{j^*}, W\right)$ kgs of item j^*.

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- Replace this weight by $\min(w_{j^*}, W)$ kgs of item j^*.

Greedy-choice Property: Why?

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$$x_1 + x_2 + \cdots + x_n = \min(w_{j^*}, W)$$
- Replace this weight by $\min(w_{j^*}, W)$ kgs of item j^*.
- Total weight does not change. Total value does not decrease because value/kg of j^* is maximum. So, knapsack stays optimal.

Strategy for Greedy Algorithm

- Use greedy-choice property to put $\min(w_{j^*}, W)$ kgs of item j^* in knapsack.
- If knapsack weighs W kgs, we are done.
- Otherwise, use optimal substructure to solve subproblem where all of item j^* is removed and knapsack weight limit is $W - w_{j^*}$.

Iterative greedy algorithm

```
ITER-FRAC-KNAPSACK(v, w, W):
    valperkg ← [1,2, … , n]
    Sort valperkg using comparison operator ≼ where i ≼ j if v[i]/w[i] ≤ v[j]/w[j]
    for i = n to 1:
        if W = 0: break
        j ← valperkg[i]
        k ← min(w[j], W)
        print "k kgs of item j"
        W ← W − k
    return
```

Iterative greedy algorithm

ITER-FRAC-KNAPSACK(v, w, W):

$valperkg \leftarrow [1,2,\ldots,n]$

Sort $valperkg$ using comparison operator $\preccurlyeq$ where $i \preccurlyeq j$ if $\frac{v[i]}{w[i]} \leq \frac{v[j]}{w[j]}$

for $i = n$ to 1:

if $W = 0$: **break**

$j \leftarrow valperkg[i]$

$k \leftarrow \min(w[j], W)$

print "k kgs of item j"

$W \leftarrow W - k$

return

Total time = $\boldsymbol{O(n \log n)}$

Paradigm for greedy algorithms

1. Cast the problem where we have to make a choice and are left with one subproblem to solve.

2. Prove that there is always an optimal solution to the original problem that makes the greedy choice, so the greedy choice is safe.

3. Use optimal substructure to show that we can combine an optimal solution to the subproblem with the greedy choice to get an optimal solution to the original problem.

The Simplest Greedy Algorithm

- **Problem**: Given n items with positive weights $w_1, \dots, w_n$, find a set S of k items that maximizes the total weight of items in S.

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- **Optimal Substructure**:
  - Suppose S is any optimal solution, and S contains item i.
  - **Claim**: $S - \{i\}$ is optimal for the sub-problem with i-th item removed and size limit $k - 1$.
  - **Cut & Paste Proof**: If T is optimal for the sub-problem and weighs more than $S - \{i\}$, then $T \cup \{i\}$ would weigh more than S. Contradiction!

The Simplest Greedy Algorithm

- **Problem**: Given n items with positive weights $w_1, \ldots, w_n$, find a set S of k items that maximizes the total weight of items in S.

- **Greedy-choice Property:**
  - Suppose the i-th item has the maximum weight
  - **Claim:** There exists an optimal solution that contains item i.
  - **Proof**: Suppose there is an optimal solution S that does not contain i. Then, replace any item in S with i. The total weight does not decrease; so new set is also optimal and contains i.

The Simplest Greedy Algorithm

- **Problem**: Given n items with positive weights $w_1, \ldots, w_n$, find a set S of k items that maximizes the total weight of items in S.

- **Greedy Algorithm:**
  - Output the top k heaviest items.
  - **Correctness:** By greedy-choice property, can assume that heaviest item in S**.** By optimal substructure, can combine with solution to remaining subproblem.
  - Via sorting: $O(n \log n)$. Via QuickSelect: $O(n)$.

Minimum spanning trees

Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \rightarrow \mathsf{R}^+$.

- For simplicity, assume that all edge weights are distinct. (CLRS covers the general case.)

Minimum spanning trees

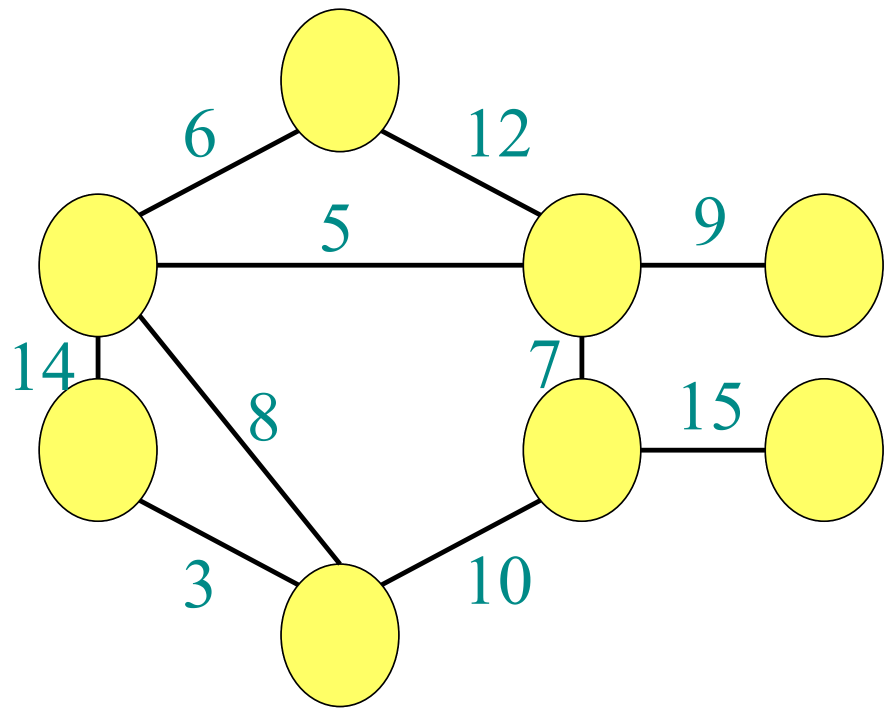
Input: A connected, undirected graph $G = (V, E)$ with weight function $w : E \to \mathrm{R}$.

- For simplicity, assume that all edge weights are distinct.

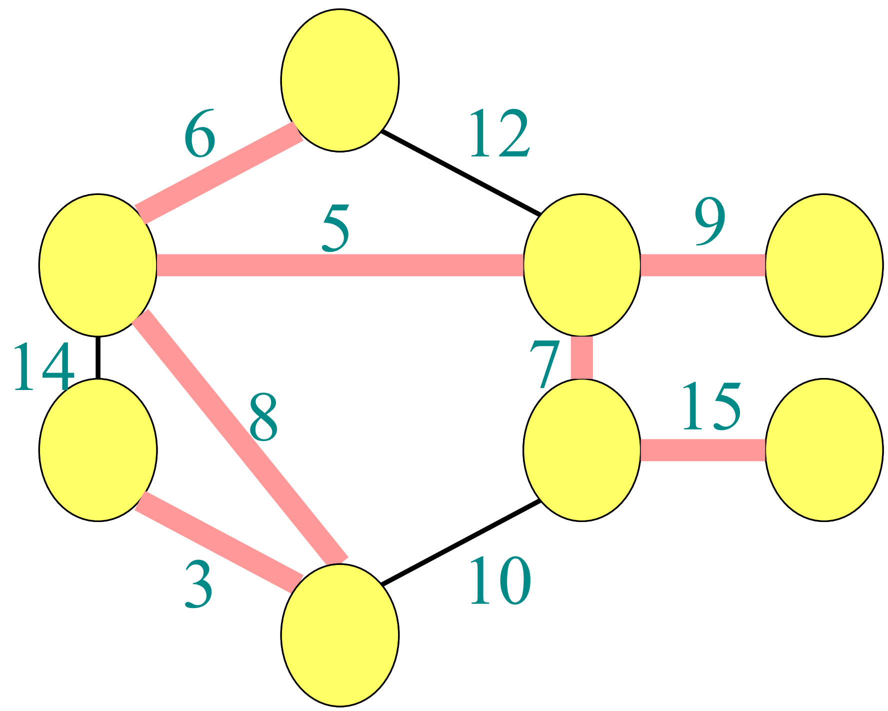
Output: A ***spanning tree*** T — a tree that connects all vertices — of minimum weight:

$$w(T) = \sum_{(u,v) \in T} w(u,v).$$

Example of MST



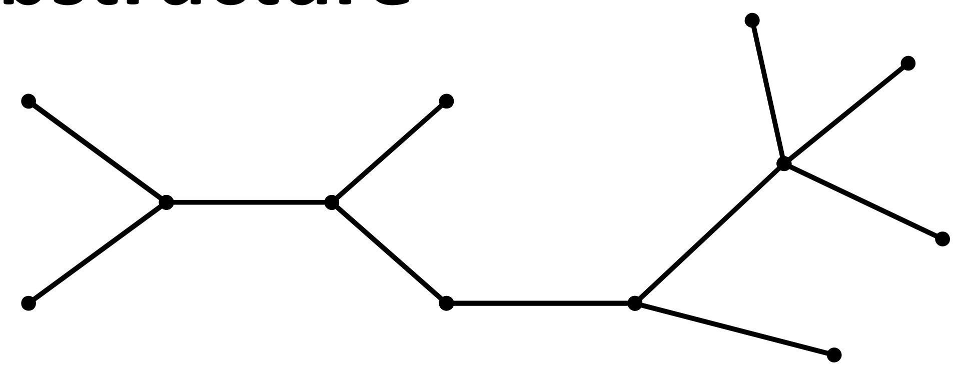
Example of MST



Optimal substructure

MST T:

(Other edges of G are not shown.)



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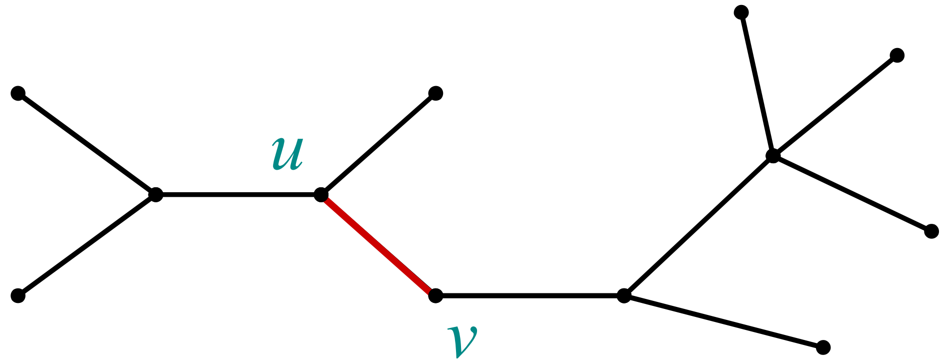

Remove any edge $(u, v) \in T$.

Optimal substructure

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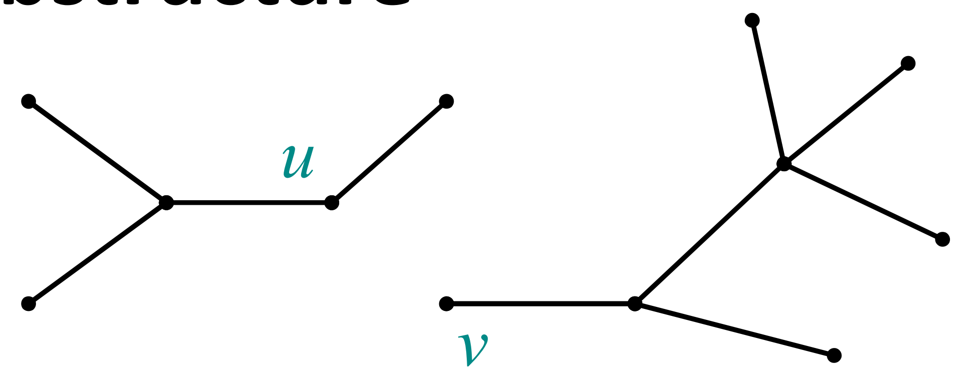

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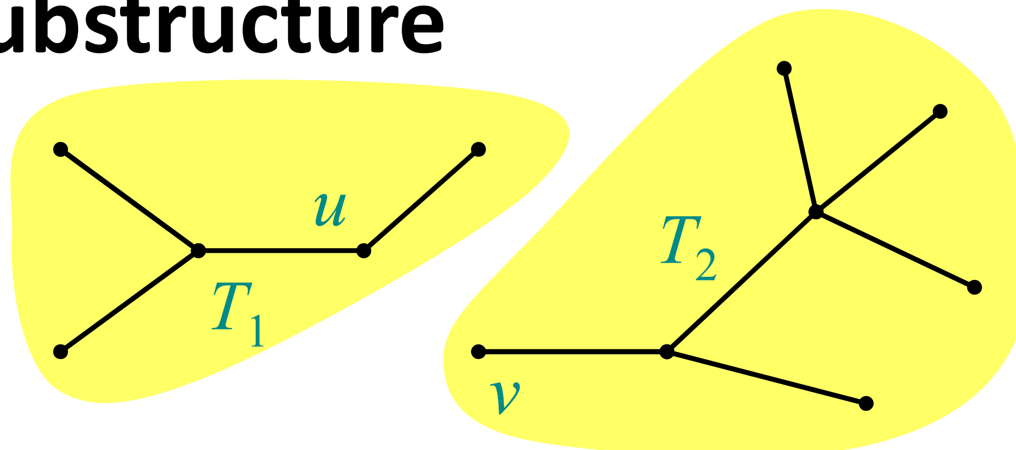

Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.

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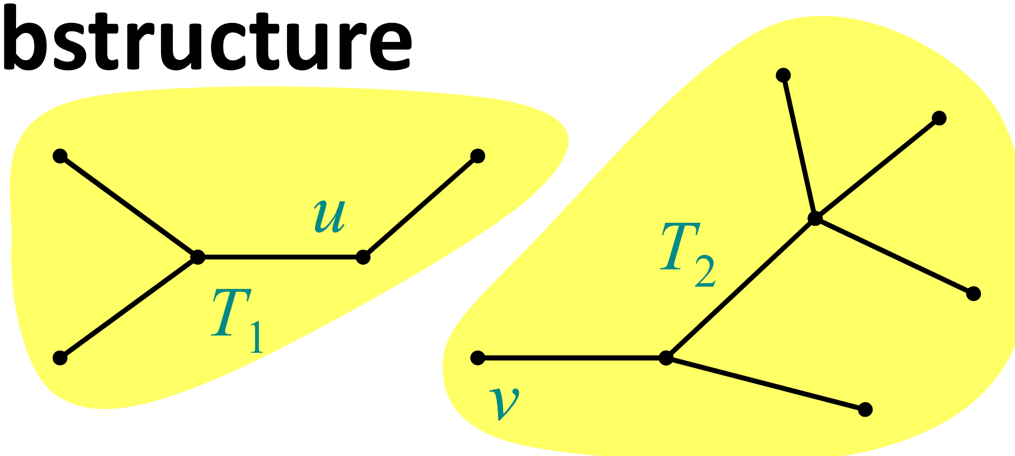

Remove any edge $(u, v) \in T$. Then, T is partitioned into two subtrees T_1 and T_2.

Theorem. The subtree T_1 is an MST of $G_1 = (V_1, E_1)$, the subgraph of G ***induced*** by the vertices of T_1:

$$V_1 = \text{vertices of } T_1,$$
$$E_1 = \{ (x, y) \in E : x, y \in V_1 \}.$$

Similarly for T_2.

Proof of optimal substructure

Proof. Cut and paste:

$$w(T) = w(u, v) + w(T_1) + w(T_2).$$

If T_1' were a lower-weight spanning tree than T_1 for G_1, then $T' = \{(u, v)\} \cup T_1' \cup T_2$ would be a lower-weight spanning tree than T for G. $\square$

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Do we also have overlapping subproblems?

- Yes.

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Do we also have overlapping subproblems?

- Yes.

Great, then dynamic programming may work!

- Yes, but MST exhibits another powerful property which leads to an even more efficient algorithm.

Hallmark for “greedy” algorithms

Greedy-choice property

A locally optimal choice is globally optimal.

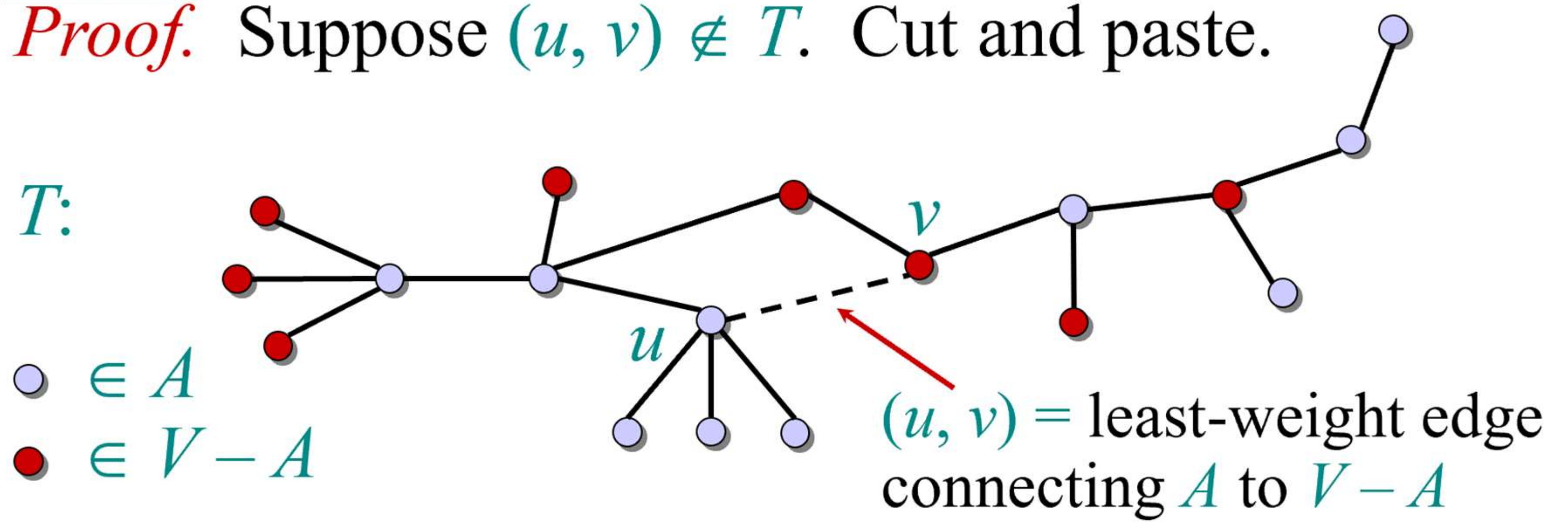
Hallmark for “greedy” algorithms

Greedy-choice property
A locally optimal choice is globally optimal.

Theorem. Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V - A$. Then, $(u, v) \in T$.

Proof of theorem

Proof. Suppose $(u, v) \notin T$. Cut and paste.



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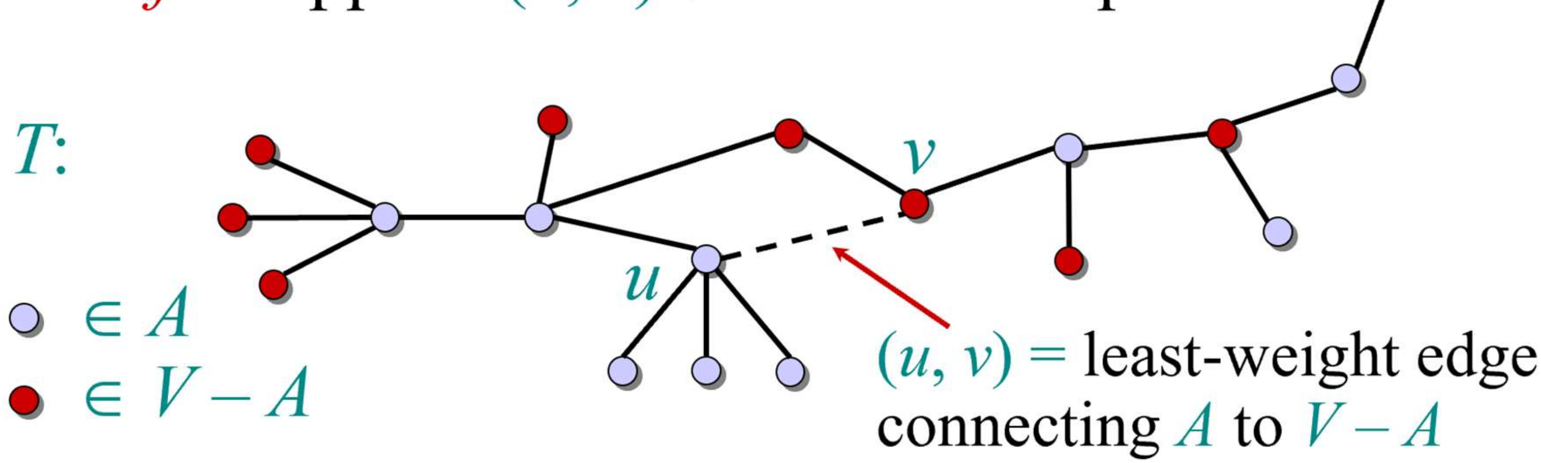

Consider the unique simple path from u to v in T.

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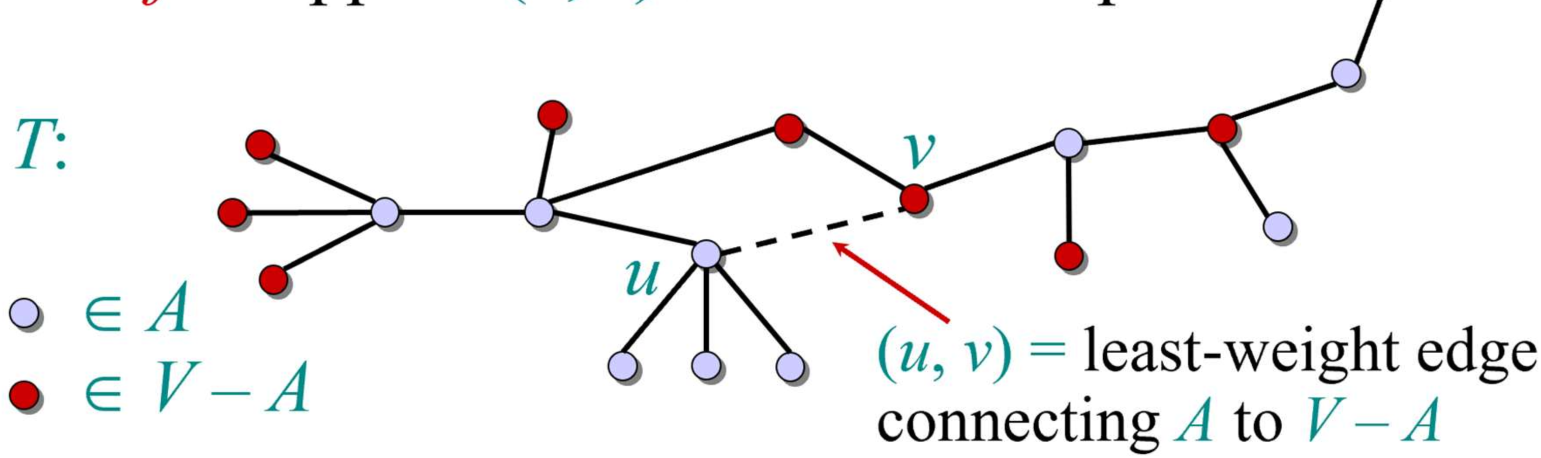

Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

Proof of theorem

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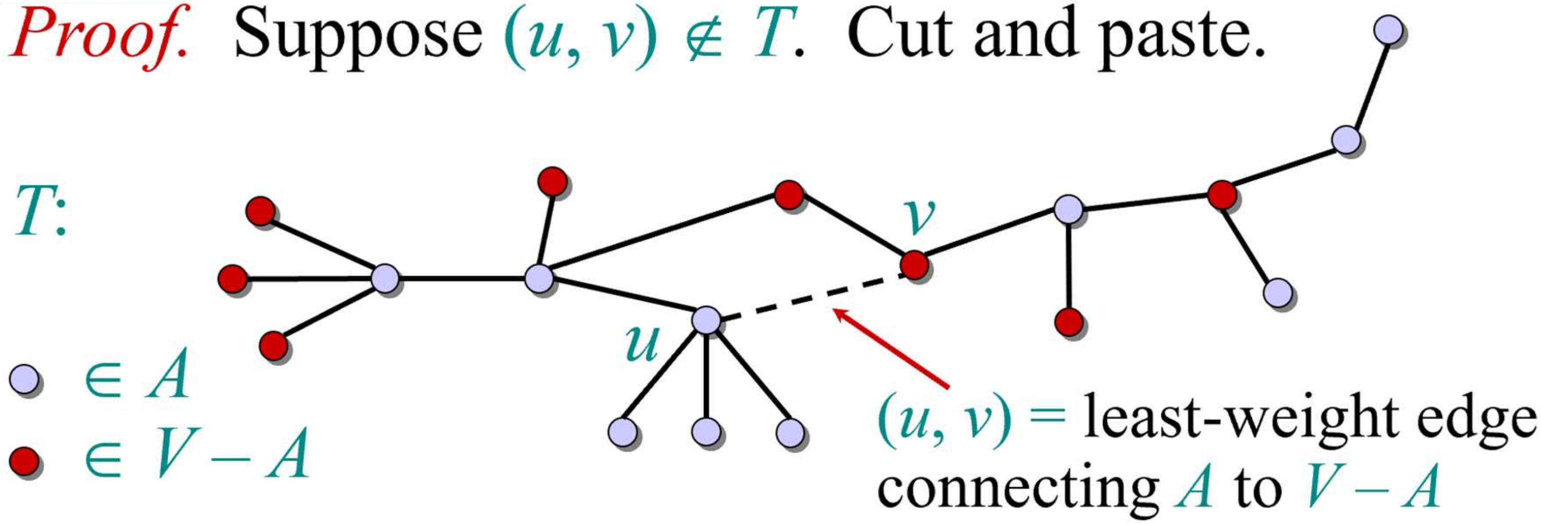


Consider the unique simple path from u to v in T.

Swap (u, v) with the first edge on this path that connects a vertex in A to a vertex in $V - A$.

A lighter-weight spanning tree than T results. $\square$

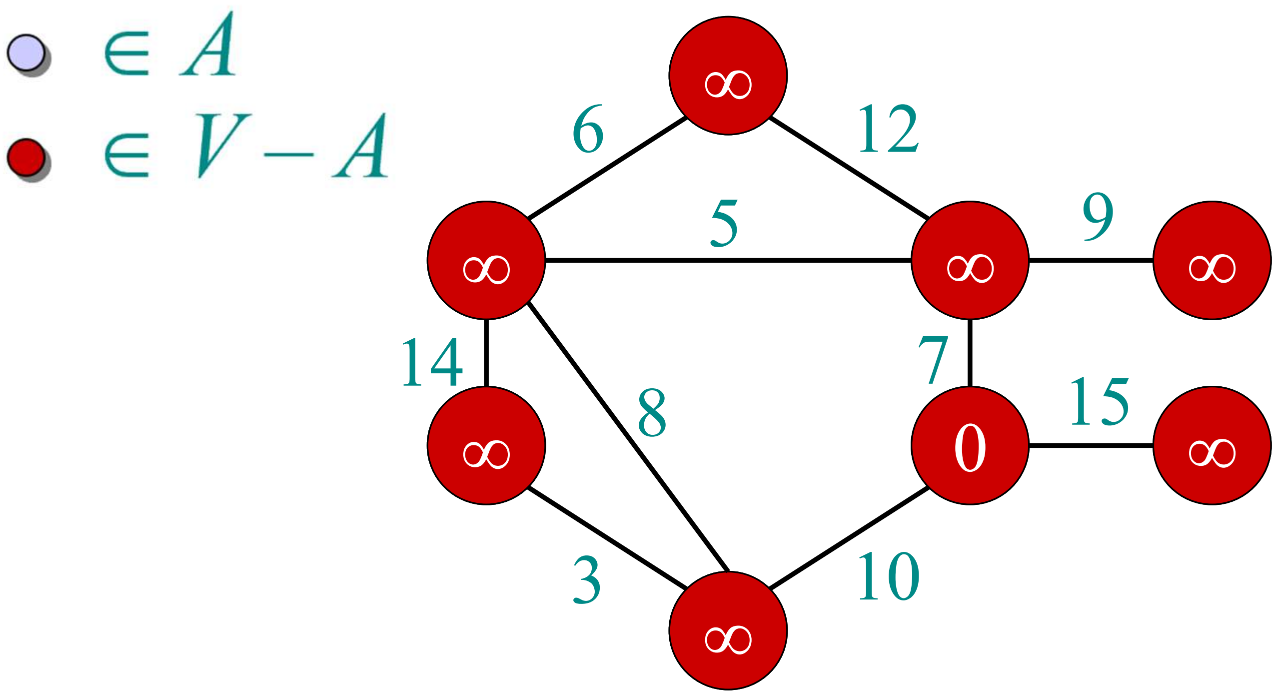
Prim's algorithm

IDEA: Build the tree one vertex at a time. At each step, add a least-weight edge from the tree to some vertex outside the tree.

$Q \leftarrow V$
$key[v] \leftarrow \infty$ for all $v \in V$
$key[s] \leftarrow 0$ for some arbitrary $s \in V$
while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    for each $v \in Adj[u]$
      do if $v \in Q$ and $w(u, v) < key[v]$
        then $key[v] \leftarrow w(u, v)$ ⊳ DECREASE-KEY
          $\pi[v] \leftarrow u$

At the end, $\{(v, \pi[v])\}$ forms the MST.

Example of Prim's algorithm



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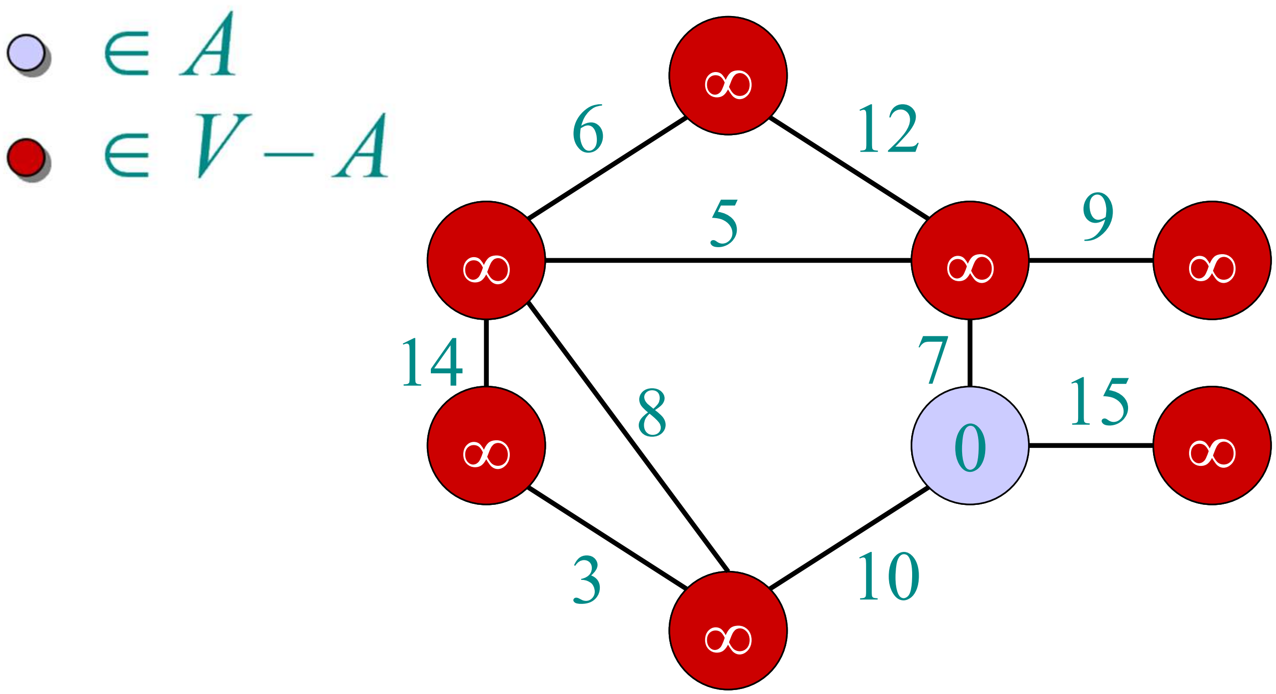

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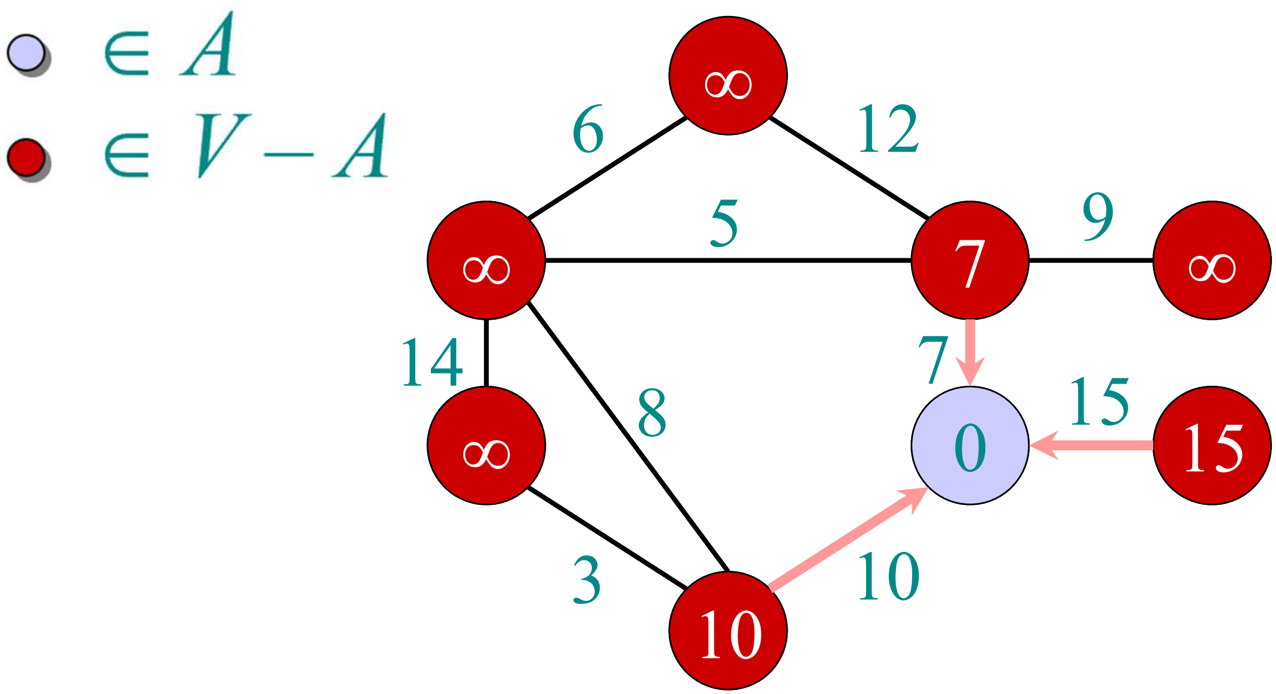

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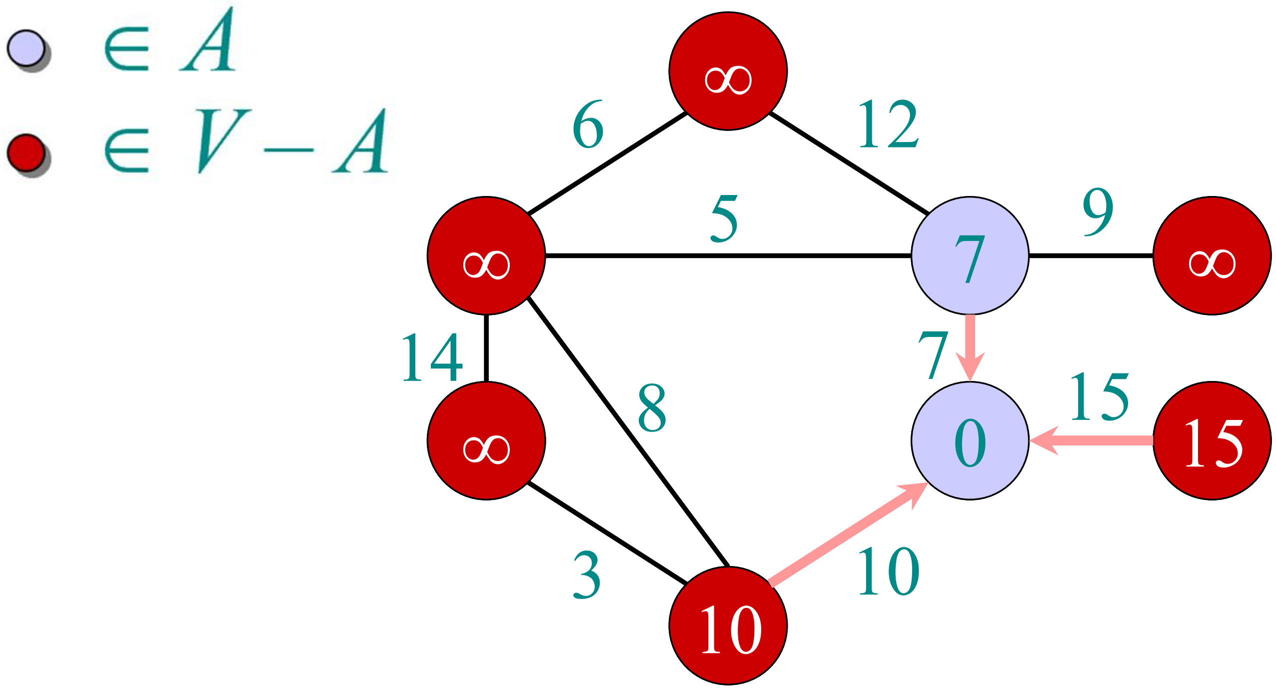

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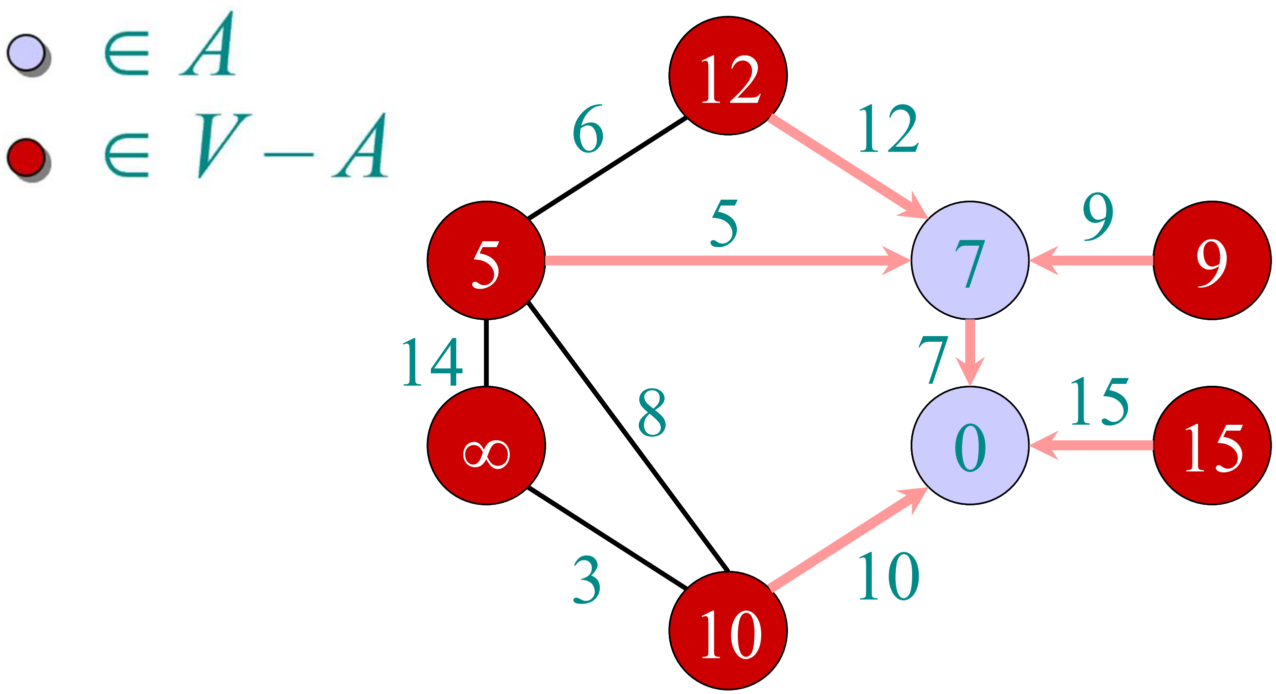

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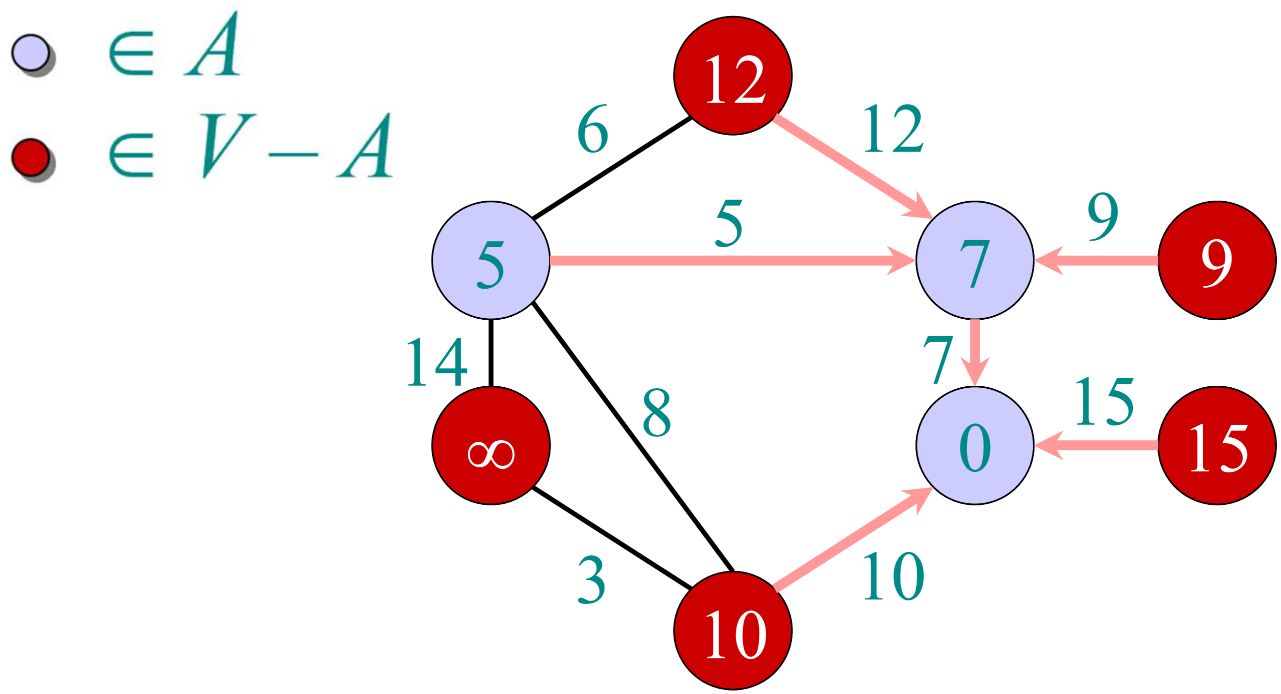

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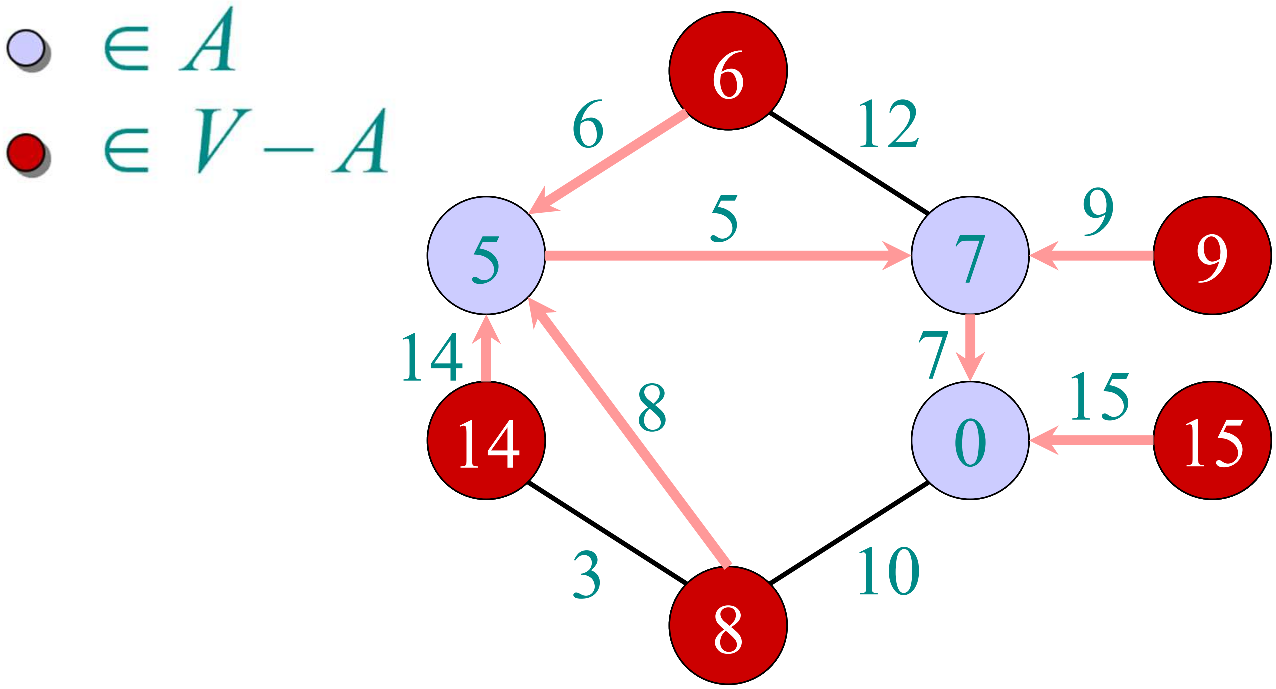

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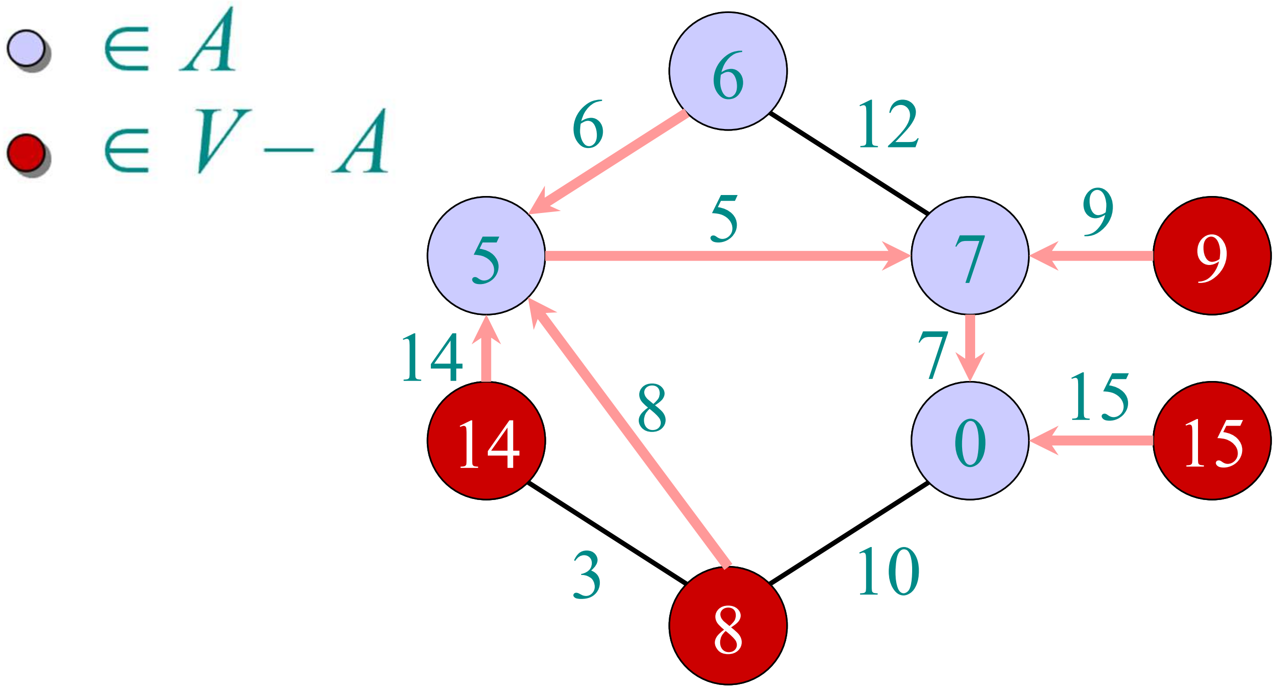

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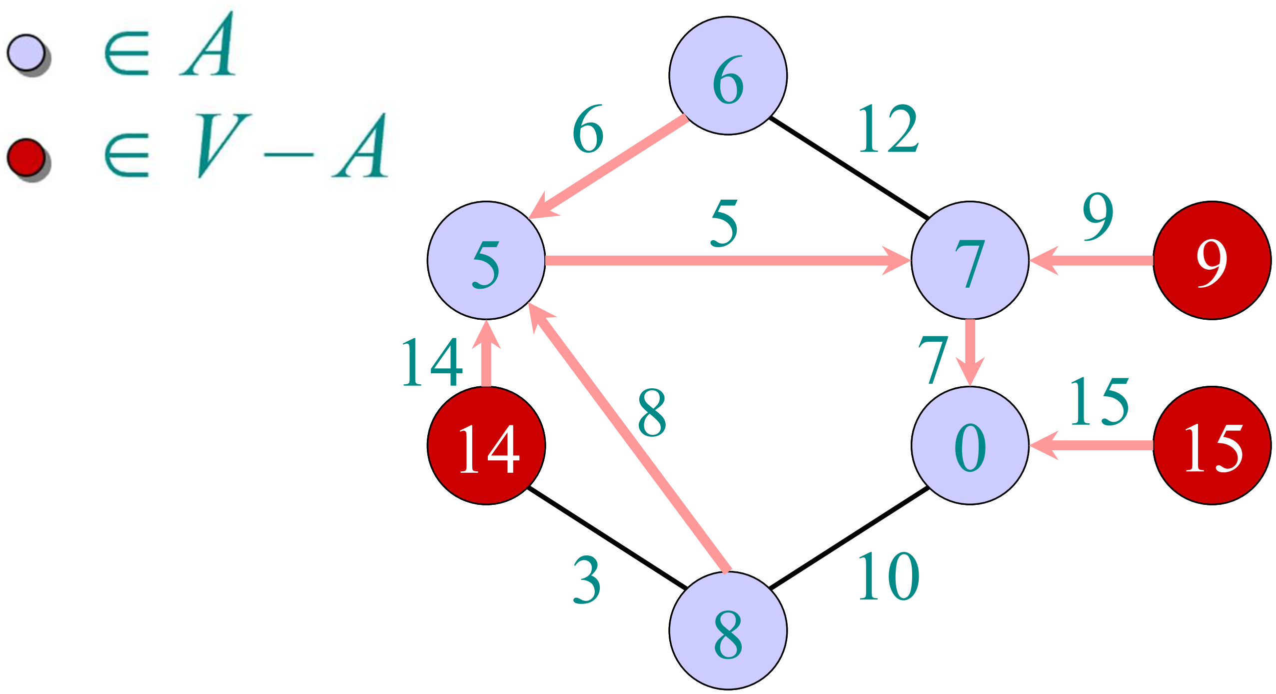

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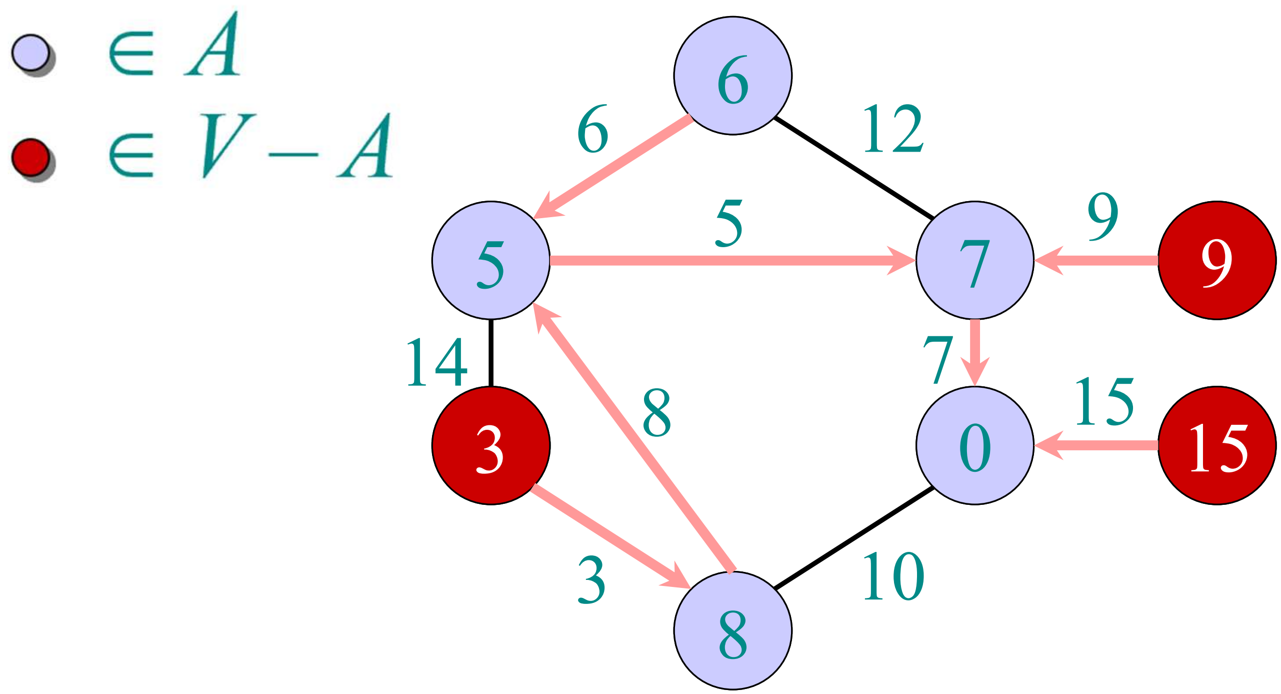

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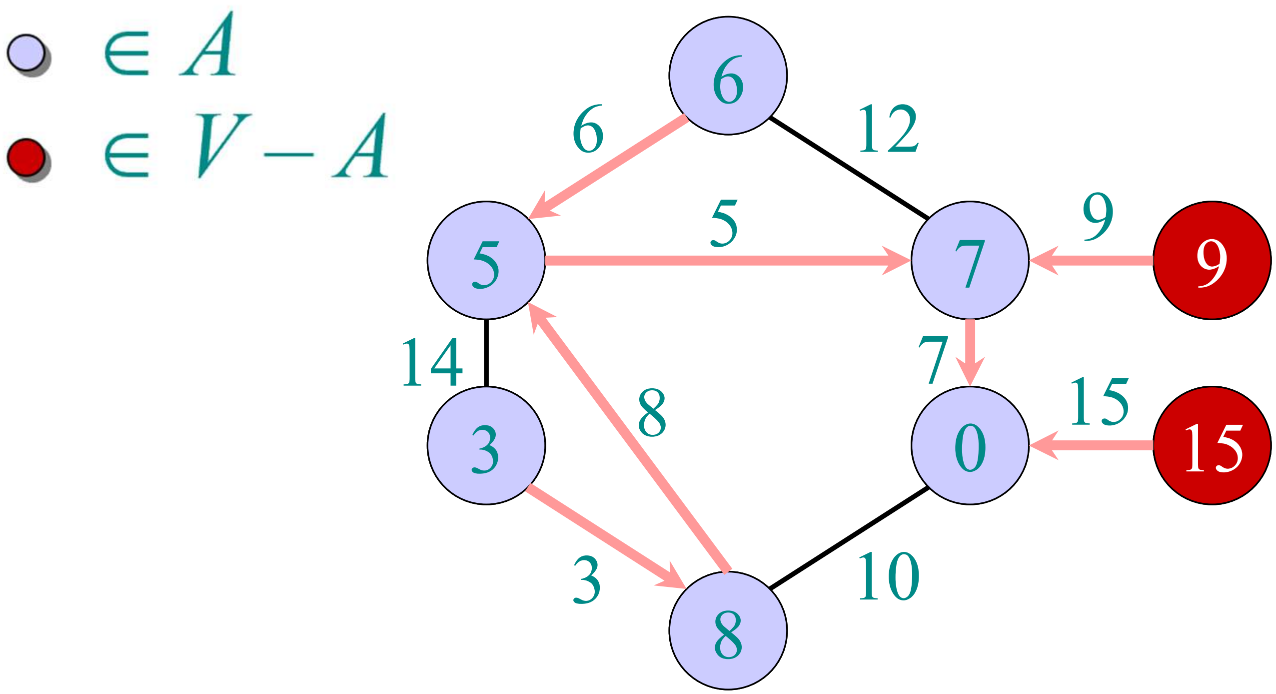

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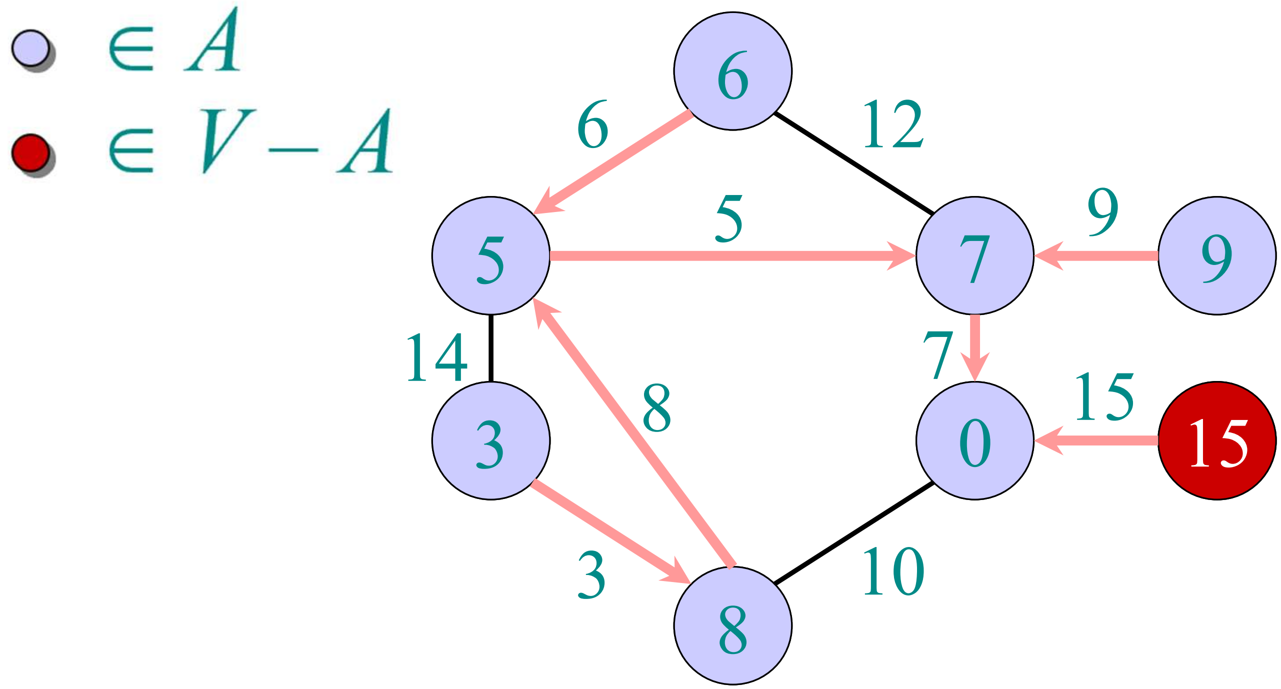

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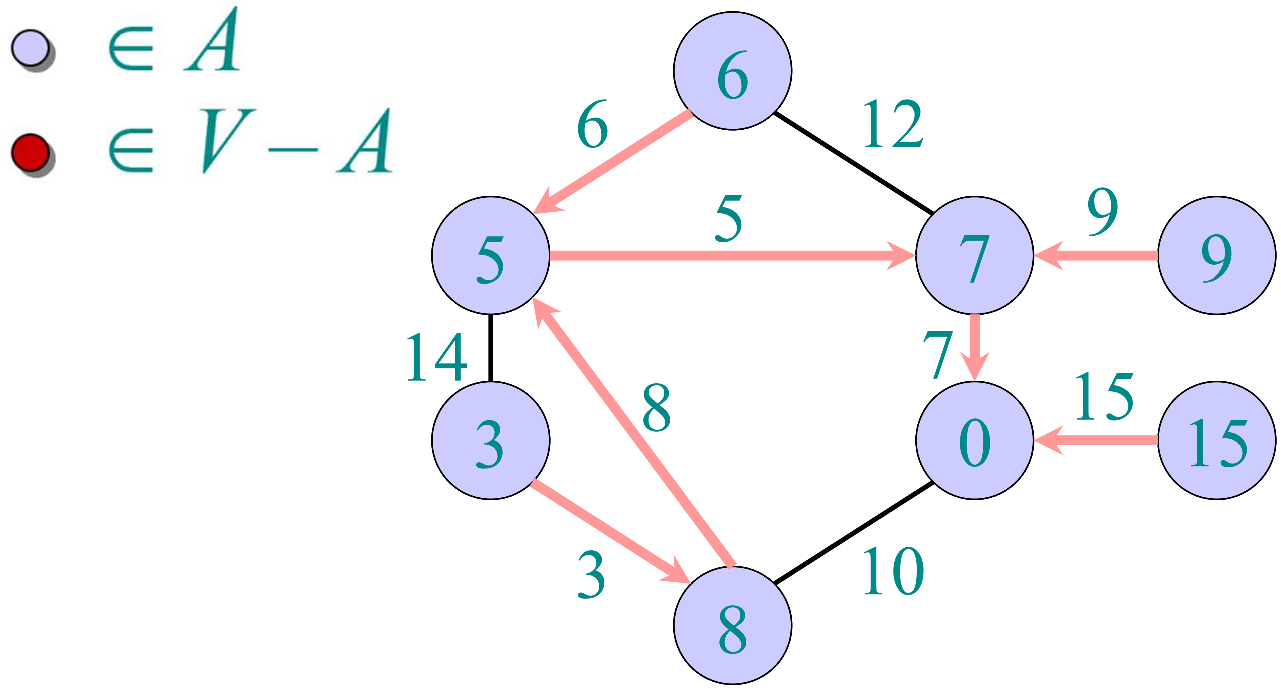


MST algorithms

Kruskal's algorithm (see CLRS Section 23.2):

- Also a greedy algorithm
- At each step, add a least-weight edge that does not cause a cycle to form

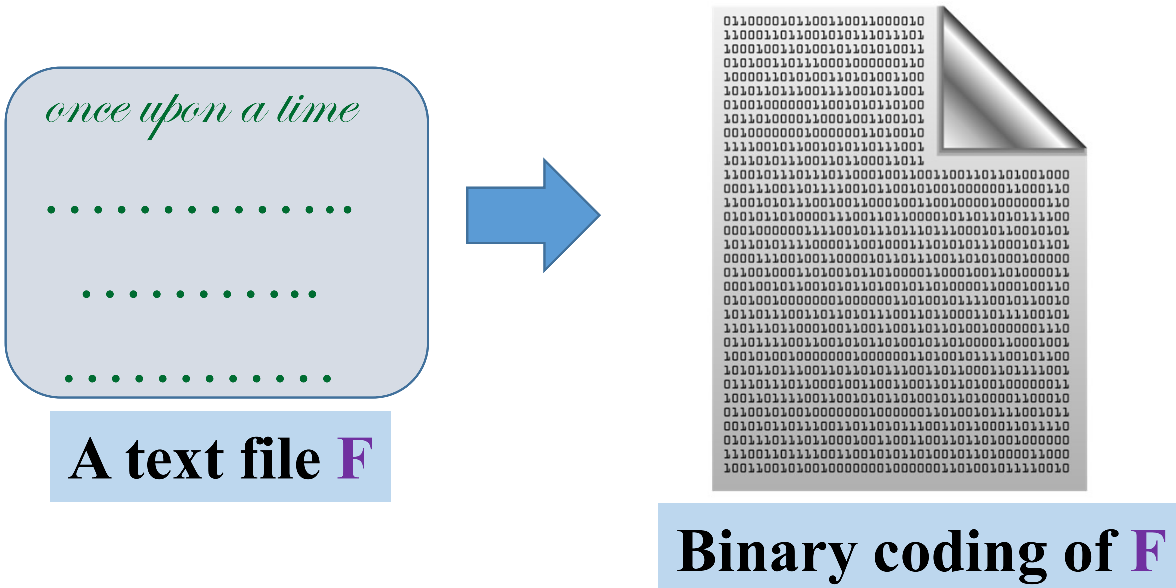
Huffman Code

Applications in data compression, …

Binary coding

Alphabet set A : $\{a_1, a_2, \ldots, a_n\}$

A text file: a sequence of alphabets



Question: How many bits needed to encode a text file with m characters?

Answer: $m \cdot \lceil \log_2 n \rceil$ bits.

Fixed length coding

Alphabet set $\boldsymbol{A}$: $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$

Question: What is a binary coding of $\boldsymbol{A}$?

Answer: $\boldsymbol{\gamma}: \boldsymbol{A} \rightarrow$ **binary strings**

Question: What is a **fixed length** coding of $\boldsymbol{A}$?

Answer: each alphabet ← a unique binary string of length $\lceil \log_2 \boldsymbol{n} \rceil$.

Question: How to decode a **fixed length binary** coding?

Answer: **Easy** ☺

$$\mathbf{0100|1010|0000|1011|} \ldots$$

Fixed length coding

Alphabet set $\boldsymbol{A}$: $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$

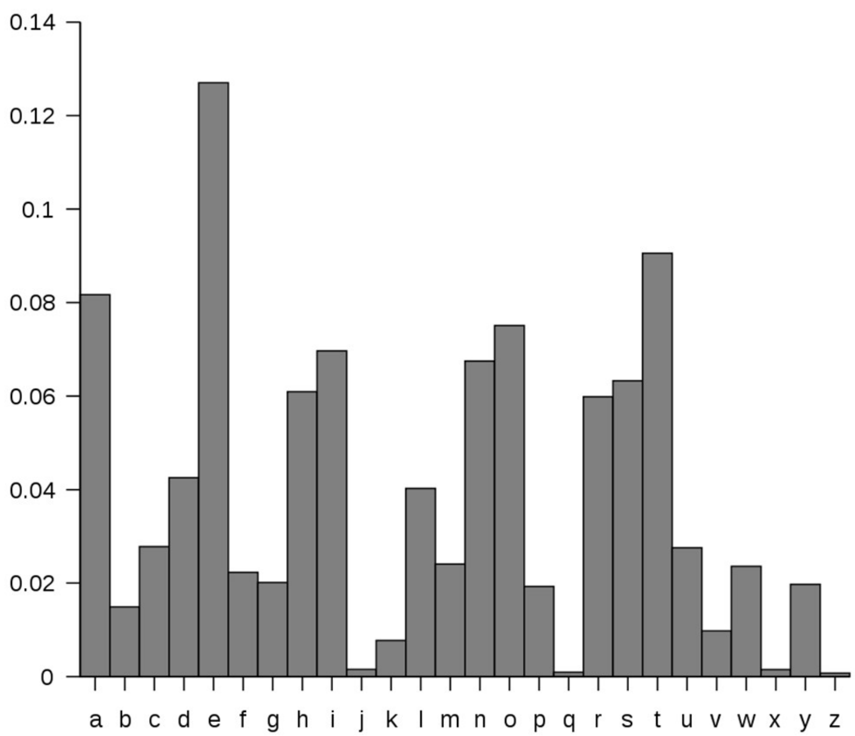
Question: Can we use fewer bits to store alphabet set $\boldsymbol{A}$?

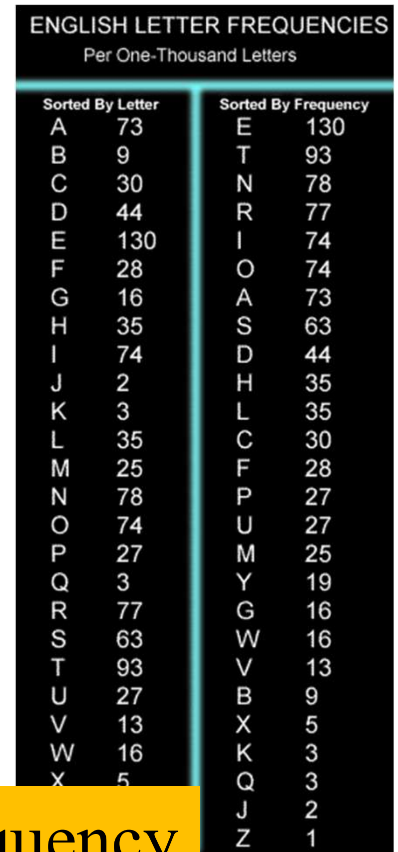
Answer: No.

Question: Can we use fewer bits to store a file?

Answer: **Yes**

Huge variation in the frequency of alphabets in a text





ENGLISH LETTER FREQUENCIES

Per One-Thousand Letters

Sorted By Letter		Sorted By Frequency	
A	73	E	130
B	9	T	93
C	30	N	78
D	44	R	77
E	130	I	74
F	28	O	74
G	16	A	73
H	35	S	63
I	74	D	44
J	2	H	35
K	3	L	35
L	35	C	30
M	25	F	28
N	78	P	27
O	74	U	27
P	27	M	25
Q	3	Y	19
R	77	G	16
S	63	W	16
T	93	V	13
U	27	B	9
V	13	X	5
W	16	K	3
X	5	Q	3
		J	2
		Z	1

http://en.wikipedia.org/wiki/Letter_frequency

Huge variation in the frequency of alphabets in a text

Question: How to exploit variation in the frequencies of alphabets?

Answer:

More frequent alphabets 🡐 coding with **shorter bit string**

Less frequent alphabets 🡐 coding with **longer bit string**

Variable length encoding

Alphabets	Frequency f	Encoding γ
a	0.45	0
b	0.18	10
c	0.15	110
d	0.12	101
e	0.10	111

Average bit length per symbol using γ:

$$\mathbf{ABL}(\gamma) = \sum_{x \in A} f(x) \cdot |\gamma(x)|$$

$= \mathbf{0.45} \times \mathbf{1} + \mathbf{0.18} \times \mathbf{2} + (\mathbf{0.15} + \mathbf{0.12} + \mathbf{0.10}) \times \mathbf{3}$

$= \mathbf{1.92}$

There is a serious problem with the γ encoding. Can you see?

Variable length encoding

Can you fix it?

ets	Frequency f	Encoding γ
a	0.45	0
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Average bit length per symbol using $\boldsymbol{\gamma}$:

$$\mathbf{ABL}(\boldsymbol{\gamma}) = \sum_{x \in A} f(x) \cdot |\boldsymbol{\gamma}(x)|$$

$= \mathbf{0.45} \times \mathbf{1} + \mathbf{0.18} \times \mathbf{2} + (\mathbf{0.15} + \mathbf{0.12} + \mathbf{0.10}) \times \mathbf{3}$

$= \mathbf{1.92}$

Question: How will you decode 01010111?

Answer: $\boldsymbol{abbe}$ or $\boldsymbol{adae}$ ☹

Question: What is the source of this ambiguity?

Answer: $\boldsymbol{\gamma}(\boldsymbol{b})$ is a prefix of $\boldsymbol{\gamma}(\boldsymbol{d})$.

Variable length Coding

Alphabets	Frequency f	Encoding γ
$\boldsymbol{a}$	0.45	0
$\boldsymbol{b}$	0.18	100
$\boldsymbol{c}$	0.15	110
$\boldsymbol{d}$	0.12	101
$\boldsymbol{e}$	0.10	111

Average bit length per symbol using $\boldsymbol{\gamma}$:

$$\mathbf{ABL}(\boldsymbol{\gamma}) = \sum_{x \in A} f(x) \cdot |\boldsymbol{\gamma}(x)|$$

$= \mathbf{0.45} \times \mathbf{1} + \mathbf{0.18} \times \mathbf{3} + (\mathbf{0.15} + \mathbf{0.12} + \mathbf{0.10}) \times \mathbf{3}$

$= \mathbf{2.1}$

Prefix Coding

Definition:

A coding $\boldsymbol{\gamma}(\boldsymbol{A})$ is called **prefix coding** if there do not exist $x, y \in \boldsymbol{A}$ such that

$$\boldsymbol{\gamma}(\boldsymbol{x}) \text{ is } \textbf{prefix} \text{ of } \boldsymbol{\gamma}(\boldsymbol{y})$$

Algorithmic Problem: Given a set $\boldsymbol{A}$ of $\boldsymbol{n}$ alphabets and their frequencies, compute coding $\boldsymbol{\gamma}$ such that

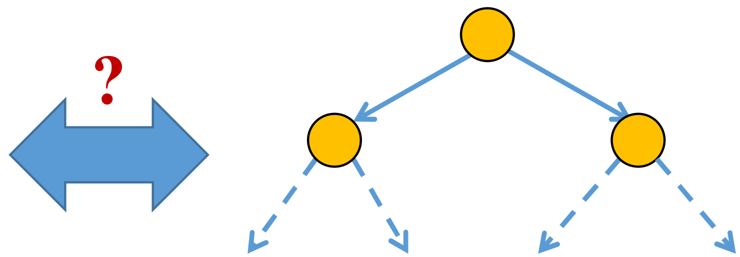
- $\boldsymbol{\gamma}$ is prefix coding
- $\mathbf{ABL}(\boldsymbol{\gamma})$ is **minimum**.

The challenge of the problem

Among all possible binary coding of ***A***, how to find the **optimal prefix coding**?

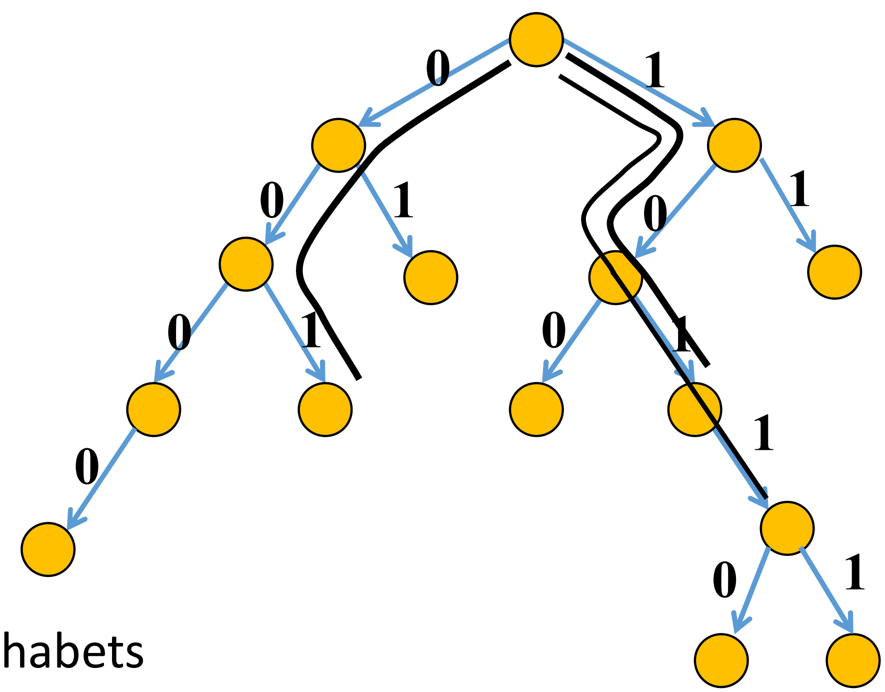
The novel idea of Huffman

Binary coding



Binary tree

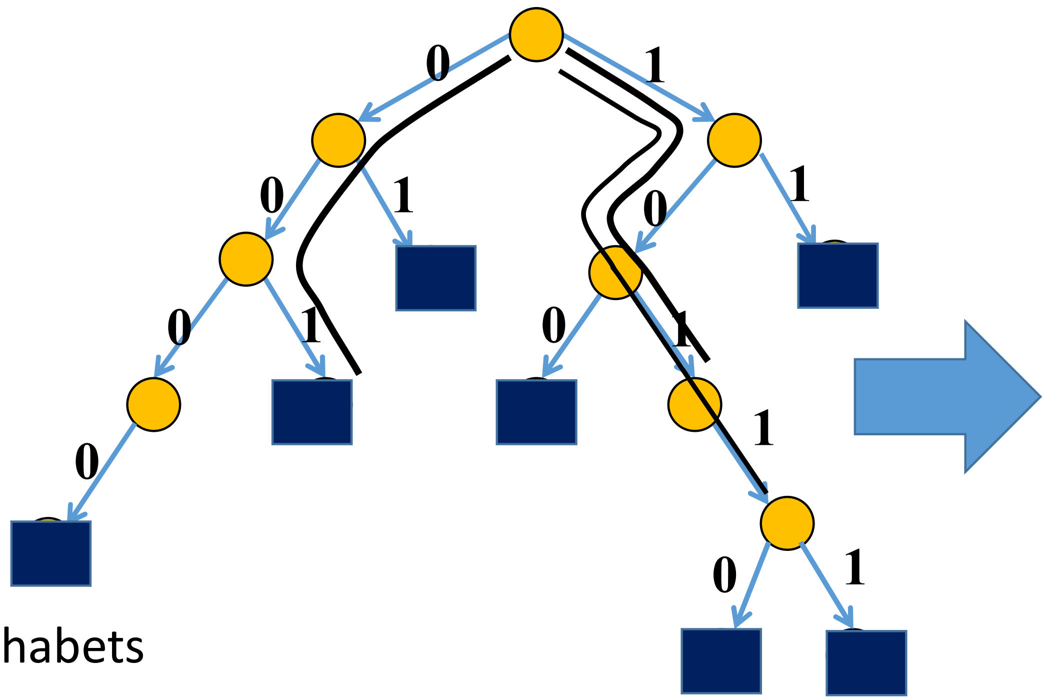
A labeled binary tree



Leaf nodes → alphabets

Code of an alphabet = **Label of path from root**

A labeled binary tree



01,
001,
0000,
11,
100,
10110,
10111

Leaf nodes → alphabets

Code of an alphabet = **Label of path from root**

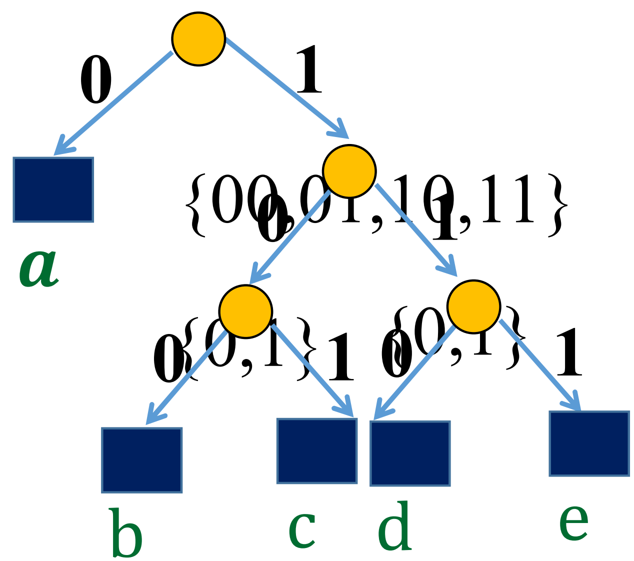
Variable length Coding

Question:

How to build the labeled tree for a prefix code ?

$$\{0,\ 100,\ 101,\ 110,\ 111\}$$

Alphabets	Frequency f	Encoding γ
a	0.45	0
b	0.18	100
c	0.15	110
d	0.12	101
e	0.10	111



Prefix code and Labelled Binary tree

Theorem:

For each prefix code of a set $\boldsymbol{A}$ of $\boldsymbol{n}$ alphabets, there exists a binary tree $\boldsymbol{T}$ on $\boldsymbol{n}$ leaves such that

- There is a bijective mapping between the **alphabets** and the **leaves.**
- The label of a path from root to a leaf node corresponds to the prefix code of the corresponding alphabet.

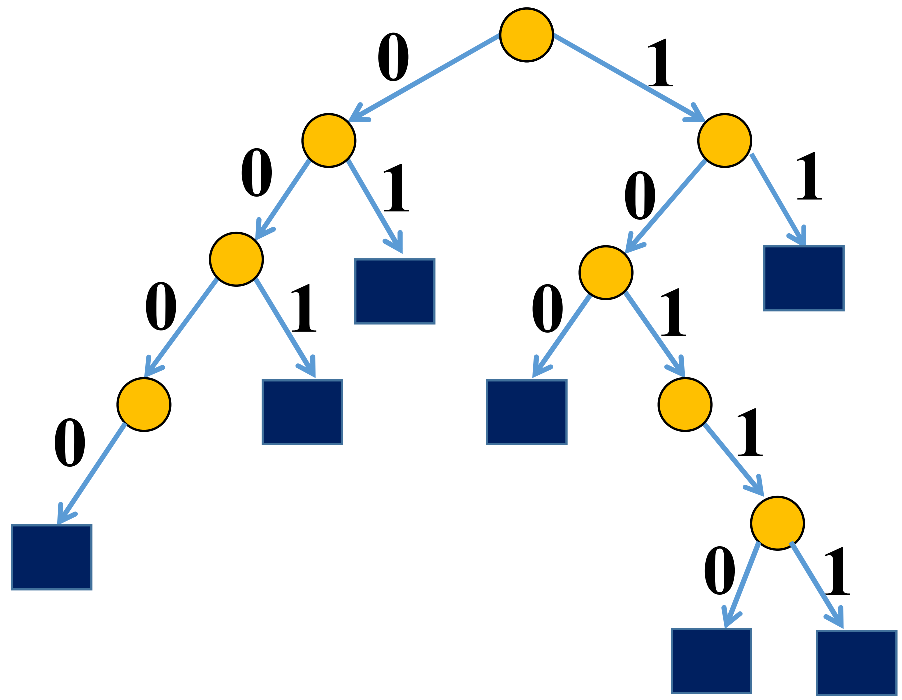
Question: Can you express **Average bit length** of $\boldsymbol{\gamma}$ in terms of its binary tree $\boldsymbol{T}$?

$$\mathbf{ABL}(\boldsymbol{\gamma}) = \sum_{x \in \boldsymbol{A}} f(x) \cdot |\boldsymbol{\gamma}(x)|$$

$$= \sum_{x \in \boldsymbol{A}} f(x) \cdot |\mathbf{depth}_{\mathbf{T}}(x)|$$

Finding the labeled binary tree for an optimal prefix codes

Is the following prefix coding optimal ?



NO

Observations on the binary tree of an optimal prefix code

Lemma:

The binary tree corresponding to optimal prefix coding must be a **full binary tree**:

Every internal node has degree exactly 2.

Question: What next?

We need to see the influence of frequencies on an optimal binary tree.

Let $\boldsymbol{a_1}$, $\boldsymbol{a_2}$,…, $\boldsymbol{a_n}$ be the alphabets of $\boldsymbol{A}$ in non-decreasing order of their frequencies.

Observations on the binary tree of an optimal prefix code

Intuitively, **more frequent** alphabets should be **closer to the root** and

less frequent alphabets should be **farther from the root**.

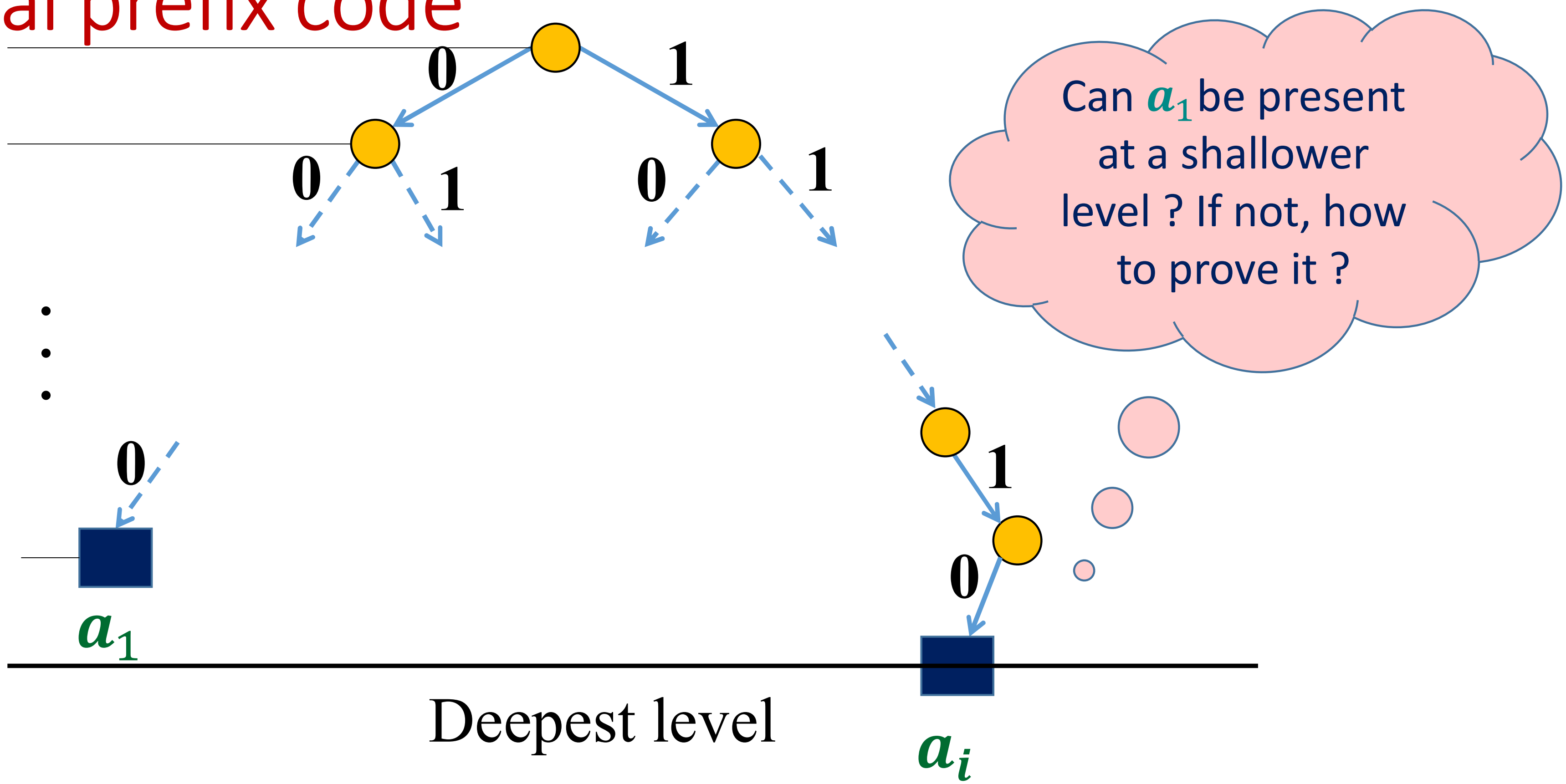
But how to organize them to achieve optimal prefix code ?

- We shall now make some simple observations about the structure of the binary tree corresponding to the optimal prefix codes.

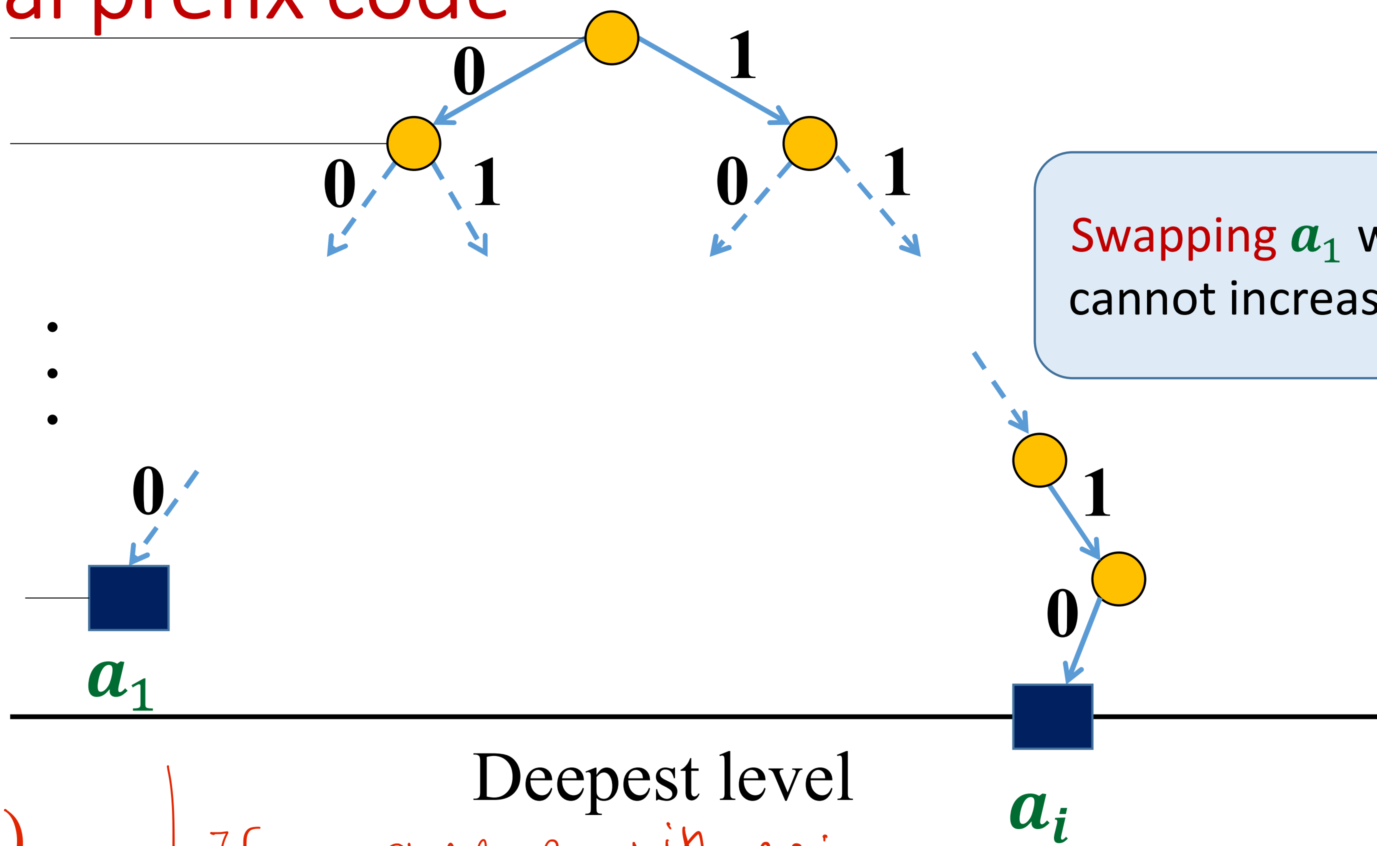
- These observations will be about some local structure in the tree.

- Nevertheless, these observations will play a crucial role in the design of a binary tree with optimal prefix code for given $\boldsymbol{A}$.

Observations on the binary tree of an optimal prefix code



Observations on the binary tree of an optimal prefix code



Swapping a_1 with a_i cannot increase **ABL.**

$f(a_1) \le f(a_i)$

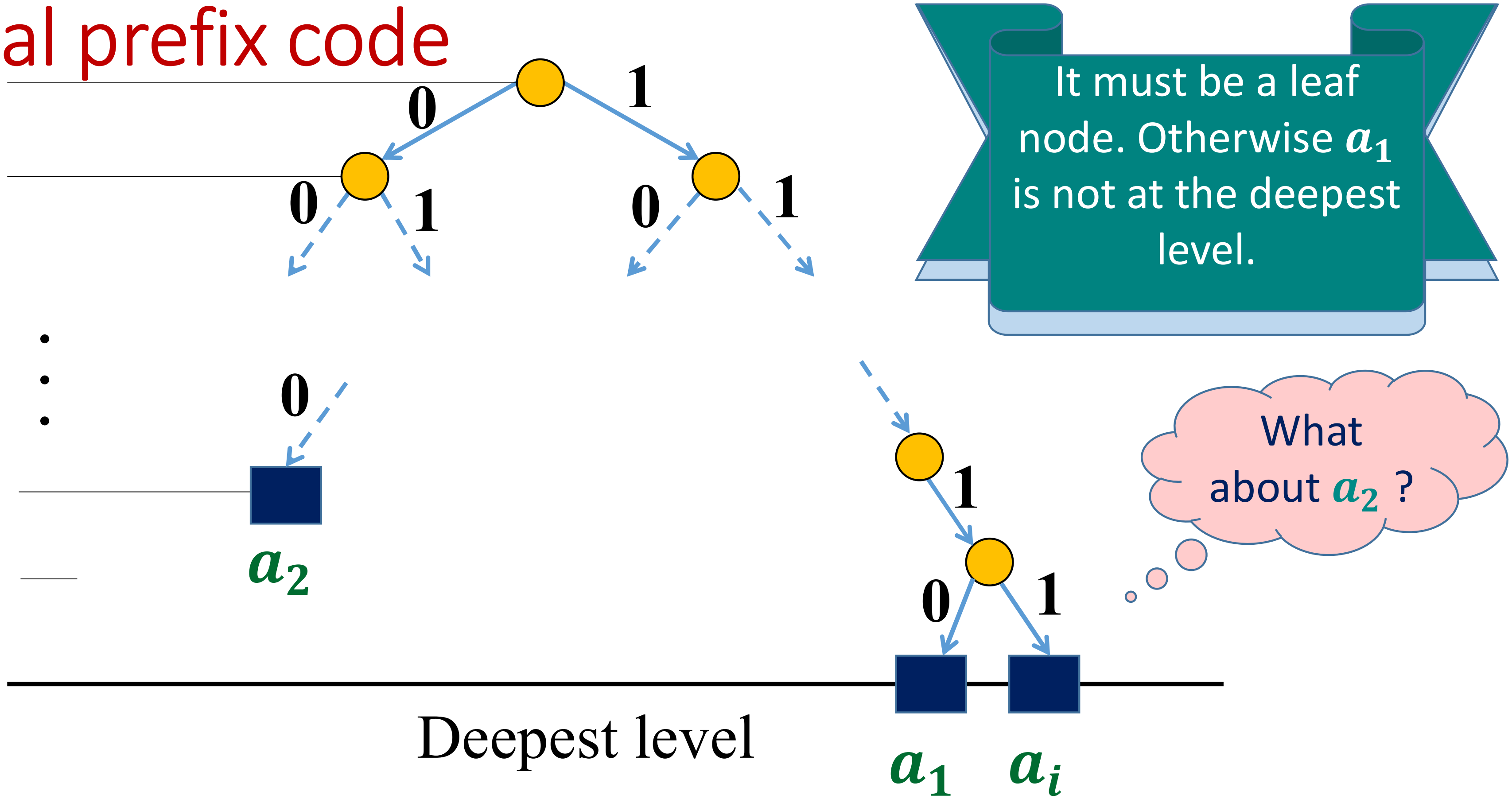
$|y(a_1)| < |y(a_i)|$

If we swap a_1 with a_i:

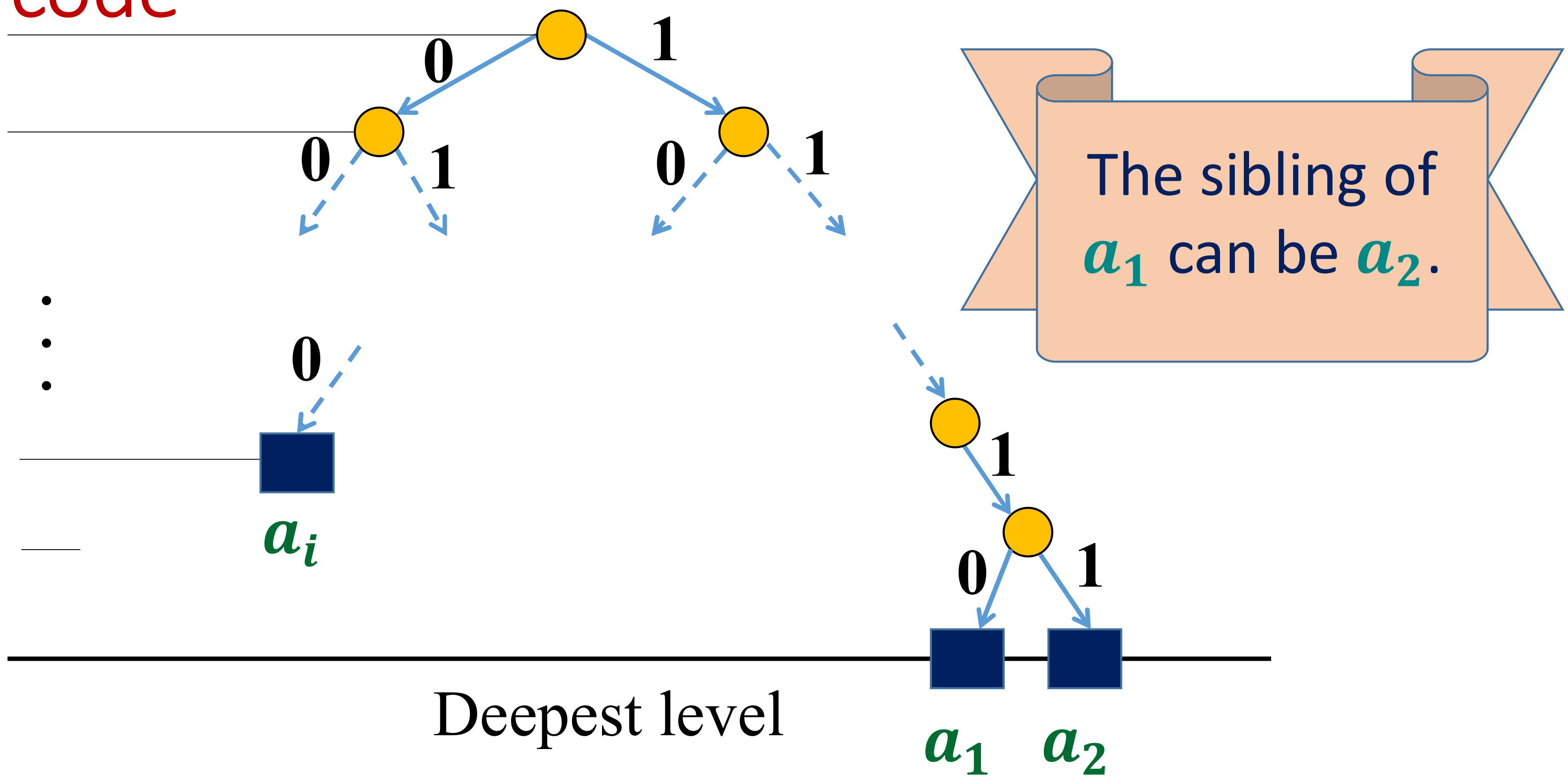
$$ABL_{old} - ABL_{new} = f(a_1)|y(a_1)| + f(a_i)|y(a_i)| - f(a_1)|y(a_i)| - f(a_i)|y(a_1)|$$

$$= (f(a_1) - f(a_i))(|y(a_1)| - |y(a_i)|) \geq 0$$

Observations on the binary tree of an optimal prefix code



Observations on the binary tree of an optimal prefix code



An important observation

Lemma: There *exists an optimal* prefix coding in which $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ appear as siblings in the corresponding labeled binary tree.

Important note: It is inaccurate to claim that "In every optimal prefix coding, $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ appear as siblings in the labeled binary string."

But algorithmic implication of the Lemma mentioned above is quite important:

➔ We just need to focus on that binary tree of optimal prefix coding in which $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ appear as siblings.

This lemma is a powerful hint to the design of optimal prefix code.

See CLRS Section 16.3 for full details

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