W09: Amortisation

CS3230 AY21/22 Sem 2

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Amortisation Introduction

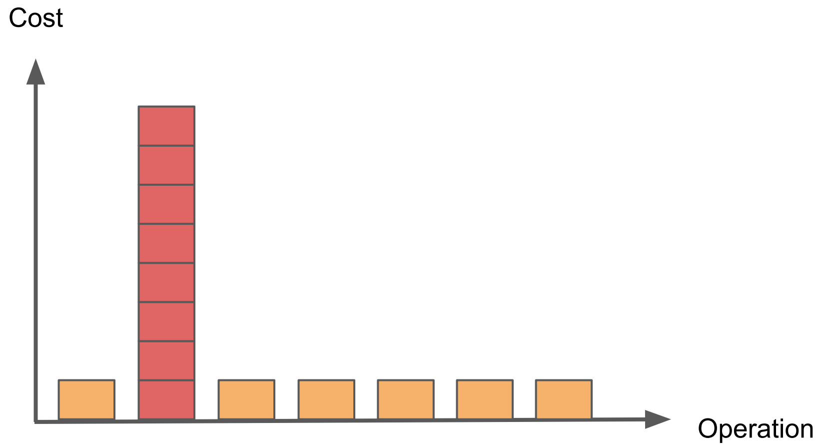
Motivation

- In analysing a sequence of operations on data structure, the easy way out is for you to always take the **worst case running time** of **each** operation

Motivation

- In analysing a sequence of operations on data structure, the easy way out is for you to always take the **worst case running time** of **each** operation

- But! You might be **overcounting**! What if some operations actually take **way less** cost?



Amortisation

Common techniques:

- *Aggregate* method
- *Accounting* method (Banker)
- *Potential* method (Physicist)

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Goal of Amortisation:
To give a guarantee of average performance of each operation **over all operations** so far

Amortisation

Common techniques:

- *Aggregate* method
- *Accounting* method (Banker)
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Note:

- Average-Case analysis is averaging over the random inputs
- Amortisation here has **no randomness**, it averages over total cost of all operations

Goal of Amortisation:

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Amortisation

Common techniques:

- *Aggregate* method (not discussed in this tut. Not as flexible as the other two)
- *Accounting* method (Banker)
- *Potential* method (Physicist)

Note:

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- Amortisation here has **no randomness**, it averages over total cost of all operations

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To give a guarantee of average performance of each operation **over all operations** so far

Accounting Method

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- Every day, you need to spend a bit -- maybe for food, or other needs
  - But you **set aside some money every day too** - $5

I am fully aware rent is not this cheap. Numbers are for illustration purposes only

Accounting Method

Analogy -- Living Expenses:

- Let's say you have daily allowance - $15
- Every day, you need to spend a bit -- maybe for food, or other needs
  - But you **set aside some money every day too** - $5
- At the end of the month: pay rent for $120!
  - Expensive -- it's more than your daily allowance :<

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- Every day, you need to spend a bit -- maybe for food, or other needs
  - But you **set aside some money every day too** - $5
- At the end of the month: pay rent for $120!
  - Expensive -- it's more than your daily allowance :<
  - But! You set aside some money every day. Pay the rent with that!
  - Now rent is "cheap" -- because your savings paid for it and it doesn't take anything from today's daily allowance

Accounting Method

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- Idea: Impose an **extra charge** on **inexpensive operations**, and use it to pay for a later expensive operation
- Charge *i*-th operation a fictitious **amortised cost** $c(i)$, where $1 = 1 unit of time [analogy: your daily allowance]
- The amortised cost will be used for that operation in **two** ways:
  - Immediately use a portion to pay for the **true (actual) cost** $t(i)$ [analogy: daily food]
  - Defer the cost by storing it in a **bank** [analogy: money you set aside]

Accounting Method

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  - Immediately use a portion to pay for the **true (actual) cost** $t(i)$ [analogy: daily food]
  - Defer the cost by storing it in a **bank** [analogy: money you set aside]
- When the expensive operation comes, use money from the bank instead! [analogy: paying rent with savings]

Living Expenses

Scenario: You just started life. Bank contains no money

	What you have to pay for today	Daily Allowance	How much you save
Day	*Actual Cost*	*Amortised Cost*	*To the bank*

Living Expenses

Day 1: You bought Chicken Rice for $10. And you saved $5

	What you have to pay for today	Daily Allowance	How much you save
Day	*Actual Cost*	*Amortised Cost*	*To the bank*
1	*$10*	*$15*	*$5*

Living Expenses

Day 2: You try another food that also costs $10.
Also saved $5

Day	Actual Cost (What you have to pay for today)	Amortised Cost (Daily Allowance)	To the bank (How much you save)
1	$10	$15	$5
2	$10	$15	$5

Living Expenses

Day 3: Not hungry, but bought groceries for $5

	What you have to pay for today	Daily Allowance	How much you save
Day	*Actual Cost*	*Amortised Cost*	*To the bank*
1	*$10*	*$15*	*$5*
2	*$10*	*$15*	*$5*
3	*$5*	*$15*	*$10*

Living Expenses

Day 30: After a sequence of operation, let’s say you have $150 in the bank!

	What you have to pay for today	Daily Allowance	How much you save
Day	*Actual Cost*	*Amortised Cost*	*To the bank*
1	*$10*	*$15*	*$5*
2	*$10*	*$15*	*$5*
3	*$5*	*$15*	*$10*
...	...	...	...
30	*$10*	*$15*	*$5*

Living Expenses

Day 31: Rent payment of $120 comes! You still need to buy food for $10

	What you have to pay for today	Daily Allowance	How much you save
Day	*Actual Cost*	*Amortised Cost*	*To the bank*
1	*$10*	*$15*	*$5*
2	*$10*	*$15*	*$5*
3	*$5*	*$15*	*$10*
...	...	...	...
30	*$10*	*$15*	*$5*
31	*$120 + $10*	*$15*	

Living Expenses

Use the saved-up money first

	What you have to pay for today	Daily Allowance	How much you save
Day	*Actual Cost*	*Amortised Cost*	*To the bank*
1	*$10*	*$15*	*$5*
2	*$10*	*$15*	*$5*
3	*$5*	*$15*	*$10*
...	...	...	...
30	*$10*	*$15*	*$5*
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Living Expenses

The rent is "cheap" (compared to daily allowance) -- the thrifty you in the past paid for it! So focus on today's expenses

	What you have to pay for today	Daily Allowance	How much you save
Day	*Actual Cost*	*Amortised Cost*	*To the bank*
1	*$10*	*$15*	*$5*
2	*$10*	*$15*	*$5*
3	*$5*	*$15*	*$10*
...	...	...	...
30	*$10*	*$15*	*$5*
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Living Expenses

Only costs $10 like usual, but you are ready to pay $15 -- $5 to the bank!

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Day	*Actual Cost*	*Amortised Cost*	*To the bank*
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Living Expenses

Computing Total Costs: It's okay to have excess money in the bank!

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...	...	...	...
30	*$10*	*$15*	*$5*
31	*$120 + $10*	*$15*	*$5*
TOTAL	*$430*	*$465*	

Suppose 31 days = $310
and rent = $120

Living Expenses

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Total Actual Cost ≤ Total Amortised Cost

Total Actual Cost ≤ 465 units

As long as bank never goes negative

Living Expenses

Computing Total Costs: It’s okay to have excess money in the bank!

$$\sum_{i=1}^{n} t(i) \le \sum_{i=1}^{n} c(i)$$

Total Actual Cost ≤ Total Amortised Cost

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Total Actual Cost ≤ Total Amortised Cost
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Living Expenses

Computing Total Costs: It’s okay to have excess money in the bank!

$$\sum_{i=1}^{n} t(i) \leq \sum_{i=1}^{n} c(i)$$

Total Actual Cost ≤ Total Amortised Cost
If Total Amortised Cost = O(n),
Then Total Actual Cost = O(n)

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Total Actual Cost ≤ Total Amortised Cost
Total Actual Cost ≤ 465 units
As long as bank never goes negative

Question 1: T/F on Amortisation

Question 1

Which of the following statements is **false**?

- The amortized cost for insert in dynamic tables is $\Theta(1)$.
- In the accounting method, the amortized cost $\hat{c}_i$ is always greater than the actual cost c_i of an operation.
- $\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$ where $\hat{c}_i$ and c_i are the amortized and actual costs of the i-th operation respectively.

Question 1 (Ans)

Option A: True

Charge \$3:

- \$1 for the insert
- \$2 to the bank

- The amortized cost for insert in dynamic tables is $\Theta(1)$.
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Option A: True

Charge $3:

- $1 for the insert
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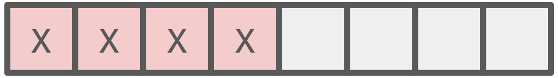
Insert Amortised

```
Let’s say we *just* doubled the table
```



Note: Ignore costs of allocation and deleting elements. Only count cost of inserting and transferring (demo purposes)

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Option A: True

Charge $3:

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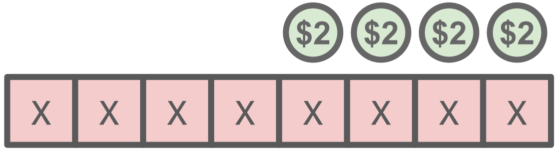
Insert Amortised

Then let’s say we insert 4 more times. Each one of them stores $2 to the bank



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Question 1 (Ans)

Option A: True

Charge $3:

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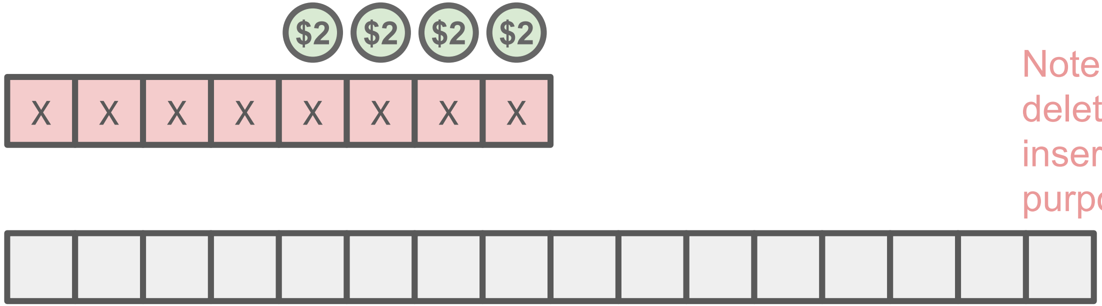
Insert Amortised

```
When we insert one more time, need to double!
```



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Option A: True

Charge $3:

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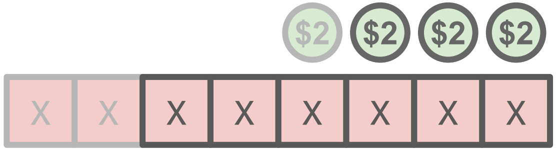
Insert Amortised

Cost of transferring -- paid by the bank. An element pays for two transfers!



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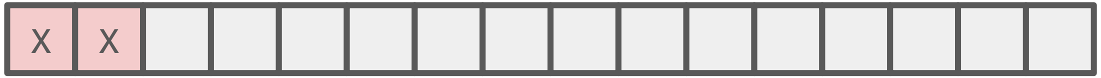

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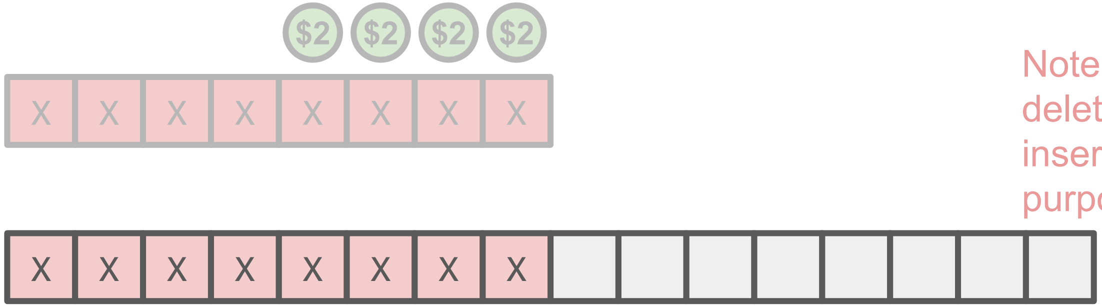

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Option A: True

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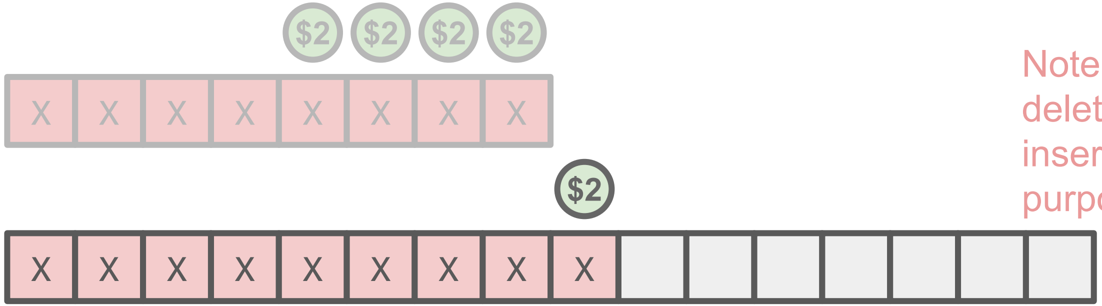
Insert Amortised

`Do the actual insert!`



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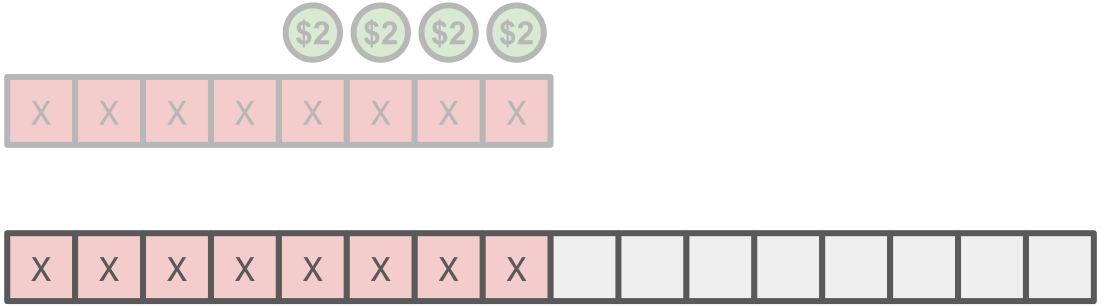
Option B: False

Recall insertion example -- our amortised cost for the operation is $3, but the **actual cost is more** than $3! (Probably something like 9 here. Copy 8 elements and insert the actual element)



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Option C: True

Total amortised cost must be larger than sum of actual costs

$$\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \hat{c}_i$$

$$0 \leq \sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i$$

Question 2:
Fancier Queue (Accounting Method)

Question 2

- Consider a data structure that is based on a queue with these operations:
  - `enqueue(a)` - Add element `a` into the queue
  - `dequeue()` - Dequeue a single element from the queue
  - `delete(k)` - Dequeue `k` elements from the queue
  - `add(A)` - enqueue all elements in `A`

Question 2

```
Start off with an empty queue
```

- `enqueue(a)` - Add element `a` into the queue
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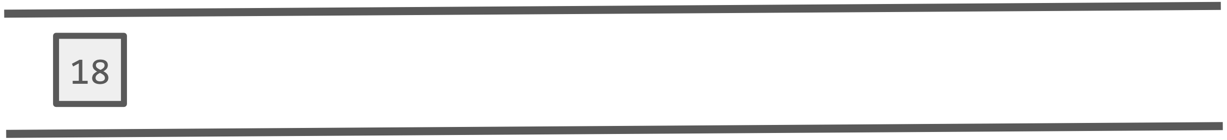
Question 2

`enqueue(18)`



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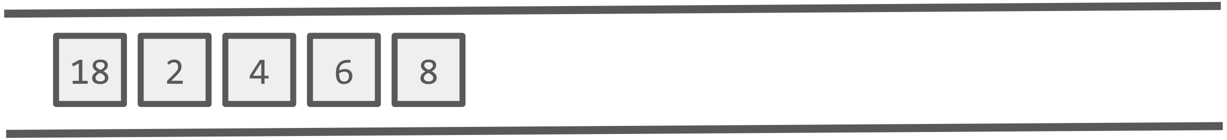
Question 2

`add([2, 4, 6, 8])`



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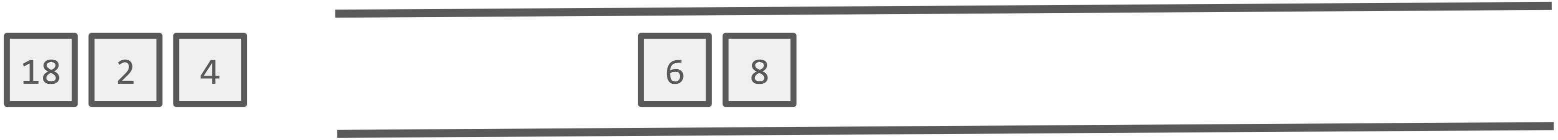
Question 2

`delete(3)`



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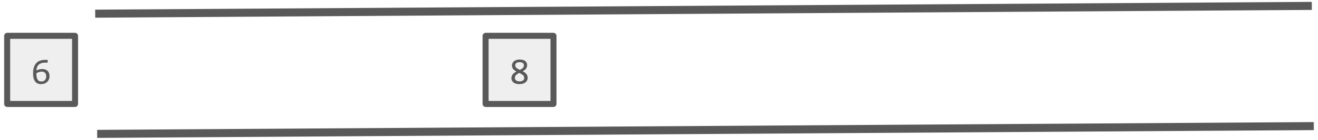
Question 2

`dequeue()`



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- Consider a data structure that is based on a queue with these operations:
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- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised *O(1)* time
  - `add(A)` runs in amortised *O(|A|)* time

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- Consider a data structure that is based on a queue with these operations:
  - `enqueue(a)` - Add element `a` into the queue
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- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised *O(1)* time
  - `add(A)` runs in amortised *O(|A|)* time
- **Using accounting method**, prove these time complexities
  - To use accounting method, you need to state the charge (this is the amortised cost)

Question 2

- Consider a data structure that is based on a queue with four operations:
  - ENQUEUE(a): Add the element a into the queue
  - DEQUEUE(): Dequeue a single element from the queue
  - DELETE(k): Dequeue k elements from the queue
  - ADD(A): Enqueue all elements in A
- **Claim**: ENQUEUE, DEQUEUE and DELETE run in amortized O(1) time while ADD runs in amortized O(|A|) time.
- Using accounting method, can you show that these time complexities are correct?
- (Please state the charge for each operation.)

Question 2 Idea

- Consider a data structure that is based on a queue with these operations:
  - `enqueue(a)` - Add element `a` into the queue
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- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised $O(1)$ time
  - `add(A)` runs in amortised $O(|A|)$ time

- Naive analysis of delete is $O(k)$ time -- but we now want amortised $O(1)$ time!

Question 2 Idea

- Consider a data structure that is based on a queue with these operations:
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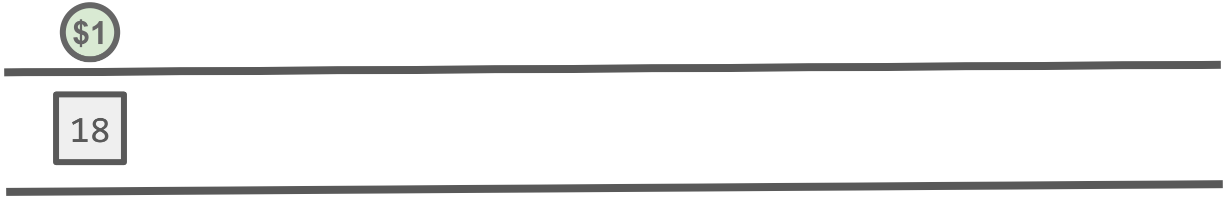
- Naive analysis of delete is $O(k)$ time -- but we now want amortised $O(1)$ time!
- Pre-charge other operations to “pay in advance” for deletion

Question 2

`enqueue(18)`

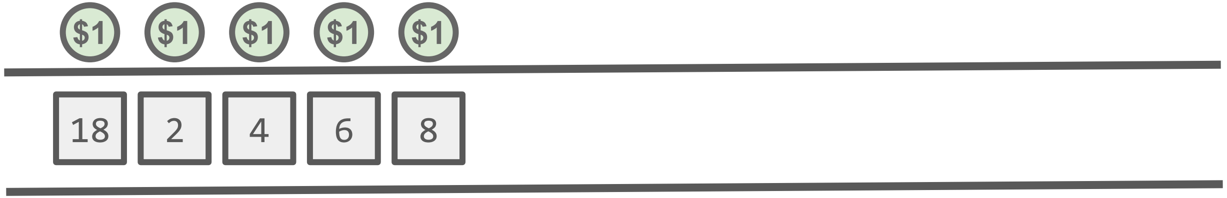


- `enqueue(a)` - Charge $2:
  - $1 for the actual insertion
  - $1 in the bank, for future dequeue

- `enqueue(a)` - Add element `a` into the queue
- `dequeue()` - Dequeue a single element from the queue
- `delete(k)` - Dequeue `k` elements from the queue
- `add(A)` - enqueue all elements in `A`

Question 2

`add([2, 4, 6, 8])`



- `add(a)` - Charge $2|A|:
  - Consists of |A| enqueues. Each charged $2
    - $1 for the actual insertion
    - $1 in the bank, for future dequeue

- `enqueue(a)` - Add element `a` into the queue
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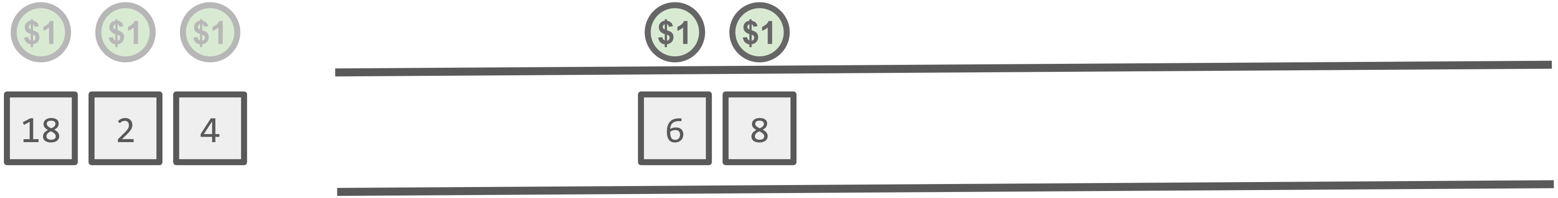
Question 2

`delete(3)`

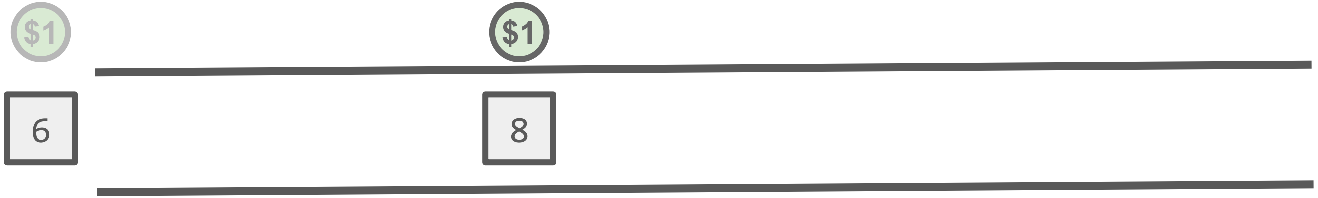


- `delete(k)` - Charge **$0**:
  - The actual cost is **paid for already**
  - k elements deleted with $k from the bank

- `enqueue(a)` - Add element `a` into the queue
- `dequeue()` - Dequeue a single element from the queue
- `delete(k)` - Dequeue `k` elements from the queue
- `add(A)` - enqueue all elements in `A`

Question 2

`dequeue()`



- `dequeue()` - Charge **$0**:
  - The actual cost is **paid for already**
  - Delete using $1 from bank

- `enqueue(a)` - Add element `a` into the queue
- `dequeue()` - Dequeue a single element from the queue
- `delete(k)` - Dequeue `k` elements from the queue
- `add(A)` - enqueue all elements in `A`

One final thing to argue

- **Does the bank ever go negative?**

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- **Does the bank ever go negative?**

- After each insertion of element, $1 is associated to each element in the queue
- Hence when we dequeue that element, we can use the $1 in the bank associated to that element. i.e. there is always enough money in the bank

Q2 Summary

Operation	*Actual Cost*	*Amortised Cost*	*Effect on bank*
`enqueue(a)`	*$1*	*$2*	*$1*
`add(A)`	*$|A|*	*$2|A|*	*$|A|*
`dequeue()`	*$1*	*$0*	*-$1*
`delete(k)`	*$k*	*$0*	*-$k*

- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised *O(1)* time
  - `add(A)` runs in amortised *O(|A|)* time

Q2 Summary

Tip: It's okay if you decide to "overcharge". e.g. Say amortised cost is \$5. Don't need to be "tight"! Sometimes overcharging makes it easier to argue things

Operation	*Actual Cost*	*Amortised Cost*	*Effect on bank*
`enqueue(a)`	*\$1*	*\$2*	*\$1*
`add(A)`	*\$|A|*	*\$2|A|*	*\$|A|*
`dequeue()`	*\$1*	*\$0*	*-\$1*
`delete(k)`	*\$k*	*\$0*	*-\$k*

- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised $O(1)$ time
  - `add(A)` runs in amortised $O(|A|)$ time

Potential Method

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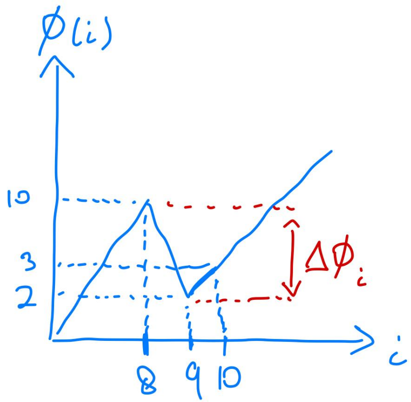

Not to scale

$\phi(8) = 10$, $\phi(9) = 2$, $\phi(10) = 3$

$\Delta\phi_9 = 2 - 10 = -8$ (Change in potential after 9^{th} op)

$\Delta\phi_{10} = 3 - 2 = 1$ (Change in potential after 10^{th} op)

Potential Method Analogy (Source)

- Water tank:

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- Drawing a lot of water is a “cheap” operation, because of the water pressure (the potential energy) pushing it out for you [Similar to accounting method. Where your ‘stored up energy’ will pay for the ‘expensive operation’]

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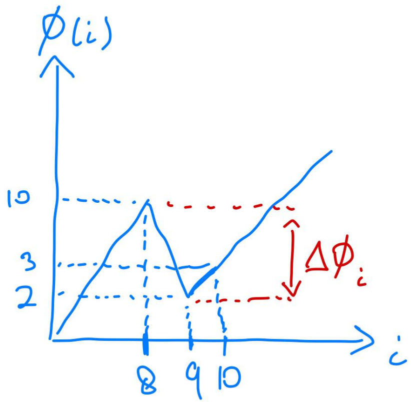
Sorry, not to scale :p



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Game plan to show why potential method works:

1. Define an amortised cost $c(i)$ that is the actual cost $t(i)$ but adjusted
2. Show that SUM of amortised cost ≥ SUM of actual cost
3. Conclude that SUM of amortised cost is $O(f(n))$.
   So we know that SUM of actual cost is $O(f(n))$

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In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

- t(i) is small, then we want $\Delta\phi_i$ to be positive and small. So $c(i)$ is still small
- t(i) is **large**, then we want $\Delta\phi_i$ to be **negative** and large! So $c(i)$ is **also small**

Potential Method

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Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i\text{-}1) = t(i) + \Delta\phi_i$

Here, we will argue the case for $\phi(0) = 0$ only

Goal: Show that SUM of amortised cost ≥ SUM of actual cost

Potential Method

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$$\sum_{i=1}^{n} c(i) = \sum_{i=1}^{n} \left(t(i) + \phi(i) - \phi(i-1) \right)$$

Potential Method

By our definition of c(i)

Goal: Show that SUM of amortised cost ≥ SUM of actual cost

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$$\sum_{i=1}^{n} c(i) = \sum_{i=1}^{n} \left(t(i) + \phi(i) - \phi(i-1)\right)$$

$$\begin{aligned} = &\ t(n) + \phi(n) - \phi(n-1) \\ &+ t(n-1) + \phi(n-1) - \phi(n-2) \\ &\ \vdots \\ &+ t(2) + \phi(2) - \phi(1) \\ &+ t(1) + \phi(1) - \phi(0) \end{aligned}$$

Potential Method

Expanding the sums!

Goal: Show that SUM of amortised cost ≥ SUM of actual cost

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Potential Method

These terms cancel! (This is called 'Telescoping')

Goal: Show that SUM of amortised cost ≥ SUM of actual cost

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$$= \left(\sum_{i=1}^{n} t(i)\right) + \phi(n) - \phi(0)$$

Potential Method

Sum up over t(i), and the remaining not-cancelled $\phi(n)$ and $\phi(0)$

Goal: Show that SUM of amortised cost ≥ SUM of actual cost

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Potential Method

Copying over from prev slide

Goal: Show that SUM of amortised cost ≥ SUM of actual cost

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Potential Method

Restriction on $\phi(0) = 0$

Goal: Show that SUM of amortised cost ≥ SUM of actual cost

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$$\geq \sum_{i=1}^{n} t(i) \qquad [\phi(i) \geq 0]$$

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Restriction on $\phi(i) \geq 0$

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Goal: Show that SUM of amortised cost ≥ SUM of actual cost

Shown!

Potential Method

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   So we know that SUM of actual cost is $O(f(n))$

Since $\sum_{i=1}^{n} c(i) \geq \sum_{i=1}^{n} t(i)$,

If $\sum_{i=1}^{n} c(i) = O(f(n))$

then $\sum_{i=1}^{n} t(i) = O(f(n))$

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If $\sum_{i=1}^{n} c(i) = O(f(n))$

then $\sum_{i=1}^{n} t(i) = O(f(n))$

It means that all we need to bound is just the **total amortised cost**

And we bound the **total actual cost** *for free!*

Potential Method - Recipe

- Try to select a suitable $\boldsymbol{\phi}$, so that for the costly operation, $\Delta\boldsymbol{\phi}_i$ is negative to such an extent that it *nullifies* or *reduces* the effect of actual cost.

Potential Method - Recipe

- Try to select a suitable $\boldsymbol{\phi}$, so that for the costly operation, $\Delta\boldsymbol{\phi}_i$ is negative to such an extent that it *nullifies* or *reduces* the effect of actual cost.
- **Question**: How to find such a suitable potential function $\boldsymbol{\phi}$?

Try to view carefully the costly operation and see if there is some quantity that is "**decreasing**" during the operation.

Question 3:
Fancier Queue (Potential Method)

Question 3

- Consider a data structure that is based on a queue with these operations:
  - `enqueue(a)` - Add element `a` into the queue
  - `dequeue()` - Dequeue a single element from the queue
  - `delete(k)` - Dequeue `k` elements from the queue
  - `add(A)` - enqueue all elements in `A`
- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised *O(1)* time
  - `add(A)` runs in amortised *O(|A|)* time
- **Using potential method**, prove these time complexities
  - To use potential method, state the potential function

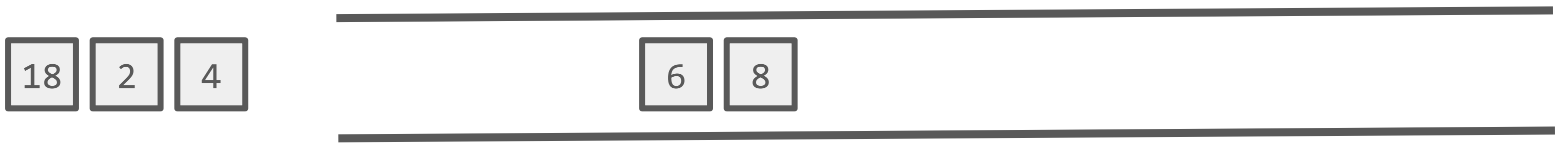
Question 3

- Consider a data structure that is based on a queue with four operations:
  - ENQUEUE(a): Add the element a into the queue
  - DEQUEUE(): Dequeue a single element from the queue
  - DELETE(k): Dequeue k elements from the queue
  - ADD(A): Enqueue all elements in A
- **Claim**: ENQUEUE, DEQUEUE and DELETE run in amortized $O(1)$ time while ADD runs in amortized $O(|A|)$ time.
- Using Potential method, can you show that these time complexities are correct?
- (Please state your potential function.)

Question 3 Idea

- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised *O(1)* time
  - `add(A)` runs in amortised *O(|A|)* time

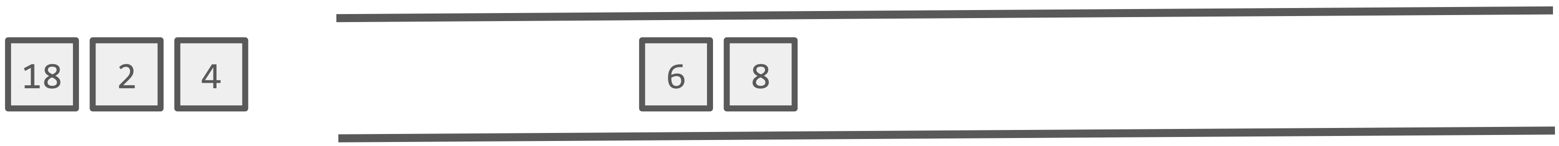
- The delete is our **costly** operation that we want to "nullify" - what's decreasing during deletion?



Question 3 Idea

- **Claim:**
  - `enqueue(a), dequeue(), delete(k)` runs in amortised *O(1)* time
  - `add(A)` runs in amortised *O(|A|)* time

- The delete is our **costly** operation that we want to “nullify” - what’s decreasing during deletion?
- **The number of elements in the queue!**



Question 3 Solution

- ϕ is the **Potential function** associated with the algorithm / data structure
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- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = number of elements in the queue after i^{th} operation

Question 3 Solution

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = number of elements in the queue after i^{th} operation

For `enqueue(a)`:

- Actual cost $t(i) = 1$

Question 3 Solution

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = number of elements in the queue after i^{th} operation

For `enqueue(a)`:

- Actual cost $t(i) = 1$
- **One new element**: $\phi(i) - \phi(i - 1) = \Delta\phi_i = 1$

Question 3 Solution

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = number of elements in the queue after i^{th} operation

For `enqueue(a)`:

- Actual cost $t(i) = 1$
- **One new element**: $\phi(i) - \phi(i - 1) = \Delta\phi_i = 1$
- Amortised Cost $c(i) = t(i) + \Delta\phi_i = 1 + 1 = 2$

Question 3 Solution

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = number of elements in the queue after i^{th} operation

For `dequeue()`:

- Actual cost $t(i) = 1$
- **One LESS element**: $\phi(i) - \phi(i - 1) = \Delta\phi_i = -1$
- Amortised Cost $c(i) = t(i) + \Delta\phi_i = 1 + (-1) = 0$

Question 3 Solution

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i\text{-}1)$ = potential **difference** going from $(i\text{-}1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i\text{-}1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = number of elements in the queue after i^{th} operation

For `delete(k)`:

- Actual cost $t(i) = k$
- **k LESS element**: $\phi(i) - \phi(i - 1) = \Delta\phi_i = -k$
- Amortised Cost $c(i) = t(i) + \Delta\phi_i = k + (-k) = 0$

Question 3 Solution

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i\text{-}1)$ = potential **difference** going from $(i\text{-}1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i\text{-}1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = number of elements in the queue after i^{th} operation

For `add(A)`:

- Actual cost $t(i) = |A|$
- **|A| new element**: $\phi(i) - \phi(i - 1) = \Delta\phi_i = |A|$
- Amortised Cost $c(i) = t(i) + \Delta\phi_i = |A| + |A| = 2|A|$

Potential Method -- to fulfill

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ = *number of elements in the queue after* i^{th} *operation*

Potential Method -- to fulfill

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$ *= number of elements in the queue after* i^{th} *operation*

In the beginning, we have empty queue → $\phi(0) = 0$

Never do we have negative elements → $\phi(i) \geq 0$

Question 4: Dynamic Table (Delete-only)

Question 4

```
def DynamicTableDeleteOnly(T):
n = number of elements in T
delete last element x from T
n -= 1
if n == 0:
  free(T)
else:
  if n == size(T)/2:
    T' = create_table(n/2)
    copy all elements from T to T'
    free(T)
    T = T'
```

The idea:

- Start with a full table
- Once number of elements is half the elements it can hold, halve it

Note: we do not call any other operations on `T`

Show that amortised cost of `DynamicTableDeleteOnly` is O(1)

Question 4

Delete x from T;

$n \leftarrow n - 1$;

If ($n = 0$)

  free(T);

Else

  If(n = size(T)/2)

  { $T' \leftarrow$ createTable($n/2$);

   copy(T,T');

   free(T);

   $T \leftarrow T'$

  }

Note, T is the dynamic table that supports only deletions.

Using Potential method show that the amortized cost of each Deletion operation is O(1).

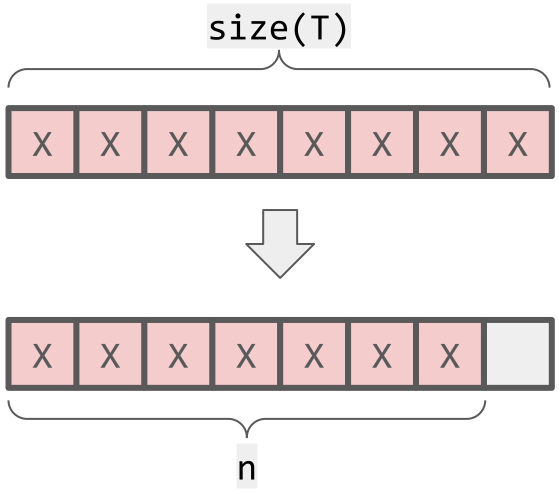
(State your potential function.)

Question 4 Idea

There are two possibilities when deleting:

Case 1: Table doesn't shrink (just delete)

This is cheap!



Question 4 Idea

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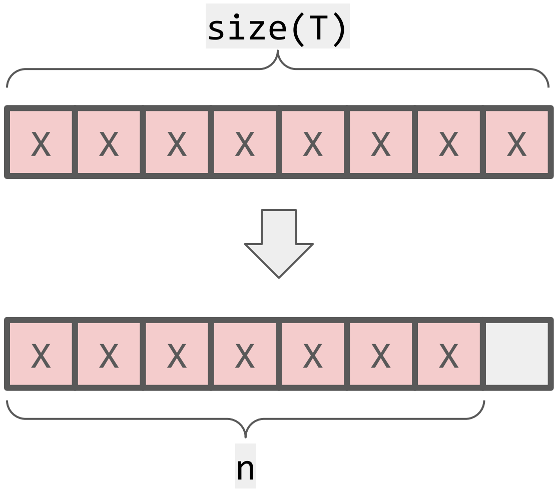
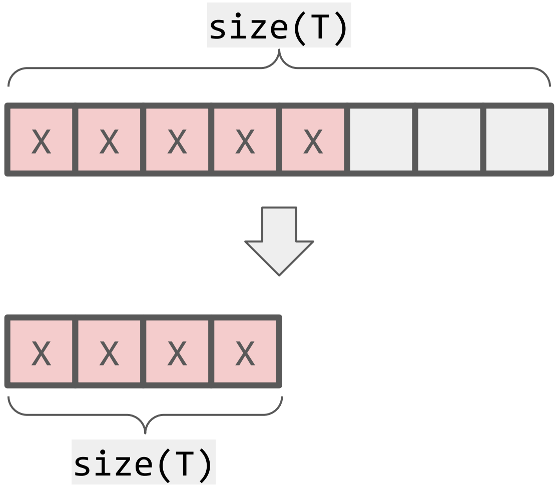


Case 2: Table **shrinks**

This operation is expensive!



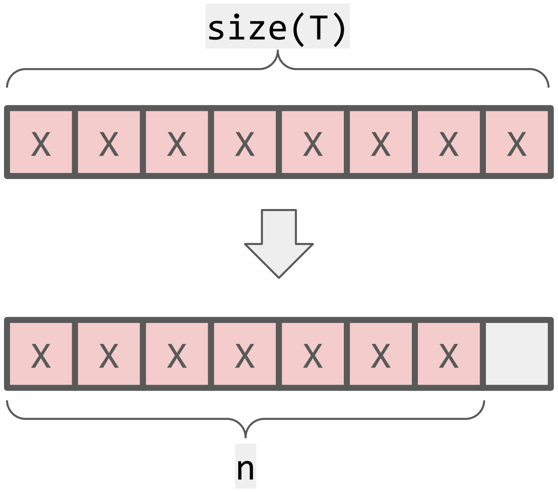
Look for a “decreasing” quantity during expensive operation.
Potential function?

Question 4 Idea

There are two possibilities when deleting:

Case 1: Table doesn’t shrink (just delete)

This is cheap!



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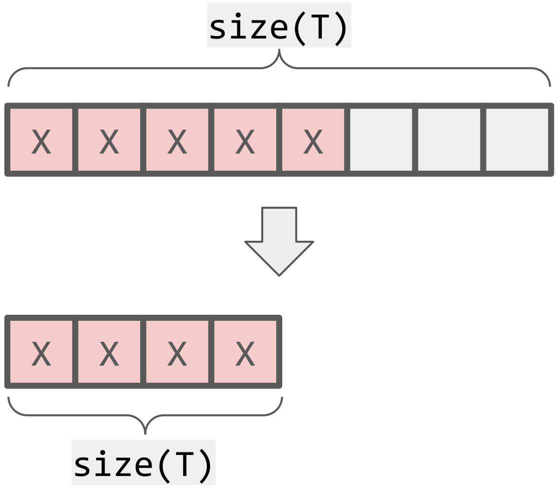


Question 4 Idea

Look for a "decreasing" quantity during expensive operation.
Potential function? `size(T) - n`
The number of empty slots!

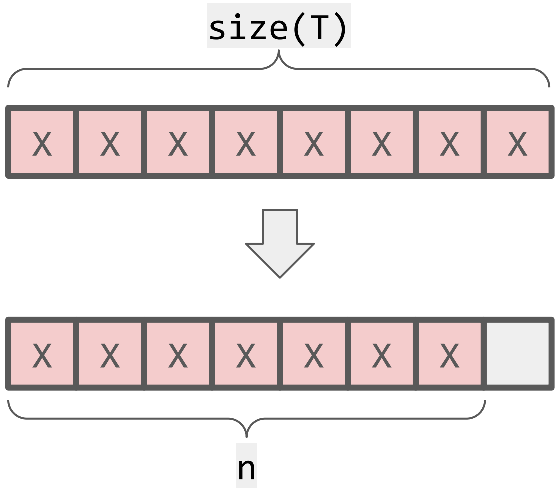
There are two possibilities when deleting:

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

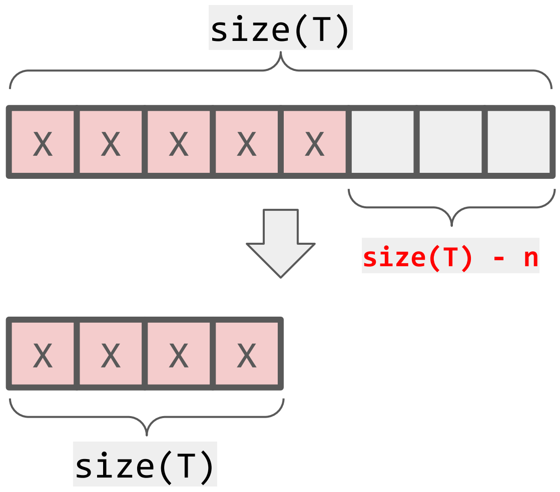
Case 1: Table doesn't shrink (just delete)

This is cheap!



Case 2: Table **shrinks**

This operation is expensive!



Question 4 Idea

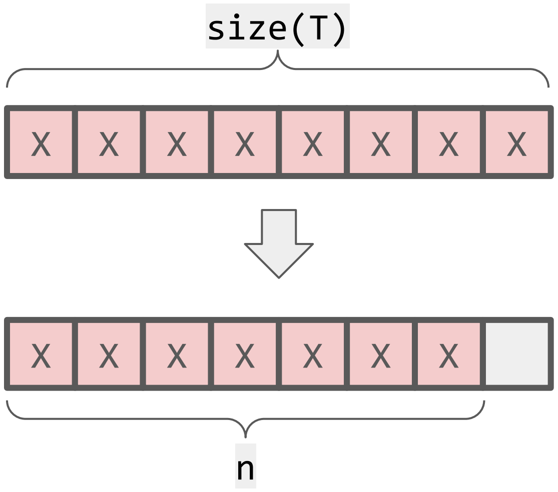
Look for a “decreasing” quantity during expensive operation.
Potential function? `size(T) - n`
The number of empty slots!
Verify that it satisfies these properties

Additionally:
- $\phi(0) = 0$
- $\phi(i) \geq 0$

There are two possibilities when deleting:

Case 1: Table doesn’t shrink (just delete)

This is cheap!



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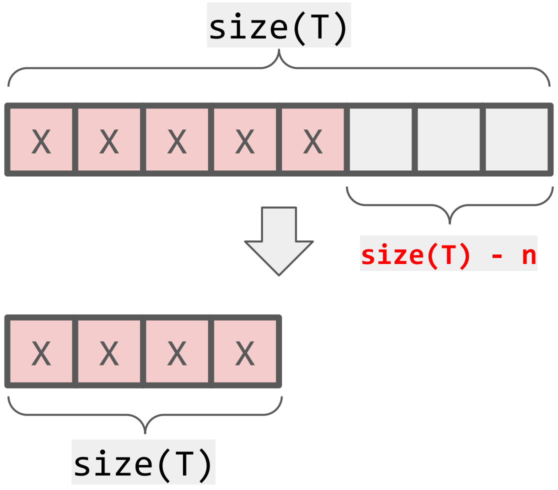
This operation is expensive!



Question 4 Soln

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

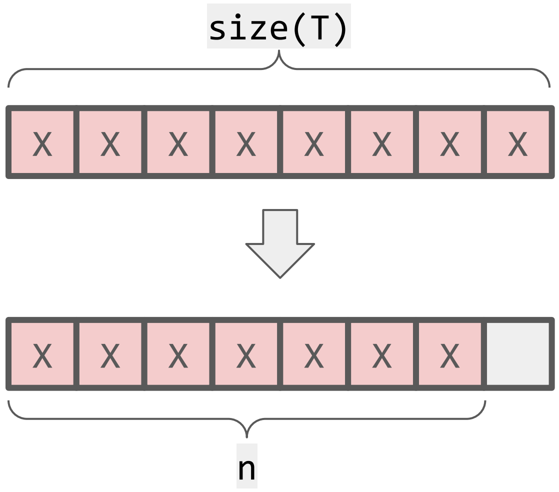
Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

Case 1: Table doesn't shrink (just delete)

This is cheap!



Actual Cost	$\Delta\phi_i$	*Amortised Cost*

Case 1

Remember:

amortised = actual + potential difference

Question 4 Soln

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
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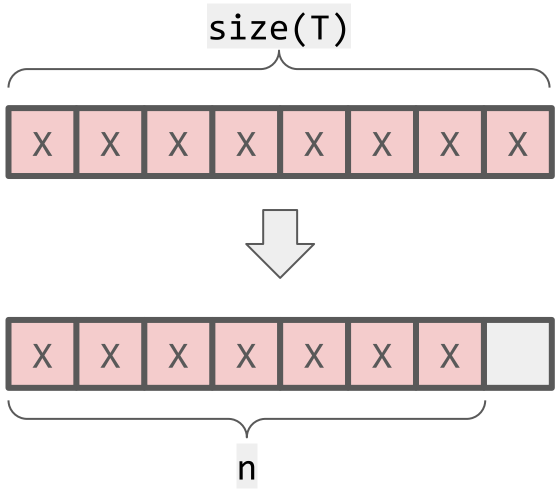
$\phi(i)$: `size(T) - n`

Case 1: Table doesn't shrink (just delete)

This is cheap!



Actual Cost	$\Delta\phi_i$	*Amortised Cost*
1		

Case 1

```
Actual Cost = 1 because we just delete
that element
```

Question 4 Soln

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- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
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- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

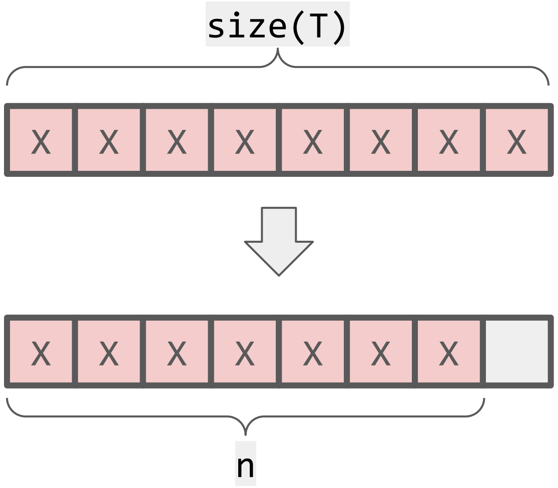
Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

Case 1: Table doesn't shrink (just delete)

This is cheap!



Actual Cost	$\Delta\phi_i$	*Amortised Cost*
1	1	

Case 1

```
Potential difference is 1 - 0 = 1
Recall: potential is empty slots in table
```

Question 4 Soln

- ϕ is the **Potential function** associated with the algorithm / data structure
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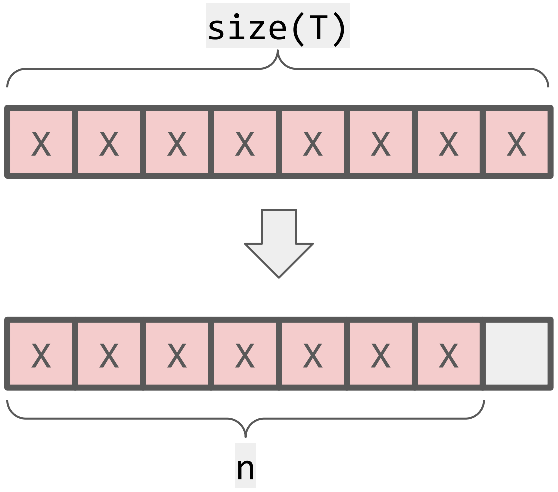
Additionally:
- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

Case 1: Table doesn't shrink (just delete)

This is cheap!



Actual Cost	$\Delta\phi_i$	*Amortised Cost*
1	1	2

Case 1

`Amortised Cost is 1 + 1 = 2`

Question 4 Soln

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- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

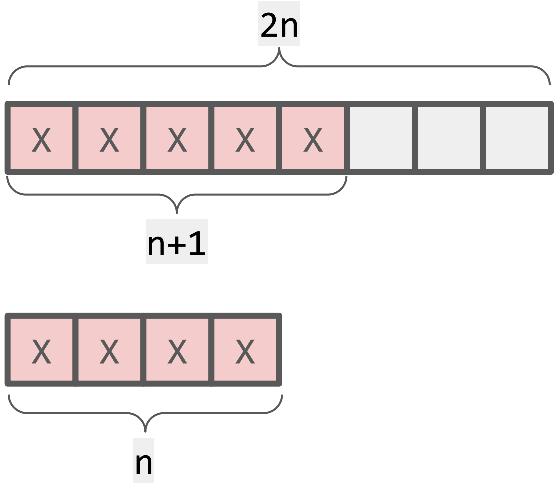
Additionally:
- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

Actual Cost	$\Delta\phi_i$	*Amortised Cost*

Case 2: Table **shrinks**

This operation is expensive!



Case 2

For simplicity, I will base number of elements to the final number of elements

Question 4 Soln

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i\text{-}1)$ = potential **difference** going from $(i\text{-}1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i\text{-}1) = t(i) + \Delta\phi_i$

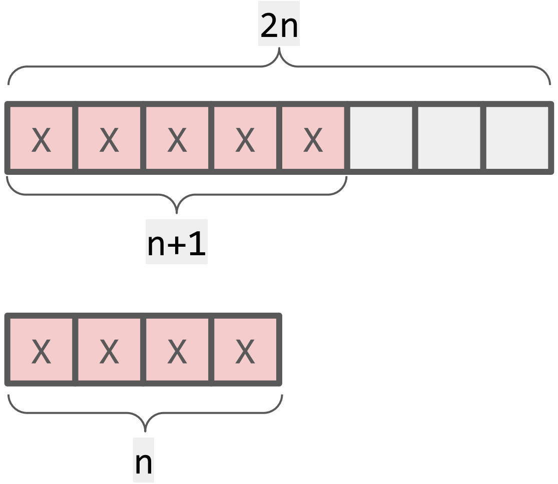
Additionally:
- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

Actual Cost	$\Delta\phi_i$	*Amortised Cost*
$n+1$		

Case 2: Table **shrinks**

This operation is expensive!



Case 2

Actual cost is n+1, because we delete one of the elements, and copy the rest

Question 4 Soln

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

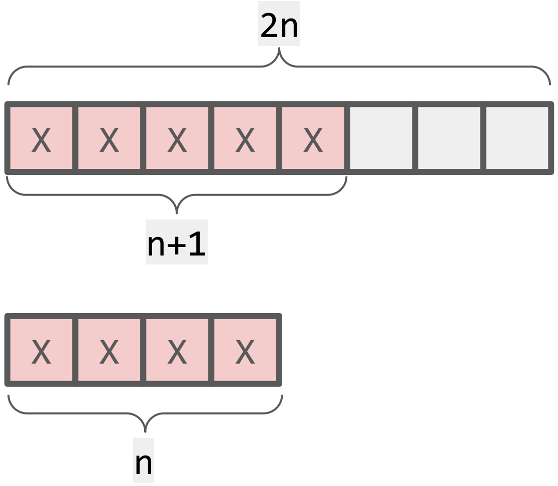
Actual Cost	$\Delta\phi_i$	*Amortised Cost*
$n+1$	$1-n$	

Case 2

```
Potential After: 0
Potential Before: (2n) - (n+1) = n-1
Potential Difference = (0) - (n-1) = 1-n
```

Case 2: Table **shrinks**

This operation is expensive!



Question 4 Soln

- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

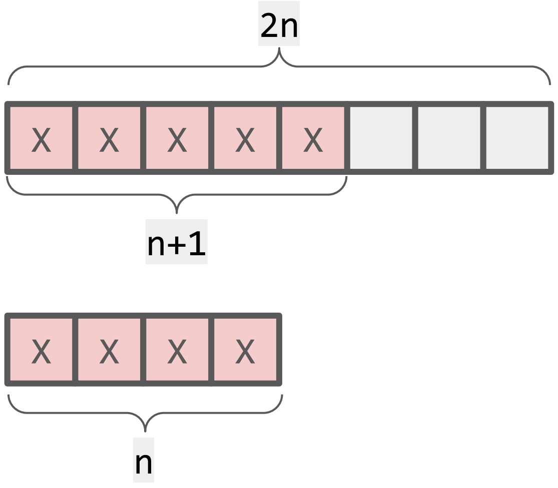
- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

Actual Cost	$\Delta\phi_i$	*Amortised Cost*
$n+1$	$1-n$	2

Case 2: Table **shrinks**

This operation is expensive!



Case 2

`Amortised Cost: (n+1) + (1-n) = 2`

Q4 Summary

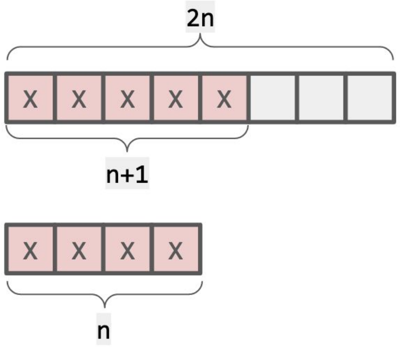
- ϕ is the **Potential function** associated with the algorithm / data structure
- $\phi(i)$ is the **Potential at the end of the i^{th} operation**
- $\Delta\phi_i = \phi(i) - \phi(i-1)$ = potential **difference** going from $(i-1)^{th}$ to i^{th} operation
- Amortised cost $c(i)$ and true cost $t(i)$
- In Potential Method, definition of $c(i) = t(i) + \phi(i) - \phi(i-1) = t(i) + \Delta\phi_i$

Additionally:

- $\phi(0) = 0$
- $\phi(i) \geq 0$

$\phi(i)$: `size(T) - n`

Cases	Actual Cost	$\Delta\phi_i$	Amortised Cost
Case 1: No shrink	*1*	*1*	*2*
Case 2: Shrinking	*n+1*	*1-n*	*2*



Thus, the amortised cost of this deletion operation is O(1)