CS3230 Assignment 2

Zhuang Jianning
A0241561M

1. (1 point) Indicate, for each pair of expressions (A, B) in the table below, whether A is O, o, Ω, ω, or Θ of B. Your answer should be in the form of the table with "yes" or "no" written in each box. No proof is required.

	A	B	O	o	Ω	ω	Θ
(a)	$n^3 + 4n$	$(\lg n)^{2022}$	no	no	yes	yes	no
(b)	n^9	1.01^n	yes	yes	no	no	no
(c)	$n^{1.5}$	$n \lg n$	no	no	yes	yes	no
(d)	2^n	3^n	yes	yes	no	no	no
(e)	$\lg(n^4)$	$\lg(n^8)$	yes	no	yes	no	yes
(f)	n^{10}	$n^{\lg n}$	yes	yes	no	no	no

a) $(\lg n)^k = o(n^d) \quad \forall k, d > 0$ (by lemma 2.2.4 i))

$(\lg n)^{2022} = o(n^3 + 4n)$

$n^3 + 4n = \omega((\lg n)^{2022})$ (by complementarity property)

b) $n^d = o(a^n) \quad \forall d > 0, a > 1$ (by lemma 2.2.4 ii))

$n^9 = o(1.01^n)$

c) $\lim_{n\to\infty} \frac{n^{1.5}}{n \lg n} = \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{\lg n} \overset{\text{L'Hopital}}{=} \lim_{n\to\infty} \frac{\frac{1}{2} n^{-\frac{1}{2}}}{\lg e / n} = \lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{2 \lg e} = \infty$

$n^{1.5} = \omega(n \lg n)$

d) $\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

$2^n = o(3^n)$

e) $\lim_{n\to\infty} \frac{\lg(n^4)}{\lg(n^8)} = \lim_{n\to\infty} \frac{4 \lg n}{8 \lg n} = \frac{1}{2}$

$\lg(n^4) = \Theta(\lg(n^8))$

f) $\lim_{n\to\infty} \frac{n^{10}}{n^{\lg n}} = \lim_{n\to\infty} n^{10 - \lg n} = \lim_{n\to\infty} e^{\ln(n^{10-\lg n})} = \lim_{n\to\infty} e^{\overbrace{(10-\lg n)\ln n}^{-\infty}} = 0$

$n^{10} = o(n^{\lg n})$

2.

suppose $f(n) = O(g(n))$

Is it always true that $2^{f(n)} = O(2^{g(n)})$?

Proof by counter example:

let $f(n) = 2n$ $\quad 2n = O(n)$

$g(n) = n$

However, $\lim_{n \to \infty} \frac{2^{f(n)}}{2^{g(n)}} = \lim_{n \to \infty} \frac{2^{2n}}{2^n} = \lim_{n \to \infty} 2^n = \infty$

$2^{f(n)} = \omega(2^{g(n)}) \quad \neq O(2^{g(n)})$

$\therefore$ It is not always true that $2^{f(n)} = O(2^{g(n)})$ when $f(n) = O(g(n))$

3.

$f(2) = 2$

$f(n) = 2f(\frac{n}{2}) + n \qquad n = 2^i$ for all integer $i \geq 1$

Prove by induction that $f(n) = n\lg n$

Base case: $f(2) = 2 = 2\lg 2$ //

Inductive step: Assume $f(2^k) = 2^k \lg 2^k$

$f(2^{k+1}) = 2f(\frac{2^{k+1}}{2}) + 2^{k+1}$

$= 2f(2^k) + 2^{k+1}$

$= 2\cdot(2^k \cancel{\lg 2^k} k) + 2^{k+1}$

$= k \cdot 2^{k+1} + 2^{k+1}$

$= (k+1)2^{k+1}$

$= 2^{k+1} \lg 2^{k+1}$ //

conclusion: For all integers $i \geq 1 \Rightarrow n = 2^i$ powers of 2, $f(n) = n\lg n$ //