CS3230 2021/22 Semester 2 Midterm Practice

1. True or False? Recall that lg denotes the logarithm with base 2.

   (a) $2^n = \Theta(2^{2n})$
   (b) $\ln(n^2) = O(\lg n)$
   (c) $2^{\sqrt{\lg n}} = \omega(\lg n)$
   (d) $\sum_{i=1}^{n} \frac{n}{i} = o(n^2)$
   (e) $\lg n = \Omega(1)$

2. Solve the following recurrences by providing tight asymptotic bounds.

   (a) $T(n) = 3T(n/4) + \sqrt{n}$
   (b) $T(n) = T(n^{1/5}) + \lg n$
   (c) $T(n) = T(0.7n) + T(0.2n) + n$

3. Let $B : \{0,1\}^4 \to \{0,1\}$ be defined by $B(x_1, x_2, x_3, x_4) = (x_1 \texttt{ and } x_2) \texttt{ or } (x_3 \texttt{ and } x_4)$. Prove that any deterministic algorithm for computing B must query all four input bits.

4. Show that at least $2n-1$ comparisons are needed to merge two sorted arrays $A = [A_1, A_2, \ldots, A_n]$ and $B = [B_1, B_2, \ldots, B_n]$ into one sorted array by any (deterministic) comparison-based algorithm.

   (**Hint**: Recall that your goal is to come up with two pairs of inputs (A, B) and (A', B') that have different mergings but which cannot be distinguished by an algorithm making at most $2n-2$ comparisons. Take $A = [1, 3, 5, \ldots, 2n-1]$ and $B = [2, 4, \ldots, 2n]$. Define A' and B' based on how the algorithm acts on A and B.)

5. Let $\mathcal{H}$ be a universal family of hash functions mapping a universe $\mathcal{U}$ to $\{1, \ldots, M\}$. Let x and y be two different elements of $\mathcal{U}$. Are the following always true or not?

   (a)
   $$\Pr_{h \in \mathcal{H}}[h(x) = 1] \leq \frac{1}{M}$$

   (b)
   $$\Pr_{h \in \mathcal{H}}[h(x) = h(y) = 1] \leq \frac{1}{M^2}$$

6. Suppose you are throwing n balls into two bins, labeled A and B. Each ball goes into bin A with probability 1/2 and bin B with probability 1/2, and the balls are thrown independently. Let N_A be the total number of balls in bin A after all n balls have been thrown.

   (a) Let X_i be the indicator random variable that equals 1 when the i-th ball falls in bin A and equals 0 otherwise. What is $\mathbb{E}[X_i]$? What is $\mathbb{E}[X_i^2]$? What is $\mathbb{E}[X_i X_j]$ for $i \neq j$?
   (b) Compute $\mathbb{E}[N_A]$ and $\mathbb{E}[N_A^2]$. (**Hint**: Write N_A in terms of the indicator random variables $X_1, \ldots, X_n$.)