CS3230 Midterm Q2

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2. a)

$$T(n) = T(n-1) + 3230n$$
$$= T(n-2) + 3230(n-1) + 3230n$$
$$= \vdots$$
$$= T(1) + 3230(n + n-1 + \cdots 2)$$
$$= 3230\left(\frac{n(n+1)}{2}\right)$$
$$= \Theta(n^2)$$

b) Using master theorem

$a=7,\ b=2 \Rightarrow n^{\log_b a} \approx n^{2.8073}$ $\qquad f(n) = n^3$

$f(n) = \Omega(n^{\log_b a + \varepsilon})$ for $\varepsilon = 0.1 > 0$

$7\left(\frac{n}{2}\right)^3 = \frac{7}{8}n^3 \le cn^3$ for $c = \frac{7}{8}$

$\therefore\ T(n) = \Theta(n^3)$

c) Using master theorem

$a=1,\ b=\frac{3}{2} \Rightarrow n^{\log_b a} = n^0$ $\qquad f(n) = \lg n + 1$

$f(n) = \Theta(n^0 \lg^1 n)$

$\therefore\ T(n) = \Theta(\lg^2 n)$

d)

$T(n) = \Theta(\sqrt{n})$

e) Let X be the rv representing number of bins with 3 balls.

X_i be indicator rv such that $X_i = 1$ if ith bin has 3 balls, 0 otherwise

$$E[X_i] = P(X_i = 1) = P(i\text{th bin has 3 balls}) = \binom{n}{3} \frac{1}{n}^3 \left(1-\frac{1}{n}\right)^{n-3}$$

Expected fraction

$$E\left[\frac{X}{n}\right] = \frac{1}{n} E[X] = \frac{1}{n} \sum_{i=1}^{n} E[X_i]$$

$$= \frac{1}{n} \cdot n \cdot \binom{n}{3} \frac{1}{n}^3 \left(1-\frac{1}{n}\right)^{n-3}$$

$$= \binom{n}{3}\left(\frac{1}{n}\right)^3 \left(\frac{n-1}{n}\right)^{n-3}$$

As $n \to \infty$ $\lim_{n\to\infty} \binom{n}{3}\left(\frac{1}{n}\right)^3\left(\frac{n-1}{n}\right)^{n-3} = \frac{1}{6} \cdot \frac{1}{e}$

$$= \underline{\underline{\frac{1}{6e}}}$$

f) At each stage, $P[X = x_1] = \prod_{k=2}^{n} \frac{k^2-1}{k^2}$

After $k = 2022$ $P[X = x_1] = \prod_{k=2}^{2022} \frac{k^2-1}{k^2}$

k	2	3	4	5	6	7
	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{5}{8}$	$\frac{9}{15}$	$\frac{14}{24}$	$\frac{20}{35}$

$\to$ after $k=2$ numerator $+ i$, denominator $= i^2 - 1$

$k = 2022 \Rightarrow \frac{2+3+4+\dots+2021}{2021^2-1} = \underline{\underline{\frac{2043230}{2021^2-1}}} = \frac{2023}{4044}$,