

# Chapter 3

## Two-dimensional Random Variables and Conditional Probability Distributions

# Overview

- 2 dimensional random variables
- Joint probability functions for discrete random variables
- Joint probability density functions for continuous random variables
- Marginal distributions
- Conditional distributions
- Independent random variables
- Expectation

## 3.1 Two Dimensional Random Variables

- There are many experiment situations in which more than one random variable will be of interest to an investigator.
- For example, the investigator may be interested in studying the **height** (H) and **weight** (W) of a person chosen from a certain population.
- Another researcher may be interested in the **hardness** (H) and **tensile strength** (T) of a piece of cold-drawn copper

# Two Dimensional Random Variables (Continued)

## Definition 3.1

- Let  $E$  be an experiment and  $S$  a sample space associated with  $E$ .
- Let  $X$  and  $Y$  be two functions each assigning a real number to each  $s \in S$ .
- We call  $(X, Y)$  a **two-dimensional random variable**. (Sometimes called a **random vector**).

# Two Dimensional Random Variables (Continued)

## Range Space

$$R_{X,Y} = \{(x, y) \mid x = X(s), y = Y(s), s \in S\}.$$

The above definition can be extended to more than two random variables.

## Definition 3.2

- Let  $X_1, X_2, \dots, X_n$  be  $n$  functions each assigning a real number to every outcome  $s \in S$ . We call  $(X_1, X_2, \dots, X_n)$  an  **$n$ -dimensional random variable**. (or an  **$n$ -dimensional random vector**).

# Two Dimensional Random Variables (Continued)

## Definition 3.3

1.  $(X, Y)$  is a two-dimensional **discrete** random variable if the possible values of  $(X(s), Y(s))$  are **finite or countable infinite**.

i.e. the possible values of  $(X(s), Y(s))$  may be represented as  
 $(x_i, y_j), i = 1, 2, 3, \dots; j = 1, 2, 3, \dots$

2.  $(X, Y)$  is a two-dimensional **continuous** random variable if the possible values of  $(X(s), Y(s))$  can **assume all values in some region** of the Euclidean plane  $\mathbb{R}^2$ .

# Example 1

- Consider a television set to be serviced.
- Let  $X$  represent the age to the nearest year of the set and  $Y$  represent the number of defective components in the set.
- $(X, Y)$  is a discrete 2-dimensional random variable.
- Then the set of possible values for  $(X, Y)$  is  $R_{X,Y} = \{(x, y): x = 0, 1, 2, \dots; y = 0, 1, 2, \dots, n\}$ , where  $n$  is the total number of components in the television set.
- $(X, Y) = (5, 3)$  means the television set is 5 years old and has 3 defective components.

## Example 2

- A fast food restaurant operates a drive-up facility and a walk-up window.
- On a randomly selected day, let  $X$  = the proportion of time that the **drive-up facility** is in use (at least one customer is being served or waiting to be served) and  $Y$  = the proportion of the time that the **walk-up window** is in use.
- Then the set of possible values for  $(X, Y)$  is
$$R_{X,Y} = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$
- $(X, Y)$  is a **continuous** 2-dimensional random variable.



## 3.2 Joint Probability Density Function

- As in the one-dimensional random variable case, we would like to have a number associated to the probability or probability density of a 2-dimensional random variable to take on a certain value.

## 3.2.1 Joint Probability Function for Discrete RVs

### Definition 3.4

- Let  $(X, Y)$  be a 2-dimensional **discrete** random variable defined on the sample space of an experiment. With each possible value  $(x_i, y_j)$ , we associate a number  $f_{X,Y}(x_i, y_j)$  representing  $\Pr(X = x_i, Y = y_j)$  and satisfying the following conditions:
  - $f_{X,Y}(x_i, y_j) \geq 0$  for all  $(x_i, y_j) \in R_{X,Y}$ .
  - $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} f_{X,Y}(x_i, y_j) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Pr(X = x_i, Y = y_j) = 1$  (3.1)

# Joint Probability Function (Continued)

- The function  $f_{X,Y}(x, y)$  defined for all pairs of values  $(x_i, y_j) \in R_{X,Y}$  is called the **joint probability function of  $(X, Y)$** .
- Let  $A$  be any set consisting of pairs of  $(x, y)$  values. Then the probability  $\Pr((X, Y) \in A)$  is defined by summing the joint probability function over pairs in  $A$  :

$$\Pr((X, Y) \in A) = \sum \underbrace{\sum}_{(x,y) \in A} f_{X,Y}(x, y)$$

# Example 1

- Find the value of  $k$  so that the function given by  $f_{X,Y}(x, y) = kxy$  for  $x = 1, 2, 3$ , and  $y = 1, 2, 3$ , can serve as a **joint probability function**.

## Solution

$$R_{X,Y} = \{(x, y) \mid x = 1, 2, 3, \text{ and } y = 1, 2, 3\}.$$

$$f(1,1) = k, f(1,2) = 2k, f(1,3) = 3k,$$

$$f(2,1) = 2k, f(2,2) = 4k, f(2,3) = 6k,$$

$$f(3,1) = 3k, f(3,2) = 6k, f(3,3) = 9k.$$

# Example 1 (Continued)

By (3.1) on p3-10, we obtain

$$\sum_{x=1}^3 \sum_{y=1}^3 f_{X,Y}(x, y) = 1$$

$$\Leftrightarrow 1k + 2k + 3k + 2k + 4k + 6k + 3k + 6k + 9k = 1$$

$$\Leftrightarrow k = \frac{1}{36}.$$

## Example 2

- A company has 2 production lines, A and B, which produces at most 5 and 3 machines, respectively.
- Assume that the number of machines produced is a random variable.
- Let  $(X, Y)$  represent the 2-dimensional random variable yielding the numbers of machines produced by Line A and Line B respectively on a given day.
- The joint probability function,  $f_{X,Y}(x, y)$ , of  $(X, Y)$  is given on next slide.
- What is the probability that more chips are produced by Line A than by Line B on a given day?

## Example 2 (Continued)

The table below gives the joint probability function for  $(X, Y)$ .

y	x						Row Total
	0	1	2	3	4	5	
0	0	0.01	0.02	0.05	0.06	0.08	0.22
1	0.01	0.03	0.04	0.05	0.05	0.07	0.25
2	0.02	0.03	0.05	0.06	0.06	0.07	0.29
3	0.02	0.04	0.03	0.04	0.06	0.05	0.24
Column Total	0.05	0.11	0.14	0.20	0.23	0.27	1

- Each entry represents  $f_{X,Y}(x_i, y_j) = \Pr(X = x_i, Y = y_j)$ .
- For example,  $f_{X,Y}(2, 3) = \Pr(X = 2, Y = 3) = 0.03$ .

# Solution to Example 2

Let  $B = \{X > Y\}$ .

$$\begin{aligned}
 \Pr(B) &= \Pr(X > Y) \\
 &= \Pr[(X, Y) = (1, 0) \text{ or } (X, Y) = (2, 0) \text{ or} \\
 &\quad (X, Y) = (2, 1) \text{ or } \cdots \text{ or } (X, Y) = (5, 3)] \\
 &= \Pr[(X, Y) = (1, 0)] + \Pr[(X, Y) = (2, 0)] \\
 &\quad + \cdots + \Pr[(X, Y) = (5, 3)] \\
 &= f_{X,Y}(1, 0) + f_{X,Y}(2, 0) + f_{X,Y}(2, 1) + \cdots + f_{X,Y}(5, 3) \\
 &= 0.01 + 0.02 + 0.04 + 0.05 + \cdots + 0.06 + 0.05 \\
 &= 0.73.
 \end{aligned}$$



# Example 3

- In a group of 9 executives of a certain company, 4 are married, 3 have never married and 2 are divorced.
- Three of the executives are to be randomly selected for promotion.
- Let  $X$  denote the number of married executives and  $Y$  the number of never married executives among the three selected for promotion.
- Find the joint probability function of  $X$  and  $Y$ .

# Solution to Example 3

- The number of ways to select 3 executives out of 9 executives for promotion is

$$\binom{9}{3}$$

- For  $x, y = 0, 1, 2, 3$  such that  $1 \leq x + y \leq 3$ , the number of ways to select  $x$  executives from 4 married executives,  $y$  executives from 3 never married executives and the rest from 2 divorced executives is

$$\binom{4}{x} \binom{3}{y} \binom{2}{3-x-y}$$

# Solution to Example 3 (Continued)

- Therefore

$$\begin{aligned}
 f_{X,Y}(x, y) &= \Pr(X = x, Y = y) \\
 &= \frac{\binom{4}{x} \binom{3}{y} \binom{2}{3-x-y}}{\binom{9}{3}}
 \end{aligned}$$

for  $x, y = 0, 1, 2, 3$  such that  $1 \leq x + y \leq 3$

and  $f_{X,Y}(x, y) = 0$  otherwise.

# Solution to Example 3 (Continued)

The above p.f. are given explicitly in the following table.

$x$	$y$				Row Total
	0	1	2	3	
0	0	3/84	6/84	1/84	10/84
1	4/84	24/84	12/84	0	40/84
2	12/84	18/84	0	0	30/84
3	4/84	0	0	0	4/84
Column Total	20/84	45/84	18/84	1/84	1

## 3.2.2 Joint pdf for Continuous RVs

- Let  $(X, Y)$  be a 2-dimensional **continuous** random variable assuming all values in some region  $R$  of the Euclidean plane,  $\mathbb{R}^2$ .
- $f_{X,Y}(x, y)$  is called a **joint probability density function** if it satisfies the following conditions:

# Joint pdf for Continuous RVs (Continued)

1.  $f_{X,Y}(x, y) \geq 0$  for all  $(x, y) \in R_{X,Y}$ .

2.

$$\iint_{(x,y) \in R_{X,Y}} f_{X,Y}(x, y) dx dy = 1$$

or

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1.$$

# Example 1

Suppose that the two-dimensional continuous random variable  $(X, Y)$  have the joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Find  $\Pr(X + Y \geq 1)$ .

# Solution to Example 1

- First check if  $f_{X,Y}(x, y)$  is a joint p.d.f.
- It is obvious that  $f_{X,Y}(x, y) \geq 0$  for all  $(x, y)$ .
- Check that  $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$ .

$$\begin{aligned}
 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy &= \int_0^2 \int_0^1 \left( x^2 + \frac{xy}{3} \right) dx dy \\
 &= \int_0^2 \left[ \frac{x^3}{3} + \frac{x^2 y}{6} \right]_{x=0}^1 dy = \int_0^2 \left( \frac{1}{3} + \frac{y}{6} \right) dy = \left[ \frac{y}{3} + \frac{y^2}{12} \right]_{y=0}^2 \\
 &= 2/3 + 4/12 = 1.
 \end{aligned}$$



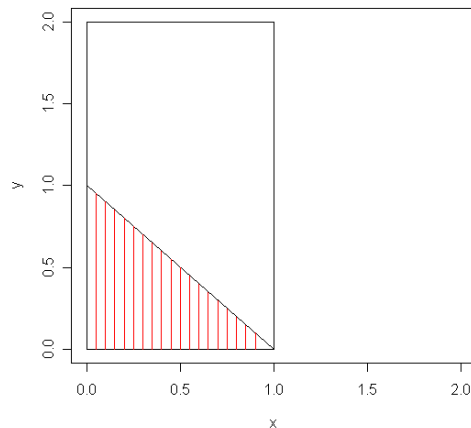
# Solution to Example 1 (Continued)

- Let  $A = \{X + Y \geq 1\}$ . Then  $A' = \{X + Y < 1\}$ .

$$\Pr(A) = 1 - \Pr(A')$$

$$= 1 - \iint_{x+y < 1} f_{X,Y}(x,y) dx dy$$

$$= 1 - \int_0^1 \int_0^{1-x} \left( x^2 + \frac{xy}{3} \right) dy dx$$



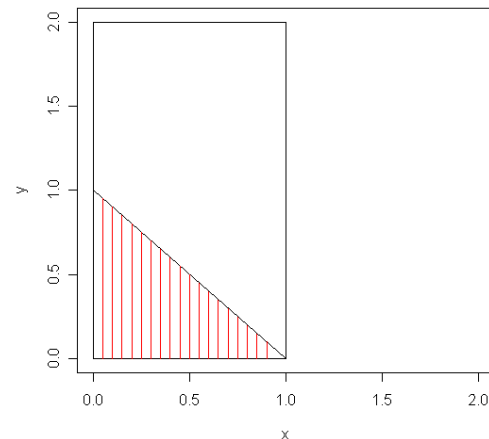
# Solution to Example 1 (Continued)

$$\Pr(A) = 1 - \Pr(A')$$

$$= \dots$$

$$= 1 - \int_0^1 \int_0^{1-x} \left( x^2 + \frac{xy}{3} \right) dy dx$$

- The integration limits of  $y$  are based on the facts that  $0 \leq y \leq 2$  and  $0 < y < 1 - x$  for a fixed  $x$  with  $0 \leq x \leq 1$ .



# Solution to Example 1 (Continued)

- Hence

$$\begin{aligned}
 \Pr(A) &= 1 - \int_0^1 \left[ x^2 y + \frac{xy^2}{6} \right]_{y=0}^{1-x} dx \\
 &= 1 - \int_0^1 x^2(1-x) + \frac{1}{6}x(1-x)^2 dx \\
 &= 1 - \left[ \frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{6} \left( \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right) \right]_{x=0}^1 \\
 &= 1 - \frac{7}{72} = \frac{65}{72}.
 \end{aligned}$$

## Example 2

- If the joint p.d.f. of  $(X, Y)$  is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{12}{13}x(x+y), & \text{for } 0 \leq x \leq 1, 1 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Define  $A = \{(x, y) : 0 < x < 1/2, 1 < y < 2\}$ .
- Find  $\Pr((X, Y) \in A)$ .

# Solution to Example 2

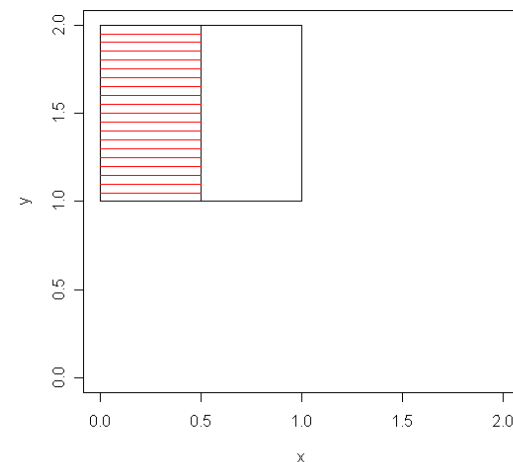
$$\Pr((X, Y) \in A) = \Pr(0 < X < 1/2, 1 < Y < 2)$$

$$= \int_1^2 \int_0^{1/2} \frac{12}{13} x (x + y) dx dy$$

$$= \frac{12}{13} \int_1^2 \left[ \frac{x^3}{3} + \frac{x^2 y}{2} \right]_{x=0}^{1/2} dy$$

$$= \frac{12}{13} \int_1^2 \frac{1}{24} + \frac{y}{8} dy$$

$$= \frac{1}{26} \left[ y + \frac{3y^2}{2} \right]_{y=1}^2 = \frac{11}{52}.$$



## 3.3 Marginal and Conditional Probability Distributions

### 3.3.1 Marginal probability distributions

#### Definition 3.6

- Let  $(X, Y)$  be a 2-dimensional discrete (or continuous) random variable with joint probability function (or joint probability density function)  $f_{X,Y}(x, y)$ .
- The **marginal probability distributions** of  $X$  and  $Y$  are respectively given by:

# Marginal Distributions (Continued)

- For **discrete** case,

$$f_X(x) = \sum_y f_{X,Y}(x, y) \quad \text{and} \quad f_Y(y) = \sum_x f_{X,Y}(x, y)$$

- For **continuous** case,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

# Example 1

- Refer to the example 2 in Section 3.2.1 on p3-15.
- Find the marginal probability distributions of  $X$  and  $Y$ .

## Solution

To find the marginal distribution of  $X$ ,  $f_X(x)$

$$f_X(0) = \sum_{y=0}^3 f_{X,Y}(0, y) = 0 + 0.01 + 0.02 + 0.02 = 0.05.$$

$$f_X(1) = \sum_{y=0}^3 f_{X,Y}(1, y) = 0.01 + 0.03 + 0.03 + 0.04 = 0.11.$$



# Example 1 (Continued)

- Similarly we obtain the values of the marginal distribution of  $f_X(x)$  for  $x = 2, 3, 4$  and  $5$ .

$x$	0	1	2	3	4	5
$f_X(x) = \Pr(X = x)$	0.05	0.11	0.14	0.20	0.23	0.27

Note:

- $f_X(x) \geq 0$  for  $x = 0, 1, 2, 3, 4, 5$
- $\sum_{x=0}^5 f_X(x) = 1$

# Example 1 (Continued)

To find the marginal distribution of  $Y$ ,  $f_Y(y)$

$$f_Y(0) = \sum_{x=0}^5 f_{X,Y}(x, 0) = 0 + 0.01 + 0.02 + 0.05 + 0.06 + 0.08$$

$$= 0.22.$$

$$f_Y(1) = \sum_{x=0}^5 f_{X,Y}(x, 1) = 0.01 + 0.03 + 0.04 + 0.05 + 0.07$$

$$= 0.25.$$

# Example 1 (Continued)

- Similarly we obtain the other values of the marginal distribution of  $f_Y(y)$  for  $y = 2$  and 3.

$y$	0	1	2	3
$f_Y(y) = \Pr(Y = y)$	0.22	0.25	0.29	0.24

# Example 1 (Continued)

- $f_{X,Y}(x,y)$ ,  $f_X(x)$  and  $f_Y(y)$  are displayed in the following table

$y$	$x$						$f_Y(y)$
	0	1	2	3	4	5	
0	0	0.01	0.02	0.05	0.06	0.08	0.22
1	0.01	0.03	0.04	0.05	0.05	0.07	0.25
2	0.02	0.03	0.05	0.06	0.06	0.07	0.29
3	0.02	0.04	0.03	0.04	0.06	0.05	0.24
$f_X(x)$	0.05	0.11	0.14	0.20	0.23	0.27	1

## Example 2

- Refer to example 2 in Section 3.2.2 on p3-28.
- The joint p.d.f. of  $(X, Y)$  is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{12}{13}x(x + y), & \text{for } 0 \leq x \leq 1, 1 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the marginal distributions of  $X$  and  $Y$ .

# Solution to Example 2

- For  $0 \leq x \leq 1$ ,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy = \int_1^2 \frac{12}{13} x(x + y) dy \\ &= \frac{12}{13} \left[ x^2 y + \frac{1}{2} x y^2 \right]_{y=1}^2 \\ &= \frac{6}{13} x(2x + 3). \end{aligned}$$

- For  $x < 0$  or  $x > 1$ ,  $f_X(x) = 0$

since  $f_{X,Y}(x, y) = 0$  and  $\int_{-\infty}^{\infty} 0 dy = 0$ .

# Solution to Example 2 (Continued)

- For  $1 \leq y \leq 2$ ,

$$\begin{aligned}
 f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = \int_0^1 \frac{12}{13} x(x+y) dx \\
 &= \frac{12}{13} \left[ \frac{x^3}{3} + \frac{1}{2} x^2 y \right]_{x=0}^1 \\
 &= \frac{2}{13} (2 + 3y).
 \end{aligned}$$

- For  $y < 1$  or  $y > 2$ ,  $f_Y(y) = 0$

since  $f_{X,Y}(x,y) = 0$  and  $\int_{-\infty}^{\infty} 0 dx = 0$ .

## 3.3.2 Conditional Distribution

### Definition 3.7

- Let  $(X, Y)$  be a discrete (or continuous) 2-dimensional random variable with joint probability function (or p.d.f.)  $f_{X,Y}(x, y)$ .
- Let  $f_X(x)$  and  $f_Y(y)$  be the marginal probability functions of  $X$  and  $Y$  respectively.



# Conditional Distribution (Continued)

## Definition 3.7 (Continued)

- Then **the conditional distribution of  $Y$  given that  $X = x$**  is given by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}, \quad \text{if } f_X(x) > 0,$$

for each  $x$  within the range of  $X$ .

# Conditional Distribution (Continued)

## Definition 3.7 (Continued)

- Similarly, the **conditional probability distribution of  $X$  given  $Y = y$**  is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}, \quad \text{if } f_Y(y) > 0,$$

for each  $y$  within the range of  $Y$ .

# Remarks

1. The conditional p.f.'s (p.d.f.'s) satisfy all the requirements for a 1-dimensional p.f. (p.d.f.). Thus, we have

(a) For a fixed  $y$ ,

$$f_{X|Y}(x|y) \geq 0$$

and for a fixed  $x$ ,

$$f_{Y|X}(y|x) \geq 0.$$

# Remarks (Continued)

1. (b)

For discrete r.v.'s,

$$\sum_x f_{X|Y}(x|y) = 1 \quad \text{and} \quad \sum_y f_{Y|X}(y|x) = 1.$$

For continuous r.v.'s

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} f_{Y|X}(y|x) dy = 1.$$

# Remarks (Continued)

2. For  $f_X(x) > 0$ ,

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)f_X(x).$$

For  $f_Y(y) > 0$ ,

$$f_{X,Y}(x, y) = f_{X|Y}(x|y)f_Y(y).$$

# Example 1

- Suppose an experiment consists of 3 tosses of a fair coin with each outcome being equally likely.
- Let  $X$  be the number of head on the last flip and  $Y$  the total number of heads for the 3 tosses.
- Find the conditional distribution of  $Y$  given  $X = 1$ .

# Example 1 (Continued)

Outcome	HHH	THH	HTH	HHT	TTH	THT	HTT	TTT
$(x, y)$	(1,3)	(1,2)	(1,2)	(0,2)	(1,1)	(0,1)	(0,1)	(0,0)
$f_{X,Y}(x, y)$	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

- The joint probability distribution of  $(X, Y)$  is given in the following table:

$x$	$y$				$f_X(x)$
	0	1	2	3	
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
$f_Y(y)$	1/8	3/8	3/8	1/8	1

# Example 1 (Continued)

**Note:** Summing across the rows gives  $f_X(x)$  and summing across the columns gives  $f_Y(y)$ .

$$f_{Y|X}(0|1) = \frac{f_{X,Y}(1, 0)}{f_X(1)} = \frac{0}{1/2} = 0.$$

$$f_{Y|X}(1|1) = \frac{f_{X,Y}(1, 1)}{f_X(1)} = \frac{1/8}{1/2} = \frac{1}{4}.$$

$$f_{Y|X}(2|1) = \frac{f_{X,Y}(1, 2)}{f_X(1)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

$$f_{Y|X}(3|1) = \frac{f_{X,Y}(1, 3)}{f_X(1)} = \frac{1/8}{1/2} = \frac{1}{4}.$$



# Example 1 (Continued)

Therefore the conditional distribution of  $Y$  given  $X = 1$  is

$y$	0	1	2	3
$f_{Y X}(y 1)$	0	1/4	1/2	1/4

**Note:**

$$\sum_{y=0}^3 f_{Y|X}(y|1) = 1.$$

## Example 2

- Refer to Example 1 in Section 3.2.1 on p3-12.

$$f_{X,Y}(x,y) = \frac{1}{36}xy, \quad \text{for } x = 1, 2, 3, \text{ and } y = 1, 2, 3.$$

- Find  $f_X(x)$  and  $f_{Y|X}(y|x)$ .

## Example 2 (Continued)

### Solution

$$\begin{aligned} f_X(x) &= \sum_{y=1}^3 \frac{1}{36} xy = \frac{x}{36} \left( \sum_{y=1}^3 y \right) \\ &= \frac{x}{36} (1 + 2 + 3) = \frac{x}{6}, \quad \text{for } x = 1, 2, 3. \end{aligned}$$

and  $f_X(x) = 0$  for other values of  $X$ .

## Example 2 (Continued)

For  $x = 1, 2$  or  $3$ ,

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{xy/36}{x/6} = \frac{y}{6}, \end{aligned}$$

for  $y = 1, 2, 3$ ,

and 0 otherwise.

## Example 3

Suppose  $(X, Y)$  has the joint p.d.f

$$f_{X,Y}(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find  $f_X(x)$  and  $f_Y(y)$ .
- (b) Find  $f_{Y|X}(y|x)$  and  $f_{X|Y}(x|y)$ .

# Solution to Example 3

(a) For  $0 \leq x \leq 1$ ,

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_0^2 \left( x^2 + \frac{xy}{3} \right) dy \\ &= \left[ x^2 y + \frac{xy^2}{6} \right]_{y=0}^2 = 2x^2 + \frac{2}{3}x. \end{aligned}$$

For  $x < 0$  and  $x > 1$ ,  $f_X(x) = 0$ ,

since  $f_{X,Y}(x,y) = 0$  and  $\int_{-\infty}^{\infty} 0 dy = 0$ .

# Solution to Example 3 (Continued)

(a) (Continued)

Hence

$$f_X(x) = \begin{cases} 2x^2 + \frac{2}{3}x, & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 3 (Continued)

(a) (Continued)

For  $0 \leq y \leq 2$ ,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx = \int_0^1 \left( x^2 + \frac{xy}{3} \right) dx \\ &= \left[ \frac{x^3}{3} + \frac{x^2 y}{6} \right]_{x=0}^1 = \frac{1}{3} + \frac{1}{6} y. \end{aligned}$$

For  $y < 0$  and  $y > 2$ ,  $f_Y(y) = 0$ ,

since  $f_{X,Y}(x, y) = 0$  and  $\int_{-\infty}^{\infty} 0 dx = 0$ .



# Solution to Example 3 (Continued)

(a) (Continued)

Hence

$$f_Y(y) = \begin{cases} \frac{1}{3} + \frac{1}{6}y, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 3 (Continued)

(b) For  $0 \leq x \leq 1$ ,

$$\begin{aligned}
 f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
 &= \begin{cases} \frac{x^2 + xy/3}{2x^2 + 2x/3}, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} \frac{3x + y}{2(3x + 1)}, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

# Solution to Example 3 (Continued)

For example

- When  $x = 1$ , then

$$f_{Y|X}(y|1) = (3 + y)/8, \text{ for } 0 \leq y \leq 2 \text{ and}$$

$$f_{Y|X}(y|1) = 0, \text{ otherwise.}$$

- When  $x = 0.5$ , then

$$f_{Y|X}(y|0.5) = (3 + 2y)/10, \text{ for } 0 \leq y \leq 2 \text{ and}$$

$$f_{Y|X}(y|0.5) = 0, \text{ otherwise.}$$

# Solution to Example 3 (Continued)

(b) For  $0 \leq y \leq 2$ ,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \begin{cases} \frac{x^2 + xy/3}{(2+y)/6}, & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{2x(3x+y)}{2+y}, & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 3 (Continued)

For example

- When  $y = 1$ , then

$$f_{X|Y}(x|1) = [2x(3x + 1)]/3, \text{ for } 0 \leq x \leq 1 \text{ and}$$

$$f_{X|Y}(x|1) = 0, \text{ otherwise.}$$

- When  $y = 0.5$ , then

$$f_{X|Y}(x|0.5) = [2x(6x + 1)]/5, \text{ for } 0 \leq x \leq 1 \text{ and}$$

$$f_{X|Y}(x|0.5) = 0, \text{ otherwise.}$$

# Example 4

- A fast food restaurant operates a drive-up facility and a walk-up window.
- On a randomly selected day, let  $X$  = the proportion of time that the drive-up facility is in use (at least one customer is being served or waiting to be served) and  $Y$  = the proportion of the time that the walk-up window is in use.
- Suppose that the joint p.d.f. of  $(X, Y)$  is given by

$$f_{X,Y}(x, y) = \begin{cases} \frac{6}{5}(x + y^2) & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

## Example 4 (Continued)

(a) Find the probability that neither facility is busy more than one-quarter of the time.

i.e. Find  $\Pr(0 < X < 1/4, 0 < Y < 1/4)$ .

(b) Find the **probability distribution** of busy time for the drive-up facility without reference to the walk-up window. i.e. Find  $f_X(x)$ .

Hence find the **probability** that the drive-up facility is busy more than one-quarter of the time but less than three quarters of the time.

i.e. Find  $\Pr(1/4 < X < 3/4)$ .

## Example 4 (Continued)

(c) Given that the drive-up facility is busy 80% of the time, what is the probability that the walk-in facility is busy at most half the time?

i.e. Find  $\Pr(Y \leq 1/2 \mid X = 4/5)$ .

(d) Given that the drive-up facility is busy 80% of the time, what is the expected proportion of time that the walk-in facility is busy?

i.e. Find  $E(Y \mid X = 4/5)$ .



# Solution to Example 4

$$\begin{aligned}
 \text{(a)} \quad & \Pr\left(0 \leq X \leq \frac{1}{4}, 0 \leq Y \leq \frac{1}{4}\right) \\
 &= \int_0^{1/4} \int_0^{1/4} \frac{6}{5} (x + y^2) dx dy \\
 &= \frac{6}{5} \left[ \int_0^{1/4} \int_0^{1/4} x dx dy + \int_0^{1/4} \int_0^{1/4} y^2 dx dy \right] \\
 &= \frac{6}{5} \left[ \int_0^{1/4} \left[ \frac{x^2}{2} \right]_{x=0}^{1/4} dy + \int_0^{1/4} [xy^2]_{x=0}^{1/4} dy \right]
 \end{aligned}$$

# Solution to Example 4

(a) (Continued)

$$\begin{aligned}
 &= \frac{6}{5} \left[ \int_0^{1/4} \frac{1}{2(4)^2} dy + \int_0^{1/4} \frac{y^2}{4} dy \right] \\
 &= \frac{6}{5} \left\{ \left[ \frac{y}{32} \right]_{y=0}^{1/4} + \left[ \frac{y^3}{12} \right]_{y=0}^{1/4} \right\} \\
 &= \frac{7}{640} = 0.0109.
 \end{aligned}$$

# Solution to Example 4 (Continued)

(b)

$$\begin{aligned} f_X(x) &= \int_0^1 \frac{6}{5} (x + y^2) dy = \frac{6}{5} \left[ xy + \frac{y^3}{3} \right]_{y=0}^1 \\ &= \frac{6}{5}x + \frac{2}{5}. \end{aligned}$$

for  $0 \leq x \leq 1$

and 0 otherwise.

# Solution to Example 4 (Continued)

(b) (Continued)

$$\begin{aligned}\Pr\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right) &= \int_{1/4}^{3/4} \left(\frac{6}{5}x + \frac{2}{5}\right) dx \\ &= \left[\frac{3}{5}x^2 + \frac{2}{5}x\right]_{x=1/4}^{3/4} = \frac{1}{2}.\end{aligned}$$

# Solution to Example 4 (Continued)

(c)

$$\begin{aligned} f_{Y|X}\left(y\left|\frac{4}{5}\right.\right) &= \frac{f_{X,Y}(4/5, y)}{f_X(4/5)} \\ &= \frac{6[(4/5) + y^2]/5}{(6/5)(4/5) + (2/5)} \\ &= \frac{3(4 + 5y^2)}{17}. \end{aligned}$$

for  $0 < y < 1$  and 0 otherwise.

# Solution to Example 4 (Continued)

(c) (Continued)

Hence

$$\begin{aligned}
 \Pr\left(Y \leq \frac{1}{2} \mid X = \frac{4}{5}\right) &= \int_{-\infty}^{1/2} f_{Y|X}\left(y \mid \frac{4}{5}\right) dy \\
 &= \int_0^{1/2} \frac{3}{17} (4 + 5y^2) dy \\
 &= \frac{3}{17} \left[ 4y + \frac{5}{3} y^3 \right]_{y=0}^{1/2} = \frac{53}{136} = 0.3897.
 \end{aligned}$$

# Solution to Example 4 (Continued)

(d)

$$\begin{aligned} E\left(Y \middle| X = \frac{4}{5}\right) &= \int_{-\infty}^{\infty} y f_{Y|X}\left(y \middle| \frac{4}{5}\right) dy \\ &= \int_0^1 \frac{6}{34} y(4 + 5y^2) dy \\ &= \frac{6}{34} \left[ 2y^2 + \frac{5}{4} y^4 \right]_{y=0}^1 = \frac{39}{68} = 0.5735. \end{aligned}$$

# Example 5

Let  $X$  and  $Y$  be **uniformly distributed** over the triangle with the boundaries:  $0 \leq x \leq y, 0 \leq y \leq 2$ .

- (a) Find the joint p.d.f. of  $(X, Y)$ ,
- (b) Find  $f_X(x)$  and  $f_Y(y)$ .
- (c) Find  $f_{Y|X}(y|x)$  and  $f_{X|Y}(x|y)$ .
- (d) Find  $\Pr(X \leq 1/2 \mid Y = 1)$
- (e) Find  $\Pr(X \leq 1, Y \leq 1)$ .

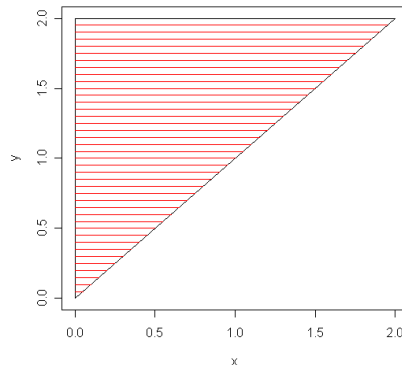


# Solution to Example 5

- Since  $f_{X,Y}(x, y)$  is uniform over the triangle bounded by  $0 \leq x \leq y, 0 \leq y \leq 2$ , therefore

$$f_{X,Y}(x, y) = k \quad \text{for } 0 \leq x \leq y, 0 \leq y \leq 2.$$

- Note: The area bounded by  $0 \leq x \leq y, 0 \leq y \leq 2$  is  $(1/2)(2)(2) = 2$ .



# Solution to Example 5 (Continued)

(a) (Continued)

$$\begin{aligned}\int_0^2 \int_0^y k \, dx \, dy &= \int_0^2 [kx]_{x=0}^y \, dy \\ &= \int_0^2 ky \, dy = \left[ \frac{ky^2}{2} \right]_0^2 = 2k.\end{aligned}$$

Hence

$$\int_0^2 \int_0^y k \, dx \, dy = 1 \Leftrightarrow 2k = 1 \Leftrightarrow k = \frac{1}{2}.$$

# Solution to Example 5 (Continued)

(a) (Continued)

Therefore,

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{2}, & \text{for } 0 \leq x \leq y, 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 5 (Continued)

(b)

$$\begin{aligned}
 f_X(x) &= \begin{cases} \int_x^2 \frac{1}{2} dy, & \text{for } 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} \left[ \frac{y}{2} \right]_{y=x}^2, & \text{for } 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} \frac{1}{2} (2 - x), & \text{for } 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

# Solution to Example 5 (Continued)

(b) (Continued)

$$f_Y(y) = \begin{cases} \int_0^y \frac{1}{2} dx, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \frac{y}{2}, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 5 (Continued)

(c) For  $0 \leq x \leq 2$ ,

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \frac{1/2}{(2-x)/2}, & \text{for } x \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{2-x}, & \text{for } x \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 5 (Continued)

(c) (Continued)

For  $0 \leq x \leq 2$ ,

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2-x}, & \text{for } x \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

For example, when  $x = 1.5$ ,

$$f_{Y|X}(y|1.5) = \frac{1}{2-1.5} = 2, \text{ for } 1.5 \leq y \leq 2 \text{ and}$$

$$f_{Y|X}(y|1.5) = 0, \text{ otherwise.}$$

# Solution to Example 5 (Continued)

(c) (Continued)

For  $0 \leq y \leq 2$ ,

$$\begin{aligned}
 f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
 &= \begin{cases} \frac{1/2}{y/2}, & \text{for } 0 \leq x \leq y, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} 1/y, & \text{for } 0 \leq x \leq y, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$



# Solution to Example 5 (Continued)

(c) (Continued)

For  $0 \leq y \leq 2$ ,

$$f_{X|Y}(x|y) = \begin{cases} 1/y, & \text{for } 0 \leq x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

For example, when  $y = 0.5$ ,

$$f_{X|Y}(x|0.5) = 2 \quad \text{for } 0 \leq x \leq 0.5 \text{ and}$$

$$f_{X|Y}(x|0.5) = 0, \text{ otherwise.}$$

# Solution to Example 5 (Continued)

(d) From (c), we have  $f_{X|Y}(x|1) = 1$  for  $0 \leq x \leq 1$  and 0 otherwise.

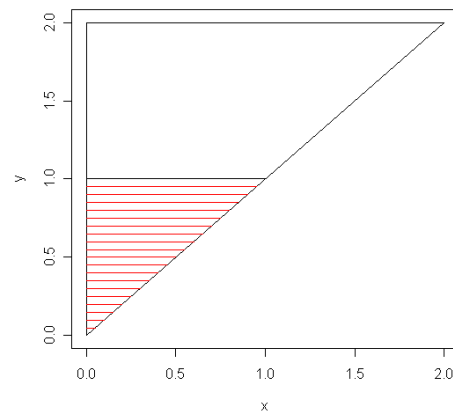
Therefore

$$\begin{aligned}\Pr\left(X \leq \frac{1}{2} \mid Y = 1\right) &= \int_{-\infty}^{1/2} f_{X|Y}(x|1) dx \\ &= \int_0^{1/2} 1 \, dx = \frac{1}{2}.\end{aligned}$$

# Solution to Example 5 (Continued)

(e)

$$\begin{aligned}
 \Pr(X \leq 1, Y \leq 1) &= \int_{-\infty}^1 \int_{-\infty}^1 f_{X,Y}(x, y) dx \, dy \\
 &= \int_0^1 \int_0^y \frac{1}{2} dx \, dy = \int_0^1 \frac{y}{2} dy \\
 &= \left[ \frac{1}{2} \left( \frac{y^2}{2} \right) \right]_0^1 = \frac{1}{4}.
 \end{aligned}$$



## 3.4 Independent Random Variables

### 3.4.1 Definition of Independent RVs

#### Definition

- Random variables  $X$  and  $Y$  are **independent** if and only if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y), \quad \text{for all } x, y.$$

#### Extension:

- Random variables  $X_1, X_2, \dots, X_n$  are independent if and only if

$$f_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

**for all  $x_i, i = 1, \dots, n$ .**

# Remark

- The product of 2 positive functions  $f_X(x)$  and  $f_Y(y)$  means a function which is positive on a **product space**.

- That is, if

$$f_X(x) > 0, \text{ for } x \in A_1 \quad \text{and}$$

$$f_Y(y) > 0, \text{ for } y \in A_2$$

then  $f_X(x)f_Y(y) > 0, \text{ for } (x, y) \in A_1 \times A_2.$

# Example 1

1. The joint p.d.f.  $f_{X,Y}(x, y)$  is given as follows.

$x$	$y$			$f_X(x)$
	1	3	5	
2	0.1	0.2	0.1	0.4
4	0.15	0.3	0.15	0.6
$f_Y(y)$	0.25	0.5	0.25	1

Are  $X$  and  $Y$  independent?

# Solution to Example 1

$$f_X(2)f_Y(1) = 0.4(0.25) = 0.1 = f_{X,Y}(2, 1).$$

Similarly, we have

$$f_X(2)f_Y(3) = 0.4(0.5) = 0.2 = f_{X,Y}(2, 3).$$

$$f_X(2)f_Y(5) = 0.4(0.25) = 0.1 = f_{X,Y}(2, 5).$$

$$f_X(4)f_Y(1) = 0.6(0.25) = 0.15 = f_{X,Y}(4, 1).$$

$$f_X(4)f_Y(3) = 0.6(0.5) = 0.3 = f_{X,Y}(4, 3).$$

$$f_X(4)f_Y(5) = 0.6(0.25) = 0.15 = f_{X,Y}(4, 5).$$

Since  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$  for **all**  $(x, y)$ , hence  $X$  and  $Y$  are **independent**.

## Example 2

- Refer to example 1 in Section 3.2.1 on p3-12.

$$f_{X,Y}(x, y) = \frac{xy}{36}$$

for  $x = 1, 2, 3$ , and  $y = 1, 2, 3$ .

- Are  $X$  and  $Y$  independent?



## Example 2 (Continued)

### Solution

$$\begin{aligned} f_X(x) &= \sum_{y=1}^3 \frac{1}{36} xy = \frac{x}{36} \sum_{y=1}^3 y \\ &= \frac{x}{36} (1 + 2 + 3) = \frac{1}{6}x \quad \text{for } x = 1, 2, 3, \end{aligned}$$

and 0 otherwise.

## Example 2 (Continued)

- Similarly

$$\begin{aligned} f_Y(y) &= \sum_{x=1}^3 \frac{1}{36} xy = \frac{y}{36} \sum_{x=1}^3 x \\ &= \frac{y}{36} (1 + 2 + 3) = \frac{1}{6}y \quad \text{for } y = 1, 2, 3, \end{aligned}$$

and 0 otherwise.

## Example 2 (Continued)

- Hence

$$f_{X,Y}(x, y) = \frac{1}{36} xy = f_X(x)f_Y(y) = \left(\frac{x}{6}\right)\left(\frac{y}{6}\right)$$

for **all**  $x, y = 1, 2, 3$ .

- Therefore  $X$  and  $Y$  are independent.

# Example 3

- $X$  and  $Y$  are 2 **independent** random variables with

$$f_X(x) = e^{-x}, \quad \text{for } x \geq 0 \text{ and}$$

$$f_Y(y) = e^{-y}, \quad \text{for } y \geq 0.$$

- What is  $f_{X,Y}(x, y)$ ?

## Example 3 (Continued)

### Solution

- Since  $X$  and  $Y$  are independent, therefore

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)f_Y(y) = \begin{cases} e^{-x}e^{-y}, & \text{for } x \geq 0 \text{ and } y \geq 0, \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} e^{-(x+y)}, & \text{for } x \geq 0 \text{ and } y \geq 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

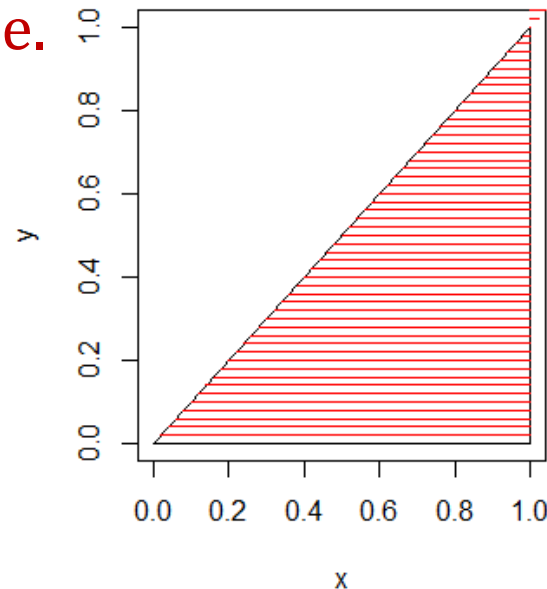
# Example 4

- Given that

$$f_{X,Y}(x,y) = \begin{cases} 2(x+y), \\ 0, \end{cases}$$

for  $0 \leq x \leq 1, 0 < y < x$ ,  
otherwise.

- are  $X$  and  $Y$  independent?



# Solution to Example 4

- $f_X(x)$  is given by

$$\begin{aligned}
 f_X(x) &= \begin{cases} \int_0^x 2(x+y)dy, & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} 2 \left[ xy + \frac{y^2}{2} \right]_{y=0}^x, & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} 3x^2, & \text{for } 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

# Solution to Example 4 (Continued)

- $f_Y(y)$  is given by

$$\begin{aligned}
 f_Y(y) &= \begin{cases} \int_y^1 2(x+y)dx, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} 2 \left[ \frac{x^2}{2} + yx \right]_{x=y}^1, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} 1 + 2y - 3y^2, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$



# Solution to Example 4 (Continued)

- Since
  - $f_{X,Y}(x, y) = 2(x + y)$  for  $0 < x < 1$  and  $0 < y < x$
  - $f_X(x) = 3x^2$  for  $0 < x < 1$
  - $f_Y(y) = 1 + 2y - 3y^2$  for  $0 < y < 1$ .
- Therefore  $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$  for some  $x$  and  $y$
- Hence  $X$  and  $Y$  are not independent.
- Note that the region for which  $f_{X,Y}(x, y) > 0$  is not a rectangle and cannot be expressed as the product of 2 intervals.

# Solution to Example 4 (Continued)

Alternatively, if the region for which  $f_{X,Y}(x, y) > 0$  is not a rectangle, then we look for a point

1. in the product space of the interval for which  $f_X(x) > 0$  (i.e.  $0 < x < 1$ ) and
2. the interval for which  $f_Y(y) > 0$  (i.e.  $0 < y < 1$ )
3. But not in the region for which  $f_{X,Y}(x, y) > 0$

e.g. Consider  $(x, y) = (0.6, 0.8)$ . Since  $x (= 0.6) < y (= 0.8)$ , therefore  $(0.6, 0.8)$  lies outside the region for which  $f_{X,Y}(x, y) > 0$ . On the other hand,  $x = 0.6$  lies in the interval  $0 < x < 1$  and  $y = 0.8$  lies in the interval  $0 < y < 1$ .

# Solution to Example 4 (Continued)

- Consider the point  $(x, y) = (0.6, 0.8)$  (note:  $y > x$ )
  - $f_X(0.6) = 3(0.6)^2 = 1.08 > 0$  (Refer to p3.95)
  - $f_Y(0.8) = 1 + 2(0.8) - 3(0.8)^2 = 0.68 > 0$  (Refer to p3.96)
  - $f_{X,Y}(0.6, 0.8) = 0$  (Refer to p3.94)
- Therefore  $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$  for  $(x, y) = (0.6, 0.8)$
- Hence  $X$  and  $Y$  are not independent

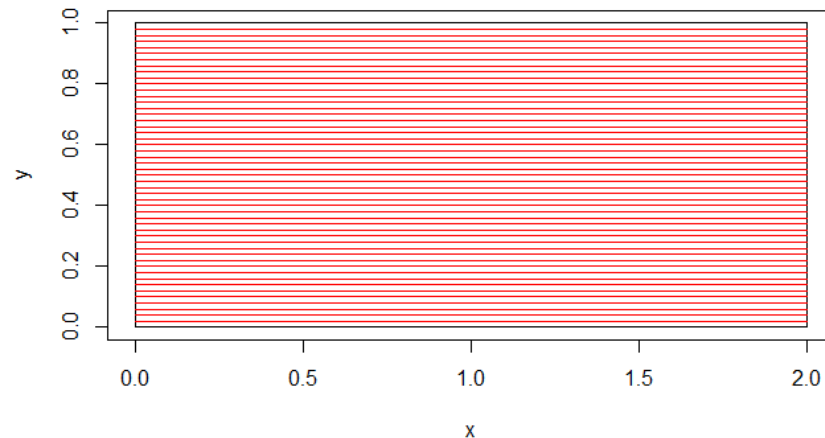
# Example 5

- Given that

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3} x(1+y), & \text{for } 0 < x < 2, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

for  $0 < x < 2, 0 < y < 1$ ,  
 otherwise.

- are  $X$  and  $Y$  independent?



# Solution to Example 5

- $f_X(x)$  is given by

$$f_X(x) = \begin{cases} \int_0^1 \frac{x}{3} (1+y) dy, & \text{for } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{x}{3} \left[ y + \frac{y^2}{2} \right]_{y=0}^1, & \text{for } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{2}x, & \text{for } 0 < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 5 (continued)

- $f_Y(y)$  is given by

$$\begin{aligned}
 f_Y(y) &= \begin{cases} \int_0^2 \frac{x}{3} (1+y) dx, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} \frac{(1+y)}{3} \left[ \frac{x^2}{2} \right]_{x=0}^2, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} \frac{2}{3} (1+y), & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

# Solution to Example 5 (Continued)

- Since

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

for  $0 < x < 2, 0 < y < 1$ ,

- therefore  $X$  and  $Y$  are **independent**.

# Example 6

- Given that

$$f_{X,Y}(x, y) = \begin{cases} x + \frac{3}{2}y^2, & \text{for } 0 < x < 1, 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- are  $X$  and  $Y$  independent?



# Solution to Example 6

- $f_X(x)$  is given by

$$f_X(x) = \begin{cases} \int_0^1 \left( x + \frac{3}{2} y^2 \right) dy, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \left[ xy + \frac{3}{2} \frac{y^3}{3} \right]_{y=0}^1, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} x + \frac{1}{2}, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

# Solution to Example 6 (Continued)

- $f_Y(y)$  is given by

$$\begin{aligned}
 f_Y(y) &= \begin{cases} \int_0^1 \left( x + \frac{3}{2}y^2 \right) dx, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} \left[ \frac{x^2}{2} + \frac{3}{2}y^2x \right]_{x=0}^1, & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \\
 &= \begin{cases} \frac{1}{2}(1 + 3y^2), & \text{for } 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

# Solution to Example 6 (Continued)

- Since

$$f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$$

for  $0 < x < 1, 0 < y < 1$ ,

- therefore  $X$  and  $Y$  are **not** independent.

# 3.5 Expectation

## Definition 3.5.1

- The expectation of  $g(X, Y)$  is defined as

$$E[g(X, Y)] = \begin{cases} \sum_x \sum_y g(x, y) f_{X,Y}(x, y), & \text{for Discrete RV's,} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dx dy, & \text{for Cont. RV's} \end{cases}$$

# A Special Case

- Let  $g(X, Y) = (X - \mu_X)(Y - \mu_Y)$ . This leads to the definition of covariance between two random variables.

## Definition 3.5.2

- Let  $(X, Y)$  be a bivariate random vector with joint p.f. (or p.d.f.)  $f_{X,Y}(x, y)$ , then the **covariance** of  $(X, Y)$  is defined as
$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

# A Special Case (Continued)

- For **discrete** case

$$\begin{aligned} \text{Cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= \sum_x \sum_y (x - \mu_X)(y - \mu_Y) f_{X,Y}(x, y) \end{aligned}$$

- For **continuous** case

$$\begin{aligned} \text{Cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y) f_{X,Y}(x, y) dx dy \end{aligned}$$

# Remarks

1.  $Cov(X, Y) = E(XY) - \mu_X \mu_Y$ .
2. If  $X$  and  $Y$  are independent, then  $Cov(X, Y) = 0$ . However  $Cov(X, Y) = 0$  does not imply  $X$  and  $Y$  are independent.
3.  $Cov(aX + b, cY + d) = ac Cov(X, Y)$
4.  $V(aX + bY) = a^2V(X) + b^2V(Y) + 2ab Cov(X, Y)$

# Correlation coefficient

## Definition 3.5.2

- The **correlation coefficient** of  $X$  and  $Y$ , denoted by  $Cor(X, Y)$ ,  $\rho_{X,Y}$  or  $\rho$ , is defined by

$$\rho_{X,Y} = \frac{Cov(X, Y)}{\sqrt{V(X)} \sqrt{V(Y)}}$$



# Remarks on Correlation coefficient

1.  $-1 \leq \rho_{X,Y} \leq 1$ .
2.  $\rho_{X,Y}$  is a measure of the degree of **linear** relationship between  $X$  and  $Y$ .
3. If  $X$  and  $Y$  are independent, then  $\rho_{X,Y} = 0$ .  
On the other hand,  $\rho_{X,Y} = 0$  does **not** imply independence.

# Example 1

- Refer to Example 1 in Section 3.3.2 on p3-47. The joint distribution of  $(X, Y)$  is given by

$x$	$y$				$f_X(x)$
	0	1	2	3	
0	1/8	1/4	1/8	0	1/2
1	0	1/8	1/4	1/8	1/2
$f_Y(y)$	1/8	3/8	3/8	1/8	1

# Example 1 (Continued)

- (a) Find  $E(Y - X)$ .
- (b) Find  $Cov(X, Y)$ .
- (c) Find  $\rho_{X,Y}$ .
- (d) Find  $E(Y \mid X = 1)$ .

# Solution to Example 1

(a)

$$E(Y - X) = (0 - 0)(1/8) + (1 - 0)(1/4) + (2 - 0)(1/8) \\ + \cdots + (3 - 1)(1/8) = 1.$$

Or

$$E(Y - X) = E(Y) - E(X) = 1.5 - 0.5 = 1.$$

(See part (b))

# Solution to Example 1 (Continued)

(b)

$$E(XY) = (0)(0)(1/8) + (0)(1)(1/4) + (0)(2)(1/8) \\ + \dots + (1)(3)(1/8) = 1.$$

$$E(X) = 0(1/2) + 1(1/2) = 0.5.$$

$$E(Y) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = 1.5.$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) \\ = 1 - (0.5)(1.5) = 0.25.$$

# Solution to Example 1 (Continued)

$$(c) \quad V(X) = [0^2(1/2) + 1^2(1/2)] - (0.5)^2 = 0.25.$$

$$\begin{aligned} V(Y) &= [0^2(1/8) + 1^2(3/8) + 2^2(3/8) + 3^2(1/8)] - 1.5^2 \\ &= 3 - 2.25 = 0.75. \end{aligned}$$

$$\begin{aligned} \rho_{X,Y} &= \frac{\text{Cov}(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{0.25}{\sqrt{(0.25)(0.75)}} \\ &= \frac{1}{\sqrt{3}} = 0.5774. \end{aligned}$$

# Solution to Example 1 (Continued)

(d) The conditional distribution of  $Y$  given  $X = 1$  is

$y$	1	2	3
$f_{Y X}(y   1)$	1/4	1/2	1/4

$$E(Y | X = 1) = 1(1/4) + 2(1/2) + 3(1/4) = 2.$$

(Refer to the conditional distribution on p3-49)

## Example 2

Refer to Example 3 in Section 3.3.2 on p3-53. The joint p.d.f. of  $(X, Y)$  is given by

$$f_{X,Y}(x, y) = \begin{cases} x^2 + \frac{xy}{3}, & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find  $Cov(X, Y)$ .
- (b) Find  $E(Y \mid X = 1/2)$ .



# Solution to Example 2

(a)

$$\begin{aligned}
 E(XY) &= \int_0^2 \int_0^1 xy \left( x^2 + \frac{xy}{3} \right) dx dy \\
 &= \int_0^2 \left[ y \frac{x^4}{4} + \frac{y^2 x^3}{9} \right]_{x=0}^1 dy = \int_0^2 \left( \frac{y}{4} + \frac{y^2}{9} \right) dy \\
 &= \left[ \frac{y^2}{8} + \frac{y^3}{27} \right]_{y=0}^2 = \frac{43}{54}.
 \end{aligned}$$

# Solution to Example 2 (Continued)

(a)

$$E(X) = \int_0^1 x \left( 2x^2 + \frac{2}{3}x \right) dx \quad (\text{Refer to p3-55})$$

$$= \left[ \frac{2x^4}{4} + \frac{2x^3}{9} \right]_{x=0}^1 = \frac{13}{18}.$$

$$E(Y) = \int_0^2 y \left( \frac{1}{3} + \frac{y}{6} \right) dy \quad (\text{Refer to p3-57})$$

$$= \left[ \frac{y^2}{6} + \frac{y^3}{18} \right]_{y=0}^2 = \frac{10}{9}.$$

# Solution to Example 2 (Continued)

Hence

$$\begin{aligned} \text{Cov}(X, Y) &= E(XY) - E(X)E(Y) = \frac{43}{54} - \left(\frac{13}{18}\right)\left(\frac{10}{9}\right) \\ &= -\frac{1}{162}. \end{aligned}$$

# Solution to Example 2 (Continued)

(b) From p3-58, the conditional distribution of  $Y$  given  $X = 1/2$  is given by

$$f_{(Y|X)}\left(y\middle|\frac{1}{2}\right) = \begin{cases} \frac{3+2y}{10}, & \text{for } 0 \leq y \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$E\left(Y\middle|X = \frac{1}{2}\right) = \int_0^2 y \left(\frac{3+2y}{10}\right) dy = \frac{1}{10} \left[ 3\frac{y^2}{2} + 2\frac{y^3}{3} \right]_{y=0}^2 = \frac{17}{15}.$$