Constraint Satisfaction Problems: Generalising Goal Search II

CS3243: Introduction to Artificial Intelligence – Lecture 6

13 February 2023

Contents

Reference: AIMA 4th Edition, Section 5.1-5.3

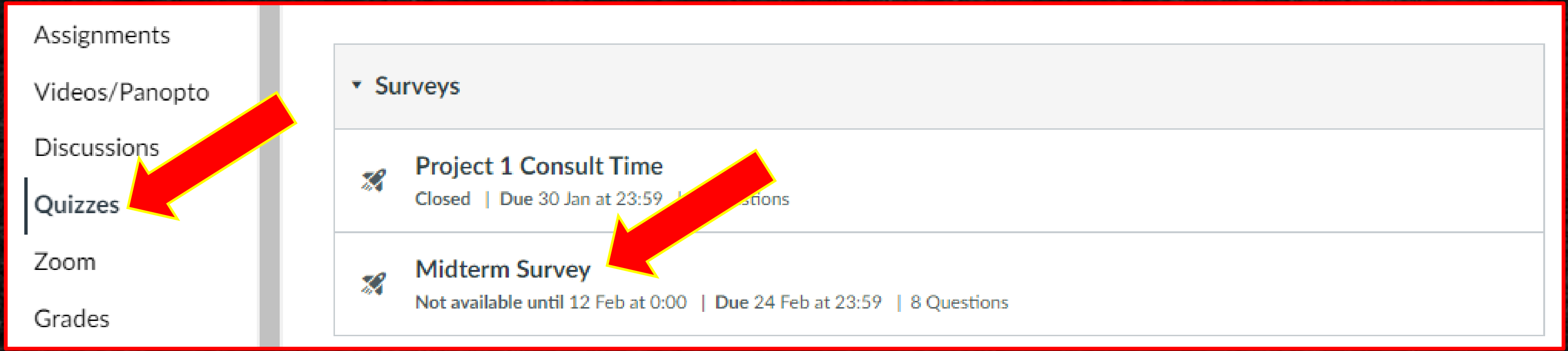
Administrative Matters

Project 2 & Midterm Examination

- Project 2
  - Released at the end of this week
  - Due in Week 9
  - Implement goal search via local search and CSP
  - Consultation TBA
- Midterm Examination
  - Week 7 Lecture Slot
    - 27 February (Monday), 1030-1130 hrs
    - Conducted in-person at the MPSH1a
  - Duration: 60 minutes
  - Topics: Lectures 1-5 (i.e., everything up to local search – excludes CSPs)

Midterm Survey

- Particulars
  - Now Open
    - Closes next Friday (24 February), 2359 hrs
  - Anonymous Survey
  - Access at Canvas > CS3243 > Quizzes > Midterm Survey



- Please help us to improve the course by providing feedback

Upcoming…

- Deadlines
  - TA4 (released last week)
    - *Due in your Week 6 tutorial session*
    - *Submit the a physical copy (more instructions on the Tutorial Worksheet)*
  - Prepare for the tutorial!
    - Participation marks = 5%
  - Project 1 (released Week 3 – 25 January)
    - *Due this Sunday (19 February), 2359 hrs*

Recap on CSPs

Formulating CSPs

- State representation
  - Variables: $X = \{x_1, \ldots, x_n\}$
  - Domains: $D = \{d_1, \ldots, d_k\}$
    - Such that x_i has a domain d_i
  - Initial state: all variables unassigned
  - Intermediate state: partial assignment
- Goal test
  - Constraints: $C = \{c_1, \ldots, c_m\}$
    - Defined via a constraint language
      - Algebra, Logic, Sets
    - Each c_i corresponds to a requirement on some subset of X
- Actions, costs and transition
  - Assignment of values (within domain) to variables
  - Costs are not utilised
  - Objective is a ***complete*** and ***consistent*** assignment
    - Find a legal assignment $(y_1, \ldots, y_n)$
      - $y_i \in d_i$ for all $i \in [n]$
    - Complete: all variables assigned values
    - Consistent: all constraints C satisfied

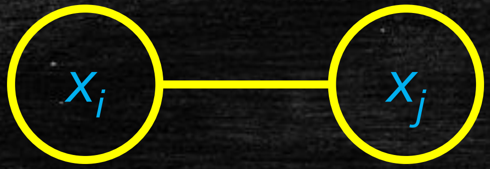
More on Constraints

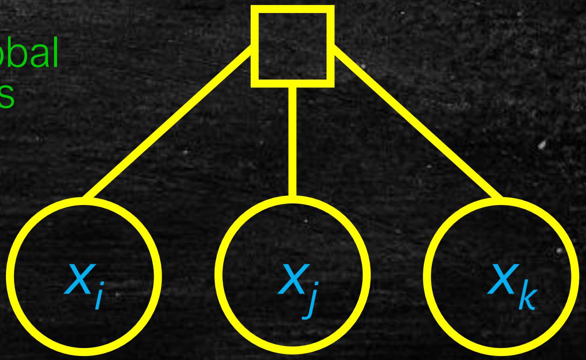
- A language is necessary to express the constraints
  - Arithmetic
  - Sets (of legal values)
  - Logic
  - For example, x_1 greater than x_2 given $d = \{1, 2, 3\}$ may be written
    - $\langle (x_1, x_2), x_1 > x_2 \rangle$
    - $\langle (x_1, x_2), \{ (2, 1), (3, 1), (3, 2) \} \rangle$
- Each constraint, c_i,
  - Describes the necessary relationship, ***rel***, between a set of variables, ***scope***
    - For the example above, ***scope*** $= (x_1, x_2)$. ***rel*** $= x_1 > x_2$
- Types of constraints
  - Unary: | ***scope*** | = 1
  - Binary: | ***scope*** | = 2
  - Global: | ***scope*** | > 2 (i.e., higher-order constraints)

Drawing Constraint Graphs and Hypergraphs

- Constraint graphs represent the constraints in a CSP
  - Simple Vertex: variable 

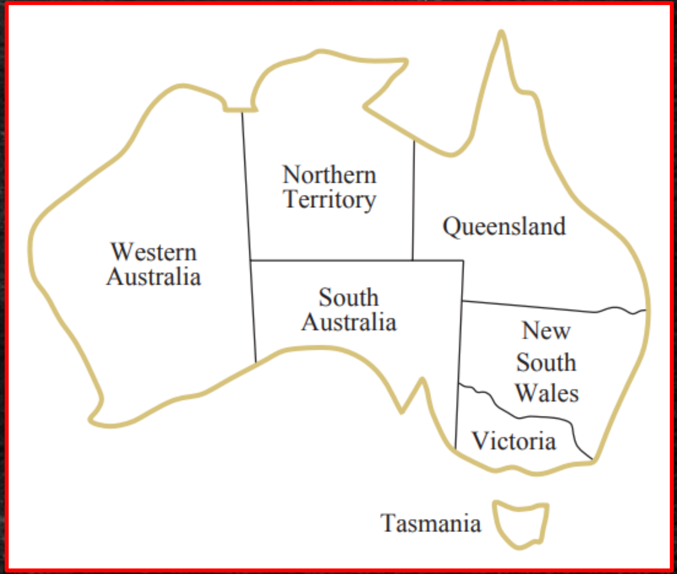
  - Linking Vertex: for global constraints 
  - Edge: links all variables in the scope of a constraint (***rel***)
    - Unary constraints 

    - Binary constraints 

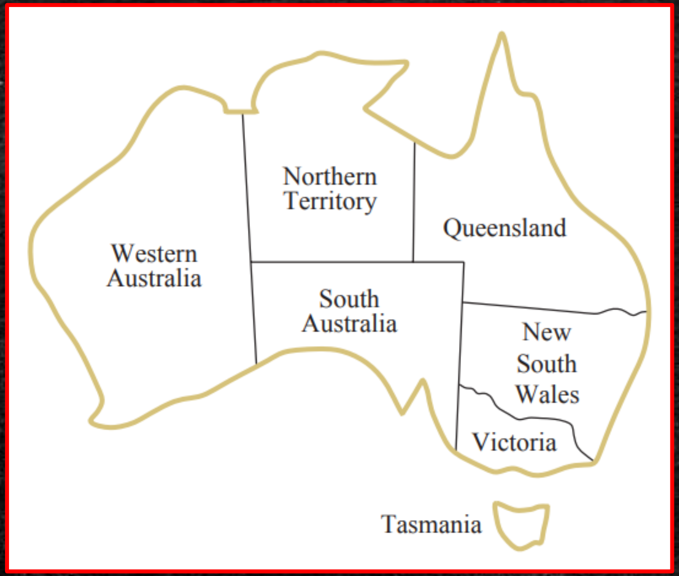
    - Binary/Global constraints 

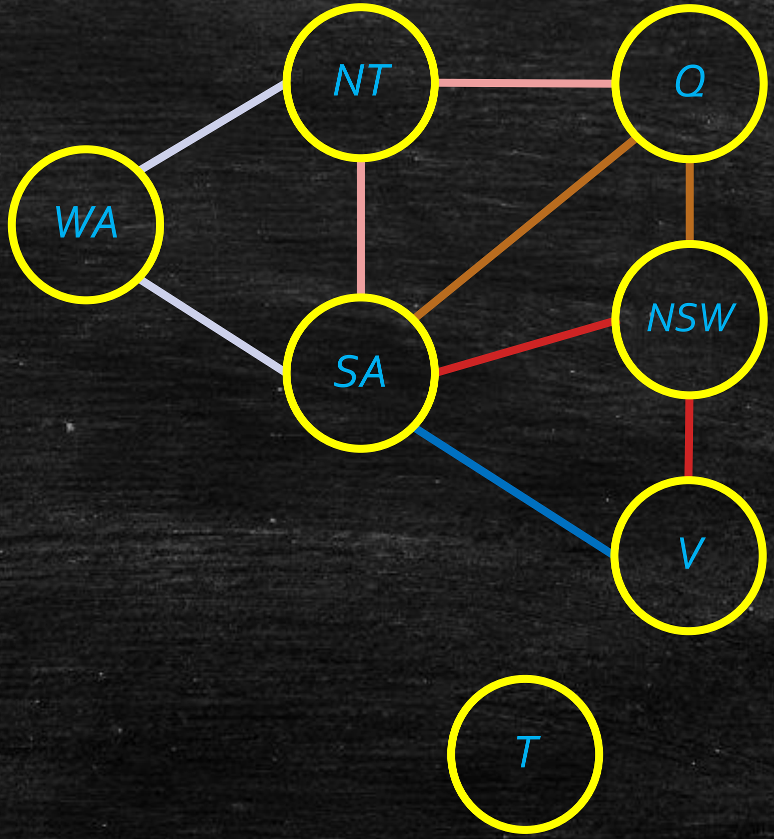
CSP Formulation Example: Graph Colouring

- Colour each state of Australia such that no two adjacent states share the same colour
- Variables
  – $X = \{ WA, NT, Q, NSW, V, SA, T \}$
- Domains
  – $d_i = \{$ Red, Green, Blue $\}$
- Constraints
  – $\forall(x_i, x_j) \in E$, $\mathtt{\textbf{colour}}(x_i) \neq \mathtt{\textbf{colour}}(x_j)$



Constraint Graph for Example: Graph Colouring





Backtracking Algorithm for CSPs

function BACKTRACKING-SEARCH(*csp*) **returns** a solution or *failure*
  return BACKTRACK(*csp*, { })

function BACKTRACK(*csp*, *assignment*) **returns** a solution or *failure*
  if *assignment* is complete **then return** *assignment*
  var ← SELECT-UNASSIGNED-VARIABLE(*csp*, *assignment*)
  for each *value* **in** ORDER-DOMAIN-VALUES(*csp*, *var*, *assignment*) **do**
    if *value* is consistent with *assignment* **then**
      add {*var* = *value*} to *assignment*
      inferences ← INFERENCE(*csp*, *var*, *assignment*)
      if *inferences* ≠ *failure* **then**
        add *inferences* to *csp*
        result ← BACKTRACK(*csp*, *assignment*)
        if *result* ≠ *failure* **then return** *result*
        remove *inferences* from *csp*
      remove {*var* = *value*} from *assignment*
  return *failure*

Determine the variable to assign to | 1

Determine the value to assign | 2

Trying to determine if the chosen assignment will lead to a terminal state | 3

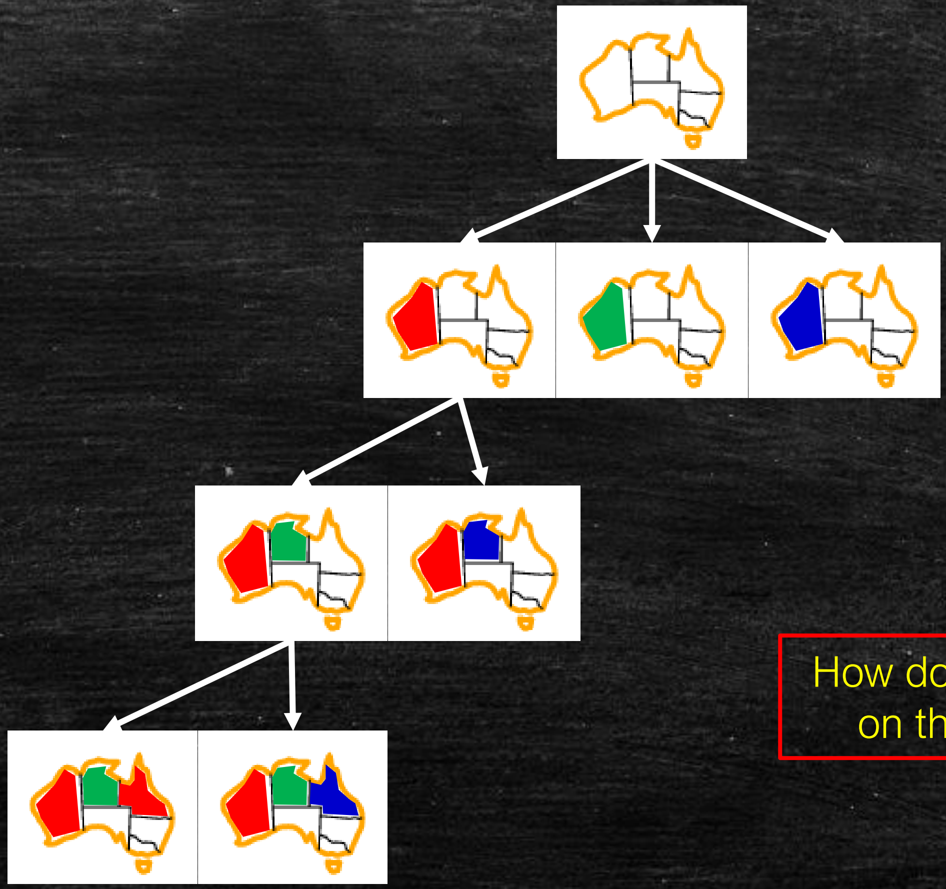
Continues recursively as long as the ***assignment*** is ***viable***

General purpose heuristics for 1, 2 and 3 can lead to improved search efficiency

Variable-Order Heuristics

Backtracking Example: Graph Colouring

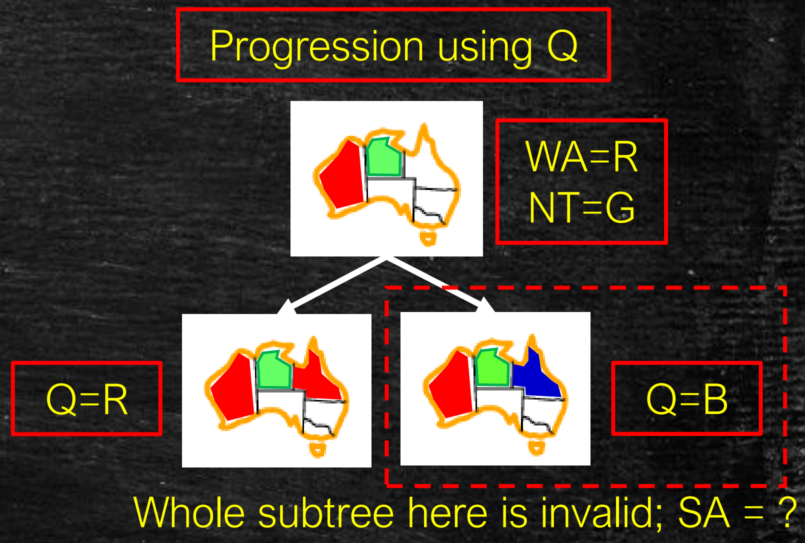
- During backtracking search
  - Assign variables in some order
    - For example:
      - WA
      - NT
      - Q
      - etc ...

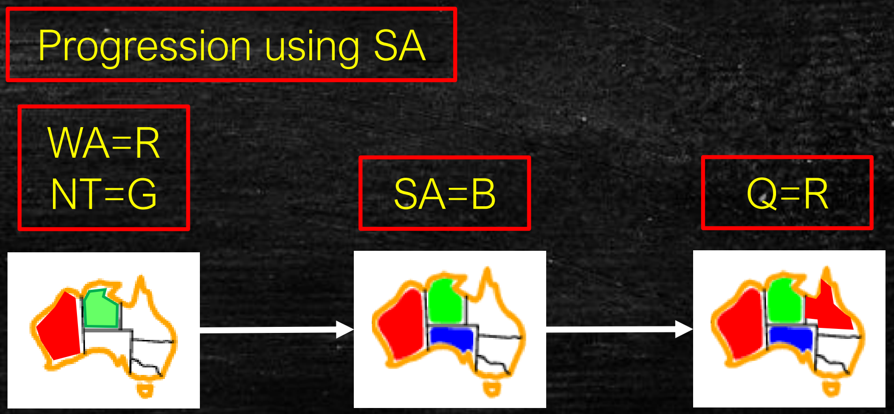


How do we decide on the order?

Minimum-Remaining-Values Heuristic

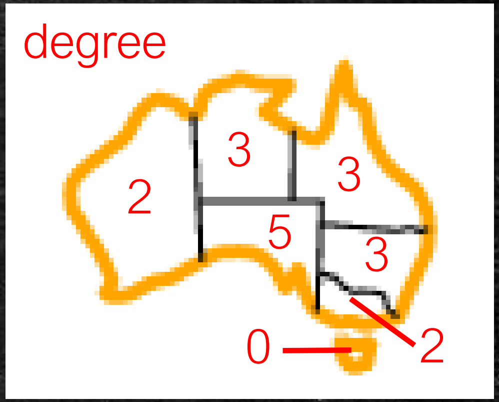
- Choose the variable with fewest legal values
  - Minimum-remaining-values (MRV) heuristic
  - Also considered the most constrained variable
  - Smallest consistent domain size among unassigned variables
- General idea
  - Places larger subtrees closer to the root
    - Any invalid state found prunes a larger subtree
  - Eliminate larger subtrees earlier
- Consider the Australia Colouring problem
  - Suppose we start with WA = R and NT = G
    - Remaining values: SA = 1, Q = 2, Rest = 3
    - MRV suggest selecting SA, what if we use Q?
- MRV usually performs better than static or random ordering





Degree Heuristic

- MRV heuristic requires tie-breaking
  - E.g., at initial state all variables have same RV
- Tie-break with degree heuristic
  - Picks variable with most constraints relative to unassigned variables
- General idea
  - By selecting variable that restricts the most number of other variables, we reduce b
- Consider the Australia Colouring problem
  - Using MRV with degree for tie-breaking
    - Initially state
      - all states have same RV; tie-break with degree
      - chosen variable is SA
    - After assigning to SA, any assignment leads to a solution without any backtracking



Recommended variable selection: MRV, then degree, then random

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     if value is consistent with assignment then
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         add inferences to csp
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         if result ≠ failure then return result
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       remove {var = value} from assignment
  return failure
```

Determine the variable to assign to 1

Determine the value to assign 2

Trying to determine if the chosen assignment will lead to a terminal state 3

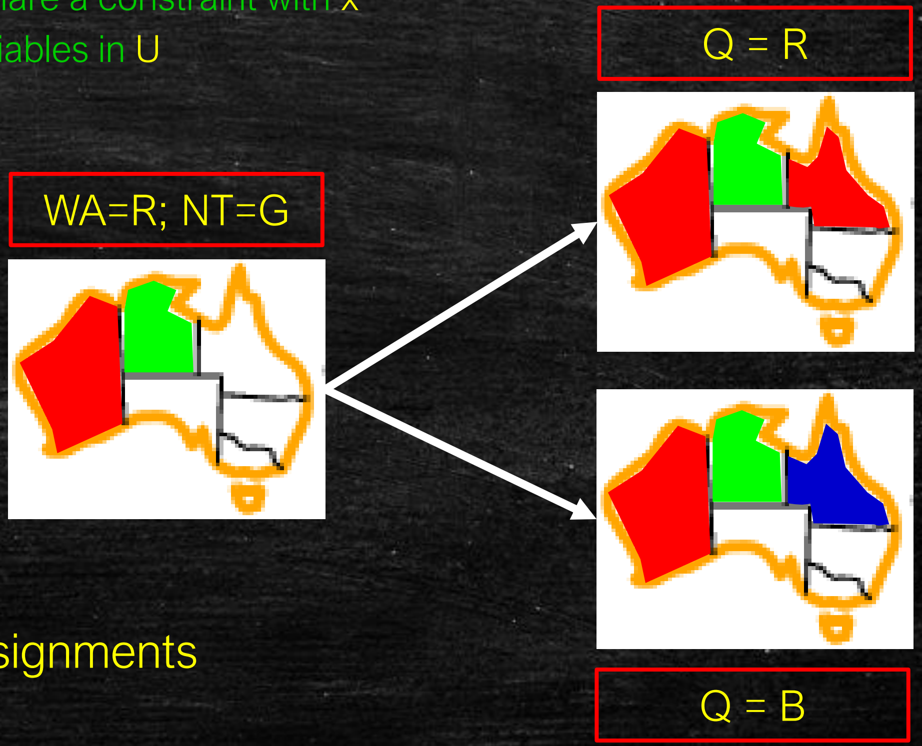
Continues recursively as long as the ***assignment*** is ***viable***

General purpose heuristics for 1, 2 and 3 can lead to improved search efficiency

Value-Order Heuristics

Least-Constraining-Value Heuristic

- Choose the value that rules out the fewest choices
  - Least-constraining-value (LCV) heuristic
    - Given assignment of value v to variable x'
    - Determine set of unassigned variables U = {x_a, x_b, ...} that share a constraint with x'
    - Pick v that maximises sum of consistent domain sizes of variables in U
- General idea
  - Avoid failure (i.e., avoid empty domains)
- Consider the Australia Colouring problem
  - Suppose we start with WA = R and NT = G
  - Suppose our next choice is Q
    - Q = R leaves SA = B
    - Q = B leaves SA = None
    - Select Q = R (since it constrains 1 less than Q = B)
- Tries to leave maximum flexibility for subsequent assignments



Why Different Strategies with Variables & Values?

- With variables: fail-first
  - Every variable must be assigned to arrive at a solution
  - Must look at all variables
  - Fail-first strategy on average leads to fewer successful assignments to backtrack over
- With values: fail-last
  - Only one solution required
  - May not have to look at some values
  - If all solutions required, then value-ordering irrelevant

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Determine the variable to assign to | 1

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Continues recursively as long as the ***assignment*** is ***viable***

General purpose heuristics for 1, 2 and 3 can lead to improved search efficiency

Inference in CSPs

Avoiding Failure

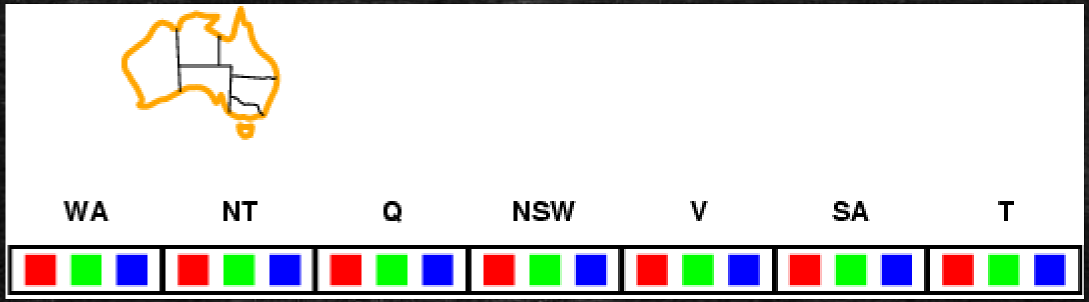
- With certain states, we know we are heading for failure

- Searching such subtrees is a waste of computation
- How can we detect these as early as possible?

Forward Checking

Forward Checking

- Track remaining legal values for unassigned variables
- Terminate search when any variable has no legal values



Forward Checking

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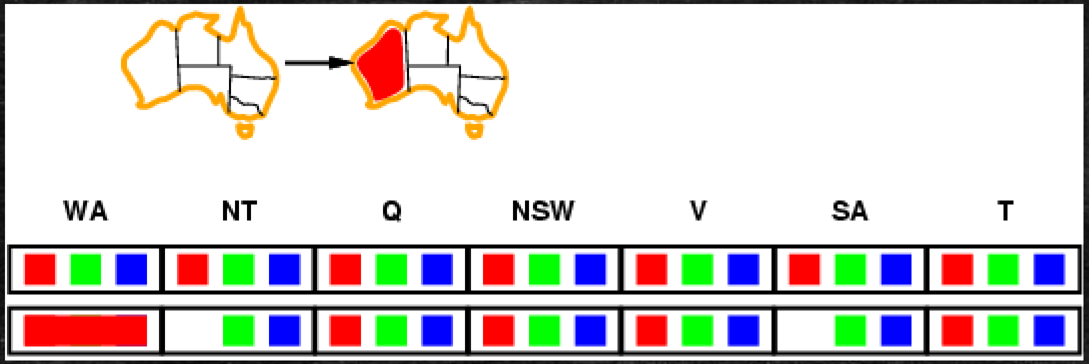

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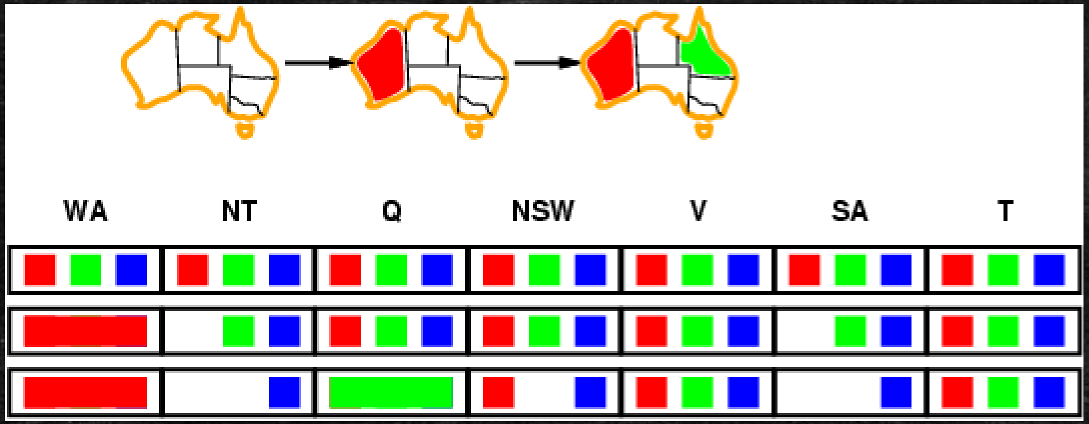

Forward Checking

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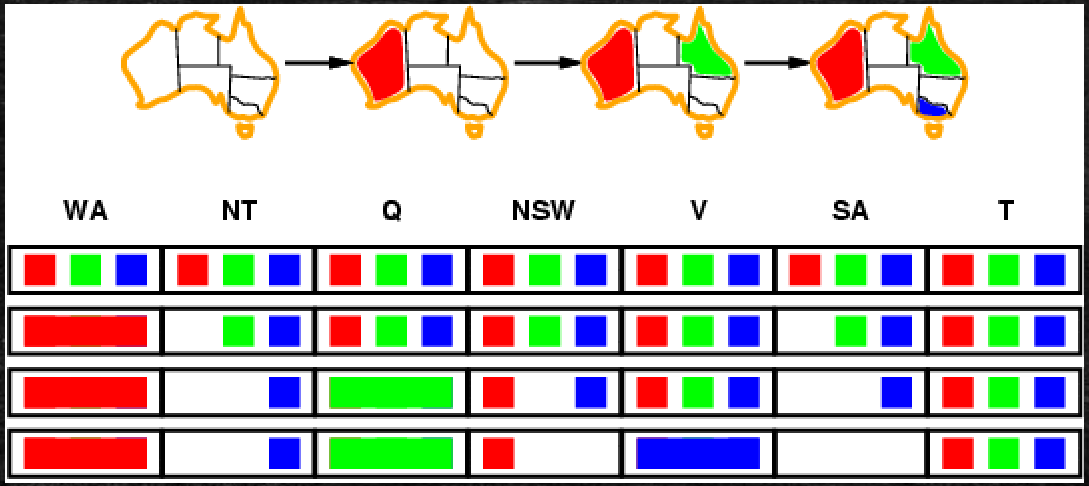
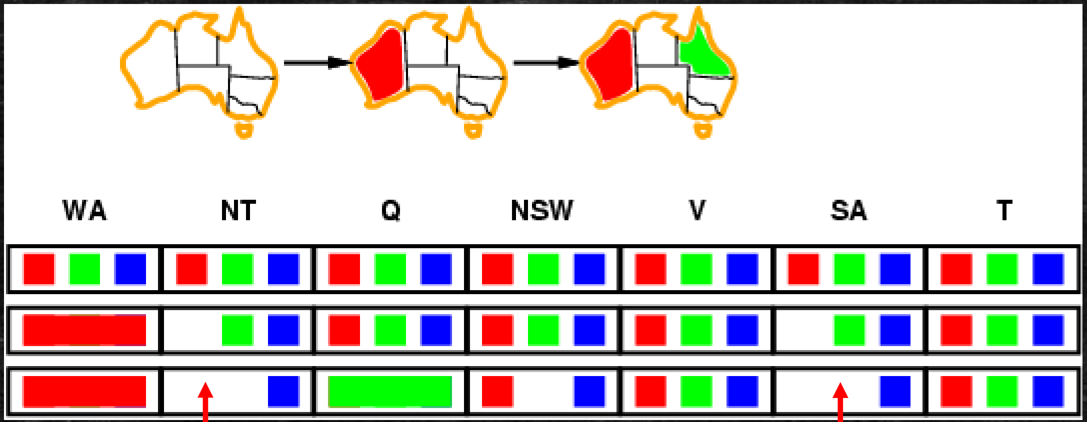


The Issue with Forward Checking

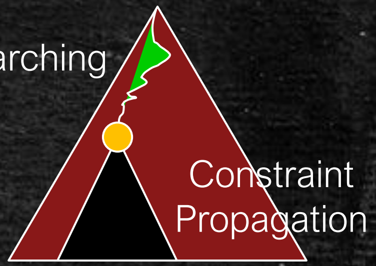
- Problem: forward checking propagates information from assigned to unassigned variables, but *does not provide early detection for all failures*



NT and SA cannot both be blue

Solution: ensuring consistency via constraint propagation

Constraint Propagation



- Inference step to ensure local consistency of ALL variables
  - Traverse constraint graph to ensure variable at each node is consistent
    - Eliminate all values in variable's domain that are not consistent with linked constraints
- Node-consistent
  - A single variable is node-consistent if its domain is consistent with related unary constraints
- Arc-consistent (i.e., edge-consistent)
  - A single variable is arc-consistent if its domain is consistent with related binary constraints

All global constraints may be transformed into binary constraints via hidden variable encoding (use a new variable whose domain contains legal n-tuples domain values for variables in global constraint) or dual encoding (general idea described in AIMA pp.168-169)

For CS3243: assume all questions requiring algorithm traces done over unary/binary constraints only (global constraints only present in CSP formulation questions)

Questions about the Lecture?

- Was anything unclear?
- Do you need to clarify anything?

- Ask on Archipelago
  - Specify a question
  - Upvote someone else's question

Invitation Link (Use NUS Email --- starts with E)
https://archipelago.rocks/app/resend-invite/91207333004

Node Consistency

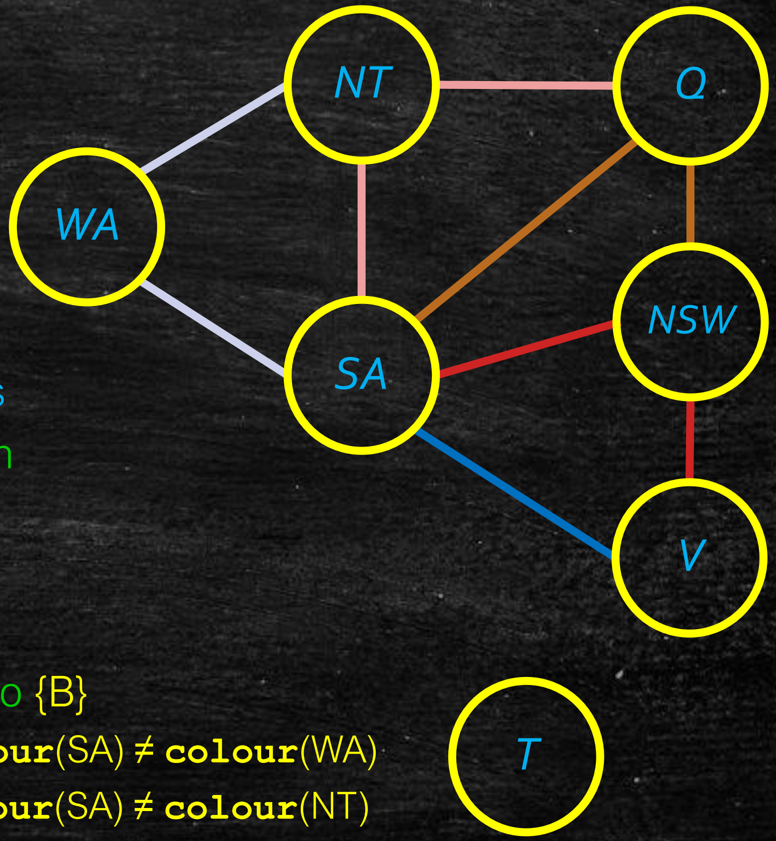
Node Consistency

- Node-consistent
  - A single variable is node-consistent if its domain is consistent with related unary constraints
- Trivial to ensure node consistency
  - Only concerned with unary constraints
  - For each variable
    - Eliminate domain values inconsistent with unary constraints
  - Perform the above as a pre-processing step (i.e., before application of backtracking)

Arc Consistency

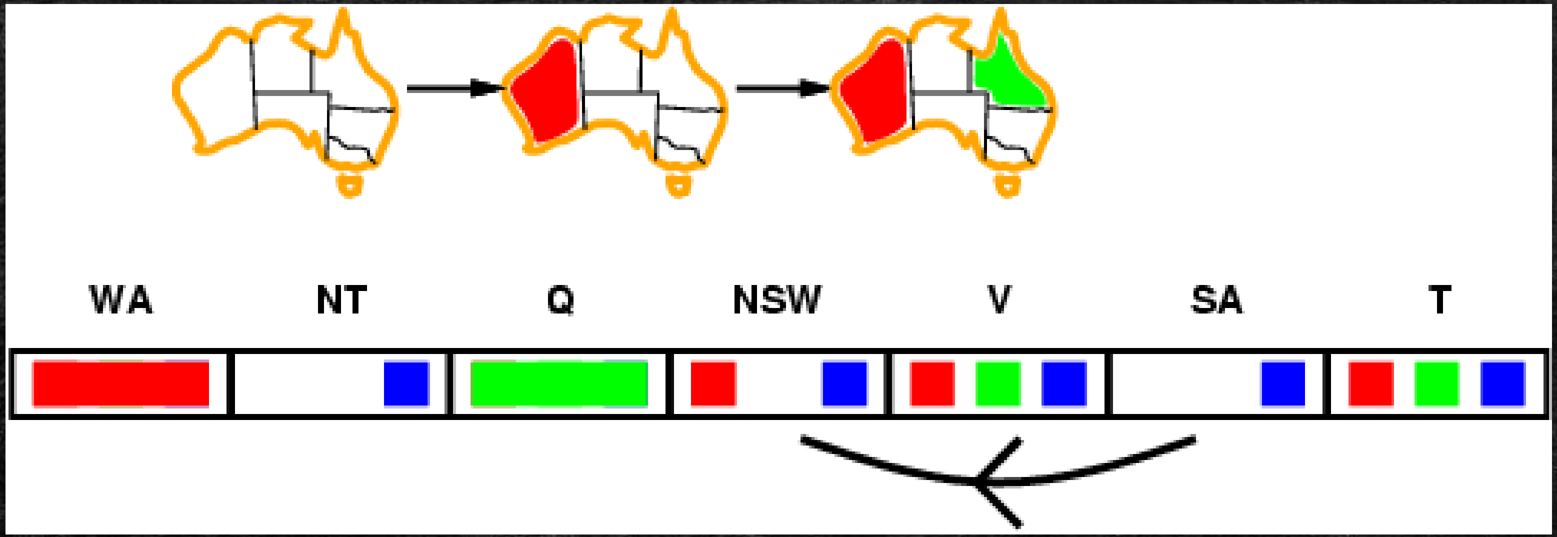
Arc Consistency

- Arc-consistent (i.e., edge-consistent)
  - A single variable is arc-consistent if its domain is consistent with related binary constraints
- Ensuring arc consistency (AC)
  - For each variable
    - Eliminate domain values inconsistent with binary constraints
    - Variable domain value must have partnering domain value in other variable that will satisfy constraint
  - Consider the Australia Colouring problem
    - Assume D_{WA} = {R}, D_{NT} = {G}, domains of rest = {R,G,B}
    - To ensure arc consistency at SA, reduce D_{SA} from {R,G,B} to {B}
      - Eliminate SA = R since there is no value in D_{WA} that satisfies **colour**(SA) ≠ **colour**(WA)
      - Eliminate SA = G since there is no value in D_{NT} that satisfies **colour**(SA) ≠ **colour**(NT)



Arc Consistency

X_i is arc-consistent *wrt* X_j (i.e., the arc (X_i, X_j) is consistent) *iff* for every value $x \in D_i$ there exists some value $y \in D_j$ that satisfies the binary constraint on the arc (X_i, X_j)



To maintain AC, remove any value from the target variable if it makes a constraint impossible to satisfy

Arc (SA, NSW) is consistent since the constraint linking SA and NSW is still satisfied when NSW = R

Arc Consistency

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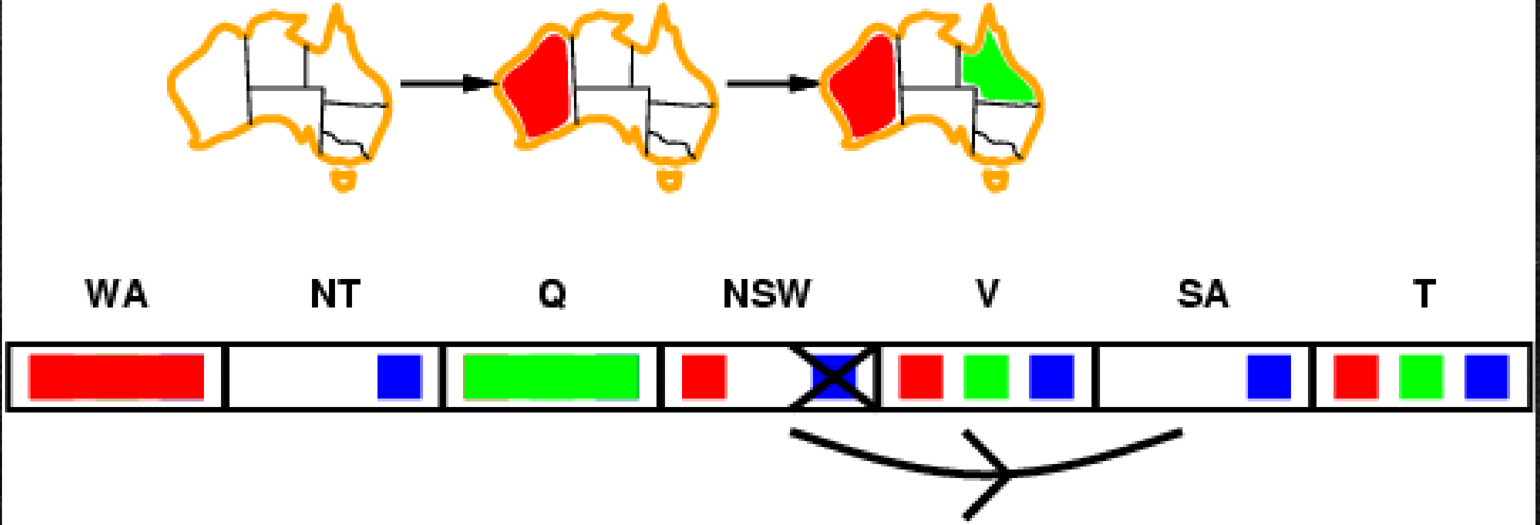


Arcs are directed

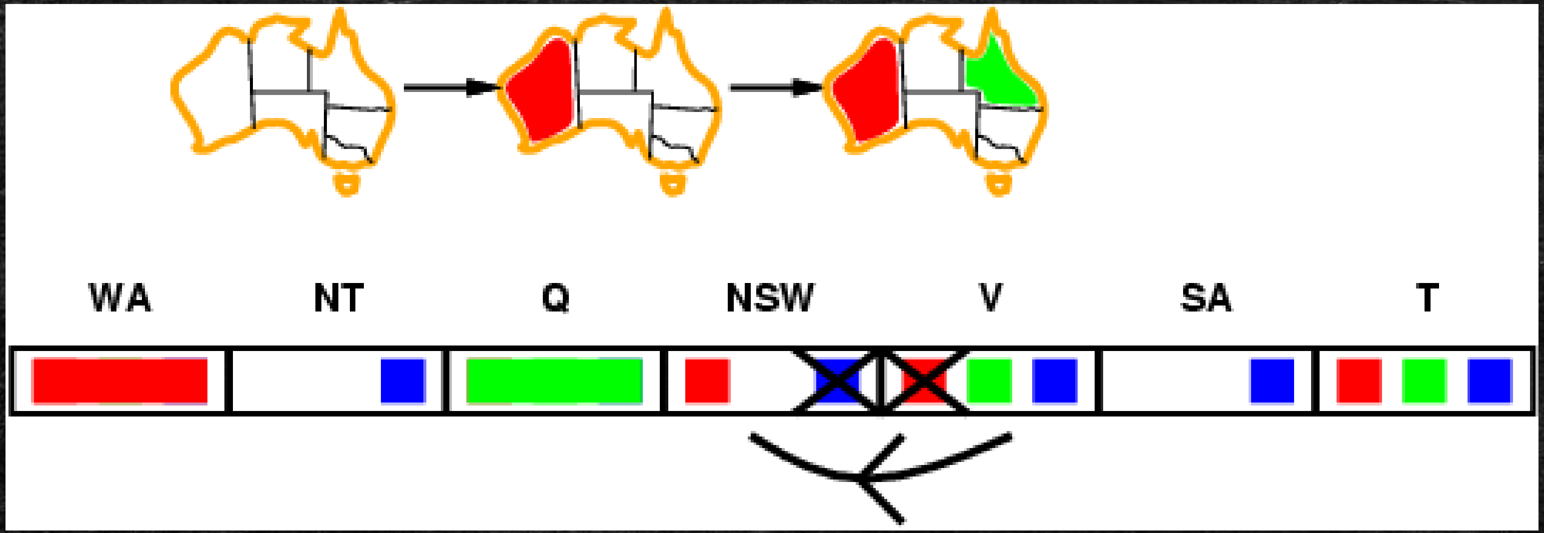
A binary constraint becomes two arcs

Arc (NSW, SA) is originally not consistent

It becomes consistent after deleting NSW = B

Arc Consistency

X_i is arc-consistent *wrt* X_j (i.e., the arc (X_i, X_j) is consistent) *iff* for every value $x \in D_i$ there exists some value $y \in D_j$ that satisfies the binary constraint on the arc (X_i, X_j)



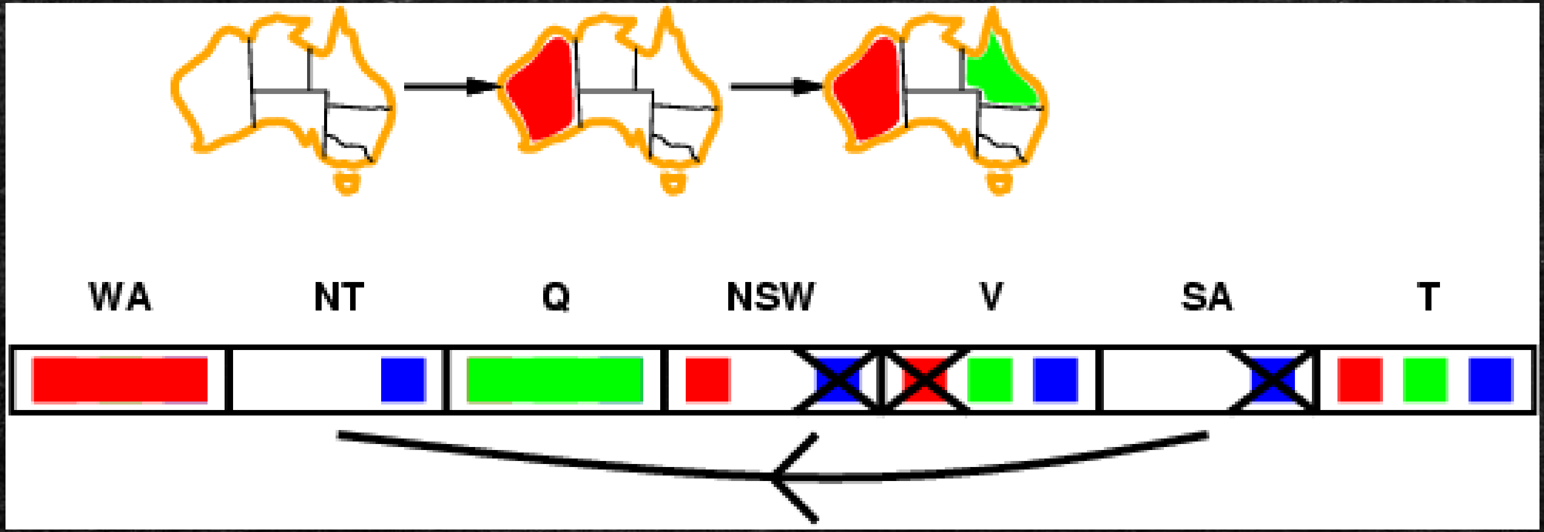
If X_i loses a value, neighbours of X_i need to be (re)checked

After deleting NSW = B, Arc (V, NSW) is originally not consistent

It becomes consistent after deleting V = R

Arc Consistency

X_i is arc-consistent *wrt* X_j (i.e., the arc (X_i, X_j) is consistent) *iff* for every value $x \in D_i$ there exists some value $y \in D_j$ that satisfies the binary constraint on the arc (X_i, X_j)



Arc consistency propagation detects failure earlier than forward checking

Terminal state detection requires the check on ALL arcs

Can be run within backtracking (after each variable assignment), or as a pre-processing step before backtracking begins

Arc Consistency Example 1

- Example
  - $D_a = \{1,3,5\}$
  - $D_b = \{2,4,6\}$
  - $X_a > X_b$

X_a $\{\cancel{1},3,5\}$

arc (X_a, X_b)

arc consistent

X_b $\{2,4,6\}$

X_a $\{3,5\}$

arc (X_b, X_a)

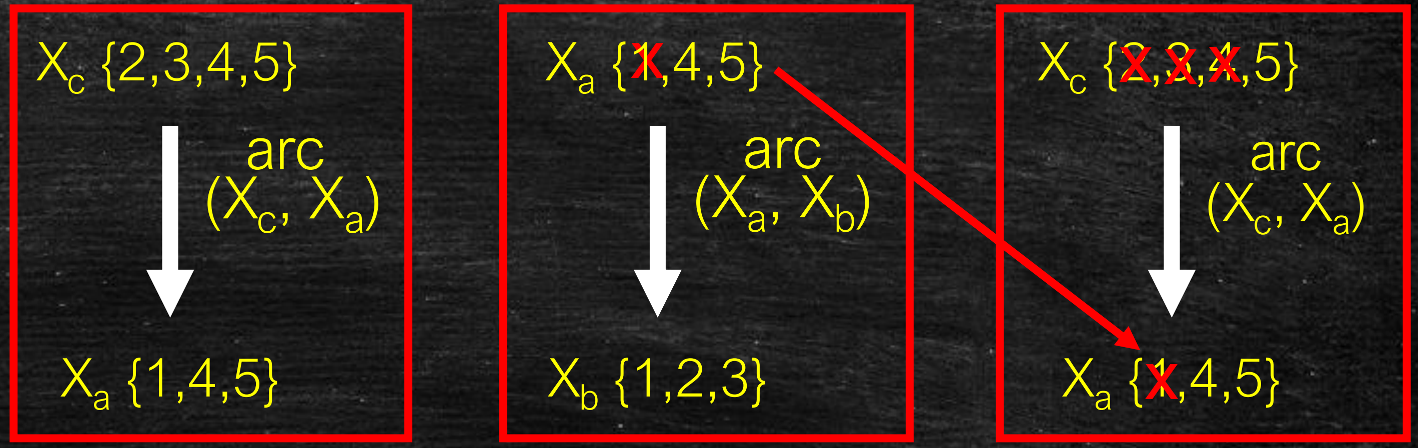
arc consistent

X_b $\{2,4,\cancel{6}\}$

- Should arc (X_a, X_b) be checked again after the update to X_b?
  - No
  - Notice that 1 in D_a was not required to satisfy any value (2,4,6) from D_b
  - Notice that 6 in D_b was not required to satisfy any value (1,3,5) from D_a

Arc Consistency Example 2

- Example
  - $D_a = \{1,4,5\}$
  - $D_b = \{1,2,3\}$
  - $D_c = \{2,3,4,5\}$
  - $X_c > X_a$
  - $X_a > X_b$



arc consistent　　arc consistent　　arc consistent

- Constraint propagation may result in a chain-reaction of domain reductions

Sudoku Chain Reaction

- Consider the following Sudoku puzzle
  - **`Alldiff`** constraint on middle box reduces domain of red square {3,4,5,6,9}
  - Column constraint further reduces it to {4}
  - Box and column constraints on the orange square reduce its domain to {4, 7}
  - Red square further reduces it to {7}
  - Blue box now has domain {1} since the rest of the column is defined

		3		2		6		
9			3		5			1
		1	8		6	4		
		8	1		2	9		
								8
		6	7		8	2		
		2	6		9	5		
8			2		3			9
		5		1		3		

AC-3 Algorithm

```
function AC-3(csp) returns false if an inconsistency is found and true otherwise
  queue ← a queue of arcs, initially all the arcs in csp

  while queue is not empty do
    (Xi, Xj) ← POP(queue)
    if REVISE(csp, Xi, Xj) then
      if size of Di = 0 then return false
      for each Xk in Xi.NEIGHBORS - {Xj} do
        add (Xk, Xi) to queue
  return true

function REVISE(csp, Xi, Xj) returns true iff we revise the domain of Xi
  revised ← false
  for each x in Di do
    if no value y in Dj allows (x,y) to satisfy the constraint between Xi and Xj then
      delete x from Di
      revised ← true
  return revised
```

Initialise a queue containing all arcs (both directions for each binary constraint)

Each time a variable Xi's domain is updated add all arcs corresponding to binary constraints with other variable (not Xi) as target (except the one that just caused the revision)

Eliminating domain values of the target variable X_i relative to the other variable X_j in the binary constraint

Time Complexity of AC-3

- CSPs have at most $2.{}^{n}C_2$ or $O(n^2)$ directed arcs (given n variables)
- Each arc (X_i, X_j) can be inserted at most d times because X_i has at most d values to delete (given domain size d)
  - Checking consistency of an arc (REVISE function) takes $O(d^2)$ time
- AC-3 time complexity:
  - $O(n^2 \times d \times d^2) = O(n^2d^3)$

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Maintaining Arc Consistency (MAC)

- AC usage
  - Pre-processing step before backtracking begins
    - Reduces domain sizes, so reduces size of search tree
  - After each variable assignment within backtracking
    - Inference to update domains
    - Checks for terminal state (i.e., if any domain empty)
      - Backtrack from terminal states
    - Only initialise queue with arcs of neighbouring unassigned variables (relative to current node)

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Appendix

AC-3 Algorithm

function AC-3(*csp*) **returns** false if an inconsistency is found and true otherwise
  queue ← a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ POP(*queue*)
    if REVISE(*csp*, X_i, X_j) **then**
      if size of $D_i = 0$ **then return** *false*
      for each X_k **in** X_i.NEIGHBORS $\boxed{- \{X_j\}}$ **do**
        add (X_k, X_i) to *queue*
  return *true*

function REVISE(*csp*, X_i, X_j) **returns** true iff we revise the domain of X_i
  revised ← *false*
  for each x **in** D_i **do**
    if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j **then**
      delete x from D_i
      revised ← *true*
  return *revised*

Why not enqueue arc (X_j, X_i) ?

On initialisation, we enqueue all arcs

So we have either already checked arc (X_j, X_i) or else it is in the queue

If it is on the queue, then we would check it anyway, so no need to enqueue it again

If we have already checked it, then arc (X_j, X_i) was consistent, and we need to know if any change of the domain of X_i could cause arc (X_j, X_i) to become inconsistent

However, we know that all values removed from X_i did not have values in X_j that satisfied the constraint; therefore, there must not be any value in X_j that requires that value in X_i either

Once both arcs corresponding to a binary constraint have been checked, there is no need to propagate the checking further