Uncertainty II

CS3243: Introduction to Artificial Intelligence – Lecture 10

27 March 2023

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1. Uncertainty Fundamentals
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Reference: AIMA 4th Edition, Section 12; 13.1-13.2

Uncertainty Fundamentals

Recap on Conditional Probabilities & Bayes' Rule

- $\Pr[A \mid B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$ assuming that $\Pr[B] > 0$

Note:
Pr[A | B] = Pr[A ∧ B] / Pr[B] --- (1)
Pr[B | A] = Pr[B ∧ A] / Pr[A] --- (2)

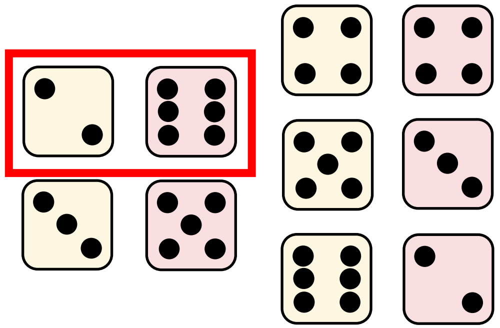
Also, we know:
Pr[A ∧ B] = Pr[B ∧ A] --- (3)

Substituting (3) into (2), we have:
Pr[A ∧ B] = Pr[B | A] . Pr[A] --- (4)

Substituting (4) into (1), we have Bayes' Rule:
P[A|B] = (P[B|A].P[A]) / P[B]

- Bayes' rule: $\Pr[\,A \mid B\,] = \frac{\Pr[B|A]\,\Pr[A]}{\Pr[B]}$

- Example: $\Pr[X_1 = 2 \mid X_1 + X_2 = 8] = \frac{\overbrace{\Pr[X_1{+}X_2{=}8|X_1{=}2]}^{1/6}\,.\,\overbrace{\Pr[X_1{=}2]}^{1/6}}{\underbrace{\Pr[X_1{+}X_2{=}8]}_{5/36}} = \frac{1}{5}$



X_1 = a: Die 1 rolls a
X_2 = b: Die 2 rolls b
a, b ∈ {1,2,3,4,5,6}

Chain Rule

$$\Pr[A \mid B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

- With more than two events, we have

$$\Pr[R_1 \wedge \cdots \wedge R_k] = \Pr[\,(R_1 \wedge \cdots \wedge R_{k-1}) \wedge R_k\,]$$
$$= \Pr[\,R_k \mid R_{k-1} \wedge \cdots \wedge R_1\,]\,.\,\Pr[\,R_{k-1} \wedge \cdots \wedge R_1\,]$$

- And by induction, we have

$$\Pr[R_1 \wedge R_2 \wedge \cdots \wedge R_k] = \prod_{j=1,\ldots,k} \Pr\big[\,R_j \mid R_1 \wedge \cdots R_{j-1}\,\big]$$

- Example:

$$\begin{aligned}\Pr[A \wedge B \wedge C \wedge D] &= \Pr[D|C \wedge B \wedge A].\Pr[C \wedge B \wedge A]\\ &= \Pr[D|C \wedge B \wedge A].\Pr[C|B \wedge A].\Pr[B \wedge A]\\ &= \Pr[D|C \wedge B \wedge A].\Pr[C|B \wedge A].\Pr[B \wedge A]\\ &= \Pr[D|C \wedge B \wedge A].\Pr[C|B \wedge A].\Pr[B|A].\Pr[A]\end{aligned}$$

Independence

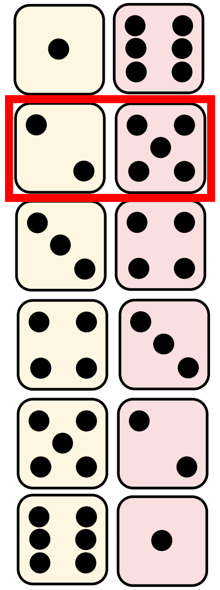
- A and B are independent if $\Pr[A \wedge B] = \Pr[A]\,.\Pr[B]$
- Equivalent to $\Pr[\,A \mid B\,] = \Pr[A]$
- Knowing B adds no information about A
  - B does not further categorise or classify A into sub-categories

Recall that:
P[A|B] = P[A∧B] / P[B]
P[B|A] = P[B∧A] / P[A]

- Example: rolling two dice

$$\Pr[\,X_1 = 2 \mid X_1 + X_2 = 7\,] = ?$$

$$= \frac{\overset{1\,/\,36}{\Pr[X_1=2 \wedge X_1+X_2=7]}}{\underset{6\,/\,36}{\Pr[X_1+X_2=7]}} = \frac{1}{6}$$



Since $\Pr[X_1 = 2] = 1/6$
$[X_1 = 2]$ and $[X_1 + X_2 = 7]$ are independent

Knowing $[X_1 + X_2 = 7]$ does quantify the probability of $[X_1 = 2]$ differently – no new information about $[X_1 = 2]$ is added

Bayesian Inference

Performing Inference via Bayes' Rule

- Instead of inferring statements in the form

"is α true given KB?"

i.e., $R_1 \land \cdots \land R_k \Rightarrow \alpha$?

- We infer statement of the form

"What is the likelihood of an event α given the probabilities of other events?"

i.e., $\Pr[\alpha \mid R_1 \land \cdots \land R_k] = ?$

Naturally occurs in everyday situations …
… e.g., $\Pr$[COVID-19 | Fever ∧ Cough ∧ …]

Inference by Enumeration

- Assuming we have the joint probability distribution

	toothache		¬toothache	
	catch	¬catch	catch	¬catch
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

- For any proposition (event) X, sum the atomic events y where X holds: $\Pr[X] = \sum_{y \in X} \Pr[X = y]$

- $\Pr[\text{toothache}] = 0.108 + 0.016 + 0.012 + 0.064 = 0.2$

Inference by Enumeration

- Assuming we have the joint probability distribution

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	catch	¬catch	catch	¬catch
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

- For any proposition (event) X, sum the atomic events y where X holds: $\Pr[X] = \sum_{y \in X} \Pr[X = y]$

- $\Pr[\neg\text{cavity} \mid \text{toothache}] = \dfrac{\Pr[\neg\text{cavity} \wedge \text{toothache}]}{\Pr[\text{toothache}]}$

$$= \frac{0.016+0.064}{0.2} = 0.4$$

$\Pr[\text{toothache}]$
$= 0.108 + 0.016 + 0.012 + 0.064$
$= 0.2$

Power of Independence

- We have n random variables, $X_1, \dots, X_n$, with domains of size d
  - How big is their joint distribution table?
    - $\underbrace{d \times d \times \dots \times d}_{n \text{ times}} = d^n$

- Suppose that the n random variables, $X_1, \dots, X_n$, are independent
  - How big is the joint distribution table now?
    - $\underbrace{d + d + \dots + d}_{n \text{ times}} = dn$

If A and B are independent:
$\Pr[A \wedge B] = \Pr[A] \,.\, \Pr[B]$
$\Pr[\, A \mid B \,] = \Pr[A]$

We no longer need to know the joint probabilities – e.g., $\Pr[\, A \mid B \,]$

General Idea Behind Bayesian Networks

- Independence is good (if we can find it)
  - Less information (i.e., probabilities) to determine and store
  - Less to enumeration in order to determine probabilities

- Bayesian Networks try to work with some independence

Conditional Independence & Bayes' Rule

Conditional Independence

- Suppose that we test for COVID-19 using two tests
  - Antigen Rapid Test (ART): $A \in \{T, F\}$
  - Breathalyser Test: $B \in \{T, F\}$

As both tests are taken by the same patient, the outcomes of both tests are dependent on whether that patient is ill

- Are they fully independent?
  - Tests were conducted independently
  - Only related by the underlying sickness

But when we assume a patient is ill, the associated probabilities of both tests (on patients who are ill) are now independent!

- A, B are independent ***given*** knowledge of underlying cause S = sick!
  - $\Pr[A \wedge B \mid S] = \Pr[A \mid S].\Pr[B \mid S]$

Conditional Independence

- A joint distribution for n Boolean random variables results in $2^n - 1$ entries

Note that we have $2^n - 1$ since all the values sum to 1 and we can deduce the last value from the rest

- With conditional independence, writing out a full joint distribution using the chain rule becomes

$$\Pr[\, T_1 \wedge \cdots \wedge T_{n-1} \wedge S \,] = \Pr[\, T_1 \mid S \,].\Pr[\, T_2 \mid S \,].\ \ldots\ .\Pr[\, T_{n-1} \mid S \,].\Pr[\, S \,]$$

Notice that we need to store $n - 1$ conditional probabilities for each value of S and (assuming each T_i and S are Booleans), and thus $2(n-1)$ conditional probabilities

With conditional independence: linear

From the example above: $2(n-1) + 1$

Conditional independence is more robust and common than absolute independence

Bayes' Rule & Conditional Independence

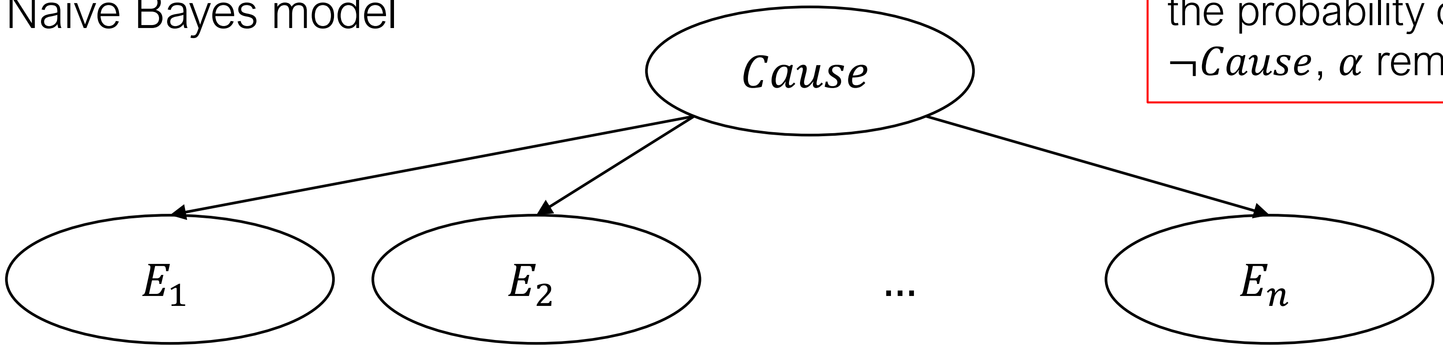
- A cause can have several conditionally independent effects
  - Cause: heavy rain
  - Conditionally independent effects: Alice brings umbrella, Bob brings umbrella, ...

$$\Pr[Cause \mid E_1, \ldots, E_n] = \frac{\Pr[Cause]\Pr[E_1, \ldots, E_n | Cause]}{\Pr[E_1, \ldots, E_n]}$$

$$= \frac{\Pr[Cause]}{\Pr[E_1, \ldots, E_n]} . \prod_i \Pr[E_i \mid Cause] = \alpha . \Pr[Cause] . \prod_i \Pr[E_i \mid Cause]$$

When comparing, for example, the probability of $Cause$ and $\neg Cause$, α remains constant

  - This is the Naive Bayes model



Normalisation under Naive Bayes Algorithm

- Example
  - Suppose we are trying to diagnose a disease in a patient X
    - 70% of the population is $healthy$
    - 20% are $carrier$
    - 10% are $sick$
  - A test will come back $positive$ with the following probability
    - $\Pr[Test(X) = positive \mid X = healthy] = 0.1$
    - $\Pr[Test(X) = positive \mid X = carrier] = 0.7$
    - $\Pr[Test(X) = positive \mid X = sick] = 0.9$
  - Three tests are run (independently) with the following results
    - Two $positive$ (on tests 1 and 2)
    - One $negative$ (on test 3)
  - What is the most likely value for X?

$$\boxed{\begin{array}{l}\Pr[X = healthy] = 0.7\\ \Pr[X = carrier] = 0.2\\ \Pr[X = sick] = 0.1\end{array}}$$

Normalisation

- Example
  - What is the most likely value for X?
  - Need to determine $\Pr[X \mid T_1 = T_2 = 1, T_3 = 0] = \frac{\Pr[X]}{\Pr[T_1=T_2=1,T_3=0]} . \Pr[T_1 = T_2 = 1, T_3 = 0 \mid X]$
  - Notice that $\frac{1}{\Pr[T_1=T_2=1,T_3=0]}$ is constant over each X in $\Pr[X \mid T_1 = T_2 = 1, T_3 = 0]$
  - So only compute $\Pr[X] . \Pr[T_1 = T_2 = 1, T_3 = 0 \mid X]$ for all X
  - As defined earlier, we let $\alpha = \frac{1}{\Pr[T_1=T_2=1,T_3=0]}$

$\Pr[Test(X) = positive \mid X = healthy] = 0.1$
$\Pr[Test(X) = positive \mid X = carrier] = 0.7$
$\Pr[Test(X) = positive \mid X = sick] = 0.9$

$\Pr[X = healthy] = 0.7$
$\Pr[X = carrier] = 0.2$
$\Pr[X = sick] = 0.1$

$$\Pr[X = healthy \mid TestResults] = \alpha \times \underbrace{0.7}_{\Pr[X]} \times \underbrace{0.1}_{\Pr[T_1 = 1 \mid X]} \times \underbrace{0.1}_{\Pr[T_2 = 1 \mid X]} \times \underbrace{0.9}_{\Pr[T_3 = 0 \mid X]} = 0.0063\alpha$$

$$\Pr[X = carrier \mid TestResults] = \alpha \times 0.2 \times 0.7 \times 0.7 \times 0.3 = \boxed{0.0294\alpha}$$

$$\Pr[X = sick \mid TestResults] = \alpha \times 0.1 \times 0.9 \times 0.9 \times 0.1 = 0.0081\alpha$$

Conditional Probability Tables

- Given the chain rule and conditional independence assumption

$$\Pr[\,Cause \mid Effect\,] = \frac{\Pr[Cause]}{\Pr[Effect]} . \Pr[Effect \mid Cause]$$

$$= \alpha . \Pr[Cause] . \prod_{i=1,\ldots,k} \Pr[\,Effect_i \mid Cause\,]$$

- We only need the Conditional Probability Table (CPT) with
  - Each $\Pr[\,Effect_i \mid Cause\,]$
  - $\Pr[Cause]$
  - i.e., $k+1$ entries (assuming k effects)

Bayesian Networks

Representing Bayesian Networks (BN)

- Represent joint distributions via a graph
  - Vertices are random variables
  - An edge from X to Y → X directly influences Y

    some correlation – assume X causes Y

Note that the chain rule implies no cycles (BN is a DAG)

- A conditional distribution for each node given its parents

$$\Pr[\, X \mid Parents(X) \,]$$

Edges link dependent variables

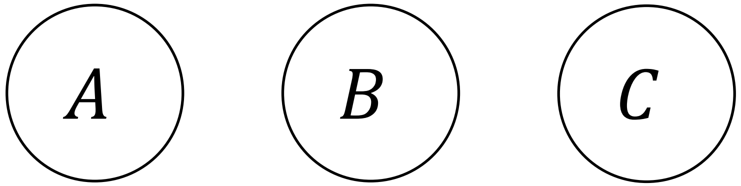
We want lower problem complexity – i.e., fewer parents

- In the simplest case, conditional distribution can be represented as a conditional probability table (CPT)
  - CPTs in the BN are the distribution over X for each combination of parent values

Note that CS3243 will only cover the specification and application of BN (not implementation)

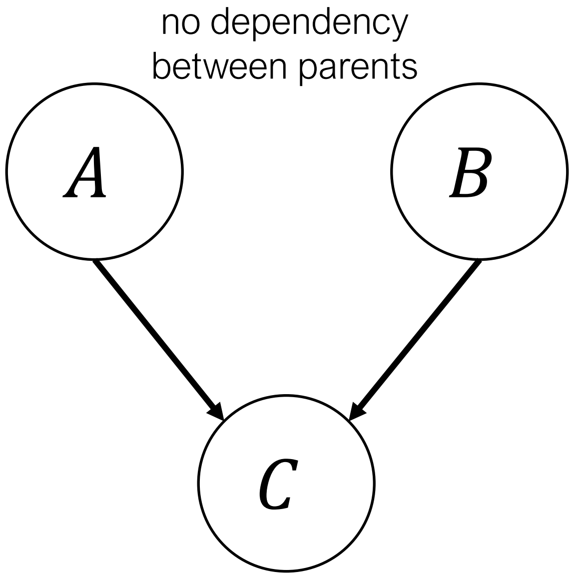
Relationships: Independent Events

- $\Pr[A \wedge B \wedge C] = \Pr[C] \Pr[A] \Pr[B]$



Relationships: Independent Causes

- $\Pr[A \land B \land C] = \Pr[\, C \mid A, B \,] \Pr[A] \Pr[B]$

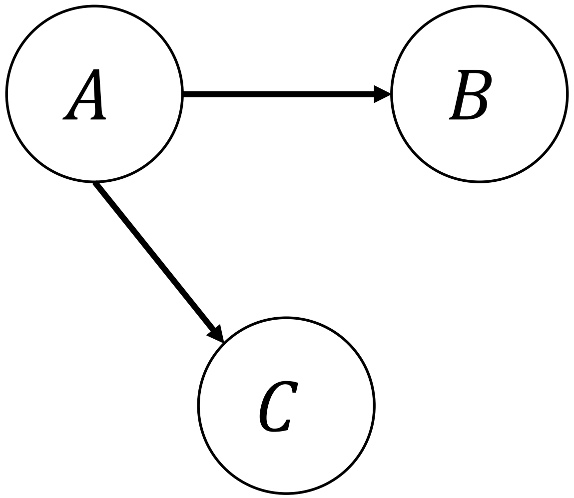


"I can be late either because of rain or because I was sick"

In logic: $A \lor B \to C$

Relationships: Conditionally Independent Effects

- $\Pr[A \wedge B \wedge C] = \Pr[\,C \mid A\,] \Pr[B \mid A] \Pr[A]$

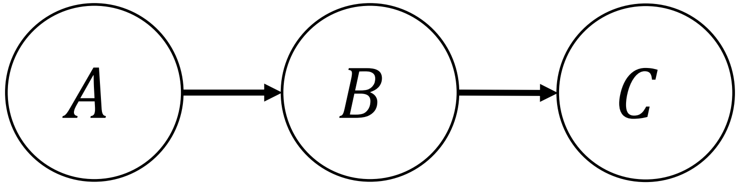


"A disease can cause two independent tests to be positive"

In logic: $A \rightarrow B; A \rightarrow C$

Relationships: Causal Chain

- $\Pr[A \land B \land C] = \Pr[C \mid B] \Pr[B \mid A] \Pr[A]$



Bayesian Networks Example

Example Context

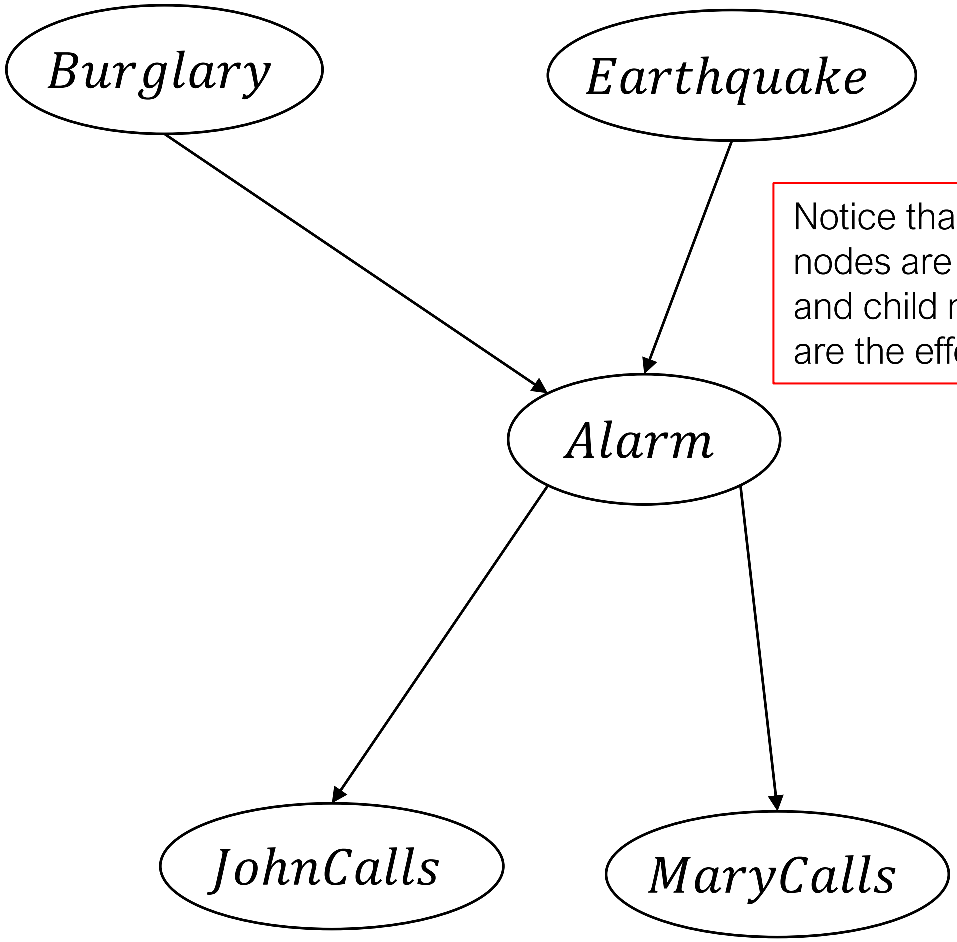
- You are out of the house...
  - J: your neighbour John calls to say your house alarm is ringing
  - M: another neighbour Mary does not call
  - A, E: Alarm sometimes set off by minor earthquake
  - B: Is there a burglar?

- Five binary variables
  - Joint distribution table size is 2^5 - 1

- Use domain knowledge to construct a Bayesian Network
  - Define the dependencies between variables
  - Fewer dependencies → fewer probabilities (i.e., smaller CPTs)

Example Bayesian Network

- Some domain knowledge...
  - The alarm is triggered by a burglary or earthquake
    - Independent Causes
  - John and Mary aren't friendly with each other; assume they do not check with each other before calling
  - They are mindful of privacy, so, would not directly observe a burglary at your house
  - They never notice earthquakes
    - Conditionally Independent Effects



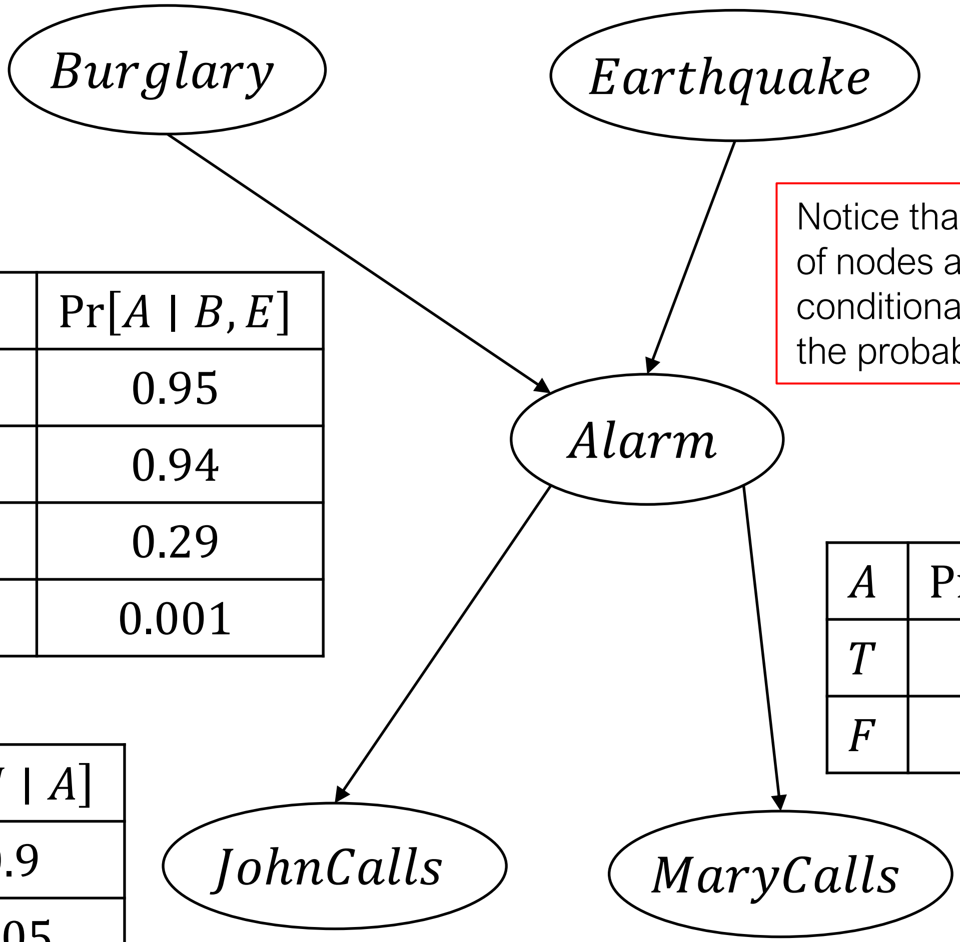
Notice that parent nodes are causes and child nodes are the effects

Example Bayesian Network

- Some domain knowledge...
  - We know the crime rate in the neighbourhood, which gives $\Pr[B]$
  - We know the likelihood of earthquakes where you live, which gives $\Pr[E]$
  - The alarm company provided us with the statistics of the alarm system, which gives $\Pr[A \mid B, E]$
  - Based on experience, we also know how likely John and Mary are to call when the alarm sounds, which gives $\Pr[J \mid A]$ and $\Pr[M \mid A]$ respectively

$\Pr[B]$
0.001

$\Pr[E]$
0.002



Notice that parents of nodes are in the conditional part of the probabilities

B	E	$\Pr[A \mid B, E]$
T	T	0.95
T	F	0.94
F	T	0.29
F	F	0.001

A	$\Pr[J \mid A]$
T	0.9
F	0.05

A	$\Pr[M \mid A]$
T	0.7
F	0.01

Example Bayesian Network

- From the context, we know
  - J = True
  - M = False
- We want to know if
  - B = True
- $\Pr[\, B = 1 \mid J = 1 \wedge M = 0 \,]$

$$= \frac{\Pr[B = 1 \wedge J = 1 \wedge M = 0]}{\Pr[J = 1 \wedge M = 0]}$$

- $\Pr[J, M, A, B, E] =$

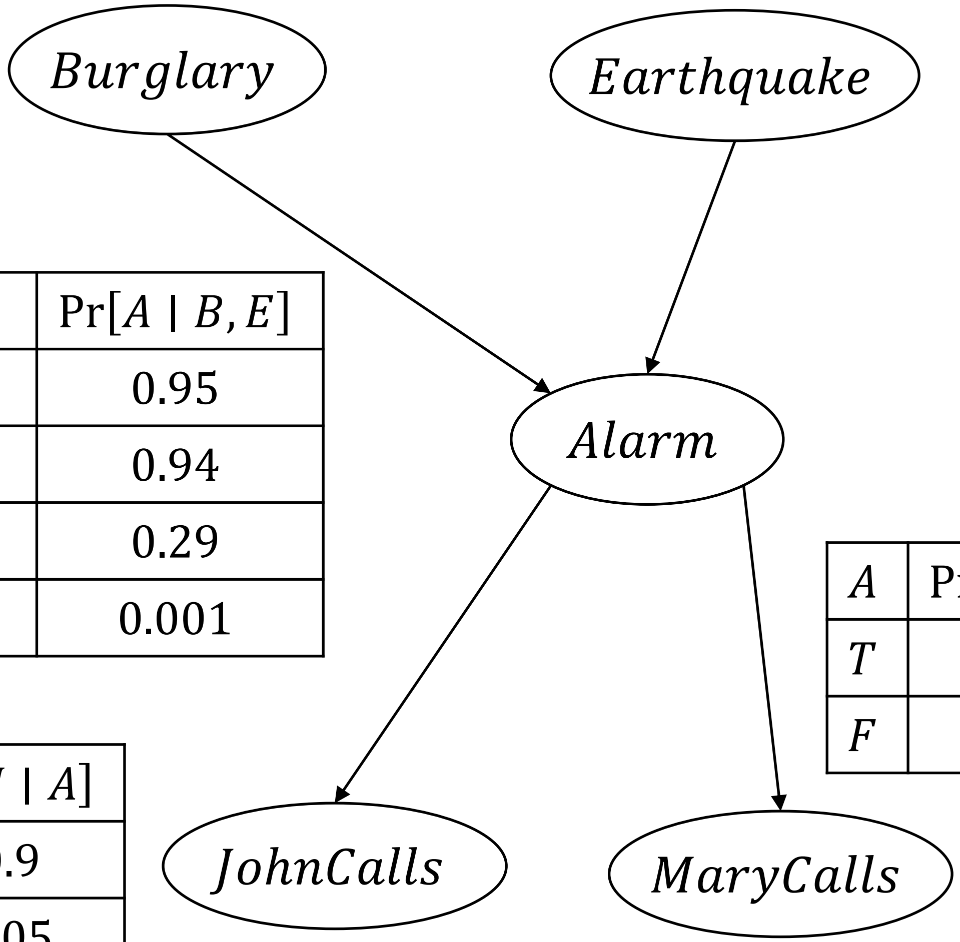
$$\Pr[\, J \mid A \,] . \Pr[\, M \mid A \,] .$$
$$\Pr[\, A \mid B, E \,] . \Pr[B] \Pr[E]$$

Recall that: $\Pr[X \mid Y] = \dfrac{\Pr[X \wedge Y]}{\Pr[Y]}$

Need to also consider $A \wedge E$; so, 4 versions

Given chain rule and conditional dependencies

Remove α from consideration



$\Pr[B]$
0.001

$\Pr[E]$
0.002

B	E	$\Pr[A \mid B, E]$
T	T	0.95
T	F	0.94
F	T	0.29
F	F	0.001

A	$\Pr[J \mid A]$
T	0.9
F	0.05

A	$\Pr[M \mid A]$
T	0.7
F	0.01

Example Bayesian Network

- Calculate $\Pr[\, B = 1 \mid J = 1 \wedge M = 0 \,]$ for
  - $A = 0, E = 0$
  - $A = 1, E = 0$
  - $A = 0, E = 1$
  - $A = 1, E = 1$
- Use $\Pr[J, M, A, B, E]$
  $= \Pr[\, J \mid A \,] \Pr[\, M \mid A \,] \Pr[\, A \mid B, E \,] \Pr[B] \Pr[E]$
- For example, for $A = 1, E = 0$, we have

$$\Pr[B = 1, J = 1, M = 0, A = 1, E = 0]$$
$$= \Pr[\, j \mid a \,] \Pr[\neg m \mid a] \Pr[\, a \mid b, \neg e \,] \Pr[b] \Pr[\neg e]$$
$$= 0.9 \times 0.3 \times 0.94 \times 0.001 \times 0.998 \simeq 0.000253$$

- Compare the sum against similar calculations for $B = 0$

$\Pr[B]$
0.001

$\Pr[E]$
0.002

B	E	$\Pr[A \mid B, E]$
T	T	0.95
T	F	0.94
F	T	0.29
F	F	0.001

A	$\Pr[J \mid A]$
T	0.9
F	0.05

A	$\Pr[M \mid A]$
T	0.7
F	0.01

Compactly Represent Joint Distributions

- Conditional probability for Boolean variable X with k Boolean parents has 2^k rows
  - All possible parent values

- Each row requires one number p for $X = True$

- If each variable has $\leq k$ parents, network representation requires $\mathcal{O}(n2^k)$ values
  - Full joint distribution has $\mathcal{O}(2^n)$ values

Recall that we wanted fewer parents; this is why

- For burglary network, $1 + 1 + 2 + 2 + 4 = 10$ numbers as compared to $2^5 - 1 = 31$ numbers for full joint distribution

We will not delve further into the implementational details of Bayesian Networks for CS3243 – please review Chapter 13 as further reading for more details

Coping with Uncertainty: Decision-Theoretic Agents

Recap on the Issues with Uncertainty

- Deterministic environment
  - No uncertainty
  - Planning is possible
- Stochastic environment
  - Uncertainty in Transition Model
    - Taking an action can lead to different intermediate states
  - Planning is difficult
    - Must now account for all possible intermediate states
    - Model-based agents operating in real-time?
      - Even if not planning, must still model transitions

We now know how to account for uncertainty – i.e., define the likelihood of an intermediate state, but how should we use this?

The Decision-Theoretic Agent

- Rationality in the face of uncertainty
  - Probability Theory – accounting for uncertainty
  - Utility Theory – accounting for value (subjective and dependent on specific agent)
  - Decision Theory = Probability Theory + Utility Theory
- General idea
  - Agent is rational only *iff* it chooses actions that maximise utility
  - Maximum Expected Utility (MEU) Principle
    - Pick action with highest utility weighted over probable outcomes

The Decision-Theoretic Agent

function DT-AGENT(*percept*) **returns** an *action*
  persistent: *belief_state*, probabilistic beliefs about the current state of the world
    action, the agent's action

  update *belief_state* based on *action* and *percept*
  calculate outcome probabilities for actions,
    given action descriptions and current *belief_state*
  select *action* with highest expected utility
    given probabilities of outcomes and utility information
  return *action*

Agent determines an action with each percept (not planning)

Maintains BELIEF STATE – i.e., what it currently believes about the problem environment

Accounting for the uncertainty – e.g., via a Bayesian Network – to determine the likelihood of intermediate states given actions

For each action, a_i, that can be taken at the current state s', determine

$$\sum_{s \in T} utility(s) \times probability(s)$$

where T denotes the set of states that one may transition to from state s' when action a_i is taken

Choose the action that yields the highest value