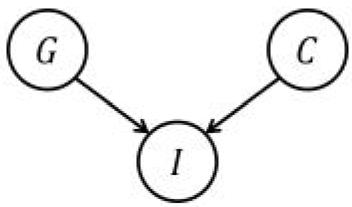
National University of Singapore
School of Computing
CS3243 Introduction to AI

Tutorial 9: Uncertainty and Bayesian Networks (Solutions)

1. Suppose that having both $good\ grades$ (G) and $good\ communication\ skills$ (C) will increase your chances of $performing\ well\ in\ your\ interview$ (I). We thus have the following belief network.



Suppose further that $\Pr[G = 1] = 0.7$, $\Pr[C = 1] = 0.2$, and the probability table of I given G and C is as follows.

G	C	$\Pr[I = 1 \| G, C]$
1	1	0.9
1	0	0.5
0	1	0.5
0	0	0.1

(a) What is the probability that Alice, who has poor grades and communication skills, performs well on her interview?

(b) What is the probability that Bob is a student with great communication skills, assuming we do not know anything about him?

(c) What is the probability that a student has good communication skills, given that he or she has performed well in an interview? Are good communication skills independent of good performance in an interview?

Solution:

(a) Based on the Bayesian network, we have $\Pr[I = 1|G = 0, C = 0] = 0.1$.

(b) As we know nothing else about Bob, the probability is simply $\Pr[C = 1] = 0.2$.

(c) We want to determine $\Pr[C = 1|I = 1]$.

$$\Pr[C = 1|I = 1] = \frac{\Pr[C = 1, I = 1]}{\Pr[I = 1]}$$

$$= \frac{\sum_{g \in Dom(G)} \Pr[G = g, C = 1, I = 1]}{\sum_{g \in Dom(G), c \in Dom(C)} \Pr[G = g, C = c, I = 1]}$$

$$= \frac{0.3 \times 0.2 \times 0.5 + 0.7 \times 0.2 \times 0.9}{0.3 \times 0.8 \times 0.1 + 0.3 \times 0.2 \times 0.5 + 0.7 \times 0.8 \times 0.5 + 0.7 \times 0.2 \times 0.9}$$

$$= \frac{0.156}{0.46}$$

$$= 0.339$$

The probability of having good communication skills given a good interview performance is greater than the probability of having good communication skills without any information about interview performance. Thus, having good communication skills is not independent of interview performance.

2. Assume that 2% of the population in a country carry a particular virus. A test kit developed by a pharmaceutical firm is able to detect the presence of the virus from a patient's blood sample. The firm claims that the test kit has a high accuracy of detection in terms of the following conditional probabilities obtained from their quality control testing:

$$\Pr[\text{the kit shows positive}|\text{the patient is a carrier}] = 0.998$$
$$\Pr[\text{the kit shows negative}|\text{the patient is not a carrier}] = 0.996$$

(a) Given that a patient is tested to be positive using this kit, what is the posterior belief that he is not a carrier? Give your answer to 3 decimal places.

Solution: Let X and $\neg X$ represent the test kit shows positive and negative, respectively. Let Y and $\neg Y$ represent the patient is a carrier and not a carrier, respectively. Then,

$$\begin{aligned}\Pr[Y] =& 0.02 \Rightarrow \Pr[\neg Y] = 0.98\\ \Pr[X|Y] =& 0.998\\ \Pr[\neg X|\neg Y] =& 0.996 \Rightarrow \Pr[X|\neg Y] = 0.004\end{aligned}$$

Applying Bayes' Rule,

$$\begin{aligned}\Pr[\neg Y|X] =& \frac{\Pr[X|\neg Y]\Pr[\neg Y]}{\Pr[X]} = \frac{\Pr[X|\neg Y]\Pr[\neg Y]}{\Pr[X|Y]\Pr[Y] + \Pr[X|\neg Y]\Pr[\neg Y]}\\ =& \frac{0.004 \times 0.98}{0.998 \times 0.02 + 0.004 \times 0.98} \simeq 0.164\end{aligned}$$

There is an approximately 16.4% chance that when the test kit shows positive, the patient is not a carrier.

(b) Suppose that the patient doesn't entirely trust the result offered by the first kit (perhaps because it has expired) and decides to use another test kit. If the patient is again tested to be positive using this second kit, what is the (updated) likelihood that he is not a carrier? You can assume conditional independence between results of different test kits given the patient's state of virus contraction. Give your answer to 4 decimal places.

Solution: We model the two tests as X_1 and X_2 (both independent instantiations of the original test X).

$$\begin{aligned}\Pr[\neg Y \mid X_1 \wedge X_2] =& \frac{\Pr[\neg Y \wedge X_1 \wedge X_2]}{\Pr[X_1 \wedge X_2]} = \frac{\Pr[X_1 \mid \neg Y \wedge X_2]\Pr[X_2 \mid \neg Y]\Pr[\neg Y]}{\Pr[X_1 \wedge X_2]} \quad (1)\\ =& \frac{\Pr[X_1 \mid \neg Y]\Pr[X_2 \mid \neg Y]\Pr[\neg Y]}{\Pr[X_1 \wedge X_2]} \quad (2)\end{aligned}$$

The transition from (1) to (2) is due to the fact that X_2 is conditionally independent of X_1 given Y, so $\Pr[X_2 \mid \neg Y \wedge X_1] = \Pr[X_2 \mid \neg Y]$. Next, by the law of total probability:

$$\begin{aligned}(2) =& \frac{\Pr[X_1 \mid \neg Y]\Pr[X_2 \mid \neg Y]\Pr[\neg Y]}{\Pr[X_1 \wedge X_2 \wedge Y] + \Pr[X_1 \wedge X_2 \wedge \neg Y]} \quad (3)\\ =& \frac{\Pr[X_1 \mid \neg Y]\Pr[X_2 \mid \neg Y]\Pr[\neg Y]}{\Pr[X_1 \mid Y]\Pr[X_2 \mid Y]\Pr[Y] + \Pr[X_1 \mid \neg Y]\Pr[X_2 \mid \neg Y]\Pr[\neg Y]} \quad (4)\end{aligned}$$

The transition from (3) to (4) is similar to the one we used to transition from (1) to (2). Finally, we note that X_1 and X_2 are both distributed as X, so

$$(4) = \frac{\Pr[X \mid \neg Y]^2 \Pr[\neg Y]}{\Pr[X \mid Y]^2 \Pr[Y] + \Pr[X \mid \neg Y]^2 \Pr[\neg Y]} \tag{5}$$

$$= \frac{0.004^2 \times 0.98}{0.004^2 \times 0.98 + 0.998^2 \times 0.02} \simeq 0.0008 \tag{6}$$

3. Assume that the following conditional probabilities are available:

P(Wet_Grass | Sprinkler $\wedge$ Rain)	0.95
P(Wet_Grass | Sprinkler $\wedge$ $\neg$ Rain)	0.9
P(Wet_Grass | $\neg$ Sprinkler $\wedge$ Rain)	0.8
P(Wet_Grass | $\neg$ Sprinkler $\wedge$ $\neg$ Rain)	0.1
P(Sprinkler | Rainy_Season)	0.0
P(Sprinkler | $\neg$ Rainy_Season)	1.0
P(Rain | Rainy_Season)	0.9
P(Rain | $\neg$ Rainy_Season)	0.1
P(Rainy_Season)	0.7

Construct a Bayesian network and determine the probability

P(Wet_Grass $\wedge$ Rainy_Season $\wedge$ $\neg$ Rain $\wedge$ $\neg$ Sprinkler).

Solution: Let RS, S, R, and WG denote Rainy_Season, Sprinkler, Rain, and Wet_Grass respectively. First, let us observe the following lemma

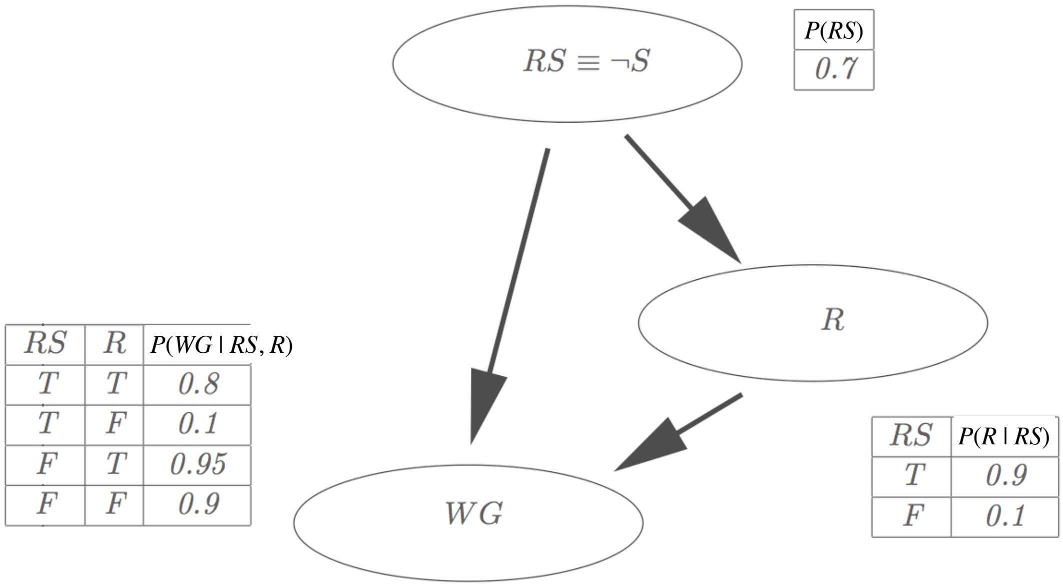
Lemma 1. *Given two random boolean variables A and B, if* $\Pr[A \mid B] = 0$ *and* $\Pr[A \mid \neg B] = 1$ *then* $\Pr[A] = 1 - \Pr[B]$*; in fact,* $A \equiv \neg B$.

Proof. It suffices to show that $\Pr[A] = \Pr[\neg B]$ since both random variables are boolean.

$$\Pr[A] = \Pr[A \wedge B] + \Pr[A \wedge \neg B] = \Pr[A \mid B] \Pr[B] + \Pr[A \mid \neg B] \Pr[\neg B]$$

Substituting $\Pr[A \mid B] = 1$ and $\Pr[A \mid \neg B] = 0$ into the expression we obtain that $\Pr[A] = \Pr[\neg B]$. Note that since $\Pr[A \mid B] = 0$, we know that $A = 0$ whenever $B = 1$, and since $\Pr[A \mid \neg B] = 1$ we know that $A = 1$ whenever $B = 0$; this is essentially saying that $A \equiv \neg B$. $\square$

According to the above lemma we have that $S \equiv \neg RS$.



$P(RS)$
0.7

RS	$P(R \mid RS)$
T	0.9
F	0.1

RS	R	$P(WG \mid RS, R)$
T	T	0.8
T	F	0.1
F	T	0.95
F	F	0.9

$$\begin{aligned}\Pr[WG \wedge RS \wedge \neg R \wedge \neg S] &= \Pr[WG \mid RS \wedge \neg R \wedge \neg S]\Pr[RS \wedge \neg R \wedge \neg S]\\ &= \Pr[WG \mid RS \wedge \neg R \wedge \neg S]\Pr[\neg R \mid RS \wedge \neg S]\Pr[RS \wedge \neg S]\end{aligned} \tag{7}$$

Since RS and $\neg S$ are equivalent, $\Pr[RS \wedge \neg S] = \Pr[RS]$. Thus, (7) equals

$$\Pr[WG \mid RS \wedge \neg R \wedge \neg S]\Pr[\neg R \mid RS \wedge \neg S]\Pr[RS] = 0.1 \times (1 - 0.9) \times 0.7 = 0.007$$

4. An expert system called PROSPECTOR for use in geological exploration makes use of an inference mechanism similar to a Bayesian Network. The following are two modified versions of its rule patterns:

   If E1
   Then H1 (P(H1 | E1), P(H1 | ¬E1))

   If E2
   and E3
   Then H2 (P(H2 | E2 ∧ E3), P(H2 | E2 ∧ ¬E3), P(H2 | ¬E2 ∧ E3), P(H2 | ¬E2 ∧ ¬E3))

   The following is a hypothetical set of PROSPECTOR's rules (where we also use two letters to represent propositions for your easy working later)

   > If the igneous rocks in the region have a fine to medium grain size (Gr)
   > Then they have a porphyritic texture (Tx) (0.6, 0.2)
   >
   > If the igneous rocks in the region have a fine to medium grain size (Gr)
   > and they have a porphyritic texture (Tx)
   > Then the region is a hypabyssal environment (Hy) (0.88, 0.76, 0.52, 0.02)
   >
   > If the region is a hypabyssal environment (Hy)
   > Then the region has a favourable level of erosion (Er) (0.75, 0.12)
   >
   > If the region has a favourable level of erosion (Er)
   > Then the region is favourable for copper deposits (Cu) (0.92, 0.03)

   Assume that a geologist could only ascertain with probability 0.15 that a region's igneous rocks have a fine to medium grain size.

   (a) Construct a Bayesian network based on the above rules.

   (b) Determine the probability that this region is favourable for copper deposits and has a favourable level of erosion, given that the region (1) has large grain size igneous rocks, (2) has non-porphyritic texture rocks, and (3) is a hypabyssal environment.

Solution: We wish to compute the probability that $Cu \wedge Er$ occurs, given the event $Gr \wedge \neg Tx \wedge Hy$. In other words, we need to compute

$$\Pr[Cu \wedge Er \mid Gr \wedge \neg Tx \wedge Hy]$$

First, the Bayesian network we construct is described in Figure 1.

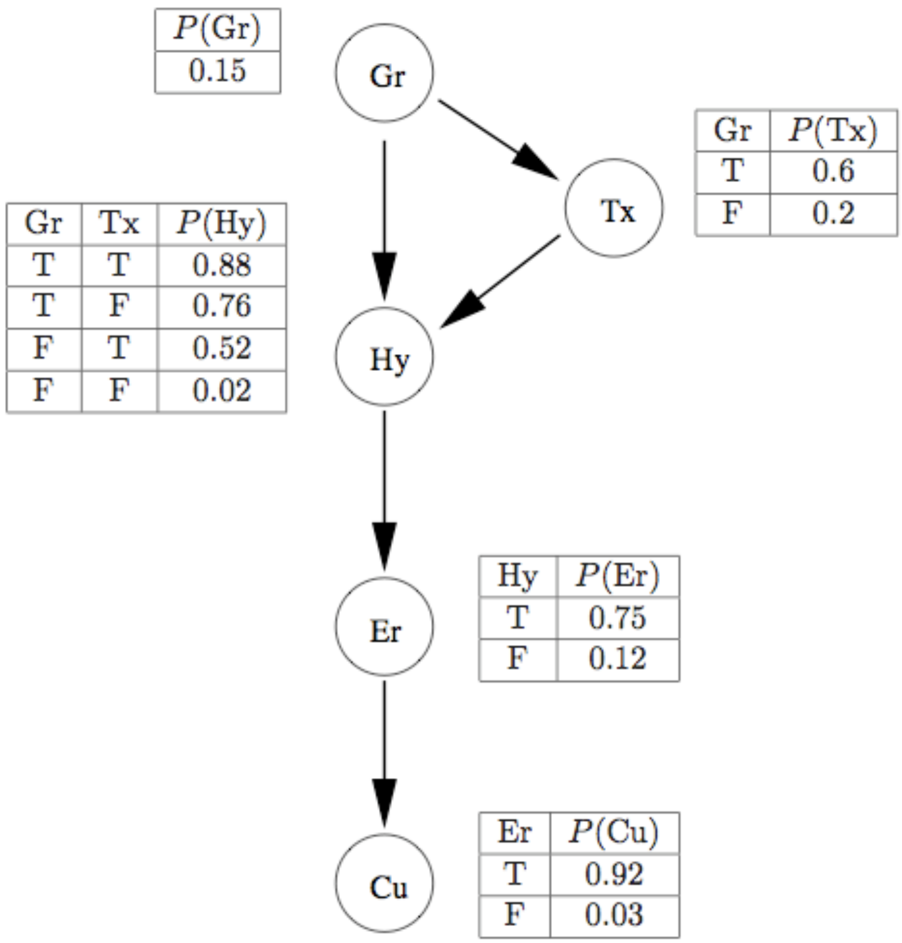


Figure 1: The Bayesian network for the PROSPECTOR system in Question 3

Next, recall that the conditional probability of a variable X depends only on $Parents(X)$; thus:

$$\Pr[Cu \mid Er \wedge \neg Gr \wedge \neg Tx \wedge Hy] = \Pr[Cu \mid Er]$$
$$\Pr[Er \mid \neg Gr \wedge \neg Tx \wedge Hy] = \Pr[Er \mid Hy]$$

We observe that for any three events A, B, C:

$$\begin{aligned}
\Pr[A \wedge B \mid C] &= \frac{\Pr[A \wedge B \wedge C]}{\Pr[C]} = \frac{\Pr[A \wedge B \wedge C]}{\Pr[B \wedge C]} \times \frac{\Pr[B \wedge C]}{\Pr[C]} \\
&= \Pr[A \mid B \wedge C] \times \Pr[B \mid C]
\end{aligned}$$

Thus,

$$\begin{aligned}
\Pr[Cu \wedge Er \mid \neg Gr \wedge \neg Tx \wedge Hy] &= \Pr[Cu \mid Er \wedge \neg Gr \wedge \neg Tx \wedge Hy] \Pr[Er \mid \neg Gr \wedge \neg Tx \wedge Hy] \\
&= \Pr[Cu \mid Er] \times \Pr[Er \mid Hy] = 0.92 \times 0.75 \simeq 0.69
\end{aligned}$$